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Powerfully Nilpotent p-groups

Williams, James

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James L I Williams

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Contents

Abstract

In this thesis we introduce a special class of powerful p-groups that we call *powerfully nilpotent groups*. These groups are finite p-groups which possess a central series of a special kind. To these groups we can attach the notion of a powerful nilpotence class and this leads naturally to a classification in terms of an "ancestry tree" and powerful coclass. We show that there are finitely many powerfully nilpotent groups for any given powerful coclass. We also develop the general theory for powerfully nilpotent groups. For odd primes p we classify the powerfully nilpotent groups of rank 2 and also the powerfully nilpotent groups of order at most p^6. We determine the growth of powerfully nilpotent groups of exponent p^2 and order p^n where p is odd. The number of these is $f(n) = p^{\alpha n^3 + o(n^3)}$ where $\alpha = \frac{9+4\sqrt{2}}{394}$. For the larger class of all powerful groups of exponent p^2 and order p^n, where p is odd, the number is $p^{\frac{2}{27}n^3 + o(n^3)}$. Thus here the class of powerfully nilpotent p-groups is large, while being sparse within the larger class of powerful p-groups. We show that many characteristic subgroups of powerful p-groups are powerfully nilpotent, and investigate when the Omega subgroups of the Agemo subgroups of a powerful p-group are powerfully nilpotent. The study of these groups gives an example of characteristic subgroups of powerful p-groups which are powerfully nilpotent without being strongly powerful.

1. Introduction

1.1. Overview

In this thesis we introduce a special class of powerful p-groups, which we term "Powerfully Nilpotent" groups. These groups possess a special kind of central series. We develop the theory of powerful nilpotence and powerful nilpotency class. We utilise the idea of an ancestry tree and powerful coclass, which leads to a weak classification of these groups. One of the main results of this thesis is that for any given powerful coclass, there are only finitely many groups.

We now include a brief overview of each chapter:

- In Chapter 2 we introduce what it means for a group to be powerfully nilpotent, and develop the main structure theory for powerfully nilpotent groups. In particular we develop a notion of powerful coclass, and show that the rank and exponent of a powerfully nilpotent group are both bounded in terms of the powerful coclass. It follows that there are only finitely many groups of a given powerful coclass. We also show that all powerfully nilpotent groups must have a presentation of a particular form, which then enables us to determine the growth of the powerful nilpotent groups of exponent p^2. The notion of a "maximal tail" of a powerfully nilpotent group is introduced and a conjecture about tail length is posed. The material in this chapter is a joint work with Gunnar Traustason, and the bulk of the material in this chapter forms the joint author paper "Powerfully Nilpotent Groups", which has been published in the Journal of Algebra [TW19].

- In Chapter 3 we classify the powerfully nilpotent groups of rank 2, for p odd. We enumerate these groups and obtain a closed formula for the exact number of groups of a given order. Finally we consider the ancestry tree and classify the children and parents of rank 2 powerfully nilpotent groups. In doing so we are able to identify when infinite branches on the ancestry tree occur and what these infinite branches must look like. The material in this chapter is a joint work with Gunnar Traustason, and is currently being prepared for publication.

- In Chapter 4 we show that for any powerful p-group G, the subgroup $\Omega_i(G^{p^j})$ is powerfully nilpotent for all $i, j \geq 1$ when p is an odd prime, and for $i \geq 1$, $j \geq 2$ when $p = 2$. We provide an example to show why this modification is needed in the case $p = 2$. Furthermore we obtain a bound on the powerful nilpotency class of $\Omega_i(G^{p^j})$. At the time of writing, this work has been accepted with minor changes to the International Journal of Group Theory [Wil18].

- In Chapter 5 we demonstrate how the computer algebra system "GAP" can be used to form conjectures and find examples. We prove some results about the powerful coclass of specific families of powerfully nilpotent groups.

- In Chapter 6 we classify all powerfully nilpotent groups of order less than or equal to p^6 for odd primes p. The material in this chapter is a joint work with Gunnar Traustason, and is currently being prepared for publication.
- In Chapter 7 we outline the algorithm we use to determine whether or not a group is powerfully nilpotent, and provide a GAP implementation.
- In Chapter 8 we discuss some open problems and further areas for investigation.

During this project extensive use has been made of the Computer Algebra System, "GAP" (Groups Algorithm Procedures) [GAP18]. Data obtained from GAP has enabled us to form conjectures, or to find counterexamples. In writing this thesis I have aimed to maintain this hands-on "experimental" feel, by including tables of data and examples which motivate the theory or prove that results are sharp.

1.2. Main Results

The main results of this thesis are as follows:

- Bounding the rank and exponent of any powerfully nilpotent group in terms of the powerful coclass. Thus in particular we are able to deduce that there are only finitely many groups of any given powerful coclass.
- Determining the growth for the powerfully nilpotent groups of exponent p^2 and order p^n. In particular we show that the number of these is $f(n) = p^{\alpha n^3+o(n^3)}$ where $\alpha = \frac{9+4\sqrt{2}}{394}$. A similar argument shows that for powerful p-groups of exponent p^2 and order p^n the growth is $p^{\frac{2}{27}n^3+o(n^3)}$. Thus we see that although the class of powerfully nilpotent groups is large, it is sparse when compared to the larger class of powerful groups.
- Classifying the powerfully nilpotent groups of rank 2 and all powerfully nilpotent groups of order at most p^6 for odd primes p.
- Showing that for any powerful p-group G, the subgroup $\Omega_i(G^{p^j})$ is powerfully nilpotent for all $i, j \geq 1$ when p is an odd prime, and for $i \geq 1$, $j \geq 2$ when $p = 2$. In particular this gives an example of a characteristic subgroup of a powerful p-group which is powerfully nilpotent but not strongly powerful.

1.3. Preliminaries

In this section we recap some results about p-groups and powerful p-groups, which will be used in the rest of the thesis.
For the original paper introducing the notion of a powerful p-group, see [LM87]. For a textbook introduction to powerful p-groups, and for the proofs which we omit, see [DDSMS03] and [Khu98].
First we set up notation and terminology. For a group G, we denote the centre of G by $Z(G)$, the commutator subgroup of G by G', and G^n denotes the subgroup generated

by all nth powers of elements of G. For a p-group G, the group G^{p^i} is sometimes denoted as $\mho_i(G)$ and is known as the ith Agemo subgroup of G. The ith Omega subgroup of G, denoted $\Omega_i(G)$, is the subgroup generated by all elements of G whose order divides p^i. The exponent of G is denoted by $\exp G$, and will often be written as p^e for some integer $e \geq 0$. If we have a group G, a normal subgroup $N \trianglelefteq G$ and an element $a \in G$, we may denote the image of a in G/N as $\overline{a}$.

Definition 1. A subgroup N of a finite p-group G is *powerfully embedded* in G if $N^p \geq [N, G]$ (for $p = 2$, if $N^4 \geq [N, G]$). A finite p-group G is *powerful* if it is powerfully embedded in itself, that is if $[G, G] \leq G^p$ (for $p = 2$, $[G, G] \leq G^4$).

Proposition 2. *If M and N are powerfully embedded subgroups in a finite p-group G, then*
(a) $[M, N]$ is powerfully embedded in G;
(b) M^p is powerfully embedded in G;
(c) MN is powerfully embedded in G.

Proof. See [Khu98], Theorem 11.4. □

Theorem 3. *If G is a powerful p-group, then the subgroup G^p coincides with the set $\{x^p | x \in G\}$ of p^{th} powers of elements of G.*

Proof. See [Khu98], Theorem 11.9. □

Lemma 4. *Suppose that N is powerfully embedded in G where G is a finite p-group. Suppose $N = \langle S \rangle^G$ for some $S \subseteq N$. Then $N = \langle S \rangle$.*

Proof. For each $x \in S$ and $g \in G$, we have $[x, g] \in N^p = \Phi(N)$. Hence $N = \langle S, \Phi(N) \rangle = \langle S \rangle$. □

The following result is used many times, often without explicit mention.

Lemma 5 (Shalev's interchange lemma)**.** *If M, N are powerfully embedded subgroups of a finite p-group G, then $[M^{p^i}, N^{p^j}] = [M, N]^{p^{i+j}}$ for all $i, j \in \mathbb{N}$.*

Proof. See [Sha93] or [Khu98] Lemma 11.12. □

Proposition 6. *Suppose that G is a 2-generator powerful group. Then G is metacyclic.*

Proof. Suppose $G = \langle a, b \rangle$. Using Proposition 2 we see that G' is powerfully embedded in G. It follows that $G' = \langle [a, b] \rangle^G = \langle [a, b] \rangle$, by Lemma 4. Choose r maximal so that $[a, b] \in G^{p^r}$. Let $[a, b] = u^{p^r} \in G^{p^r}$. Let $N = \langle u \rangle$. Then N is cyclic, and normal in G because it contains G'. Furthermore, notice that if $u \in G^p$, then we could write $u = g^p$ for some $g \in G$, and then $[a, b] = g^{p^{r+1}}$, contradicting the maximality of r. Thus we must have $u \notin G^p$, that is, u is not in the Frattini subgroup of G. Hence there exists some $v \in G$ such that $G = \langle u, v \rangle$, with G/N cyclic. □

2. Powerful Nilpotence

2.1. Motivation

One motivation for the thesis is as follows. When studying a type of linear structure known as a Symplectic Alternating Algebra, it was observed that there was a bijective correspondence between Symplectic Alternating Algebras and a certain class $\mathcal{C}$ of powerful 3-groups. The symplectic alternating algebras which are nilpotent are in bijective correspondence with a subclass of $\mathcal{C}$. It turns out that all groups in this subclass exhibit a property which we call *powerful nilpotence*. However this property of powerful nilpotence is not limited to this subclass, and in fact can be studied as a property in its own right. The goal of this thesis is to study this as a general property of groups and to better understand what it means for a group to be powerfully nilpotent.
For more details on Symplectic Alternating Algebras and the class $\mathcal{C}$ of groups, see [ST15, Tra08, TM08].

2.2. Basic results

We begin this section by introducing the notion of powerful nilpotence and developing the basic theory of powerfully nilpotent groups. Let G be a finite p-group where p is any prime, not necessarily odd.

Definition 7. Let $H \leq K \leq G$. Consider an ascending chain of subgroups:

$$H = H_0 \leq H_1 \leq \cdots \leq H_n = K.$$

If this chain has the property that, for each $i \in \{1, \dots n\}$, we have that $[H_i, G] \leq H_{i-1}^p$ then we call this a *powerfully central ascending chain of length n in G.*
Similarly consider a descending chain of subgroups of G:

$$K = H_1 \geq H_2 \geq \cdots \geq H_{n+1} = H.$$

If this chain has the property that, for each $i \in \{1, \dots n\}$ we have that $[H_i, G] \leq H_{i+1}^p$then we call this a *powerfully central descending chain of length n in G.* We call chains with either of these properties *powerfully central.*

Definition 8. A powerful p-group G is *powerfully nilpotent* if it has a powerfully central ascending chain of the form:

$$\{1\} = H_0 \leq H_1 \leq \cdots \leq H_n = G.$$

If G is powerfully nilpotent then the *powerful nilpotence class* of G is the shortest length that a powerfully central chain of G can have.

Remark 9. Notice that for p odd, the requirement that G be powerful is not needed, since if G has some powerfully central ascending chain

$$1 = H_0 \leq H_1 \leq \cdots \leq H_n = G,$$

then in particular we have that $[G,G] \leq H_{n-1}^p \leq G^p$. Thus the group G is powerful. Similarly, each subgroup in the chain is powerfully embedded in itself, and hence powerful.

Example 10. $G = \{1\}$ is powerfully nilpotent and has powerful nilpotency class 0.

Example 11. Let G be any non-trivial abelian p-group. Then we have that $\{1\} \leq G$ is an ascending chain of length 1 and $[G,G] = \{1\} \leq G^p$. Hence G is powerfully nilpotent with powerful nilpotency class 1. Conversely suppose G is of powerful nilpotence class 1. Then G must have a powerfully central ascending chain of length 1, and the chain must be exactly $1 \leq G$ with $[G,G] \leq 1$. Thus a non trivial p-group G is powerfully nilpotent of class 1 if and only if it is abelian.

Remark 12. It turns out that every powerful 2-group is powerfully nilpotent. To see why this is the case, consider a 2-group G of exponent 2^e and the chain of subgroups:

$$\{1\} = G^{2^e} \leq G^{2^{e-1}} \leq \cdots \leq G^2 \leq G.$$

This chain is powerfully central, because by the Interchange Lemma 5, we know that $[G, G^{2^k}] = [G,G]^{2^k} \leq (G^{2^2})^{2^k} = (G^{2^{k+1}})^2$. Notice also that if $e \geq 2$ then $[G^{2^{e-2}}, G] = [G,G]^{2^{e-2}} \leq G^{2^e} = \{1\}$ and thus $G^{2^{e-2}} \leq Z(G)$ and the powerful class of G is at most $e-1$. Having seen that all powerful 2-groups are powerfully nilpotent, we will normally assume in the following that a given prime p is odd except otherwise stated.

In a similar way as for standard nilpotence, there exists a fastest ascending powerfully central series.

Definition 13. Let G be a finite p-group. We define the *higher powerful centres* of G recursively by:

$$\begin{aligned} \hat{Z}_0(G) &= \{1\} \\ \hat{Z}_{i+1}(G) &= \{g \in G : [g,x] \in \hat{Z}_i(G)^p \, \forall x \in G\} \text{ for } i \geq 0. \end{aligned}$$

We call the ascending chain formed by the higher powerful centres the *upper powerfully central series of G.*

The examples we have seen so far have been abelian. For $p = 3$ the smallest non-abelian examples occur for order p^4 where there are two distinct powerfully nilpotent groups. An example of one such group is below.

Example 14. Consider the 2-generator group, $G = \langle a, b | a^{27} = 1, b^3 = 1, [b,a] = a^9 \rangle$. The centre of this group is $Z(G) = \langle a^3 \rangle$, and it is easy to see that the series $\{1\} \leq Z(G) \leq G$ is powerfully central.

Proposition 15. *The centre of a non-abelian, powerfully nilpotent group G cannot have exponent p.*

Proof. We argue by contradiction and suppose G is a non-abelian powerfully nilpotent group with $Z(G)^p = 1$. Consider the upper powerfully central series for G

$$1 < Z(G) < \hat{Z}_2(G) \leq \cdots \leq G.$$

In particular, as G is not abelian, we know that $G \neq Z(G)$ and so there exists some element $x \in \hat{Z}_2(G)$ which is not central. However as this series is powerfully central we have that $[x, G] \leq Z(G)^p = \{1\}$ since the exponent of the centre is p. Thus in fact x is central, giving a contradiction. □

Proposition 16. *The upper powerfully central series is characteristic in G.*

Proof. This can be proved by induction. It is clear that $\hat{Z}_1(G) = Z(G)$ is characteristic in G. Suppose that $\hat{Z}_k(G)$ is characteristic in G. Observe that $\frac{\hat{Z}_{k+1}(G)}{\hat{Z}_k(G)^p} = Z(\frac{G}{\hat{Z}_k(G)^p})$ is characteristic in $\frac{G}{\hat{Z}_k(G)^p}$ and $\hat{Z}_k(G)^p$ is characteristic in G. It follows that $\hat{Z}_{k+1}(G)$ is characteristic in G because characteristicity is quotient transitive. □

Lemma 17. *Suppose that a finite p-group G has a powerfully ascending chain of length n:*

$$1 = H_0 \leq H_1 \leq \cdots \leq H_n = G.$$

Then for each $i \in \{0, \ldots, n\}$ we have that $H_i \leq \hat{Z}_i(G)$.

Proof. We prove this by induction on i. If $i = 0$, then in this case $H_0 = 1 = \hat{Z}_0(G)$ and the claim holds. Suppose that the claim holds for $i = k$ and so $H_k \leq \hat{Z}_k(G)$. We have that $[G, H_{k+1}] \leq H_k^p \leq \hat{Z}_k(G)^p$ and thus $H_{k+1} \leq \hat{Z}_{k+1}(G)$. □

Corollary 18. *A finite p-group G is powerfully nilpotent if and only if the upper powerfully central series terminates in a finite number of steps. In this case the powerful nilpotency class of G is then the smallest integer c such that $\hat{Z}_c(G) = G$.*

Proposition 19. *Any refinement of a powerfully central series is also powerfully central.*

Proof. Suppose G has a powerfully central series

$$1 = H_0 \leq H_1 \leq \cdots \leq H_r = G.$$

Suppose that $H_i \leq K \leq H_{i+1}$ for some $i \in \{0, \ldots, r-1\}$. We will show that the chain

$$1 = H_0 \leq H_1 \leq \cdots \leq H_i \leq K \leq H_{i+1} \leq \cdots \leq H_r = G$$

is powerfully central. It is clear that $H_{j-1} \leq H_j$ is powerfully central in G for $j \in \{2, \ldots, i, i+2, \ldots, r\}$ because the series we had to begin with was powerfully central. It remains to show that $H_i \leq K$ and that $[K, G] \leq H_i^p$ and similarly that $K \leq H_{i+1}$ and that $[H_{i+1}, G] \leq K^p$. We have that $[H_{i+1}, G] \leq H_i^p$, and so since $K \leq H_{i+1}$ we have that $[K, G] \leq [H_{i+1}, G] \leq H_i^p$. Next, we show that $[H_{i+1}, G] \leq K^p$. Note that $H_i \leq K$ and so $H_i^p \leq K^p$. Then $[H_{i+1}, G] \leq H_i^p \leq K^p$. □

For nilpotent p-groups of order p^n it is known that the maximal (standard) nilpotency class is $n-1$. We prove a similar upper bound on powerful nilpotency class below, although we note that empirical evidence obtained using GAP suggests that this bound can be significantly improved.

Proposition 20. *Let G be a powerfully nilpotent group of order $|G| = p^n$, with $n \geq 3$. Then the powerful nilpotency class of G is not greater than $n-2$.*

Proof. Using the result of Proposition 19 that the refinement of a powerfully central chain is powerfully central, we can write down a powerfully central series, such that each term is of index p in the next:

$$1 \leq \langle a_1 \rangle \leq \langle a_1, a_2 \rangle \leq \cdots \leq \langle a_1, a_2, \ldots, a_n \rangle = G.$$

Note that then we must have $a_1^p = 1$. Next we use that $\langle a_1 \rangle \leq \langle a_{1,}a_2 \rangle$ is powerfully central in G, that is $[\langle a_1, a_2 \rangle, G] \leq \langle a_1 \rangle^p$. Let $\langle a_1, a_2 \rangle = H$. Then $[H, G] \leq \langle a_1^p \rangle = 1$. Hence the term $\langle a_1 \rangle$ is superfluous. Next we consider the term $\langle a_1, \ldots, a_{n-1} \rangle$. Call this group K. Then $[K, G]$ is generated by $\langle [a_i, a_j] : 1 \leq i \leq n-1, 1 \leq j \leq n \rangle$ and $[G, G] = \langle [a_i, a_j] : 1 \leq i \leq n, 1 \leq j \leq n \rangle$. But notice that $[a_n, a_n] = 1$ and so in fact $[G, G] = \langle [a_i, a_j] : 1 \leq i \leq n-1, 1 \leq j \leq n \rangle = [K, G]$. We thus see that $[K, G] \leq \langle a_1^p, \ldots, a_{n-2}^p \rangle$, but then because $[K, G] = [G, G]$ we have $[G, G] \leq \langle a_1^p, \ldots, a_{n-2}^p \rangle$. Hence the term K is redundant. Thus we know that $1 \leq \langle a_1, a_2 \rangle \leq \cdots \leq \langle a_1, \ldots, a_{n-2} \rangle \leq G$ is a powerfully central series for G of length $n-2$. ☐

Corollary 21. *For a powerfully nilpotent group G of order p^n with $|Z(G)| = p^k$ and powerful nilpotency class c, we have the bound $c \leq n-k$.*

Proof. Consider the upper powerfully central series of G. Then the longest possible chain is $1 \leq Z(G) \leq \hat{Z}_2(G) \leq \cdots \leq \hat{Z}_{c-1}(G) \leq \hat{Z}_c(G) = G$. However by the proposition above, we know that $|\hat{Z}_{c-1}(G)| \leq p^{n-2}$ and also given $|Z(G)| = p^k$, we have at least $(k-1)+1 = k$ redundant terms and so $c \leq n-k$. ☐

Proposition 22. *The powerful nilpotence property is preserved under homomorphism. Moreover the powerful nilpotency class of the homomorphic image is less than or equal to that of the group.*

Proof. Let $1 = H_0 \leq H_1 \leq \cdots \leq H_n = G$ be any powerfully central chain for G, that is $[H_i, G] \leq H_{i-1}^p$ for $i = 1, \ldots, n$ and let φ be some homomorphism of G. Then observe that $\varphi([H_i, G]) = [\varphi(H_i), \varphi(G)]$ and $\varphi(H_{i-1}^p) = \varphi(H_{i-1})^p$. Thus it follows that the series $\varphi(1) \leq \varphi(H_1) \leq \cdots \leq \varphi(H_n) = \varphi(G)$ is powerfully central. ☐

In particular it follows that powerful nilpotence is preserved under taking quotients.

Remark 23. The powerful nilpotence property is not preserved under taking subgroups. Examples can be found readily using the GAP package, for example the group with GAP SmallGroup Id $(81, 14)$ is powerfully nilpotent but contains 9 subgroups that are not powerfully nilpotent. This is the smallest (for $p = 3$) counterexample.

Remark 24. It also should be noted that this is not the only way that subgroups behave badly. In fact it is possible for a subgroup to have a higher powerful nilpotency class than that of the group, for example the group with GAP SmallGroup Id $(2187, 5302)$ is of powerful nilpotency class 2, but it contains subgroups of powerful nilpotency class 3.

So far we have largely focussed on the similarities between nilpotence and powerful nilpotence. We now look at one of the fundamental ways in which the theory differs, namely that there seems to be no natural way to construct a canonical fastest descending powerfully central series.

Example 25. Let p be an odd prime. Consider the group $G = \langle a, b, c, d \,|\, a^p = b^p = c^{p^2} = 1,\ [b, a] = c^p \rangle$. This group is of powerful nilpotency class 2. We can find two powerfully central descending chains for G which descend at the same rate:

$$C_1 : G \geq \langle c \rangle \geq 1$$
$$C_2 : G \geq \langle dc \rangle \geq 1.$$

Notice that taking the intersection of the corresponding terms in these chains does not give a powerfully central chain, because $\langle c \rangle \cap \langle dc \rangle = \langle c^p \rangle$ is of exponent p.

This examples shows that there is no natural fastest descending series.
Now we shall introduce the notion of a strongly powerful group. We will see that certain characteristic subgroups of powerful p-groups are in fact strongly powerful, and that strongly powerful groups are necessarily powerfully nilpotent.

Definition 26. We say that a finite p-group G is strongly powerful if $[G, G] \leq G^{p^2}$.

Notice how the group in Example 14 above is strongly powerful. We will see later that every rank 2 powerfully nilpotent group must be strongly powerful.

Proposition 27. *If G is strongly powerful, then G is powerfully nilpotent.*

Proof. Follows from the same argument as Remark 12. □

Remark 28. Let G be a finite p-group and let N be a subgroup of the Frattini subgroup $\Phi(G)$, where N is powerfully embedded in G. Then N is strongly powerful and thus powerfully nilpotent. To see this, notice that

$$[N, N] \leq [N, [G, G]G^p] \leq [N, G, G][N, G]^p \leq [N^p, G][N, G]^p \leq [N, G]^p \leq N^{p^2}.$$

This implies in particular that if G is powerful then $\Phi(G)$, and more generally, G^{p^i} for $i \geq 1$ are powerfully nilpotent. The same is true for the proper terms of the derived and lower central series.

Remark 29. Let p be an odd prime. Consider any powerfully nilpotent group with a powerfully central chain $\{1\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$. Then in particular $[H_i, G] \leq H_i^p$ and each term of the chain is powerfully embedded in G. Notice also that each term is powerfully nilpotent.

Notice that if G is strongly powerful then $[G, G] \leq G^{p^2}$ and so G/G^{p^2} is abelian. Conversely if G/G^{p^2} is abelian then we must have that $[G, G] \leq G^{p^2}$ and so the group is strongly powerful.
However, it turns out that G/G^{p^2} being powerfully nilpotent is sufficient for G to be powerfully nilpotent.

Proposition 30. *Let G be any finite p-group of exponent p^e where $e \geq 2$. If G/G^{p^2} is powerfully nilpotent, then G is powerfully nilpotent. Furthermore if G/G^{p^2} has powerful class m, then the powerful class of G is at most $(e-1)m$.*

Proof. Notice first that, as G/G^{p^2} is powerful, we have $[G, G] \leq G^p$ when p is odd and $[G, G] \leq G^4$ when $p = 2$. Thus G is powerful. We have seen above in Remark 12 that a powerful $2-$group is powerfully nilpotent and that the powerful class is then at most $e - 1$.

We can thus assume that p is odd. Suppose

$$G/G^{p^2} = \overline{H}_0 \geq \overline{H}_1 \geq \cdots \geq \overline{H}_{m-1} \geq \{\overline{1}\}$$

is the upper powerfully central series for G/G^{p^2}, where $\overline{H}_i = H_i/G^{p^2}$ with $G^{p^2} \leq H_i \leq G$. Since G is powerful, we have $[G^p, G] \leq G^{p^2}$ and thus $G^p \leq H_{m-1}$. Hence

$$G = H_0 \geq H_1 \geq \cdots \geq H_m = G^p$$

is powerfully central. Consider the descending chain

$$\begin{array}{cccccc} G = & H_0 \geq & H_1 \geq & \ldots \geq & H_m = & G^p \\ G^p = & H_0^p \geq & H_1^p \geq & \ldots \geq & H_m^p = & G^{p^2} \\ & \vdots & & & & \\ G^{p^{e-2}} = & H_0^{p^{e-2}} \geq & H_1^{p^{e-2}} \geq & \ldots \geq & H_m^{p^{e-2}} = & G^{p^{e-1}}. \end{array}$$

We know already that the first line gives us a powerfully central chain and, as a result, $H_0, \ldots, H_m$ are powerfully embedded in G. Using this fact and the Interchange Lemma, we have that

$$[H_i^{p^k}, G] = [H_i, G]^{p^k} \leq H_{i+1}^{p^{k+1}} = (H_{i+1}^{p^k})^p$$

for $0 \leq i \leq m-1$ and $0 \leq k \leq e-2$. It follows that we have a powerfully central chain. Notice also that $[H_{m-1}^{p^{e-2}}, G] = [H_{m-1}, G]^{p^{e-2}} \leq (G^{p^2})^{p^{e-2}} = \{1\}$ and thus $H_{m-1}^{p^{e-2}} \leq Z(G)$. It follows from this and the powerfully central chain above that the powerful class of G is at most $(e-1)m$. □

For powerful p-groups, it is known that all subgroups of G are powerful if and only if all 2-generator subgroups are powerful. (See the proof of Theorem 3.1 in [LM87].) In fact for powerfully nilpotent groups, a similar result holds.

Proposition 31. *Given a powerfully nilpotent group G, all subgroups of G are themselves powerfully nilpotent if and only if every 2-generator subgroup of G is powerfully nilpotent.*

Proof. ($\Rightarrow$) Suppose all subgroups of G are themselves powerfully nilpotent, then in particular the 2-generator subgroups will be.
($\Leftarrow$) Conversely suppose that every 2-generator subgroup of G is powerfully nilpotent. Consider some arbitrary t-generator subgroup $H = \langle a_1, \ldots, a_t \rangle$. For $t = 1$ we have H is cyclic, and for $t = 2$ the result follows by our assumption, thus we can assume $t \geq 3$. Consider the 2-generator subgroups of H of the form $K = \langle a_i, a_j \rangle$ where $1 \leq i < j \leq t$. These are powerfully nilpotent by assumption. As K is of rank 2, it must be strongly powerful (see Proposition 71). Thus in particular $[a_i, a_j] \in K^{p^2} \leq H^{p^2}$. It follows that $[H, H] \leq H^{p^2}$ and so H is strongly powerful and thus powerfully nilpotent. □

Remark 32. Observe from the proof above that any subgroup of such a group must be strongly powerful, and so we have for a powerfully nilpotent group G:

G has the property that all of its subgroups are powerfully nilpotent $\Longleftrightarrow$
G has the property that every rank 2 subgroup is powerfully nilpotent $\Longleftrightarrow$
G has the property that every subgroup is strongly powerful

In particular, the powerfully nilpotent groups for which being powerfully nilpotent is subgroup closed are precisely the same groups as the strongly powerful groups for which all subgroups are strongly powerful.

The following example shows that it is not necessary for subgroups of strongly powerful groups to be strongly powerful (or indeed powerful) themselves.

Example 33. $G = \langle a, b, c | a^{27} = b^9 = c^3 = 1, [b, c] = a^{18} \rangle$. G is a strongly powerful group. The subgroup $H = \langle b, c, a^9 \rangle = \langle b, c \rangle$ is rank 2 and is not powerful, which can be clearly seen since $H^3 = \langle b^3 \rangle$ does not contain H'.

Remark 34. The proof above also shows that a finite p-group of rank r is powerfully nilpotent if it has generators $a_1, \ldots, a_r$ such that any pair of these generate a powerfully nilpotent group.

2.3. The ancestry tree and powerful coclass

In this section we begin to develop a theory of "Powerful Coclass", in some ways analogous to the existing theory of coclass for p-groups. We will see how powerfully nilpotent groups can be associated with a tree, which in a way forms a classification as every powerfully nilpotent group appears exactly once in this tree. In later sections we will see how the rank and exponent of a powerfully nilpotent group can be bounded in terms of the powerful coclass and it will follow that for any given powerful coclass there are only finitely many groups.

Definition 35. Let G be a powerfully nilpotent p-group of powerful class c and order p^n. We define the powerful coclass of G to be the number $n - c$.

We denote the powerful coclass of a group G by $d(G)$. We now give a construction of the ancestry tree. Let p be a fixed prime. The *vertices* of the ancestry tree are all the powerfully nilpotent p-groups (one for each isomorphism class). We join two vertices G and H by a directed edge from H to G if and only if $H \cong \frac{G}{Z(G)^p}$ and G is not abelian. Notice that this implies that $Z(G)^p \neq \{1\}$ by Proposition 15 and thus the powerful class of G is one more than that of H. We then also say that G is a *direct ancestor* of H or that H is a *direct descendant* of G, and we write $H \to G$. Observe that the descendant of a group is determined uniquely, whereas a group can have a number of (even infinitely many) direct ancestors. More generally, if for powerfully nilpotent groups H and G there is a chain

$$H = H_0 \to H_1 \to \cdots \to H_n = G,$$

then we say that G is an *ancestor* of H and that H is a *descendant* of G. Notice that if H has powerful class c then H_i has powerful class $c + i$. We will see later an example of a family of groups with infinitely many direct ancestors and a family of groups with infinitely many generations of ancestors. So each generation up the tree corresponds to an increase in powerful nilpotency class by 1. We now consider what happens to the powerful coclass.

Proposition 36. *Let H be a powerfully nilpotent group and let G be a direct ancestor of H. Then $d(G) \geq d(H)$.*

Proof. Suppose H is a powerfully nilpotent p-group of order $p^{n(H)}$ and powerful class $c(H)$. Then G is a direct ancestor of H and has powerful class $c(G) = c(H) + 1$ and order $p^{n(G)} = \left|\frac{G}{Z(G)^p}\right| \cdot |Z(G)^p| = |H| \cdot |Z(G)^p| = p^{n(H)+k}$ where $|Z(G)^p| = p^k$. Our definition ensures that G cannot be abelian, and so by Proposition 15 we know that in this case $Z(G)$ must be of exponent strictly greater than p, and so $k \geq 1$.
Thus $d(G) = n(H) + k - (c(H) + 1) = (n(H) - c(H)) + (k - 1)$ and so $d(G) \geq d(H)$, where $d(H)$ is the powerful coclass of H. Notice that we have equality if and only if $|Z(G)^p| = p$. □

Remark 37. We remark that the reason we specify that G is non-abelian in the definition of direct descendant G of H, is to avoid having two generations of groups of class 1, where each proper abelian group has as a descendant the trivial group.

The following example demonstrates that a group can have infinitely many generations of ancestors.

Example 38. Let $G_n := \langle a^{p^n} = 1, b^{p^n} = 1, [b,a] = a^{p^2} \rangle$, and $n \in \mathbb{N}$ with $n \geq 2$. G_n is powerfully nilpotent with powerful nilpotency class $n - 1$ and order p^{2n}. The powerful coclass is $n + 1$. Notice that such a group G_n is a semi direct product $C_{p^n} \ltimes C_{p^n}$ and so we do not need to carry out consistency checks. We also have that $\frac{G_n}{Z(G_n)^p} \cong G_{n-1}$, (see Lemma 72 for a result on determining the centre of a powerfully nilpotent group of rank 2).

The following example demonstrates that a group can have infinitely many direct ancestors (parents).

Example 39. For $n \geq 3$ let $G_n := \langle a, b | a^{p^n} = 1, b^p = 1, [a,b] = a^{p^{n-1}} \rangle$. Then G_n is a direct ancestor to $C_{p^2} \times C_p$.

A key result of this thesis is that there are only finitely many groups of any given powerful coclass. Informally, moving up the ancestry tree there must be a point at which the powerful coclass increases. To prove this, we show that the rank and exponent of any given powerfully nilpotent group G are bounded in terms of the powerful coclass of G.
Our first task is to prove that the rank of G is bounded in terms of the powerful coclass.

2.4. Bounding the rank in terms of the powerful coclass

Lemma 40. *Let $\{1\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$ be any powerfully central chain for a powerfully nilpotent p-group G, such that $|H_k^p| = p^k$ for each $k = 0, \ldots, n$. Suppose that $H_j^p = H_{j-1}^p$, for some $1 \leq j \leq n$. Then there exists $x \in H_j \backslash H_{j-1}$ such that $x^p = 1$.*

Proof. If $j = 1$ this is obvious. Suppose then that $j \geq 2$. Pick first any $x \in H_j \backslash H_{j-1}$. Then as $H_j^p = H_{j-1}^p$ we have that $x^p = y^p$ for some $y \in H_{j-1}$. Now x commutes with y modulo H_{j-2}^p. Hence

$$(xy^{-1})^p = x^p y^{-p} z^p = z^p,$$

for some $z \in H_{j-2}$. By replacing x by xy^{-1} we can now assume that $x^p \in H_{j-2}^p$. If $j = 2$ we are done, otherwise a similar argument shows that we can replace x by a new element in $H_j \backslash H_{j-1}$ such that $x^p \in H_{j-3}^p$. Continuing like this we see that we can eventually pick our x such that $x^p = 1$. □

Proposition 41. *Let $\{1\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$ be any powerfully central chain for a powerfully nilpotent p-group G, where $|H_k| = p^k$ for $k = 0, \ldots, n$. We can then choose $a_1, \ldots, a_n \in G$ such that $H_i = \langle a_1, \ldots, a_i \rangle$ and such that $a_i^p = 1$ if and only if $H_i^p = H_{i-1}^p$ for $i = 1, \ldots, n$.*

Proof. Follows directly from Lemma 40. □

Definition 42. Let G be a powerfully nilpotent group of order p^n and let $\mathcal{L}$ be an ascending powerfully central chain

$$\{1\} = H_0 \leq H_1 \leq \cdots \leq H_n = G$$

where $|H_j| = p^j$ for $0 \leq j \leq n$. The number $S_{\mathcal{L}}(G) = |\{H_0^p, H_1^p, \ldots, H_n^p\}|$ is called the *pth power length* of $\mathcal{L}$.

Remark. Intuitively we can think of $S_{\mathcal{L}}(G) - 1$ as the number of "jumps" in the chain $\{1\} = H_0^p \leq H_1^p \leq \cdots \leq H_n^p = G^p$.

Lemma 43. *If $\mathcal{L}$ is any powerfully central chain as above then $|G^p| = p^{S_{\mathcal{L}}(G)-1}$.*

Proof. Suppose $\mathcal{L}$ is the ascending central chain

$$\{1\} = H_0 \leq H_1 \leq \cdots \leq H_n = G.$$

Let $t = s_{\mathcal{L}}(G) - 1$ and suppose that

$$\{1\} = H_0^p = \cdots = H_{j_1-1}^p < H_{j_1}^p = \cdots = H_{j_2-1}^p < \cdots < H_{j_t}^p = \cdots = H_n^p = G^p.$$

From Proposition 41 we know that we can pick elements $a_1, \ldots, a_n \in G$ such that $H_i = \langle a_1, \ldots, a_i \rangle$ and that $a_{j_k}^p \notin H_{j_k-1}^p$. Notice that $a_{j_k}^p \in H_{j_k-1}$ and thus $a_{j_k}^{p^2} \in H_{j_k-1}^p$, and so it follows that $H_{j_k}^p / H_{j_k-1}^p$ is a cyclic group of order p. Notice that we have also used the fact that $H_{j_k}^p = \langle a_{j_k}^p, H_{j_k-1}^p \rangle$.This is a well known property of powerful p-groups (see [LM87] Corollary 1.9). Thus

$$|G^p| = \frac{|H_n^p|}{|H_{n-1}^p|} \cdots \frac{|H_1^p|}{|H_0^p|} = \frac{|H_{j_t}^p|}{|H_{j_t-1}^p|} \cdots \frac{|H_{j_1}^p|}{|H_{j_1-1}^p|} = p^t.$$

□

Proposition 44. *Let G be a powerfully nilpotent p-group of rank r and order p^n. Then $s_{\mathcal{L}}(G) = n - r + 1$.*

Proof. As G^p is the Frattini subgroup of G, it follows that

$$p^r = \left| \frac{G}{G^p} \right| = \frac{p^n}{p^{s_{\mathcal{L}}(G)-1}}.$$

The result follows from this and the previous lemma. □

Remark 45. Notice that $s_{\mathcal{L}}(G)$ does not depend on the choice of chain $\mathcal{L}$, and thus we denote this number $s(G)$ and call it the *pth power length* of G.

Lemma 46. *Let G be a powerfully nilpotent group of powerful class $c \geq 2$. Then*

$$[G, G] = [\hat{Z}_c(G), G] > [\hat{Z}_{c-1}(G), G] > \cdots > [\hat{Z}_1(G), G] = \{1\}$$

and

$$G^p = \hat{Z}_c(G)^p \geq \hat{Z}_{c-1}(G)^p > \cdots > \hat{Z}_1(G)^p > \hat{Z}_0(G)^p = \{1\}.$$

In particular $|G^p| \geq |[G, G]| \geq p^{c-1}$.

Proof. Suppose $2 \leq j \leq c$. If $[\hat{Z}_j(G), G] = [\hat{Z}_{j-1}(G), G]$, then $[\hat{Z}_j(G), G] \leq \hat{Z}_{j-2}(G)^p$ and thus we get the contradiction that $\hat{Z}_j(G) \leq \hat{Z}_{j-1}(G)$. The proof of the latter strict inequalities is similar. Let $1 \leq j \leq c-1$. If $\hat{Z}_j(G)^p = \hat{Z}_{j-1}(G)^p$ then $[\hat{Z}_{j+1}(G), G] \leq \hat{Z}_{j-1}(G)^p$ and thus $\hat{Z}_{j+1}(G) = \hat{Z}_j(G)$ that gives the contradiction that the powerful class of G is at most $j \leq c-1$. □

Theorem 47. *Let G be a powerfully nilpotent p-group with rank r, powerful class c and order p^n. Then $r \leq n - c + 1$.*

Proof. Refining the upper powerfully central series we get an ascending powerfully central series

$$\{1\} = H_0 < H_1 < \cdots < H_n = G,$$

where $|H_i| = p^i$ for $i = 0, \ldots, n$, and where $\{H_0^p, \ldots, H_n^p\}$ contains $\hat{Z}_0(G)^p$,$\hat{Z}_1(G)^p$,...,$\hat{Z}_{c-1}(G)^p$. By Lemma 46 these c groups are distinct and thus $c \leq s(G) = n - r + 1$. Hence $r \leq n - c + 1$. □

2.5. Presentations of powerfully nilpotent groups

In this section we show that every powerfully nilpotent group can be described by a presentation of a specific form.
Let G be any powerfully nilpotent p-group of exponent p^e, order p^n and rank r. Suppose that

$$G = H_0 > H_1 > \cdots > H_r = G^p \tag{2.1}$$

is a powerfully central series as given in the proof of Proposition 30. We then have $G = \langle a_1, \ldots, a_r \rangle$ for any choice of elements $a_1, \ldots, a_r \in G$, where $H_i = \langle a_{i+1}, \ldots, a_r \rangle G^p$ for $i = 0, \ldots, r$. We would like to choose $a_1, \ldots, a_r$ so that the choice best reflects the structure of the group G.
As we have seen in the proof of proposition 30 , we then get a powerfully central series

$$\begin{array}{cccccc}
G = & H_0 \geq & H_1 \geq & \ldots \geq & H_r = & G^p \\
G^p = & H_0^p \geq & H_1^p \geq & \ldots \geq & H_r^p = & G^{p^2} \\
 & \vdots & & & & \\
G^{p^{e-1}} = & H_0^{p^{e-1}} \geq & H_1^{p^{e-1}} \geq & \ldots \geq & H_r^{p^{e-1}} = & G^{p^e} = \{1\}.
\end{array}$$

Omitting repetitions, we obtain a chain where for each $k = 0, \dots, e-1$, the length of the subchain between G^{p^k} and $G^{p^{k+1}}$ is equal to rank(G^{p^k}). Notice that, as in the proof of Lemma 43, we have been using here implicitly [LM87] Corollary 1.9. Writing the groups in ascending order we obtain a chain of the form:

$$\{1\} = K_0 < K_1 < \cdots < K_n = G$$

where $|K_i| = p^i$ for $i = 0, \dots, n$. However we can modify this still further.
For $0 \leq i \leq r-1$ we have $H_i = H_{i+1}\langle a_{i+1}\rangle$ and by Lemma 40 we can choose our generators $a_1, \dots, a_r$ such that $H_i^p = H_{i+1}^p$ if and only if $a_{i+1}^p = 1$. Having done this we can then move all the generators that are of order p to the front of the others, (keeping the previous order unchanged otherwise) and we still have that (2.1) gives us a powerfully central series. We can thus assume that for some $0 \leq s \leq r$ we have $a_1^p = \cdots = a_s^p = 1$ and that $G^p = H_s^p > H_{s+1}^p > \cdots > H_r^p = G^{p^2}$.
Notice that the rank of G^p is the number of jumps and thus the number of generators that have order at least p^2. We have $G^p = \{1\}$ if $s = r$ and otherwise $0 \leq s < r$ and $\{1\} < G^p = H_s^p$. Suppose we are in the latter situation. Using again Lemma 40 we see that for $s \leq j \leq r-1$ we have $H_j^{p^2} = H_{j+1}^{p^2}$ if and only if there exists $x = y^p \in H_j^p \backslash H_{j+1}^p$ such that $x^p = 1$. This happens if and only if there is $y \in H_j \backslash H_{j+1}$ such that the order of y is p^2.We can thus choose our generators such that furthermore $H_j^{p^2} = H_{j+1}^{p^2}$ if and only if $a_{j+1}^{p^2} = 1$. Notice again that the rank of G^{p^2} is the number of generators that have order at least p^3. Continuing in this manner, considering next $H_0^{p^3} \geq H_1^{p^3} \geq \cdots \geq H_r^{p^3} = G^{p^4}$and then p^4th powers and so on, we eventually arrive at a set of generators $a_1, \dots, a_r$ with some specific properties.
If for $0 \leq i \leq r$ we let $s(i)$ be the number of generators of order p^i then $|G^{p^{i-1}}/G^{p^i}| = p^{s(i)+s(i+1)+\cdots+s(e)}$. Using the fact that $[G^{p^i}, G] \leq G^{p^{i+1}}$ it is easy to see inductively that every element in G can be written of the form $a_1^{l_1} \dots a_r^{l_r}$.We thus see that:

$$G = \langle a_1\rangle \cdot \langle a_2\rangle \dots \langle a_r\rangle$$

where

$$\begin{aligned} |G| &= |G/G^p| \cdot |G^p/G^{p^2}| \dots |G^{p^{r-1}}/G^{p^r}| \\ &= p^{s(1)+\cdots+s(e)} p^{s(2)+\cdots+s(e)} \dots p^{s(e)} \\ &= p^{s(1)} p^{2s(2)} \dots p^{es(e)} \\ &= o(a_1) \dots o(a_r). \end{aligned}$$

Notice that the number of generators of order p^i is an invariant of the group, namely $s(i) = \text{rank}G^{p^{i-1}} - \text{rank}G^{p^i}$.

Theorem 48. *Let G be any powerfully nilpotent p-group of rank r, exponent p^e and order p^n. Then we can choose our generators $a_1, \dots, a_r$ such that $|G| = o(a_1) \cdot o(a_2) \dots o(a_r)$ and such that for $H_0 = G = \langle a_1, \dots, a_r\rangle G^p$, $H_1 = \langle a_2, \dots, a_r\rangle G^p$, ..., $H_r = G^p$ we get a powerfully central chain*

$$\begin{aligned} G &= H_0 > H_1 > \cdots > H_r = G^p \\ G^p &= H_0^p \geq H_1^p \geq \cdots \geq H_r^p = G^{p^2} \\ &\vdots \\ G^{p^{e-1}} &= H_0^{p^{e-1}} \geq H_1^{p^{e-1}} \geq \cdots \geq H_r^{p^{e-1}} = G^{p^e} = \{1\}. \end{aligned}$$

Moreover if we write the groups of this chain in ascending order without repetitions, then we get a powerfully central chain of the form

$$\{1\} = K_0 < K_1 < \cdots < K_n = G$$

where $|K_i| = p^i$ *for* $i = 1, \ldots, n$ *and where the groups* $G, G^p, \ldots G^{p^e} = \{1\}$ *are included. Furthermore the generators satisfy relations of the form*

$$[a_i, a_j] = a_1^{m_1(i,j)} a_2^{m_2(i,j)} \ldots a_r^{m_r(i,j)}$$

for $1 \le j < i \le r$ *where all the power indices are divisible by* p *and where furthermore* $p^2 | m_k(i,j)$ *when* $k \le i$*.* G *is then the largest finite* p*-group satisfying these relations and the relations*

$$a_1^{o(a_1)} = \cdots = a_r^{o(a_r)}.$$

The number of generators of any given order p^i *is also an invariant for the group* G*.*

Proof. Most of this has been proved already. The fact that G satisfies relations of the form given follows from the fact that

$$G = H_0 > H_1 > \cdots > H_r = G^p$$

is powerfully central. Let $H = \langle b_1, \ldots, b_r \rangle$ be the largest finite p-group satisfying the given relations in the variables $b_1, \ldots, b_r$. The relations then imply that H is powerful and that the chain

$$\langle b_1, \ldots, b_r \rangle H^p > \langle b_2, \ldots, b_r \rangle H^p > \cdots > \langle b_r \rangle H^p > H^p$$

is powerfully central and thus H is powerfully nilpotent by Proposition 30. It follows that $H = \langle b_1 \rangle \cdot \langle b_2 \rangle \ldots \langle b_r \rangle$(a property that holds in fact for all powerful p-groups, see [LM87] Theorem 1.1) and thus $|H| \le o(b_1) \ldots o(b_r) = o(a_1) \ldots o(a_r) = |G|$. As G is clearly a homomorphic image of H it follows that $|H| = |G|$ and thus $H \cong G$. □

Definition 49. Consider a presentation of a group $G = \langle b_1, \ldots, b_r \rangle$ satisfying relations of the form

$$[b_i, b_j] = b_1^{m_1(i,j)} b_2^{m_2(i,j)} \ldots b_r^{m_r(i,j)}$$

for $1 \le j < i \le r$ where all the power indices are divisible by p and where furthermore $p^2 | m_k(i,j)$ when $k \le i$. To these we add relations of the form

$$b_1^{p^{e_1}} = 1, b_2^{p^{e_2}} = 1, \ldots, b_r^{p^{e_r}} = 1$$

for some positive integers $e_1, \ldots, e_r$. We call such a presentation a *powerfully nilpotent presentation.*

Remark 50. By Theorem 48 we know that every powerfully nilpotent group has a *powerfully nilpotent presentation.*

Definition 51. We say that a powerfully nilpotent presentation is *consistent* if the largest finite p-group H satisfying the relations given in Definition 49 has order $p^{e_1} p^{e_2} \ldots p^{e_r}$.

2.6. Bounding the exponent by the powerful coclass

Lemma 52. *Let G be any powerful p-group and let $a, b \in G$. For any integer $k \geq 0$ we have*

$$[a^{p^k}, b] = [a, b]^{p^k} = [a, b^{p^k}]$$

modulo $[G^{p^{k+1}}, G]$.

Proof. First suppose that p is odd. We prove this by induction on k. For $k = 0$ this is clear. Now suppose $k \geq 1$ and that the result holds for smaller values of k. Using the Interchange Lemma 5 we observe that $[G^{p^{k-1}}, G, G]^p = [[G^{p^{k-1}}, G]^p, G]$, $[G^{p^{k-1}}, G, G, G]$, $[G^{p^k}, G, G]$ and $[G^{p^k}, G]^p$ are all contained in $[G^{p^{k+1}}, G]$. We will be using this fact in the following calculations. By the induction hypothesis we have that $[a^{p^{k-1}}, b] = [a, b]^{p^{k-1}} u$ for some $u \in [G^{p^k}, G]$. Let $x = a^{p^{k-1}}$. Then, calculating modulo $[G^{p^{k+1}}, G]$, we have

$$\begin{aligned}
[a^{p^k}, b] &= [x^p, b] \\
&= [x, b]^p [x, b, x]^{\binom{p}{2}} \\
&= [a^{p^{k-1}}, b]^p \\
&= ([a, b]^{p^{k-1}} u)^p \\
&= [a, b]^{p^k} u^p \\
&= [a, b]^{p^k}.
\end{aligned}$$

This proves the induction step and thus completes the proof when p is odd. When $p = 2$, the induction step is easier as $[G, G] \leq G^4$ and thus $[G^{p^{k-1}}, G, G] \leq [G^{p^{k+1}}, G]$. Using this fact, the calculations as above show that, modulo $[G^{p^{k+1}}, G]$ we have that $[a^{p^k}, b] = [a, b]^{p^k}$. □

As $[G^{p^{k+1}}, G] \leq G^{p^{k+2}}$ we get in particular that $[a^{p^k}, b] = [a, b]^{p^k} = [a, b^{p^k}]$ modulo $G^{p^{k+2}}$.

Lemma 53. *Let G be a powerful p-group and suppose $G = \langle a_1, \ldots, a_r \rangle$ and that $G^{p^k} = \langle a_{i_1}^{p^k}, \ldots, a_{i_s}^{p^k} \rangle$ for some $1 \leq i_1 < i_2 < \cdots < i_s \leq r$. Then*

$$[\langle a_{i_j}^{p^k} \rangle, G][G^{p^{k+1}}, G] = \langle [a_{i_j}^{p^k}, a_{i_1}], \ldots, [a_{i_j}^{p^k}, a_{i_s}] \rangle [G^{p^{k+1}}, G].$$

Proof. It suffices to show that for $m \in \{1, \ldots, r\}$ we have

$$[a_{i_j}^{p^k}, a_m] \in \langle [a_{i_j}^{p^k}, a_{i_1}], \ldots, [a_{i_j}^{p^k}, a_{i_s}] \rangle [G^{p^{k+1}}, G].$$

Now $a_m^{p^k} = a_{i_1}^{p^k e_1} a_{i_2}^{p^k e_2} \ldots a_{i_s}^{p^k e_s}$ for some integers $e_1, \ldots, e_s$. Thus, calculating modulo $[G^{p^{k+1}}, G]$ and using Lemma 52, we have

$$\begin{aligned}
[a_{i_j}^{p^k}, a_m] &= [a_{i_j}, a_m^{p^k}] \\
&= [a_{i_j}, a_{i_1}^{p^k e_1} \ldots a_{i_s}^{p^k e_s}] \\
&= [a_{i_j}, a_{i_1}^{p^k}]^{e_1} \ldots [a_{i_j}, a_{i_s}^{p^k}]^{e_s} \\
&= [a_{i_j}^{p^k}, a_{i_1}]^{e_1} \ldots [a_{i_j}^{p^k}, a_{i_s}]^{e_s}.
\end{aligned}$$

□

Corollary 54. *Let G be a powerful p-group. If the rank of G^{p^k} is 1 then $G^{p^k} \leq Z(G)$.*

Proof. Suppose $G^{p^k} = \langle c^{p^k} \rangle$. By Lemma 53 we then have

$$[G^{p^k}, G] = \langle [c^{p^k}, c] \rangle [G^{p^{k+1}}, G] = [G^{p^{k+1}}, G].$$

But then $[G^{p^k}, G] = [G^{p^k}, G]^p = [G^{p^k}, G]^{p^2} = \cdots = \{1\}$. □

Proposition 55. *Let G be a powerfully nilpotent group and suppose that G^{p^k} has rank $s \geq 2$. Then there exists a chain*

$$G^{p^k} = H_0 > H_1 > \cdots > H_{s-1} = G^{p^{k+1}}$$

that is powerfully centralised by G.

Proof. Suppose G has rank r. By Theorem 48, we can pick our generators $a_1, \ldots, a_r$ such that for $K_0 = \langle a_1, \ldots, a_r \rangle G^p$, $K_1 = \langle a_2, \ldots, a_r \rangle G^p$,$\ldots$,$K_r = G^p$ we have a chain that is powerfully centralised by G. Furthermore the chain

$$G^{p^k} = K_0^{p^k} \geq K_1^{p^k} \geq \cdots \geq K_r^{p^k} = G^{p^{k+1}},$$

is also powerfully centralised by G. Omitting repetitions we get a chain of the form

$$G^{p^k} = \langle a_{i_1}^{p^k}, \ldots, a_{i_s}^{p^k} \rangle G^{p^{k+1}} > \langle a_{i_2}^{p^k}, \ldots, a_{i_s}^{p^k} \rangle G^{p^{k+1}} > \cdots > \langle a_{i_s}^{p^k} \rangle G^{p^{k+1}} > G^{p^{k+1}}$$

for some $1 \leq i_1 < i_2 < \cdots < i_s \leq r$. We can do better than this. Using Lemma 52 and Lemma 53, we have

$$\begin{aligned}[\langle a_{i_1}^{p^k} \rangle, G] G^{p^{k+2}} =& \langle [a_{i_1}^{p^k}, a_{i_2}], \ldots, [a_{i_1}^{p^k}, a_{i_s}] \rangle G^{p^{k+2}} \\ =& \langle [a_{i_1}, a_{i_2}^{p^k}], \ldots, [a_{i_1}, a_{i_s}^{p^k}] \rangle G^{p^{k+2}}.\end{aligned}$$

As $[G^{p^{k+1}}, G] \leq G^{p^{k+2}}$, this implies that

$$[G^{p^k}, G] \leq [\langle a_{i_2}^{p^k}, \ldots, a_{i_s}^{p^k} \rangle G^{p^{k+1}}, G] G^{p^{k+2}} \leq (\langle a_{i_3}^{p^k}, \ldots, a_{i_s}^{p^k} \rangle)^p G^{p^{k+2}}.$$

We thus have that

$$\begin{aligned}G^{p^k} = \langle a_{i_1}^{p^k}, \ldots, a_{i_s}^{p^k} \rangle G^{p^{k+1}} &> \langle a_{i_3}^{p^k}, \ldots, a_{i_s}^{p^k} \rangle G^{p^{k+1}} \\ &> \langle a_{i_4}^{p^k}, \ldots, a_{i_s}^{p^k} \rangle G^{p^{k+1}} > \cdots > \langle a_{i_s}^{p^k} \rangle G^{p^{k+1}} > G^{p^{k+1}}\end{aligned}$$

is powerfully central. □

Theorem 56. *Let G be a powerfully nilpotent group of order p^n, powerful class c and exponent p^e. Then*

$$e \leq n - c + 1.$$

Proof. This is clear when G is cyclic, and so we can assume that the rank of G is at least 2. Let k be the largest non-negative integer such that the rank of G^{p^k} is greater than or equal to 2. Let r_i be the rank of G^{p^i} for $i = 0, 1, \ldots, k$ and let $p^{n_0} = |G^{p^{k+1}}|$. Notice then that

$$e = k + 1 + n_0$$

and

$$n = r_0 + r_1 + \cdots + r_k + n_0.$$

By Proposition 55 for each $0 \leq j \leq k$ there exists a descending chain

$$G^{p^j} = H_0 > H_1 > \cdots > H_{r_j - 1} = G^{p^{j+1}}$$

that is powerfully centralised by G. Adding up for $j = 0, 1, \ldots, k$ and using the fact from Corollary 54 that $G^{p^{k+1}} \leq Z(G)$, we get a central chain of total length $(r_0 - 1) + (r_1 - 1) + \ldots + (r_k - 1) + 1$.
Hence $c \leq (r_0 - 1) + (r_1 - 1) + \cdots + (r_k - 1) + 1$. We conclude that

$$\begin{aligned} n - c &\geq n - [(r_0 - 1) + \cdots + (r_k - 1) + 1] \\ &\geq (r_0 + r_1 + \cdots + r_k + n_0) - [(r_0 - 1) + \cdots + (r_k - 1) + 1] \\ &= k + n_0 \\ &= e - 1. \end{aligned}$$

Hence $e \leq n - c + 1$. □

Thus we obtain one of the central results of this thesis.

Theorem 57. *For each prime p and non-negative integer d, there are finitely many powerfully nilpotent p-groups of powerful coclass d.*

Proof. Let G be a powerfully nilpotent group of order p^n, rank r and exponent p^e and powerful coclass d. Then $n \leq re \leq (d+1)(d+1)$ and thus the order of G is bounded by the powerful coclass. □

The Table **2.1** shows the number of groups for each given coclass, for $p = 3$. The question marks indicate the data is not available or beyond the limits of our computations.

	Coclass	0	1	2	3	4	5	6	7	8	**Total**
Order											
3^1		1									1
3^2			2								2
3^3				3							3
3^4				2	5						7
3^5					6	7					13
3^6					5	17	11				33
3^7					3	28	52	15			98
3^8						53	176	169	22		420
	Total	1	2	5	19	?	?	?	?		

Table 2.1.: Table Showing Number Of PN Groups By Coclass And Order ($p = 3$)

2.7. Growth of powerfully nilpotent groups of exponent p^2

Let p be a given odd prime. In this section we describe the growth of powerfully nilpotent p-groups of exponent p^2 in terms of the order. Let x, y be fixed non negative integers. Let $G = \langle a_1, a_2, \ldots, a_{y+x}\rangle$be a powerfully nilpotent group of rank $r = y + x$ and order p^n where $n = y + 2x$. Here we are assuming furthermore that $o(a_1) = \ldots = o(a_y) = p$ and $o(a_{y+1}) = \ldots = o(a_{y+x}) = p^2$ and that these satisfy a powerfully nilpotent presentation

$$[a_i, a_j] = a_{i+1}^{p\alpha_{i+1}(i,j)} a_{i+2}^{p\alpha_{i+2}(i,j)} \ldots a_{y+x}^{p\alpha_{y+x}(i,j)},$$

for $1 \le j < i \le y + x$ with $0 \le \alpha_k(i, j) \le p - 1$.
Taking into account that $a_1^p = \cdots = a_y^p = 1$ and that $y = n - 2x$, we see that for $1 \le j < i \le y = n - 2x$ there are p^x choices for $[a_i, a_j]$. Notice that there are $\binom{y}{2} = \binom{n-2x}{2}$ such pairs (i, j). For $y + 1 \le i \le y + x$ and $1 \le j < i$ there are p^{x+y-i} possible values for $[a_i, a_j]$ and for each such i there are $i - 1$ such pairs (i, j). Thus for a fixed n and x the total number of powerfully nilpotent presentations is $p^{h(x)}$, where

$$p^{h(x)} = p^{x\binom{n-2x}{2}+\sum_{i=1}^{x}(y+i-1)(x-i)}$$

$$\begin{aligned}
h(x) &= x\binom{n-2x}{2} + \sum_{i=1}^{x}(y+i-1)(x-i) = x\binom{n-2x}{2} + \sum_{i=1}^{x}(n-2x+i-1)(x-i) \\
&= x\binom{n-2x}{2} + \sum_{i=1}^{x}((n-2x-1)x + i(3x-n+1) - i^2) \\
&= x\binom{n-2x}{2} + (3x-n+1)\sum_{i=1}^{x} i - \sum_{i=1}^{x} i^2 + x^2(n-2x-1) \\
&= \frac{x}{2}(n-2x-1)(n-2x) + (3x-n+1)\frac{1}{2}x(x+1) \\
&\qquad - \frac{1}{6}x(x+1)(2x+1) + x^2(n-2x-1) \\
&= \frac{1}{6}x(7x^2 - 9(n-1)x + 3n^2 - 6n + 2).
\end{aligned}$$

For a fixed n, we seek the maximal value that $h(x)$ can take. Note also that $0 \le x \le \lfloor \frac{n}{2} \rfloor$.Differentiating $h(x)$ with respect to x yields

$$h'(x) = \frac{7}{2}x^2 + 3(1-n)x + (\frac{n^2}{2} - n + \frac{1}{3}),$$

with roots

$$x(n)_{\pm} = \frac{3(n-1)}{7} \pm \sqrt{\frac{2}{49}(n-1)^2 + \frac{1}{21}}.$$

As we have a positive cubic equation, we know that the turning point $x(n)_-$ corresponds to a local maximum. We note that for the endpoints of the interval, we have $h(0) = 0$, and that for suitably large n we have that

$$\left\lfloor \frac{n}{2} \right\rfloor < x(n)_+,$$

and hence the interval ends before the second turning point. That means that $h(x(n)_-)$ is the global maximal value across the interval. To aid notation, let $X = x(n)_-$.The maximum value is

$$h(X) = \frac{1}{6}X(3n^2 - 9nX - 6n + 7X^2 + 9X + 2),$$

and from this we can see the growth is proportional to n^3, so we divide by this to obtain

$$\frac{h(X)}{n^3} = \frac{X}{2n} - \frac{3X^2}{2n^2} - \frac{X}{n^2} + \frac{7X^3}{6n^3} + \frac{3X^2}{2n^3} + \frac{X}{3n^3}.$$

Now notice that $lim_{n\to\infty}(\frac{X}{n}) = \frac{3-\sqrt{2}}{7}$, and hence that

$$lim_{n\to\infty}\left(\frac{h(X)}{n^3}\right) = \frac{1}{2}\left(\frac{3-\sqrt{2}}{7}\right) - \frac{3}{2}\left(\frac{3-\sqrt{2}}{7}\right)^2 + \frac{7}{6}\left(\frac{3-\sqrt{2}}{7}\right)^3 = \frac{9+4\sqrt{2}}{294}.$$

Let $\alpha = \frac{9+4\sqrt{2}}{294}$ and let n be fixed. For any $0 \leq x \leq \lfloor \frac{n}{2} \rfloor$, let $\mathcal{P}(n, x)$ be the collection of all powerfully nilpotent presentations as above. It is not difficult to see that those presentations are consistent and thus the resulting group is of exponent p^2, order p^n and rank $n - x$. Furthermore $a_1^p = \cdots = a_{n-2x}^p = 1$ and $a_{n-2x+1}^{p^2} = \cdots = 1$. We have just seen that if $x(n)$ is chosen such that the number of presentations is maximal then

$$|\mathcal{P}(n, x(n))| = p^{\alpha n^3 + o(n^3)}$$

where $\alpha = \frac{9+4\sqrt{2}}{294}$. Let $\mathcal{P}_n$ be the total number of powerfully nilpotent presentations where $0 \leq x \leq \lfloor \frac{n}{2} \rfloor$. Then

$$\mathcal{P}_n = \left|\mathcal{P}(n, 0) \cup \mathcal{P}(n, 1) \cup \cdots \cup \mathcal{P}(n, \lfloor \frac{n}{2} \rfloor)\right|$$

and thus

$$\begin{aligned} p^{\alpha n^3 + o(n^3)} = |\mathcal{P}(n, x(n))| \leq& |\mathcal{P}_n| = |\mathcal{P}(n, 0)| + \cdots + |\mathcal{P}(n, \lfloor \frac{n}{2} \rfloor)| \\ \leq& n \cdot |\mathcal{P}(n, x(n))| = p^{\alpha n^3 + o(n^3)}. \end{aligned}$$

This shows that

$$|\mathcal{P}_n| = p^{\alpha n^3 + o(n^3)}.$$

We want to show that this also gives us the growth of powerfully nilpotent groups of exponent p^2 with respect to the order p^n. Clearly $p^{\alpha n^3 + o(n^3)}$ gives us an upper bound. We want to show that this is also a lower bound. Let $x = x(n)$ be as above. Let $a_1, \ldots, a_{n-x}$ be a set of generators for a powerfully nilpotent group G where $a_1^p = \cdots = a_{n-2x}^p = 1$ and $a_{n-2x+1}^{p^2} = \cdots = a_{n-x}^{p^2} = 1$ and where we have chosen the generators to get the maximum possible number of generators of order p. Notice that $\langle a_1, \ldots, a_{n-2x} \rangle G^p = G[p] = \{g \in G : g^p = 1\}$ and this is thus a characteristic subgroup of G. It will be useful to consider a larger class of presentations for powerfully nilpotent groups of exponent p^2 and order p^n where we still require $o(a_1) = \cdots = o(a_{n-2x}) = p$ and $o(a_{n-2x+1}) = \cdots = o(a_{n-x}) = p^2$. We let $Q(n, x) = Q(n, x(n))$ be the collection of all presentations with the additional commutator relations

$$[a_i, a_j] = a_{n-2x+1}^{p\alpha_{n-2x+1}(i,j)} \cdots a_{n-x}^{p\alpha_{n-x}(i,j)}.$$

The presentation will be included in $Q(n,x)$ provided the resulting group is a powerfully nilpotent group G of exponent p^2 and order p^n. Notice then that $G^p \leq Z(G)$ and as a result the commutator relations above only depend on the cosets $\bar{a}_1 = a_1G^p, \ldots, \bar{a}_{n-x} = a_{n-x}G^p$ and not on the exact values of $a_1, \ldots, a_{n-x}$. Consider the vector space $V = G/G^p = \mathbb{Z}_p\bar{a}_1 + \cdots + \mathbb{Z}_p\bar{a}_{n-x}$. Recall that $\langle a_1, \ldots, a_{n-2x}\rangle G^p$ is a characteristic subgroup of G. Let $W = \mathbb{Z}_p\bar{a}_1 + \cdots + \mathbb{Z}_p\bar{a}_{n-2x}$ be the corresponding subspace of V. Then let

$$H = \{\phi \in GL(n-x,p) : \phi(W) = W\}.$$

There is now a natural action from H on $Q(n,x)$. Suppose we have some presentation with generators $a_1, \ldots, a_{n-x}$ as above. Let $\phi \in H$ and suppose that $\bar{a}_i^{\phi} = \beta_1(i)\bar{a}_1 + \cdots + \beta_{n-x}(i)\bar{a}_{n-x}$. We then get a new presentation in $Q(n,x)$ for G with respect to the generators $b_1, \ldots, b_{n-x}$, where $b_i = a_1^{\beta_1(i)} \ldots a_{n-x}^{\beta_{n-x}(i)}$.
Suppose there are l powerfully nilpotent groups of exponent p^2 and order p^n where $|G^p| = p^x$.Pick powerfully nilpotent presentations $p_1, \ldots, p_l \in \mathcal{P}(n,x)$ for these. Let q be any powerfully nilpotent presentation in $\mathcal{P}(n,x)$ of a group K with generators $b_1, \ldots, b_{n-x}$. Then q is a presentation for an isomorphic group G with presentation p_i and generators $a_1, \ldots, a_{n-x}$. Let $\phi : K \to G$ be an isomorphism and let $\psi : K/K^p \to G/G^p$ be the corresponding linear automorphism. This gives us a linear automorphism $\tau \in H$ induced by $\tau(\bar{a}_i) = \psi(\bar{b}_i)$. Thus $q = p_i^{\tau}$. Hence the orbits of $p_1, \ldots, p_l$ cover $\mathcal{P}(n,x)$ and thus

$$\mathcal{P}(n,x) \subseteq p_1^H \cup p_2^H \cup \cdots \cup p_l^H.$$

From this we get

$$p^{\alpha n^3 + o(n^3)} = |\mathcal{P}(n,x)| \leq |p_1^H| + \cdots + |p_l^H| \leq l \cdot |H| \leq l \cdot p^{n^2},$$

and it follows that $l \geq p^{\alpha n^3 + o(n^3)}$. We obtain the following result.

Theorem 58. *Let p be an odd prime. The number of powerfully nilpotent groups of exponent p^2 and order p^n is $p^{\alpha n^3 + o(n^3)}$, where $\alpha = \frac{9+4\sqrt{2}}{394}$.*

Remark 59. Using a similar analysis one can estimate the growth of all powerful p-groups of exponent p^2 and order p^n, where p is odd. This turns out to be

$$p^{\frac{2}{27}n^3 + o(n^3)}.$$

Thus while the powerfully nilpotent p-groups of exponent p^2 and order p^n are very numerous, they are sparse within the larger class of all powerful p-groups of exponent p^2 and order p^n as n tends to infinity.

2.8. Groups with maximal tail

Definition 60. Let G be a powerfully nilpotent p-group and let k be the largest non-negative integer such that

$$p = |Z(G)^p| = \left|\frac{\hat{Z}_2(G)^p}{\hat{Z}_1(G)^p}\right| = \cdots = \left|\frac{\hat{Z}_k(G)^p}{\hat{Z}_{k-1}(G)^p}\right|.$$

We refer to $\hat{Z}_k(G)^p$ as the *tail* of G and k as the *length of the tail.*

Remark 61. If G has a tail of length k then G,$G/\hat{Z}_1(G)^p$,$G/\hat{Z}_2(G)^p,\dots$,$G/\hat{Z}_k(G)^p$ all have the same powerful coclass.

Now let G be any powerfully nilpotent p-group of rank r and exponent p^e. By Theorem 48 we can find $a_1, a_2, \dots, a_r \in G$ such that, for

$$K_0 = \langle a_1, a_2, \dots, a_r\rangle G^p, K_1 = \langle a_2, \dots, a_r\rangle G^p, \dots, K_r = G^p,$$

we have that the chain

$$\begin{aligned} G =& K_0 > K_1 > \cdots > K_r = G^p \\ G^p =& K_0^p \geq K_1^p \geq \cdots \geq K_r^p = G^{p^2} \\ &\vdots \\ G^{p^{e-1}} =& K_0^{p^{e-1}} \geq K_1^{p^{e-1}} \geq \cdots \geq K_r^{p^{e-1}} = G^{p^e} = \{1\} \end{aligned}$$

is powerfully central in G.

Lemma 62. *Rewrite the above chain in ascending order without repetitions. Then suppose the chain up to and including G^p is*

$$\{1\} = M_0 < M_1 < \cdots < M_t = G^p.$$

We have that $M_j \leq \hat{Z}_j(G)^p$ for $j = 0, \dots, t$. Also if the tail of G is $\hat{Z}_k(G)^p$ then $M_i = \hat{Z}_i(G)^p$ for $i = 0, \dots, k$.

Proof. We prove the first part by induction on $0 \leq j \leq t$. This is obvious when $j = 0$. Now suppose that $j \geq 1$ and that the result holds for smaller values of j. Let q be the largest and then, for that q, let i be the largest such that $M_j = K_i^{p^q}$. Then $0 \leq i \leq r-1$ and $K_{i+1}^{p^q} = M_{j-1}$. Thus

$$[K_i^{p^{q-1}}, G] \leq (K_{i+1}^{p^{q-1}})^p = M_{j-1} \leq \hat{Z}_{j-1}(G)^p$$

by the induction hypothesis. Hence $K_i^{p^{q-1}} \leq \hat{Z}_j(G)$ and thus $M_j = K_i^{p^q} \leq \hat{Z}_j(G)^p$. This finishes the inductive proof. The second part follows from the first part and the fact that $|\hat{Z}_i(G)^p| = p^i$ for $i = 0, \dots, k$. □

Proposition 63. *Let G be a powerfully nilpotent p-group with a tail of length k. Suppose that $G^{p^{i+1}} \leq \hat{Z}_k(G)^p$ for some non-negative integer i. If rank(G^{p^i})≥ 2 we have*

$$\text{rank}(G^{p^i}) > \text{rank}(G^{p^{i+1}}).$$

Proof. Suppose the rank of G^{p^i} is s. We know that the following chain is powerfully centralised by G

$$G^{p^i} = K_0^{p^i} \geq K_1^{p^i} \geq \cdots \geq K_r^{p^i} = G^{p^{i+1}}.$$

Omitting repetitions we get a chain

$$G^{p^i} = \langle a_{i_1}^{p^i}, \dots, a_{i_s}^{p^i}\rangle G^{p^{i+1}} > \langle a_{i_2}^{p^i}, \dots, a_{i_s}^{p^i}\rangle G^{p^{i+1}} > \cdots > \langle a_{i_s}^{p^i}\rangle G^{p^{i+1}} > G^{p^{i+1}}$$

with $1 \leq i_1 < i_2 < \cdots < i_s \leq r$. As we saw in the proof of Proposition 55 we can omit the second term and we still have a chain powerfully centralised by G. All the terms are then in particular powerfully embedded in G. If $G^{p^i} \geq E > F \geq G^{p^{i+1}}$ where E, F

are consecutive terms in the chain, then $[E^p, G] = [E, G]^p \leq (F^p)^p$. Thus we get the chain

$$G^{p^i} = \langle a_{i_1}^{p^i}, \ldots, a_{i_s}^{p^i} \rangle G^{p^{i+1}} > \langle a_{i_3}^{p^i}, \ldots, a_{i_s}^{p^i} \rangle G^{p^{i+1}} > \cdots > \langle a_{i_s}^{p^i} \rangle G^{p^{i+1}} > G^{p^{i+1}}$$

$$G^{p^{i+1}} = \langle a_{i_1}^{p^{i+1}}, \ldots, a_{i_s}^{p^{i+1}} \rangle G^{p^{i+2}} \geq \langle a_{i_3}^{p^{i+1}}, \ldots, a_{i_s}^{p^{i+1}} \rangle G^{p^{i+2}} \geq \cdots \geq \langle a_{i_s}^{p^{i+1}} \rangle G^{p^{i+2}} \geq G^{p^{i+2}}$$

that is powerfully centralised by G. Let $E = G^{p^i}$ and $F = \langle a_{i_3}^{p^i}, \ldots, a_{i_s}^{p^i} \rangle G^{p^{i+1}}$. By Lemma 62 we know that $F^p = K_{i_3-1}^{p^{i+1}}$ is equal to $\hat{Z}_t(G)^p$ for some $0 \leq t \leq k$. As $[E, G] \leq F^p = \hat{Z}_t(G)^p$, it follows that $E \leq \hat{Z}_{t+1}(G)$ and thus $E^p \leq \hat{Z}_{t+1}(G)^p$. If $E^p = F^p$ then $[E^p : F^p] = 1$. Otherwise $\hat{Z}_{t+1}(G)^p \geq E^p > F^p = \hat{Z}_t(G)^p$. As $E^p = G^{p^{i+1}} \leq \hat{Z}_k(G)^p$ we know from Lemma 62 that $[E^p : F^p] \leq [\hat{Z}_{t+1}(G)^p : \hat{Z}_t(G)^p] = p$ and it follows that the rank of $G^{p^{i+1}}$ is at most $s - 1$. □

Theorem 64. *Let G be a powerfully nilpotent group of rank $r \geq 2$ and with tail of size at least 1. Let f be the largest non-negative integer such that G^{p^f} has rank at least 2. Let i be the smallest non-negative integer such that $G^{p^{i+1}}$ is contained in the tail of G. Then $i \leq f + 1$ and if $i < f + 1$, then*

$$\operatorname{rank}(G^{p^i}) > \operatorname{rank}(G^{p^{i+1}}) > \cdots > \operatorname{rank}(G^{p^{f+1}}).$$

Furthermore the length of the tail of G is at most $1 + \frac{r(r-1)}{2}$.

Proof. By Corollary 54 we know that $G^{p^{f+2}} \leq Z(G)^p$ and thus $G^{p^{f+2}}$ is contained in the tail of G. Therefore $i+1 \leq f+2$ and then $i \leq f+1$. Notice that if $i = f+1$ then G^{p^i} is cyclic. If $i < f+1$ we must have that the rank of G^{p^i} is at least 2. It then follows directly from Proposition 63 that $\operatorname{rank}(G^{p^i}) > \operatorname{rank}(G^{p^{i+1}}) > \cdots > \operatorname{rank}(G^{p^{f+1}})$. For the last part, notice that the scenario where the length of the tail would potentially be largest is when the tail is equal to G^p and where $\operatorname{rank}(G) = r$, $\operatorname{rank}(G^p) = r-1, \ldots, \operatorname{rank}(G^{p^{r-1}}) = 1$. By Corollary 54 we then have that $G^{p^{r-1}} \leq Z(G)$. For there to be a tail of length greater than 0 we then need $|Z(G)^p| = p$ and thus $|G^{p^{r-1}}| \leq p^2$. Thus the size of tail cannot be longer than

$$|G^p| = |G^p/G^{p^2}| \cdot |G^{p^2}/G^{p^3}| \ldots |G^{p^{r-2}}/G^{p^{r-1}}| \cdot |G^{p^{r-1}}| \leq p^{(r-1)+(r-2)+\cdots+2+2}.$$

□

Definition 65. Let G be a powerfully nilpotent p-group. We say that G has *maximal tail* if the tail of G is G^p.

Take care to notice that a group having maximal tail does not necessarily mean that it has a long tail.

Example 66. This example demonstrates a family of groups with tail length 1, but with maximal tail.
For $r \geq 4$ let G be the group given by the following presentation:

$$\langle a_1, \ldots, a_r | a_1^9 = 1, a_2^3 = 1, a_3^3 = 1 \ldots, a_r^3 = 1, [a_3, a_2] = a_1^3 \rangle$$

with the non specified commutator relations trivial. This gives rise to a group of order p^{r+1}, and powerful nilpotency class 2. Notice that $\hat{Z}_1^p(G) = G^p$. Thus the group has maximal tail, but the tail length is 1.

Definition 67. Let G be a powerfully nilpotent p-group. We say that a subgroup H is powerfully hypercentral if there exists a descending chain of subgroups

$$H = H_0 > H_1 > \cdots > H_n = \{1\}$$

such that $[H_i, G] \leq H_{i+1}^p$ for $i = 0, \ldots, n-1$.

Theorem 68. *Let G be a powerfully nilpotent group of rank $r \geq 2$ that has maximal tail. Suppose that G has order p^n, powerful class c and exponent p^e. Let t be the length of the tail.*
(a) We have that $t = n - r$ and $c - 1 \leq t \leq c$. It follows also that $n - c \leq r \leq n - c + 1$.
(b) We have $t \leq 1 + \frac{r(r-1)}{2}$ and $n \leq 1 + \frac{r(r+1)}{2}$.
(c) We have $\text{rank}(G) > \text{rank}(G^p) > \cdots > \text{rank}(G^{p^{e-2}})$.
(d) We have $G/\hat{Z}_i(G)^p$ has maximal tail for all non-negative integers i.
(e) If $H \leq G$ is powerfully hypercentral, then $H^p = \hat{Z}_i(G)^p$ for some integer $i \geq 0$.

Proof. (a) As $|G^p| = p^{n-r}$, it is clear that $t = n - r$. From Lemma 46 we know that $\hat{Z}_{c-2}(G)^p < \hat{Z}_{c-1}(G)^p \leq \hat{Z}_c(G)^p = G^p$. As the tail is G^p we see from this that $c - 1 \leq t \leq c$.
(b) The first inequality follows from Theorem 64 and the second follows from the fact that $|G| = |G^p| \cdot |G/G^p| = p^t p^r$.
(c) If $\text{rank}(G^{p^{e-1}}) \geq 2$, this follows from Theorem 64. Now suppose that $\text{rank}(G^{p^{e-1}}) = 1$. Let f be the largest non-negative integer such that G^{p^f} has rank at least 2. Then $G^{p^{f+1}}$ is cyclic and non-trivial and by Corollary 54 we then know that $G^{p^{f+1}} \leq Z(G)$. As $|Z(G)^p| = p$ it follows that $G^{p^{f+3}} = \{1\}$ and thus $e \leq f + 3$ which implies that $e - 2 \leq f + 1$. The result now follows from Theorem 64.
(d) This follows straight from that fact that $\hat{Z}_i(G)^p \leq G^p$ and that G has maximal tail.
(e) If $H^p = \{1\}$ then $H^p = \hat{Z}_0(G)^p$. Suppose that $H^p \neq \{1\}$. By our assumptions there is a descending chain

$$H = H_0 > H_1 > \cdots > H_n = \{1\},$$

where $[H_i, G] \leq H_{i+1}^p$ for $i = 0, \ldots n-1$. Let j be smallest such that $H_j^p = \{1\}$. Notice that $j \geq 1$. As $[H_{j-1}, G] \leq H_j^p = \{1\}$ we have $H_{j-1} \leq Z(G)$. Then $\{1\} < H_{j-1}^p \leq Z(G)^p$ and, as $|Z(G)^p| = p$, it follows that $H_{j-1}^p = Z(G)^p$. Replacing H, G by $H/Z(G)^p$, $G/Z(G)^p$, we see inductively that $H^p = \hat{Z}_j(G)^p$ for some j. □

We know from Theorem 68 that if G has maximal tail then $n - c \leq r \leq n - c + 1$. Let G be a powerfully nilpotent group of rank $r \geq 2$, powerful class c, order p^n and exponent p^e where $r = n - c + 1$. Notice that $|G^p| = p^{n-r} = p^{c-1}$. From Lemma 46 we know that

$$\{1\} = \hat{Z}_0(G)^p < \hat{Z}_1(G)^p < \cdots < \hat{Z}_{c-1}(G)^p.$$

Hence $\hat{Z}_{c-1}(G)^p = G^p$ and G has maximal tail of length $c - 1$. If on the other hand $r = n - c$ and G has maximal tail, then the length of the tail must be c and this happens if and only if $\hat{Z}_{c-1}(G)^p < \hat{Z}_c(G)^p$.
We know from Theorem 68 that if G has maximal tail, then the length of the tail is $t \leq 1 + r(r-1)/2$. We also know from the proof of Theorem 64 that in order for the upper bound to be attained, we need $\text{rank}(G) = r$, $\text{rank}(G^p) = r - 1, \ldots, \text{rank}(G^{p^{r-2}}) = 2$

and that $G^{p^{r-1}}$ is cyclic of order p^2. In particular we must have that $e = r + 1$. As $n - c \leq r \leq n - c + 1$ and $e \leq n - c + 1$ this can only happen if

$$r = n - c \text{ and } e = n - c + 1. \tag{2.2}$$

Conversely, G has maximal tail of length $1 + r(r-1)/2$ if $\hat{Z}_{c-1}(G)^p < \hat{Z}_c(G)^p$ and (2.2) holds. To see this notice first that $r = n - c$ and $\hat{Z}_{c-1}(G)^p < \hat{Z}_c(G)^p$ imply that $|\hat{Z}_c(G)^p| \geq p^c = p^{n-r} = |G^p|$ and thus $G^p = \hat{Z}_c(G)^p$. Hence G has maximal tail. From Theorem 68 we then have that

$$\text{rank}(G) = r, \text{rank}(G^p) = r - 1, \ldots, \text{rank}(G^{p^{r-1}}) = rank(G^{p^{e-2}}) = 1$$

and thus, as $e = r+1$, $G^{p^{r-1}}$ is cyclic of order p^2. It follows then that $|G^p| = p^{1+r(r-1)/2}$ and thus $t = 1 + r(r-1)/2$.
We conjecture that for each $r \geq 1$ there exists a group G with maximal tail of length $t = 1 + r(r-1)/2$. The following examples demonstrate this for $r \leq 5$.

Example 69. For $r \leq 4$, let $G(r) = \langle x, a_1, \ldots, a_{r-1} \rangle$ where the following relations hold

$$x^{p^{r+1}} = 1, a_1^p = a_2^{p^2} = \cdots = a_{r-1}^{p^{r-1}} = 1,$$

$$[a_1, x] = a_2^p, [a_2, x] = a_3^p, \ldots, [a_{r-2}, x] = a_{r-1}^p, [a_{r-1}, x] = x^{p^2}.$$

Then G has maximal tail of length $t = 1 + r(r-1)/2$.

Unfortunately this recipe does not work for higher r without modification, although we believe that a modification in line with the one given in the following example will work for all r.

Example 70. Let $p = 5$, and $G = \langle x, a, b, c, d \rangle$ with relations

$$x^{p^6} = 1, a^p = b^{p^2} = c^{p^3} = d^{p^4} = 1,$$

$$[a, x] = b^p, [b, x] = c^p, [c, x] = d^p, [d, x] = x^{p^2}, [c, d] = c^{25} d^{375}.$$

Then G has maximal tail of length 11.

3. Powerfully Nilpotent Groups of Rank 2

3.1. A classification

In this section we provide a classification for the non-abelian powerfully nilpotent p-groups of rank 2 where p is an odd prime. Let G be a non-abelian, powerfully nilpotent group of rank 2. First recall from Proposition 6 that such a group must be metacyclic. For a non-abelian group G of rank 2 we have the following:

Proposition 71. *G is powerfully nilpotent if and only if G is strongly powerful.*

Proof. If G is strongly powerful then it is powerfully nilpotent from Proposition 6. Conversely suppose that G is powerfully nilpotent. Let $H = \frac{G}{G^{p^2}}$ and observe that H is powerful. If $\exp(H) < p^2$ then it follows that H is abelian and we are done. Now suppose $\exp(H) = p^2$, then $\exp(\Phi(H)) = p$, and by Proposition 15 $\exp(Z(H)) \geq p^2$ and so there must exist $\alpha \in Z(H)$ such that $\alpha \in H \backslash \Phi(H)$. That is α is a generator of H, but then as the rank of H is at most 2 it follows that H must be abelian. □

It follows from this that all rank 2 powerfully nilpotent groups are strongly powerful. Let r be the largest integer such that $[G, G] \leq G^{p^r}$. Note that since G is not abelian, $[G, G] \neq \{1\}$ and so by the proposition above we have that $r \geq 2$. Then there exists $n > r$ such that $|[G, G]| = p^{n-r}$. As $[G, G]$ is a cyclic subgroup of G^{p^r} and G is powerful, $[G, G]$ must have a generator of the form a^{p^r} and $o(a) = p^n$. Notice that $a \in G \backslash G^p$, for if $a \in G^p$, then $[G, G] = \langle a^{p^r} \rangle \leq G^{p^{r+1}}$, contradicting the maximality of r. Note also that $\langle a \rangle$ is normal in G as $G' \leq \langle a \rangle$. Let m be a positive integer such that $p^m = \left|\frac{G}{\langle a \rangle}\right| = \frac{|G|}{p^n}$. It follows that $|G| = p^{n+m}$. For any $b \in G \backslash \langle a \rangle G^p$ (i.e. any other generator), we have $b^{p^m} \in \langle a \rangle$, say $b^{p^m} = a^{p^l}$, with $1 \leq l \leq n$. Among all choices of a and b as above, make a choice so that l is maximal. We may then assume we have a presentation of the form: $a^{p^n} = 1$, $b^{p^m} = a^{p^l}$ and $[a, b] = a^{p^r}$. Note that one might need to adjust the powers to assume this clean form, see the Appendix A.1.

Lemma 72. $Z(G) = \langle a^{p^{n-r}}, b^{p^{n-r}} \rangle$.

Proof. Observe that $a^e b^f \in Z(G) \Leftrightarrow [a^e, b] = 1$ and $[a, b^f] = 1$. Thus we can consider separately which elements a^e commute with b and then which elements b^f commute with a. First we find which elements a^e commute with b. Notice that $(a^e)^b = a^{e+ep^r} = a^e \Leftrightarrow p^n | ep^r \Leftrightarrow p^{n-r} | e \Leftrightarrow a^e \in \langle a^{p^{n-r}} \rangle$. Now let $f = p^k x$, where $p \nmid x$. Note that $b^{p^k x} \in C_G(a) \Leftrightarrow b^{p^k} \in C_G(a)$ and hence we can assume $f = p^k$. Then $a = a^{b^{p^k}} = a^{(1+p^r)^{p^k}} = a^{1+p^{r+k}+tp^{r+k+1}}$ for some $t \in \mathbb{Z} \Leftrightarrow n \leq r + k \Leftrightarrow k \geq n - r \Leftrightarrow b^f \in \langle b^{p^{n-r}} \rangle$. □

Hence we must have $m, l \geq n - r$, as $b^{p^m}, a^{p^l} \in Z(G)$. Notice also that either $l = n$ in which case the group is a semidirect product, or else we must have $m > l$. To see this, suppose for contradiction that $m \leq l < n$. Modulo $\langle a^{p^{l+1}} \rangle$, we have $(ba^{-p^{l-m}})^{p^m} = b^{p^m} a^{-p^l} = 1$ and thus $(ba^{-p^{l-m}})^{p^m} \in \langle a^{p^{l+1}} \rangle$ contradicting the maximality of l. Finally, for the non-split groups we must have $r < l$. Suppose for contradiction that $l \leq r$. Then $a^{p^r} = (a^{p^l})^{p^{r-l}} = (b^{p^m})^{p^{r-l}} = b^{p^{m+r-l}} \implies [a, b] = b^{p^{m+r-l}}$. Now notice that $m + r - l > r$, because we have that $m > l$. Then this means that $G' \leq G^{p^{r+1}}$, contradicting the maximality of r. We thus arrive at two types of presentations:

I. Semidirect products: $G(n, m, r) := \langle a, b | a^{p^n} = 1, b^{p^m} = 1, [a, b] = a^{p^r} \rangle$ with parameters

$$m \geq n - r$$
$$2 \leq r \leq n - 1.$$

II. Non-semidirect products: $G(n, m, l, r) := \langle a, b | a^{p^n} = 1, b^{p^m} = a^{p^l}, [a, b] = a^{p^r} \rangle$ with parameters

$$m > l \geq n - r$$
$$2 \leq r \leq n - 1$$
$$n - r \leq l \leq n - 1$$
$$r < l.$$

For (II) these parameter constraints can be simplified to the system:

$$2 \leq r < l \leq n - 1$$
$$n - r \leq l < m.$$

Remark 73. The process used in the classification above gives a canonical way to go from a group to a presentation with parameterisation (n, m, l, r). If two groups are the same, they will have the same characteristic features and so we will get the same canonical parameterisation from applying the process. Similarly if two groups are different, we must get different presentations. Conversely, the arguments below show that given a presentation of type (I) or (II), that presentation does determine the structure of the group in the way we have claimed above. First note that it is easy to see from the presentations that the groups are powerful (since $r \geq 2$).

1) For the split case (type I) it is clear that r is the largest positive integer such that $G' \leq G^{p^r}$, for the non-split case notice that $l > r$ and so $a^{p^r} \notin \langle a^{p^l} \rangle$.

2) We see from the presentation that $[G, G] = \langle a^{p^r} \rangle$ and $o(a^{p^r}) = \frac{o(a)}{p^r} = p^{n-r}$.

3) $G/\langle a \rangle$ is cyclic of order p^m and so $p^m = |G|/p^n$.

4)If we are in (II) then $G^{p^m} = \langle a^{p^m}, a^{p^l} \rangle = \langle a^{p^l} \rangle$ and so clearly $p^{n-l} = |G^{p^m}|$.

Thus we see we have a one to one correspondence between parameters and groups, and in particular different parameters encode different groups.

Remark 74. Notice that the groups of type (I) and (II) are distinct. Suppose for contradiction that a group of type (II) could also be written as a semidirect product as in (I): $G = \langle c \rangle \rtimes \langle d \rangle$ with $o(c) = p^n$ and $o(d) = p^m$. Then $G^{p^m} = \langle c^{p^m}, d^{p^m} \rangle = \langle c^{p^m}, 1 \rangle = \langle c^{p^m} \rangle$ and $\exp(G^{p^m}) = p^{n-m}$ or $G^{p^m} = \{1\}$ if $m > n$. However from (II) we have $G^{p^m} = \langle a^{p^m}, b^{p^m} \rangle = \langle a^{p^m}, a^{p^l} \rangle$, but $l < m$ and so $G^{p^m} = \langle a^{p^l} \rangle \neq 1$ and thus $\exp(G^{p^m}) = p^{n-l}$. We obtain a contradiction since $p^{n-m} \neq p^{n-l}$ as $l < m$. We can assume that n and m in the second presentation are the same as in the first, as by the previous remark we see they correspond to structural properties of the group.

Remark 75. These presentations are consistent and do define groups of order p^{n+m}. For groups of type (I) this is clear since they are semidirect products. For groups of type (II) note that they can be constructed in the following way. Suppose we want to construct a group G with $a^{p^n} = 1, b^{p^m} = a^{p^l}, [a,b] = a^{p^r}$, where the parameters satisfy $m > l \geq n-r$ and $2 \leq r < l \leq n-1$. Consider a group H with presentation $\langle \bar{a}, \bar{b} | \bar{a}^{p^n} = 1, \bar{b}^{p^{m+n-l}} = 1, [\bar{a}, \bar{b}] = \bar{a}^{p^r} \rangle$, and note that this satisfies the constraints for a type (I) group since we have that: $m+n-l \geq m \geq n-r$ (since $1 \leq n-l$). Consider the subgroup $\langle \bar{b}^{p^m} \bar{a}^{-p^l} \rangle$. Note that as $m, l \geq n-r$ we have that $\bar{b}^{p^m} \bar{a}^{-p^l} \in Z(H)$ and so the subgroup is normal. Then G can be realised as the quotient $H / \langle \bar{b}^{p^m} \bar{a}^{-p^l} \rangle$.

3.2. A formula for powerful class

In this part we derive formulas for the powerful nilpotency class of a non-abelian powerfully nilpotent group G of rank 2. For groups of type (I), consider a group G with $\langle a, b | a^{p^n} = 1, b^{p^m} = 1, [a,b] = a^{p^r} \rangle$. By Lemma 72, we know that $Z(G) = \langle a^{p^{n-r}}, b^{p^{n-r}} \rangle$ and so $Z(G)^p = \langle a^{p^{n-r+1}}, b^{p^{n-r+1}} \rangle$. Note that for type (I) groups there is the option that $m = n-r$ and in this case $Z(G)^p = \langle a^{p^{n-r+1}} \rangle$, in what follows we deal with $m > n-r$ but the calculations for $m = n-r$ are similar. Quotient G by $Z(G)^p$ to obtain $\bar{G} : \bar{a}^{p^{n-r+1}} = 1, \bar{b}^{p^{n-r+1}} = 1, [\bar{a}, \bar{b}] = \bar{a}^{p^r}$and then reapply the lemma to get

$$\hat{Z}_2(G) = \begin{cases} \langle a^{p^{n-2r+1}}, b^{p^{n-2r+1}} \rangle & \text{if } n-2r+1 > 0, \\ G & \text{if } n-2r+1 \leq 0. \end{cases}$$

Applying this process repeatedly we see that

$$\hat{Z}_j(G) = \begin{cases} \langle a^{p^{n-1+j(1-r)}}, b^{p^{n-1+j(1-r)}} \rangle & \text{if } n-1+j(1-r) > 0, \\ G & \text{if } n-1+j(1-r) \leq 0. \end{cases}$$

Then the powerful class is the smallest integer j such that $n-1+j(1-r) \leq 0$, which is equivalent to saying that j is the smallest integer such that $j \geq \frac{n-1}{r-1}$. Hence the powerful class is $\left\lceil \frac{n-1}{r-1} \right\rceil$.

We next consider the groups of type (II). The groups of type (II) must be split into two cases, depending on the value of the parameter m. Suppose we have a group of type (II) of the form $a^{p^n} = 1, b^{p^m} = a^{p^l}, [a,b] = a^{p^r}$, and recall the bounds on the parameters: $2 \leq r < l \leq n-1$ and $n-r \leq l < m$. By Lemma 72, we know that $Z(G) = \langle a^{p^{n-r}}, b^{p^{n-r}} \rangle$, and so it follows that $Z(G)^p = \langle a^{p^{n-r+1}}, b^{p^{n-r+1}} \rangle$. If $n-r+1 < m$, then the case here is the same as for (I) above, and by the same process we find that the powerful class is $\left\lceil \frac{n-1}{r-1} \right\rceil$. We now consider the second case where $n-r+1 = m$. In this case we have that $Z(G)^p = \langle a^{p^{n-r+1}}, b^{p^m} \rangle = \langle a^{p^{n-r+1}}, a^{p^l} \rangle$. By the inequality conditions, we have that $n-r \leq l < m = n-r+1$, and so we must have that $l = n-r$ and so in fact $Z(G)^p = \langle a^{p^{n-r}} \rangle$. We now repeat the same process as before, quotienting by the p^{th} power of the centre, and finding the centre of this quotient group, to work our way along the upper powerful central series. Taking the quotient of G by $Z(G)^p$ we find $\bar{G} = \langle \bar{a}, \bar{b} | \bar{a}^{p^{n-r}} = 1, \bar{b}^{p^m} = 1, [\bar{a}, \bar{b}] = \bar{a}^{p^r} \rangle$and

$$\hat{Z}_2(G) = \begin{cases} \langle a^{p^{n-2r}}, b^{n-2r} \rangle & \text{if } n-2r > 0, \\ G & \text{if } n-2r \leq 0. \end{cases}$$

Continuing in this manner it follows that for $j \geq 2$:

$$\hat{Z}_j(G) = \begin{cases} \langle a^{p^{n-jr+(j-2)}}, b^{p^{n-jr+(j-2)}} \rangle & \text{if } n - jr + (j-2) > 0, \\ G & \text{if } n - jr + (j-2) \leq 0. \end{cases}$$

In this case the powerful class is the smallest integer j such that $n - jr + (j-2) \leq 0$. Equivalently, the smallest integer $j \geq \frac{n-2}{r-1}$, that is $j = \left\lceil \frac{n-2}{r-1} \right\rceil$. Hence we have the following:

Proposition 76. *For a non-abelian, rank 2, powerfully nilpotent group G of type (I), or of type (II) with $m > n - r + 1$,*

$$\textit{the powerful class of } G \textit{ is } \left\lceil \frac{n-1}{r-1} \right\rceil.$$

If G is of type (II) with $m = n - r + 1$ then:

$$\textit{the powerful class of } G \textit{ is } \left\lceil \frac{n-2}{r-1} \right\rceil.$$

Remark 77. It follows from this that for a powerfully nilpotent group G of rank 2 and order p^N, we have $c \leq \lfloor \frac{N}{2} \rfloor$, where c is the powerful class of G. Furthermore, this bound is sharp, as can be seen from the following examples.

Example 78. Let n be an even integer. The group $G = \langle a, b | \ a^{3^n} = 1, \ b^{3^{n-2}} = 1, \ b^a = ba^9 \rangle$ has powerful class $n - 1$ and is a group of order $N = 2n - 2$. Similarly $G = \langle a, b | \ a^{3^n} = 1, \ b^{3^{n-1}} = 1 \ , b^a = ba^9 \rangle$ has powerful class $n - 1$ and is a group of order $N = 2n - 1$.

3.3. Counting the split groups with respect to order

In this section we count the split groups (those of type (I)), with respect to order. That is, for a given $x \geq 4$, we produce a formula for the number of type (I) groups of order p^x. Recall our presentation for the groups of type (I): $\langle a, b | a^{p^n} = 1, b^{p^m} = 1, [a, b] = a^{p^r} \rangle$. We take the inequality conditions on our parameters, and obtain an equivalent system of inequalities that is easier to count. We label our initial inequalities (1), (2) and (3) with $m \geq_{(1)} n - r$ and $2 \leq_{(2)} r \leq_{(3)} n - 1$. Let $x = n + m$ and $y = n - r$. From (1) $m \geq n - r = y \Leftrightarrow x = m + n \geq y + n \Leftrightarrow x - 2 \geq y + (n-2)$. From (2) $2 \leq r \Leftrightarrow -2 \geq -r \Leftrightarrow n - 2 \geq n - r = y$. Furthermore observe $n - 2 \leq n + 1 - 2 \leq n + m - 2 = x - 2$ as we must have $m \geq 1$. From (3) $r \leq n - 1 \Leftrightarrow 1 \leq y$. Thus by combining these we obtain the equivalent conditions:

$$1 \leq_{(4)} y \leq_{(5)} n - 2 \leq_{(6)} x - 3$$
$$2 \leq_{(7)} y + (n-2) \leq_{(8)} x - 2.$$

For a given order p^x, note that n and x determine m, and that n and y determine r. Thus to enumerate all type (I) groups of order p^x we must count all allowable pairs (n, y). First we look at conditions on y. By (4) we have that $y \geq 1$, and condition (7) does not strengthen that. From adding (5) and (8) we obtain that $2y + (n-2) \leq (x-2) + (n-2)$ and so it follows that that $y \leq \frac{x-2}{2}$ and as y must be an integer, we obtain $y \leq \lfloor \frac{x-2}{2} \rfloor$.

We must now count the number of allowed values for n that can be paired with each y. From (6) and (8) we see that $n-2 \leq \min(x-2-y, x-3)$. We always have $x-2-y \leq x-3$ and so we deduce the bound $n-2 \leq x-2-y$. From (5) and (7) we see that $\max(y, 2-y) \leq n-2$, and because of the condition that $y \geq 1$ we see that $n-2 \geq y$. Thus we count

$$\sum_{y=1}^{\lfloor \frac{x-2}{2} \rfloor} |\{n : y \leq n-2 \leq x-2-y\}| = \sum_{y=1}^{\lfloor \frac{x-2}{2} \rfloor} (x-1-2y) = \lfloor \frac{x-2}{2} \rfloor \cdot (x-1) - 2 \cdot \lfloor \frac{x-2}{2} \rfloor \cdot \lfloor \frac{x}{2} \rfloor \cdot \frac{1}{2}$$

and then we obtain the following formula.

Proposition 79. The number of (non-abelian) split groups of order p^x is

$$\begin{cases} (\frac{x-2}{2})^2 & \textit{if } x \textit{ is even,} \\ \frac{(x-3)(x-1)}{4} & \textit{if } x \textit{ is odd.} \end{cases}$$

3.4. Counting the non-split groups with respect to order

In this section we find a formula for the number of non-split (type (II)) groups of order p^x. Recall our presentation for these types of groups is $\langle a, b | a^{p^n} = 1, b^{p^m} = a^{p^l}, [a,b] = a^{p^r} \rangle$ with $2 \leq r < l \leq n-1$ and $n - r \leq l < m$. As before, we now transform our system of inequalities into an equivalent form that will be easier to work with. Let $x = n + m$ and $y = n - r$. We label our initial system (1), (2), (3), (4) and $(*)$:

$$n - r \leq_{(1)} l <_{(2)} m$$
$$2 \leq_{(*)} r <_{(3)} l \leq_{(4)} n - 1$$

For (1) note that $y = n - r \leq l$. From (2), $l < m \Leftrightarrow n + l < x \Leftrightarrow n < x - l \Leftrightarrow n - 1 \leq x - l - 2$. From (3), $r < l \Leftrightarrow -r > -l \Leftrightarrow n - r > n - l \Leftrightarrow y > n - l \Leftrightarrow y \geq n - l + 1$. Condition (4) is unchanged. Now we turn our attention to condition $(*)$, that is $2 \leq r$. This follows as a consequence of (1) and (4) if $n - r < l$, because then we have $y = n - r < l \leq n - 1$ and so $r \geq 2$. However if $n - r = l$ then we must explicitly avoid $l = n - 1$ otherwise $r = 1$. Thus we obtain the equivalent system with two cases, which we have labelled (α) and (β):

$$\begin{array}{ccc} l < n-1 \leq x-2-l & & l = n-1 \\ n-1-l+2 \leq y \leq l & \text{or} & n-1-l+2 \leq y \leq l-1 \Leftrightarrow 2 \leq y \leq l-1 \\ (\alpha) & & (\beta). \end{array}$$

Finally we make the following observation, as $l > 2$ we have $l \geq 3$. Furthermore as $l \leq n-1$ and $l < m$ we have by summing these that $2l < x - 1$ and it follows that $l \leq \lfloor \frac{x-2}{2} \rfloor$. Thus a consequence of our conditions is $3 \leq l \leq \lfloor \frac{x-2}{2} \rfloor$.
First consider the case (β). In this case for each l the number of values for y that can be paired with each l is $l-2$. Thus we have the contribution from (β) as $\sum_{l=3}^{\lfloor \frac{x-2}{2} \rfloor} l - 2$. Now consider the case (α). For each value l, $n-1$ can take all values such that $l < n - 1 \leq x - 2 - l$. Then for each pair $(l, n-1)$ we seek to count the number of

values for y such that $n-1-l+2 \leq y \leq l$. However observe that we get a non empty set of solutions only if $l \geq n-1-l+2$, equivalently, $n-1 \leq 2l-2$. Thus we must split (α) into two cases:
Case (a). $x-2-l < 2l-2 \Leftrightarrow x < 3l$, this forces $n-1 \leq x-2-l < 2l-2$ and so we have solutions for all $n-1$.
Case (b). If $x-2-l \geq 2l-2$ then $x \geq 3l \Leftrightarrow \lfloor \frac{x}{3} \rfloor \geq l$.
Thus counting we obtain the following:

$$
\begin{aligned}
&\sum_{l=3}^{\lfloor \frac{x}{3} \rfloor} \left[\left(\sum_{n-1=l+1}^{2l-2} (2l-n) \right) + (l-2) \right] + \sum_{l=\lfloor \frac{x}{3} \rfloor+1}^{\lfloor \frac{x-2}{2} \rfloor} \left[\sum_{n-1=l+1}^{x-2-l} (2l-n) + (l-2) \right] \\
&= \sum_{l=1}^{\lfloor \frac{x-6}{3} \rfloor} \left[\left(\sum_{n-1=l+3}^{2l+2} (2l-n+4) \right) + l \right] + \sum_{l=1}^{\lfloor \frac{x-6}{2} \rfloor} \left[\left(\sum_{n-1=l+3}^{x-4-l} (2l-n+4) \right) + l \right] \\
&\qquad - \sum_{l=1}^{\lfloor \frac{x-6}{3} \rfloor} \left[\left(\sum_{n-1=l+3}^{x-4-l} (2l-n+4) \right) + l \right] \\
&= \sum_{l=1}^{\lfloor \frac{x-6}{3} \rfloor} \left[\left(\sum_{n-1=l+3}^{2l+2} (2l+3)-(n-1)) \right) + l \right] + \sum_{l=1}^{\lfloor \frac{x-6}{2} \rfloor} \left[\left(\sum_{n-1=l+3}^{x-4-l} ((2l+3)-(n-1)) \right) + l \right] \\
&\qquad - \sum_{l=1}^{\lfloor \frac{x-6}{3} \rfloor} \left[\left(\sum_{n-1=l+3}^{x-4-l} \Big((2l+3)-(n-1)\Big) \right) + l \right] \\
&= \sum_{l=1}^{\lfloor \frac{x-6}{3} \rfloor} \left[(2l+3)l - \frac{(3l+5)l}{2} + l \right] - \sum_{l=1}^{\lfloor \frac{x-6}{3} \rfloor} \left[(2l+3)(x-6-2l) - \frac{(x-1)(x-6-2l)}{2} + l \right] \\
&\qquad + \sum_{l=1}^{\lfloor \frac{x-6}{2} \rfloor} \left[(2l+3)(x-6-2l) - \frac{(x-1)(x-6-2l)}{2} + l \right] \\
&= \sum_{l=1}^{\lfloor \frac{x-6}{2} \rfloor} \left[\frac{(x-6-2l)}{2}(4l+7-x) + l \right] + \sum_{l=1}^{\lfloor \frac{x-6}{3} \rfloor} \left[\frac{(l+1)l}{2} + l \right] \\
&\qquad - \sum_{l=1}^{\lfloor \frac{x-6}{3} \rfloor} \left[\frac{(x-6-2l)}{2}(4l+7-x) + l \right] \\
&= \sum_{l=1}^{\lfloor \frac{x-6}{2} \rfloor} \left[\frac{(x-6)(7-x)}{2} + l(3x-18) - 4l^2 \right] + \sum_{l=1}^{\lfloor \frac{x-6}{3} \rfloor} \left[\frac{(x-6)(x-7)}{2} + l\left(\frac{-6x+39}{2} \right) + \frac{9}{2}l^2 \right] \\
&= \left\lfloor \frac{x-6}{2} \right\rfloor \frac{(x-6)(7-x)}{2} + \left\lfloor \frac{x-6}{2} \right\rfloor \left\lfloor \frac{x-4}{2} \right\rfloor \frac{(3x-18)}{2} \\
&\qquad - \frac{4\lfloor \frac{x-6}{2} \rfloor \lfloor \frac{x-4}{2} \rfloor (2\lfloor \frac{x-6}{2} \rfloor + 1)}{6} + \left\lfloor \frac{x-6}{3} \right\rfloor \frac{(x-6)(x-7)}{2} \\
&\qquad + \left\lfloor \frac{x-6}{3} \right\rfloor \left\lfloor \frac{x-3}{3} \right\rfloor \left(\frac{-6x+39}{4} \right) + \frac{9}{2} \frac{\lfloor \frac{x-6}{3} \rfloor \lfloor \frac{x-3}{3} \rfloor (2\lfloor \frac{x-6}{3} \rfloor + 1)}{6}.
\end{aligned}
$$

Dealing with the six cases modulo 6 we obtain the following formulas for the number of type (II) groups of order p^x:

$$\begin{aligned}
\frac{(x-6)}{72}(x^2-24) &\text{ if } x \equiv 0\,(mod\,6),\\
\frac{(x-7)}{72}(x^2+x-26) &\text{ if } x \equiv 1\,(mod\,6),\\
\frac{1}{72}(x^3-6x^2-24x+136) &\text{ if } x \equiv 2\,(mod\,6),\\
\frac{(x-6)}{72}(x^2-33) &\text{ if } x \equiv 3\,(mod\,6),\\
\frac{(x-4)}{72}(x^2-2x-32) &\text{ if } x \equiv 4\,(mod\,6),\\
\frac{(x-5)}{72}(x^2-x-38) &\text{ if } x \equiv 5\,(mod\,6).
\end{aligned}$$

We now can obtain a formula for the number of rank 2 powerfully nilpotent groups of order p^x. For a given x we add the number of abelian rank 2 groups of this order ($\lfloor \frac{x}{2} \rfloor$) with the number of split and non-split groups.

Proposition 80. *The number of rank 2 groups which are powerfully nilpotent of order p^x for $x \geq 4$ are:*

$$\begin{array}{ll}
\dfrac{x^3+12x^2-60x+216}{72} & if\ N\equiv 0(mod\ 6)\\
\dfrac{x^3+12x^2-69x+200}{72} & if\ N\equiv 1(mod\ 6)\\
\dfrac{x^3+12x^2-60x+208}{72} & if\ N\equiv 2(mod\ 6)\\
\dfrac{x^3+12x^2-69x+216}{72} & if\ N\equiv 3(mod\ 6)\\
\dfrac{x^3+12x^2-60x+200}{72} & if\ N\equiv 4(mod\ 6)\\
\dfrac{x^3+12x^2-69x+208}{72} & if\ N\equiv 5(mod\ 6)
\end{array}$$

3.5. Children and parents

3.5.1. Children of rank 2 groups

In this section we find presentations for the children of a given rank 2 powerfully nilpotent group. To begin we consider children of semidirect products. That is, given a group $G = \langle a, b |\, a^{p^n} = 1, b^{p^m} = 1, [a,b] = a^{p^r}\rangle$ where the conditions $m \geq n-r$ and $2 \leq r \leq n-1$ hold, we find a presentation for $G/Z(G)^p$. We remark that by Lemma 72 we know that $Z(G)^p = \langle a^{p^{n-r+1}}, b^{p^{n-r+1}}\rangle$. From the conditions on our presentations, we know that $m \geq n-r$ and thus we see that we have two cases depending on whether $b^{p^m} \in Z(G)^p$. These are

<u>I.(A) Children of semidirect products where the parent has $m > n-r$:</u>

$$\langle a, b |\, a^{p^{n-r+1}} = 1,\ b^{p^{n-r+1}} = 1,\ [a,b] = a^{p^r}\rangle.$$

I.(B) Children of semidirect products where the parent has $m = n - r$:

$$\langle a, b|\, a^{p^{n-r+1}} = 1,\ b^{p^{n-r}} = 1,\ [a, b] = a^{p^r}\rangle.$$

Note that in both cases the child is abelian if and only if $r \geq n - r + 1$.
Next we look at children of non-semidirect products. That is, the parent group is of the form $G = \langle a, b|\, a^{p^n} = 1, b^{p^m} = a^{p^l}, [a, b] = a^{p^r}\rangle$, where the conditions $2 \leq r < l \leq n - 1$ and $n - r \leq l < m$ hold. As before we have two cases, this time depending on the relationship between $n - r$ and l. From the parameters we know that $n - r \leq l$ and the point to note is that if $n - r = l$ then $Z(G)^p = \langle a^{p^{l+1}}, b^{p^{l+1}}\rangle$. But $l + 1 < m$ and so $b^{p^m} = a^{p^l} \in Z(G)^p$. Thus we have
II.(A) Children of non-semidirect products where the parent has $n - r < l$:

$$\langle a, b|\, a^{p^{n-r+1}} = 1,\ b^{p^{n-r+1}} = 1,\ [a, b] = a^{p^r}\rangle.$$

Note that this is abelian if and only if $r \geq n - r + 1$.
II.(B) Children of non-semidirect products where the parent has $n - r = l$:

$$\langle a, b|\, a^{p^{n-r}} = 1,\ b^{p^{n-r+1}} = 1, [a, b] = a^{p^r}\rangle.$$

Note that in this case the child is abelian if and only if $r \geq n - r$, but notice that this never happens because we have the condition that $r < l$ and in this case $l = n - r$.

Remark 81. Notice that in all cases the child group is a semidirect product. In other words, the non-semidirect products have no parents.

3.5.2. Parents of rank 2 groups

Now in this section we consider the converse question, given a powerfully nilpotent group of rank 2, what are its parents with parameters in terms of our classification. The method is to look at the presentation of the child group, and use the characterisation of Section 3.5.1 to "work backwards" and deduce the parent. Once we have a presentation for the parent, we must check the parameters satisfy our required conditions.

3.5.2.1. Parents of abelian groups

First, we consider parents of abelian groups. By considering the results of Section 3.5.1 we see that the abelian groups that have parents are
(a) $a^{p^{\overline{n}}} = 1$, $b^{p^{\overline{n}}} = 1$, $\overline{n} \geq 2$.
(b) $a^{p^{\overline{n}+1}} = 1$, $b^{p^{\overline{n}}} = 1$, $\overline{n} \geq 1$.
Note that in (a) $\overline{n} = n - r + 1$ and in (b) $\overline{n} = n - r$, where n and r are parameters of the parent. The conditions on $\overline{n}$ follow by observing that we must have $n - r \geq 1$ in the parent group. The semidirect parents of (a) are of the form

$$a^{p^{\overbrace{\overline{n} + r - 1}^{n}}} = 1,\ b^{p^m} = 1,\ [a, b] = a^{p^r}$$

with $r \geq \overline{n}$ (since the child is abelian) and $m \geq \overline{n}$ (since $m > n - r = \overline{n} - 1$). We now check that our parameter conditions are satisfied. Notice that $2 \leq \overline{n} \leq r \leq r + \overline{n} - 2 = n - 1$ so $2 \leq r \leq n - 1$ and also that $m \geq \overline{n} = n - r + 1 > n - r$. Note that there are infinitely many parents of this form.

The non-semidirect parents of (a) are of the form

$$a^{p^{\overbrace{\overline{n}+r-1}^{n}}} = 1,\ b^{p^m} = a^{p^l},\ [a,b] = a^{p^r}$$

such that $r + (\overline{n} - 2) \geq l > r \geq \overline{n}$ and $m > l \geq \overline{n}$. We note that $r \geq \overline{n}$ is necessary since the child is abelian, and $n-r < l \Leftrightarrow \overline{n} \leq l$ is because this is a child of type II.(A) in Section 3.5.1. The other conditions must be enforced for the parameterisation to be valid. Indeed, notice that $2 \leq \overline{n} \leq r < l \leq r+\overline{n}-2 = n-1$ and $n-r = \overline{n}-1 < l < m$. Finally we consider parents of groups of type (b). Notice that such a parent must be semidirect of the form

$$a^{p^{\overline{n}+r}} = 1,\ b^{p^{\overline{n}}} = 1,\ [a,b] = a^{p^r},$$

with $r \geq \overline{n}+1$ (since the child group is abelian). Checking the parameter conditions, we see that $r \geq \overline{n}+1 \geq 2$, and in this case also $m = n-r$ and so $m \geq n-r$ is satisfied. Notice that $r \leq n-1 = \overline{n}+r-1$ is clear since $\overline{n} \geq 1$.

3.5.2.2. Parents of (non-abelian) semidirect products

By considering the results in Section 3.5.1 we see that the semidirect products that have parents are
(a) $a^{p^{\overline{n}}} = 1,\ b^{p^{\overline{n}}} = 1,\ [a,b] = a^{p^r}$ with $2 \leq r \leq \overline{n}-1$ and $\overline{n} \geq 3$,
(b) $a^{p^{\overline{n}+1}} = 1,\ b^{p^{\overline{n}}} = 1,\ [a,b] = a^{p^r}$ with $2 \leq r \leq \overline{n}$ and $\overline{n} \geq 2$,
(c) $a^{p^{\overline{n}}} = 1,\ b^{p^{\overline{n}+1}} = 1,\ [a,b] = a^{p^r}$ with $2 \leq r \leq \overline{n}-1$ and $\overline{n} \geq 3$.
Note the conditions on $\overline{n}$ and r stop the group from being abelian. The semidirect parents of (a) are

$$a^{p^{\overline{n}+r-1}} = 1,\ b^{p^m} = 1,\ [a,b] = a^{p^r}$$

with $m \geq \overline{n}$, (since this is case I.(A)). Notice that the parameter conditions hold:

$$2 \leq r \leq \overline{n}-1 < n-1,$$
$$m \geq \overline{n} = n-r+1 > n-r,$$

and furthermore notice that there exists infinitely many parents. The non-semidirect parents of (a) are:

$$a^{p^{\overline{n}+r-1}} = 1,\ b^{p^m} = a^{p^l},\ [a,b] = a^{p^r}$$

with $m > l \geq \overline{n}$ and $l \leq r+\overline{n}-2$. Notice that $2 \leq r \leq n-1 < l \leq r+\overline{n}-2 = n-1$ and also that $n-r = \overline{n}-1 < l < m$ and thus the parameter conditions hold. Again notice that there are infinitely many parents of this type. Now for case (b), parents will be semidirect (of type I.(B)), and in fact there is exactly one parent group:

$$a^{p^{\overline{n}+r}} = 1,\ b^{p^{\overline{n}}} = 1,\ [a,b] = a^{p^r}.$$

Notice that $m = \overline{n} = n-r$ and $2 \leq r \leq \overline{n} = n-r < n-1$ and so the required conditions hold. Finally we move on to case (c). The parents of (c) are of type II.(B), of the form

$$a^{p^{\overline{n}+r}} = 1,\ b^{p^m} = a^{p^{\overline{n}}},\ [a,b] = a^{p^r},$$

with $m > \overline{n}$. Notice that $2 \leq r \leq \overline{n}-1 < \overline{n} = l < \overline{n}+r-1 = n-1$ and $n-r = \overline{n} \leq \overline{n} = l < m$. In this case there are infinitely many parents.

3.5.3. Further up the tree and infinite branches

We have seen in Example 38 that there exist groups that have infinitely many ancestors. In this section we explain the situation for the infinite "branch" of the ancestry tree for the rank 2 powerfully nilpotent groups. It follows from the analysis above that the only groups that have grandparents are

$$a^{p^{\overline{n}}} = 1,\ b^{p^{\overline{n}}} = 1,\ [a,b] = 1,\ \overline{n} \geq 2$$

$$a^{p^{\overline{n}}} = 1,\ b^{p^{\overline{n}}} = 1,\ [a,b] = a^{p^r},\ \overline{n} \geq 3,\ 2 \leq r \leq \overline{n} - 1.$$

In particular, if there is an infinite branch, it must start with an abelian group of the first type, with all subsequent groups being of the second type. We then have the following:

Proposition 82. *For rank 2 powerfully nilpotent groups, the infinite branches are precisely*

$$\begin{matrix} a^{p^{\overline{n}}=1} & & a^{p^{\overline{n}+r-1}} = 1 & & a^{p^{\overline{n}+i(r-1)}} = 1 & \\ b^{p^{\overline{n}}} = 1 & \leftarrow & b^{p^{\overline{n}+r-1}} = 1 & \leftarrow \cdots \leftarrow & b^{p^{\overline{n}+i(r-1)}} = 1 & \leftarrow \ldots . \\ 2 \leq \overline{n} \leq r & & [a,b] = a^{p^r} & & [a,b] = a^{p^r} & \end{matrix}$$

4. Omegas of Agemos in Powerful Groups

4.1. Introduction

It is well known that for a powerful p-group G, the i^{th} Agemo subgroup, $\mho_i(G) = G^{p^i}$, coincides with the set of $p^{i^{th}}$ powers, and that this subgroup is itself powerful [LM87, Corollary 1.2, Proposition 1.7]. We have seen that for a powerful p-group G the groups G^{p^i}, for $i \geq 1$, are powerfully nilpotent. In some sense dual to the Agemo subgroups are the Omega subgroups, $\Omega_i(G)$. For a powerful p-group G these Omega subgroups are studied in [FA07].
We have observed how powerfully nilpotent groups often occur as characteristic subgroups of powerful groups. For example in Remark 28 we saw that the proper terms of the derived and lower central series of a powerful group G are powerfully nilpotent. One aim of this chapter is to further motivate the study of the relationship between powerful groups and the powerfully nilpotent groups within them, by showing another important class of characteristic subgroups of powerful groups to be powerfully nilpotent.
In this chapter we prove that for an odd prime p and a powerful p-group G, the Omega subgroups of any proper Agemo subgroup are powerfully nilpotent (and hence powerful), and moreover we can obtain a bound on the powerful nilpotency class. We obtain a similar result for $p = 2$ with a small modification. We remark that it follows from Theorem 1.1 in [GSJZ04], that $\Omega_i(G^p)$ is powerful. However in what follows we give an elementary proof of this fact.
In [FA07, Theorem 1] the following theorem is proved. We make extensive use of this theorem in this chapter. In keeping with [FA07], for a p-group G and $x \in G$, we define the meaning of the inequality $o(x) \leq p^i$ with $i < 0$ to be that $x = 1$. Similarly we define $\Omega_i(G) = \{1\}$ for $i < 0$.

Theorem 83 (Fernández-Alcober). *Let G be a powerful p-group. Then, for every $i \geq 0$:*
(i) If $x, y \in G$ and $o(y) \leq p^i$, then $o([x, y]) \leq p^i$.
(ii) If $x, y \in G$ are such that $o(x) \leq p^{i+1}$ and $o(y) \leq p^i$, then $o([x^{p^j}, y^{p^k}]) \leq p^{i-j-k}$ for all $j, k \geq 0$.
(iii) If p is odd, then $\exp \Omega_i(G) \leq p^i$.
(iv) If $p = 2$, then $\exp \Omega_i(T) \leq 2^i$ for any subgroup T of G which is cyclic over G^2. In particular, $\exp \Omega_i(G^2) \leq 2^i$.

4.2. Omega subgroups of agemo subgroups

The natural place to start when considering Omega subgroups of powerful p-groups is $\Omega_i(G)$. However it is not true in general that $\Omega_i(G)$ is powerful and such counterexamples are easy to find. Consider the following example.

Example 84. The 3-group

$$G = \langle a, b, c | a^3 = b^3 = c^9 = [c, b] = [c, a] = 1, [b, a] = c^3 \rangle$$

is powerful (in fact it is powerfully nilpotent), but $\Omega_1(G) = \langle a, b, c^3 \rangle$ is not powerful.

Thus we turn our attention to $\Omega_i(G^p)$. First we shall use Theorem 83 to prove that for a powerful p-group G, elements in G^p of order p commute with each other and with elements in G^p of order p^2.

Lemma 85. *Let G be a powerful p-group. Let $g_1, g_2 \in G^p$ where $o(g_1) = p$ and $o(g_2) \leq p^2$. Then $[g_1, g_2] = 1$.*

Proof. As G is powerful, we know that elements of G^p are pth powers, and so we may assume $g_1 = a^p, g_2 = b^p$ for $a, b \in G$ where $o(a) = p^2$ and $o(b) \leq p^3$. Using Theorem 83(ii) and taking $x = b$, $y = a$ and $i = 2$ we see that $o([x^p, y^p]) \leq p^{2-1-1} = 1$, hence $[g_3, g_1] = 1$. It follows that the elements in G^p of order p commute with the elements in G^p of order p^2. □

Notice that from this we obtain that $\Omega_1(G^p)$ is abelian. The next result is needed in the proof of Proposition 87, although it is also of independent interest in the context of better understanding the relationship between Agemo and Omega subgroups in powerful p-groups.

Proposition 86. *Let G be a powerful p-group. Then $\left(\Omega_i(G^{p^k})\right)^{p^j} \leq \Omega_{i-j}(G^{p^{k+j}})$ and $\exp(\Omega_i(G^{p^k})^{p^j}) \leq p^{i-j}$ for $i, j \geq 0$ and $k \geq 1$.*

Proof. Consider an element $x \in \left(\Omega_i(G^{p^k})\right)^{p^j}$. This element can be written in the form $g_1^{p^j} \cdots g_t^{p^j}$ where $g_l \in \Omega_i(G^{p^k})$ for each $l \in \{1, \ldots, t\}$. Note that $g_l \in G^{p^k}$ and so $g_l^{p^j} \in G^{p^{k+j}}$. Using Theorem 83(iii) if p is odd and Theorem 83(iv) if $p = 2$, it follows that the order of each g_l is at most p^i. Then the order of each $g_l^{p^j}$ is at most p^{i-j}. Thus each $g_l^{p^j} \in \Omega_{i-j}(G^{p^{k+j}})$. As $\Omega_{i-j}(G^{p^{k+j}})$ is a group, it is closed under taking products and so $x = g_1^{p^j} \cdots g_t^{p^j} \in \Omega_{i-j}(G^{p^{k+j}})$. Hence $\left(\Omega_i(G^{p^k})\right)^{p^j} \leq \Omega_{i-j}(G^{p^{k+j}})$. Then by Theorem 83(iii) if p is odd and Theorem 83(iv) if $p = 2$, we obtain that $\exp(\Omega_i(G^{p^k})^{p^j}) \leq p^{i-j}$. □

We now consider the case where p is an odd prime. We seek to show that $\Omega_i(G^p)$ is powerfully nilpotent for all $i \geq 1$. Recall by Proposition 30 that for any p-group G we have that G is powerfully nilpotent if and only if G/G^{p^2} is powerfully nilpotent. Thus in what follows we consider $H = \frac{\Omega_i(G^p)}{(\Omega_i(G^p))^{p^2}}$, for some powerful p-group G. Let $K = \frac{G}{(\Omega_i(G^p))^{p^2}}$. Notice that K and K^p are powerful and that $H \leq K^p$.

Proposition 87. *H is a powerful group.*

Proof. The exponent of H is at most p^2, and so it follows from Lemma 85 that all elements of order p are central. We thus only need to consider commutators between elements of order p^2. Since $H \leq K^p$, we can thus assume these commutators are of the form $[a^p, b^p]$ where $o(a) = p^3 = o(b)$. Applying Theorem 83(ii) with $x = a$, $y = b$ and $i = 3$ we see that $o([a^p, b^p]) \leq p$. Since K is powerful, we have that $[a^p, b^p] \in [K^p, K^p] = [K, K]^{p^2} \leq K^{p^3}$, and hence there exists some $g \in K$ such that $[a^p, b^p] = g^{p^3}$, where g has order at most p^4. Let $g = x\left(\Omega_i(G^p)^{p^2}\right)$. Then $x^{p^4} \in \Omega_i(G^p)^{p^2}$, which is of exponent at most p^{i-2}, by Proposition 86. Hence $o(x) \leq p^{4+i-2} = p^{i+2}$. Then x^{p^2} has order at most p^i and so $x^{p^2} \in \Omega_i(G^{p^2})$. Then $g^{p^3} = x^{p^3}\left(\Omega_i(G^p)^{p^2}\right) \in \left(\frac{\Omega_i(G^{p^2})}{\Omega_i(G^p)^{p^2}}\right)^p \leq H^p$. Thus H is powerful. □

Lemma 88. *H is powerfully nilpotent of powerful nilpotency class at most 2; in particular $H \geq \frac{\Omega_i(G^{p^2})}{\Omega_i(G^p)^{p^2}} \geq 1$ is a powerfully central chain..*

Proof. We will show that $H \geq \frac{\Omega_i(G^{p^2})}{\Omega_i(G^p)^{p^2}} \geq 1$ is a powerfully central chain. In the proof of Proposition 87 we saw that $[H, H] \leq \left(\frac{\Omega_i(G^{p^2})}{\Omega_i(G^p)^{p^2}}\right)^p$. We now show that $\frac{\Omega_i(G^{p^2})}{\Omega_i(G^p)^{p^2}} \leq Z(H)$; to do this we will show that $[\Omega_i(G^{p^2}), \Omega_i(G^p)] \leq \Omega_i(G^p)^{p^2}$. Consider $[g^{p^2}, h^p]$ for $g, h \in G$ with $o(g) \leq p^{i+2}$ and $o(h) \leq p^{i+1}$. Using Theorem 83(ii) we obtain that $o([g^{p^2}, h^p]) \leq p^{i-2}$. As $[g^{p^2}, h^p] \in G^{p^4}$ we may write $[g^{p^2}, h^p] = k^{p^4}$ for some $k \in G$. Then $o(k^{p^2}) \leq p^i$ and so $[g^{p^2}, h^p] = (k^{p^2})^{p^2} \in \Omega_i(G^p)^{p^2}$. Thus $\frac{\Omega_i(G^{p^2})}{\Omega_i(G^p)^{p^2}} \leq Z(H)$. Hence it follows that $H \geq \frac{\Omega_i(G^{p^2})}{\Omega_i(G^p)^{p^2}} \geq 1$ is a powerfully central chain. □

Using Lemma 88 and Proposition 30 one can obtain a powerfully central chain for $\Omega_i(G^p)$ of length $2i - 1$. However a shorter chain is possible. The following Lemma will be used to reduce the length of the chain.

Lemma 89. $[\Omega_i(G^{p^2})^{p^j}, \Omega_i(G^p)] \leq \Omega_i(G^{p^2})^{p^{j+2}}$ *for $i \geq 1$ and $j \geq 0$.*

Proof. By Proposition 86 we know that $[\Omega_i(G^{p^2})^{p^j}, \Omega_i(G^p)] \leq [\Omega_{i-j}(G^{p^{2+j}}), \Omega_i(G^p)]$, hence it suffices to show that $[\Omega_{i-j}(G^{p^{2+j}}), \Omega_i(G^p)] \leq \Omega_i(G^{p^2})^{p^{j+2}}$. Consider $g, h \in G$ with $o(g) \leq p^{i+2}$ and $o(h) \leq p^{i+1}$ then $g^{p^{2+j}} \in \Omega_{i-j}(G^{p^{2+j}})$ and $h^p \in \Omega_i(G^p)$. Using Theorem 83(ii) we obtain that $o([g^{p^{2+j}}, h^p]) \leq p^{i-j-2}$. Also notice that $[g^{p^{2+j}}, h^p] \in G^{p^{4+j}}$, and hence we may write $[g^{p^{2+j}}, h^p] = k^{p^{4+j}}$ for some $k \in G$, where $o(k^{p^{4+j}}) \leq p^{i-j-2}$. It follows that $k^{p^2} \in \Omega_i(G^{p^2})$ and $[g^{p^{2+j}}, h^p] = k^{p^{4+j}} \in \Omega_i(G^{p^2})^{p^{j+2}}$. Hence $[\Omega_{i-j}(G^{p^{2+j}}), \Omega_i(G^p)] \leq \Omega_i(G^{p^2})^{p^{j+2}}$. □

Note that if $j > i$ in the above, the inclusion still holds, with both sides of the inequality being equal to the trivial group.

Theorem 90. *If G is a powerful p-group where p is an odd prime, then $\Omega_i(G^p)$ is powerfully nilpotent for all $i \geq 1$ and the powerful nilpotency class of $\Omega_i(G^p)$ is at most i.*

Proof. As we observed above, for $i = 1$ the group is abelian, thus we may assume $i \geq 2$. Note that if $p^e = \exp(\Omega_i(G^p)) < p^2$ then by Lemma 85 it follows the group is abelian and so of powerful class 1 and so the claim holds in this case. If $\exp(\Omega_i(G^p)) = p^2$ then $H \cong \Omega_i(G^p)$ and so the claim follows by Lemma 88. Thus we may assume that $e > 2$ and $i \geq 2$. In Lemma 88 we saw that $H = \frac{\Omega_i(G^p)}{(\Omega_i(G^p))^{p^2}}$ has a powerfully central chain $H \geq \frac{\Omega_i(G^{p^2})}{\Omega_i(G^p)^{p^2}} \geq 1$. Then by Proposition 30 we have the following powerfully central chain for $\Omega_i(G^p)$:

$$\begin{array}{lll} \Omega_i(G^p) & \geq \Omega_i(G^{p^2}) & \geq \Omega_i(G^p)^p \\ \Omega_i(G^p)^p & \geq \Omega_i(G^{p^2})^p & \geq \Omega_i(G^p)^{p^2} \\ & \vdots & \\ \Omega_i(G^p)^{p^{e-2}} & \geq \Omega_i(G^{p^2})^{p^{e-2}} & \geq 1 \end{array}$$

Now using Lemma 89 we see that the terms $\Omega_i(G^p)^{p^j}$ for $j \in \{1, \ldots, e-2\}$ are redundant. Noting that by Theorem 83(iii) we have that $\exp \Omega_i(G^p) \leq p^i$, we obtain the following powerfully central chain for $\Omega_i(G^p)$ of length at most i.

$$\Omega_i(G^p) \geq \Omega_i(G^{p^2}) \geq \Omega_i(G^{p^2})^p \geq \cdots \geq \Omega_i(G^{p^2})^{p^{i-2}} \geq 1.$$

□

Recall that for a powerful p-group G, we have that G^{p^j} is powerful for all $j \geq 0$. Given a powerful group G, applying Theorem 90 to G^{p^j} gives that $\Omega_i(G^{p^{j+1}})$ is powerfully nilpotent for all $i \geq 1$. Thus we have that for a powerful p-group G, where p is an odd prime, all Omega subgroups of the proper Agemo subgroups are powerfully nilpotent. Note also that the same bound on the powerful nilpotency class which we saw in Theorem 90 holds in this general case.

Theorem 91. *Let G be a powerful p-group for an odd prime p. Then $\Omega_i(G^{p^j})$ is powerfully nilpotent for $i, j \geq 1$. The powerful nilpotency class of $\Omega_i(G^{p^j})$ is at most i.*

We now turn to the case $p = 2$. Due to the modification in the definition of a powerful 2-group, namely the requirement that $G' \leq G^{2^2}$, the arguments used above would require us to show that the group H is abelian. However, this is not true in general. Below we exhibit an example of a powerful 2-group such that $\Omega_2(G^p)$ is not powerful, and so we see that Theorem 90 cannot hold in its current form for $p = 2$.

Example 92. Consider the 2-group

$$G = \langle a, b, c | a^{2^3} = 1, b^{2^3} = 1, c^{2^{10}} = 1, [a, c] = 1, [b, c] = 1, [a, b] = c^{2^7} \rangle.$$

One can check either by hand or with GAP, that this is a consistent presentation defining a group of order 2^{16}. Clearly G is powerful and so $G^2 = \langle a^2, b^2, c^2 \rangle$. Consider $\Omega_2(G^2)$; this subgroup contains everything in G^2 of order less than or equal to 4. In particular it contains a^2, b^2 and c^{2^8}. Notice $[a^2, b^2] = c^{2^9}$. Hence $\Omega_2(G^2)$ is not abelian, but then it cannot be powerful for it has exponent at most 4 (Theorem 83(iv)) and any powerful group of exponent at most 4 is abelian.

Also note that in the example above, the prime $p = 2$ can be replaced with $p = 3$ to give a consistent presentation for a powerfully nilpotent group of order 3^{16}, where the property still holds that $\Omega_2(G^3)$ is not abelian. Thus in particular $\Omega_2(G^3)$ is not strongly powerful, yet is still powerfully nilpotent. Thus for p odd we see that the subgroups $\Omega_i(G^p)$ are an example of characteristic subgroups of a powerful group G which are powerfully nilpotent but not necessarily strongly powerful. This is in contrast to the subgroups G^{p^i} for $i \geq 1$, and the proper terms of the derived and lower central series of G, which are all strongly powerful [TW19]. Furthermore observe that $\Omega_2(G^3)$ has powerful nilpotency class 2 and so the bound from Theorem 90 is attained.
For the case $p = 2$ we make the following modification - instead of looking at $\Omega_i(G^p)$ we look at $\Omega_i(G^{p^2})$.

Theorem 93. *If G is a powerful 2-group, then $\Omega_i(G^4)$ is powerfully nilpotent for all $i \geq 1$ and furthermore for $i > 1$ the powerful nilpotency class of $\Omega_i(G^4)$ is at most $i-1$, for $i = 1$ the powerful class is 1.*

Proof. Consider $\tilde{H} = \frac{\Omega_i(G^4)}{(\Omega_i(G^4))^4}$. We will show that $\tilde{H}$ is abelian. By Lemma 85 we only need to consider commutators between elements of order 4. Let $\tilde{K} = G/(\Omega_i(G^4))^4$, and notice that $\tilde{K}$ and $\tilde{K}^4$ are powerful and $\tilde{H} \leq \tilde{K}^4$. We only need to consider commutators of the form $[a^4, b^4]$ where $o(a) = 2^4$ and $o(b) = 2^4$. However by Theorem 83(ii), setting $i = 4$ yields that $o([a^{2^2}, b^{2^2}]) \leq p^{4-2-2}$ and thus the commutator is trivial. It follows that $\tilde{H}$ is abelian. Suppose that $\exp(\Omega_i(G^4)) = p^e$. If $e = 1$ then $\Omega_i(G^4)$ is abelian and so of powerful nilpotency class 1, otherwise by Proposition 30 the powerful class of $\Omega_i(G^4)$ is at most $e-1$. Since $\Omega_i(G^{p^2}) \leq \Omega_i(G^p)$, by Theorem 83(iv) we obtain that $e \leq i$ and so the result follows. □

As in the odd case, we can apply the above theorem to G^{2^j} to obtain the following.

Theorem 94. *Let G be a powerful 2-group, then $\Omega_i(G^{2^j})$ is powerfully nilpotent for all $i \geq 1$, $j \geq 2$. Furthermore for $i > 1$ the powerful nilpotency class of $\Omega_i(G^{2^j})$ is at most $i - 1$. For $i = 1$ the powerful nilpotency class is 1.*

5. Properties of Specific Families of Powerfully Nilpotent Groups

5.1. Overview

A large part of this project has involved using the computer algebra system GAP to help generate conjectures or to find counterexamples. The previous chapters of the thesis have presented the results in their final form, but in this chapter we aim to retain some of the experimental feel.

5.2. Groups of exponent p^2

In Theorem 58 we determined the growth rate of the powerfully nilpotent groups of exponent p^2. For small orders we are able to find exactly how many such groups exist, in this case by using the Computer Algebra System GAP.
Below we include a table containing the number of groups of exponent p^2 and order p^n, for the prime $p = 3$ for different powerful coclasses. This data was obtained using GAP.

	Coclass	1	2	3	4	5	6	7	8	**Total**
Order										
3^2		1								1
3^3			1							1
3^4			1	2						3
3^5				2	2					4
3^6				1	5	3				9
3^7					6	8	3			17
3^8					13	27	19	4		63
	Total	1	2	5	26					

Table 5.1.: Number of Powerfully Nilpotent Groups of exponent p^2 of a given coclass

We saw in Theorem 57 that there are only finitely many groups of a given coclass, and so we can gain insight into how and when the coclass changes by understanding this table. In particular in this section we prove in Theorem 102 that for powerfully nilpotent groups of exponent p^2 and any given powerful coclass m, groups of this coclass exist for m generations. The inspiration for this result came from observations of the experimental data above. For example the length of the rows is $1, 1, 2, 2, 3, 3, 4$ and so one might wonder whether this pattern continues (see Theorem 101). Similarly in the columns we have one entry for coclass 1, two entries for coclass 2 and three entries for coclass 3 and so again we might ask whether this pattern continues (see Theorem 102).

Answering these questions gives a better insight into the ancestry tree for this family of groups, as well as allowing us to better understand powerful coclass.

Proposition 95. *For a powerfully nilpotent, non-cyclic, group G of exponent p^2, we have that $c \leq r-1$ where c is the powerful class of G and r is the rank of G.*

Proof. If G is abelian, then we are done. Now suppose G is not abelian and that $G = \langle a_1, \ldots, a_r \rangle$. Observe that $[G, G^p] \leq G^{p^2} = 1$ and so G^p is central. Moreover as it is of exponent p we know it is strictly contained within the centre by Proposition 15. Also recall that the index of the first term of the upper powerful central series in G is at least p^2. Hence the length of the upper powerful central series will not exceed the length of the following powerfully central chain:

$$G \geq \langle a_1, \ldots, a_{r-2} \rangle G^p \geq \langle a_1, \ldots, a_{r-3} \rangle G^p \geq \cdots \geq \langle a_1 \rangle G^p \geq 1$$ □

Remark 96. Recall from Theorem 47 the inequality $r \leq n-c+1$. Then combining this and the proposition above yields that $c \leq \frac{n}{2}$.

Proposition 97. *For powerfully nilpotent groups of exponent p^2, and a given order p^n, the maximal powerful nilpotency class that can occur is $\left\lfloor \frac{n}{2} \right\rfloor$, and hence there are at most $\left\lfloor \frac{n}{2} \right\rfloor$ different powerful coclasses that can occur.*

Proof. If G is abelian the bound is clear. Now suppose G is not abelian, so in particular is not cyclic.
By the remark above we know that $r \leq n-c+1$ and by Proposition 95 we also know that $c+1 \leq r$. Thus combining we obtain $2c \leq n$ and so $c \leq \frac{n}{2}$. As the powerful class must be an integer, this is the same as $c \leq \left\lfloor \frac{n}{2} \right\rfloor$. Hence there are at most $\left\lfloor \frac{n}{2} \right\rfloor$ different powerful classes, and hence coclasses, that can occur for the order p^n. □

Proposition 98. *Let $m \geq 1$ be some powerful coclass. Then in terms of the ancestor tree, m can exist as a powerful coclass for at most m generations. In other words, if the first time the coclass m occurs is for a powerfully nilpotent group of order p^n, then there do not exist groups of order greater than or equal to p^{n+m} which have powerful coclass m.*

Proof. Suppose for contradiction this is not true, and so for some integer $\delta \geq 0$ there exists a group G of order $p^{n+m+\delta}$ with powerful coclass m. This means that G has powerful class $n+\delta$. Then by Proposition 97 we must have that $n+\delta \leq \frac{n+m+\delta}{2}$, that is, $n+\delta \leq m$. This is a contradiction, because the first time the coclass m occurs is for a group of order p^n and so $n > m$. Then certainly $n+\delta > m$. □

We can think of Proposition 98 as telling us how "deep" a column in Table 5.1 can be and Proposition 97 tells us how "wide" a row can be.
Now we prove that for a given order p^n we have precisely $\left\lfloor \frac{n}{2} \right\rfloor$ different powerful coclasses. In terms of the table above, we want to show we have no gaps. We will introduce the following notation: Let $f(n,m)$ be the number of powerfully nilpotent groups of exponent p^2 with order p^n and powerful coclass m. Let G be a group of order p^n and powerful coclass m, then $G \times C_p$ has order p^{n+1} and coclass $m+1$. Hence by the above observation we get:

Lemma 99. *$f(n,m) \leq f(n+1, m+1)$.*

Note that the fact $G \times C_p \cong G_2 \times C_p \implies G \cong G_2$ follows as a corollary of the Krull-Remak-Schmidt theorem.
Lemma 99 is *almost* enough to deduce that precisely $\left\lfloor \frac{n+1}{2} \right\rfloor$ different coclasses occur for p^{n+1}, given that precisely $\left\lfloor \frac{n}{2} \right\rfloor$ different coclasses occur for p^n. However, as can be seen from the table above, Lemma 99 and knowledge of the row corresponding to 3^5 does not guarantee that there exists a group of order 3^6 and powerful coclass 2. Thus we will exhibit a family of powerfully nilpotent groups of exponent p^2 and maximal powerful class (and so minimal powerful coclass). Considering specific examples found using GAP, we can deduce a general presentation for such groups. Thus we consider a family of groups of maximal powerful class (and so minimal powerful coclass).

Example 100. For $n \geq 1$ let $G_{2n} = \langle g_1, \ldots, g_{2n} | \mathcal{R}_1 \rangle$ and $G_{2n+1} = \langle g_1, \ldots, g_{2n+1} | \mathcal{R}_2 \rangle$, with

$$\begin{aligned}\mathcal{R}_1 = \{ & g_1^p = 1, g_2^p = 1, \\ & g_3^p = g_{3+(n-1)}, \ldots, g_{n+1}^p = g_{2n}, \\ & g_{3+(n-1)}^p = 1, \ldots, g_{2n}^p = 1, \\ & [g_2, g_1] = g_3^p,\ [g_3, g_1] = g_4^p, \ldots, [g_n, g_1] = g_{n+1}^p \}\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}_2 = \{ & g_1^p = 1, \\ & g_2^p = g_{2+n}, \ldots, g_{n+1}^p = g_{2n+1}, \\ & g_{n+2}^p = 1, \ldots, g_{2n+1}^p = 1, \\ & [g_2, g_1] = g_3^p,\ [g_3, g_1] = g_4^p, \ldots, [g_n, g_1] = g_{n+1}^p \}.\end{aligned}$$

The group G_{2n} is a semidirect product $(C_p \times \underbrace{C_{p^2} \times \cdots \times C_{p^2}}_{n-1}) \ltimes C_p$ and thus of order p^{2n}, and similarly G_{2n+1} is a semidirect product of order p^{2n+1}. The upper powerful central series of G_{2n} is

$$\begin{aligned}1 < Z(G) = \langle g_{n+1}, g_n^p, \ldots, g_3^p \rangle < \langle g_{n+1}, g_n, g_{n-1}^p, \ldots, g_3^p \rangle < \ldots \\ \cdots < \langle g_{n+1}, \ldots, g_3 \rangle < \langle g_{n+1}, \ldots, g_3, g_2, g_1 \rangle = G\end{aligned}$$

and that for G_{2n+1} is similar. These groups have powerful class n, and so powerful coclass n and $n+1$ respectively. Note that this example shows the bound on powerful class from Proposition 97 is sharp.

Theorem 101. *There are precisely $\left\lfloor \frac{n}{2} \right\rfloor$ different powerful coclasses for groups of order p^n and exponent p^2.*

Proof. Suppose n is even, then $n = 2k$ for some $k \in \mathbb{N}$. We know by Proposition 97 that at most $\left\lfloor \frac{n}{2} \right\rfloor = k$ powerful coclasses can occur. The groups $G_2 \times (C_p)^{2k-2}$,$G_4 \times (C_p)^{2k-4}, \ldots, G_{2k} \times (C_p)^0$ are k distinct groups with powerful coclasses $2k-1, \ldots, k$. Similarly if n is odd we can write it as $n = 2k+1$ for some $k \in \mathbb{N}$ and then the k groups $G_3 \times (C_p)^{2k-2}$,$G_5 \times (C_p)^{2k-4}$,$\ldots$,$G_{2k+1} \times (C_p)^0$ are distinct and have powerful coclasses $k+1, \ldots, 2k$. ∎

A similar argument to that used above can show that a given coclass m exists for *precisely* m generations.

Theorem 102. *Let $m \in \mathbb{N}$, then there exist precisely m orders where groups of powerful coclass m occur.*

Proof. By Proposition 98 we know there are at most m such orders. To see there are exactly this many, consider the m distinct groups $G_{2\cdot 1} \times (C_p)^{m-1}$,$G_{2\cdot 2} \times (C_p)^{m-2}$,$\ldots$,$G_{2\cdot m} \times (C_p)^0$. □

Based on the work in this section, we can now understand the patterns we observed in the table of experimental data. The n^{th} row of the table has length precisely $\left\lfloor \frac{n}{2} \right\rfloor$ because for groups of exponent p^2 and order p^n there are precisely $\left\lfloor \frac{n}{2} \right\rfloor$ powerful coclasses (or powerful classes) that can occur, and in fact these coclasses must be precisely $n - \left\lfloor \frac{n}{2} \right\rfloor, \ldots, n-1$. The smallest coclass shifts "to the right" every two rows (this is the behaviour of the floor $\frac{n}{2}$ function for odd/even values). The m^{th} column of the table has depth m, and this is because as we saw for powerfully nilpotent groups of exponent p^2 and coclass m, this coclass exists for m generations. Thus now we have an understanding far beyond what we could compute in GAP.

5.3. Groups of nilpotency class 2

Noting now that the non-abelian powerfully nilpotent groups of exponent p^2 are of nilpotency class 2 (that is $G' \leq Z(G)$), we wonder if in a similar way to above, we are able to better understand the coclass and powerful classes for these groups. As we are restricting the length of the *standard* upper central series, it is natural to ask whether this imposes some bound on the *powerful nilpotency class,* but the family of examples in Example 100 demonstrates that this is not the case.
We remark that with a knowledge of the groups of class $\mathcal{C}$, that is those groups corresponding to the nilpotent symplectic alternating algebras, this is not surprising, as these are groups of class 3 but unbounded powerful class.
We now obtain a sharp bound on the powerful nilpotency class of a class 2 group G of order p^n.

Theorem 103. *For a powerfully nilpotent group G of class 2, powerful nilpotency class c and order p^n, we have that $c \leq \frac{n}{2}$.*

Proof. Let G have rank r. By the results of Section 2.5 we can pick $a_1, \ldots, a_r$ such that

$$G = \langle a_1, \ldots, a_r \rangle > \langle a_3, \ldots, a_r \rangle G^p > \cdots > \langle a_r \rangle G^p > G^p \geq [G,G] > 1$$

is a powerfully central chain. Note that $G^p \geq [G,G] > 1$ is powerfully central follows from the fact the group is of class 2. In particular $[G^p, G] = [G,G]^p$ and $[G,G] \leq Z(G)$. By considering the chain above, it follows that $c \leq r+1$. Let $\left|\frac{G^p}{[G,G]}\right| = p^k$ and $|[G,G]| = p^d$. Note that $d \geq c-1$ by Lemma 46. We now consider three cases.
For the first case, suppose $c \leq r-1$. Then $n = r+k+d \geq c+1+k+c-1 \geq 2c$.

For the second case, suppose $c = r$. Then as above we have $n = r+k+d \geq c+k+c-1 = 2c-1+k$. If $k \geq 1$ then $n \geq 2c$. Thus we can assume $k = 0$. Then $G^p = [G, G]$. It follows that

$$1 = \gamma_3(G) = [G^p, G] = [G, G]^p.$$

Thus $G^{p^2} = 1$, and so we can ignore the last 2 terms of the powerfully central chain above, and thus $c \leq r - 1$, a contradiction.
For the final case, suppose $c = r + 1$, and note that then we cannot have $k = 0$. The reason for this is that if $k = 0$ we have $G^p = [G, G]$ and so the chain above becomes a chain of length r, contradicting the assumption that $c = r + 1$. Thus we may assume $k \geq 1$. We have

$$n = r + k + d \geq c - 1 + k + c - 1 = 2c - 2 + k.$$

If $k \geq 2$, then $n \geq 2c$. We can assume that $k = 1$. Then $\left|\frac{G^p}{[G,G]}\right| = p \implies G^p = \langle a^p \rangle [G, G]$ for some $a \in G$. Let $1 \leq i < j \leq r$. Then

$$[a_i, a_j]^p = [a_i^p, a_j] = [a^{p\alpha}u, a_j]$$

where $\alpha \in \mathbb{Z}$ and $u \in [G, G]$. However the group is class 2 and so

$$[a^{p\alpha}u, a_j] = [a^\alpha, a_j^p] = [a^\alpha, a^{p\beta}v] = 1$$

where $\beta \in \mathbb{Z}$ and $v \in [G, G]$. Hence $[G, G]^p = 1$, then $G^p \leq Z(G)$ and hence we obtain the contradiction $c \leq r$. □

As powerful groups of exponent p^2 have nilpotency class 2, the examples from Section 5.2 show that this bound is sharp. The same arguments used in the previous section yield that for powerfully nilpotent groups of real nilpotency class 2, a given powerful coclass m exists for at most $m-1$ generations. Furthermore, for a given order p^n there are precisely $\left\lfloor \frac{n}{2} \right\rfloor - 1$ distinct powerful coclasses that occur. Note well here the change from the previous case where we now introduce a minus 1, this is because abelian groups are not class 2. Finally we remark that similar arguments work for any family in which one can obtain a tight bound on the powerful class in terms of the order p^n. For example for powerfully nilpotent groups of rank 2.

6. Classification

In this chapter we classify the powerfully nilpotent p-groups of order at most p^6 for p an odd prime. Unless otherwise stated, p is an odd prime.

6.1. Preliminaries

By the results of Section 2.5, in particular Theorem 48, we know that for any powerfully nilpotent group G of rank r and generating set $a_1, \ldots, a_r$, in some sense the number of generators of any given order is an invariant of the group. Thus we can introduce the notion of the *type* of a powerfully nilpotent group G.

Definition 104. We say that a powerfully nilpotent p-group G is of type $(n_1, \ldots n_r)$ if there exists a generating set $\{a_1, \ldots, a_r\}$ for G where $o(a_i) = p^{n_i}$ and $n_1 + \cdots + n_r = n$, $n_1 \geq n_2 \geq \cdots \geq n_r$. In this case we say that $\{a_1, \ldots, a_r\}$ is a generating set *corresponding to this type.*

For the benefit of notation, if a number n_i within the type is repeated t times, then we may write $(\ldots,_t n_i, \ldots)$. For example $(2, 1, 1, 1)$ could be written as $(2,_3 1)$. We also need the following basic results from number theory.

Proposition 105. *Let p be an odd prime and let N be the set of non-quadratic residues modulo p, and let S be the set of non-zero quadratic residues modulo p. Then*
(i) $|N| = |S|$.
(ii) Multiplying a quadratic non-residue by a quadratic residue gives a quadratic non-residue.
(iii) For $s \in S$, the map $f_s : N \to N$ defined by $f_s(n) = ns$ is a bijection.
(iv) If $s_1, s_2 \in S$ and $s_1 \neq s_2$ then we have that $f_{s_1}(n) \neq f_{s_2}(n)$ for any $n \in N$. In particular for any $n_1, n_2 \in N$ there exists $s \in S$ such that $f_s(n_2) = n_1$.

Proof. See Appendix A.2. □

6.2. Structural results

In this section we collect together results which shall be used in the classification in Section 6.3. When classifying the groups of order at most p^6, we shall see that many of the types we must consider are of the form $(n,_t 1)$. We will now classify the powerfully nilpotent groups of type $(n,_t 1)$ for $n \geq 2$, $t \geq 0$. Let G be a powerfully nilpotent group with a generating set $\{a, b_1, \ldots, b_t\}$ corresponding to the type $(n,_t 1)$. We make first some observations. First observe that $G^p = \langle a^p \rangle$ and so $G^p \leq Z(G)$ by Corollary 54. Then $\gamma_3(G) = 1$ as $\gamma_3(G) \leq [G^p, G] = 1$ and also $[G, G]^p = [G^p, G] = 1$, that is,

$\exp G' = p$. Next observe that $\Omega_1(G) = \langle a^{p^{n-1}}, b_1, \ldots, b_t\rangle$ and so $[G,G] \leq \Omega_1(G) \cap G^p = \langle a^{p^{n-1}}\rangle$. Also note that $G[p^{n-1}] = \langle a^p, b_1, \ldots, b_t\rangle$.
Consider $W = \frac{\langle b_1,\ldots,b_t\rangle G^p}{G^p}$ as a vector space, and in the natural way we obtain an alternating form from the commutator operation. That is, $(b_iG^p, b_jG^p) = \lambda$ if $[b_i, b_j] = a^{\lambda p^{n-1}}$ for $0 \leq \lambda < p$. As is known from linear algebra, we can then write

$$W = \langle b_1G^p, b_2G^p\rangle \oplus \cdots \oplus \langle b_{2s-1}G^p, b_{2s}G^p\rangle \oplus W^{\perp},$$

where $W^{\perp} = \{w \in W \mid (w,u) = 0 \,\forall u \in W\} = \langle b_{2s+1}G^p, \ldots, b_tG^p\rangle$, and in terms of the commutator operation, that

$$\begin{aligned}[b_{2i-1}, b_{2i}] &= a^{p^{n-1}} \text{ for } i = 1, \ldots, s \qquad (6.1)\\ [b_i, b_j] &= 1 \text{ otherwise for } i < j.\end{aligned}$$

We now consider two cases depending on whether or not $Z(G) \leq G[p^{n-1}]$.
Case 1: $Z(G) \not\leq G[p^{n-1}]$. This means there is some element $a^r u \in Z(G)$ with $u \in \langle b_1, \ldots, b_t\rangle$ and $0 < r < p$. Notice $(a^r u)^{p^i} = (a^r)^{p^i}(u)^{p^i}[u,a]^{\frac{p^i(p^i-1)}{2}}$ and so $o(a^r u) = o(a) = p^n$. Thus $\{a^r u, b_1, \ldots, b_t\}$ is a generating set for G of type $(n,_t 1)$ and so we can replace a by $a^r u$ and so without loss of generality we can assume that $a \in Z(G)$. The power relations of the group are determined by the type, and the non-trivial commutator relations (6.1) depend only on what s is. Thus we get $\left\lfloor \frac{t}{2} \right\rfloor + 1$ such groups depending on the value of $s \in \{0, 1, \ldots, \left\lfloor \frac{t}{2} \right\rfloor\}$.
Remark 106. If $n = 2$, then we must have this situation, because we cannot have $Z(G) \leq G[p^{2-1}] = \Omega_1(G)$ as this would mean that the exponent of the centre would be p. Then as we know that would mean the group would have to be abelian, but then we obtain $p = \exp Z(G) = \exp G = p^2$, a contradiction.

Case 2: $Z(G) \leq G[p^{n-1}]$. Note in this case that G cannot be abelian. Suppose that for some b_i, $i \in \{1, \ldots, t\}$ that $[a, b_i] = a^{p^{n-1}\alpha}$. Then $[ab_j^{\beta}, b_i] = a^{p^{n-1}\alpha}[b_j, b_i]^{\beta}$ and we see that for a suitable choice of j and β we obtain $[ab_j^{\beta}, b_i] = 1$. Hence we may assume without loss of generality that

$$[a, b_1] = \cdots = [a, b_{2s}] = 1.$$

Notice that $a \notin Z(G) \leq G[p^{n-1}]$, as $o(a) = p^n$ and $\exp(G[p^{n-1}]) = p^{n-1}$. Thus we can assume that $t > 2s$ and for some $i \in \{2s+1, \ldots, t\}$ that $[a, b_i] = a^{\lambda p^{n-1}} \neq 1$ for $\lambda \in \mathbb{Z}_p$. Replacing b_i with $b_i^{\lambda^{-1}}$ and relabelling b_i if necessary, we can assume that $[a, b_{s+1}] = a^{p^{n-1}}$. Then in a similar way to before, we can assume that

$$[a, b_{2s+2}] = \cdots = [a, b_t] = 1.$$

Then $G = \langle b_1, b_2, \ldots, b_{2s}, b_{2s+1}, a\rangle \times \langle b_{2s+2}, \ldots, b_t\rangle$ and the only non-trivial commutator relations are $[b_{2i-1}, b_{2i}] = a^{p^{n-1}}$ for $i \in \{1, \ldots, s\}$ and $[a_1, b_{2s+1}] = a^{p^{n-1}}$. Thus we obtain $\left\lfloor \frac{t-1}{2} \right\rfloor + 1$ groups, one for each $s \in \{0, \ldots, \left\lfloor \frac{t-1}{2} \right\rfloor\}$. Thus we obtain the following theorem.

Theorem 107. *The number of powerfully nilpotent groups of type* $(2,_t 1)$ *for* $t \geq 0$ *is* $\left\lfloor \frac{t}{2} \right\rfloor + 1$. *The number of powerfully nilpotent groups of type* $(n,_t 1)$ *for* $n > 2$ *is* $t + 1$.

Remark 108. We will parameterise the families of groups from the previous argument as follows:

- $A(n,t,s)$ for $n \geq 1$, $t \geq 0$, $s \in \{0, \dots, \lfloor \frac{t}{2} \rfloor\}$ is the group $G = \langle a, b_1, \dots, b_t \rangle$ with the following relations:

$$\begin{aligned} &a^{p^n} = b_1^p = \cdots = b_t^p = 1, \\ &[b_{2i-1}, b_{2i}] = a^{p^{n-1}} \text{ for } i = 1, \dots, s, \\ &[b_i, b_j] = 1 \text{ otherwise for } i < j, \\ &[a, b_i] = 1 \text{ for all } i. \end{aligned}$$

  Note then that if $s = 0$ we get here the abelian group $C_{p^n} \times \underbrace{C_p \times \cdots \times C_p}_{t}$.

- $B(n,t,s)$ for $n \geq 3$, $t > 2s$, $s \in \{0, \dots, \left\lfloor \frac{t-1}{2} \right\rfloor\}$ is the group $G = \langle a, b_1, \dots, b_t \rangle$ with the following relations:

$$\begin{aligned} &a^{p^n} = b_1^p = \cdots = b_t^p = 1, \\ &[b_{2i-1}, b_{2i}] = a^{p^{n-1}} \text{ for } i = 1, \dots, s, \\ &[b_i, b_j] = 1 \text{ otherwise for } i < j, \\ &[a, b_1] = \cdots = [a, b_{2s}] = [a, b_{2s+2}] = \cdots = [a, b_t] = 1, \\ &[a, b_{2s+1}] = a^{p^{n-1}}. \end{aligned}$$

Lemma 109. *All powerfully nilpotent groups of order at most p^3 are abelian.*

Proof. This is clearly true for p and p^2. For order p^3, the largest rank that can occur is 3. By the results of Chapter 3 we can see that the first non-abelian rank 2 powerfully nilpotent group occurs at order p^4 and so the rank 2 groups of order at most 3 are abelian. Finally there is only one possibility for a rank 3 group, occurring for p^3, but then the exponent of the group must be p and so $G' \leq G^p = 1$. □

Lemma 110. *There are precisely two groups with presentations of the form $\langle a, b, c | a^p = b^{p^2} = c^{p^3} = 1, [a, b] = c^{\lambda p^2}, [b, c] = 1, [a, c] = b^p \rangle$ where $0 < \lambda < p$. The group obtained depends on whether or not λ is a square modulo p.*

Proof. First we will show that given a presentation of this form, the presentation can be assumed to be of one of two canonical forms depending on whether or not λ is a square modulo p.
Case 1. Suppose that $\lambda = \mu^2 \mod p$. Let $\hat{a} = a^{\mu^{-1}}, \hat{b} = b^{\mu^{-1}}, \hat{c} = c$. Then $[\hat{a}, \hat{b}] = [a, b]^{\mu^{-2}} = c^{p^2} = \hat{c}^{p^2}$, $[\hat{b}, \hat{c}] = [b, c]^{\mu^{-1}} = 1$ and $[\hat{a}, \hat{c}] = [a, c]^{\mu^{-1}} = b^{\mu^{-1}p} = \hat{b}^p$. Hence we may assume a presentation with commutator relations of the form $a^p = b^{p^2} = c^{p^3} = 1$ and $[a, b] = c^{p^2}, [b, c] = 1, [a, c] = b^p$.
Case 2. Conversely suppose that λ is not a square modulo p. Let n_1 be the smallest non-square modulo p. Then by Proposition 105 we can find an s which is a square modulo p and which has the property that $s \cdot \lambda = n_1$. As s is a square we may write $s = \mu^2$. Then similar to before, let $\hat{a} = a^{\mu}$,$\hat{b} = b^{\mu}$ and $\hat{c} = c$. Then $[\hat{b}, \hat{c}] = [b, c]^{\mu} = 1$, $[\hat{a}, \hat{b}] = [a, b]^{\mu^2} = c^{\mu^2 \lambda p^2} = c^{n_1 p^2}$ and $[\hat{a}, \hat{c}] = [a, c]^{\mu} = b^{\mu p} = \hat{b}^p$. Thus we can assume a presentation of the form $a^p = b^{p^2} = c^{p^3} = 1$ $[a, b] = c^{n_1 p^2}$, $[b, c] = 1$, $[a, c] = b^p$.
Next we show that the groups defined by these two presentations are not isomorphic. Let G_1 be the group given by the presentation $\langle a, b, c | a^p = b^{p^2} = c^{p^3} =$

$1, [a,b] = c^{p^2}, [b,c] = 1, [a,c] = b^p\rangle$ and let G_2 be the group given by the presentation $\langle A,B,C|A^p = B^{p^2} = C^{p^3}, [A,B] = C^{n_1p^2}, [B,C] = 1, [A,C] = B^p\rangle$. Suppose for contradiction that an isomorphism does exist between these two groups, then we can find a set of generators of G_1 which satisfy the relations of G_2. Noting that G_1 is powerful and G_1^p is central, we may assume that we have $A = a^{\lambda_1}u^p$, $B = b^{\lambda_2}v^p$ and $C = c^{\lambda_3}w^p$ for some $u,v,w \in G_1$. First consider the relation $[A,C] = B^p$. From this we see that $[a^{\lambda_1}u^p, c^{\lambda_3}w^p] = (b^{\lambda_2}v^p)^p$ and expanding gives $[a,c]^{\lambda_1\lambda_3} = b^{\lambda_2 p}v^{p^2}$. Notice $v^{p^2} \in G_1^{p^2} = \langle c^{p^2}\rangle$ and $v^{p^2} = c^{\mu p^2}$ for some μ, and so we have $b^{\lambda_1\lambda_3 p} = b^{\lambda_2 p}c^{\mu p^2}$. Thus we can deduce that

$$\lambda_1\lambda_3 \equiv \lambda_2 \mod p. \tag{6.2}$$

Similarly we can analyse $[A,B] = C^{n_1p^2}$. We see that $[a^{\lambda_1}u^p, b^{\lambda_2}v^p] = c^{\lambda_1\lambda_2p^2} = c^{n_1\lambda_3p^2} = (c^{\lambda_3}w^p)^{n_1p^2} = C^{n_1p^2}$. Hence from this we obtain that

$$\lambda_1\lambda_2 \equiv n_1\lambda_3 \mod p. \tag{6.3}$$

We can rearrange equation (6.3) to obtain that $\lambda_2 \equiv n_1\lambda_1^{-1}\lambda_3 \mod p$, and then substituting this into equation (6.2) yields $\lambda_1\lambda_3 \equiv n_1\lambda_1^{-1}\lambda_3 \mod p$. Hence we obtain that $\lambda_1^2 \equiv n_1 \mod p$, which contradicts that $n_1$is not a square modulo p. Thus the groups G_1 and G_2 cannot be isomorphic.

□

Theorem 111. *There are exactly* 8 *powerfully nilpotent groups of type* $(3,2,1)$ *for each odd prime* p*.*

Proof. By Section 2.5 and Theorem 48 we know that we can assume that there is a powerfully central chain where the generator of order p is at the "front" of the chain, and so we have two cases to consider depending on whether or not the generator of order p^2 appears next. We can visualise this as

a	b	c
	b^p	c^p
		c^{p^2}

or

a	c	b
	c^p	b^p
	c^{p^2}	

.

Formally, if $o(a) = p$, $o(b) = p^2$ and $o(c) = p^3$ then G must have a powerfully central chain of one of the following forms (not necessarily exclusive):

$$\text{Type A. } \langle a,b,c\rangle \geq \langle b,c\rangle \geq \langle b^p,c\rangle \geq \langle b^p,c^p\rangle \geq \langle c^p\rangle \geq \langle c^{p^2}\rangle \geq 1$$

$$\text{Type B. } \langle a,b,c\rangle \geq \langle c,b\rangle \geq \langle c^p,b\rangle \geq \langle c^p,b^p\rangle \geq \langle c^{p^2},b^p\rangle \geq \langle c^{p^2}\rangle \geq 1.$$

Note that no attempt has been made to remove redundant terms.

First we suppose that G is a powerfully nilpotent group *with a chain of type A*. Notice that $[G,G] \leq G^{p^2}$. To see this use the fact that the chain is powerfully central, and also that $o([a,b]) \leq p$ by Theorem 83. It follows that $G^p \leq Z(G)$, $\gamma_3(G) = 1$, $[G^p,G] = [G,G]^p = G^{p^3} = 1$, $\Omega_1(G) = \langle a,b^p,c^{p^2}\rangle$ and $G[p^2] = \langle a,b,c^p\rangle$. Considering the powerfully central chain A we see the commutators must be of the forms $[a,b] = c^{p^2r}$, $[b,c] = c^{p^2s}$, $[a,c] = c^{p^2t}$ for some r,s and t. First we shall consider different cases for groups with chains of type A, based on structural differences of the group, thus it is clear to see there is no isomorphism between groups of different cases.

Case 1. $Z(G) \not\leq G[p^2]$. Hence we can assume that $c \in Z(G)$. We thus either have $[a,b]=1$ or $[a,b] \neq 1$ in which case we may assume $[a,b]=c^{p^2}$. Hence we obtain the following groups:

1. $C_p \times C_{p^2} \times C_{p^3}$.

2. $a^p = b^{p^2} = c^{p^3} = 1$,$[a,c]=[b,c]=1$,$[a,b]=c^{p^2}$.

Case 2. $Z(G) \leq G[p^2]$ but $Z(G) \not\leq \Omega_1(G)G^p = \langle a, b^p, c^p \rangle$. Thus we can assume that $b \in Z(G)$ and obtain the group

3. $a^p = b^{p^2} = c^{p^3} = 1$, $[a,c]=c^{p^2}$, $[a,b]=[b,c]=1$.

Case 3. $Z(G) \leq \Omega_1(G)G^p = \langle a, b^p, c^p \rangle$ but $Z(G) \not\leq G^p = \langle a^p, b^p, c^p \rangle$ so we can assume that $a \in Z(G)$, so we obtain the group

4. $a^p = b^{p^2} = c^{p^3} = 1$, $[b,c]=c^{p^2}$, $[a,b]=[a,c]=1$.

Case 4. $Z(G)=G^p$. This case cannot occur. To see why not, consider the three relations $[a,b],[b,c],[a,c]$. At least 2 must be non-trivial in this case, and all commutators are of the form $c^{\lambda p^2}$. However if 2 commutators are of this form, we could replace the generators in such a way so as to make one of the relations trivial, and so in fact the centre must contain either a, b or c. Note that we can also see this by considering the vector space G/G^p of odd dimension 3 and the associated alternating form obtained in the natural way from the commutator relation.
Notice that all the groups found above while considering a chain of type A are strongly powerful. Now consider groups with a chain of type B. If a group with such a chain was to be strongly powerful, in fact the only options for the commutator relations are the same as those we have already seen. Thus to find the remaining groups we can assume that G *has a chain of type* B *and is not strongly powerful*. Hence in what follows we consider different cases for groups with a chain of type B and which are not strongly powerful. The cases are again based on structural differences. For the cases in which more than one group occurs, if it is not clear that the groups are distinct, we prove explicitly that no isomorphism exists.
By considering the chain of type B we see our commutators must satisfy $[a,b] \in \langle c^{p^2}, b^{p^2} \rangle = \langle c^{p^2} \rangle$, $[b,c] \in \langle c^{p^2} \rangle$, $[a,c] \in \langle c^{p^2}, b^p \rangle$. Note that G' has exponent p and it follows that the group has real class at most 2, to see this consider $[G',G] \leq [G^p,G] = (G')^p = 1$. Notice also that for G to not be strongly powerful, we must have that $[a,c] = b^{\lambda p} c^{\mu p^2}$ with $0 < \lambda < p$. Notice also that c^p is central and so let $\hat{b} = b^{\lambda} c^{\mu p}$. Then $[a,c] = b^{\lambda p} c^{\mu p^2} = \hat{b}^p$. Thus we can assume that $[a,c]=b^p$ and then we only have $[a,b]$ and $[b,c]$ to determine. Note that as before $\Omega_1(G) = \langle a, b^p, c^{p^2} \rangle$and $G[p^2] = \langle a, b, c^p \rangle$.
Case 1. $Z(G) \not\leq G[p^2]$ and hence we can assume that $c \in Z(G)$, which contradicts $[a,c] \neq 1$ and so this case cannot occur.
Case 2. $Z(G) \leq G[p^2]$ but $Z(G) \not\leq \Omega_1(G)G^p = \langle a, b^p, c^p \rangle$. Thus we must have that there is some element $ba^{\lambda} c^{\mu p^2}$ in $Z(G)$, but then since $c^{p^2} \in Z(G)$ we have in fact $ba^{\lambda} \in Z(G)$ and then replace b with ba^{λ}. Notice this does not alter the relation $[a,c]=b^p$. Thus we obtain the following group:

5. $a^p = b^{p^2} = c^{p^3} = 1$,$[a,c]=b^p$,$[a,b]=[b,c]=1$.

Case 3. $Z(G) \leq \Omega_1(G)G^p = \langle a, b^p, c^p \rangle$ but $Z(G) \nleq G^p$. Then we can assume $a \in Z(G)$ but this is a contradiction.
Case 4. $Z(G) = G^p$. We will find that there are three distinct groups for this case. First we consider the form the upper powerfully central series must take. We must have that $\hat{Z}_1(G) = \langle a^p, b^p, c^p \rangle = \langle b^p, c^p \rangle$. Then we must have that $[\hat{Z}_2(G), G] \leq \hat{Z}_1^p = \langle c^{p^2} \rangle$. Consider some element $a^r b^s c^t \in \hat{Z}_2(G)$, we will show that in fact we can assume $r = t = 0$. Consider the commutator $[a^r b^s c^t, c] = [a, c]^r [b, c]^s = b^{rp} c^{\lambda p^2}$ for some λ. It is clear that for this commutator to be in $\langle c^{p^2} \rangle$ we must have $p|r$ and so we can assume $r = 0$. In a similar way by expanding the commutator $[a^r b^s c^t, a]$ we obtain $[b, a]^s [c, a]^t = c^{p^2 \lambda} b^{-tp}$ for some λ. Thus we must have $p|t$, and so we can assume our element is of the form $b^s c^{ph}$ but then since $c^p \in Z(G) \leq \hat{Z}_2(G)$ we see that in fact $b \in \hat{Z}_2(G)$. Hence for the upper powerfully central series to ascend, the only choice is that $\hat{Z}_2(G) = \langle b, c^p \rangle$ and so at least one of $[a, b]$ or $[b, c]$ must be non-trivial. Notice also that $\hat{Z}_3(G)$ is then forced to be $\langle a, b, c \rangle$ since $[a, G]$ and $[c, G]$ are in $\hat{Z}_2(G)^p$. We now consider cases depending on whether or not $\Omega_2(G)' = 1$.
Case i. Suppose that $\Omega_2(G)' = 1$. That is, suppose $[a, b] = 1$. Then we have $[b, c] = c^{\lambda p^2}$ for some $0 < \lambda < p$ and $[a, c] = b^p$. Let $\hat{b} = b^{\lambda^{-1}}$ and $\hat{a} = a^{\lambda^{-1}}$ and $\hat{c} = c$. Then $[\hat{a}, \hat{b}] = 1$, $[\hat{b}, \hat{c}] = [b^{\lambda^{-1}}, c] = [b, c]^{\lambda^{-1}} = c^{p^2} = \hat{c}^{p^2}$ and $[\hat{a}, \hat{c}] = [a, c]^{\lambda^{-1}} = (b^p)^{\lambda^{-1}} = \hat{b}^p$. Hence we obtain

6. $\langle a^p = b^{p^2} = c^{p^3} = 1[a, b] = 1, [b, c] = c^{p^2} [a, c] = b^p \rangle$.

Case ii. Suppose that $\Omega_2(G)' \neq 1$. Then we must have $[a, b] = c^{\lambda p^2}$ for some $0 < \lambda < p$.
Case ii (a). If also we have that $[b, c] = c^{\mu p^2}$ for some $0 < \mu < p$ then our relations are $[a, b] = c^{\lambda p^2}$, $[b, c] = c^{\mu p^2}$ and $[a, c] = b^p$. We will now show that if we are in this case we can in fact choose the generators to obtain $[b, c] = 1$. For clarity we break the process into two parts. First let $\hat{c} = c$, $\hat{b} = b^{\mu^{-1}}$ and $\hat{a} = a^{\mu^{-1}}$. Then $[\hat{b}, \hat{c}] = [b^{\mu^{-1}}, c] = (c^{\mu p^2})^{\mu^{-1}} = c^{p^2} = \hat{c}^{p^2}$. Also $[\hat{a}, \hat{c}] = [a^{\mu^{-1}}, c] = (b^{\mu^{-1}})^p = \hat{b}^p$. Finally $[\hat{a}, \hat{b}] = [a^{\mu^{-1}}, b^{\mu^{-1}}] = c^{\lambda \mu^{-2} p^2}$ and hence we can assume a presentation of the form $[a, b] = c^{\gamma p^2}$, $[b, c] = c^{p^2}$ and $[a, c] = b^p$. Now for the second step, let $\hat{c} = c^\gamma a$, $\hat{b} = b$ and $\hat{a} = a^{\gamma^{-1}}$. Then $[\hat{a}, \hat{b}] = [a^{\gamma^{-1}}, b] = [a, b]^{\gamma^{-1}} = c^{\gamma^{-2} p^2} = \hat{c}^{\delta p^2}$ for some δ, $[\hat{b}, \hat{c}] = [b, c^\gamma a] = [b, c]^\gamma [b, a] = c^{\gamma p^2} c^{-\gamma p^2} = 1$, and $[\hat{a}, \hat{c}] = [a^{\gamma^{-1}}, c^\gamma a] = [a, c] = b^p = \hat{b}^p$. Hence we can suppose that the the presentation has $[b, c] = 1$, $[a, c] = b^p$ and $[a, b] = c^{\delta p^2}$ for some δ. Thus we only need consider the next case.
Case ii (b). Suppose G has a presentation of the form $a^p = b^{p^2} = c^{p^3}$ and $[a, b] = c^{\lambda p^2}$, $[b, c] = 1$ and $[a, c] = b^p$. Then by Lemma 110 we see that there are two distinct groups with a presentation of this type:

7. $\langle a, b, c | a^p = b^{p^2} = c^{p^3} = 1, [a, b] = c^{p^2}, [b, c] = 1, [a, c] = b^p \rangle$.
8. $\langle a, b, c | a^p = b^{p^2} = c^{p^3} = 1, [a, b] = c^{n_1 p^2}, [b, c] = 1, [a, c] = b^p \rangle$ where n_1 is the smallest non-square modulo p.

This then exhausts all cases. Thus we see in total we have eight distinct groups.

□

Lemma 112. *There are two groups of type* $(2, 2, 2)$.

Proof. If the group is abelian then we have $C_{p^2}\times C_{p^2}\times C_{p^2}$. Now suppose that the group G of type $(2,2,2)$ is not abelian. Notice that since G is powerful, $[G^p,G]\leq G^{p^2}=1$ and so $G^p\leq Z(G)$. Let the generators be a,b,c and each be of order p^2. As the exponent of the centre of a non-abelian powerfully nilpotent group must be at least p^2, we can assume that $a\in Z(G)$ so that $\langle a,b^p,c^p\rangle\leq Z(G)$. Notice that neither b nor c can be in the centre or else the group would be abelian and so we must have exactly that $Z(G)=\langle a,b^p,c^p\rangle$. Then the only non-trivial commutator relation is between b and c. We must have that $[b,c]\in Z(G)^p=\langle a^p\rangle$, and we may assume that $[b,c]=a^p$. Thus we have the following group $G=\langle a,b,c\mid a^{p^2}=b^{p^2}=c^{p^2}=1,\ [a,b]=[a,c]=1,[b,c]=a^p\rangle$. We may ask whether this presentation is consistent and whether such a group really does exist; notice in this case that $G\cong(C_{p^2}\times C_{p^2})\rtimes C_{p^2}$. Hence up to isomorphism we have two groups. ☐

Lemma 113. *There are two groups of type* $(2,2,1)$.

Proof. Let G be a powerfully nilpotent group of type $(2,2,1)$ with generators a,b,c with $o(a)=p$, $o(b)=o(c)=p^2$. By the results of Section 2.5 we know that we can move the generator of order p to the front of the chain, and so G must have a powerfully central chain as follows:

$$\langle a,b,c\rangle\geq\langle b,c\rangle\geq\langle b^p,c\rangle\geq\langle b^p,c^p\rangle\geq\langle c^p\rangle\geq\{1\}.$$

From this we can read off that $[a,b]\in\langle b^p,c\rangle^p=\langle c^p\rangle$, $[a,c]\in\langle b^p,c^p\rangle^p=\{1\}$ and $[b,c]\in\langle b^p,c^p\rangle^p=1$. If $[a,b]=1$ then we have the abelian group $C_p\times C_{p^2}\times C_{p^2}$. If $[a,b]=c^{\lambda p}$ for $0<\lambda<p$, then we may replace a by $a^{\lambda^{-1}}$ and so can assume that we have $[a,b]=c^p$. Thus we have the group with presentation $\langle a,b,c|a^p=1,b^{p^2}=c^{p^2}=1,[a,b]=c^p,[a,c]=[b,c]=1\rangle$. Notice also that this group can be realised as a semidirect product, and so such a group does exist. Hence there are two groups. ☐

Lemma 114. *There are four groups of type* $(2,2,1,1)$.

Proof. Consider a group of type $(2,2,1,1)$ with generators a,b,c,d of orders p,p,p^2,p^2 respectively. By Section 2.5 and Theorem 48 we know there is a powerfully central chain:

$$\langle a,b,c,d\rangle\geq\langle b,c,d\rangle\geq\langle c,d\rangle\geq\langle c^p,d\rangle\geq\langle c^p,d^p\rangle\geq\langle d^p\rangle\geq\{1\}.$$

Using the fact that this is a powerfully central chain, we can read off the commutators: $[a,d]=[b,d]=[c,d]=1$, and $[a,c],[b,c]\in\langle d^p\rangle$ and $[a,b]\in\langle c^p,d^p\rangle$. Notice also that such a group has real nilpotency class at most 2, since $[G^p,G]\leq G^{p^2}=1$. We split the classification into two cases depending on whether or not the commutators $[a,c]$ and $[b,c]$ are both trivial.
Case 1. Suppose $[a,c]$ and $[b,c]$ are both trivial. Then the only commutator relation that remains to be determined is $[a,b]$. If this is trivial, then we obtain the abelian group $C_{p^2}\times C_{p^2}\times C_p\times C_p$.
If $[a,b]$ is not trivial then we must have that $[a,b]=c^{\lambda_1 p}d^{\lambda_2 p}$. If $\lambda_1=0$ and $\lambda_2\neq 0$ then replace d with d^{λ_2}, if $\lambda_1\neq 0$ and $\lambda_2=0$ then replace c with c^{λ_1}, and if $\lambda_1\neq 0$ and $\lambda_2\neq 0$ let $c=c^{\lambda_1}d^{\lambda_2}$. Thus we may assume that the presentation is

1. $\langle a,b,c,d|a^p=b^p=c^{p^2}=d^{p^2}=1,\ [a,b]=c^p,\ [a,c]=[a,d]=[b,c]=[b,d]=[c,d]=1\rangle$.

Case 2. Suppose $[a,c]$ and $[b,c]$ are not both trivial, say that $[a,c] = d^{\lambda_1 p}$ and $[b,c] = d^{\lambda_2 p}$. Without loss of generality we may assume that $[b,c] = d^p$ and $[a,c] = 1$, with the other established relations unchanged. To see this, if $\lambda_1 = 0$ and $\lambda_2 \neq 0$ then replace d with d^{λ_2}, if $\lambda_1 \neq 0$ and $\lambda_2 = 0$ then let $\hat{d} = d^{\lambda_1}$.If both $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ then replace a by $a^{\lambda_1^{-1}}$ and b with $b^{\lambda_2^{-1}}$, to get that $[a,c] = d^p$ and $[b,c] = d^p$. Then replace a with $\hat{a} = ab^{-1}$ to get that $[\hat{a},b] = [a,c][b,c]^{-1} = 1$. Thus we may assume without loss of generality that $[a,d] = [b,d] = [c,d] = [a,c] = 1$ and $[b,c] = d^p$ and $[a,b]$ is still to be determined. If $[a,b] = 1$ we obtain the following group:

2. $\langle a,b,c,d | a^p = b^p = c^{p^2} = d^{p^2} = 1, [a,b] = [a,c] = [a,d] = [b,d] = [c,d] = 1, [b,c] = d^p \rangle$.

If $[a,b] \neq 1$, then we can suppose $[a,b] = c^{\lambda_1 p} d^{\lambda_2 p}$, with $0 \leq \lambda_1, \lambda_2 < p$ and not both equal to zero. If $\lambda_1 = 0$ and $\lambda_2 \neq 0$ then replace d with d^{λ_2} and c with c^{λ_2}. Then we get that $[a,b] = [b,c] = d^p$, but then $[b,ac] = 1$ and by replacing c with ac we can move to the case where both $[a,c]$ and $[b,c]$ are trivial, which we have dealt with above. If $\lambda_2 = 0$ and $\lambda_1 \neq 0$ we can replace c with c^{λ_1} and d with d^{λ_1}, to obtain $[a,b] = c^p$. If both $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ then replace c with $c^{\lambda_1} d^{\lambda_2}$ and d with d^{λ_1}. Thus we see that we may assume $[a,b] = c^p$. Hence we obtain

3. $\langle a,b,c,d | a^p = b^p = c^{p^2} = d^{p^2} = 1, [a,b] = c^p, [b,c] = d^p, [a,c] = [a,d] = [b,d] = [c,d] = 1 \rangle$.

This exhausts all possibilities. Note that to see that these groups do exist, i.e. that the presentations are consistent, notice all these groups can be described as direct and semidirect products. Alternatively as the groups are class 2, one could check the consistency of the presentations in a straightforward way.
To see that these groups are not isomorphic, notice that the centre has a different structure in each case. For the abelian group we clearly have $Z(G) \cong C_{p^2} \times C_{p^2} \times C_p \times C_p$. For the second group, we have $Z(G) \cong \langle c,d \rangle \cong C_{p^2} \times C_{p^2}$. For the third group we have $Z(G) \cong \langle a, c^p, d \rangle \cong C_p \times C_p \times C_{p^2}$. For the fourth group we have $Z(G) \cong \langle c, d^p \rangle \cong C_{p^2} \times C_p$. □

6.3. Classification

We first outline how we approach the classification. By the results of Section 2.5, we know that the *type* is an invariant of a group. For groups of order p^n the possible types are all partitions of n. Thus we consider each order p^n and then for each partition of n we consider how many groups there are of that type.

6.3.1. Order p

There is only one group of this type, the cyclic group:

- C_p.

6.3.2. Order p^2

The possible types are (2) and $(1,1)$. It is well known that groups of order p^2 are abelian. Thus there are two distinct groups:

- C_{p^2},
- $C_p \times C_p$.

6.3.3. Order p^3

By Lemma 109 we know all powerfully nilpotent groups of this order are abelian. We thus have the following groups:

- C_{p^3},
- $C_p \times C_{p^2}$,
- $C_p \times C_p \times C_p$.

6.3.4. Order p^4

The partitions of 4 are $(1,1,1,1)$, $(2,1,1)$, $(3,1)$, $(2,2)$, (4).
For type $(1,1,1,1)$ the exponent is p and so the group must be abelian:

- $C_p \times C_p \times C_p \times C_p$.

For type $(2,1,1)$ we use Theorem 107 to see that there are two groups and that these are

- $A(2,2,0) \cong C_{p^2} \times C_p \times C_p$,
- $A(2,2,1)$.

For type $(3,1)$, first we see that there is an abelian group of this type:

- $C_{p^3} \times C_p$.

By consulting the results of Chapter 3 we see that for type $(3,1)$ we also have a non-abelian semidirect product

- $G(3,1,2) = \langle a, b | a^{p^3} = b^p = 1, [a,b] = a^{p^2} \rangle$.

For type $(2,2)$ there is only an abelian group:

- $C_{p^2} \times C_{p^2}$.

Finally for type (4) there is the cyclic group:

- C_{p^4}.

Thus we find that there are 7 powerfully nilpotent groups of order p^4. 5 of the groups are abelian and 2 are non-abelian. The non-abelian groups have powerful nilpotency class 2.

6.3.5. Order p^5

The partitions of 5 are $(1,1,1,1,1)$, $(2,1,1,1)$, $(2,2,1)$, $(3,1,1)$, $(3,2)$, $(4,1)$ and (5). For the type $(1,1,1,1,1)$ the only group is

- $C_p \times C_p \times C_p \times C_p \times C_p$.

For the type $(2,1,1,1)$, by Theorem 107, we know that there are 2 groups:

- $A(2,3,0) \cong C_{p^2} \times C_p \times C_p \times C_p$,
- $A(2,3,1)$.

For the type $(2,2,1)$ by Lemma 113 we know there are two groups:

- $C_{p^2} \times C_{p^2} \times C_p$,
- $\langle a, b, c \mid a^{p^2} = b^{p^2} = c^p = 1, [a,b] = 1, [a,c] = 1, [b,c] = a^p \rangle$.

For the type $(3,1,1)$ we can use Theorem 107, to see that there are 3 groups of this type.
These are

- $A(3,2,0)$,
- $A(3,2,1)$,
- $B(3,2,0)$.

For the type $(3,2)$ the abelian group of this type is

- $C_{p^3} \times C_{p^2}$.

Now for the non-abelian case. Any group of type $(3,2)$ will be of rank 2, and so we can use the classification of Chapter 3 to write down all such groups. It follows from the classification that there is only one group. It is a semidirect product with the following presentation:

- $\langle a, b | a^{p^3} = b^{p^2} = 1,\ [a,b] = a^{p^2} \rangle$.

For the type $(4,1)$ we again consult the results of Chapter 3. There is one abelian group and one non-abelian semidirect product:

- $C_{p^4} \times C_p$,
- $G(4,1,3) = \langle a, b | a^{p^4} = b^p = 1, [a,b] = a^{p^3} \rangle$.

Finally for type (5) we have the cyclic group.

- C_{p^5}.

Thus we find there are 13 powerfully nilpotent groups of order p^5, 7 of the groups are abelian, 6 of the groups are of powerful nilpotency class 2.

6.3.6. Order p^6

The partitions of 6 are $(1,1,1,1,1,1)$, $(2,1,1,1,1)$, $(2,2,1,1)$, $(2,2,2)$, $(3,1,1,1)$, $(3,2,1)$, $(3,3)$, $(4,1,1)$, $(4,2)$, $(5,1)$, (6).
For type $(1,1,1,1,1,1)$ there is one group:

- $C_p \times C_p \times C_p \times C_p \times C_p \times C_p$.

For type $(2,1,1,1,1)$, using Theorem 107, we see that there are 3 groups and that they are

- $A(2,4,0) \cong C_{p^2} \times C_p \times C_p \times C_p \times C_p$,
- $A(2,4,1)$,
- $A(2,4,2)$.

For type $(2,2,1,1)$, by Lemma 114 there are 4 distinct groups. They are:

- $C_{p^2} \times C_{p^2} \times C_p \times C_p$,
- $\langle a,b,c,d | a^p = b^p = c^{p^2} = d^{p^2} = 1, [a,b] = [a,c] = [a,d] = [b,d] = [c,d] = 1, [b,c] = d^p \rangle$,
- $\langle a,b,c,d | a^p = b^p = c^{p^2} = d^{p^2} = 1, [a,b] = c^p, [b,c] = d^p, [a,c] = [a,d] = [b,d] = [c,d] = 1 \rangle$,
- $\langle a,b,c,d | a^p = b^p = c^{p^2} = d^{p^2} = 1, [a,b] = c^p, [b,c] = [a,c] = [a,d] = [b,d] = [c,d] = 1 \rangle$.

For type $(2,2,2)$ we know by Lemma 112 that there are 2 distinct groups:

- $C_{p^2} \times C_{p^2} \times C_{p^2}$,
- $\langle a,b,c \mid a^{p^2} = b^{p^2} = c^{p^2} = 1, [a,b] = [a,c] = 1, [b,c] = a^p \rangle$.

For type $(3,1,1,1)$ there are 4 distinct groups by Theorem 107, and by the argument in the proof we know that these groups are

- $A(3,3,0) \cong C_p \times C_p \times C_p \times C_{p^3}$,
- $A(3,3,1)$,
- $B(3,3,0)$,
- $B(3,3,1)$.

For type $(3,2,1)$ there are 8 distinct groups, as were found in Theorem 111.
Next we consider type $(4,1,1)$. Using Theorem 107 we know that there are 3 distinct groups of this type and that these are as follows:

- $A(4,2,0)$,
- $A(4,2,1)$,

- $B(4, 2, 0)$.

For the types $(3, 3)$, $(2, 4)$ and $(1, 5)$ we use the classification from Chapter 3 and we have the following groups.
The groups of type $(3, 3)$ are

- $G(3, 3, 2) = \langle a, b \mid a^{p^3} = b^{p^3} = 1,\ [a, b] = a^{p^2} \rangle$,
- $C_{p^3} \times C_{p^3}$.

The groups of type $(1, 5)$ are

- $G(5, 1, 4) = \langle a, b \mid a^{p^5} = b^p = 1,\ [a, b] = a^{p^4} \rangle$,
- $C_{p^5} \times C_p$.

The groups of type $(2, 4)$ are

- $G(4, 2, 2) = \langle a, b \mid a^{p^4} = b^{p^2} = 1,\ [a, b] = a^{p^2} \rangle$,
- $G(4, 2, 3) = \langle a, b \mid a^{p^4} = b^{p^2} = 1,\ [a, b] = a^{p^3} \rangle$,
- $C_{p^3} \times C_{p^2}$.

Finally we have type (6):

- C_{p^6}.

7. Programming in GAP

The GAP package [GAP18] has been used extensively in this project to find examples and counterexamples. Below we outline the algorithm we use to determine whether a group is powerfully nilpotent and the mathematical basis for the algorithm. We then display a table, computed using GAP, of the number of powerfully nilpotent groups that exist for given orders. Finally we provide a simple GAP implementation. The aim of this chapter is to enable a future researcher to immediately be able to investigate these groups using GAP, and moreover to keep a record of time-consuming computations we have undertaken in GAP (in particular determining the number of powerfully nilpotent groups of order 3^8).

7.1. Computing the upper powerful central series

We give pseudocode below to determine the upper powerful central series for a finite p-group G. Note that we are not assuming that G is powerfully nilpotent.

Input: A finite p-group G (where p is an odd prime)
$M_0 = \{1_G\}$
$M_1 = Z(G)$
$i = 1$
While $M_i \neq M_{i-1}$ **do**
Define $\varphi_i : G \to \frac{G}{M_i^p}$ to be the natural homomorphism.
$H_{i+1} := Z\left(\frac{G}{M_i^p}\right)$ and then $M_{i+1} := \varphi_i^{-1}(H_{i+1})$
$i = i + 1$
od;
#We are outside the while loop, so in particular $M_i = M_{i-1}$
return ($M_0, \ldots, M_{i-1}$)
#This is the Upper Powerful Central Series of G

Claim 115. $M_i = \hat{Z}_i(G)$ for $i \in \{0, \ldots, i-1\}$.

Proof. We use induction on i. It is clear that M_0 and M_1 are $\hat{Z}_0(G)$ and $\hat{Z}_1(G)$ respectively. Now suppose the claim holds for $i \leq k$ and consider $M_{k+1} = \varphi_k^{-1}\left(Z\left(\frac{G}{M_k^p}\right)\right)$ and by the inductive hypothesis we know $M_k^p = \hat{Z}_k^p(G)$. If $m \in M_{k+1}$ then $\varphi_k(m) \in Z\left(\frac{G}{\hat{Z}_k^p}\right)$, that is $[m, G] \leq \hat{Z}_k(G)^p$ and hence $M_{k+1} \leq \hat{Z}_{k+1}(G)$. Conversely suppose $z_1 \in \hat{Z}_{k+1}(G)$. Then for any $g \in G$ we have $[z_1, g] = z_2^p$ where $z_2 \in \hat{Z}_k(G)$, but then we see that $[\varphi_k(z_1), \varphi(g)] = \varphi(z_2^p) = \bar{1}$, in other words $\varphi_k(z_1) \in Z\left(\frac{G}{\hat{Z}_k^p}\right) = Z\left(\frac{G}{M_k^p}\right)$, thus $\hat{Z}_{k+1}(G) \leq M_{k+1}$. □

As the groups we are dealing with are finite, this series must eventually reach a repetition. If the repetition is the group G, then the upper powerfully central series has successfully ascended, and so the group is powerfully nilpotent, of class $i-1$. If the upper powerful central series reaches a repetition before G, then it will never ascend and so G is not powerfully nilpotent.
We remark that as all powerful 2-groups are in fact powerfully nilpotent, determining whether a 2-group is powerfully nilpotent is straightforward, just check that $[G,G] \leq G^4$. If the upper powerful central series of a 2-group is required, it is straightforward to adapt the above algorithm, however in the course of this project we have been mainly concerned with odd primes p.

7.2. Number of powerfully nilpotent groups of small order

For reference we include the following table, which shows the number of powerfully nilpotent groups for given orders. Note how for $p^n > p^6$ the number of groups is dependent on the prime, whereas we showed in Section 6.3 that this is not the case for $p^n \leq p^6$.

p		3	5	7	11	13	17	19	23
p^4	Abelian	5	5	5	5	5	5	5	5
	non-abelian	2	2	2	2	2	2	2	2
	Total	7	7	7	7	7	7	7	7
p^5	Abelian	7	7	7	7	7	7	7	7
	non-abelian	6	6	6	6	6	6	6	6
	Total	13	13	13	13	13	13	13	13
p^6	Abelian	11	11	11	11	11	11	11	11
	non-abelian	22	22	22	22	22	22	22	22
	Total	33	33	33	33	33	33	33	33
p^7	Abelian	15	15	15	15				
	non-abelian	**83**	**85**	**87**	**91**				
	Total	98	100	102	106				
p^8	Abelian	22							
	non-abelian	398							
	Total	420							

The table above was found by using the libraries of groups that are within GAP, and then filtering these groups to find those that are powerfully nilpotent. For p^n with $n \leq 7$ we used the Small Groups Library, and for 3^8 we used the package SglPPow. There is a natural question for further research here, is it possible to construct powerfully nilpotent groups from the ground up? See Section 8.2 for more on this.

7.3. A GAP implementation

The code below is a basic implementation, in the language GAP, of methods to determine whether or not a group is powerfully nilpotent, to determine its powerful class and upper powerfully central series.

```
# GAP Code For Powerfully Nilpotent Groups
# Author James Williams

# This function returns the Upper Powerful Central Series for
# a finite p-group note that if G is not powerfully nilpotent
# this series will not terminate at G. The series returned is
# in the form of an ascending list, where the first term will
# be the trivial subgroup.
UpperPowerfulCentralSeries := function(G)
local chain, p, im, hom, lifted;
if(IsFinite(G)=false or IsPGroup(G)=false) then
  return fail;
fi;
 p:=PrimePGroup(G);
 chain:=[Group(Identity(G))];
 while(true) do
     if Order(chain[Length(chain)])=1 then
            hom:= NaturalHomomorphismByNormalSubgroup(G,chain[
                Length(chain)]);
     else
          hom:= NaturalHomomorphismByNormalSubgroup(G,Agemo(
              chain[Length(chain)],p));
     fi;
          im := Image(hom);
          lifted:=  PreImages(hom, Center(im));
          #Have we reached the top
          if(Order(lifted)=Order(chain[Length(chain)])) then
            break;
          else
            Add(chain,lifted);
          fi;
   od;
  return chain;
end;

IsPowerfullyNilpotent := function(G)
local upcs;
# Return true if G is powerfully nilpotent and false
# otherwise. This function calls UpperPowerfulCentralSeries
# and then determines whether or not the series has
# completely ascended.
```

```
upcs := UpperPowerfulCentralSeries(G);
if(upcs[Length(upcs)] = G) then
  return true;
else
  return false;
fi;
end;

PowerfulNilpotencyClass := function(G)
# Returns the powerful nilpotency class of a group G.
if( IsPowerfullyNilpotent(G)) then
  return Length(UpperPowerfulCentralSeries(G))-1;
else
  return fail; #If G is not Powerfully Nilpotent
fi;
end;
```

8. Future Work and Further Questions

This section discusses some open questions, conjectures and areas for future research.

8.1. Developing understanding of longest tail and maximal powerful class

Let p be a fixed prime, and let $M_p(n)$ denote the maximal powerful class that occurs for a group of order p^n. Then $M_p(n) \leq M_p(n+1)$ (for instance take a group of maximal class of order p^n and a direct product with C_p). We also note that $M_p(n+1) - M_p(n) \leq 1$, that is, the maximal class can increase by at most 1 between orders. To see this, suppose that $M_p(n) = c$ but $M_p(n+1) = c+1+m$ where $m \geq 1$. Let G be such a group of order p^{n+1} with powerful class $c+1+m$. Then $G/Z(G)^p$ has class $c+m > c$ but $|G/Z(G)^p| \leq p^n$, hence a contradiction.
For a given order p^n, the smallest coclass that occurs is $n - M_p(n)$. If there existed an $N \in \mathbb{N}$ such that we had $M_p(n+1) = M_p(n) + 1$ for all $n \geq N$, then we see that the smallest coclass that occurs will always be $N - M_p(N)$. However by Theorem 57 we know that there are only finitely many groups of any given coclass, so this can never occur. Hence for any $N \in \mathbb{N}$, there exists $n \geq N$ such that $M_p(n) = M_p(n+1)$. It is interesting to try to understand when the increases in the powerful class and increases in the powerful coclass occur. For groups of rank 2 or of exponent p^2 we have a good understanding of the situation, because we have tight bounds on the powerful class in terms of the order. The general case is more elusive.
Recall the definition of a group of maximal tail introduced in Section 2.8. At the end of that section we discussed finding examples to show that the bound on the maximal length of a tail was sharp, and recall that our presentation was not consistent in general. Intuitively it makes sense that these groups with the longest tails will have maximal powerful class, and empirically in GAP we see that this seems to be true. Note however that for the same order, there can exist groups of this same class but with different ranks. Better understanding these groups of longest maximal tail is closely connected to understanding what is the maximal powerful class that can occur for a given order, or rephrased - what is the minimal powerful coclass that can occur.
Indeed Table 8.1 below shows, for groups of order up to 3^8 the maximal class that can occur. This was obtained using GAP.
In Section 2.8, the groups of longest tail which we were trying to find would have order p^n where $n = 1 + \frac{r(r+1)}{2}$ and where r is the rank of the group, and tail length $1 + \frac{r(r-1)}{2}$. Note that their orders would be $n = 1, 2, 4, 7, 11, 16\ldots$ and that we have concrete examples for these existing up to and including rank 5 for order p^{16} . In Table 8.2 below, the columns corresponding to the groups of longest tail are in bold font. Observe that these orders correspond exactly with the points in the table where the

Order $3^n, n$:	1	2	3	4	5	6	7	8
Maximal Powerful Class	1	1	1	2	2	3	4	4
Minimal Powerful Coclass	0	1	2	2	3	3	3	4

Table 8.1.: Empircal Data

class no longer increases. The table below shows what we ***conjecture*** the situation to be like for larger n. In particular, we conjecture that for a positive integer d, the minimal powerful coclass is d for d columns.

Order $3^n, n$:	**1**	**2**	3	**4**	5	6	**7**	8	9	10	**11**	12	13	14	15	**16**	17
Maximal Powerful Class	**1**	**1**	1	**2**	2	3	**4**	4	5	6	**7**	7	8	9	10	**11**	11
Minimal Powerful Coclass	**0**	**1**	2	**2**	3	3	**3**	4	4	4	**4**	5	5	5	5	**5**	6

Table 8.2.: Conjectured values for $n \geq 9$

8.2. Generating powerfully nilpotent groups

During this project we were able to classify the powerfully nilpotent groups of rank 2. As such, given an arbitrary order, we are able to directly and immediately construct all groups of this order. Furthermore as they are parameterised by four numbers, we do not even need to realise them as groups in a computer algebra system, as all the information we need about them and their structure is obtainable just from these four numbers. However to explore powerfully nilpotent groups in general, we have taken a top down approach, whereby using the SmallGroupsLibrary in GAP we have built all groups of a certain order, and then filtered out the groups we want. Not only is this inefficient, but it means we are limited by the size of library. A future area of research is to try to construct these groups “bottom up”, perhaps working up the powerful coclass tree, in analogy with the well known p-group generation algorithm [O’B90].

8.3. Which powerfully nilpotent groups appear as subgroups of powerful groups

We have seen that several characteristic subgroups of powerful groups are in fact powerfully nilpotent, for example the Frattini subgroup of a powerful p-group is always strongly powerful. It would be interesting to know whether every strongly powerful group appears as the Frattini subgroup of some powerful group, and furthermore does an understanding of the (strongly) powerfully nilpotent group $\Phi(G)$ lead to a better understanding of the powerful group G.

A. Appendix

A.1. Adjusting the presentation of a metacyclic group

Lemma 116. *If p is an odd prime then $\mathbb{Z}^*_{p^n}$ is cyclic of order $(p-1)p^{n-1}$.*

Proof. Let $B = \{[a] \in \mathbb{Z}^*_{p^n} : a \cong 1 \mod p\} = \{[1 + a_1 p + \cdots + a_{n-1}p^{n-1}] : 0 \leq a_j \leq p-1\}$. Thus $|B| = p^{n-1}$, B is the (unique) Sylow $p-$subgroup and $\mathbb{Z}^*_{p^n} = B \times A$ where $|A| = p-1$. To see this is the order of A, consider the homomorphism $\phi : \mathbb{Z}^*_{p^n} \to \mathbb{Z}^*_p$, given by $[a]_{p^n} \mapsto [a]_p$, which has kernel B and thus

$$\mathbb{Z}^*_p = \operatorname{Im}\phi \cong \frac{\mathbb{Z}^*_{p^n}}{\ker\phi} = \frac{B \times A}{B} \cong A.$$

Note then that it follows A is cyclic (it is well known that $\mathbb{Z}^*_p$ is cyclic). It remains to show that B is cyclic by showing that $B = \langle[1+p]\rangle$.We show by induction that

$$(1+p)^{p^m} \equiv 1 \mod p^{m+1}, \tag{A.1}$$
$$(1+p)^{p^m} \not\equiv 1 \mod p^{m+2}, \tag{A.2}$$

$m = 0$: We have $1 + p \cong 1 \mod p$ but $1 + p \not\cong 1 \mod p^2$.
Then for the inductive step, suppose the claim holds for m, we will show this implies it holds for $m+1$. Assume $(1+p)^{p^m} = 1 + kp^{m+1}$ (for A.1) and where $p \nmid k$ (for A.2). Then

$$(1+p)^{p^{m+1}} = (1+kp^{m+1})^p = 1 + p \cdot k \cdot p^{m+1} + \binom{p}{2}k^2p^{2m+2} + \binom{p}{3}k^3p^{3m+3} + \dots$$

$$\begin{aligned} &= 1 + kp^{m+2} + sp^{m+3} \\ &\equiv 1 \mod p^{m+2} \\ &\not\equiv 1 \mod p^{m+3} \text{ as } p \nmid k. \end{aligned}$$

In particular $(1+p)^{p^{n-1}} \cong 1 \mod p^n$ and $(1+p)^{p^{n-2}} \not\cong 1 \mod p^n$, so the order is exactly p^{n-1}, so we have a cyclic group. Then $\mathbb{Z}^*_{p^n} = B \times A$ must be cyclic as the product of two coprime cyclic groups. □

We now explain why we can assume the presentation to be of the form:

$$a^{p^n} = 1, b^{p^m} = a^{p^l}, [a,b] = a^{p^r}.$$

First, if we had $b^{sp^m} = a^{p^l}$ then we could replace b by b^s. The part that requires more work is to show if $[a,b] = a^{p^r s}$ with $p \nmid s$ then we can adjust into the required form. We remark that $a^b = a[a,b] = a^{1+p^r s}$ and notice that $o([1+p^r s]) = p^{n-r}$ in $\mathbb{Z}^*_{p^n}$, (one can use induction to show $(1+p^r s)^{p^m} \cong 1 \mod p^{r+m}$ and $(1+p^r s)^{p^m} \ncong 1 \mod p^{r+m+1}$, in a similar way to the proof above). But $\mathbb{Z}_{p^n}^*$ has exactly one cyclic subgroup of order p^{n-r}, namely $\langle[1+p^r]\rangle$, and hence $[1+p^r] = [1+p^r s]^t$ for some t where $p \nmid t$ (since both generators, and so order must stay the same). Now $a^{b^t} = a^{(1+p^r s)^t} = a^{1+p^r} \implies [a, b^t] = a^{p^r}$. Thus, if we were given a presentation of the form:

$$a^{p^n} = 1, b^{p^m} = a^{p^l}, [a, b] = a^{p^r s}$$

then we know that $a^{p^n} = 1, (b^t)^{p^m} = (a^t)^{p^l}, [a, b^t] = a^{p^r}$. Hence it follows $(a^t)^{p^n} = 1, (b^t)^{p^m} = (a^t)^{p^l}, [a^t, b^t] = (a^t)^{p^r}$ and so letting $\bar{a} = a^t$ and $\bar{b} = b^t$ we obtain a presentation of the desired form.

A.2. Quadratic residues

In Chapter 6 we used some elementary results about quadratic residues. In this section we outline a proof for these results. For completeness we restate the proposition below.

Proposition 105. *Let p be an odd prime and let N be the set of non-quadratic residues modulo p, and let S be the set of non-zero quadratic residues modulo p. Then*
(i) $|N| = |S|$.
(ii) Multiplying a quadratic non-residue by a quadratic residue gives a quadratic non-residue.
(iii) For $s \in S$, the map $f_s : N \to N$ defined by $f_s(n) = ns$ is a bijection.
(iv) If $s_1, s_2 \in S$ and $s_1 \neq s_2$ then we have that $f_{s_1}(n) \neq f_{s_2}(n)$ for any $n \in N$. In particular for any $n_1, n_2 \in N$ there exists $s \in S$ such that $f_s(n_2) = n_1$.

Proof. (i) We define the map $\varphi : \mathbb{F}_p^\times \to \mathbb{F}_p^\times$ by $x \mapsto x^2$. This map is a homomorphism. Notice that $\operatorname{Im}\varphi$ is the set of quadratic residues. Notice also that as p is a prime $\mathbb{Z}_p$ is a field and so in particular we have that

$$x^2 = 1 \iff 0 = (x-1)(x+1) \iff x = \pm 1$$

and so $\ker\varphi = \{+1, -1\}$. By the Homomorphism Theorems we obtain that $|\operatorname{Im}\varphi| = \frac{p-1}{2}$. Thus $|S| = \frac{p-1}{2}$ and so it follows that $|N| = (p-1) - \frac{p-1}{2} = |S|$.
(ii) Notice that $\frac{\mathbb{F}_p^\times}{\operatorname{Im}\varphi} \cong C_2 = \{+1, -1\}$. In particular the coset $\operatorname{Im}\varphi$ containing the quadratic residues corresponds to the identity element and the other coset containing the quadratic non-residues corresponds to the non-identity element.
(iii) Observe that for $s \in S$, we have that $s \neq 0$ and so as $\mathbb{F}_p$ is a field, there exists an inverse $s^{-1} \in \mathbb{F}_p$. Then the map $f_{s^{-1}}$ is an inverse to f_s and so we have that f_s is bijective.
(iv) If for $s_1, s_2 \in S$ and $n_1 \in N$ we had that $s_1 n_1 \equiv s_2 n_1 \mod p$ then again using the fact that inverses exist in $\mathbb{F}_p$ we would obtain that $s_1 \equiv s_2 \mod p$. In particular it follows that for a given $n_2 \in N$, the map $f_s(n_2)$ takes a different value in N for each $s \in S$. As $|S| = |N|$ the result follows. ☐

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