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OPTIMAL CONTROL ON THE DOUBLY INFINITE TIME AXIS FOR WELL-POSED LINEAR SYSTEMS*

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Abstract. We study the problem of existence of weak right or left or strong coprime factorizations in H-infinity over the right half-plane of an analytic function defined and uniformly bounded on some right half-plane. We give necessary and sufficient conditions for the existence of such coprime factorizations in terms of an optimal control problem over the doubly infinite continuous time axis. In particular, we show that an equivalent condition for the existence of a strong coprime factorization is that both the control and the filter algebraic Riccati equation (of an arbitrary well-posed realization) have a solution (in general unbounded and not even densely defined) and that a coupling condition involving these two solutions is satisfied.

Key words. Riccati equation, linear quadratic optimal control, infinite-dimensional system, coprime factorization, input-output stabilization, state feedback

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1. Introduction.

This is the second article in a series of articles where we consider the relationships between linear quadratic optimal control in continuous time, the factorization approach to control theory, and algebraic Riccati equations. The corresponding discrete-time results were obtained in [6, 7, 8]. We refer the reader to the introduction of [9], the first article in the series, for the motivation for and an overview of this project and how it fits within the wider literature.

In [9] we considered a very general class of infinite-dimensional control systems. In this article, we specialize to the case of well-posed linear systems [10, 12, 11], a class of infinite-dimensional control systems which has been very well studied over the last few decades.

In the case of a well-posed transfer function (i.e., a function which is analytic and uniformly bounded on some open right half-plane), it is natural to require that the inverse of the "denominator" in a left or right factorization is also well-posed [11, section 8.3], a condition which was (naturally) not imposed in [9], where we considered transfer functions which need not be well-posed. To obtain equivalences in the well-posed case akin to those obtained in [9] between existence of factorizations and solvability conditions for the linear quadratic optimal control problem and for algebraic Riccati equations, some additional "uniformity" assumptions must be made in the latter two contexts as well.

The remainder of this article is organized as follows. In section 2 we review that part of the theory of well-posed linear systems which is needed in this article. Section 3 shows that the notion of (past and future) trajectories as used in [9] is consistent with the standard notion of trajectories for well-posed linear systems. In section 4 we expand on the theory of Riccati equations developed in [9]. Section 5 briefly considers

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well-posed right factorizations and the relation with Riccati equations. In section 6 we turn to the linear quadratic optimal control problem on $[0,\infty)$ and link this to right factorizations and Riccati equations. For a function which has a well-posed right factorization, in section 7 we construct a realization with very nice properties. The various strands are pulled together in section 8, where we give several necessary and sufficient conditions for a function to have a well-posed right factorization. In section 9 we consider (mainly through utilizing duality) the linear quadratic optimal control problem on $(-\infty,0]$ and left factorizations. Finally, in section 10, we consider doubly coprime factorizations and relate this to the linear quadratic optimal control problem on $(-\infty,\infty)$.

2. Well-posed linear systems. In this section we very briefly review the concept of a well-posed linear system. We do this from the "operator node" point of view so as to most easily connect to [9]. We refer to [11] for more background on well-posed linear systems and in particular for alternative (but equivalent) viewpoints to this theory.

The following is [9, Definition 2.1].

DEFINITION 2.1. *By an* operator node *on a triple of Hilbert spaces* $(\mathcal{X},\mathcal{U},\mathcal{Y})$ *we mean a (possibly unbounded) linear operator* $S\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ *with the following properties. We decompose* S *into* $S = \left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right]$*, where* $A\&B = P_{\mathcal{X}}S\colon \operatorname{dom}(S) \to \mathcal{X}$ *and* $C\&D = P_{\mathcal{Y}}S\colon \operatorname{dom}(S) \to \mathcal{Y}$*. We denote* $\operatorname{dom}(A) = \left\{x \in \mathcal{X} \,\middle|\, \left[\begin{smallmatrix}x\\ 0\end{smallmatrix}\right] \in \operatorname{dom}(S)\right\}$*, define* $A\colon \operatorname{dom}(A) \to \mathcal{X}$ *by* $Ax = A\&B\left[\begin{smallmatrix}x\\ 0\end{smallmatrix}\right]$*, and require the following conditions to hold:*

(i) *S is closed as an operator from $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{Y}\end{smallmatrix}\right]$ (with domain* $\operatorname{dom}(S)$*).*

(ii) *$A\&B$ is closed as an operator from $\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right]$ to $\mathcal{X}$ (with domain* $\operatorname{dom}(S)$*).*

(iii) *A has a nonempty resolvent set, and* $\operatorname{dom}(A)$ *is dense in $\mathcal{X}$.*

(iv) *For every $u \in \mathcal{U}$ there exists a $x \in \mathcal{X}$ with $\left[\begin{smallmatrix}x\\ u\end{smallmatrix}\right] \in \operatorname{dom}(S)$.*

We call S a system node *if, in addition, A is the generator of a C_0 semigroup. The* growth bound *of a system node is defined as the growth bound of the semigroup.*

Remark 2.2. By [11, Lemma 4.7.7], Definition 2.1 is equivalent to [11, Definition 4.7.2].

We recall some basic properties of operator nodes from [11] which were also already considered in [9, section 2]. Let $\Sigma := \left(\left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right]; \mathcal{X},\mathcal{U},\mathcal{Y}\right)$ be an operator node. We define $\mathcal{X}^1 := \operatorname{dom}(A)$ with the graph norm of A and $\mathcal{X}^1_* := \operatorname{dom}(A^*)$ with the graph norm of A^* and let $\mathcal{X}^{-1}$ be the dual of $\mathcal{X}^1_*$ when we identify the dual of $\mathcal{X}$ with itself. Then $\mathcal{X}^1 \subset \mathcal{X} \subset \mathcal{X}^{-1}$ with continuous and dense embeddings, and the operator A has a unique extension to an operator $A|_{\mathcal{X}} = (A^*)^* \in \mathcal{B}(\mathcal{X};\mathcal{X}^{-1})$ (with the same spectrum as A), where we interpret A^* as an operator in $\mathcal{B}(\mathcal{X}^1_*;\mathcal{X})$. The operator $A \in \mathcal{B}(\mathcal{X}^1,\mathcal{X})$ is called the *main operator* of Σ. The operator $A\&B$ (with $\operatorname{dom}(A\&B) = \operatorname{dom}\left(\left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right]\right)$) can be extended to an operator $\left[\begin{matrix}A|_{\mathcal{X}} & B\end{matrix}\right] \in \mathcal{B}(\left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right];\mathcal{X}^{-1})$ (this follows from Remark 2.2). The operator $B \in \mathcal{B}(\mathcal{U},\mathcal{X}^{-1})$ is called the *control operator* of Σ. The operator $C\colon \mathcal{X}^1 \to \mathcal{Y}$ defined by $Cx = C\&D\left[\begin{smallmatrix}x\\ 0\end{smallmatrix}\right]$ is called the *observation operator* of Σ. For any $\lambda \in \rho(A)$ we have that $\left[\begin{smallmatrix}(\lambda-A|_{\mathcal{X}})^{-1}B\\ 1_{\mathcal{U}}\end{smallmatrix}\right]$ maps $\mathcal{U}$ into $\operatorname{dom}\left(\left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right]\right)$. The *transfer function* of Σ is the operator-valued function

$$\widehat{\mathfrak{D}}(\lambda) = C\&D\begin{bmatrix}(\lambda - A|_{\mathcal{X}})^{-1}B\\ 1_{\mathcal{U}}\end{bmatrix}, \qquad \lambda \in \rho(A). \tag{2.1}$$

We denote $\mathbb{C}^+_\alpha := \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > \alpha\}$, $\mathbb{C}^+ := \mathbb{C}^+_0$, $\mathbb{R}^+ := [0,\infty)$, and $\mathbb{R}^- := (-\infty,0]$. Furthermore, $\mathcal{U}$, $\mathcal{Y}$, and $\mathcal{X}$ will always denote Hilbert spaces.

Let $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be an operator node, and assume that $\rho(A)$ contains some right half-plane. By $\rho_{+\infty}(A)$ we denote the (connected) component of $\rho(A) \cap \mathbb{C}^+$ which is unbounded to the right.

DEFINITION 2.3. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be an operator node, and let* $I \subset \mathbb{R}$ *be an interval.*

(i) *A triple* $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C^1(I;\mathcal{X}) \\ C(I;\mathcal{U}) \\ C(I;\mathcal{Y}) \end{smallmatrix}\right]$ *is called a* classical trajectory *of* Σ *if for all* $t \in I$

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right), \qquad \begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \tag{2.2}$$

(ii) *A triple* $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} C(I;\mathcal{X}) \\ L^2_{\mathrm{loc}}(I;\mathcal{U}) \\ L^2_{\mathrm{loc}}(I;\mathcal{Y}) \end{smallmatrix}\right]$ *is called a* generalized trajectory *of* Σ *if there exists a sequence of classical trajectories of* Σ *which converges to* $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ *in* $\left[\begin{smallmatrix} C(I;\mathcal{X}) \\ L^2_{\mathrm{loc}}(I;\mathcal{U}) \\ L^2_{\mathrm{loc}}(I;\mathcal{Y}) \end{smallmatrix}\right]$.

If $I = \mathbb{R}^+$, *then we add the adjective "future" (i.e.,* classical future trajectory *and* generalized future trajectory*), and if* $I = \mathbb{R}^-$, *then we add the adjective "past" (i.e.,* classical past trajectory *and* generalized past trajectory*).*

PROPOSITION 2.4. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be a system node. Then for all* $x_0 \in \mathcal{X}$ *and* $u \in W^{1,2}_{\mathrm{loc}}(0,\infty;\mathcal{U})$ *with* $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in D(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right])$ *there exists a unique classical future trajectory of* Σ *with* $x(0) = x_0$.

Proof. This is [11, Lemma 4.7.8]. □

DEFINITION 2.5. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be an operator node. Then* Σ *is called* well-posed *if* Σ *is a system node and for all* $T > 0$ *there exists a* $M > 0$ *such that for all classical future trajectories there holds*

$$\|x(T)\|^2_{\mathcal{X}} + \|y\|^2_{L^2(0,T;\mathcal{Y})} \le M\left(\|x_0\|^2_{\mathcal{X}} + \|u\|^2_{L^2(0,T;\mathcal{U})}\right).$$

Remark 2.6. Definition 2.5 is adapted from [11, Theorem 4.7.13].

PROPOSITION 2.7. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be a well-posed operator node. Then for all* $x_0 \in \mathcal{X}$ *and* $u \in L^2_{\mathrm{loc}}(0,\infty;\mathcal{U})$ *there exists a unique generalized future trajectory with* $x(0) = x_0$.

Proof. This follows from Proposition 2.4 by using density combined with well-posedness. □

3. Future and past stable trajectories and behaviors. In [9] we used different notions of past and future trajectories than those defined in Definition 2.3. In this section we show that these notions are, however, consistent (see Lemma 3.5 for the case of future trajectories and Lemma 3.9 for the case of past trajectories). The following two definitions correspond to [9, Definition 3.2] and define the notions of future trajectories and the future behavior as it was used in [9].

DEFINITION 3.1. *Let* φ *be an analytic* $\mathcal{B}(\mathcal{U};\mathcal{Y})$*-valued function defined on some open subset* Ω *of* $\mathbb{C}^+$. *By the* stable future Ω-behavior *of* φ *we mean the set of all pairs* $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} L^2(\mathbb{R}^+;\mathcal{U}) \\ L^2(\mathbb{R}^+;\mathcal{Y}) \end{smallmatrix}\right]$ *which satisfy*

$$\hat{y}(\lambda) = \varphi(\lambda)\hat{u}(\lambda), \qquad \lambda \in \Omega, \tag{3.1}$$

where $\hat{u}$ *and* $\hat{y}$ *are the Laplace transforms of* u *and* y, *respectively. We denote this set by* $\mathfrak{W}^0_+(\Omega)$ *and call* u *the* input component *and* y *the* output component *of a pair* $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in \mathfrak{W}^0_+(\Omega)$.

DEFINITION 3.2. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be an operator node with main operator* A*, observation operator* C*, and transfer function* $\widehat{\mathfrak{D}}$*, and let* Ω *be an open subset of* $\rho(A) \cap \mathbb{C}^+$.

(i) *By the set of* stable future Ω-trajectories *of* Σ *we mean the set of all triples* $\left[\begin{smallmatrix} x_0 \\ u \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ L^2(\mathbb{R}^+;\mathcal{U}) \\ L^2(\mathbb{R}^+;\mathcal{Y}) \end{smallmatrix}\right]$ *which satisfy*

$$\hat{y}(\lambda) = \widehat{\mathfrak{D}}(\lambda)\hat{u}(\lambda) + C(\lambda - A)^{-1}x_0, \qquad \lambda \in \Omega, \tag{3.2}$$

where $\hat{u}$ *and* $\hat{y}$ *are the Laplace transforms of* u *and* y*, respectively. We denote this set by* $\mathfrak{W}_+(\Omega)$ *and call* x_0 *the* initial state*,* u *the* input component*, and* y *the* output component *of a triple* $\left[\begin{smallmatrix} x_0 \\ u \\ y \end{smallmatrix}\right] \in \mathfrak{W}_+(\Omega)$.

(ii) *By the* stable future Ω-behavior *of* Σ *we mean the stable future* Ω*-behavior of its transfer function* $\widehat{\mathfrak{D}}$.

Remark 3.3. The notion of a stable future Ω-trajectory and the stable future Ω-behavior of Σ is independent of the choice of Ω to the following extent. If $\rho(A) \cap \mathbb{C}^+$ is connected, then $\mathfrak{W}_+(\Omega_1) = \mathfrak{W}_+(\Omega_2)$ and $\mathfrak{W}_+^0(\Omega_1) = \mathfrak{W}_+^0(\Omega_2)$ for all pairs of open subsets Ω_1 and Ω_2 of $\rho(A) \cap \mathbb{C}^+$. That this is true follows from (3.2) by using analytic continuation. If $\rho(A) \cap \mathbb{C}^+$ is not connected, then only the following weaker statement is true: $\mathfrak{W}_+(\Omega_1) = \mathfrak{W}_+(\Omega_2)$ and $\mathfrak{W}_+^0(\Omega_1) = \mathfrak{W}_+^0(\Omega_2)$ whenever Ω_1 and Ω_2 are both contained in the same (connected) component of $\rho(A) \cap \mathbb{C}^+$. In the remainder of this article, we shall refer to this type of independence as "independence within each (connected) component of $\rho(A) \cap \mathbb{C}^+$."

In the well-posed case it is natural to consider generalized trajectories in the sense of Definition 2.3 instead of Ω-trajectories.

DEFINITION 3.4. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be a well-posed operator node.*

(i) *By the set of* stable future trajectories *of* Σ *we mean the set of all triples* $\left[\begin{smallmatrix} x(0) \\ u \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ L^2(\mathbb{R}^+;\mathcal{U}) \\ L^2(\mathbb{R}^+;\mathcal{Y}) \end{smallmatrix}\right]$*, where* $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ *is a generalized future trajectory of* Σ*. We denote this set by* $\mathfrak{W}_+$ *and call* x_0 *the* initial state*,* u *the* input component*, and* y *the* output component *of a triple* $\left[\begin{smallmatrix} x_0 \\ u \\ y \end{smallmatrix}\right] \in \mathfrak{W}_+$.

(ii) *By the* stable future behavior *of* Σ *we mean the set of all pairs* $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} L^2(\mathbb{R}^+;\mathcal{U}) \\ L^2(\mathbb{R}^+;\mathcal{Y}) \end{smallmatrix}\right]$ *for which* $\left[\begin{smallmatrix} 0 \\ u \\ y \end{smallmatrix}\right] \in \mathfrak{W}_+$*. We denote this set by* $\mathfrak{W}_+^0$ *and call* u *the* input component *and* y *the* output component *of a pair* $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in \mathfrak{W}_+^0$.

For well-posed systems there is a close connection between Definitions 3.2 and 3.4.

LEMMA 3.5. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be a well-posed operator node with main operator* A*. Let* Ω *be an open subset of* $\rho_{+\infty}(A)$*. Then* $\mathfrak{W}_+ = \mathfrak{W}_+(\Omega)$ *and* $\mathfrak{W}_+^0 = \mathfrak{W}_+^0(\Omega)$.

Proof. We denote the growth bound of Σ by α and let $\alpha_+ = \max\{\alpha, 0\}$. Then $\mathbb{C}_{\alpha_+}^+ \subset \rho_{+\infty}(A)$.

Assume first that $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ is a classical future trajectory of Σ with $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} L^2(\mathbb{R}^+;\mathcal{U}) \\ L^2(\mathbb{R}^+;\mathcal{Y}) \end{smallmatrix}\right]$. Since Σ has growth bound α, for every $\beta > \alpha_+$ we have that there exists a $M > 0$ such that for all $t \geq 0$ there holds $\|x(t)\| \leq M e^{\beta t}$. It follows that $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ is Laplace transformable, and we obtain from (2.2) that for $\lambda \in \mathbb{C}^+_\beta$

$$\begin{bmatrix} \lambda\hat{x}(\lambda) - x(0) \\ \hat{y}(\lambda) \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \end{bmatrix}.$$

This is equivalent to (see, e.g., [2])

$$\begin{bmatrix} \hat{x}(\lambda) \\ \hat{y}(\lambda) \end{bmatrix} = \begin{bmatrix} (\lambda - A)^{-1}x(0) + (\lambda - A|_{\mathcal{X}})^{-1}B\hat{u}(\lambda) \\ C(\lambda - A)^{-1}x(0) + \widehat{\mathfrak{D}}(\lambda)\hat{u}(\lambda) \end{bmatrix}. \tag{3.3}$$

Since $\beta > \alpha_+$ was arbitrary, we obtain the above equality for all $\lambda \in \mathbb{C}^+_{\alpha_+}$, and since $\rho_{+\infty}(A)$ is connected, by analytic continuation (3.3) holds for all $\lambda \in \rho_{+\infty}(A)$. In particular, (3.3) holds for all $\lambda \in \Omega$, and thus $\left[\begin{smallmatrix} x(0) \\ u \\ y \end{smallmatrix}\right] \in \mathfrak{W}_+(\Omega)$.

Next suppose that $\left[\begin{smallmatrix} x_0 \\ u \\ y \end{smallmatrix}\right] \in \mathfrak{W}_+$. Then $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} L^2(\mathbb{R}^+;\mathcal{U}) \\ L^2(\mathbb{R}^+;\mathcal{Y}) \end{smallmatrix}\right]$, and there exists a generalized future trajectory $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ of Σ with $x(0) = x_0$. For each $n \in \mathbb{Z}^+$, define

$$\begin{bmatrix} x_n(t) \\ u_n(t) \\ y_n(t) \end{bmatrix} := \frac{1}{n} \int_t^{t+1/n} \begin{bmatrix} x(\tau) \\ u(\tau) \\ y(\tau) \end{bmatrix} d\tau, \qquad t \in \mathbb{R}^+.$$

By [2] each $\left[\begin{smallmatrix} x_n \\ u_n \\ y_n \end{smallmatrix}\right]$ is a classical future trajectory of Σ, and by standard properties of approximate identities (see, e.g., [3]), $\left[\begin{smallmatrix} u_n \\ y_n \end{smallmatrix}\right] \to \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ in $\left[\begin{smallmatrix} L^2(\mathbb{R}^+;\mathcal{U}) \\ L^2(\mathbb{R}^+;\mathcal{Y}) \end{smallmatrix}\right]$ and $\mathrm{e}^{-\lambda t}x_n(t) \to \mathrm{e}^{-\lambda t}x(t)$ uniformly on $\mathbb{R}^+$ for every $\lambda \in \mathbb{C}^+_{\alpha_+}$. Since the solutions $\left[\begin{smallmatrix} x_n \\ u_n \\ y_n \end{smallmatrix}\right]$ are classical, the equations (3.3) hold with $\left[\begin{smallmatrix} \hat{x} \\ \hat{u} \\ \hat{y} \end{smallmatrix}\right]$ replaced by $\left[\begin{smallmatrix} \hat{x}_n \\ \hat{u}_n \\ \hat{y}_n \end{smallmatrix}\right]$. The Laplace transforms $\left[\begin{smallmatrix} \hat{x}_n(\lambda) \\ \hat{u}_n(\lambda) \\ \hat{y}_n(\lambda) \end{smallmatrix}\right]$ converge to $\left[\begin{smallmatrix} \hat{x}(\lambda) \\ \hat{u}(\lambda) \\ \hat{y}(\lambda) \end{smallmatrix}\right]$ as $n \to \infty$ for every $\lambda \in \mathbb{C}^+_{\alpha_+}$. In addition $x_n(0) \to x(0) = x_0$ in $\mathcal{X}$ as $n \to \infty$. This implies that (3.3) holds with $x(0) = x_0$ for every $\lambda \in \mathbb{C}^+_{\alpha_+}$ and therefore, by analytic continuation, for all $\lambda \in \rho_{+\infty}(A)$. In particular, (3.3) holds with $x(0) = x_0$ for all $\lambda \in \Omega$, and thus $\left[\begin{smallmatrix} x_0 \\ u \\ y \end{smallmatrix}\right] \in \mathfrak{W}_+(\Omega)$. This proves that $\mathfrak{W}_+ \subset \mathfrak{W}_+(\Omega)$.

Conversely, suppose that $\left[\begin{smallmatrix} x_0 \\ u \\ y \end{smallmatrix}\right] \in \mathfrak{W}_+(\Omega)$, i.e., $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} L^2(\mathbb{R}^+;\mathcal{U}) \\ L^2(\mathbb{R}^+;\mathcal{Y}) \end{smallmatrix}\right]$, and (3.2), holds for all $\lambda \in \Omega$. Let $\left[\begin{smallmatrix} x \\ u \\ y_1 \end{smallmatrix}\right]$ be the generalized future trajectory of Σ with initial state x_0 and input function u (existence and uniqueness of which follows from Proposition 2.7). Then $\left[\begin{smallmatrix} x_0 \\ u \\ y_1 \end{smallmatrix}\right] \in \mathfrak{W}_+ \subset \mathfrak{W}_+(\Omega)$. Consequently, it follows from (3.2) that $\hat{y}_1(\lambda) = \hat{y}(\lambda)$ for all $\lambda \in \Omega$. It follows from the uniqueness theorem for Laplace transforms that $y_1 = y$. Thus, $\left[\begin{smallmatrix} x_0 \\ u \\ y \end{smallmatrix}\right] \in \mathfrak{W}_+$. This proves that $\mathfrak{W}_+(\Omega) \subset \mathfrak{W}_+$, and consequently $\mathfrak{W}_+(\Omega) = \mathfrak{W}_+$.

That also $\mathfrak{W}^0_+(\Omega) = \mathfrak{W}^0_+$ follows from Definitions 3.2 and 3.4 and the fact that $\mathfrak{W}_+(\Omega) = \mathfrak{W}_+$. ∎

The following two definitions correspond to [9, Definition 3.8] and define the notions of past trajectories and the past behavior used in [9].

DEFINITION 3.6. *Let φ be an analytic $\mathcal{B}(\mathcal{U};\mathcal{Y})$-valued function defined on some open subset Ω of $\mathbb{C}^+$. For each $\lambda \in \mathbb{C}^+$ we denote the function $t \mapsto \mathrm{e}^{\lambda t}$, $t \in \mathbb{R}^-$, by* $\mathbf{e}_\lambda$.

(i) *By the* classical exponential past Ω-behavior *of* φ *we mean*

$$\mathfrak{V}^0_-(\Omega) := \operatorname{span}\left\{ \begin{bmatrix} \mathbf{e}_\lambda u_0 \\ \mathbf{e}_\lambda \varphi(\lambda) u_0 \end{bmatrix} \,\middle|\, \lambda \in \Omega,\ u_0 \in \mathcal{U} \right\} \subset \begin{bmatrix} L^2(\mathbb{R}^-;\mathcal{U}) \\ L^2(\mathbb{R}^-;\mathcal{Y}) \end{bmatrix}.$$

We call u *the* input component *and* y *the* output component *of a pair* $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in \mathfrak{V}^0_-(\Omega)$.

(ii) *By the (generalized)* stable past Ω-behavior *of* φ *we mean the closure in* $\left[\begin{smallmatrix} L^2(\mathbb{R}^-;\mathcal{U}) \\ L^2(\mathbb{R}^-;\mathcal{Y}) \end{smallmatrix}\right]$ *of* $\mathfrak{V}^0_-(\Omega)$. *We denote this set by* $\mathfrak{W}^0_-(\Omega)$.

Definition 3.7. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be an operator node with main operator* A, *control operator* B, *and transfer function* $\widehat{\mathfrak{D}}$, *and let* Ω *be an open subset of* $\rho(A) \cap \mathbb{C}^+$.

For each $\lambda \in \mathbb{C}^+$ *we denote the function* $t \mapsto \mathrm{e}^{\lambda t}$, $t \in \mathbb{R}^-$, *by* $\mathbf{e}_\lambda$.

(i) *By the set of* classical stable past exponential Ω-trajectories *of* Σ *we mean*

(3.4)
$$\mathfrak{V}_-(\Omega) := \operatorname{span}\left\{ \begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1} B u_0 \\ \mathbf{e}_\lambda u_0 \\ \mathbf{e}_\lambda \widehat{\mathfrak{D}}(\lambda) u_0 \end{bmatrix} \,\middle|\, \lambda \in \Omega,\ u_0 \in \mathcal{U} \right\} \subset \begin{bmatrix} \mathcal{X} \\ L^2(\mathbb{R}^-;\mathcal{U}) \\ L^2(\mathbb{R}^-;\mathcal{Y}) \end{bmatrix}.$$

We call x_0 *the* final state, u *the* input component, *and* y *the* output component *of a triple* $\left[\begin{smallmatrix} x_0 \\ u \\ y \end{smallmatrix}\right] \in \mathfrak{V}_-(\Omega)$.

(ii) *By the set of* generalized stable past Ω-trajectories *of* Σ *we mean the closure in* $\left[\begin{smallmatrix} \mathcal{X} \\ L^2(\mathbb{R}^-;\mathcal{U}) \\ L^2(\mathbb{R}^-;\mathcal{Y}) \end{smallmatrix}\right]$ *of* $\mathfrak{V}_-(\Omega)$. *We denote this set by* $\mathfrak{W}_-(\Omega)$.

(iii) *By the* classical exponential past Ω-behavior *of* Σ *we mean the classical exponential past* Ω*-behavior of its transfer function* $\widehat{\mathfrak{D}}$.

(iv) *By the* stable past Ω-behavior *of* Σ *we mean the stable past* Ω*-behavior of its transfer function* $\widehat{\mathfrak{D}}$.

In the well-posed case it is natural to consider generalized trajectories in the sense of Definition 2.3 which "vanish at $-\infty$" instead of past Ω-trajectories.

Definition 3.8. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be a well-posed operator node.*

(i) *The notation* $\mathfrak{V}_-$ *stands for the set of all* $\left[\begin{smallmatrix} x(0) \\ u \\ y \end{smallmatrix}\right]$, *where* $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ *is a generalized past trajectory of* Σ *with compact support.*

(ii) *By the set of* generalized stable past trajectories *of* Σ *we mean the closure in* $\left[\begin{smallmatrix} \mathcal{X} \\ L^2(\mathbb{R}^-;\mathcal{U}) \\ L^2(\mathbb{R}^-;\mathcal{Y}) \end{smallmatrix}\right]$ *of* $\mathfrak{V}_-$. *We denote this set by* $\mathfrak{W}_-$.

(iii) *The notation* $\mathfrak{V}^0_-$ *stands for the set of all* $\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} L^2(\mathbb{R}^-;\mathcal{U} \\ L^2(\mathbb{R}^-;\mathcal{Y}) \end{smallmatrix}\right]$ *(with compact support) with the property that* $\left[\begin{smallmatrix} x_0 \\ u \\ y \end{smallmatrix}\right] \in \mathfrak{V}_-$ *for some* $x_0 \in \mathcal{X}$.

(iv) *By the* stable past behavior *of* Σ *we mean the closure in* $\left[\begin{smallmatrix} L^2(\mathbb{R}^-;\mathcal{U}) \\ L^2(\mathbb{R}^-;\mathcal{Y}) \end{smallmatrix}\right]$ *of* $\mathfrak{V}^0_-$. *We denote this set by* $\mathfrak{W}^0_-$.

For well-posed systems there is a close connection between Definitions 3.7 and 3.8.

Lemma 3.9. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be a well-posed operator node, and let* Ω *be an open subset of* $\rho_{+\infty}(A)$. *Then* $\mathfrak{W}_- = \mathfrak{W}_-(\Omega)$ *and* $\mathfrak{W}^0_- = \mathfrak{W}^0_-(\Omega)$.

Proof. Define $\Omega^* := \{\lambda : \bar{\lambda} \in \Omega\}$ and $\Sigma^\dagger := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]^*; \mathcal{X}, \mathcal{Y}, \mathcal{U}\right)$. We will add a qualifier to the various sets of trajectories to indicate whether they are considered for the operator node Σ or for its adjoint $\Sigma^\dagger$. By [9, Lemma 3.16] we have that $\mathfrak{W}_-(\Omega;\Sigma)$ is the annihilator of $\mathfrak{W}_+(\Omega^*;\Sigma^\dagger)$ (with respect to the duality pairing given there) and that $\mathfrak{W}^0_-(\Omega;\Sigma)$ is the annihilator of $\mathfrak{W}^0_+(\Omega^*;\Sigma^\dagger)$. By [11, section 6.2], we have that $\mathfrak{W}_-(\Sigma)$ is the annihilator of $\mathfrak{W}_+(\Sigma^\dagger)$ and that $\mathfrak{W}^0_-(\Sigma)$ is the annihilator of $\mathfrak{W}^0_+(\Sigma^\dagger)$. From Lemma 3.5 and uniqueness of annihilators we obtain the desired result. ∎

4. Riccati equations. In [9] we used the concept of a normalized solution of a Riccati equation. It is often, however, more convenient to replace the normalization condition by a (more general) invertibility assumption. In this section we first recall the concept of a normalized solution from [9] (Definition 4.1), then introduce the alternative solution notion (Definition 4.2) and subsequently show that these two solution notions are consistent (Lemma 4.3). Finally, we show that the feedback operator which appears in the definition of the Riccati equation is (up to multiplication by a unitary operator) uniquely determined by the solution of the Riccati equation (Lemma 4.6).

The following is [9, Definition 5.1].

DEFINITION 4.1. *Let $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be an operator node with main operator A and control operator B, and let $\lambda \in \rho(A) \cap \mathbb{C}^+$. By a λ-normalized solution of the continuous time* control Riccati equation *induced by $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ we mean a form q on $\mathcal{X}$ with the following properties:*

(i) *q is a closed nonnegative sesquilinear symmetric form on $\mathcal{X}$ with domain $\mathcal{Z}$.*
(ii) *$(\lambda - A)^{-1}\mathcal{Z} \subset \mathcal{Z}$.*
(iii) *$(\lambda - A|_{\mathcal{X}})^{-1}B\mathcal{U} \subset \mathcal{Z}$.*
(iv) *There exists an operator $[K\&F]_\lambda \colon \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \mathcal{U}$ with*

$$\text{(4.1)} \qquad \operatorname{dom}([K\&F]_\lambda) = \left\{ \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \right) \middle| \begin{array}{l} x_0 \in \mathcal{Z} \text{ and} \\ A\&B \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{Z} \end{array} \right\}$$

and a self-adjoint operator $W_\lambda \in \mathcal{B}(\mathcal{U})$ such that the following identity holds:
(4.2)
$$\begin{aligned} &2\mathrm{Re}\, q\left[A\&B \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, x_0 \right] + \left\| C\&D \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\|^2_{\mathcal{Y}} + \|u_0\|^2_{\mathcal{U}} \\ &\quad = \left\langle [K\&F]_\lambda \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, W_\lambda [K\&F]_\lambda \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\rangle_{\mathcal{U}}, \quad \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \operatorname{dom}([K\&F]_\lambda), \end{aligned}$$

and

$$\text{(4.3)} \qquad [K\&F]_\lambda \begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1}B \\ 1_{\mathcal{U}} \end{bmatrix} = -1_{\mathcal{U}}.$$

It will be convenient to replace the normalization condition (4.3) in Definition 4.1 by an invertibility condition. The resulting concept of a Riccati equation is formalized in Definition 4.2. Subsequently, in Lemma 4.3, we show that this concept is essentially the same as that in Definition 4.1.

DEFINITION 4.2. *Let $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be an operator node with main operator A and control operator B, and let Ω be an open subset of $\rho(A) \cap \mathbb{C}^+$. By an Ω-solution of the continuous time* control Riccati equation *induced by $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ we mean a form q on $\mathcal{X}$ with the following properties:*

(i) *q is a closed nonnegative sesquilinear symmetric form on $\mathcal{X}$ with domain $\mathcal{Z}$.*
(ii) *There exists an operator $K\&F\colon \left[\begin{smallmatrix}\mathcal{X}\\ \mathcal{U}\end{smallmatrix}\right] \to \mathcal{U}$ with domain given by*

$$\operatorname{dom}(K\&F) = \left\{ \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \operatorname{dom}\left(\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\right) \middle| \begin{array}{l} x_0 \in \mathcal{Z} \text{ and} \\ A\&B \left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{Z} \end{array} \right\} \tag{4.4}$$

such that the following identity holds:

$$\begin{aligned} &2\mathrm{Re}\, q\left[[A\&B]\begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, x_0\right] + \left\| C\&D \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\|_{\mathcal{Y}}^2 + \|u_0\|_{\mathcal{U}}^2 \\ &\qquad = \left\| K\&F \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\|_{\mathcal{U}}^2, \qquad \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \in \operatorname{dom}(K\&F). \end{aligned} \tag{4.5}$$

(iii) *For all $\lambda \in \Omega$ the following conditions hold:*
   (a) $(\lambda - A)^{-1}\mathcal{Z} \subset \mathcal{Z}$;
   (b) $(\lambda - A|_{\mathcal{X}})^{-1}B\mathcal{U} \subset \mathcal{Z}$;
   (c) the operator

$$\mathsf{F}(\lambda) := K\&F \begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1}B \\ 1_{\mathcal{U}} \end{bmatrix} \tag{4.6}$$

   is bounded and boundedly invertible.

An Ω-solution $q_{\min}$ is called the minimal *Ω-solution if $q_{\min} \le q$ for all Ω-solutions q (the inequality $q_{\min} \le q$ meaning that $D(q) \subset D(q_{\min})$ and $q_{\min}[x_0, x_0] \le q[x_0, x_0]$ for all $x_0 \in D(q)$).*

LEMMA 4.3. *Let $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be an operator node with main operator A and control operator B.*

(i) *Let Ω be an open subset of $\rho(A) \cap \mathbb{C}^+$, and let q be an Ω-solution of the continuous time control Riccati equation with corresponding operator $K\&F$. Then for any $\lambda \in \Omega$, q is a λ-normalized solution of the continuous time control Riccati equation with $[K\&F]_\lambda := -\mathsf{F}(\lambda)^{-1}K\&F$ and $W_\lambda := \mathsf{F}(\lambda)^*\mathsf{F}(\lambda)$.*
(ii) *Conversely, let $\lambda \in \rho(A) \cap \mathbb{C}^+$ and q be a λ-normalized solution of the continuous time control Riccati equation with corresponding operators $[K\&F]_\lambda$ and W_λ, and let Ω be an open subset of the (connected) component of $\rho(A) \cap \mathbb{C}^+$ which contains λ. Then q is an Ω-solution of the continuous time control Riccati equation with corresponding operator $K\&F := -W_\lambda^{1/2}[K\&F]_\lambda$.*

Proof. (i) Assume that q is an Ω-solution of the continuous time control Riccati equation, where Ω is an open subset of $\rho(A) \cap \mathbb{C}^+$. Parts (i), (ii), and (iii) of Definition 4.1 are clearly satisfied. From the above definition of $[K\&F]_\lambda$, the fact that $\mathsf{F}(\lambda)$ is invertible, and (4.4) we obtain (4.1). From the definitions of $[K\&F]_\lambda$ and W_λ we have for $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in \operatorname{dom}([K\&F]_\lambda) = \operatorname{dom}(K\&F)$ that

$$\begin{aligned} &\left\langle [K\&F]_\lambda \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, W_\lambda [K\&F]_\lambda \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\rangle_{\mathcal{U}} \\ &\qquad = \left\langle \mathsf{F}(\lambda)^{-1} K\&F \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, \mathsf{F}(\lambda)^*\mathsf{F}(\lambda)\mathsf{F}(\lambda)^{-1} K\&F \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\rangle_{\mathcal{U}} = \left\| K\&F \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\|_{\mathcal{U}}^2, \end{aligned}$$

so that (4.2) follows from (4.5). We also obtain (4.3) since

$$[K\&F]_\lambda \begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1}B \\ 1_{\mathcal{U}} \end{bmatrix} = -\mathsf{F}(\lambda)^{-1} K\&F \begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1}B \\ 1_{\mathcal{U}} \end{bmatrix} = -\mathsf{F}(\lambda)^{-1}\mathsf{F}(\lambda) = -1_{\mathcal{U}},$$

where we have used (4.6).

(ii) Now assume that q is an λ-normalized solution where $\lambda \in \rho(A) \cap \mathbb{C}^+$. Let Ω^0 be the (connected) component of $\rho(A) \cap \mathbb{C}^+$ which contains λ. Part (i) of Definition 4.2 is clearly satisfied. We obtain (4.4) from the definition of $K\&F$, (4.1), and the fact that, by [9, Theorem 5.6], W_λ is boundedly invertible. We obtain (4.5) from the fact that

$$\left\| K\&F \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\|_{\mathcal{U}}^2 = \left\| W_\lambda^{1/2} [K\&F]_\lambda \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\|_{\mathcal{U}}^2 = \left\langle [K\&F]_\lambda \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}, W_\lambda [K\&F]_\lambda \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \right\rangle_{\mathcal{U}}$$

and (4.2). We have

$$\mathsf{F}(\lambda) = K\&F \begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1} B \\ 1_{\mathcal{U}} \end{bmatrix} = -W_\lambda^{1/2} [K\&F]_\lambda \begin{bmatrix} (\lambda - A|_{\mathcal{X}})^{-1} B \\ 1_{\mathcal{U}} \end{bmatrix} = W_\lambda^{1/2},$$

where in the last equality we have used (4.3). It follows that for the λ specified in the statement of the lemma, we have part (iii) of Definition 4.2. However, by [9, Theorem 5.9] we have that q is a β-normalized solution for all $\beta \in \Omega^0$. Therefore, (iii) of Definition 4.2 in fact holds for all $\lambda \in \Omega^0$, and consequently also for all $\lambda \in \Omega$. $\square$

Remark 4.4. It follows from Lemma 4.3 that the notion of an Ω-solution of the continuous time Riccati equation is independent of the choice of Ω within each (connected) component of $\rho(A) \cap \mathbb{C}^+$ (in the same sense as in Remark 3.3).

The following technical lemma will be used in the proof of Lemma 4.6.

LEMMA 4.5. *Assume that $T_1, T_2 : \mathcal{H} \to \mathcal{U}$ are surjective operators with common domain $\mathcal{Z}$ which satisfy $\|T_1 x\| = \|T_2 x\|$ for all $x \in \mathcal{Z}$. Then there exists a unitary operator $W \in \mathcal{B}(\mathcal{U})$ such that $T_2 = W T_1$.*

Proof. Let $x_1, x_2 \in \mathcal{Z}$ be such that $T_1 x_1 = T_1 x_2$. Then $T_1(x_1 - x_2) = 0$ and therefore, by the assumed equality of norms, $T_2(x_1 - x_2) = 0$. Hence, $T_2 x_1 = T_2 x_2$.

Let $y \in \mathcal{U}$. By surjectivity there exists a $x \in \mathcal{Z}$ such that $y = T_1 x$. Define $Wy = T_2 x$. By the above paragraph, this is well-defined (i.e., does not depend on the choice of x). Since $\|Wy\| = \|T_2 x\| = \|T_1 x\| = \|y\|$, this operator W is an isometry. We clearly have $T_2 = W T_1$. Since T_2 is surjective, this implies that also W is surjective, and since W is also an isometry, we obtain that W is unitary. $\square$

LEMMA 4.6. *Let $\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]; \mathcal{X}, \mathcal{U}, \mathcal{Y} \right)$ be an operator node, let Ω be an open subset of $\rho(A) \cap \mathbb{C}^+$, let q be an Ω-solution of the continuous time control Riccati equation induced by $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]$, and let $K\&F$ be an operator satisfying the conditions in Definition 4.2. Then the operator $K\&F$ is determined uniquely by q, Ω, and $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]$ up to the multiplication by a unitary operator in $\mathcal{U}$ to the left in the following sense:*

(i) *if $K\&F$ is an operator satisfying the conditions in Definition 4.2 and if W is a unitary operator in $\mathcal{U}$, then $WK\&F$ is also an operator satisfying the conditions in Definition 4.2;*

(ii) *if $K\&F_1$ and $K\&F_2$ are two operators which satisfy the conditions in Definition 4.2, then there exists a unitary operator W in $\mathcal{U}$ such that $K\&F_2 = WK\&F_1$.*

Proof. The first statement is clear. So assume that $K\&F_1$ and $K\&F_2$ are two operators which satisfy the conditions in Definition 4.2. From (4.4) we have that $K\&F_1$ and $K\&F_2$ have the same domain, and by (4.5) we have that $\|K\&F_2 \left[\begin{smallmatrix} x \\ u \end{smallmatrix} \right]\| = \|K\&F_1 \left[\begin{smallmatrix} x \\ u \end{smallmatrix} \right]\|$ for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix} \right]$ in this domain. It follows from part (iiic) of Definition 4.2 that $K\&F_1$ and $K\&F_2$ are surjective. Lemma 4.5 with $T_1 := K\&F_1$, $T_2 := K\&F_2$, $\mathcal{H} := \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix} \right]$, and $\mathcal{Z}$ the common domain of $K\&F_1$ and $K\&F_2$ then gives the result. $\square$

5. Right factorizations. The following definition adds an extra well-posedness condition on M^{-1} to [9, Definition 5.8] which is relevant in the well-posed case (conditions (i)–(iii) below are the same as in [9, Definition 5.8]).

DEFINITION 5.1. *Let φ be an analytic $\mathcal{B}(\mathcal{U};\mathcal{Y})$-valued function defined on some open subset Ω of $\mathbb{C}^+$.*

(i) *φ has a* right $H^\infty(\mathbb{C}^+)$ factorization valid in Ω *if there exist two functions $\mathsf{M} \in H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{U}))$ and $\mathsf{N} \in H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{U};\mathcal{Y}))$ such that $\mathsf{M}(\lambda)$ has a bounded inverse and $\varphi(\lambda) = \mathsf{N}(\lambda)\mathsf{M}(\lambda)^{-1}$ for all $\lambda \in \Omega$.*

(ii) *The factorization in* (i) *is* normalized *if $\left[\begin{smallmatrix}\mathsf{N}\\ \mathsf{M}\end{smallmatrix}\right]$ is* inner*; i.e., the multiplication by $\left[\begin{smallmatrix}\mathsf{N}\\ \mathsf{M}\end{smallmatrix}\right]$ is an isometric operator from $H^2(\mathbb{C}^+;\mathcal{U})$ to $H^2(\mathbb{C}^+;\left[\begin{smallmatrix}\mathcal{Y}\\ \mathcal{U}\end{smallmatrix}\right])$.*

(iii) *The factorization in* (i) *is* weakly (right) coprime *if the range of the multiplication operator in* (ii) *is equal to the Laplace transform of the future behavior $\mathfrak{W}^0_+(\Omega)$ defined in Definition* 3.1.

(iv) *The factorization in* (i) *is* well-posed *if there exists some $\alpha \geq 0$ such that $\mathsf{M}(\lambda)$ has a bounded inverse for all $\lambda \in \mathbb{C}^+_\alpha$ and $\mathsf{M}^{-1} \in H^\infty(\mathbb{C}^+_\beta;\mathcal{B}(\mathcal{U}))$ for all $\beta > \alpha$.*

(v) *If the factorization in* (i) *is well-posed, then the* growth bound *of this factorization is the infimum over all α for which the condition in* (iv) *holds. (If the factorization is not well-posed, then its growth bound is $+\infty$.)*

The following lemma shows how the minimal solution of the control Riccati equation gives rise to a normalized weakly coprime right $H^\infty(\mathbb{C}^+)$ factorization (which need not be well-posed in general).

LEMMA 5.2. *Let $\Sigma := \left(\left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y}\right)$ be an operator node with main operator A and transfer function $\widehat{\mathfrak{D}}$. Let Ω be an open set which is contained in some (connected) component of $\rho(A) \cap \mathbb{C}^+$. Assume that there exists a minimal Ω-solution q of the continuous time control Riccati equation induced by $\left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right]$. Let $K\&F$ be an operator satisfying the conditions in Definition* 4.2 *and define F by* (4.6). *Define*

$$\mathsf{M}(\lambda) := \mathsf{F}(\lambda)^{-1}, \qquad \mathsf{N}(\lambda) := \widehat{\mathfrak{D}}(\lambda)\mathsf{M}(\lambda), \qquad \lambda \in \Omega. \tag{5.1}$$

Then M and N can be extended to H^∞-functions over $\mathbb{C}^+$, and $\widehat{\mathfrak{D}} = \mathsf{N}\mathsf{M}^{-1}$ is a normalized weakly coprime right $H^\infty(\mathbb{C}^+)$ factorization of $\widehat{\mathfrak{D}}$ valid in Ω.

Proof. This follows from [9, Theorem 5.10 part (ii)]; the details are as follows. By Remark 4.4 we may, without loss of generality, assume that Ω is connected (we may, e.g., replace Ω by the component of $\rho(A) \cap \mathbb{C}^+$ which contains Ω). Fix $\alpha \in \Omega$. By Lemma 4.3, solutions of the Riccati equations according to Definitions 4.1 and 4.2 coincide, and therefore q coincides with the q in [9, Theorem 5.10]. Let $[K\&F]_\alpha$ and W_α be as in Definition 4.1 (by [9, Theorem 5.6 part (ii)] these operators are uniquely determined by Σ, q and α). The operator $\mathbf{F}_\alpha(\lambda)$ appearing in [9, Theorem 5.10] is

$$\mathbf{F}_\alpha(\lambda) := [K\&F]_\alpha \begin{bmatrix}(\lambda - A|_{\mathcal{X}})^{-1}B\\ 1_{\mathcal{U}}\end{bmatrix}.$$

From Lemma 4.3 and the uniqueness up to a unitary operator of $K\&F$ from Lemma 4.6 we obtain that $W_\alpha^{1/2}\mathbf{F}_\alpha(\lambda) = -W\mathsf{F}(\lambda)$ for some unitary W.

From [9, Theorem 5.10 part (ii)] we have that

$$\mathsf{M}_\alpha(\lambda) := -[W_\alpha^{1/2}\mathbf{F}_\alpha(\lambda)]^{-1}, \qquad \mathsf{N}_\alpha(\lambda) := \widehat{\mathfrak{D}}(\lambda)\mathsf{M}_\alpha(\lambda), \qquad \lambda \in \Omega, \tag{5.2}$$

have the properties desired of M and N. By the above relation between $\mathbf{F}_\alpha$ and F we have $\mathsf{M}(\lambda) = \mathsf{M}_\alpha(\lambda)W$. It then follows that $\mathsf{N}(\lambda) = \mathsf{N}_\alpha(\lambda)W$. From this we see that M and N also have the desired properties. ∎

6. The future optimal control problem. As in [9] (but now for the well-posed case), we obtain in this section equivalence of (i) a "cost condition" for the future optimal control problem being satisfied, (ii) solvability of the control Riccati equation, and (iii) existence of a weakly coprime right factorization. In comparison to [9], each of these three equivalent statements has an additional "uniformity" condition. The above equivalence is precisely formulated in Theorem 6.10. The first part of this section (up to and including Lemma 6.6) briefly recalls relevant notions from [9]. Definition 6.7 introduces the relevant "uniform" version of the cost condition.

DEFINITION 6.1. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be an operator node with main operator* A*, and let* Ω *be an open subset of* $\rho(A) \cap \mathbb{C}^+$.

(i) *A vector* $x_0 \in \mathcal{X}$ *is said to have* finite future Ω-cost *if it is the initial state of a generalized stable future* Ω*-trajectory of* Σ*. The* future Ω-cost *of such a vector* x_0 *is the infimum of the future cost functional*

$$J_{\mathrm{fut}}(x_0, u) = \int_0^\infty \left(\|u(t)\|^2_{\mathcal{U}} + \|y(t)\|^2_{\mathcal{Y}}\right) dt \tag{6.1}$$

over all generalized stable future Ω*-trajectories* $\left[\begin{smallmatrix} x_0 \\ u \\ y \end{smallmatrix}\right]$ *of* Σ*. We denote this cost by* $\|x_0\|^2_{\mathrm{fut},\Omega}$.

(ii) *If* Σ *is well-posed, then a vector* $x_0 \in \mathcal{X}$ *is said to have* finite future cost *if it is the initial state component of a stable future trajectory. The* future cost *of such a vector* x_0 *is the infimum of the future cost functional* (6.1) *over all generalized stable future trajectories* $\left[\begin{smallmatrix} x_0 \\ u \\ y \end{smallmatrix}\right]$ *of* Σ*. We denote this cost by* $\|x_0\|^2_{\mathrm{fut}}$.

Remark 6.2. By [9, Theorem 3.7], the infimum in part (i) of Definition 6.1 is actually achieved by a unique minimizing generalized stable future Ω-trajectory of Σ, and $\|\cdot\|^2_{\mathrm{fut},\Omega}$ is a closed quadratic form in $\mathcal{X}$. By Remark 3.3, $\|\cdot\|^2_{\mathrm{fut},\Omega}$ is independent of Ω in the following sense: If Ω_1 and Ω_2 are two open subsets $\rho(A) \cap \mathbb{C}^+$ both of which are contained in the same (connected) component of $\rho(A) \cap \mathbb{C}^+$, then $\|\cdot\|^2_{\mathrm{fut},\Omega_1} = \|\cdot\|^2_{\mathrm{fut},\Omega_2}$. An analogous result is true for well-posed systems: The infimum in part (ii) of Definition 6.1 is achieved by a unique minimizing generalized stable future trajectory of Σ, and $\|\cdot\|^2_{\mathrm{fut}}$ is a closed quadratic form in $\mathcal{X}$. (The proof is essentially the same as the proof of the Ω-version.)

Parts (i) and (ii) of Definition 6.1 are related to each other by the following lemma.

LEMMA 6.3. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be a well-posed operator node with main operator* A*, and let* Ω *be an open subset of* $\rho_{+\infty}(A)$*. Then a vector* $x_0 \in \mathcal{X}$ *has a finite future cost if and only if* x_0 *has a finite future* Ω*-cost, and* $\|x_0\|^2_{\mathrm{fut},\Omega} = \|x_0\|^2_{\mathrm{fut}}$.

Proof. This follows from Lemma 3.5. ∎

The following is essentially [9, Definition 5.7] (see Remark 6.5 for the connection).

DEFINITION 6.4. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be an operator node with main operator* A *and control operator* B*, and let* Ω *be an open subset of* $\rho(A) \cap \mathbb{C}^+$.

(i) Σ *satisfies the* input finite future Ω-cost condition *if* $(\lambda - A|_{\mathcal{X}})^{-1} B u_0$ *has a finite future* Ω*-cost for every* $\lambda \in \Omega$ *and every* $u_0 \in \mathcal{U}$.

(ii) Σ *satisfies the* state finite future Ω-cost condition *if every initial state in* $\mathcal{X}$ *has a finite future* Ω*-cost.*

Remark 6.5. In this remark we assume that the subset Ω in Definition 6.4 is contained in some (connected) component of $\rho(A) \cap \mathbb{C}^+$. Then it follows from [9, Theorem 5.9] that $(\lambda - A|_{\mathcal{X}})^{-1}Bu_0$ has a finite future Ω-cost for *every* $\lambda \in \Omega$ and every $u_0 \in \mathcal{U}$ if and only if $(\lambda - A|_{\mathcal{X}})^{-1}Bu_0$ has a finite future Ω-cost for *some* $\lambda \in \Omega$ and every $u_0 \in \mathcal{U}$. Thus, in this case it is possible to replace "every $\lambda \in \Omega$" by "some $\lambda \in \Omega$" in condition (i) above.

Under the same additional assumption on Ω, if Σ satisfies the input finite future Ω-cost condition, then $\|\cdot\|^2_{\mathrm{fut},\Omega}$ is the minimal Ω-solution of the control algebraic Riccati equation by [9, Theorem 5.9] (combined with Lemma 4.3). Conversely, if the control algebraic Riccati equation has an Ω-solution, then Σ satisfies the input finite future Ω-cost condition by [9, Theorem 5.9] (combined with Lemma 4.3).

The following result was never explicitly stated in [9] but follows easily from the results presented there. We recall that a sesquilinear form q on $\mathcal{X}$ is called *bounded* if its domain equals $\mathcal{X}$ and there exists a $M > 0$ such that $|q[x_0, z_0]| \le M\|x_0\|_{\mathcal{X}}\|z_0\|_{\mathcal{X}}$ for all $x_0, z_0 \in \mathcal{X}$.

LEMMA 6.6. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be an operator node with main operator* A*, and let* Ω *be an open subset of a connected subset of* $\rho(A) \cap \mathbb{C}^+$*. The following are equivalent:*

(i) Σ *satisfies the state finite future* Ω*-cost condition.*

(ii) *The quadratic form* $\|\cdot\|^2_{\mathrm{fut},\Omega}$ *giving the optimal future* Ω*-cost is bounded.*

(iii) *The control Riccati equation has a bounded* Ω*-solution.*

If these equivalent conditions hold, then $\|\cdot\|^2_{\mathrm{fut},\Omega}$ *is equal to the minimal nonnegative* Ω*-solution of the control Riccati equation.*

Proof. Since the state finite future Ω-cost condition trivially implies the input finite future Ω-cost condition, we have by [9, Theorem 5.9] combined with Lemma 4.3 that (i) implies that $\|\cdot\|^2_{\mathrm{fut},\Omega}$ is equal to the minimal nonnegative Ω-solution of the control Riccati equation. Using [9, Theorem 5.9] combined with Lemma 4.3 we also obtain that (iii) implies that $\|\cdot\|^2_{\mathrm{fut},\Omega}$ is equal to the minimal nonnegative Ω-solution of the control Riccati equation.

(i) $\Longrightarrow$ (ii) follows since $\|\cdot\|^2_{\mathrm{fut},\Omega}$ is closed by [9, Lemma 3.6], and since by the state finite future Ω-cost condition it is everywhere defined, it must then be bounded.

(ii) $\Longrightarrow$ (i) is trivial.

(ii) $\Longrightarrow$ (iii). We have already shown that if (ii) holds, then so does (i). We have also already seen that then $\|\cdot\|^2_{\mathrm{fut},\Omega}$ is the minimal nonnegative Ω-solution of the control Riccati equation. Since by assumption $\|\cdot\|^2_{\mathrm{fut},\Omega}$ is bounded, (iii) holds.

(iii) $\Longrightarrow$ (ii). We saw above that if (iii) holds, then $\|\cdot\|^2_{\mathrm{fut},\Omega}$ is the minimal nonnegative Ω-solution of the control Riccati equation. Since existence of a bounded Ω-solution of the control Riccati equation implies that the minimal nonnegative Ω-solution is also bounded, it follows that $\|\cdot\|^2_{\mathrm{fut},\Omega}$ is bounded. ∎

The following strengthens [9, Definition 5.7] to the notion relevant in the well-posed case. Note that what is added is an estimate on the size of the cost (see Remark 6.8 for further comments on this).

DEFINITION 6.7. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be an operator node with main operator* A *and control operator* B*, and let* Ω *be an open subset of* $\rho(A) \cap \mathbb{C}^+$*.* Σ *is said to satisfy the* uniform input finite future Ω-cost condition *if* Σ *satisfies the input finite future* Ω*-cost condition and if there exist constants* $\alpha \ge 0$ *and* $M > 0$ *such that* $\mathbb{C}^+_\alpha \subset \Omega$ *and*

$$\left\|(\lambda - A)^{-1}Bu_0\right\|^2_{\mathrm{fut},\Omega} \le \frac{M}{\mathrm{Re}(\lambda)}\|u_0\|^2, \qquad u_0 \in \mathcal{U},\ \lambda \in \mathbb{C}^+_\alpha. \tag{6.2}$$

Remark 6.8. Condition 6.7 can be interpreted as a strengthened version of the condition

$$\left\|(\lambda - A)^{-1}Bu_0\right\|^2_{\mathrm{fut},\Omega} \le \frac{M}{\mathrm{Re}(\lambda)}\left(\|u_0\|^2 + \|\widehat{\mathfrak{D}}(\lambda)u_0\|^2\right), \qquad u_0 \in \mathcal{U},\ \lambda \in \mathbb{C}^+_\alpha, \tag{6.3}$$

which has the following interpretation. For each $\lambda \in \mathbb{C}^+_\alpha$ and $u_0 \in \mathcal{U}$ the past cost of the classical stable past exponential trajectory $\left[\begin{smallmatrix} x_0 \\ u \\ y \end{smallmatrix}\right] := \left[\begin{smallmatrix} (\lambda - A|_{\mathcal{X}})^{-1}Bu_0 \\ \mathbf{e}_\lambda u_0 \\ \mathbf{e}_\lambda \widehat{\mathfrak{D}}(\lambda)u_0 \end{smallmatrix}\right]$ in (3.4) is equal to

$$J_{\mathrm{past}}(x_0, u) = \int_{-\infty}^0 \left(\|u(t)\|^2_{\mathcal{U}} + \|y(t)\|^2_{\mathcal{Y}}\right) dt = \frac{1}{\mathrm{Re}(\lambda)}\left(\|u_0\|^2 + \|\widehat{\mathfrak{D}}(\lambda)u_0\|^2\right).$$

Therefore, (6.3) says that the optimal future cost of the initial state $(\lambda - A)^{-1}Bu_0$ is bounded by a constant times the past cost it takes to reach that state with input $\mathbf{e}_\lambda u_0$.

Clearly (6.2) implies (6.3). If Σ is well-posed and the growth bound of Σ is at most α, then $\widehat{\mathfrak{D}}$ is uniformly bounded on $\mathbb{C}^+_\alpha$, and the converse implication holds as well.

Whereas it is immediately clear that the state finite future Ω-cost condition implies the input finite future Ω-cost condition, it is not immediately clear that it implies the uniform input finite future cost condition. The following lemma shows that in the well-posed case this is in fact true.

LEMMA 6.9. *Let $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be a well-posed operator node with main operator A, and let Ω be an open subset of $\rho_{+\infty}(A)$ which contains some right half-plane. If Σ satisfies the state finite future cost condition, then Σ also satisfies the uniform input finite future Ω-cost condition.*

Proof. By Lemma 6.3, the assumption that Σ satisfies the state finite future cost condition implies that Σ satisfies the state future Ω-cost condition and therefore the input finite future Ω-cost condition as well.

Fix any $\alpha \ge 0$ such that the growth bound of Σ is less than $\alpha - 1$ and such that $\mathbb{C}^+_\alpha \subset \Omega$. By [11, Proposition 4.2.9], there exists a $M_0 > 0$ such that

$$\left\|(\lambda - A)^{-1}Bu_0\right\|^2_{\mathcal{X}} \le \frac{M_0}{\mathrm{Re}(\lambda) - \alpha + 1}\|u_0\|^2_{\mathcal{U}}, \qquad u_0 \in \mathcal{U},\ \lambda \in \mathbb{C}^+_\alpha.$$

Since $\mathrm{Re}(\lambda)/(\mathrm{Re}(\lambda) - \alpha + 1) \le \max\{1, \alpha\}$ for all $\lambda \in \mathbb{C}^+_\alpha$, this implies that

$$\left\|(\lambda - A)^{-1}Bu_0\right\|^2_{\mathcal{X}} \le \frac{M_1}{\mathrm{Re}(\lambda)}\|u_0\|^2_{\mathcal{U}}, \qquad u_0 \in \mathcal{U},\ \lambda \in \mathbb{C}^+_\alpha, \tag{6.4}$$

where $M_1 = \max\{1, \alpha\}M_0$. From Lemma 6.6 we obtain that $\|\cdot\|^2_{\mathrm{fut},\Omega}$ is bounded; i.e., there exists a $M_2 > 0$ such that

$$\|z\|^2_{\mathrm{fut},\Omega} \le M_2\|z\|^2, \qquad z \in \mathcal{X}.$$

In particular,

$$\|(\lambda - A)^{-1}Bu_0\|^2_{\mathrm{fut},\Omega} \le M_2\|(\lambda - A)^{-1}Bu_0\|^2, \qquad u_0 \in \mathcal{U},\ \lambda \in \mathbb{C}^+_\alpha. \tag{6.5}$$

Combining (6.4) and (6.5) we get (6.2) with $M := M_1M_2$. Thus, the uniform input finite future Ω-cost condition holds. ∎

THEOREM 6.10. *Let $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be an operator node with main operator A and transfer function $\widehat{\mathfrak{D}}$. Assume that $\rho(A)$ contains some right half-plane, and let Ω be an open subset of $\rho_{+\infty}(A)$ which contains some right half-plane. Then the following conditions are equivalent:*

(i) *Σ satisfies the uniform input finite future Ω-cost condition, and $\widehat{\mathfrak{D}}$ is uniformly bounded on some right half-plane.*

(ii) *The control Riccati equation for Σ has an Ω-solution for which the function F in (4.6) is uniformly bounded on some right half-plane.*

(iii) *The control Riccati equation for Σ has a unique minimal Ω-solution, and the function F in (4.6) corresponding to this solution is uniformly bounded on some right half-plane.*

(iv) *$\widehat{\mathfrak{D}}$ has a well-posed normalized weakly coprime right $H^\infty(\mathbb{C}^+)$ factorization valid in Ω.*

Proof. We first show that each of the conditions (i), (ii), and (iv) implies that there exists a minimal nonnegative Ω-solution of the control Riccati equation. Indeed, according to [9, Theorem 5.9] conditions (i), (ii), and (iv) are equivalent if we drop the word "uniform" and the uniform boundedness condition on $\widehat{\mathfrak{D}}$ in (i), drop the uniform boundedness condition on F in (ii), and drop the word "well-posed" in (iv), and these three equivalent weaker conditions imply that the control Riccati equation has a minimal Ω-solution. Thus, under all four conditions in the theorem we have a minimal Ω-solution q of the control Riccati equation.

Let $\lambda \in \Omega$ and $u_0 \in \mathcal{U}$. Substituting $\left[\begin{smallmatrix} (\lambda-A)^{-1}Bu_0 \\ u_0 \end{smallmatrix}\right]$ in the control Riccati equation gives

$$2\mathrm{Re}(\lambda)\, q\big[(\lambda - A)^{-1}Bu_0, (\lambda - A)^{-1}Bu_0\big] + \|\widehat{\mathfrak{D}}(\lambda)u_0\|^2 + \|u_0\|^2 = \|\mathsf{F}(\lambda)u_0\|^2. \tag{6.6}$$

This substitution is allowed since $\left[\begin{smallmatrix} (\lambda-A)^{-1}Bu_0 \\ u_0 \end{smallmatrix}\right] \in \mathrm{dom}\left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]\right)$ and we have that both $(\lambda - A)^{-1}Bu_0 \in \mathrm{dom}(q)$ and $A\&B \left[\begin{smallmatrix} (\lambda-A)^{-1}Bu_0 \\ u_0 \end{smallmatrix}\right] = \lambda(\lambda - A)^{-1}B \in \mathrm{dom}(q)$. We use (6.6) to complete the proof.

(i) $\iff$ (iii). We recall from Lemma 6.6 that $\|\cdot\|^2_{\mathrm{fut},\Omega}$ is equal to the minimal nonnegative Ω-solution of the control Riccati equation. From (6.6) with $q[(\lambda - A)^{-1}Bu_0, (\lambda - A)^{-1}Bu_0] = \|(\lambda - A)^{-1}Bu_0\|^2_{\mathrm{fut},\Omega}$ we see that F is uniformly bounded on some right half-plane if and only if (a) $\widehat{\mathfrak{D}}$ is uniformly bounded on the same right half-plane and (b) condition (6.3) holds on the same right half-plane.

(iii) $\Longrightarrow$ (ii). This is trivial.

(ii) $\Longrightarrow$ (i). This follows from (6.6) since $\|\cdot\|^2_{\mathrm{fut},\Omega}$ is the minimal Ω-solution of the control Riccati equation, and hence $\|(\lambda - A)^{-1}Bu_0\|^2_{\mathrm{fut},\Omega} \le q[(\lambda - A)^{-1}Bu_0, (\lambda - A)^{-1}Bu_0]$.

(iii) $\Longrightarrow$ (iv) follows from Lemma 5.2.

(iv) $\Longrightarrow$ (iii). Let (N, M) be a well-posed normalized weakly coprime right factorization of $\widehat{\mathfrak{D}}$. Since a normalized weakly coprime right factorization is unique up to multiplication by a unitary operator, we obtain using Lemma 5.2 that there exists a $U \in \mathcal{B}(\mathcal{U})$ unitary such that $\mathsf{F}(\lambda)^{-1} := \mathsf{M}(\lambda)U$ for all $\lambda \in \Omega$. Since M^{-1} is assumed to be uniformly bounded on some right half-plane, it follows that F has the same property. ∎

7. LQ future normalized realizations. In this section we construct a realization with particularly nice properties for a function which has a well-posed right

$H^\infty(\mathbb{C}^+)$ factorization. This realization is analogous to an "output normalized realization" [11, section 9.5] (relevant for $H^\infty(\mathbb{C}^+)$ functions) and to an "optimal realization" [11, section 11.8], [1] (relevant for contractive $H^\infty(\mathbb{C}^+)$ functions). (All these realizations are unique up to a unitary similarity transformation in the state space.)

Definition 7.1. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be an operator node with main operator* A*, and let* Ω *be an open subset of* $\rho(A) \cap \mathbb{C}^+$*. Then* Σ *is called* LQ future Ω-normalized *if*

(i) Σ *is* Ω*-controllable in the sense that* $\bigvee_{\lambda\in\Omega} \operatorname{img}\left((\lambda - A)^{-1}B\right) = \mathcal{X}$*;*

(ii) Σ *satisfies the state finite future* Ω*-cost condition, and for each* $x_0 \in \mathcal{X}$ *the optimal future* Ω*-cost of* x_0 *is equal to* $\|x_0\|^2_{\mathcal{X}}$*.*

If Σ *is well-posed, then it is called* LQ future normalized if

(i′) Σ *is controllable (in the sense of* [11, Definition 9.1.2]*);*

(ii′) Σ *satisfies the state finite future cost condition, and for each* $x_0 \in \mathcal{X}$ *the optimal future cost of* x_0 *is equal to* $\|x_0\|^2_{\mathcal{X}}$*.*

Remark 7.2. The notion "LQ future Ω-normalized" is independent of Ω within each (connected) component of $\rho(A) \cap \mathbb{C}^+$ (in the same sense as in Remark 3.3). See also Remarks 4.4 and 6.2.

We also note that the definitions of LQ future normalized and LQ future Ω-normalized are consistent in the sense that a well-posed operator node is LQ future normalized if and only if it is LQ future Ω-normalized for some (equivalently, for all) open subset Ω of $\rho_{+\infty}(A)$. This follows from Lemma 6.3 (for equivalence of (ii) and (ii′)) and [11, Corollarly 9.6.5] (for equivalence of (i) and (i′)).

The following lemma shows uniqueness (up to a unitary similarity transformation in the state space) of LQ future Ω-normalized realizations of a given transfer function.

Lemma 7.3. *For* $j \in \{1, 2\}$*, let* $\Sigma_j := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]_j; \mathcal{X}_j, \mathcal{U}, \mathcal{Y}\right)$ *be an operator node with main operator* A_j*. Assume that* $\rho(A_1) \cap \rho(A_2) \cap \mathbb{C}^+$ *is nonempty, and let* Ω *be an open subset of* $\rho(A_1) \cap \rho(A_2) \cap \mathbb{C}^+$*. Further assume that the restrictions of the transfer functions of* Σ_1 *and* Σ_2 *to* Ω *are equal. If* Σ_1 *and* Σ_2 *are LQ future* Ω*-normalized, then they are unitarily similar (i.e., there exists a unitary* $U \in \mathcal{B}(\mathcal{X}_1, \mathcal{X}_2)$ *such that* $\left[\begin{smallmatrix} U & 0 \\ 0 & 1_{\mathcal{U}} \end{smallmatrix}\right] S_1 = S_2 \left[\begin{smallmatrix} U & 0 \\ 0 & 1_{\mathcal{U}} \end{smallmatrix}\right]$*).*

Proof. Let $\beta \in \Omega$, let $j \in \{1, 2\}$, and consider the (internal) Cayley transform with parameter β of Σ_j (as defined in, e.g., [9, section 4]) and denote this by $\mathbf{\Sigma}_j^\beta$. From [9, Theorem 4.5] we obtain that $\mathbf{\Sigma}_j^\beta$ satisfies the discrete-time equivalent of (ii) in Definition 7.1. The proofs of [11, Lemmas 9.6.3 and 12.2.6] show that $\mathbf{\Sigma}_j^\beta$ is controllable. Hence, $\mathbf{\Sigma}_j^\beta$ is discrete-time LQ future normalized (as defined in [8, Definition 2.8]) noting that observability follows from the fact that the norm equals the optimal future cost.

On a neighborhood of zero, the transfer functions of $\mathbf{\Sigma}_1^\beta$ and $\mathbf{\Sigma}_2^\beta$ are equal. From [8, Lemma 2.11] we conclude that $\mathbf{\Sigma}_1^\beta$ and $\mathbf{\Sigma}_2^\beta$ are unitarily similar. It follows that Σ_1 and Σ_2 are unitarily similar as well. ∎

The following theorem uses the notion of a strongly stabilizable well-posed linear system from [11, Definition 8.2.4], that of a controllable well-posed linear system from [11, Definition 9.1.2], and that of a minimal well-posed linear system from [11, Definition 9.1.2].

Theorem 7.4. *Let* φ *be an analytic* $\mathcal{B}(\mathcal{U}; \mathcal{Y})$*-valued function defined on some right half-plane. Then*

(i) *φ has a well-posed LQ future normalized realization Σ if and only if φ has a well-posed right $H^\infty(\mathbb{C}^+)$ factorization valid in some right half-plane.*

If the above equivalent conditions hold, then the realization Σ of φ in (i) has the following additional properties:

(ii) *Σ is minimal.*

(iii) *Σ is determined uniquely by φ, up to a unitary similarity transformation in the state space.*

(iv) *Denote the growth bound of Σ by ω_Σ. Then $\max\{\omega_\Sigma, 0\} = \max\{\omega_\varphi, 0\}$, where ω_φ is the growth bound of an arbitrary normalized weakly coprime right $H^\infty(\mathbb{C}^+)$ factorization (N, M) of φ.*

(v) *Σ is strongly stabilizable.*

(vi) *If a generalized future trajectory $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ of Σ satisfies $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in L^2(\mathbb{R}^+; \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right])$, then $x(t) \to 0$ as $t \to \infty$ (in particular, x is bounded).*

Proof. We first show that every function φ which has a well-posed right $H^\infty(\mathbb{C}^+)$ factorization valid in some right half-plane has a well-posed LQ future normalized realization.

Suppose that φ has a well-posed right $H^\infty(\mathbb{C}^+)$ factorization. Then φ also has a well-posed normalized weakly coprime right $H^\infty(\mathbb{C}^+)$ factorization (N, M) by [5, Theorem 1.1]. Since $\left[\begin{smallmatrix} \mathsf{N} \\ \mathsf{M} \end{smallmatrix}\right]$ is inner, it has a minimal well-posed strongly stable energy-preserving realization by, e.g., [11, Theorem 11.8.1 (i)]. We denote this operator node by $\Sigma^\frown = (S^\frown; \mathcal{X}, \mathcal{U}, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right])$. We note that the transfer function from the input to the second output of $\Sigma^\frown$ is M, which by assumption has an inverse which is uniformly bounded on some right half-plane $\mathbb{C}^+_\alpha$, where $\alpha \geq 0$. By [11, Theorems 6.6.1 and 10.3.5], we obtain a well-posed operator node $\Sigma_{\text{ext}} = \big(S_{\text{ext}}; \mathcal{X}, \mathcal{U}, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]\big)$ with growth bound at most α by considering the second output of $\Sigma^\frown$ as the input of Σ_{ext} and the input of $\Sigma^\frown$ as the second output of Σ_{ext}. We have the following relation between generalized future trajectories of $\Sigma^\frown$ and Σ_{ext}: $\left[\begin{smallmatrix} x \\ w \\ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \end{smallmatrix}\right]$ is a generalized future trajectory of $\Sigma^\frown$ if and only if $\left[\begin{smallmatrix} x \\ u \\ \left[\begin{smallmatrix} y \\ w \end{smallmatrix}\right] \end{smallmatrix}\right]$ is a generalized future trajectory of Σ_{ext}. We define the system $\Sigma = \big(S; \mathcal{X}, \mathcal{U}, \mathcal{Y}\big)$ by dropping the second output of Σ_{ext}. We will show that this Σ has the properties claimed in the theorem. It follows from the above that Σ is well-posed with growth bound at most α.

We next show that the system Σ constructed above satisfies condition (vi). Since the state and output of a well-posed system are uniquely determined by the initial state and input, there is a one-to-one correspondence between the trajectories of Σ and the trajectories of Σ_{ext}; i.e., if $\left[\begin{smallmatrix} x \\ u \\ \left[\begin{smallmatrix} y \\ w \end{smallmatrix}\right] \end{smallmatrix}\right]$ is a generalized future trajectory of Σ_{ext}, then $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ is a generalized future trajectory of Σ, and, conversely, if $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ is a generalized future trajectory of Σ, then there exists a unique $w \in L^2_{\text{loc}}(\mathbb{R}^+; \mathcal{U})$ such that $\left[\begin{smallmatrix} x \\ u \\ \left[\begin{smallmatrix} y \\ w \end{smallmatrix}\right] \end{smallmatrix}\right]$ is a generalized future trajectory of Σ_{ext}. As we noticed above, there is also a one-to-one correspondence between the trajectories of Σ_{ext} and the trajectories of $\Sigma^\frown$. However, we also need a one-to-one correspondence between *stable* generalized future trajectories, which can be established as follows. Let $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ be a stable generalized future trajectory of Σ, so that $u \in L^2(\mathbb{R}^+; \mathcal{U})$ and $y \in L^2(\mathbb{R}^+; \mathcal{Y})$. Let $\left[\begin{smallmatrix} x \\ w \\ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \end{smallmatrix}\right]$ be the corresponding generalized future trajectory of $\Sigma^\frown$. We shall prove that $\left[\begin{smallmatrix} x \\ w \\ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \end{smallmatrix}\right]$ is stable as well, i.e., that additionally $w \in L^2(\mathbb{R}^+;\mathcal{U})$. We can write the trajectory as the sum of two trajectories: $\left[\begin{smallmatrix} x \\ w \\ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_1 \\ 0 \\ \left[\begin{smallmatrix} y_1 \\ u_1 \end{smallmatrix}\right] \end{smallmatrix}\right] + \left[\begin{smallmatrix} x_2 \\ w \\ \left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right] \end{smallmatrix}\right]$, where $x_1(0) = x(0)$ and the input function of the first of these trajectories is zero, and $x_2(0) = 0$. Since $\Sigma^{\frown}$ is strongly stable, we have $x_1(t) \to 0$ as $t \to \infty$, and since $\Sigma^{\frown}$ is strongly stable and energy preserving, by, e.g., [11, Theorem 11.3.4], we have $\left[\begin{smallmatrix} y_1 \\ u_1 \end{smallmatrix}\right] \in L^2(\mathbb{R}^+;\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right])$. From the assumption that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in L^2(\mathbb{R}^+;\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right])$ and the just established $\left[\begin{smallmatrix} y_1 \\ u_1 \end{smallmatrix}\right] \in L^2(\mathbb{R}^+;\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right])$ we obtain that $\left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right] \in L^2(\mathbb{R}^+;\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right])$. Since $x_2(0) = 0$, we have $\left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathfrak{N} \\ \mathfrak{M} \end{smallmatrix}\right] w$, where $\left[\begin{smallmatrix} \mathfrak{N} \\ \mathfrak{M} \end{smallmatrix}\right]$ is the causal shift-invariant operator with symbol $\left[\begin{smallmatrix} \mathsf{N} \\ \mathsf{M} \end{smallmatrix}\right]$. Since (N, M) is weakly right coprime, from $\left[\begin{smallmatrix} y_2 \\ u_2 \end{smallmatrix}\right] \in L^2(\mathbb{R}^+;\left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right])$ we obtain $w \in L^2(\mathbb{R}^+;\mathcal{U})$. Since $\Sigma^{\frown}$ is strongly stable and energy preserving, by [11, Theorem 11.3.5] it is strongly input/state stable (in the sense of [11, Definition 8.1.1 (iib)]), and since the input w giving rise to x_2 is in $L^2(\mathbb{R}^+;\mathcal{U})$, it follows that $x_2(t) \to 0$ as $t \to \infty$. We conclude that $x(t) = x_1(t) + x_2(t) \to 0$ as $t \to \infty$. Hence, we obtain that the constructed Σ satisfies (vi).

We now prove that Σ satisfies condition (ii′) in Definition 7.1. Let $\left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right]$ be a stable generalized future trajectory of Σ. By the above, there exists a unique w such that $\left[\begin{smallmatrix} x \\ w \\ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \end{smallmatrix}\right]$ is a stable generalized future trajectory of $\Sigma^{\frown}$. Since $\Sigma^{\frown}$ is energy preserving, we obtain for all $t \geq 0$

$$\|x(t)\|^2_{\mathcal{X}} + \int_0^t \|y(\tau)\|^2_{\mathcal{Y}}\, d\tau + \int_0^t \|u(\tau)\|^2_{\mathcal{U}}\, d\tau = \|x(0)\|^2_{\mathcal{X}} + \int_0^t \|w(\tau)\|^2_{\mathcal{U}}\, d\tau. \tag{7.1}$$

Letting $t \to \infty$ and using that $x(t) \to 0$ by the above established (vi), we obtain

$$\int_0^\infty \|y(\tau)\|^2_{\mathcal{Y}}\, d\tau + \int_0^\infty \|u(\tau)\|^2_{\mathcal{U}}\, d\tau = \|x(0)\|^2_{\mathcal{X}} + \int_0^\infty \|w(\tau)\|^2_{\mathcal{U}}. \tag{7.2}$$

From this we see that the infimum over all stable generalized future trajectories of Σ of $\int_0^\infty \|y(\tau)\|^2_{\mathcal{Y}}\, d\tau + \int_0^\infty \|u(\tau)\|^2_{\mathcal{U}}\, d\tau$ is obtained for $w = 0$ and equals $\|x(0)\|^2_{\mathcal{X}}$. Therefore, we obtain condition (ii′) in Definition 7.1.

We now prove that Σ is controllable (this is condition (i′) in Definition 7.1). We have that $\Sigma^{\frown}$ is controllable (in the sense of [11, Definition 9.1.2]). By [11, Lemma 9.9.2] (where the first input space is taken to be the trivial vector space) we then obtain that Σ_{ext} is controllable. Since dropping an output does not affect controllability, it follows that Σ is controllable.

According to Definition 7.1, Σ is a well-posed LQ future normalized realization of φ.

Conversely, suppose that Σ is a well-posed LQ future normalized realization of φ. We proceed to prove that φ has a well-posed right $H^\infty(\mathbb{C}^+)$-factorization valid in some right half-plane and that this realization has the additional properties (ii)–(vi). In the remainder of the proof we denote the main operator of Σ by A, the control operator by B, the transfer function by $\widehat{\mathfrak{D}}$, and the growth bound of Σ by ω_Σ.

We begin by proving (ii). If $\left[\begin{smallmatrix} x \\ 0 \\ 0 \end{smallmatrix}\right]$ is a generalized future trajectory of Σ, then the optimal future cost of $x(0)$ is clearly zero, and from condition (ii′) in Definition 7.1 we then obtain that $\|x(0)\|^2_{\mathcal{X}} = 0$, so that $x = 0$. Hence, Σ is observable. A well-posed system which is both controllable and observable is minimal.

We next prove that φ has a well-posed right $H^\infty(\mathbb{C}^+)$-factorization valid in some right half-plane. Let $\alpha > \max\{\omega_\Sigma, 0\}$, and denote $\Omega := \mathbb{C}^+_\alpha$. By Lemma 6.6 combined with Definition 7.1 and Remark 7.2, the inner-product in $\mathcal{X}$ is the minimal Ω-solution of the continuous time control Riccati equation (with domain $\mathcal{X}$). Hence, we have that there exists an operator $K\&F : \mathrm{dom}(S) \to \mathcal{U}$ such that

$$2\mathrm{Re}\left\langle [A\&B]\begin{bmatrix} x \\ u \end{bmatrix}, x\right\rangle + \left\| C\&D \begin{bmatrix} x \\ u \end{bmatrix}\right\|_{\mathcal{Y}}^2 + \|u\|_{\mathcal{U}}^2 = \left\| K\&F \begin{bmatrix} x \\ u \end{bmatrix}\right\|_{\mathcal{U}}^2, \quad \begin{bmatrix} x \\ u \end{bmatrix} \in \mathrm{dom}\,(S), \tag{7.3}$$

and such that the operator $\mathsf{F}(\lambda) := K\&F \left[\begin{smallmatrix} (\lambda - A|_{\mathcal{X}})^{-1}B \\ 1_{\mathcal{U}} \end{smallmatrix}\right]$ has a bounded inverse for all $\lambda \in \Omega$. From Lemma 5.2 we obtain that $\mathsf{M}(\lambda) := \mathsf{F}(\lambda)^{-1}$, $\mathsf{N}(\lambda) := \varphi(\lambda)\mathsf{M}(\lambda)$ gives rise to a normalized weakly coprime right $H^\infty(\mathbb{C}^+)$ factorization of $\widehat{\mathfrak{D}}$. From (7.3) we see that $K\&F$ is continuous with respect to the graph norm of S and therefore $\Sigma_{\mathrm{ext}} := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \\ K\&F \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right]\right)$ is a system node. We now prove that Σ_{ext} is well-posed. Let $\left[\begin{smallmatrix} x \\ u \\ \left[\begin{smallmatrix} y \\ w \end{smallmatrix}\right] \end{smallmatrix}\right]$ be a classical trajectory of Σ_{ext}. From (7.3) we obtain by integrating that (7.1) holds. Since Σ is well-posed, for all $T > 0$ there exists a $M > 0$ such that for all $t \in [0, T]$

$$\|x(t)\|_{\mathcal{X}}^2 + \int_0^t \|y(\tau)\|_{\mathcal{Y}}^2\, d\tau \le M\left(\|x(0)\|_{\mathcal{X}}^2 + \int_0^t \|u(\tau)\|_{\mathcal{U}}^2\, d\tau\right). \tag{7.4}$$

From (7.1) we obtain

$$\int_0^t \|w(\tau)\|_{\mathcal{U}}^2\, d\tau \le \|x(t)\|_{\mathcal{X}}^2 + \int_0^t \|y(\tau)\|_{\mathcal{Y}}^2\, d\tau + \int_0^t \|u(\tau)\|_{\mathcal{U}}^2\, d\tau,$$

which combined with (7.4) gives

$$\|x(t)\|_{\mathcal{X}}^2 + \int_0^t \|y(\tau)\|_{\mathcal{Y}}^2\, d\tau + \int_0^t \|w(\tau)\|_{\mathcal{U}}^2\, d\tau \le (2M+1)\left(\|x(0)\|_{\mathcal{X}}^2 + \int_0^t \|u(\tau)\|_{\mathcal{U}}^2\, d\tau\right),$$

which shows that Σ_{ext} is well-posed. The growth bound of Σ_{ext} is the same as the growth bound ω_Σ of Σ (equal to the growth bound of the evolution semigroup of Σ). In particular, this implies that the transfer function F from the input to the second output of Σ_{ext} is bounded in $\mathbb{C}^+_\alpha$. Since $\mathsf{F} = \mathsf{M}^{-1}$, this implies that M^{-1} is bounded in $\mathbb{C}^+_\alpha$. Consequently, the factorization (N, M) of $\widehat{\mathfrak{D}}$ is well-posed, and the growth bound of this factorization is at most α. Since α is an arbitrary number satisfying $\alpha > \max\{\omega_\Sigma, 0\}$, we see that the growth bound of the factorization (N, M) is at most $\max\{\omega_\Sigma, 0\}$. This proves that φ has a well-posed right $H^\infty(\mathbb{C}^+)$-factorization valid in some right half-plane (and also proves one-half of (iv)).

We next prove (v). As we noticed above, the transfer function from the input to the second output of Σ_{ext} equals F, whose inverse M is well-posed. By [11, Theorem 6.6.1] we obtain a well-posed operator node $\Sigma^{\curvearrowleft} = (S^{\curvearrowleft}; \mathcal{X}, \mathcal{U}, \left[\begin{smallmatrix} \mathcal{Y} \\ \mathcal{U} \end{smallmatrix}\right])$ by considering the second output of Σ_{ext} as input of $\Sigma^{\curvearrowleft}$ and the input of Σ_{ext} as the second output of $\Sigma^{\curvearrowleft}$. The transfer function of $\Sigma^{\curvearrowleft}$ is $\left[\begin{smallmatrix} \mathsf{N} \\ \mathsf{M} \end{smallmatrix}\right]$. From (7.1) we obtain that $\Sigma^{\curvearrowleft}$ is energy preserving. Since Σ is controllable, Σ_{ext} is controllable and using [11, Lemma 9.9.2], $\Sigma^{\curvearrowleft}$ is controllable. From [11, Theorem 11.3.3] we then obtain that $\Sigma^{\curvearrowleft}$ is additionally strongly stable and observable. Therefore, $\Sigma^{\curvearrowleft}$ has the properties assumed in the first part of this proof; additionally, Σ, Σ_{ext}, and $\Sigma^{\curvearrowleft}$ are related as in that first part of this proof. By [11, Chapter 7], the operator $K\&F$ is an admissible state feedback for

Σ with closed-loop system $\Sigma^{\frown}$. Since $\Sigma^{\frown}$ is well-posed and strongly stable, it follows that Σ is strongly stabilizable, i.e. that (v) holds.

We note that (iii) follows from Lemma 7.3.

In the first part of the proof we showed that the system Σ constructed there satisfes condition (vi). It therefore follows from (iii) that *all* well-posed LQ future normalized systems Σ must satisfy (vi).

The only property left to be established is (iv). All normalized weakly comprime right $H^\infty(\mathbb{C}^+)$ factorizations of φ are determined uniquely up to the multiplication from the right by an unitary operator, and hence they all have the same growth bound, which we may denote by ω_φ. Likewise, all well-posed LQ future normalized realizations Σ of φ have the same growth bound since they are unitarily similar. We denote this common growth bound by ω_Σ. It follows from the construction in the first part of the proof that $\max\{\omega_\Sigma, 0\} \le \max\{\omega_\varphi, 0\}$, and as we saw above, also the converse inequality is true. Thus, $\max\{\omega_\Sigma, 0\} = \max\{\omega_\varphi, 0\}$. ∎

The following lemma gives a necessary and sufficient condition for a LQ future Ω-normalized operator node to be well-posed (and hence LQ future normalized).

Lemma 7.5. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be an operator node with main operator* A *and transfer function* $\widehat{\mathfrak{D}}$*. Then the two following conditions are equivalent:*

(i) Σ *is well-posed and LQ future normalized.*

(ii) *The following conditions hold:*

(a) $\rho(A)$ *contains some right half-plane;*

(b) Σ *is LQ future* Ω*-normalized for some (or, equivalently, for every) open subset* Ω *of* $\rho_{+\infty}(A)$*;*

(c) $\widehat{\mathfrak{D}}$ *has a well-posed right* $H^\infty(\mathbb{C}^+)$ *factorization valid in* Ω *(with* Ω *as in* (b)*).*

Proof. Suppose first that Σ is well-posed and LQ future normalized. Then (a) holds. By Remark 7.2, Σ is LQ future Ω-normalized for every open subset Ω of $\rho_{+\infty}(A)$. By Theorem 7.4, $\widehat{\mathfrak{D}}$ has a well-posed right $H^\infty(\mathbb{C}^+)$ factorization valid in some right half-plane. By analytic continuation, this factorization is actually valid in $\rho_{+\infty}(A)$ and hence also valid in every open subset Ω of $\rho_{+\infty}(A)$.

Conversely, suppose that conditions (a)–(c) in (ii) hold (where we in (b) fix Ω to be *some* open subset of $\rho_{+\infty}(A)$). Since $\widehat{\mathfrak{D}}$ has a well-posed right $H^\infty(\mathbb{C}^+)$ factorization valid in Ω, it also has a well-posed normalized weakly right coprime $H^\infty(\mathbb{C}^+)$ factorization (N, M) valid in Ω (cf. the proof of Theorem 7.4). By analytic continuation, $\widehat{\mathfrak{D}}(\lambda)\mathsf{M}(\lambda) = \mathsf{N}(\lambda)$ for all $\lambda \in \rho_{+\infty}(A)$, and consequently the factorization $\widehat{\mathfrak{D}}(\lambda) = \mathsf{N}(\lambda)\mathsf{M}(\lambda)^{-1}$ is valid everywhere in $\rho_{+\infty}(A)$ where $\mathsf{M}(\lambda)$ is invertible. The well-posedness assumption on the factorization means that $\mathsf{M}(\lambda)$ is invertible in some right half-plane, and thus the factorization $\widehat{\mathfrak{D}}(\lambda) = \mathsf{N}(\lambda)\mathsf{M}(\lambda)^{-1}$ is also valid in some right half-plane $\mathbb{C}^+_\alpha$.

By Theorem 7.4, $\widehat{\mathfrak{D}}$ has a well-posed LQ future normalized realization Σ_1, and by Remark 7.2, Σ_1 is also LQ future $\mathbb{C}^+_\alpha$-normalized. By Lemma 7.3, Σ and Σ_1 are unitarily similar. Since Σ_1 is well-posed and LQ future normalized, also Σ is therefore well-posed and LQ future normalized. ∎

8. Realization theory. By collecting several results from the previous sections, we obtain the following theorem.

Theorem 8.1. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be an operator node with main operator* A *and transfer function* $\widehat{\mathfrak{D}}$*. Assume that* $\rho(A)$ *contains some right half-plane, let*

Ω be an open subset of $\rho_{+\infty}(A)$ which contains some right half-plane, and denote the restriction of $\widehat{\mathfrak{D}}$ to Ω by φ. Then the following conditions are equivalent:

(i) *Σ satisfies the uniform input finite future Ω-cost condition, and φ is uniformly bounded on some right half-plane.*
(ii) *The control Riccati equation for Σ has an Ω-solution for which the function F in (4.6) is uniformly bounded on some right half-plane.*
(iii) *The control Riccati equation for Σ has an Ω-solution, and the function F in (4.6) corresponding to the minimal Ω-solution is uniformly bounded on some right half-plane.*
(iv) *φ has a well-posed realization for which the control Riccati equation has a bounded $\mathbb{C}^+_\alpha$-solution for some $\alpha \geq 0$.*
(v) *φ has a well-posed realization which satisfies the state finite future cost condition.*
(vi) *φ has a well-posed stabilizable realization.*
(vii) *φ has a well-posed strongly stabilizable realization.*
(viii) *φ has a well-posed LQ future normalized realization.*
(ix) *φ has an well-posed right $H^\infty(\mathbb{C}^+)$ factorization.*
(x) *φ has a well-posed normalized weakly coprime right $H^\infty(\mathbb{C}^+)$ factorization.*

Proof. (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (x) follows from Theorem 6.10.

(x) $\Longrightarrow$ (ix) is trivial.

(ix) $\Longrightarrow$ (viii) follows from Theorem 7.4.

(viii) $\Longrightarrow$ (vii) follows since the LQ future normalized realization is well-posed and strongly stabilizable by Theorem 7.4.

(vii) $\Longrightarrow$ (vi) is trivial.

(vi) $\Longrightarrow$ (v) follows since any stabilizable realization satisfies the state finite future cost condition.

(v) $\Longleftrightarrow$ (iv) follows from Lemmas 6.3 and 6.6 with Ω replaced by $\mathbb{C}^+_\alpha$, where α is taken to be large enough so that $\mathbb{C}^+_\alpha$ is contained in the resolvent set of the main operator.

(v) $\Longrightarrow$ (x) follows from Lemma 6.9 and Theorem 6.10 applied to the realization in (v). ∎

Remark 8.2. We note that the equivalence of (v), (vi), (vii), (ix), and (x) in Theorem 8.1 had already been proven by Mikkola in [5]. In [4] he also proved that those conditions are equivalent to some modified version of (iv) involving integral Riccati equations.

9. The past optimal control problem and left factorizations. In this section we consider the past optimal control problem and left factorizations. Several results follow in a relatively straightforward way from previous sections by duality.

DEFINITION 9.1. *Let $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be an operator node with main operator A, and let Ω be an open subset of $\rho(A) \cap \mathbb{C}^+$. By an Ω-solution of the continuous time* filter Riccati equation *induced by $\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, we mean an Ω^*-solution of the continuous time control Riccati equation induced by the adjoint system $\Sigma^\dagger = \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]^*; \mathcal{X}, \mathcal{Y}, \mathcal{U}\right)$, where $\Omega^* := \{\lambda \in \mathbb{C} : \bar{\lambda} \in \Omega\}$.*

DEFINITION 9.2. *Let $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be an operator node with main operator A, and let Ω be an open subset of $\rho(A) \cap \mathbb{C}^+$.*

(i) *A vector $x_0 \in \mathcal{X}$ is said to have* finite past Ω-cost *if it is the final state component of a generalized stable past Ω-trajectory. The* past Ω-cost *of such* *a vector x_0 is the infimum of the past cost functional*

$$J_{\text{past}}(x_0,u) = \int_{-\infty}^{0} \left(\|u(t)\|_{\mathcal{U}}^2 + \|y(t)\|_{\mathcal{Y}}^2\right) dt \tag{9.1}$$

over all generalized stable past Ω-trajectories $\left[\begin{smallmatrix} x_0 \\ u \\ y \end{smallmatrix}\right]$ of Σ. We denote this cost by $\|x_0\|_{\text{past},\Omega}^2$.

(ii) *If Σ is well-posed, then a vector $x_0 \in \mathcal{X}$ is said to have* finite past cost *if it is the final state component of a stable past trajectory. The* past cost *of such a vector x_0 is the infimum of the past cost functional* (9.1) *over all generalized stable past trajectories $\left[\begin{smallmatrix} x_0 \\ u \\ y \end{smallmatrix}\right]$ of Σ. We denote this cost by $\|x_0\|_{\text{past}}^2$.*

Remark 9.3. By [9, Theorem 3.12], the infimum in part (i) of Definition 9.2 is actually achieved by a unique minimizing generalized stable past Ω-trajectory of Σ, and $\|\cdot\|_{\text{past},\Omega}^2$ is a closed quadratic form in $\mathcal{X}$. Also the infimum in part (ii) of Definition 9.2 is achieved by a unique minimizing generalized stable past trajectory of Σ, and $\|\cdot\|_{\text{past}}^2$ is a closed quadratic form in $\mathcal{X}$ as well. By Lemma 3.9, if Σ is well-posed, and if Ω is an open subset of $\rho_{+\infty}(A)$, then $x_0 \in \mathcal{X}$ has a finite past Ω-cost if and only if x_0 has a finite past cost, and $\|\cdot\|_{\text{past},\Omega}^2 = \|\cdot\|_{\text{past}}^2$.

The following definition is essentially a reformulation of [9, Definition 6.2] (the connection is similar to what is mentioned in Remark 6.5 in connection to the future optimal control problem).

DEFINITION 9.4. *Let $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be an operator node with main operator A and observation operator C, and let Ω be an open subset of $\rho(A) \cap \mathbb{C}^+$.*

(i) *Σ satisfies the* output coercive past Ω-cost condition *if for every $\lambda \in \Omega$ there exists a constant $M > 0$ such that*

$$\left\|C(\lambda - A)^{-1}x_0\right\|_{\mathcal{Y}}^2 \le M\|x_0\|_{\text{past},\Omega}^2 \tag{9.2}$$

for every $x_0 \in \mathcal{X}$ with a finite past Ω-cost.

(ii) *Σ satisfies the* state coercive past Ω-cost condition *if there exists a constant $M > 0$ such that*

$$\|x_0\|_{\mathcal{X}}^2 \le M\|x_0\|_{\text{past},\Omega}^2 \tag{9.3}$$

for every $x_0 \in \mathcal{X}$ with a finite past Ω-cost.

The following result was never explicitly stated in [9] but follows easily from the results presented there.

LEMMA 9.5. *Let $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be an operator node with main operator A, and let Ω be an open subset of a connected subset of $\rho(A) \cap \mathbb{C}^+$. The following are equivalent:*

(i) *Σ satisfies the state coercive past Ω-cost condition.*

(ii) *The quadratic form $\|\cdot\|_{\text{past},\Omega}^2$ giving the optimal past Ω-cost is bounded away from zero.*

(iii) *The filter Riccati equation has a bounded Ω-solution.*

If these equivalent conditions hold, then $\|\cdot\|_{\text{past},\Omega}^2$ is equal to the inverse of the minimal nonnegative Ω-solution of the filter Riccati equation (in the sense of [9, Lemma 3.17]*).*

Proof. The proof is analogous to the proof of Lemma 6.6 with [9, Theorem 5.9] replaced by [9, Theorem 6.5]. ∎

The following strengthens the notion of output coercive past Ω-cost condition.

DEFINITION 9.6. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be an operator node with main operator* A *and observation operator* C*, and let* Ω *be an open subset of* $\rho(A) \cap \mathbb{C}^+$*.* Σ *is said to satisfy the* uniform output coercive past Ω-cost condition *if* Σ *satisfies the output coercive past* Ω*-cost condition and there constants* $\alpha \geq 0$ *and* $M > 0$ *such that* $\mathbb{C}^+_\alpha \subset \Omega$ *and*

$$\left\|C(\lambda - A)^{-1}x_0\right\|^2_{\mathcal{Y}} \leq \frac{M}{\operatorname{Re}(\lambda)} \|x_0\|^2_{\text{past},\Omega}, \qquad \lambda \in \mathbb{C}^+_\alpha, \tag{9.4}$$

for every $x_0 \in \mathcal{X}$ *with a finite past* Ω*-cost.*

Thus, Definition 9.6 imposes an extra uniformity condition in some right half-plane on the constant M in (9.2).

The following lemma is the "uniform" equivalent of [9, Lemma 6.3].

LEMMA 9.7. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be an operator node, and let* Ω *be an open subset of* $\rho(A) \cap \mathbb{C}^+$*. Then* Σ *satisfies the uniform output coercive past* Ω*-cost condition for some constants* $\alpha \geq 0$ *and* $M > 0$ *if and only if the adjoint system* $\Sigma^\dagger = (\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]^*; \mathcal{X}, \mathcal{Y}, \mathcal{U})$ *satisfies the uniform input finite future* Ω^**-cost condition for the same constants* α *and* M*, where* $\Omega^* := \{z \in \mathbb{C} : \bar{z} \in \Omega\}$*.*

Proof. First assume that the uniform output coercive past Ω-cost condition for Σ holds, and let $\alpha \geq 0$ and $M > 0$ be as in Definition 9.6. By [9, Theorem 3.18] we have for all $x_0 \in \mathcal{X}$ with finite future Ω^*-cost for $\Sigma^\dagger$ that

$$\|x_0\|_{\text{fut},\Omega^*} = \sup_{\|z_0\|_{\text{past},\Omega} \leq 1} \left|\langle x_0, z_0\rangle_{\mathcal{X}}\right|.$$

Applying this with $x_0 := (\lambda - A)^{-*}C^*y_0$, where $y_0 \in \mathcal{Y}$ and $\lambda \in \Omega^*$ (by [9, Lemma 6.3], this x_0 indeed has finite future cost for $\Sigma^\dagger$), we obtain

$$\begin{aligned} \left\|(\lambda - A)^{-*}C^*y_0\right\|_{\text{fut},\Omega^*} &= \sup_{\|z_0\|_{\text{past},\Omega} \leq 1} \left|\langle(\lambda - A)^{-*}C^*y_0, z_0\rangle_{\mathcal{X}}\right| \\ &= \sup_{\|z_0\|_{\text{past},\Omega} \leq 1} \left|\langle y_0, C(\lambda - A)^{-1}z_0\rangle_{\mathcal{Y}}\right| \leq \|y_0\|_{\mathcal{Y}} \sup_{\|z_0\|_{\text{past},\Omega} \leq 1} \|C(\lambda - A)^{-1}z_0\|_{\mathcal{Y}}. \end{aligned}$$

By the uniform output coercive past Ω-cost condition for Σ we then obtain for $\lambda \in \mathbb{C}^+_\alpha$

$$\left\|(\lambda - A)^{-*}C^*y_0\right\|^2_{\text{fut},\Omega^*} \leq \frac{M}{\operatorname{Re}(\lambda)} \|y_0\|^2_{\mathcal{Y}},$$

which shows that the uniform input finite future Ω^*-cost condition for $\Sigma^\dagger$ holds.

Now assume that the uniform input finite future Ω^*-cost condition for $\Sigma^\dagger$ holds, and let $\alpha \geq 0$ and $M > 0$ be as in Definition 6.7 (applied to $\Sigma^\dagger$). Let x_0 have finite past Ω-cost for Σ. By [9, Theorem 3.18] we have

$$\|x_0\|_{\text{past},\Omega} = \sup_{\|z_0\|_{\text{fut},\Omega^*} \leq 1} |\langle x_0, z_0\rangle_{\mathcal{X}}|.$$

Take $z_0 := \sqrt{\frac{\operatorname{Re}(\lambda)}{M}}(\lambda - A)^{-*}C^*y_0$, where $\lambda \in \mathbb{C}^+_\alpha$ and $y_0 \in \mathcal{Y}$ satisfies $\|y_0\|_{\mathcal{Y}} \leq 1$. From the uniform input finite future Ω^*-cost condition for $\Sigma^\dagger$ we then obtain that $\|z_0\|_{\text{fut}^\dagger} \leq 1$. Hence,

$$\|x_0\|_{\text{past},\Omega} \geq \sqrt{\frac{\operatorname{Re}(\lambda)}{M}} \left|\langle x_0, (\lambda - A)^{-*}C^*y_0\rangle_{\mathcal{X}}\right| = \sqrt{\frac{\operatorname{Re}(\lambda)}{M}} \left|\langle C(\lambda - A)^{-1}x_0, y_0\rangle_{\mathcal{Y}}\right|.$$

Since $y_0 \in \mathcal{Y}$ with $\|y_0\|_{\mathcal{Y}} \le 1$ was arbitrary, we then obtain

$$\|x_0\|_{\text{past},\Omega} \ge \sqrt{\frac{\mathrm{Re}(\lambda)}{M}} \sup_{\|y_0\|_{\mathcal{Y}}\le 1} \left|\langle C(\lambda - A)^{-1}x_0, y_0\rangle_{\mathcal{Y}}\right| = \sqrt{\frac{\mathrm{Re}(\lambda)}{M}}\, \left\|C(\lambda - A)^{-1}x_0\right\|_{\mathcal{Y}}.$$

This precisely shows that the uniform output coercive past Ω-cost condition for Σ holds. ∎

The following is the left version of Definition 5.1 and the well-posed version of [9, Definition 6.4].

Definition 9.8. *Let φ be an analytic $\mathcal{B}(\mathcal{U};\mathcal{Y})$-valued function defined on some open subset Ω of $\mathbb{C}^+$.*

(i) *φ has a* left $H^\infty(\mathbb{C}^+)$ factorization valid in Ω *if there exist two functions $\widetilde{\mathsf{M}} \in H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{Y}))$ and $\widetilde{\mathsf{N}} \in H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{U};\mathcal{Y}))$ such that $\widetilde{\mathsf{M}}(\lambda)$ has a bounded inverse and $\varphi(\lambda) = \widetilde{\mathsf{M}}(\lambda)^{-1}\widetilde{\mathsf{N}}(\lambda)$ for all $\lambda \in \Omega$.*

(ii) *The factorization in* (i) *is called* normalized *if the operator*

$$\begin{bmatrix}\hat{u}\\ \hat{y}\end{bmatrix} \mapsto P_{H^2(\mathbb{C}^-;\mathcal{Y})}\begin{bmatrix}-\widetilde{\mathsf{N}} & \widetilde{\mathsf{M}}\end{bmatrix}\begin{bmatrix}\hat{u}\\ \hat{y}\end{bmatrix} : \begin{bmatrix}H^2(\mathbb{C}^-;\mathcal{U})\\ H^2(\mathbb{C}^-;\mathcal{Y})\end{bmatrix} \to H^2(\mathbb{C}^-;\mathcal{Y})$$

is co-isometric.

(iii) *The factorization in* (i) *is* weakly (left) coprime *if the kernel of the operator in* (ii) *is equal to the (past time) Laplace transform of the stable past behavior $\mathfrak{W}^0_-(\Omega)$ defined in Definition* 3.6.

(iv) *The factorization in* (i) *is* well-posed *if there exists some $\alpha \ge 0$ such that $\widetilde{\mathsf{M}}(\lambda)$ has a bounded inverse for all $\lambda \in \mathbb{C}^+_\alpha$ and $\widetilde{\mathsf{M}}^{-1} \in H^\infty(\mathbb{C}^+_\beta;\mathcal{B}(\mathcal{Y}))$ for all $\beta > \alpha$.*

(v) *If the factorization in* (i) *is well-posed, then the* growth bound *of this factorization is the infimum over all α for which the condition in* (iv) *holds. (If the factorization is not well-posed, then its growth bound is $+\infty$.)*

Definition 9.9. *Let $\Sigma := \left(\left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y}\right)$ be an operator node with main operator A, and let Ω be an open subset of $\rho(A)\cap\mathbb{C}^+$. Then Σ is called* LQ past Ω-normalized *if*

(i) *Σ is Ω-observable in the sense that $\cap_{n=0}^\infty \ker(C(\lambda - A)^{-n}) = \{0\}$ for some $\lambda \in \Omega$;*

(ii) *Σ satisfies the state coercive past Ω-cost condition and for each $x_0 \in \mathcal{X}$ the optimal past Ω-cost of x_0 is equal to $\|x_0\|^2_{\mathcal{X}}$.*

If Σ is well-posed, then it is called LQ past normalized if

(i′) *Σ is observable (in the sense of* [11, Definition 9.1.2]*);*

(ii′) *Σ satisfies the state coercive past cost condition and for each $x_0 \in \mathcal{X}$ the optimal past cost of x_0 is equal to $\|x_0\|^2_{\mathcal{X}}$.*

Remark 9.10. Remark 7.2 with the obvious substitutions applies to "LQ past normalized" as well.

The following follows from Theorem 7.4 by duality.

Theorem 9.11. *Let φ be an analytic $\mathcal{B}(\mathcal{U};\mathcal{Y})$-valued function defined on some right half-plane. Then*

(i) *φ has a well-posed LQ past normalized realization Σ if and only if φ has a well-posed left $H^\infty(\mathbb{C}^+)$ factorization valid in some right half-plane.*

If the above equivalent conditions hold, then the realization Σ of φ in (i) has the following additional properties:

(ii) Σ *is minimal.*

(iii) Σ *is determined uniquely by* φ*, up to a unitary similarity transformation in the state space.*

(iv) *Denote the growth bound of* Σ *by* ω_Σ*. Then* $\max\{\omega_\Sigma, 0\} = \max\{\omega_\varphi, 0\}$*, where* ω_φ *is the growth bound of an arbitrary normalized weakly coprime left* $H^\infty(\mathbb{C}^+)$ *factorization of* φ*.*

(v) Σ *is strongly* $*$*-detectable; i.e., there exists an output injection operator which makes the closed-loop system obtained by output injection strongly co-stable (in the sense that its dual system is strongly stable).*

The following follows from Theorem 8.1 and duality using Lemma 9.7.

THEOREM 9.12. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be an operator node with main operator* A *and transfer function* $\widehat{\mathfrak{D}}$*. Assume that* $\rho(A)$ *contains some right half-plane, let* Ω *be an open subset of* $\rho_{+\infty}(A)$ *which contains some right half-plane, and denote the restriction of* $\widehat{\mathfrak{D}}$ *to* Ω *by* φ*. Then the following conditions are equivalent:*

(i) Σ *satisfies the uniform output coercive past* Ω*-cost condition, and* φ *is uniformly bounded on some right half-plane.*

(ii) *The control Riccati equation for* $\Sigma^\dagger$ *has an* Ω^**-solution for which the function* F *in* (4.6) *is uniformly bounded on some right half-plane.*

(iii) *The control Riccati equation for* $\Sigma^\dagger$ *has an* Ω^**-solution, and the function* F *in* (4.6) *corresponding to the minimal* Ω^**-solution is uniformly bounded on some right half-plane.*

(iv) φ *has a well-posed realization for which the filter Riccati equation has a bounded* $\mathbb{C}^+_\alpha$*-solution for some* $\alpha \geq 0$*.*

(v) φ *has a well-posed realization which satisfies the state coercive past cost condition.*

(vi) φ *has a well-posed detectable realization.*

(vii) φ *has a well-posed strongly* $*$*-detectable realization.*

(viii) φ *has a well-posed LQ past normalized realization.*

(ix) φ *has an well-posed left* $H^\infty(\mathbb{C}^+)$ *factorization.*

(x) φ *has a well-posed normalized weakly coprime left* $H^\infty(\mathbb{C}^+)$ *factorization.*

10. Doubly coprime factorizations. In this section we consider doubly coprime factorizations and as in [9] relate it to an optimal control problem on the whole real axis.

The following are [9, Definition 7.1 and 7.2].

DEFINITION 10.1. *Let* q *and* r *be two closed symmetric nonnegative sesquilinear forms on the Hilbert space* $\mathcal{X}$*. Then we say that* r dominates q *if* $\mathrm{dom}\,(r) \subset \mathrm{dom}\,(q)$ *and there exists a constant* $M > 0$ *such that* $q[x,x] \leq Mr[x,x]$ *for all* $x \in \mathrm{dom}\,(r)$*.*

DEFINITION 10.2. *Let* $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ *be an operator node with main operator* A*, and let* Ω *be an open subset of* $\rho(A) \cap \mathbb{C}^+$*.*

(i) Σ *is said to satisfy the* past Ω-cost dominance condition *if the optimal future* Ω*-cost* $\|\cdot\|^2_{\mathrm{fut},\Omega}$ *is dominated by the optimal past* Ω*-cost* $\|\cdot\|^2_{\mathrm{past},\Omega}$*.*

(ii) *If* Σ *is well-posed, then it is said to satisfy the* past cost dominance condition *if the optimal future cost* $\|\cdot\|^2_{\mathrm{fut}}$ *is dominated by the optimal past cost* $\|\cdot\|^2_{\mathrm{past}}$*.*

Remark 10.3. The past Ω-cost dominance condition and the past cost dominance condition are consistent by Remarks 6.2 and 9.3.

The following result on the past cost dominance condition and duality had not been considered in [9].

LEMMA 10.4. *Let $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be an operator node with main operator A, and let Ω be an open subset of $\rho(A) \cap \mathbb{C}^+$. If Σ satisfies the past Ω-cost dominance condition, then $\Sigma^\dagger$ satisfies the past Ω^*-cost dominance condition.*

Proof. Let $M > 0$ be such that $\|z\|_{\mathrm{fut},\Omega} \le M\|z\|_{\mathrm{past},\Omega}$ for all z with finite past cost for Σ. By [9, Theorem 3.18] we have that the domain of $\|\cdot\|^2_{\mathrm{past},\Omega^*}$ for $\Sigma^\dagger$ is characterized by

$$D(\|\cdot\|^2_{\mathrm{past}^\dagger,\Omega^*}) = \{z^\dagger \in \mathcal{X} : \sup_{\|z\|_{\mathrm{fut},\Omega} \le 1} |\langle z, z^\dagger \rangle| < \infty\}$$

and that the domain of $\|\cdot\|^2_{\mathrm{fut}^\dagger,\Omega^*}$ is characterized by

$$D(\|\cdot\|^2_{\mathrm{fut}^\dagger,\Omega^*}) = \{z^\dagger \in \mathcal{X} : \sup_{\|z\|_{\mathrm{past},\Omega} \le 1} |\langle z, z^\dagger \rangle| < \infty\}.$$

For $z^\dagger \in \mathcal{X}$ we have

$$\sup_{\|z\|_{\mathrm{past},\Omega} \le 1} |\langle z, z^\dagger \rangle_{\mathcal{X}}| \le \sup_{\|z\|_{\mathrm{fut},\Omega} \le M} |\langle z, z^\dagger \rangle_{\mathcal{X}}| \le M \sup_{\|\tilde{z}\|_{\mathrm{fut},\Omega} \le 1} |\langle \tilde{z}, z^\dagger \rangle_{\mathcal{X}}|.$$

Hence, $D(\|\cdot\|^2_{\mathrm{past}^\dagger,\Omega^*}) \subset D(\|\cdot\|^2_{\mathrm{fut}^\dagger,\Omega^*})$. We further see from the above calculation using that

$$\|z^\dagger\|_{\mathrm{fut}^\dagger,\Omega^*} = \sup_{\|z\|_{\mathrm{past}} \le 1} |\langle z, z^\dagger \rangle|, \quad \|z^\dagger\|_{\mathrm{past}^\dagger,\Omega^*} = \sup_{\|z\|_{\mathrm{fut},\Omega} \le 1} |\langle z, z^\dagger \rangle|,$$

that for $z^\dagger \in D(\|\cdot\|^2_{\mathrm{past}^\dagger,\Omega^*})$

$$\|z^\dagger\|_{\mathrm{fut}^\dagger,\Omega^*} \le M\|z^\dagger\|_{\mathrm{past}^\dagger,\Omega^*}.$$

Hence, the past Ω^*-cost dominance condition for $\Sigma^\dagger$ holds. ∎

The following is the "uniform" equivalent of [9, Lemma 7.3].

LEMMA 10.5. *Let $\Sigma := \left(\left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]; \mathcal{X}, \mathcal{U}, \mathcal{Y}\right)$ be a well-posed operator node. If Σ satisfies the past cost dominance condition, then it satisfies both the uniform input finite future cost condition and the uniform output coercive past cost condition.*

Proof. Let $\alpha > 0$ be such that $\mathbb{C}^+_\alpha \subset \rho(A)$, and define $\Omega := \mathbb{C}^+_\alpha$. By Remarks 6.2, 9.3, and 10.3 we have that the well-posed cost conditions and the corresponding Ω-cost conditions are equivalent.

From Remark 6.8 we see that in the well-posed case, the past cost dominance condition implies the uniform input finite future cost condition. By Lemma 10.4, the past cost dominance condition for Σ with respect to Ω implies the past cost dominance condition for $\Sigma^\dagger$ with respect to Ω^*. Hence, using Remark 6.8 again, we obtain the uniform input finite future cost condition for $\Sigma^\dagger$ with respect to Ω^*. From Lemma 9.7 we then obtain the uniform output coercive past cost condition for Σ with respect to Ω. ∎

The following strengthens [9, Definition 7.4] to the notion relevant in the well-posed case. Note that what is added compared to [9, Definition 7.4] is a well-posedness assumption on the denominators.

DEFINITION 10.6. *Let φ be an analytic $\mathcal{B}(\mathcal{U}; \mathcal{Y})$-valued function defined on some open subset Ω of $\mathbb{C}^+$.*

(i) *A right* $H^\infty(\mathbb{C}^+)$ *factorization* $\left[\begin{smallmatrix}\mathsf{M}\\ \mathsf{N}\end{smallmatrix}\right]$ *valid in* Ω *is* strongly coprime *if there exist two functions* $\widetilde{\mathsf{X}} \in H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{U}))$ *and* $\widetilde{\mathsf{Y}} \in H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{Y};\mathcal{U}))$ *such that* $\widetilde{\mathsf{X}}(\lambda)\mathsf{M}(\lambda) - \widetilde{\mathsf{Y}}(\lambda)\mathsf{N}(\lambda) = 1_{\mathcal{U}}$ *for all* $\lambda \in \mathbb{C}^+$.

(ii) *A left* $H^\infty(\mathbb{C}^+)$ *factorization* $[\widetilde{\mathsf{M}},\widetilde{\mathsf{N}}]$ *valid in* Ω *is* strongly coprime *if there exist two functions* $\mathsf{X} \in H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{Y}))$ *and* $\mathsf{Y} \in H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{U};\mathcal{Y}))$ *such that* $\widetilde{\mathsf{M}}(\lambda)\mathsf{X}(\lambda) - \widetilde{\mathsf{N}}(\lambda)\mathsf{Y}(\lambda) = 1_{\mathcal{Y}}$ *for all* $\lambda \in \mathbb{C}^+$.

(iii) φ *has a* doubly coprime $H^\infty(\mathbb{C}^+)$-factorization *valid in* Ω *if there exist functions* $\mathsf{M} \in H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{U}))$, $\mathsf{N} \in H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{U};\mathcal{Y}))$, $\widetilde{\mathsf{X}} \in H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{U}))$, $\widetilde{\mathsf{Y}} \in H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{Y};\mathcal{U}))$, $\widetilde{\mathsf{M}} \in H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{Y}))$, $\widetilde{\mathsf{N}} \in H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{U};\mathcal{Y}))$, $\mathsf{X} \in H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{Y}))$, *and* $\mathsf{Y} \in H^\infty(\mathbb{C}^+;\mathcal{B}(\mathcal{U};\mathcal{Y}))$ *such that* $\left[\begin{smallmatrix}\mathsf{M}\\ \mathsf{N}\end{smallmatrix}\right]$ *is a right* $H^\infty(\mathbb{C}^+)$ *factorization valid in* Ω, $[\widetilde{\mathsf{M}},\widetilde{\mathsf{N}}]$ *is a left* $H^\infty(\mathbb{C}^+)$ *factorization valid in* Ω *and*

$$\begin{bmatrix}\mathsf{M} & \mathsf{Y}\\ \mathsf{N} & \mathsf{X}\end{bmatrix}\begin{bmatrix}\tilde{\mathsf{X}} & -\tilde{\mathsf{Y}}\\ -\tilde{\mathsf{N}} & \tilde{\mathsf{M}}\end{bmatrix} = \begin{bmatrix}\tilde{\mathsf{X}} & -\tilde{\mathsf{Y}}\\ -\tilde{\mathsf{N}} & \tilde{\mathsf{M}}\end{bmatrix}\begin{bmatrix}\mathsf{M} & \mathsf{Y}\\ \mathsf{N} & \mathsf{X}\end{bmatrix} = \begin{bmatrix}1_{\mathcal{U}} & 0\\ 0 & 1_{\mathcal{Y}}\end{bmatrix} \tag{10.1}$$

on $\mathbb{C}^+$.

(iv) *The factorization in* (iii) *is* well-posed *if both* $\left[\begin{smallmatrix}\mathsf{M}\\ \mathsf{N}\end{smallmatrix}\right]$ *and* $[\widetilde{\mathsf{M}},\widetilde{\mathsf{N}}]$ *are well-posed.*

It is well known that any strongly coprime factorization is weakly coprime in the corresponding sense (right/left) and that a transfer function has a strongly right coprime factorization if and only if it has a strongly left coprime factorization if and only if it has a doubly coprime factorization; see, e.g., [5].

Lemma 10.7. *Let* $\alpha \ge 0$, *and define* $\Omega := \mathbb{C}^+_\alpha$. *Let* φ *be an analytic* $\mathcal{B}(\mathcal{U};\mathcal{Y})$*-valued function which is uniformly bounded on* Ω. *Then every strongly coprime right* $H^\infty(\mathbb{C}^+)$ *factorization valid in* Ω *of* φ *is well-posed.*

Proof. We will show that $\mathsf{M}^{-1} \in H^\infty(\mathbb{C}^+_\alpha;\mathcal{B}(\mathcal{U}))$, which implies well-posedness. For $\lambda \in \mathbb{C}^+$ we have by strong coprimeness that $\widetilde{\mathsf{X}}(\lambda)\mathsf{M}(\lambda) - \widetilde{\mathsf{Y}}(\lambda)\mathsf{N}(\lambda) = 1_{\mathcal{U}}$. Since $\mathsf{M}(\lambda)$ is invertible for $\lambda \in \Omega$ and $\varphi(\lambda) = \mathsf{N}(\lambda)\mathsf{M}(\lambda)^{-1}$ for $\lambda \in \Omega$, we obtain from this that $\widetilde{\mathsf{X}}(\lambda) - \widetilde{\mathsf{Y}}(\lambda)\varphi(\lambda) = \mathsf{M}(\lambda)^{-1}$ for all $\lambda \in \Omega$. Since the left-hand side is in $H^\infty(\mathbb{C}^+_\alpha;\mathcal{B}(\mathcal{U}))$, it follows that the right-hand side is. ∎

The following theorem is the well-posed equivalent of [9, Theorem 7.5] and involves the notion of the inverse of a quadratic form as defined in [9, Lemma 3.17] and the notion of a jointly stabilizable and detectable well-posed linear system from [11, Definition 8.2.4].

Theorem 10.8. *Let* $\Sigma := \left(\left[\begin{smallmatrix}A\&B\\ C\&D\end{smallmatrix}\right];\mathcal{X},\mathcal{U},\mathcal{Y}\right)$ *be an operator node with main operator* A *and transfer function* $\widehat{\mathfrak{D}}$. *Assume that* $\rho(A)$ *contains some right half-plane, let* Ω *be an open subset of* $\rho_{+\infty}(A)$ *which contains some right half-plane, and denote the restriction of* $\widehat{\mathfrak{D}}$ *to* Ω *by* φ. *Then the following conditions are equivalent:*

(i) Σ *satisfies the past* Ω*-cost dominance condition, and* φ *is uniformly bounded on some right half-plane.*

(ii) *The control Riccati equation for* Σ *has an* Ω*-solution* q *for which the function* F *in* (4.6) *is uniformly bounded on some right half-plane, the control Riccati equation for* $\Sigma^\dagger$ *has an* Ω^**-solution* p *for which the function* F *in* (4.6) *is uniformly bounded on some right half-plane, and* q *is dominated by the inverse of* p.

(iii) *The control Riccati equation for* Σ *has an* Ω*-solution* q *and the function* F *in* (4.6) *corresponding to the minimal* Ω*-solution is uniformly bounded on some* *right half-plane, the control Riccati equation for $\Sigma^\dagger$ has an Ω^*-solution p and the function F in* (4.6) *corresponding to the minimal Ω-solution is uniformly bounded on some right half-plane, and q is dominated by the inverse of p.*

(iv) *φ has a well-posed realization for which the control Riccati equation has a $\mathbb{C}^+_\alpha$-solution q for some $\alpha \geq 0$, the filter Riccati equation has a $\mathbb{C}^+_\beta$-solution q for some $\beta \geq 0$, and q is dominated by the inverse of p.*

(v) *φ has a well-posed realization which satisfies the past cost dominance condition.*

(vi) *φ has a well-posed realization for which the control Riccati equation has a bounded Ω-solution and the filter Riccati equation has a bounded Ω-solution.*

(vii) *φ has a well-posed realization which satisfies the state finite future cost condition and the state coercive past cost condition.*

(viii) *φ has a well-posed realization which is stabilizable and detectable.*

(ix) *φ has a well-posed realization which is jointly stabilizable and detectable.*

(x) *φ has a well-posed doubly coprime $H^\infty(\mathbb{C}^+)$ factorization valid in Ω.*

Proof. (x) $\Longrightarrow$ (ix) is [11, Theorem 8.4.1 (ii)].

(ix) $\Longrightarrow$ (viii) is trivial.

(viii) $\Longrightarrow$ (vii) follows since stabilizability implies the state finite future cost condition, and (by duality) therefore detectability implies the state coercive past cost condition.

(vii) $\Longrightarrow$ (vi) follows from Lemma 6.6 applied to both the realization and its dual noting that the state coercive past Ω-cost condition is equivalent to the state finite future Ω^*-cost condition for the dual by [9, Lemma 6.3].

(vi) $\Longrightarrow$ (v). Since the optimal future Ω-cost is the minimal Ω-solution to the control Riccati equation by Lemma 6.6, we have that there exists a $M_q > 0$ such that $\|z\|_{\mathrm{fut},\Omega} \leq M_q\|z\|$ for all $z \in \mathcal{X}$. Existence of a bounded Ω-solution of the filter Riccati equation is equivalent to the state coercive past Ω-cost condition by Lemma 6.6 applied to the dual system. Hence, there exists a $M_p > 0$ such that $M_p\|z\| \leq \|z\|_{\mathrm{past},\Omega}$ for all $z \in \mathcal{X}$, which are the final state of a generalized stable past Ω-trajectory of Σ. It follows that $\|z\|_{\mathrm{fut},\Omega} \leq \frac{M_q}{M_p}\|z\|_{\mathrm{past},\Omega}$ for all $z \in \mathcal{X}$, which are the final state of a generalized stable past Ω-trajectory of Σ; i.e., the past Ω-cost dominance condition holds. By Remark 10.3, this is equivalent to the past cost dominance condition.

(v) $\Longleftrightarrow$ (iv) follows from [9, Theorem 7.5] applied to this realization (and Lemma 4.3).

(v) $\Longrightarrow$ (x). That the past Ω-cost dominance condition (which by Remark 10.3 is equivalent to the past cost dominance condition) implies the existence of a doubly coprime $H^\infty(\mathbb{C}^+)$ factorization valid in Ω follows from [9, Theorem 7.5]. The additional well-posedness assumption on the realization implies through Lemma 10.7 that this factorization is well-posed.

(x) $\Longrightarrow$ (i). That the existence of a doubly coprime $H^\infty(\mathbb{C}^+)$ factorization valid in Ω of the transfer function implies that Σ satisfies the past Ω-cost dominance condition follows from [9, Theorem 7.5]. The additional well-posedness assumption on the factorization implies that φ is uniformly bounded on some right half-plane.

(i) $\Longrightarrow$ (x). That Σ satisfying the past Ω-cost dominance condition implies the existence of a doubly coprime $H^\infty(\mathbb{C}^+)$ factorization valid in Ω of its transfer function follows from [9, Theorem 7.5]. That uniform boundedness of φ on some right half-plane implies well-posedness of this factorization follows from Lemma 10.7.

(i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii). Equivalence of the past Ω-cost dominance condition with the existence of q and p combined with the dominance of q by the inverse of p follows from [9, Theorem 7.5]. The additional uniform boundedness claims follow using Theorem 6.10 applied to both Σ and $\Sigma^\dagger$. ◻

11. An example. An example without a doubly coprime factorization (with in fact a well-posed transfer function) was given in [9, section 8]. Here we give a simple PDE example which does have a doubly coprime factorization. We additionally use this example to illustrate LQ future and past normalized realizations.

Consider the PDE with boundary control:

$$\begin{aligned} &\frac{\partial w}{\partial t}(t,\xi) = \frac{\partial w}{\partial \xi}(t,\xi), \qquad t>0,\ \xi\in(0,1),\\ &w(t,1)=u(t), \qquad t>0. \end{aligned}$$

We define x by $x(t) = \xi \mapsto w(t,\xi)$, and we define the output by $y := x$. The above PDE can then be described by the operator node on $\mathcal{X} = L^2(0,1)$, $\mathcal{U} = \mathbb{R}$, $\mathcal{Y} = L^2(0,1)$ given by

$$S\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} x' \\ x \end{bmatrix}, \qquad D(S) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \begin{bmatrix} H^1(0,1) \\ \mathbb{R} \end{bmatrix} :\ x(1) = u \right\}.$$

This operator node is in fact well-posed, and $\mathbb{C}^+ \subset \rho(A)$. We will therefore take $\Omega = \mathbb{C}^+$. Similar to the calculation in [13], it is straightforward to compute that the future optimal control is zero and that the optimal future cost is given by

$$\|x_0\|^2_{\text{fut}} = \int_0^1 \xi |x_0(\xi)|^2 \, d\xi.$$

The continuous time control Riccati equation has the bounded sesquilinear form

$$q[x_0, z_0] = \int_0^1 \xi \, x_0(\xi) \, z_0(\xi) \, d\xi$$

as solution with

$$K\&F \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \sqrt{2} u_0$$

since for $\left[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}\right] \in D(S)$

$$2\int_0^1 \xi \, x_0'(\xi) \, x_0(\xi) \, d\xi + \int_0^1 |x_0(\xi)|^2 \, d\xi + |u_0|^2 = |\sqrt{2}u_0|^2.$$

The past optimal control problem has the optimal control and output

$$u(t) = \begin{cases} 0 & t<-1 \\ x_0(-t) & t \in [-1,0], \end{cases} \qquad y(t,\xi) = \begin{cases} 0 & t+\xi \notin [0,1] \\ x_0(t+\xi) & t+\xi \in [0,1], \end{cases}$$

and therefore the optimal past cost is

$$\|x_0\|^2_{\text{past}} = \int_0^1 (2-\xi) \, |x_0(\xi)|^2 \, d\xi.$$

The adjoint of S can be calculated to be

$$S^* \begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} -z' + y \\ z(1) \end{bmatrix}, \qquad D(S^*) = \left\{ \begin{bmatrix} z \\ y \end{bmatrix} \in \begin{bmatrix} H^1(0,1) \\ L^2(0,1) \end{bmatrix} :\ x(0) = 0 \right\}.$$

The continuous time filter Riccati equation has the bounded sesquilinear form

$$p[x_0, z_0] = \int_0^1 \frac{1}{2-\xi}\, x_0(\xi)\, z_0(\xi)\, d\xi$$

as solution with

$$K\&F \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \xi \mapsto \frac{1}{2-\xi} x_0(\xi) + y_0(\xi)$$

since for $\left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right] \in D(S^*)$

$$2\int_0^1 \frac{1}{2-\xi}\, [-x_0'(\xi) + y_0(\xi)]\; x_0(\xi)\, d\xi + |x_0(1)|^2 + \int_0^1 |y_0(\xi)|^2\, d\xi \\ = \int_0^1 \left| \frac{1}{2-\xi} x_0(\xi) + y_0(\xi) \right|^2\, d\xi.$$

We see that condition (vi) from Theorem 10.8 is satisfied, and therefore so are all of the other equivalent conditions mentioned in that theorem. In particular, the transfer function of S has a doubly coprime factorization. The transfer function of S can be calculated to be (see [13])

$$\widehat{\mathfrak{D}}(\lambda) = \xi \mapsto \mathrm{e}^{\lambda(\xi-1)},$$

and, similarly as in [9, section 8], using the above solutions of the Riccati equations we can calculate a normalized strongly coprime right factorization

$$\mathsf{M}(\lambda) = 1, \qquad \mathsf{N}(\lambda) = \widehat{\mathfrak{D}}(\lambda),$$

with corresponding Bezout factors

$$\widetilde{\mathsf{X}}(\lambda) = 1, \qquad \widetilde{\mathsf{Y}}(\lambda) = 0,$$

and a normalized strongly coprime left factorization

$$\widetilde{\mathsf{M}}(\lambda) y = \xi \mapsto y(\xi) - \frac{\mathrm{e}^{\lambda\xi}}{2-\xi} \int_\xi^1 \mathrm{e}^{-\lambda\theta} y(\theta)\, d\theta, \qquad \widetilde{\mathsf{N}}(\lambda) = \xi \mapsto \mathrm{e}^{\lambda(\xi-1)} \frac{1}{2-\xi},$$

with corresponding Bezout factors

$$\mathsf{X}(\lambda) y = \xi \mapsto y(\xi) + \mathrm{e}^{\lambda\xi} \int_\xi^1 \frac{\mathrm{e}^{-\lambda\theta}}{2-\theta} y(\theta)\, d\theta, \qquad \mathsf{Y}(\lambda) = 0,$$

where to obtain $\widetilde{\mathsf{N}}(\lambda)$ we solved the boundary value problem

$$\lambda x(\xi) - x'(\xi) + \frac{1}{2-\xi} x(\xi) = 0, \qquad x(1) = 1,$$

to obtain $\widetilde{\mathsf{M}}(\lambda)$ we solved the boundary value problem

$$\lambda x(\xi) - x'(\xi) + \frac{1}{2-\xi} x(\xi) = \frac{1}{2-\xi} y(\xi), \qquad x(1) = 0,$$

and to obtain $\mathsf{X}(\lambda)$ we solved the boundary value problem

$$\lambda x(\xi) - x'(\xi) = \frac{1}{2-\xi} y(\xi), \qquad x(1) = 0.$$

From the above expression for $\|x_0\|^2_{\text{past}}$ for the past cost we see that when we consider S instead on the state space

$$\mathcal{X}_{\text{past}} := L^2(0,1;(2-\xi)\,d\xi),$$

we obtain an LQ past normalized realization of the transfer function of S. Note that since the weight $2-\xi$ and its inverse are both in $L^\infty(0,1)$, we have that $x_0 \in L^2(0,1)$ if and only if $x_0 \in L^2(0,1;(2-\xi)\,d\xi)$ (but the norm of x_0 in the two spaces is different).

From the above expression for $\|x_0\|^2_{\text{fut}}$ for the future cost we see that when we consider S instead on the state space

$$\mathcal{X}_{\text{fut}} := L^2(0,1;\xi\,d\xi),$$

we obtain an LQ future normalized realization of the transfer function of S. Note that since the weight ξ is in $L^\infty(0,1)$ but its inverse is not, we have $L^2(0,1) \hookrightarrow L^2(0,1;\xi\,d\xi)$, but we do not have the reverse inclusion. For example, $x_0(\xi) = \frac{1}{\sqrt{\xi}}$ satisfies $x_0 \notin L^2(0,1)$ and $x_0 \in L^2(0,1;\xi\,d\xi)$.

For precisely those state spaces $\mathcal{X}$ for S with

$$L^2(0,1) \hookrightarrow \mathcal{X} \hookrightarrow L^2(0,1;\xi\,d\xi),$$

we have that both the finite future cost condition and the state coercive past cost condition are satisfied.

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