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Perturbing the hexagonal circle packing: A percolation perspective

Itai Benjamini[a] and Alexandre Stauffer[b,1]

[a]*Weizmann Institute of Science, Rehovot 76100, Israel. E-mail: itai.benjamini@weizmann.ac.il*
[b]*Computer Science Division, University of California, Berkeley, CA 94720, USA. E-mail: stauffer@cs.berkeley.edu*

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Abstract. We consider the hexagonal circle packing with radius $1/2$ and perturb it by letting the circles move as independent Brownian motions for time t. It is shown that, for large enough t, if Π_t is the point process given by the center of the circles at time t, then, as $t \to \infty$, the critical radius for circles centered at Π_t to contain an infinite component converges to that of continuum percolation (which was shown – based on a Monte Carlo estimate – by Balister, Bollobás and Walters to be strictly bigger than $1/2$). On the other hand, for small enough t, we show (using a Monte Carlo estimate for a fixed but high dimensional integral) that the union of the circles contains an infinite connected component. We discuss some extensions and open problems.

Résumé. Nous considérons une juxtaposition hexagonale de cercles de rayon $1/2$ et nous la perturbons en laissant les cercles évoluer comme des mouvements browniens indépendants pendant un temps t. Nous montrons que, pour t suffisamment grand, si Π_t est le processus de points donné par les centres des cercles au temps t, alors quand $t \to \infty$, le rayon critique pour que les cercles centrés en Π_t contienne une composante infinie converge vers celui de la percolation continue (qui est strictement plus grand que $1/2$ comme l'ont montré Balister, Bollobás et Walters). D'un autre coté, pour t suffisamment petit, nous montrons (à l'aide d'une estimation de Monte Carlo pour une intégrale de grande dimension) que l'union des cercles contient une composante infinie. Nous discutons aussi des généralisations et des problèmes ouverts.

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1. Introduction

Let $\mathcal{T}$ be the triangular lattice with edge length 1 and let Π_0 be the set of vertices of $\mathcal{T}$. We see Π_0 as a point process and, to avoid ambiguity, we use the term *node* to refer to the points of Π_0. Now, for each node $u \in \Pi_0$, we add a ball of radius $1/2$ centered at u, and set $R(\Pi_0)$ to be the region of $\mathbb{R}^2$ obtained by the union of these balls; more formally,

$$R(\Pi_0) = \bigcup_{x \in \Pi_0} B(x, 1/2),$$

where $B(y, r)$ denotes the closed ball of radius r centered at y. In this way, $R(\Pi_0)$ is the so-called hexagonal circle packing of $\mathbb{R}^2$; refer to [4] for more information on packings. Clearly, the region $R(\Pi_0)$ is a *connected* subset of $\mathbb{R}^2$.

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Our goal is to analyze how this set evolves as we let the nodes of Π_0 move on $\mathbb{R}^2$ according to independent Brownian motions. For any $t > 0$, let Π_t be the point process obtained after the nodes have moved for time t. More formally, for each node $u \in \Pi_0$, let $(\zeta_u(t))_t$ be an independent Brownian motion on $\mathbb{R}^2$ starting at the origin, and set

$$\Pi_t = \bigcup_{u \in \Pi_0} \big(u + \zeta_u(t)\big).$$

A natural question is whether there exists a phase transition on t such that $R(\Pi_t)$ has an infinite component for small t but has only finite components for large t.

Intuitively, for sufficiently large time, one expects that Π_t will look like a Poisson point process with intensity $2/\sqrt{3}$, which is the density of nodes in the triangular lattice. Then, for sufficiently large t, $R(\Pi_t)$ will contain an infinite component almost surely only if $R(\Phi)$ contains an infinite component where Φ is a Poisson point process of intensity $\lambda = 2/\sqrt{3}$. In the literature, $R(\Phi)$ is referred to as the Boolean model. For this model, it is known that there exists a value λ_c so that, if $\lambda < \lambda_c$, then all connected components of $R(\Phi)$ are finite almost surely [9]. On the other hand, if $\lambda > \lambda_c$, then $R(\Phi)$ contains an infinite connected component. The value of λ_c is currently unknown and depends on the radius of the balls in the definition of the region R. When the balls have radius $1/2$, it is known that λ_c satisfies $0.52 \le \lambda_c \le 3.38$ [3], Chapter 8, but these bounds do not answer whether $2/\sqrt{3}$ is smaller or larger than λ_c. However, using a Monte Carlo analysis, Balister, Bollobás and Walters [2] showed that, with 99.99% confidence, λ_c lies between 1.434 and 1.438, which are both larger than $2/\sqrt{3}$. We then have the following two theorems, whose proofs we give in Section 3.

Theorem 1.1. *If $\lambda_c > 2/\sqrt{3}$, where λ_c is the critical intensity for percolation of the Boolean model with balls of radius $1/2$, then there exists a positive t_0 so that, for all $t > t_0$, $R(\Pi_t)$ contains no infinite component almost surely.*

For the next theorem, we note that, for each t, there exists a critical radius $r_c(t)$ so that, adding balls of radius $r > r_c(t)$ centered at the points of Π_t gives that the union of these balls contains an infinite component almost surely.

Theorem 1.2. *As $t \to \infty$, we have that $r_c(t)$ converges to the critical radius for percolation of the Boolean model with intensity $2/\sqrt{3}$.*

Before proving Theorems 1.1 and 1.2 in Section 3, we devote Section 2 to prove Theorem 1.3 below, which we believe to be of independent interest. Consider a tessellation of $\mathbb{R}^2$ into regular hexagons of side length $\delta\sqrt{t}$, where δ is an arbitrarily small constant. Let I denote the set of points of $\mathbb{R}^2$ that are the centers of these hexagons. Then, for each $i \in I$, denote the hexagon with center at i by Q_i and define a Bernoulli random variable X_i with parameter p independently of the other X_j, $j \neq i$. Define

$$C(p,\delta) = \bigcup_{i \in I:\ X_i = 1} Q_i.$$

When $p > 1/2$, which is the critical probability for site percolation on the triangular lattice, we have that $C(p,\delta)$ contains a unique infinite connected component. We are now ready to state our main technical result, Theorem 1.3 below. In Section 2, we actually prove a stronger version of this theorem (Theorem 2.2), from which Theorem 1.3 follows.

Theorem 1.3. *There exists a universal constant $c > 0$ and, for any $p \in (0,1)$ that can be arbitrarily close to 1 and any arbitrarily small $\delta > 0$, there exists a $t_0 > 0$ such that, for all $t > t_0$, we can couple Π_t, $(X_i)_{i \in I}$ and a Poisson point process Φ of intensity $\frac{2}{\sqrt{3}} + c\sqrt{\delta}$ so that*

$$\Pi_t \cap C(p,\delta) \subset \Phi \cap C(p,\delta).$$

In words, Theorem 1.3 establishes that Π_t is *stochastically dominated* by a Poisson point process inside $C(p,\delta)$. As we show later in Lemma 2.1, Π_t cannot be stochastically dominated by a Poisson point process in the whole of $\mathbb{R}^2$.

We note that the opposite direction of Theorem 1.3 was established by Sinclair and Stauffer [14], Proposition 4.1, who proved that, under some conditions on the initial location of the nodes, after moving as independent Brownian motions for time t, the nodes *stochastically dominate* a Poisson point process. The result of Sinclair and Stauffer has been used and refined in [13,15], and turned out to be a useful tool in the analysis of increasing events for models of mobile nodes, such as the so-called percolation time in [13] and detection time in [15]. We expect that the ideas in our proof of Theorem 1.3 can help in the analysis of *decreasing* events, which have so far received less attention.

We remark that the proof of our Theorem 1.3 requires a more delicate analysis than that of Sinclair and Stauffer. In their case, nodes that ended up moving atypically far away during the interval $[0,t]$ could be simply disregarded as it is possible to show that the typical nodes already stochastically dominate a Poisson point process. In our setting, no node can be disregarded, regardless of how atypical its motion turns out to be. In order to solve this problem, we first consider what we call well behaved nodes, which among other things satisfy that their motion during $[0,t]$ is contained in some ball of radius $c\sqrt{t}$ for some large constant c (we defer the complete definition of well behaved nodes to Section 2). The definition of well behaved nodes is carefully specified so that any given node is likely to be well behaved and, in addition, it is possible to show that well behaved nodes are stochastically dominated by a Poisson point process inside $C(p,\delta)$. For the remaining nodes, which comprise only a small density of nodes, we use a sprinkling argument to replace them already at time 0 by a Poisson point process of low intensity. Then, even though the motion of the nodes that are not well behaved is hard to control, we use the fact that they are a Poisson point process at time 0 to show that, at time t, they stochastically dominate a Poisson point process of low intensity.

Now, for the case when the nodes of Π_0 move for only a *small* time t, we believe the following is true.

Conjecture 1.4. *There exists a $t_0 > 0$ so that, for all $t < t_0$, $R(\Pi_t)$ contains an infinite component almost surely.*

We are able to establish the conjecture above given a Monte Carlo estimate for a finite but high dimensional integral. We discuss the details in Section 4. Note that if Conjecture 1.4 is true, then a curious consequence of this and Theorem 1.2 is that $r_c(t)$ is not monotone in t.

We conclude in Section 5 with some extensions and open problems.

2. Stochastic domination

We devote this section to the proof of our main technical result, Theorem 1.3, where we study the behavior of the balls after they have moved for a time t that is sufficiently large. We will prove a stronger version of Theorem 1.3 (see Theorem 2.2 below), from which Theorem 1.3 will follow. In Section 3 we show how to use this result to establish Theorems 1.1 and 1.2.

Intuitively, as $t \to \infty$, Π_t looks like an independent Poisson point process of intensity $2/\sqrt{3}$. Since the intensity of Φ is larger than $2/\sqrt{3}$, we would like to argue that there exists a coupling between Φ and Π_t such that Φ contains Π_t. Unfortunately, this cannot be achieved in the whole of $\mathbb{R}^2$, as established by the lemma below, which gives that, for any fixed t, the probability that a sufficiently large region $S \subset \mathbb{R}^2$ contains no node of Π_t is smaller than $\exp(-(\frac{2}{\sqrt{3}} + c\sqrt{\delta})\operatorname{vol}(S))$, which is the probability that Φ has no node in S. Hence, Φ cannot stochastically dominate Π_t in the whole of $\mathbb{R}^2$.

Lemma 2.1. *Fix t sufficiently large and let S be a hexagon of side length $(\log t)\sqrt{t}$ obtained as the union of $6(\log t)^2 t$ triangles of $\mathcal{T}$. Then, there exists a positive constant c' such that*

$$\mathbf{P}(\Pi_t \cap S = \varnothing) \le \exp\bigl(-c'(\log t)^2 \operatorname{vol}(S)\bigr).$$

Proof. For simplicity we assume that $(\log t)\sqrt{t}$ is an even integer. Let x be the middle point of S and consider the hexagon S' of side length $\frac{\log t}{2}\sqrt{t}$ composed of $\frac{3(\log t)^2}{2}t$ triangles of $\mathcal{T}$ and centered at x. Note that S contains the ball $B(x, \frac{\sqrt{3}\log t}{2}\sqrt{t})$ and S' is contained in the ball $B(x, \frac{\log t}{2}\sqrt{t})$. Therefore, a node of Π_0 that is inside S' can only be outside of S at time t if it moves at least $\frac{(\sqrt{3}-1)\log t}{2}\sqrt{t} \geq \frac{\log t}{3}\sqrt{t}$. For any fixed node $u \in \Pi_0$, we have from the Gaussian tail bound (cf. Lemma A.3) that

$$\mathbf{P}\left(\left\|\zeta_u(t)\right\|_2 \geq \frac{\log t}{3}\sqrt{t}\right) \leq \frac{3}{\sqrt{2\pi}\log t}\exp\left(-\frac{(\log t)^2}{18}\right).$$

Each node of Π_0 belongs to 6 triangles of $\mathcal{T}$, then there are at least $\frac{(\log t)^2}{4}t$ nodes of Π_0 in S'. Since each of them need to move more than $\frac{\log t}{3}\sqrt{t}$ by time t for S to contain no node of Π_t, we obtain

$$\mathbf{P}(\Pi_t \cap S = \varnothing) \leq \left(\frac{3}{\sqrt{2\pi}\log t}\exp\left(-\frac{(\log t)^2}{18}\right)\right)^{((\log t)^2/4)t} \leq \exp\left(-\frac{(\log t)^4 t}{72}\right).$$

Since $\mathrm{vol}(S) = \frac{3\sqrt{3}}{2}(\log t)^2 t$, the proof is completed. □

Now we turn to the proof of Theorem 1.3. The goal is to show that Φ contains Π_t inside a percolating cluster of a suitable tessellation of $\mathbb{R}^2$. For this, we tessellate $\mathbb{R}^2$ into hexagons of side length $\delta\sqrt{t}$. We take this tessellation in such a way that no point of Π_0 lies on the edges of the hexagons; this is not crucial for the proof but simplifies the explanations in the sequel. Let $\mathcal{H}$ denote the set of these hexagons. Consider a node $v \in \Pi_0$. Let Q_i be the hexagon of $\mathcal{H}$ that contains v and let v' be a copy of v located at the same position as v at time 0. We let v' move up to time t according to a certain procedure that we will describe in a moment, and then we say that v is *well behaved* if we are able to couple the motion of v with the motion of v' so that v and v' are at the same location at time t. Recall that I is the set of points given by the centers of the hexagons in $\mathcal{H}$. For $i \in I$, we define

$$J_i = \left\{j \in I\colon \sup_{x\in Q_i, y\in Q_j} \|x-y\|_2 \leq C\delta\sqrt{t}\right\}, \tag{1}$$

where

$$C = 4\delta^{-3/2}. \tag{2}$$

For i, j such that $j \in J_i$ we say that i and j are neighbors.

Before we describe the motion of v', we will state a stronger version of Theorem 1.3. For a Poisson point process Ξ and an event E which is measurable with respect to the σ-algebra induced by Ξ, we say that E is *decreasing* if the fact that E holds for a Poisson point process Ξ implies that E holds for all $\Xi' \subseteq \Xi$. Theorem 1.3 follows from the theorem below with E_i being the trivial event that always hold regardless of Ξ.

Theorem 2.2. *There exists a universal constant $c > 0$ so that the following holds for any $p \in (0,1)$ that can be arbitrarily close to 1 and any $\delta > 0$ that can be arbitrarily small. For each $i \in I$, let E_i be a decreasing event measurable with respect to the σ-algebra induced by $\Phi \cap \bigcup_{j\in J_i} Q_j$, where Φ is a Poisson point process of intensity $\frac{2}{\sqrt{3}} + c\sqrt{\delta}$. Assume that, for any fixed $\delta > 0$, as $t \to \infty$, we have $\mathbf{P}(E_i) \to 1$, and let C' be the union of the Q_i for which E_i holds. Then, there exists a $t_0 = t_0(p,\delta) > 0$ such that, for all $t > t_0$, we can couple Π_t, $(X_i)_{i\in I}$ and Φ so that $C' \supseteq C(p,\delta)$ and*

$$\Pi_t \cap C(p,\delta) \subset \Phi \cap C(p,\delta).$$

Remark 2.3. *In Theorem 2.2 above, it is not crucial that E_i is measurable with respect to the σ-algebra induced by $\Phi \cap (\bigcup_{j\in J_i} Q_j)$. This condition is enough for our purposes, but Theorem 2.2 also holds if, for each $i \in I$, E_i depends only on a set of events E_j whose cardinality is bounded above by a constant independent of t.*

We devote the remainder of this section to prove Theorem 2.2. We start describing the motion of v'. Let f_t be the density function for the location of a Brownian motion at time t given that it starts at the origin of $\mathbb{R}^2$. We fix t and, for each $i, j \in I$ such that i and j are neighbors, we let

$$\varphi_t(i, j) = \inf_{x \in Q_i, y \in Q_j} f_t(y - x). \tag{3}$$

If i and j are not neighbors we set $\varphi_t(i, j) = 0$. Then, the motion of v' is described by first choosing a $j \in J_i$ with probability proportional to $\varphi_t(i, j)$ and then placing v' uniformly at random in Q_j. The main intuition behind this definition is that, when v is well behaved, its position inside Q_j has the same distribution as that of a node of a Poisson point process inside Q_j. Therefore, as long as the number of well behaved nodes that end up in Q_j is smaller than the number of nodes in $\Phi \cap Q_j$, we will be able to couple them with Φ. Another important feature of the definition of well behaved nodes is that, if v is well behaved and ends up moving to hexagon Q_j, then we know that, at time 0, v was in some hexagon of J_j. In particular, there is a bounded number of hexagons from which v could have moved to Q_j, which allows us to control dependencies.

Now we show that nodes are likely to be well behaved. Since the area of each hexagon of $\mathcal{H}$ is $\frac{3\sqrt{3}}{2}\delta^2 t$, we have that

$$\mathbf{P}(v \text{ is well behaved}) = \sum_{j \in J_i} \frac{3\sqrt{3}}{2}\delta^2 t \varphi_t(i, j). \tag{4}$$

The idea is that δ is sufficiently small so that f_t varies very little (i.e., f_t is essentially constant) inside any given hexagon of $\mathcal{H}$, but, at the same time, $C\delta$ is large so that the probability that v moves to an hexagon that is not in J_i is small. We can then obtain in the lemma below that the probability that v is well behaved is large.

Lemma 2.4. *Let v be a node of Π_0 located in Q_i. We have*

$$(C - 3)^2 \le |J_i| \le \frac{4}{3}C^2,$$

and, for sufficiently large t, we have

$$\mathbf{P}(v \textit{ is well behaved}) \ge 1 - 6\sqrt{\delta}.$$

Proof. For $j \notin J_i$, we know, by definition, that there exist a $x_0 \in Q_i$ and a $y_0 \in Q_j$ such that $\|x_0 - y_0\|_2 > C\delta\sqrt{t}$. Then, by the triangle inequality, we have that, for any $y \in Q_j$,

$$\|y - i\|_2 \ge C\delta\sqrt{t} - \|y - y_0\|_2 - \|i - x_0\|_2 \ge C\delta\sqrt{t} - 3\delta\sqrt{t}, \tag{5}$$

where we used the fact that, for any two points y, y_0 in the same hexagon, we have $\|y - y_0\|_2 \le 2\delta\sqrt{t}$ and, for any $x \in Q_i$ we have $\|i - x\|_2 \le \delta\sqrt{t}$. Therefore, if we add balls of radius $\delta\sqrt{t}$ centered at each $j \in J_i$, these balls cover the whole of $B(i, C\delta\sqrt{t} - 3\delta\sqrt{t})$, which yields

$$|J_i| \ge \frac{\mathrm{vol}(B(i, C\delta\sqrt{t} - 3\delta\sqrt{t}))}{\mathrm{vol}(B(0, \delta\sqrt{t}))} = (C - 3)^2.$$

For the other direction, note that if we add balls of radius $\frac{\sqrt{3}}{2}\delta\sqrt{t}$ centered at each $j \in J_i$, these balls are disjoint and their union is contained in $B(i, C\delta\sqrt{t})$, which gives

$$|J_i| \le \frac{\mathrm{vol}(B(i, C\delta\sqrt{t}))}{\mathrm{vol}(B(0, (\sqrt{3}/2)\delta\sqrt{t}))} = \frac{4}{3}C^2.$$

Now we prove the second part of the lemma. Note that, using (4) and (3), we have

$$\begin{aligned}\mathbf{P}(v \text{ is well behaved}) &= \sum_{j\in J_i} \frac{3\sqrt{3}}{2}\frac{\delta^2 t}{2\pi t}\exp\biggl(-\frac{\sup_{x\in Q_i, y\in Q_j}\|x-y\|_2^2}{2t}\biggr)\\ &\geq \sum_{j\in J_i}\int_{Q_j}\frac{1}{2\pi t}\exp\biggl(-\frac{(\|z-i\|_2+3\delta\sqrt{t})^2}{2t}\biggr)\,\mathrm{d}z, \end{aligned} \tag{6}$$

where the last step follows by the triangle inequality. Now, from (5), the ball $B(i, C\delta\sqrt{t}-3\delta\sqrt{t})$ only intersects hexagons that are neighbors of i. We denote by $S_a = [-a/2, a/2]^2$ the square of side length a, and, for any $z = (z_1, z_2) \in \mathbb{R}^2$ and $a \in \mathbb{R}_+$, we use the inequality $(\|z\|_2 + a)^2 \leq (|z_1| + a)^2 + (|z_2| + a)^2$. Then, applying (6), we obtain

$$\begin{aligned}\mathbf{P}(v \text{ is well behaved}) &\geq \int_{B(0,C\delta\sqrt{t}-3\delta\sqrt{t})}\frac{1}{2\pi t}\exp\biggl(-\frac{(\|z\|_2+3\delta\sqrt{t})^2}{2t}\biggr)\,\mathrm{d}z\\ &\geq \int_{S_{(2C\delta\sqrt{t}-6\delta\sqrt{t})/\sqrt{2}}}\frac{1}{2\pi t}\exp\biggl(-\frac{(\|z\|_2+3\delta\sqrt{t})^2}{2t}\biggr)\,\mathrm{d}z\\ &\geq \biggl(2\int_{3\delta\sqrt{t}}^{(C\delta\sqrt{t}+3(\sqrt{2}-1)\delta\sqrt{t})/\sqrt{2}}\frac{1}{\sqrt{2\pi t}}\exp\biggl(-\frac{z_1^2}{2t}\biggr)\,\mathrm{d}z_1\biggr)^2\\ &\geq \biggl(1-\frac{6\delta}{\sqrt{2\pi}}-\frac{2}{\sqrt{\pi}(C+3(\sqrt{2}-1))\delta}\exp\biggl(-\frac{\delta^2(C+3(\sqrt{2}-1))^2}{4}\biggr)\biggr)^2\\ &\geq 1-\frac{12\delta}{\sqrt{2\pi}}-\frac{4}{\sqrt{\pi}C\delta}\exp\biggl(-\frac{\delta^2C^2}{4}\biggr),\end{aligned}$$

where the second to last step follows by the standard Gaussian tail bound (cf. Lemma A.3). Then, using the definition of C in (2), we have that $\frac{4}{\sqrt{\pi}C\delta}\mathrm{e}^{-\delta^2C^2/4} = \frac{\sqrt{\delta}}{\sqrt{\pi}}\mathrm{e}^{-4\delta}$. Plugging this in the above inequality we obtain

$$\mathbf{P}(v \text{ is well behaved}) \geq 1-\sqrt{\delta}\biggl(\frac{12\sqrt{\delta}}{\sqrt{2\pi}}+\frac{1}{\sqrt{\pi}}\mathrm{e}^{-4\delta}\biggr) \geq 1-6\sqrt{\delta} \quad \text{for all } \delta\in(0,1].$$

□

We will treat the nodes that are not well behaved by means of another point process. For any point $x \in \mathbb{R}^2$, we set $q(x) = i$ if $x \in Q_i$. Then, let $g_t(x, y)$ be the density function for a node v that is not well behaved to move from x to y after time t. We have that

$$g_t(x,y) = \frac{f_t(y-x)-\varphi_t(q(x),q(y))}{\mathbf{P}(v \text{ is not well behaved})}. \tag{7}$$

For each $v \in \Pi_0$, let $N_v(\mu)$ be a Poisson random variable with mean μ, and let $\Psi_0(\mu)$ be the point process obtained by putting $N_v(\mu)$ points at v for each $v \in \Pi_0$. We set $\mathrm{e}^{-\mu} = \mathbf{P}(v \text{ is well behaved})$ and, from Lemma 2.4 and the fact that δ is sufficiently small, we henceforth assume that $\mu \leq 1$. We can then use a standard coupling argument so that $N_v(\mu) \geq 1$ if and only if v is not well behaved. The intuition is that, by replacing each node of Π_0 that is not well behaved by a Poisson number of nodes, we can exploit the thinning property of Poisson random variables to show that, as the nodes move, they are stochastically dominated by a Poisson point process.

For each $w \in \Psi_0(\mu)$, let $\xi_w(t)$ be the position of w at time t according to the density function g_t. Define $\Psi_t(\mu)$ to be the point process obtained by

$$\Psi_t(\mu) = \bigcup_{w\in\Psi_0(\mu)} \xi_w(t).$$

The following lemma gives that $\Psi_t(\mu)$ is stochastically dominated by a Poisson point process.

Lemma 2.5. *Define μ so that $\mathrm{e}^{-\mu}$ is the probability that a node of Π_0 is well behaved. There exists a universal constant $c>0$ such that, for any $\delta \in (0,1)$, if $\widetilde{\Psi}$ is a Poisson point process with intensity $c\sqrt{\delta}$, then for all sufficiently large t it is possible to couple $\widetilde{\Psi}$ with $\Psi_t(\mu)$ so that $\Psi_t(\mu) \subseteq \widetilde{\Psi}$.*

Proof. Since the nodes of $\Psi_0(\mu)$ move independently of one another, we can apply the thinning property of Poisson random variables to obtain that $\Psi_t(\mu)$ is a Poisson point process. Let $\Lambda(x)$ be the intensity of $\Psi_t(\mu)$ at $x \in \mathbb{R}^2$. By symmetry of Brownian motion and the symmetry in the motion of well behaved nodes, we have that

$$\Lambda(x) = \sum_{v \in \Pi_0} \mu g_t(v,x) = \sum_{v \in \Pi_0} \mu g_t(x,v). \tag{8}$$

Recall that, for any $z \in \mathbb{R}^2$ and $\ell > 0$, we define $z + S_\ell$ as the translation of the square $[0,\ell]^2$ so that its center is at z. Define the square R_1 as $x + S_{5\delta\sqrt{t}}$, the annulus R_2 as $(x + S_{5C\delta\sqrt{t}}) \setminus R_1$ and the region R_3 as $\mathbb{R}^2 \setminus (R_1 \cup R_2)$. We split the sum in (8) into three parts by considering the sets $P_1 = \Pi_0 \cap R_1$, $P_2 = \Pi_0 \cap R_2$ and $P_3 = \Pi_0 \cap R_3$.

We start with P_2. We can partition each hexagon of $\mathcal{H}$ into smaller hexagons of side length $\sqrt{3}/3$ such that each point of Π_0 is contained in exactly one such hexagon. This is possible since the dual lattice[2] of $\mathcal{T}$ is a hexagonal lattice of side length $\sqrt{3}/3$, so the hexagons of side length $\sqrt{3}/3$ mentioned above can be obtained by translating and rotating the dual lattice of $\mathcal{T}$. We denote by $\mathcal{H}'$ the set of hexagons of side length $\sqrt{3}/3$ obtained in this way.

For each $z \in \mathbb{R}^2$, let H_z be the hexagon that contains z in $\mathcal{H}'$. Each H_z has side length $\sqrt{3}/3$ and area $\sqrt{3}/2$. Therefore, the distance between any two points of H_z is at most $2\sqrt{3}/3$. Thus, by the triangle inequality, we have that, for any point $z \in R_2$, the hexagon $H_z \subset x + S_{5C\delta\sqrt{t}+4\sqrt{3}/3}$. Similarly, for any point $z \in R_2$, the hexagon H_z does not intersect $x + S_{5\delta\sqrt{t}-4\sqrt{3}/3}$. Let $R_1' = x + S_{5\delta\sqrt{t}-4\sqrt{3}/3}$ and $R_2' = (x + S_{5C\delta\sqrt{t}+4\sqrt{3}/3}) \setminus R_1'$, which gives that

$$\sum_{v \in P_2} \mu g_t(x,v) \le \frac{2}{\sqrt{3}} \int_{R_2'} \sup_{z' \in H_z} \mu g_t\bigl(x,z'\bigr)\,\mathrm{d}z.$$

Now, note that $\frac{\mu}{\mathbf{P}(v \text{ is not well behaved})} = \frac{\mu}{1-\mathrm{e}^{-\mu}} \le \frac{1}{1-\mu/2} \le 2$ since $\mu \le 1$. Then, using the definition of g_t from (7) and the definition of φ_t in (3), we have that

$$\begin{aligned}\sum_{v \in P_2} \mu g_t(x,v) &\le \frac{4}{\sqrt{3}} \int_{R_2'} \Bigl(\sup_{z' \in H_z} f_t\bigl(z'-x\bigr) - \varphi_t\bigl(q(x),q\bigl(z'\bigr)\bigr)\Bigr)\,\mathrm{d}z \\ &= \frac{4}{\sqrt{3}} \int_{R_2'} \Bigl(\sup_{z' \in H_z} f_t\bigl(z'-x\bigr) - \inf_{x' \in Q_{q(x)}, z'' \in Q_{q(z')}} f_t\bigl(x'-z''\bigr)\Bigr)\,\mathrm{d}z.\end{aligned}$$

Now, by the triangle inequality, we have that

$$\begin{aligned}&\bigl\|z'-x\bigr\|_2 \ge \|z-x\|_2 - \bigl\|z-z'\bigr\|_2 \ge \|z-x\|_2 - 2\sqrt{3}/3 \quad \text{and} \\ &\bigl\|x'-z''\bigr\|_2 \le \|z-x\|_2 + \bigl\|x-x'\bigr\|_2 + \bigl\|z'-z''\bigr\|_2 + \bigl\|z-z'\bigr\|_2 \le \|z-x\|_2 + 4\delta\sqrt{t} + 2\sqrt{3}/3.\end{aligned}$$

To simplify the equations we write $4\delta\sqrt{t} + 2\sqrt{3}/3 \le 5\delta\sqrt{t}$, which holds for all t sufficiently large. With this, we have

$$\begin{aligned}&\sum_{v \in P_2} \mu g_t(x,v) \\ &\quad \le \frac{4}{\sqrt{3}} \int_{R_2'} \frac{1}{2\pi t}\biggl(\exp\biggl(-\frac{(\|z-x\|_2 - 2\sqrt{3}/3)^2}{2t}\biggr) - \exp\biggl(-\frac{(\|z-x\|_2 + 5\delta\sqrt{t})^2}{2t}\biggr)\biggr)\,\mathrm{d}z.\end{aligned}$$

[2]Recall that the dual lattice of $\mathcal{T}$ is the lattice whose points are the faces of $\mathcal{T}$ and two points are adjacent if their corresponding faces in $\mathcal{T}$ have a common edge.

Note that we can write

$$\begin{aligned}&\exp\biggl(-\frac{(\|z-x\|_2-2\sqrt{3}/3)^2}{2t}\biggr)-\exp\biggl(-\frac{(\|z-x\|_2+5\delta\sqrt{t})^2}{2t}\biggr)\\&\quad=\biggl(\exp\biggl(\frac{2\sqrt{3}\|z-x\|_2-2}{3t}\biggr)-\exp\biggl(-\frac{(10\|z-x\|_2\delta\sqrt{t}+25\delta^2t)}{2t}\biggr)\biggr)\exp\biggl(-\frac{\|z-x\|_2^2}{2t}\biggr).\end{aligned}$$

Now we use that, for $z\in R_2'$, we have $\|z-x\|_2\le\frac{5\sqrt{2}C}{2}\delta\sqrt{t}+2\sqrt{6}/3$. Then, the first exponential term above is $1+\mathrm{o}(1)$ and, for the second exponential term, we can use the inequality $\mathrm{e}^{-x}\ge 1-x$, which gives, as $t\to\infty$,

$$\begin{aligned}\sum_{v\in P_2}\mu g_t(x,v)&\le\frac{4}{\sqrt{3}}\biggl(\frac{25\sqrt{2}C\delta^2+25\delta^2}{2}+\mathrm{o}(1)\biggr)\int_{R_2'}\frac{1}{2\pi t}\exp\biggl(-\frac{\|z-x\|_2^2}{2t}\biggr)\,\mathrm{d}z\\&\le c_1\sqrt{\delta}+\mathrm{o}(1)\end{aligned}\tag{9}$$

for some universal constant $c_1>0$.

For the terms of (8) where $v\in P_3$ we have that $g_t(x,v)=\frac{f_t(x,v)}{\mathbf{P}(v\text{ is not well behaved})}$. Then, let $R_3'=\mathbb{R}^2\setminus(x+S_{5C\delta\sqrt{t}-4\sqrt{3}/3})$ so that, for each $z\in R_3$, we have $H_z\subset R_3'$, which allows us to write

$$\sum_{v\in P_3}\mu g_t(x,v)\le\frac{4}{\sqrt{3}}\int_{R_3'}\sup_{z'\in H_z}f_t\bigl(x,z'\bigr)\,\mathrm{d}z\le\frac{4}{\sqrt{3}}\int_{R_3'}\frac{1}{2\pi t}\exp\biggl(-\frac{(\|z-x\|_2-2\sqrt{3}/3)^2}{2t}\biggr)\,\mathrm{d}z.$$

Now, letting $w=z-x$ and writing $w=(w_1,w_2)$ we have that

$$\bigl(\|w\|_2-2\sqrt{3}/3\bigr)^2\ge\bigl(|w_1|-2\sqrt{3}/3\bigr)^2+\bigl(|w_2|-2\sqrt{3}/3\bigr)^2-4/3,$$

which we use to bound above $\sum_{v\in P_3}\mu g_t(x,v)$ by

$$\frac{4}{\sqrt{3}}\exp\biggl(\frac{2}{3t}\biggr)\int_{(w_1,w_2)\notin S_{5C\delta\sqrt{t}-4\sqrt{3}/3}}\frac{1}{2\pi t}\exp\biggl(-\frac{(|w_1|-2\sqrt{3}/3)^2}{2t}\biggr)\exp\biggl(-\frac{(|w_2|-2\sqrt{3}/3)^2}{2t}\biggr)\,\mathrm{d}w_1\,\mathrm{d}w_2.\tag{10}$$

We break the integral above into two parts. The first part contains the terms for which either w_1 or w_2 is in $[-2\sqrt{3}/3,2\sqrt{3}/3]$, which can be bounded above by

$$\begin{aligned}&4\int_{-2\sqrt{3}/3}^{2\sqrt{3}/3}\int_{5C\delta\sqrt{t}/2-2\sqrt{3}/3}^{\infty}\frac{1}{2\pi t}\exp\biggl(-\frac{(w_1-2\sqrt{3}/3)^2}{2t}\biggr)\,\mathrm{d}w_1\,\mathrm{d}w_2\\&\quad=\frac{16\sqrt{3}}{3\sqrt{2\pi t}}\int_{5C\delta\sqrt{t}/2-2\sqrt{3}/3}^{\infty}\frac{1}{\sqrt{2\pi t}}\exp\biggl(-\frac{(w_1-2\sqrt{3}/3)^2}{2t}\biggr)\,\mathrm{d}w_1=\mathrm{o}(1).\end{aligned}\tag{11}$$

The second part we further split into two pieces: the first for $|w_1|\ge\frac{2\sqrt{3}}{3}$ and $|w_2|\ge\frac{5C\delta\sqrt{t}}{2}-\frac{2\sqrt{3}}{3}$, and the other for the converse, i.e., $|w_1|\ge\frac{5C\delta\sqrt{t}}{2}-\frac{2\sqrt{3}}{3}$ and $|w_2|\ge\frac{2\sqrt{3}}{3}$. Then we can bound above the second part by

$$\begin{aligned}&2\biggl(4\int_{2\sqrt{3}/3}^{\infty}\int_{5C\delta\sqrt{t}/2-2\sqrt{3}/3}^{\infty}\frac{1}{2\pi t}\exp\biggl(-\frac{(w_1-2\sqrt{3}/3)^2}{2t}\biggr)\exp\biggl(-\frac{(w_2-2\sqrt{3}/3)^2}{2t}\biggr)\,\mathrm{d}w_1\,\mathrm{d}w_2\biggr)\\&\quad=4\int_{5C\delta\sqrt{t}/2-2\sqrt{3}/3}^{\infty}\frac{1}{\sqrt{2\pi t}}\exp\biggl(-\frac{(w_1-2\sqrt{3}/3)^2}{2t}\biggr)\,\mathrm{d}w_1\le\frac{c_2}{C\delta}+\mathrm{o}(1)\end{aligned}\tag{12}$$

for some constant $c_2 > 0$. Summing (11) and (12), we obtain an upper bound for the integral in (10), which establishes the bound

$$\sum_{v \in P_3} \mu g_t(x, v) \le \frac{c_2}{C\delta} + \mathrm{o}(1). \tag{13}$$

Finally, for the terms in (8) with $v \in P_1$, we use that $\mu g_t(v, x) \le 2 f_t(v, x) \le \frac{1}{\pi t}$ for all v, x which gives that

$$\sum_{v \in P_1} \mu g_t(x, v) \le \frac{1}{\pi t} \frac{2}{\sqrt{3}} \left(5\delta\sqrt{t} + 4\frac{\sqrt{3}}{3}\right)^2 \le c_5 \delta^2 + \mathrm{o}(1) \tag{14}$$

for some universal constant $c_5 > 0$ and where $\frac{2}{\sqrt{3}}(5\delta\sqrt{t} + 4\frac{\sqrt{3}}{3})^2$ is an upper bound for the number of points in P_1.

Plugging (9), (13) and (14) into (8) yields

$$\Lambda(x) = \sum_{v \in \Pi_0} \mu g_t(x, v) \le c_1 \sqrt{\delta} + \frac{c_4}{C\delta} + c_5 \delta^2 + \mathrm{o}(1).$$

□

We now proceed to the proof of Theorem 2.2.

Proof of Theorem 2.2. We first prove this theorem for the case when E_i is the trivial event that holds regardless of Φ, which is precise Theorem 1.3. Then, at the end, we prove Theorem 2.2. We start by giving a high-level overview of the proof. First, we assume that all nodes of Π_0 are well behaved. Then we consider a hexagon Q_i of $\mathcal{H}$ and count the number of such well behaved nodes that are inside Q_i at time t. Note that, by the definition of well behaved nodes, given that a node is in Q_i at time t, then its location is uniformly random in Q_i. Therefore, in order to show that they are stochastically dominated by a Poisson point process, it suffices to show that there are at most as many nodes of Π_0 in Q_i at time t as nodes of the Poisson point process. This will happen with a probability that can be made arbitrarily large by setting t large enough. We then use the fact that, since nodes are considered well behaved, a node can only be in Q_i at time t if that node was inside a hexagon of J_i at time 0. Therefore, if we consider a hexagon Q_j such that $J_i \cap J_j = \varnothing$, we have that the well behaved nodes that are able to be in Q_i at time t cannot end up in Q_j. Hence, the event that the well behaved nodes in Q_i are stochastically dominated by a Poisson point process is independent of the event that the nodes in Q_j are stochastically dominated by a Poisson point process. This bounded dependency is enough to complete the analysis of well behaved nodes. On the other hand, to handle nodes that are not well behaved, we add a discrete Poisson point process at each node of Π_0 so that the probability that we add at least one node at a given $v \in \Pi_0$ is exactly the same as the probability that v is not well behaved. Thus, this discrete Poisson point process contains the set of nodes that are not well behaved. We then use Lemma 2.5 to conclude that the nodes that are not well behaved are stochastically dominated by a Poisson point process, which concludes the proof.

We now proceed to the rigorous argument. For each $v \in \Pi_0$, let $\xi'_v(t)$ be the position of v at time t given that v is well behaved, and let

$$\Pi'_t = \bigcup_{v \in \Pi_0} \xi'_v(t).$$

Note that, since $\mathrm{e}^{-\mu}$ is the probability that a node is well behaved and $\Psi_0(\mu)$ is the point process obtained by adding a random number of nodes to the points of Π_0 according to a Poisson random variable with mean μ, then there exists a coupling so that

$$\Pi_t \subseteq \Pi'_t \cup \Psi_t(\mu).$$

Lemma 2.5 establishes that $\Psi_t(\mu)$ is stochastically dominated by a Poisson point process of intensity $c_1\sqrt{\delta}$ for some universal constant $c_1 > 0$. It remains to show that Π'_t is also stochastically dominated by a Poisson point process. Unfortunately, this is not true in the whole of $\mathbb{R}^2$ as shown in Lemma 2.1. We will then consider the tessellation given by $\mathcal{H}$ and show that, for each hexagon Q_i of the tessellation with $X_i = 1$, where the X_i are defined in the paragraph preceding Theorem 1.3, Π'_t is stochastically dominated by a Poisson point process $\widetilde{\Pi}$ of intensity $(1 + \sqrt{\delta})2/\sqrt{3}$.

In order to see this, for each $i \in I$, we define a binary random variable Y_i, which is 1 if $\widetilde{\Pi}$ has more nodes in Q_i than Π'_t. Then, since each node of Π'_t is well behaved, whenever $Y_i = 1$, we can couple $\widetilde{\Pi}$ with Π'_t such that $\widetilde{\Pi} \supseteq \Pi'_t$ in Q_i. First we derive a bound for the number of nodes of Π'_t inside Q_i. For each $v \in \Pi_0$, let Z_v be the indicator random variable for $\xi'_v(t) \in Q_i$. Then, since the probability that $\xi'_v(t) \in Q_i$ is proportional to $\varphi_t(q(v), i)$, the expected number of nodes of Π'_t in Q_i is

$$\sum_{v \in \Pi_0 \cap (\bigcup_{j \in J_i} Q_j)} \mathbf{E}[Z_v] = \sum_{v \in \Pi_0 \cap (\bigcup_{j \in J_i} Q_j)} \frac{\varphi_t(q(v), i)}{M} = \sum_{v \in \Pi_0 \cap (\bigcup_{j \in J_i} Q_j)} \frac{\varphi_t(i, q(v))}{M},$$

where the last step follows by symmetry of φ_t, and M is a normalizing constant so that

$$\sum_j \varphi_t(i, j) = \sum_{j \in J_i} \varphi_t(i, j) = M \quad \text{for all } i.$$

Since the density of points of Π_0 per unit volume is $\frac{2}{\sqrt{3}}$ and the area of Q_i is $\frac{3\sqrt{3}}{2}\delta^2 t$, we have that the number of points of Π_0 in Q_j is $3\delta^2 t$ for any j. Also, note that for $v, v' \in \Pi_0 \cap Q_j$ we have $\varphi_t(i, q(v)) = \varphi_t(i, q(v'))$. Therefore, we obtain

$$\sum_{v \in \Pi_0 \cap (\bigcup_{j \in J_i} Q_j)} \mathbf{E}[Z_v] = \frac{3\delta^2 t}{M} \sum_{j \in J_i} \varphi_t(i, j) = 3\delta^2 t.$$

A simpler way to establish the equation above is using symmetry, because $3\delta^2 t$ is the number of points of Π_0 in Q_i. Since the random variables Z_v are mutually independent, we can apply a Chernoff bound for binomial random variables (cf. Lemma A.2) to get

$$\mathbf{P}\Biggl(\sum_{v \in \Pi_0 \cap (\bigcup_{j \in J_i} Q_j)} Z_v \geq (1 + \sqrt{\delta}/2) 3\delta^2 t\Biggr) \leq \exp\biggl(-\frac{2(\sqrt{\delta}/2)^2 (3\delta^2 t)^2}{3\delta^2 t |J_i|}\biggr) \leq \exp\biggl(-\frac{9\delta^3 t}{8C^2}\biggr),$$

where the last step follows from Lemma 2.4. Using a standard Chernoff bound for Poisson random variables (cf. Lemma A.1) we have

$$\mathbf{P}\bigl(\widetilde{\Pi} \text{ has less than } (1 + \sqrt{\delta}/2) 3\delta^2 t \text{ nodes in } Q_i\bigr) \leq \exp\biggl(-\frac{\delta(3\delta^2 t)}{8(1 + \sqrt{\delta})}\biggr).$$

Therefore, we obtain a constant c_2 such that

$$\mathbf{P}(Y_i = 1) \geq 1 - \exp\biggl(-\frac{c_2 \delta^3 t}{C^2}\biggr). \tag{15}$$

The random variables Y are not mutually independent. However, note that Y_i depends only on the random variables $Y_{i'}$ for which $J_{i'} \cap J_i \neq \emptyset$. This is because, for any $i \in I$, only the nodes that are inside hexagons Q_j with $j \in J_i$ can contribute to Y_i. Therefore, using Lemma 2.4, we have that Y_i depends on at most $(\frac{4}{3}C^2)^2$ other random variables Y. By having t large enough, we can make the bound in (15) be arbitrarily close to 1. This allows us to apply a result of Liggett, Schonmann and Stacey [7], Theorem 1.3, which gives that the random field $(Y_i)_{i \in I}$ stochastically dominates a field $(Y'_i)_{i \in I}$ of independent Bernoulli random variables satisfying

$$\mathbf{P}\bigl(Y'_i = 1\bigr) \geq 1 - \exp\biggl(-\frac{c_3 \delta^3 t}{C^6}\biggr)$$

for some positive constant c_3. So, with t sufficiently large, we can assure that $\mathbf{P}(Y'_i = 1)$ is larger than p in the statement of Theorem 2.2. Then, we have that, whenever $Y'_i = 1$, the Poisson point process $\widetilde{\Pi} \cup \Psi_t(\mu)$ stochastically dominates Π_t inside Q_i. Since $\widetilde{\Pi}$ and $\Psi_t(\mu)$ are independent Poisson point processes, we have that their union is also a Poisson point process of intensity no larger than

$$\frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}}\sqrt{\delta} + c_1\sqrt{\delta}. \tag{16}$$

So, by setting c properly in the definition of Φ, we can couple Φ and $\widetilde{\Pi} \cup \Psi_t(\mu)$ so that $\Phi \supseteq \widetilde{\Pi} \cup \Psi_t(\mu)$. This completes the proof for the case where E_i holds regardless of Φ. For the other case, let $\widetilde{Y}_i = Y_i \mathbf{1}\{E_i\}$. Then, for any $\varepsilon > 0$, we can set t large enough so that $\mathbf{P}(\widetilde{Y}_i = 1) \geq 1 - \mathbf{P}(Y_i = 0) - \mathbf{P}(E_i^{\mathrm{c}}) \geq 1 - \varepsilon$. Since E_i depends only on the E_j for which $j \in J_i$, we apply [7], Theorem 1.3 as before to obtain that the random field $(\widetilde{Y}_i)_{i\in I}$ stochastically dominates a field $(\widetilde{Y}'_i)_{i\in I}$ of independent Bernoulli random variables satisfying

$$\mathbf{P}\bigl(\widetilde{Y}'_i = 1\bigr) \geq 1 - \varepsilon',$$

where ε' can be made arbitrarily close to 0 by setting ε arbitrarily small. Hence we can have $1 - \varepsilon' \geq p$. With this, whenever $\widetilde{Y}'_i = 1$ we have that Φ stochastically dominates Π_t in Q_i and E_i holds, completing the proof. □

3. Large time

In this section we give the proofs of Theorems 1.1 and 1.2. Both proofs use Theorem 2.2.

Proof of Theorem 1.1. For each $i \in I$, let N_i be the set of hexagons Q_j such that Q_i and Q_j intersect. Now, let E_i be the event that the largest component of $R(\Phi \cap (\bigcup_{j\in N_i} Q_j))$ has diameter smaller than $\delta\sqrt{t}/10$. Clearly, E_i is a decreasing event. If $\lambda_{\mathrm{c}} > \frac{2}{\sqrt{3}}$, we can set δ small enough so that the intensity of Φ is smaller than λ_{c}. Then, using the exponential decay of the radii of clusters in subcritical Boolean model [11], Theorem 10.1, we obtain that $\mathbf{P}(E_i) \to 1$ as $t \to \infty$. Note that, whenever E_i holds, the region $R(\Phi)$ does not have a component that intersects two non-adjacent edges of hexagon Q_i. By setting p larger than the critical value for site percolation on the hexagonal lattice, and using Theorem 2.2, we obtain that the set of hexagons for which E_i does *not* hold has only finite connected components. Therefore, any such component must be surrounded by hexagons j for which E_j holds and $\Pi_t \cap Q_j \subseteq \Phi \cap Q_j$; hence, Q_j is not crossed by $R(\Pi_t)$. This gives that all components of $R(\Pi_t)$ are finite. □

Proof of Theorem 1.2. Let r_{c}^{λ} be the critical radius for percolation of the Boolean model with intensity λ. Let $\lambda_0 = \frac{2}{\sqrt{3}} + c\sqrt{\delta}$, where c is the constant in Theorem 2.2 so that λ_0 is an upper bound for the intensity of Φ. Then, as in the proof of Theorem 1.1, let N_i be the set of hexagons Q_j such that Q_i and Q_j intersect, and define E_i to be the event that adding balls of radius $r < r_c^{\lambda_0}$ centered at the nodes of Φ does not create a component with diameter larger than $\delta\sqrt{t}/10$ inside $\bigcup_{j\in N_i} Q_j$. Therefore, Theorem 2.2 with p larger than the critical value for site percolation on the hexagonal lattice implies that, if t is large enough, the union of the balls do not have an infinite component in the whole of $\mathbb{R}^2$. This implies that

$$\liminf_{t\to\infty} r_{\mathrm{c}}(t) \geq r_{\mathrm{c}}^{\lambda_0}.$$

Now it remain to relate $r_{\mathrm{c}}^{\lambda_0}$ with $r_{\mathrm{c}}^{2/\sqrt{3}}$. For this, we note that the Boolean model with intensity λ and radius r is equivalent (up to scaling) to the Boolean model with intensity $\bar{\lambda}$ and $\bar{r}$ provided $\lambda r^2 = \bar{\lambda}\bar{r}^2$. Therefore, for any $\varepsilon > 0$, we have that

$$r_{\mathrm{c}}^{\lambda+\varepsilon} = r_{\mathrm{c}}^{\lambda}\sqrt{\frac{\lambda}{\lambda+\varepsilon}}. \tag{17}$$

Using this we obtain that, for any $\delta > 0$,

$$\liminf_{t\to\infty} r_{\mathrm{c}}(t) \geq r_{\mathrm{c}}^{\lambda_0} = r_{\mathrm{c}}^{2/\sqrt{3}}\sqrt{\frac{2/\sqrt{3}}{2\sqrt{3}+c\sqrt{\delta}}},$$

where $c>0$ is a universal constant. Since δ can be arbitrarily close to 0, we obtain

$$\liminf_{t\to\infty} r_{\mathrm{c}}(t) \ge r_{\mathrm{c}}^{2/\sqrt{3}}. \tag{18}$$

Now, to obtain an upper bound for $r_{\mathrm{c}}(t)$, we use a result of Sinclair and Stauffer [14], Proposition 4.1 adapted to the hexagonal tessellation. We state it in the proposition below and note that this version is a special case of [15], Proposition 2.1. Here we use the sets J_i, defined in (1).

Proposition 3.1 ([14], Proposition 4.1). *Let $i \in I$ be fixed. Take S to be the union of the hexagons in J_i, and let S' be Q_i. Let Ξ be a Poisson point process with intensity $(1-\sqrt{\delta})2/\sqrt{3}$. Then, for any fixed $\delta>0$, there exists $t_0>0$ so that, for all $t>t_0$, we can couple the nodes of Π_t and Ξ in such a way that the nodes of Π_t in S' that were in S at time 0 contain the nodes of Ξ in S' with probability at least $1-\exp(-c\delta^3 t)$ for some universal positive constant c.*

Now, when the coupling of Proposition 3.1 occurs, the nodes of Π_t that are inside Q_i at time t and were inside $\bigcup_{j\in J_i} Q_j$ at time 0 stochastically dominate a Poisson point process Φ of intensity $(1-\sqrt{\delta})\frac{2}{\sqrt{3}}$. When this happens, let $Y_i=1$; otherwise we set $Y_i=0$.

For each i with $Y_i=1$, let K_i be the point process of the nodes of Π_t inside Q_i that were in $\bigcup_{j\in J_i} Q_j$ at time 0; if $Y_i=0$ set $K_i=\varnothing$. We add balls of radius $r>r_{\mathrm{c}}^{(2/\sqrt{3})(1-\sqrt{\delta})}$ centered at the nodes of $\bigcup_{i\in I} K_i$. Let $\widetilde{Y}_i$ be 1 if the largest component of the region occupied by the balls in $\bigcup_{j\in N_i} Q_j$, which we denote by X, is such that $\bigcup_{j\in N_i} Q_j \setminus X$ contains only components of diameter smaller than $\delta\sqrt{t}/10$. Otherwise, we set $\widetilde{Y}_i=0$. Note that if there exists a path of indices $j_1, j_2, \ldots$ such that, for all k, $\widetilde{Y}_{j_k}=1$ and hexagons Q_{j_k} and $Q_{j_{k+1}}$ intersect, then the region occupied by the balls in $Q_{j_1}\cup Q_{j_2}\cup\cdots$ has a connected component that intersects each hexagon Q_{j_k}. Then, using Proposition 3.1 and [12], Theorem 2, we obtain that, for any given $i\in I$, $\mathbf{P}(\widetilde{Y}_i=1)\to 1$ as $t\to\infty$. Clearly, $\widetilde{Y}_i$ depends only on the $\widetilde{Y}_j$ such that $\bigcup_{k\in N_i} J_k$ and $\bigcup_{k\in N_k} J_k$ intersect. Then, using the upper bound on the size of J_i from Lemma 2.4, and applying [7], Theorem 1.3, we obtain that the random field $(\widetilde{Y}_i)_i$ stochastically dominates a field of independent Bernoulli random variables $(Y_i')_i$ such that $\mathbf{P}(Y_i'=1)$ is larger than the critical value for site percolation in the hexagonal lattice. Consequently, since $\bigcup_{i\in I} K_i \subseteq \Pi_t$, the union of balls of radius $r>r_{\mathrm{c}}^{(2/\sqrt{3})(1-\sqrt{\delta})}$ centered at the nodes of Π_t produces an infinite component almost surely. By the scaling argument in (17) we have that $r_{\mathrm{c}}^{(2/\sqrt{3})(1-\sqrt{\delta})} = r_{\mathrm{c}}^{2/\sqrt{3}}\sqrt{\frac{2/\sqrt{3}}{2/\sqrt{3}-2\sqrt{\delta/3}}}$, which finally yields

$$\limsup_{t\to\infty} r_{\mathrm{c}}(t) \le r_{\mathrm{c}}^{(2/\sqrt{3})(1-\sqrt{\delta})} = r_{\mathrm{c}}^{2/\sqrt{3}}\sqrt{\frac{1}{1-\sqrt{\delta}}}.$$

Since δ can be arbitrarily close to 0, we obtain $\limsup_{t\to\infty} r_{\mathrm{c}}(t)\le r_{\mathrm{c}}^{2/\sqrt{3}}$, which together with (18) concludes the proof of Theorem 1.2. □

4. Short time

Now we turn our attention to the case when t is sufficiently small. We establish that, given a Monte Carlo estimate, $R(\Pi_t)$ contains an infinite component almost surely for sufficiently small t.

Consider a tessellation of $\mathbb{R}^2$ into regular hexagons of side length 50. We will denote this tessellation by $\mathcal{H}_{50}$. Instead of considering the usual tessellation, where each hexagon is obtained by the union of some triangles of $\mathcal{T}$, we will shift the hexagonal tessellation (see the illustration in Fig. 1) so that no node of Π_0 is on an edge or vertex of $\mathcal{H}_{50}$, and the edges of $\mathcal{H}_{50}$ intersect as many of the balls centered at Π_0 as possible. More formally, since a transitive lattice can be specified by a single edge, we define $\mathcal{T}$ as the triangular lattice containing an edge between the points $(0,0)$ and $(1,0)$, and for any $\ell>0$, we let $\mathcal{H}_\ell$ be the hexagonal lattice containing an edge between $(1/2,-\sqrt{3}/4)$ and $(\ell+1/2,-\sqrt{3}/4)$.

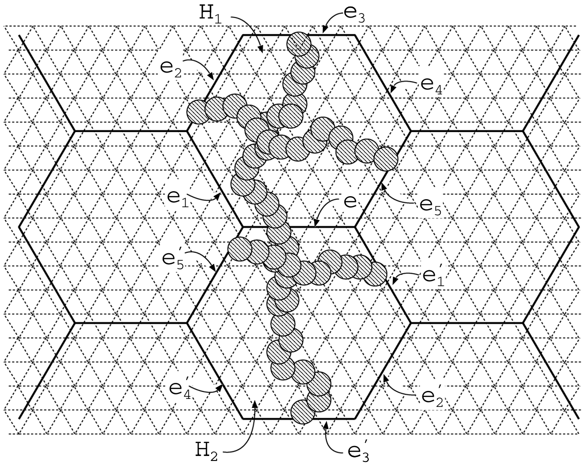

Fig. 1. The hexagonal tessellation $\mathcal{H}_4$ of $\mathbb{R}^2$ with hexagons of side length 4, and the illustration of a path of intersecting balls crossing H_1 and H_2.

Let H_1 and H_2 be two hexagons of $\mathcal{H}_{50}$ that have one edge in common, and denote this edge by e. Starting from e, denote the other edges of H_1 in clockwise direction by e_1, e_2, e_3, e_4, e_5; thus e_3 is the edge of H_1 opposite to e. Similarly, denote the other edges of H_2 in clockwise direction by $e'_1, e'_2, e'_3, e'_4, e'_5$ (refer to Fig. 1). Given any three sets $X_1, X_2, X_3 \subset \mathbb{R}^2$ and any $t > 0$, we say that $R(\Pi_t)$ has a path from X_1 to X_2 inside X_3 if there exists a sequence of nodes $u_1, u_2, \dots, u_k$ of Π_t, all of which inside X_3, such that $B(u_1, 1/2)$ intersects X_1, $B(u_k, 1/2)$ intersects X_2, and for each $i \geq 1$, the distance between u_i and u_{i+1} is at most 1. With this, we say that $R(\Pi_t)$ *crosses* H_1 and H_2 if the following three conditions hold:

1. $R(\Pi_t)$ has a path from e_3 to e'_3 inside $H_1 \cup H_2$.
2. $R(\Pi_t)$ has a path from $e_1 \cup e_2$ to $e_4 \cup e_5$ inside $H_1 \cup H_2$.
3. $R(\Pi_t)$ has a path from $e'_1 \cup e'_2$ to $e'_4 \cup e'_5$ inside $H_1 \cup H_2$.

We denote by A_t the event that $R(\Pi_t)$ crosses H_1 and H_2 with paths that also crosses H_1 and H_2 at time 0. More formally, assume that $R(\Pi_t)$ crosses H_1 and H_2 and that, for $i = 1, 2, 3$, $u_1^{(i)}, u_2^{(i)}, \dots, u_{k_i}^{(i)}$ is the path of vertices of Π_t that establishes condition i above. Then, if and only if $u_1^{(i)}, u_2^{(i)}, \dots, u_{k_i}^{(i)}$ is also a path of Π_0 establishing condition i for time 0 and all $i = 1, 2, 3$, we have that A_t holds. With this, we have the following result.

Theorem 4.1. *Suppose that there exists an* $\varepsilon_0 > 0$ *such that* $\mathbf{P}(A_{\varepsilon_0}) > 0.8639$. *Then, for all* $\varepsilon \in [0, \varepsilon_0]$, *it holds that* $R(\Pi_\varepsilon)$ *contains an infinite connected component almost surely.*

Note that, for any fixed t, verifying the condition $\mathbf{P}(A_t) > 0.8639$ resorts to solving a finite, but high dimensional integral describing the crossing probability. We were able to check the validity of this condition for $t = 0.01$ via a Monte Carlo analysis[3] with confidence 99.99%.

We start the proof with the lemma below. The application of this lemma is the main reason why we require that the paths that cause $R(\Pi_t)$ to cross H_1 and H_2 are also paths crossing H_1 and H_2 at time 0.

Lemma 4.2. *We have that* $\mathbf{P}(A_t)$ *is non-increasing with* t.

Proof. This follows by Brownian scale. For each $u \in \Pi_0$, let $(\zeta'_u(s))_s$ be a Brownian motion independent over different u. Then, consider $s' > s$, and let $\Pi'_{s'}$ be the point process $\{u + \zeta'_u(s') \colon u \in \Pi'_0\}$. Note that $\Pi'_{s'}$ has the same

[3]To obtain this Monte Carlo estimate we employed the Mersenne Twister pseudorandom number generator by Matsumoto and Nishimura [8] with period $2^{19,937} - 1$ and improved initialization scheme from January 26th, 2002.

distribution as $\Pi_{s'}$. By Brownian scale, we can couple $\zeta_u(s)$ and $\zeta'_u(s')$ via $\zeta_u(s) = \sqrt{s/s'}\zeta'_u(s')$. Let $u, v \in \Pi_0$ be such that

$$\|u - v\|_2 \le 1 \quad \text{and} \quad \left\|u + \zeta'_u(s') - v - \zeta'_v(s')\right\|_2 \le 1. \tag{19}$$

Then the distance between u and v in Π_s is

$$\left\|u + \zeta_u(s) - v - \zeta_v(s)\right\|_2 = \left\|u - v + \sqrt{s/s'}\left(\zeta'_u(s') - \zeta'_v(s')\right)\right\|_2 \le 1.$$

The last step follows since, by (19), $\|u - v + \gamma(\zeta'_u(s') - \zeta'_v(s'))\|_2 \le 1$ for $\gamma \in \{0, 1\}$ and $\sqrt{s/s'} \in (0, 1)$. This implies that, for any pair $u, v \in \Pi_0$, if the balls centered at u and v intersect in Π_0 and the balls centered at $u + \zeta'_u(s')$ and $v + \zeta'_v(s')$ intersect in $\Pi'_{s'}$, then the balls centered at $u + \zeta_u(s)$ and $v + \zeta_v(s)$ also intersect in Π_s. Hence $\mathbf{P}(A_s) \ge \mathbf{P}(A_{s'})$. □

Now we proceed to the proof of Theorem 4.1.

Proof of Theorem 4.1. We prove that if there exists an $\varepsilon_0 > 0$ such that $\mathbf{P}(A_{\varepsilon_0}) > 0.8639$, then $R(\Pi_{\varepsilon_0})$ contains an infinite connected component almost surely. Together with the definition of A_t and Lemma 4.2, this establishes that $R(\Pi_\varepsilon)$ contains an infinite connected component almost surely for all $\varepsilon \in [0, \varepsilon_0]$.

We henceforth fix a value of ε and assume that $\mathbf{P}(A_\varepsilon) > 0.8639$. We will use a renormalization argument. Consider the hexagons $\mathcal{H}_{50}$ described in the beginning of this section. Now, define the graph $L = (U, F)$ such that U is the set of points given by the centers of the hexagons and F is the set of edges between every pair of points $i, j \in U$ for which the hexagons with centers at i and j share an edge. Note that L consists of a scaling of the triangular lattice.

We now define a collection of random variables X_i for each edge $i \in F$. In order to explain the process defining X_i, let H_1 and H_2 be the hexagons whose centers are the endpoints of i. We then define $X_i = 1$ if and only if $R(\Pi_\varepsilon)$ crosses H_1 and H_2 with a path of balls that also crosses H_1 and H_2 at time 0. (The definition of crossings is given right before the statement of Theorem 4.1.) Let j be an edge such that i and j are disjoint, and let H_3 and H_4 denote the hexagons centered at the endpoints of j. Clearly, X_i and X_j are independent since the set of balls crossing H_1 and H_2 at time 0 does not intersect the set of balls crossing H_3 and H_4 at time 0. Thus, the collection $(X_i)_i$ is a so-called 1-dependent bond percolation process, with $\mathbf{P}(X_i = 1) = \mathbf{P}(A_\varepsilon) > 0.8639$. Then, we can use a result of Balister, Bollobás and Walters [2], Theorem 2, which gives that any 1-dependent bond percolation process on the square lattice with marginal probability larger than 0.8639 percolates almost surely. Since the triangular lattice contains the square lattice, we obtain that, almost surely, there exists an infinite path of consecutive edges of F with $X_i = 1$ for all i in the path.

To conclude the proof, note that, for two non-disjoint edges i and j with $X_i = X_j = 1$, we have that the crossings of the hexagons whose centers are located at the endpoints of i and j intersect. Then, the infinite path of X_i with $X_i = 1$ for all i gives an infinite connected region inside $R(\Pi_\varepsilon)$, which concludes the proof of Theorem 4.1. □

5. Extensions and open problems

In the remaining of this section we discuss extensions and open problems regarding other circle packings (Section 5.1), balls moving over graphs (Section 5.2) and critical radius for non-mobile point processes (Section 5.3).

5.1. *Other circle packings*

Let Π_0^{s} be the point process given by the vertices of the square lattice with side length 1, and let Π_t^{s} be the point process obtained by letting the nodes of Π_0^{s} move for time t according to independent Brownian motions. Note that, for any $\varepsilon > 0$, if we look at two balls of radius $1/2$ centered at two adjacent nodes of Π_0^{s}, then at time ε, the probability that these two balls intersect is strictly smaller than $1/2$, which is the critical probability for bond percolation on the square lattice [5]. This motivates our next conjecture.

Conjecture 5.1. *For any $\varepsilon > 0$, it holds that, almost surely, all components of $R(\Pi_\varepsilon^{\mathrm{s}})$ are finite.*

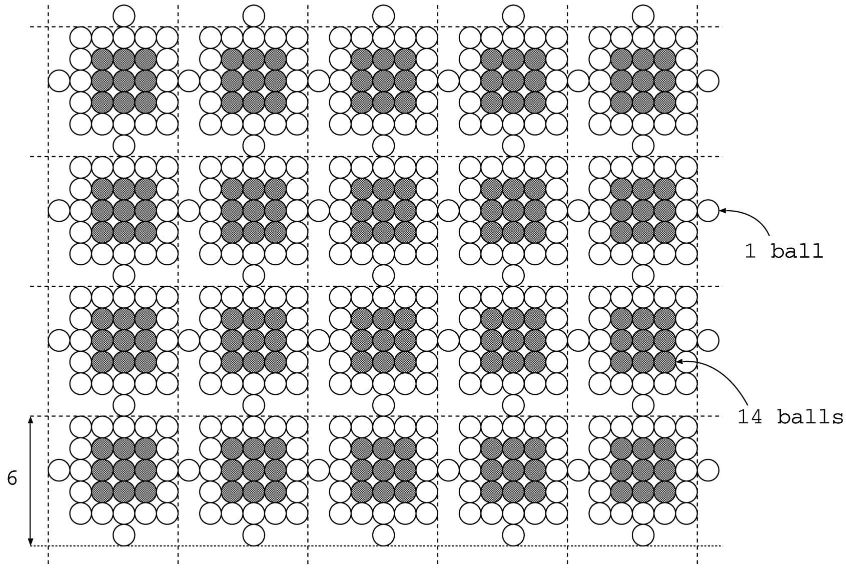

Fig. 2. An example of a non-transitive configuration of balls that is not monotone. Each solid ball represents the superposition of 14 balls and white balls represent single balls.

Now we consider the question of whether percolation is a *monotone* property. We say that a point process Π_0 is transitive if, for every two nodes $v, v' \in \Pi_0$, there exists an isometry $f\colon \Pi_0 \to \Pi_0$ such that $f(v) = v'$. The open problem below concerns the question of whether transitivity is enough to obtain monotonicity in the percolation properties of balls moving as Brownian motion.

Question 5.2. *Let Π_0 be a transitive point process so that $R(\Pi_0)$ is a connected set. Let Π_t be obtained from Π_0 by letting the nodes move as independent Brownian motions for time t. Then, if for some time t_0 we have that $R(\Pi_{t_0})$ has an infinite component almost surely, then, is it true that, for any $t < t_0$, $R(\Pi_t)$ also has an infinite component almost surely? Similarly, if for some t_1 we have that $R(\Pi_{t_1})$ contains only finite components almost surely, then, does it hold that, for any $t > t_1$, $R(\Pi_t)$ also contains only finite components almost surely?*

Remark 5.3. *Question 5.2 above is false if we drop the condition that Π_0 is transitive. For example, consider a tessellation of $\mathbb{R}^2$ into squares of side length 6 and, in each square of the tessellation, consider the configuration of balls illustrated in Fig. 2, where each ball has radius $1/2$, solid balls represent the superposition of 14 balls and white balls represent a single ball. It is easy to see that, at a sufficiently small time ε, the union of the balls will not contain an infinite component almost surely. However, the density of balls is equal to $\frac{9\times 14+18}{36} = 4$ and, as the balls move for a sufficiently large amount of time, their position will approach a Poisson point process which is known to percolate.*

Remark 5.4. *The following example, which was pointed out to us by an anonymous referee, illustrates the importance of the condition that $R(\Pi_0)$ is connected in Question 5.2. Let Π_0 be the points of the triangular lattice with edge length $1+\varepsilon$, for some sufficiently small $\varepsilon > 0$. Then, $R(\Pi_0)$ does not percolate. Also, for sufficiently small ε, our Monte Carlo estimate in Section 4 and Theorem 4.1 suggest that there exist an interval $[t_1, t_2]$ such that for all $t \in [t_1, t_2]$ we have that $R(\Pi_t)$ percolates almost surely. Then, for ε small enough and all t sufficiently large, our Theorem 1.1 and the Monte Carlo estimate of Balister, Bollobás and Walters [2] suggest that $R(\Pi_t)$ does not percolate almost surely.*

5.2. *Motion over graphs*

We now consider the case when the motion of the nodes is more restricted. First, let Π_0 be the point process given by the integer points of $\mathbb{R}$. For any node $u \in \Pi_0$, we let $u + \zeta_u(t)$ be its position at time t, where $(\zeta_u(t))_t$ is a one-dimensional Brownian motion. Now, consider a sequence of m distinct nodes $u_1, u_2, \dots, u_m$ such that $B(u_i, 1/2)$ and

$B(u_{i+1}, 1/2)$ intersect for all i. We call such a sequence of nodes as a *path*. Let ε be a sufficiently small positive constant, and consider only the nodes of Π_0 whose displacement from time 0 to time ε is smaller than $1/2$; we denote these nodes as *good* nodes. We claim that

$$\mathbf{P}(u_1, u_2, \ldots, u_m \text{ form a path at time } \varepsilon | u_i \text{ is good for all } i) = \frac{1}{m!}. \tag{20}$$

In order to see this, suppose, without loss of generality, that $u_1 < u_2 < \cdots < u_m$. For each node $u \in \Pi_0$, let $\zeta'_u(\varepsilon)$ be the displacement of u from time 0 to ε given that u is a good node. Then, in order for $B(u_1 + \zeta_{u_1}(\varepsilon), 1/2)$ to intersect $B(u_2 + \zeta_{u_2}(\varepsilon), 1/2)$ we need that $|u_1 + \zeta'_{u_1}(\varepsilon) - u_2 - \zeta'_{u_2}(\varepsilon)| \le 1$. Since u_1 and u_2 are good nodes, this condition translates to $u_2 + \zeta'_{u_2}(\varepsilon) - u_1 - \zeta'_{u_1}(\varepsilon) \le 1$, which in turn implies that $\zeta'_{u_1}(\varepsilon) \ge \zeta'_{u_2}(\varepsilon)$. Repeating this argument, we obtain the condition $\zeta'_{u_1}(\varepsilon) \ge \zeta'_{u_2}(\varepsilon) \ge \zeta'_{u_3}(\varepsilon) \ge \cdots \ge \zeta'_{u_m}(\varepsilon)$. Since the ζ' are independent and identically distributed, we have that $\mathbf{P}(\zeta'_{u_1}(\varepsilon) \ge \zeta'_{u_2}(\varepsilon) \ge \cdots \ge \zeta'_{u_m}(\varepsilon)) = 1/m!$, which establishes (20).

We now consider a more general scenario. Let G be an infinite graph that is vertex transitive and has bounded degree. We assume that each edge of G has length 1, which gives a metric over G. Let $\Pi_0(G)$ be the point process given by putting one node at each vertex of G and define $\Pi_t(G)$ as the point process obtained by letting the nodes of $\Pi_0(G)$ move for time t along the edges of G according to independent Brownian motions. Then $R(\Pi_t(G))$ is the union of balls centered at the nodes of Π_t and having radius $1/2$ with respect to the metric induced by G. We note that the probability given in (20) for any fixed path $u_1, u_2, \ldots, u_m$ of good nodes to form a path at a time ε that is sufficiently small is at most $1/m!$. This motivates our next conjecture.

Conjecture 5.5. *Let G be an infinite graph that is vertex transitive and has bounded degree. Then, for any $t > 0$, the region $R(\Pi_t(G))$ contains only finite components almost surely.*

5.3. *Critical radius of point processes*

Here we let Π be a point process over $\mathbb{R}^2$ and consider the region $R(\Pi, r)$ as the union of balls of radius r centered at the nodes of Π. In this section, we only consider point processes with unit intensity and let $r_{\mathrm{c}}(\Pi)$ be the smallest r for which $R(\Pi, r)$ contains an infinite component. It is intuitive to believe that point processes that are more organized have smaller critical radius; this is the core of our next conjecture. For more information on zeros of Gaussian analytic functions, we refer to [6].

Conjecture 5.6. *Let Π_{L} be any transitive point process with intensity* 1 (*as defined before Question* 5.2). *Let Π_{GAF} be a point process given by the zeros of a Gaussian analytic function with intensity* 1 *and Π_{P} be a Poisson point process with intensity* 1. *Then*,

$$r_{\mathrm{c}}(\Pi_{\mathrm{L}}) < r_{\mathrm{c}}(\Pi_{\mathrm{GAF}}) < r_{\mathrm{c}}(\Pi_{\mathrm{P}}).$$

Finally, consider a Poisson point process Π with intensity 1 over R^d and let r_{c} be the critical radius for percolation of balls centered at the nodes of Π. Our last open problem concerns small perturbations of the critical radius.

Question 5.7. *Let $\varepsilon > 0$ and, for each node $v \in \Pi$, let X_v be a uniform random variable over $[-\varepsilon, \varepsilon]$. For each node $v \in \Pi$, add a ball of radius $r_{\mathrm{c}} + X_v$ centered at v. Will the union of the balls contain an infinite component almost surely?*

Appendix: Standard large deviation results

We use the following standard Chernoff bounds during our proofs.

Lemma A.1 (Chernoff bound for Poisson). *Let P be a Poisson random variable with mean λ. Then, for any $0 < \varepsilon < 1$,*

$$\mathbf{P}\big(P \ge (1+\varepsilon)\lambda\big) \le \exp\left(-\frac{\lambda\varepsilon^2}{2}(1-\varepsilon/3)\right) \quad \textit{and} \quad \mathbf{P}\big(P \le (1-\varepsilon)\lambda\big) \le \exp\left(-\frac{\lambda\varepsilon^2}{2}\right).$$

Lemma A.2 (Chernoff bound for binomial, see [1], Lemma A.1.4). *Let $X_1, X_2, \dots, X_n$ be independent Bernoulli random variable such that $\mathbf{E}[X_i] = p_i$. Let $X = \sum_{i=1}^n X_i$. Then, for any $\varepsilon > 0$,*

$$\mathbf{P}\big(X \geq (1+\varepsilon)\mathbf{E}[X]\big) \leq \exp\left(-\frac{2\varepsilon^2(\mathbf{E}[X])^2}{n}\right) \quad \textit{and} \quad \mathbf{P}\big(X \leq (1-\varepsilon)\lambda\big) \leq \exp\left(-\frac{\lambda\varepsilon^2}{2}\right).$$

Lemma A.3 (Gaussian tail bound [10], Theorem 12.9). *Let X be a normal random variable with mean 0 and variance σ^2. Then, for any $R \geq \sigma$ we have that $\mathbf{P}(X \geq R) \leq \frac{\sigma}{\sqrt{2\pi}R}\exp(-\frac{R^2}{2\sigma^2})$.*

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