Global bifurcation of rotating vortex patches

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Abstract

We rigorously construct continuous curves of rotating vortex patch solutions to the two-dimensional Euler equations. The curves are large in that, as the parameter tends to infinity, the minimum along the interface of the angular fluid velocity in the rotating frame becomes arbitrarily small. This is consistent with the conjectured existence [WOZ84, Ove86] of singular limiting patches with 90° corners at which the relative fluid velocity vanishes. For solutions close to the disk, we prove that there are "Cat's eyes"-type structures in the flow, and provide numerical evidence that these structures persist along the entire solution curves and are related to the formation of corners. We also show, for any rotating vortex patch, that the boundary is analytic as soon as it is sufficiently regular.

Contents

1 Introduction

1.1 Statement of the main results

We consider the two-dimensional incompressible Euler equations, written in terms of the vorticity ω and stream function ψ as

$$\partial_t \omega + \nabla^\perp \psi \cdot \nabla \omega = 0, \qquad -\Delta \psi = \omega. \tag{1.1}$$

The fluid velocity is $\nabla^\perp \psi = (-\psi_y, \psi_x)$. A vortex patch is a (weak) solution of (1.1) with $\omega(z,t) = 1_{D(t)}(z)$ for some simply-connected region $D(t)$; see, e.g., [MB02, Section 8.3]. As is typically done, we restrict to the case where the fluid is at rest at infinity. We are interested in vortex patches for which, after moving to a (non-inertial) frame rotating with constant angular velocity Ω, the region D is stationary. The fluid velocity in the rotating frame is then $\nabla^\perp \Psi$ where the relative stream function $\Psi = \psi - \frac{1}{2}\Omega|z|^2$ solves

$$\Delta \Psi = 1_D - 2\Omega, \tag{1.2a}$$
$$\nabla(\Psi + \tfrac{1}{2}\Omega|z|^2) \to 0 \text{ as } |z| \to \infty, \tag{1.2b}$$
$$\Psi \in C^1(\mathbb{C}), \tag{1.2c}$$
$$\Psi = 0 \text{ on } \partial D. \tag{1.2d}$$

This is a free boundary problem in that the domain D and the function Ψ are both unknowns. Here and in what follows we identify $(x, y) \in \mathbb{R}^2$ with $z = x + iy \in \mathbb{C}$ whenever convenient.

Somewhat informally stated, our main existence result for (1.2) is the following.

Theorem 1.1. *For any $m \geq 2$, there exists a continuous curve $\mathscr{C}$ of rotating vortex patches with the symmetries of a regular m-gon, parametrized by $s \in [0, \infty)$, with the following properties.*

(a) (Bifurcation from the disk) *The solution at $s = 0$ is the unit disk $D = \mathbb{D}$ rotating with angular velocity $\Omega = (m-1)/2m$ and with the angular fluid velocity $\partial_r \Psi \equiv 1/2m$ on ∂D.*

(b) (Vanishing angular fluid velocity) *As $s \to \infty$,*

$$\min_{\partial D} \partial_r \Psi \to 0, \tag{1.3}$$

i.e. there are points $z(s) \in \partial D(s)$ where the angular fluid velocity becomes arbitrarily small.

(c) (Monotonicity) *For each $s > 0$, the boundary of the patch can be expressed as a polar graph $r = R(\theta)$ where R is even, $2\pi/m$-periodic, and satisfies*

$$R'(\theta) < 0 \text{ for } 0 < \theta < \frac{\pi}{m}, \quad R''(0) < 0, \quad R''\Big(\frac{\pi}{m}\Big) > 0. \tag{1.4}$$

(d) (Analyticity) *For each $s \geq 0$, the boundary ∂D (equivalently the function R above) is analytic.*

Figure 1 shows regions D satisfying (c) for various values of m. Parts (a) and (b) of Theorem 1.1 are illustrated in Figure 2. Note that the curve $\mathscr{C}$ is *global* in that it is not contained in a small neighborhood of its starting point. Indeed, $\min_{\partial D} \Psi_r = 1/2m$ at the start of $\mathscr{C}$ while the limiting value is 0. The analyticity (d) is true for any sufficiently smooth solution satisfying a non-degeneracy condition; see Theorem 5.10. For the precise sense in which $\mathscr{C}$ is a continuous curve, see Theorem 4.6 and Section 2.2.

For s sufficiently small, we also prove the existence of "Cat's eye"-type structures in the flow outside of the patch.

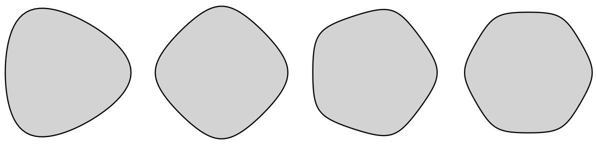

Figure 1: Patches with various symmetry classes m.

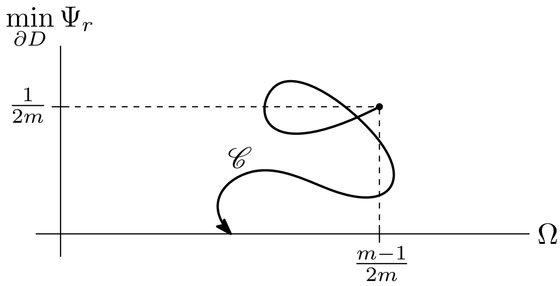

Figure 2: Sketch of the global bifurcation curve $\mathscr{C}$ constructed in Theorem 1.1.

Theorem 1.2 (Cat's eyes for small s)**.** *Let $\mathscr{C}$ be as in Theorem 1.1 and let $s > 0$ be sufficiently small. Then the phase portrait of $\dot{z} = \nabla^\perp \Psi(z)$ outside of D looks qualitatively like Figure 3. In particular, there are m saddle points, with adjacent saddle points connected by pairs of heteroclinic orbits. These heteroclinic orbits enclose regions of periodic orbits surrounding m centers. All other orbits are polar graphs $r = \tilde{r}(\theta)$.*

Based on our above results and the numerical evidence in Section 7, we make the following two conjectures:

Conjecture 1.3 (Limiting solutions)**.** *The singular solutions with $90°$ corners seen in numerics [WOZ84, Ove86] for $m \geq 3$ exist as the (weak) limits of patches along $\mathscr{C}$ as $s \to \infty$.*

Conjecture 1.4 (Persistence of Cat's eyes)**.** *The conclusion of Theorem 1.2 holds for all $s > 0$.*

A proof of Conjecture 1.3 would seem to require, among a great many other things, a positive resolution of Conjecture 1.4. This is similar to the current state of the art for steady water waves with constant vorticity; see [CSV16] and the discussion in the next subsection.

1.2 Historical discussion

In 1880, Thomson (Lord Kelvin) derived and analyzed the linear equations for small irrotational disturbances of a three-dimensional cylindrical vortex [Tho80]. For purely two-dimensional disturbances, one finds that a vortex patch with boundary $r = 1 + \varepsilon \cos m\theta$ will rotate at constant

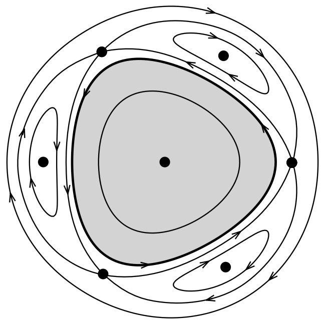

Figure 3: Phase portrait for the ODE $\dot{z} = \nabla^\perp \Psi$ showing "Cat's eyes".

angular velocity $\Omega_m = (m-1)/2m$ [Lam32, Art. 158]. Kirchhoff later discovered explicit two-dimensional solutions to the full nonlinear problem in the form of rotating ellipses [Lam32, Art. 159]. As the eccentricity vanishes, the angular velocity of these ellipses approaches Kelvin's Ω_2.

The first rigorous existence proof for nonlinear rotating vortex patches for $m \geq 3$ is due to Burbea in 1982 [Bur82]. Reformulating the problem in terms of a conformal mapping, he used the celebrated Crandall–Rabinowitz theorem [CR71] on bifurcation from a simple eigenvalue and for each m obtained a small curve of solutions close to the unit disk. In 2013, Hmidi, Mateu, and Verdera [HMV13] again used Crandall–Rabinowitz methods to construct local curves of solutions, this time showing that the boundaries ∂D are smooth. This regularity result was further improved by Castro, Córdoba, and Gómez-Serrano [CCGS16b] who in 2016 constructed a local curve of solutions with ∂D analytic.

In recent years there has been a burst of rigorous mathematical work on rotating vortex patches and related problems. In addition to the results mentioned above, there are existence proofs for rotating patches close to Kirchhoff's ellipses [CCGS16b, HM16], pairs of vortex patches [HM17], multiply-connected patches [dlHHMV16], and patches in bounded domains [dlHHHM16]. Many of these results apply not only to the Euler equations but also to the inviscid Surface Quasi-Geostrophic equations or the generalized Surface Quasi-Geostrophic equations; in this context also see [CCGSMZ14, CCGS16a].

It is important to emphasize that all of the above analytical results treat patches which are sufficiently close either to the unit disk or to some other explicit solution. Numerically, however, solutions have been found far beyond these perturbative regimes. In 1978, Deem and Zabusky [DZ78] found branches of rotating patches (which they called "V-states") with different symmetry classes m bifurcating from the unit disk. Wu, Overman, and Zabusky [WOZ84] went further along the same branches in 1984 and found singular limiting solutions with $90°$ corners; see Figure 8 on page 32. Overman [Ove86] then performed a careful asymptotic analysis near the corner of a hypothetical vortex patch (satisfying several assumptions), and confirmed analytically that either the corner is a cusp with an interior angle of 0, or that the interior angle is $90°$ as seen in the numerics. Patches bifurcating from Kirchhoff ellipses rather than the unit disk were first computed by Kamm in his thesis [Kam87]. The papers [dlHHMV16, dlHHHM16] mentioned in the previous paragraph also contain numerical results on doubly-connected vortex patches and on patches in a bounded domain.

To our knowledge, Theorem 1.1 is the first existence proof for rotating vortex patches which is global in the sense that it is not limited to a small neighborhood of an explicit solution. Our methods are inspired by global results for steady water waves, and in particular the real-analytic bifurcation techniques in [BT03]. For steady water waves, there is an analogue of Conjecture 1.3 known as the "Stokes conjecture". In the absence of vorticity, it was famously proven in a serious of papers culminating in [AFT82]. When Cat's eyes are permitted in the flow, however, the existence and nature of limiting solutions remains an important open problem; see [CSV16].

Before continuing to the outline, we lastly compare our results to the variational work of Turkington [Tur83, Tur85] in the 1980s. In [Tur83], Turkington considered *steady, non-rotating* vortex patches in a bounded domain, and in particular the singular limit as the patches become point vortices. This non-rotating problem is fundamentally different from ours: the flow no longer has Cat's eyes and the patch is simply expressed as $D = \{\Psi > 0\}$. That being said, Turkington's result is indeed global in the sense that it constructs patches with any prescribed area (less than the area of the bounded domain). The regularity of the solutions, outside of the scaling limit mentioned above, is left open, and so it is possible that some of these patches have singular boundaries. In [Tur85], Turkington considered an unbounded fluid domain with N symmetrically arranged vortex patches rotating about the origin. Restricting attention to a fixed region about each patch D, he first solved a modified variational problem for which again $D = \{\Psi > 0\}$, but was only able to guarantee that this yields a solution to the full problem in the limit as the patches approach point vortices.

1.3 Outline of the proof

1.3.1 Reformulation

We will work with Burbea's reformulation [Bur82] of the elliptic problem (1.2) in terms of conformal mappings, which we now briefly sketch; details can be found in Section 2.1 below. The first step is to rewrite (1.2) as a nonlinear integral equation for a parametrization ϕ of the patch boundary ∂D. Letting the domain of ϕ be the unit circle $\mathbb{T} \subset \mathbb{C}$ this equation can be written in the complex form

$$\operatorname{Im}\left\{\left(\Omega\overline{\phi(w)} + \frac{1}{4\pi i}\int_{\mathbb{T}} \frac{\overline{\phi(\tau)} - \overline{\phi(w)}}{\phi(\tau) - \phi(w)}\phi'(\tau)\,d\tau\right) w\phi'(w)\right\} = 0, \tag{1.5}$$

where here $\phi'(e^{it}) = -ie^{-it}\frac{d}{dt}\phi(e^{it})$ denotes the derivative of ϕ as a function of a complex variable. While this equation can be integrated once with respect to w, we prefer to work with (1.5) directly. Following Burbea, we require ϕ to be the trace of a conformal mapping Φ which takes the exterior of the unit disk $\mathbb{D} \subset \mathbb{C}$ to the exterior of the patch D. Other choices are of course possible and have their own advantages; for instance the authors in [CCGS16b] work with parametrizations of the form $\phi(w) = wR(w)$ with R real-valued.

Regardless of which parametrization is used, the relative fluid velocity $(-\Psi_y, \Psi_x)$ at $\phi(w)$ can be directly recovered from the first factor in (1.5):

$$\Omega\overline{\phi(w)} + \frac{1}{4\pi i}\int_{\mathbb{T}} \frac{\overline{\phi(w)} - \overline{\phi(\tau)}}{\phi(w) - \phi(\tau)}\phi'(\tau)\,d\tau = -(\Psi_x - i\Psi_y)(\phi(w)). \tag{1.6}$$

Since the second factor $w\phi'(w)$ in (1.5) is a normal vector to ∂D, the equation can therefore be interpreted as saying that the relative stream function Ψ is constant along ∂D.

1.3.2 The Cauchy integral operator

The nonlocal term in (1.5) can be written as $\mathcal{C}(\phi)\overline{\phi}$ where $\mathcal{C}(\phi)$ is the Cauchy integral operator

$$\mathcal{C}(\phi)\colon g \mapsto \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{g(\tau) - g(w)}{\phi(\tau) - \phi(w)}\phi'(\tau)\,d\tau \tag{1.7}$$

associated to the curve $\partial D = \phi(\mathbb{T})$. Due to its connection with double-layer potentials, this operator often appears in the study of boundary value problems for the Laplacian; see for instance [Mus72, §61]. The mapping properties of $\mathcal{C}(\phi)$ are well-understood when ϕ is $C^{1+\alpha}$ [GK92, TO16] or even merely Lipschitz [CCF$^+$78].

The dependence of $\mathcal{C}(\phi)$ on ϕ seems to be less well-studied, but Lanza de Cristoforis and Preciso [LdCP99] have proven a result which is sufficient for our purposes (Theorem 2.3 below): $\mathcal{C}$ is a real-analytic map from a particular open subset of $C^{k+\alpha}(\mathbb{T})$ to the space of bounded linear operators $C^{k+\alpha}(\mathbb{T}) \to C^{k+\alpha}(\mathbb{T})$ for any $k \geq 1$. Sijue Wu proved a related analyticity result in BMO spaces [Wu93]. The analyticity of $\mathcal{C}(\phi)$ in ϕ seems not to have been taken advantage of in the mathematical literature on rotating vortex patches. In addition to enabling us to skip tedious verifications of the regularity of the dependence of various expressions on ϕ, it enables us to use a powerful global bifurcation theory specialized to analytic operators [Dan73, BT03].

We note in passing that, by (1.6) and the properties of $\mathcal{C}(\phi)$, $\phi \in C^{k+\alpha}$ implies that $\nabla\Psi \circ \phi$ is also $C^{k+\alpha}$. This is an improvement over naive Schauder estimates based on the elliptic equation (1.2), which only give $\nabla\Psi \in C^{k-1+\alpha}(\partial D)$. There are of course other ways to obtain similar gains in regularity without recourse to conformal mappings; see for instance the "regularizing diffeomorphisms" in [Lan13, Section 2.2.2].

1.3.3 Linear and quasilinear Riemann–Hilbert problems

Rewriting (1.5) using the operator $\mathcal{C}(\phi)$, we obtain

$$0 = \operatorname{Im}\{(\Omega\overline{\phi} + \tfrac{1}{2}\mathcal{C}(\phi)\overline{\phi})w\phi'\} =: \operatorname{Im}\{A\phi'\}, \tag{1.8}$$

where, by the arguments in the previous paragraph, the coefficient $A = (\Omega\overline{\phi} + \frac{1}{2}\mathcal{C}(\phi)\overline{\phi})w$ has the same regularity as ϕ itself. Since ϕ' appears linearly, we therefore call (1.8) "quasilinear".

Freezing the coefficient A, our choice of ϕ in Section 1.3.1 means that (1.8) is a linear Riemann–Hilbert problem for the derivative Φ' of the conformal mapping Φ with trace $\phi = \Phi|_{\mathbb{T}}$. Assuming that A has winding number 0, standard formulas from the theory of Riemann–Hilbert problems then allow us to explicitly invert (1.8) to find ϕ' in terms of A:

$$\phi'(w) = \exp\left\{\frac{w}{2\pi}\int_{\mathbb{T}} \frac{1}{\tau - w}\left[\frac{1}{\tau}\arg\left(\frac{A(\tau)}{\overline{A(\tau)}}\right) - \frac{1}{w}\arg\left(\frac{A(w)}{\overline{A(w)}}\right)\right] d\tau\right\}. \tag{1.9}$$

Substituting the definition of A in (1.9) and integrating yields a fixed-point equation for ϕ with a compact right hand side. Moreover, by (1.6) we have $A = -w(\Psi_x - i\Psi_y)$, and so (1.9) allows us to convert bounds on the relative stream function Ψ into bounds on the conformal parametrization ϕ. This useful "quasilinear Riemann–Hilbert" approach to (1.5) appears to be new. While we are not aware of a general theory for nonlinear problems like (1.8), there is a theory [Weg87] for problems of the form $F(w, \operatorname{Re}\phi(w), \operatorname{Im}\phi(w)) = 0$. For a modern introduction to linear Riemann–Hilbert problems, we refer the reader to [TO16].

1.3.4 Outline of the paper

In Section 2, we collect several preliminary results and reformulate (1.2) as a nonlinear operator equation $\mathscr{F}(\phi - w, \Omega) = 0$. Here $\mathscr{F}$ is an analytic operator defined on an open subset U of a Banach space X which encodes the m-fold symmetry. In Section 3, we study the set of solutions to this equation when $\|\phi - w\|_{C^{3+\alpha}}$ is small. It consists of the "trivial" line of solutions with $\phi(w) \equiv w$ together with a sequence of analytic curves (i.e. curves with analytic parametrizations) bifurcating from this axis at discrete frequencies Ω_{nm}. The existence of these curves is well known, but the analyticity appears to be new. In Section 4, we construct a global curve $\mathscr{C}$ of solutions using the analytic global bifurcation theory of Dancer [Dan73] and Buffoni–Toland [BT03]. This curve either (i) is a closed loop or (ii) "blows up" in a certain sense as the parameter $s \to \infty$. In Section 5, we show that alternative (i) cannot happen by tracking certain "nodal properties" related to (1.4) using maximum principle and continuation arguments. We also prove Theorem 1.2 on the streamlines of solutions with small s. In Section 5.4, we show that ∂D is analytic for every solution in $\mathscr{C}$ by using a result of Kinderlehrer, Nirenberg, and Spruck [KNS78]. In Section 6, we turn to the second alternative (ii) and complete the proof of Theorem 1.1. Finally, in Section 7, we provide numerical evidence for Conjectures 1.3 and 1.4. Appendix A.1 contains some needed facts about linear Riemann–Hilbert problems, and Appendix A.2 proves several identities relating derivatives of the stream function Ψ to derivatives of the conformal parametrization ϕ.

2 Formulation and preliminaries

2.1 Reformulation as an integral equation

The equivalence of the integral equation (1.5) and the elliptic problem (1.2) is well-known [Bur82, HMV13]. In an effort to keep the presentation self-contained, we reproduce a short proof here, and also state analogues of (1.6) for higher partials of Ψ which we will need in Section 5.2.

Let $D \subset \mathbb{C}$ be a simply-connected $C^{k+\beta}$ domain for some integer $k \geq 1$ and $\beta \in (0,1)$. We know from classical potential theory that

$$\Psi(z) = \Psi^{D,\Omega}(z) := \frac{1}{2\pi}\iint_D \log|z - \zeta|\, d\zeta - \frac{\Omega}{2}|z|^2 \tag{2.1}$$

is, up to an additive constant, the unique solution to (1.2a)–(1.2c). Thus (Ψ, D, Ω) solves the full system (1.2) if and only if $\Psi = \Psi^{D,\Omega} + C$ satisfies the remaining equation (1.2d) for some constant C, i.e. if $\Psi^{D,\Omega}$ is constant on ∂D. Letting $\phi\colon \mathbb{T} \to \mathbb{C}$ be a $C^{k+\beta}$ parametrization of ∂D, this in turn is equivalent to

$$0 = \frac{d}{dt}\Psi^{D,\Omega}(\phi(e^{it})) = -\operatorname{Im}\left\{(\Psi_x^{D,\Omega} - i\Psi_y^{D,\Omega})(\phi(e^{it})) \cdot e^{it}\phi'(e^{it})\right\} \qquad \text{for } t \in \mathbb{R}, \tag{2.2}$$

where the last equality is a straightforward calculation using the chain rule. If the gradient of $\Psi = \Psi^{D,\Omega}$ restricted to ∂D is given by (1.6), then (2.2) is precisely our integral equation (1.5).

Now and for the remainder of the paper, we specialize to the case when the parametrization ϕ is the trace of a conformal mapping as follows. Let Φ be the unique conformal map which sends the exterior of the unit disk $\mathbb{D} \subset \mathbb{C}$ to the exterior of D and additionally satisfies $\Phi(\infty) = 0$ and $\Phi'(\infty) > 0$. Since D is $C^{k+\beta}$, Φ extends to a $C^{k+\beta}$ mapping $\mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus D$ [Pom92, Theorem 3.6], and its trace $\phi := \Phi|_{\mathbb{T}}$ is a $C^{k+\beta}$ parametrization of ∂D.

To simplify the formulas in the lemmas below, we make use of the complex Wirtinger derivatives $\partial_z := \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\overline{z}} := \frac{1}{2}(\partial_x + i\partial_y)$. Note that the left hand side of (1.6) is proportional to $\partial_z \Psi$.

Lemma 2.1 ([Bur82, HMV13])**.** *Let $D \subset \mathbb{C}$ be a bounded and simply-connected $C^{k+\beta}$ domain for some $k \geq 1$. Then the function $\Psi^{D,\Omega}$ defined in* (2.1) *and the conformal parametrization $\phi\colon \mathbb{T} \to \partial D$ defined above satisfy*

$$\partial_z \Psi^{D,\Omega} \circ \phi = -\tfrac{1}{4}\mathcal{C}(\phi)\overline{\phi} - \tfrac{\Omega}{2}\overline{\phi} \tag{2.3}$$

so that (1.6) *holds with $\Psi = \Psi^{D,\Omega}$. Moreover, there is a solution Ψ of* (1.2) *if and only if ϕ satisfies* (1.5)*, and in this case $\Psi = \Psi^{D,\Omega} + C$ for some constant C.*

Proof. From the above arguments, it is sufficient to prove (2.3). For $z \notin \overline{D}$, differentiating (2.1) under the integral yields

$$\partial_z \Psi^{D,\Omega}(z) = \frac{1}{4\pi}\int_D \frac{1}{z - \zeta}\, d\zeta - \frac{\Omega}{2}\overline{z}. \tag{2.4}$$

The complex form of Green's theorem,

$$\int_D \partial_{\overline{\zeta}} F(\zeta, \overline{\zeta})\, d\zeta = \frac{1}{2i}\int_{\partial D} F(\zeta, \overline{\zeta})\, d\zeta, \tag{2.5}$$

then allows us to replace the integral over D in (2.4) with an integral along ∂D:

$$\partial_z \Psi^{D,\Omega}(z) = -\frac{1}{8\pi i}\int_{\partial D} \frac{\overline{z} - \overline{\zeta}}{z - \zeta}\, d\zeta - \frac{\Omega}{2}\overline{z}. \tag{2.6}$$

Making the change of variable $z = \Phi(w)$, $\zeta = \phi(\tau)$, we are left with

$$\partial_z \Psi^{D,\Omega}(\Phi(w)) = -\frac{1}{8\pi i}\int_{\mathbb{T}} \frac{\overline{\Phi(w)} - \overline{\phi(\tau)}}{\Phi(w) - \phi(\tau)}\phi'(\tau)\, d\tau - \frac{\Omega}{2}\overline{\Phi(w)}. \tag{2.7}$$

Note that the $\overline{z}$ term in the numerator of the integrand in (2.6) and similarly the $\overline{\Phi(w)}$ term in (2.7) can be dropped thanks to the Cauchy integral theorem. Having included these terms, however, we can take the limit as z approaches ∂D in (2.6) to discover that it continues to hold for $z \in \partial D$. Similarly (2.7) holds for $w \in \mathbb{T}$, in which case we can write $\phi(w)$ for $\Phi(w)$ and obtain (2.3) as desired. ☐

Thus any solution of (1.2) with $D \in C^{k+\beta}$ gives rise to a solution ϕ of (1.5). Conversely, if $\Phi\colon \mathbb{C} \setminus D \to C$ is conformal onto its image and the $C^{k+\beta}$ restriction $\phi = \Phi|_{\mathbb{T}}$ solves (1.5), then the complement of the image of Φ is a simply-connected $C^{k+\beta}$ domain D and $\Psi^{D,\Omega} + C$ solves (1.2) for some constant C.

Lemma 2.2 (Higher partials of the stream function)**.** *In the setting of Lemma 2.1, suppose that $k \geq 3$. Then the restriction $\Psi^- = \Psi^{D,\Omega}|_{\mathbb{C}\setminus D}$ satisfies*

$$\begin{aligned}(\partial_z^2\Psi^-)\circ\phi &= \mathcal{C}(\phi)F_2(\phi),\\ (\partial_z^3\Psi^-)\circ\phi &= \mathcal{C}(\phi)F_3(\phi),\end{aligned} \tag{2.8}$$

where

$$F_2(\phi) := \frac{\overline{\phi'}}{4w^2\phi'}, \qquad F_3(\phi) := -\frac{\overline{\phi'}}{2w^3(\phi')^2} - \frac{\overline{\phi''}}{4w^4(\phi')^2} - \frac{\overline{\phi'}\phi''}{4w^2(\phi')^3}.$$

Proof. We argue exactly as in the proof of Lemma 2.1, except that now we integrate by parts in the analogue of (2.7) before taking the limit as w approaches $\mathbb{T}$. The details are postponed to Appendix A.2. □

2.2 Function spaces and open sets

For the remainder of the paper we fix the integer $m \geq 2$ describing the symmetry class of the solutions under consideration, and for the remainder of this section we fix an integer $k \geq 1$ and Hölder parameter $\beta \in (0,1)$.

Observe that (1.5) has a scaling symmetry: if ϕ is a solution then so is $\lambda\phi$ for any $\lambda \neq 0$. Following [Bur82], we will kill this symmetry by fixing the coefficient of the linear term in the Laurent expansion of Φ to be 1. We then think of (1.5) as an equation for the remainder

$$f(w) := \phi(w) - w,$$

which we require to lie in the Banach space

$$X^{k+\beta} = \left\{ f \in C^{k+\beta}(\mathbb{T}) : f(w) = \sum_{n=1}^{\infty} \frac{a_n}{w^{nm-1}},\ a_n \in \mathbb{R} \right\}.$$

The absence of w^p terms in the Fourier series of f for $p \geq 1$ guarantees that f extends to a holomorphic function F on $\mathbb{C}\setminus\mathbb{D}$ with $F'(\infty) = 0$. Note, however, that there is as of yet no guarantee that $\Phi(w) := w + F(w)$ will be conformal. The absence of the other terms in the series is equivalent to the discrete rotation symmetry

$$\phi(e^{2\pi i/m}w) = e^{2\pi i/m}\phi(w), \tag{2.9a}$$

while the fact that the coefficients are real is equivalent to the reflection symmetry

$$\phi(\overline{w}) = \overline{\phi(w)}. \tag{2.9b}$$

These symmetries correspond to the π/m-periodicity and evenness of the function R in Theorem 1.1(c).

We also define another Banach space $Y^{k-1+\beta}$, which will be the space for the right hand side of (1.5),

$$Y^{k-1+\beta} = \left\{ h \in C^{k-1+\beta}(\mathbb{T}) : h(e^{2\pi i/m}w) = h(w),\ h(\overline{w}) = -\overline{h(w)} \right\}.$$

We will not work in $X^{k+\beta}$ directly but in a convenient open subset $U^{k+\beta} = U_1^{k+\beta} \cap U_2^{k+\beta} \cap U_3^{k+\beta}$ where

$$\begin{aligned}
U_1^{k+\beta} &= \left\{ (\phi - w, \Omega) \in X^{k+\beta}\times\mathbb{R} : \inf_{\tau\neq w}\left|\frac{\phi(\tau)-\phi(w)}{\tau - w}\right| > 0 \right\},\\
U_2^{k+\beta} &= \left\{ (\phi - w, \Omega) \in U_1^{k+\beta} : \left|\Omega\overline{\phi} + \tfrac{1}{2}\mathcal{C}(\phi)\overline{\phi}\right| > 0 \right\},\\
U_3^{k+\beta} &= \left\{ (\phi - w, \Omega) \in X^{k+\beta}\times\mathbb{R} : \operatorname{Re}\left(\frac{w\phi'}{\phi}\right) > 0 \text{ on } \mathbb{T} \right\}.
\end{aligned}$$

2.2.1 Open set for the Cauchy integral.

The set $U_1^{k+\beta}$ is chosen so that the Cauchy integral operator $\mathcal{C}(\phi)$ appearing in (1.5) is well-behaved:

Theorem 2.3 (Analyticity of the Cauchy integral [LdCP99])**.**

(a) *The set* $\tilde{U}_1^{k+\beta}$ *defined by*

$$\tilde{U}_1^{k+\beta} = \left\{ \phi \in C^{k+\beta}(\mathbb{T}) : \inf_{\tau \neq w} \left| \frac{\phi(\tau) - \phi(w)}{\tau - w} \right| > 0 \right\}$$

is an open subset of $C^{k+\beta}(\mathbb{T})$*. Moreover* $\phi \in C^{k+\beta}(\mathbb{T})$ *lies in* $\tilde{U}_1^{k+\beta}$ *if and only if* ϕ *is injective and* $\phi' \neq 0$*.*

(b) *The formula* (1.7) *describes a real-analytic mapping*

$$\mathcal{C} \colon \tilde{U}_1^{k+\beta} \to \mathscr{L}(C^{k+\beta}(\mathbb{T})),$$

where $\mathscr{L}(C^{k+\beta}(\mathbb{T}))$ *is the Banach space of bounded linear maps from* $C^{k+\beta}(\mathbb{T})$ *to itself.*

Thus $(\phi - w, \Omega) \in X^{k+\beta} \times \mathbb{R}$ lies in $U_1^{k+\beta}$ if and only if ϕ is injective and satisfies $\phi' \neq 0$. This is necessary for the equivalence of (1.2) and (1.5), and also guarantees that ϕ can indeed be extended to a conformal map Φ on $\mathbb{C} \setminus \mathbb{D}$ [Pom92, p. 16].

The result in [CCF+78] is written in terms of a slightly different operator where the $g(w)$ term in (1.7) is missing and the integral is a principle value, but the difference between these two operators is a multiple of the identity; see [TO16, Section 2.2.1]. While we will always apply Theorem 2.3 to mappings ϕ which extend to holomorphic functions, we note that the theorem itself makes no such restriction.

2.2.2 Open set for Riemann–Hilbert problems.

From (1.6) we see that $(f, \Omega) \in U_2^{k+\beta}$ if and only if the relative fluid velocity $\nabla^\perp \Psi$ does not vanish on ∂D. This also guarantees that the coefficient A multiplying ϕ' in the Riemann–Hilbert problem (1.8) is non-vanishing, which will eventually enable us to apply Lemma 2.4 below with $a = A$.

Lemma 2.4 (Linear Riemann–Hilbert problems)**.** *Suppose that* $a \in C^{k-1+\beta}(\mathbb{T}, \mathbb{C})$ *has winding number* 0 *in that*

$$|a| > 0 \quad \text{and} \quad \arg a(e^{it}) \Big|_{t=0}^{t=2\pi} = 0,$$

and also that a *has the symmetry properties* $a(\overline{w}) = \overline{a(w)}$ *and* $a(e^{2\pi i/m} w) = a(w)$*. Then:*

(a) *The problem*

$$\operatorname{Im}\{a g'\} = 0 \text{ on } \mathbb{T}, \qquad g - w \in X^{k+\beta} \tag{2.10}$$

has a unique solution $g = g_0$*, whose derivative is given explicitly by*

$$g_0'(w) = \exp\left\{ \frac{w}{2\pi} \int_{\mathbb{T}} \frac{\tau^{-1}\theta(\tau) - w^{-1}\theta(w)}{\tau - w} \, d\tau \right\},$$

where here

$$\theta(w) = \arg\left(\frac{a(w)}{\overline{a(w)}} \right)$$

and the branch of the arg *function is fixed by requiring* $\theta(1) = 0$*.*

(b) *The operator*

$$L\colon X^{k+\beta} \to Y^{k-1+\beta}, \qquad g \mapsto \operatorname{Im}\{Ag'\}$$

is well-defined and invertible, with inverse operator characterized by

$$\frac{d}{dw} L^{-1} h(w) = -\frac{w g_0'(w)}{\pi} \int_{\mathbb{T}} \frac{1}{\tau - w} \left(\frac{h(\tau)}{a(\tau) g_0'(\tau) \tau} - \frac{h(w)}{a(w) g_0'(w) w} \right) d\tau. \tag{2.11}$$

Proof. We postpone the proof, which is a relatively straightforward application of classical results for Riemann–Hilbert problems (e.g. [Mus72]), to Appendix A.1. □

2.2.3 Open set for graphical boundary.

The definition of the final open set $U_3^{k+\beta}$ has a simple interpretation in terms of a polar coordinate representation $\phi = \rho e^{i\vartheta}$ of the conformal parametrization ϕ. Since we will use such a representation several times, we record its basic properties in the following elementary lemma.

Lemma 2.5 (ϕ in polar coordinates)**.** *For any $\phi \in C^{k+\beta}(\mathbb{T}, \mathbb{C} \setminus \{0\})$ we can write*

$$\phi(e^{it}) = \rho(t) e^{i\vartheta(t)} \text{ for } t \in \mathbb{R}, \tag{2.12}$$

where $\rho > 0$ and ϑ are periodic real-valued $C^{k+\beta}$ functions. Their first derivatives are

$$\vartheta'(t) = \operatorname{Re}\left(\frac{e^{it}\phi'(e^{it})}{\phi(e^{it})} \right), \qquad \rho'(t) = -\rho(t) \operatorname{Im}\left(\frac{e^{it}\phi'(e^{it})}{\phi(e^{it})} \right). \tag{2.13}$$

Proof. The existence of ρ, ϑ is standard. Differentiating (2.12) with respect to t and dividing by ϕ yields

$$\frac{ie^{it}\phi'(e^{it})}{\phi(e^{it})} = \frac{\rho'(t)}{\rho(t)} + i\vartheta'(t),$$

which has (2.13) as real and imaginary parts. □

Thus $(\phi - w, \Omega) \in U_3^{k+\beta}$ if and only if $\vartheta' > 0$, in which case $\phi(\mathbb{T})$ is a polar graph $r = R(\theta)$ for some $C^{k+\beta}$ function R. The numerical evidence [WOZ84] suggests that this is the case for all but the limiting solution, which is still graphical but loses regularity. For solutions in $U_2^{k+\beta}$, we will see that membership in $U_3^{k+\beta}$ is equivalent to the nonvanishing of the relative angular fluid velocity $\partial_r \Psi$ on ∂D.

One useful feature of the set $U_3^{k+\beta}$ is that it is completely contained in $U_1^{k+\beta}$, guaranteeing that the Cauchy integral operator is analytic.

Lemma 2.6. $U_3^{k+\beta} \subset U_1^{k+\beta}$.

Proof. Let $(\phi - w, \Omega) \in U_3^{k+\beta}$, and write $\phi(e^{it}) = \rho(t)e^{i\vartheta(t)}$ as in Lemma 2.5. By Theorem 2.3(a), it suffices to show that ϕ is injective and that $\phi' \neq 0$. The definition of $U_3^{k+\beta}$ immediately implies that $\phi' \neq 0$, while Lemma 2.5 gives $\vartheta' > 0$. If the winding number $(\vartheta(2\pi) - \vartheta(0))/2\pi = 1$, then this last fact implies that ϕ is injective. To calculate the winding number, let Φ be the holomorphic extension of ϕ to $\mathbb{C} \setminus \mathbb{D}$. From $\phi - w \in X^{k+\beta}$ we have $w\Phi'/\Phi \to 1$ as $|w| \to \infty$, and therefore

$$\frac{\vartheta(2\pi) - \vartheta(0)}{2\pi} = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\phi'(w)}{\phi(w)}\, dw = \lim_{r \to \infty} \frac{1}{2\pi i} \int_{|w|=r} \frac{\Phi'(w)}{\Phi(w)}\, dw = 1$$

as desired. □

2.3 Nonlinear operator and solution set

Setting

$$U^{k+\beta} = U_1^{k+\beta} \cap U_2^{k+\beta} \cap U_3^{k+\beta} = U_2^{k+\beta} \cap U_3^{k+\beta},$$

we define our nonlinear operator

$$\mathscr{F}^{k+\beta} \colon U^{k+\beta} \to Y^{k-1+\beta}$$

by simply rewriting the left hand side of (1.5):

$$\mathscr{F}^{k+\beta}(f,\Omega) := \operatorname{Im}\left\{\big(\Omega(\overline{w}+\overline{f}) + \tfrac{1}{2}\mathcal{C}(w+f)(\overline{w}+\overline{f})\big)w(1+f')\right\}. \tag{2.14}$$

Lemma 2.7. *The map $\mathscr{F}^{k+\beta} \colon U^{k+\beta} \to Y^{k-1+\beta}$ is well-defined and analytic. Moreover, for $(f,\Omega) \in U^{k+\beta}$, $B = \mathcal{C}(w+f)(\overline{w}+\overline{f})$ enjoys the symmetry properties $B(\overline{w}) = \overline{B(w)}$ and $B(e^{2\pi i/m}w) = e^{-2\pi i/m}B(w)$.*

Proof. The analyticity of $\mathscr{F}^{k+\beta} \colon U^{k+\beta} \to C^{k+\beta}(\mathbb{T})$ is an immediate consequence of Theorem 2.3 and the inclusion $U_1^{k+\beta} \subset U^{k+\beta}$. The symmetry properties follow from straightforward manipulations using the identities $\phi(e^{2\pi i/m}w) = e^{2\pi i/m}\phi(w)$, $\phi'(e^{2\pi i/m}w) = \phi'(w)$, and $\phi(\overline{w}) = \overline{\phi(w)}$ for $\phi = w + f$. ☐

We easily calculate that, for any $\Omega \in \mathbb{R}$,

$$\mathscr{F}^{k+\beta}(0,\Omega) = \operatorname{Im}\left\{\left(\Omega\overline{w} + \frac{1}{4\pi i}\int_{\mathbb{T}} \frac{\overline{\tau}-\overline{w}}{\tau - w}\,d\tau\right)w\right\} = \operatorname{Im}\{\Omega|w|^2 - \tfrac{1}{2}\} = 0,$$

corresponding to the fact that the unit disc $D = \mathbb{D}$ is a rotating vortex patch with angular velocity Ω. We call these "trivial" solutions and introduce the following notation.

Definition 2.8 (Trivial solutions, solution set)**.** The set of "trivial" solutions is denoted by

$$\mathscr{T}^{k+\beta} = \{(0,\Omega) : \Omega \in \mathbb{R}\} \subset U^{k+\beta},$$

and the full solution set by

$$\mathscr{S}^{k+\beta} = \{(f,\Omega) \in U^{k+\beta} : \mathscr{F}^{k+\beta}(f,\Omega) = 0\}.$$

While in the above discussion $k \geq 1$ and $\beta \in (0,1)$ have been arbitrary, in what follows we will for the most part fix $k = 3$ and $\beta = \alpha \in (0,1)$. To simplify notation, we therefore introduce the abbreviations

$$X := X^{3+\alpha}, \quad Y := Y^{2+\alpha}, \quad U := U^{3+\alpha}, \quad U_1 := U_1^{3+\alpha}, \quad U_2 := U_2^{3+\alpha}, \quad U_3 := U_3^{3+\alpha},$$
$$\mathscr{F} := \mathscr{F}^{3+\alpha}, \quad \mathscr{S} := \mathscr{S}^{3+\alpha}, \quad \mathscr{T} := \mathscr{T}^{3+\alpha}.$$

3 Local bifurcation

In this section we describe the solution set $\mathscr{S}$ near the axis $\mathscr{T}$ of trivial solutions. The main tools are the implicit function theorem and the following analytic version of the classical Crandall–Rabinowitz theorem [CR71].

Theorem 3.1 (Theorem 8.3.1 in [BT03])**.** *Let X, Y be real Banach spaces, $U \subset X \times \mathbb{R}$ an open set, and $\mathcal{F} \colon U \to Y$ a real-analytic function. Suppose that*

(a) *$\mathcal{F}(0,\lambda) = 0$ for all λ in a neighborhood of $\lambda_0 \in \mathbb{R}$;*

(b) $\mathcal{F}_x(0,\lambda_0)$ *is a Fredholm operator of index zero, with a one-dimensional kernel spanned by* $\xi_0 \in X$*; and*

(c) *the "transversality condition"* $\mathcal{F}_{x\lambda}(0,\lambda_0)\xi_0 \notin \operatorname{ran}\mathcal{F}_x(0,\lambda_0)$ *holds.*

Then $(0,\lambda_0)$ *is a bifurcation point in the following sense. There exists* $\varepsilon > 0$ *and a pair of analytic functions* $(\tilde{x},\tilde{\lambda})\colon(-\varepsilon,\varepsilon) \to \mathbb{R}\times U$ *such that*

(i) $\mathcal{F}(\tilde{x}(s),\tilde{\lambda}(s)) = 0$ *for* $s \in (-\varepsilon,\varepsilon)$*;*

(ii) $\tilde{x}(0) = 0$*,* $\tilde{\lambda}(0) = \lambda_0$*, and* $\tilde{x}'(0) = \xi_0$*; and*

(iii) *there exists an open neighborhood* V *of* $(0,\lambda_0)$ *in* $\mathbb{R}\times X$ *such that*

$$\big\{(x,\lambda) \in V : \mathcal{F}(x,\lambda) = 0,\ x \neq 0\big\} = \big\{(\tilde{x}(s),\tilde{\lambda}(s)) : 0 < |s| < \varepsilon\big\}.$$

With Theorem 3.1 in mind, we next calculate $\mathscr{F}_f(0,\Omega)$.

Lemma 3.2. *The Fréchet derivative* $\mathscr{F}_f(0,\Omega)$ *is given by*

$$\mathscr{F}_f(0,\Omega)g = \operatorname{Im}\Big\{\big(\Omega + \tfrac{1}{2}w\big)g' + \Omega w\overline{g}\Big\}. \tag{3.1}$$

Proof. Straightforward differentiation gives

$$\begin{aligned}\mathscr{F}_f(0,\Omega)g = \operatorname{Im}\Bigg\{\Bigg(&\Omega\big[\overline{w}g'(w) + \overline{g(w)}\big] + g'(w)\frac{1}{4\pi i}\int_{\mathbb{T}}\frac{\overline{\tau}-\overline{w}}{\tau - w}d\tau + \frac{1}{4\pi i}\int_{\mathbb{T}}\frac{\overline{\tau}-\overline{w}}{\tau-w}g'(\tau)d\tau \\ &+ \frac{1}{4\pi i}\int_{\mathbb{T}}\frac{\overline{g(\tau)}-\overline{g(w)}}{\tau - w}d\tau - \frac{1}{4\pi i}\int_{\mathbb{T}}\frac{(g(\tau)-g(w))(\overline{\tau}-\overline{w})}{(\tau-w)^2}d\tau\Bigg)w\Bigg\}.\end{aligned}$$

The first integral is easy to compute:

$$\int_{\mathbb{T}}\frac{\overline{\tau}-\overline{w}}{\tau - w}\,d\tau = \int_{\mathbb{T}}\frac{\frac{1}{\tau}-\frac{1}{w}}{\tau - w}\,d\tau = -\frac{1}{w}\int_{\mathbb{T}}\frac{d\tau}{\tau} = -\frac{2\pi i}{w},$$

where here we have used the identity $\overline{\tau} = 1/\tau$ for $\tau \in \mathbb{T}$. It therefore suffices to show that the remaining integrals vanish. Letting G be the holomorphic extension of g, we obtain through similar manipulations that

$$\int_{\mathbb{T}}\frac{\overline{g(\tau)}-\overline{g(w)}}{\tau - w}d\tau = \int_{\mathbb{T}}\frac{\overline{\tau}-\overline{w}}{\tau - w}g'(\tau)\,d\tau = \int_{\mathbb{T}}\frac{(g(\tau)-g(w))(\overline{\tau}-\overline{w})}{(\tau-w)^2}d\tau = -\frac{2\pi i}{w}G'(\infty) = 0,$$

where $G'(\infty)$ vanishes thanks to $g \in X$. ☐

Applying Theorem 3.1 together with the implicit function theorem in our setting we obtain the following, where

$$\Omega_{nm} := \frac{nm-1}{2nm}, \qquad n = 1,2,3,\dots$$

are the critical frequencies from Kelvin's linear analysis, and the trivial solution set $\mathscr{T}$ and full solution set $\mathscr{S}$ were introduced in Definition 2.8 and the following paragraph; see Figure 4 for an illustration.

Theorem 3.3 (Local structure)**.** *Fix* $\Omega \in \mathbb{R}$.

(i) (No bifurcation) *If* Ω *is not one of the* Ω_{nm}*, then there is a neighborhood* V *of* $(0,\Omega)$ *in* U *such that* $\mathscr{S}\cap V \subset \mathscr{T}$*.*

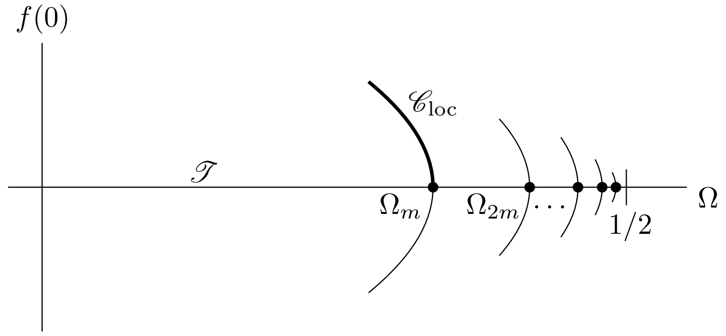

Figure 4: The local curves constructed in Theorem 3.3 near $\Omega = \Omega_m, \Omega_{2m}, \ldots$. The portion in bold is the curve $\mathscr{C}_{\mathrm{loc}}$ defined in Definition 3.4.

(ii) (Bifurcation) *For every* n, $(0, \Omega_{nm})$ *is a bifurcation point in the following sense. There exists* $\varepsilon > 0$ *and a pair of analytic functions* $(\tilde{f}, \tilde{\Omega})\colon (-\varepsilon, \varepsilon) \to U$ *with the following properties:*

   (a) $\mathscr{F}(\tilde{f}(s), \tilde{\Omega}(s)) = 0$ *for* $s \in (-\varepsilon, \varepsilon)$*;*

   (b) $\tilde{f}(0) = 0$, $\tilde{\Omega}(0) = \Omega_{nm}$, $\tilde{f}_s(0) = 1/w^{nm-1}$, *and* $\tilde{\Omega}_s(0) = 0$*; and*

   (c) *there exists an open neighborhood* V *of* $(0, \Omega_{nm})$ *in* $X \times \mathbb{R}$ *such that*

$$V \cap (\mathscr{S} \setminus \mathscr{T}) = \big\{ (\tilde{f}(s), \tilde{\Omega}(s)) : 0 < |s| < \varepsilon \big\}.$$

Proof. We have already shown in Lemma 2.7 that $\mathscr{F}$ is an analytic operator, and we have checked that $\mathscr{F}(0, \Omega) = 0$ for all Ω. Thus Lemma 4.2 in the next section implies that $\mathscr{F}_f(0, \Omega)$ is Fredholm with index 0 (this can also be verified directly). Expanding $g \in X$ in a Fourier series

$$g(w) = \sum_{k \geq 1} \frac{a_k}{w^{km-1}},$$

we see from Lemma 3.2 that the operator $\mathscr{F}_f(0, \Omega)$ in (3.1) is the Fourier multiplier

$$\mathscr{F}_f(0, \Omega) \sum_{k \geq 1} \frac{a_k}{w^{km-1}} = i \sum_{k \geq 1} km(\Omega - \Omega_{km}) a_k (\overline{w}^{km} - w^{km}). \tag{3.2}$$

If Ω is not one of the Ω_{nm}, then (3.2) shows that $\mathscr{F}_f(0, \Omega)$ has trivial kernel. Since it is Fredholm with index 0 it is therefore invertible, and (i) follows from the implicit function theorem.

So consider the trivial solution $(0, \Omega_{nm})$ for some $n \geq 1$. The strict monotonicity of the sequence $k \mapsto \Omega_{km}$ implies that the kernel of $\mathscr{F}_f(0, \Omega_{nm})$ is indeed the one-dimensional vector space spanned by $f_{nm}(w) = \overline{w}^{nm-1}$. Next we verify the transversality condition $\mathscr{F}_{f\Omega}(0, \Omega_{nm}) f_{nm} \notin \operatorname{ran} \mathscr{F}_f(0, \Omega_{nm})$. Differentiating (3.2) with respect to Ω yields

$$\mathscr{F}_{f\Omega}(0, \Omega) g(w) = 2\mathscr{F}_f(0, \Omega) g(w) = i \sum_{k \geq 1} km a_k \big(\overline{w}^{km} - w^{km}\big),$$

and so we calculate

$$\mathscr{F}_{f\Omega}(0, \Omega_{nm}) f_{nm} = inm(\overline{w}^{nm} - w^{nm}) \notin \operatorname{ran}(\mathscr{F}_f(0, \Omega_{nm}))$$

as desired. Part (ii) of the theorem now follows immediately from Theorem 3.1, except for the assertion that $\tilde{\Omega}_s(0) = 0$.

This derivative can be calculated directly using so-called "bifurcation formulas" [Kie04, Section I.6]. As is often the case for pitchfork bifurcations, however, the full calculation is unnecessary because of symmetry considerations. Set $n = 1$ and let $\ell \in Y^*$ be a linear functional with ran $\mathscr{F}_f(0,\tilde\Omega(0)) = \ker \ell$. Then the transversality condition becomes $\langle \ell, \mathscr{F}_{f\Omega}(0,\tilde\Omega(0))\tilde f_s(0)\rangle \neq 0$. Differentiating $\mathscr{F}(\tilde f(s),\tilde\Omega(s)) = 0$ with respect to s we discover that

$$\mathscr{F}_{ff}(0,\tilde\Omega(0))[\tilde f_s(0),\tilde f_s(0)] + \mathscr{F}_f(0,\tilde\Omega(0))\tilde f_{ss}(0) + 2\tilde\Omega_s(0)\mathscr{F}_{f\Omega}(0,\tilde\Omega(0))\tilde f_s(0).$$

Testing against ℓ, the $\mathscr{F}_f$ term drops out and we are left with

$$\big\langle \ell, \mathscr{F}_{ff}(0,\tilde\Omega(0))[\tilde f_s(0),\tilde f_s(0)] + 2\tilde\Omega_s(0)\mathscr{F}_{f\Omega}(0,\tilde\Omega(0))\tilde f_s(0)\big\rangle = 0$$

and hence

$$\tilde\Omega_s(0) = -\frac{1}{2}\frac{\big\langle \ell, \mathscr{F}_{ff}(0,\tilde\Omega(0))[\tilde f_s(0),\tilde f_s(0)]\big\rangle}{\big\langle \ell, \mathscr{F}_{f\Omega}(0,\tilde\Omega(0))\tilde f_s(0)\big\rangle}. \tag{3.3}$$

Now we rotate by π/m and repeat the above calculation. Consider the invertible linear maps $T_X\colon X \to X$ and $T_Y\colon Y \to Y$ defined by $T_X f(w) = e^{-i\pi/m} f(e^{i\pi/m}w)$ and $T_Y h(w) = h(e^{i\pi/m}w)$. It is straightforward to verify that $\mathscr{F}$ commutes with T_X, T_Y in that $\mathscr{F}(T_X f,\Omega) = T_Y\mathscr{F}(f,\Omega)$. In particular, $\mathscr{F}(\tilde f(s),\tilde\Omega(s)) = 0$ implies that $\mathscr{F}(T_X\tilde f(s),\tilde\Omega(s)) = 0$. The analogue of (3.3) is then

$$\tilde\Omega_s(0) = -\frac{1}{2}\frac{\big\langle \ell, \mathscr{F}_{ff}(0,\tilde\Omega(0))[T_X\tilde f_s(0),T_X\tilde f_s(0)]\big\rangle}{\big\langle \ell, \mathscr{F}_{f\Omega}(0,\tilde\Omega(0))T_X\tilde f_s(0)\big\rangle},$$

Plugging in the fact that $T_X\tilde f_s(0) = -\tilde f_s(0)$, we obtain

$$\tilde\Omega_s(0) = +\frac{1}{2}\frac{\big\langle \ell, \mathscr{F}_{ff}(0,\tilde\Omega(0))[\tilde f_s(0),\tilde f_s(0)]\big\rangle}{\big\langle \ell, \mathscr{F}_{f\Omega}(0,\tilde\Omega(0))\tilde f_s(0)\big\rangle},$$

which differs from (3.3) only in sign. Thus $\tilde\Omega_s(0) = 0$ as desired.

For $n > 1$ one can repeat the above calculation of $\tilde\Omega_s(0)$ by replacing X, Y with the subspaces X_n, Y_n with nm-fold symmetry and then use the uniqueness in (c). □

The global curve $\mathscr{C}$ which we will construct is a continuation of the local curve constructed in Theorem 3.3 that bifurcates from $(0,\Omega_m)$. As hinted at in the above proof, the portions of the curve with $s > 0$ and $s < 0$ are related by the symmetry T_X, and so there is no loss of generality in restricting to $s > 0$. Thus we make the following definition.

Definition 3.4 (The local curve $\mathscr{C}_{\mathrm{loc}}$)**.** With $\varepsilon, \tilde f, \tilde\Omega$ as in Theorem 3.3(ii) with $n = 1$, we define $\mathscr{C}_{\mathrm{loc}} \subset \mathscr{S}$ to be the portion of the bifurcation curve with $s > 0$, that is

$$\mathscr{C}_{\mathrm{loc}} := \{(\tilde f(s),\tilde\Omega(s)) : 0 < s < \varepsilon\}.$$

4 Global bifurcation

In this section we apply an abstract result on analytic global bifurcation (Theorem 4.5) to extend the local curve $\mathscr{C}_{\mathrm{loc}}$ from the previous section to a global one. In order to apply Theorem 4.5, we need to verify that the linearized operators $\mathscr{F}_f(f,\Omega)$ that we will encounter along this curve are Fredholm with index zero, and also that the curve has certain compactness properties. Both of these tasks will be accomplished by viewing $\mathscr{F}(f,\Omega) = 0$ as the quasilinear Riemann–Hilbert-type problem $\operatorname{Im}(A\phi') = 0$, applying Lemma 2.4, and using the good control we have over the coefficient A.

We first verify that the coefficient A appearing in our Riemann–Hilbert problem has winding number zero.

Lemma 4.1 (Winding number). *Suppose that $(\phi - w, \Omega) \in \mathscr{S}^{k+\beta}$ for some integer $k \geq 1$ and $\beta \in (0,1)$. Then $A := (\Omega\overline{\phi} + \frac{1}{2}\mathcal{C}(\phi)\overline{\phi})w$ has winding number 0 in that*

$$A \neq 0, \qquad \arg A(e^{it})\Big|_{t=0}^{t=2\pi} = 0 \tag{4.1}$$

for some continuous branch of arg.

Proof. By (1.8) we have $\operatorname{Im}(A\phi') = 0$, while $(\phi - w, \Omega) \in U_2^{k+\beta} \cap U_3^{k+\beta}$ implies $A \neq 0$ and $\phi' \neq 0$. Thus $A = \lambda\overline{\phi'}$ for some real-valued and non-vanishing $\lambda \in C^{k-1+\beta}(\mathbb{T})$, and it suffices to show that ϕ' has winding number zero with respect to the origin in the sense of (4.1). As in Section 2.2, let Φ be the holomorphic extension of ϕ to $\mathbb{C} \setminus \mathbb{D}$. Then $\phi - w \in X^{k+\beta}$ implies that $w\Phi''/\Phi \to 0$ as $|w| \to \infty$. Thus

$$\arg \phi'(e^{it})\Big|_{t=0}^{t=2\pi} = \lim_{r\to\infty} \arg \Phi'(re^{it})\Big|_{t=0}^{t=2\pi} = \lim_{r\to\infty} \frac{1}{2\pi i}\int_{|w|=r} \frac{\Phi''(w)}{\Phi'(w)}\,dw = 0$$

as desired. □

With Lemma 4.1 in hand, we can now use Lemma 2.4 to establish Fredholm properties for the linearized operators $\mathscr{F}_f(f,\Omega)$.

Lemma 4.2 (Fredholm index 0). *For any $(f,\Omega) \in \mathscr{S}$, the linearized operator $\mathscr{F}_f(f,\Omega)\colon X \to Y$ is Fredholm with index 0.*

Proof. By Theorem 2.3 we can write

$$\begin{aligned}\mathscr{F}(f,\Omega) &= \operatorname{Im}\left\{\big(\Omega(\overline{w} + \overline{f}) + \tfrac{1}{2}\mathcal{C}(w+f)(\overline{w}+\overline{f})\big)w(1+f')\right\} \\ &=: \operatorname{Im}\{\mathscr{A}(f,\Omega)(1+f')\},\end{aligned}$$

where $\mathscr{A}$ is analytic $U_2^{k+\beta} \to C^{k+\beta}(\mathbb{T})$ for any $k \geq 1$ and $\beta \in (0,1)$. Differentiating, we find

$$\mathscr{F}_f(f,\Omega)g = \operatorname{Im}\{\mathscr{A}(f,\Omega)g'\} + \operatorname{Im}\{(1+f')\mathscr{A}_f(f,\Omega)g\} =: L_1 g + L_2 g.$$

By Lemmas 2.7 and 4.1, $A = \mathscr{A}(f,\Omega)$ satisfies the hypotheses of Lemma 2.4 and so $L_1\colon X \to Y$ is invertible. We claim that $L_2\colon X \to Y$ is compact. Let g_n be a bounded sequence in $X = X^{3+\alpha}$, and extract a subsequence so that $g_n \to g$ in $X^{3+\alpha/2}$. The analyticity of $\mathscr{A}\colon U^{3+\alpha/2} \to C^{3+\alpha/2}(\mathbb{T})$ then guarantees that $\mathscr{A}_f(f,\Omega)g_n \to \mathscr{A}_f(f,\Omega)g_n$ in $C^{3+\alpha/2}(\mathbb{T})$ and hence also in $Y = Y^{2+\alpha}$, proving the claim. Thus $\mathscr{F}_f(f,\Omega)$ is sum of an invertible operator and a compact operator and is hence Fredholm with index 0 as desired. □

To prove the desired compactness properties for $\mathscr{S}$, we introduce a family of closed and bounded sets $E_\delta^{k+\beta} \subset U^{k+\beta}$ which exhaust $U^{k+\beta}$ as $\delta \to 0$ and which make the various conditions in the definitions of $U_1^{k+\beta}, U_2^{k+\beta}, U_3^{k+\beta}$ quantitative. Here, as always, the integer $k \geq 1$ and $\beta \in (0,1)$.

Lemma 4.3. *For any $\delta > 0$, the set $E_\delta^{k+\beta} \subset X^{k+\beta} \times \mathbb{R}$ defined by the inequalities*

$$\min_{\mathbb{T}}|\Omega\overline{\phi} + \tfrac{1}{2}\mathcal{C}(\phi)\overline{\phi}|,\ \frac{1}{1+|\Omega|+\|\phi\|_{C^{k+\beta}}},\ \frac{\pi}{2} - \max_{\mathbb{T}}\left|\arg\frac{w\phi'}{\phi}\right|,\ \min_{\mathbb{T}}|\phi'|,\ \min_{\mathbb{T}}|\phi| \geq \delta, \tag{4.2}$$

is a closed and bounded subset of $U^{k+\beta}$. Moreover, for any $(\phi - w, \Omega) \in U^{k+\beta}$ there exists $\delta > 0$ so that $(\phi - w, \Omega) \in E_\delta^{k+\beta}$.

Proof. First we prove the last statement. If $(\phi - w, \Omega) \in U^{k+\beta}$, then the existence of a bound on $|\Omega\overline{\phi} + \frac{1}{2}\mathcal{C}(\phi)\overline{\phi}|$ follows from $(\phi - w, \Omega) \in U_2^{k+\beta}$, the second bound is immediate, and the remaining bounds follow from $(\phi - w, \Omega) \in U_3^{k+\beta}$.

It remains to show that $E_\delta^{k+\beta} \subset U^{k+\beta}$ is closed and bounded. The boundedness is clear, as is the containment $E_\delta^{k+\beta} \subset U_2^{k+\beta} \cap U_3^{k+\beta}$. The containment $E_\delta^{k+\beta} \subset U_1^{k+\beta}$ then follows from Lemma 2.6. The second and final two conditions in (4.2) are clearly closed, and the third is also closed in combination with the final two since they avoid the potential singularities in the arg function. Finally, from the last three inequalities in (4.2) one can check that $\operatorname{dist}_{X^{k+\beta}\times\mathbb{R}}(E_\delta^{k+\beta}, \partial U_3^{k+\beta}) > 0$. Again by Lemma 2.6 we have $U_3^{k+\beta} \subset U_1^{k+\beta}$, and so we conclude that the closure of $E_\delta^{k+\beta}$ is contained in $U_1^{k+\beta}$. Since the mapping $(\phi - w, \Omega) \mapsto \Omega\overline{\phi} + \frac{1}{2}\mathcal{C}(\phi)\overline{\phi}$ is analytic $U_1^{k+\beta} \to C^{k+\beta}(\mathbb{T})$ by Theorem 2.3, the mapping $(\phi - w, \Omega) \mapsto \min_{\mathbb{T}}|\Omega\overline{\phi} + \frac{1}{2}\mathcal{C}(\phi)\overline{\phi}|$ is continuous, and so the first condition in (4.2) is closed. □

As the following lemma shows, solutions in $E_\delta^{1+\alpha}$ are automatically C^∞, with higher derivatives controlled by δ. The main ingredient is Lemma 2.4 on the solvability of Riemann–Hilbert problems. We will prove the analyticity of ∂D in Section 5.4.

Lemma 4.4 (Local compactness of the solution set). *For any $\delta > 0$ and $k \geq 1$, the set $\mathscr{S}^{1+\alpha} \cap E_\delta^{1+\alpha}$ is a compact subset of $C^k(\mathbb{T}) \times \mathbb{R}$. In particular, there is a constant $C(k, \delta) > 0$ so that any solution $(f, \Omega) \in E_\delta^{1+\alpha} \cap \mathscr{S}^{1+\alpha}$ satisfies*

$$\|f\|_{C^k(\mathbb{T})} < C. \tag{4.3}$$

Proof. Let $(f, \Omega) \in \mathscr{S}^{k+\beta}$ for some $k \geq 1$ and $\beta \in (0,1)$. As in the proof of Lemma 4.2, we set

$$A = \mathscr{A}(f, \Omega) = \big(\Omega(\overline{w} + \overline{f}) + \tfrac{1}{2}\mathcal{C}(w + f)(\overline{w} + \overline{f})\big)w$$

and view $\mathscr{F}^{k+\beta}(f, \Omega) = 0$ as the Riemann–Hilbert problem $\operatorname{Im}(A\phi') = 0$. Applying Lemmas 4.1 and 2.4(a), this Riemann–Hilbert problem can be explicitly solved to obtain

$$f'(w) = \exp\left\{\frac{w}{2\pi}\int_{\mathbb{T}} \frac{1}{\tau - w}\left[\frac{1}{\tau}\arg\left(\frac{A(\tau)}{\overline{A(\tau)}}\right) - \frac{1}{w}\arg\left(\frac{A(w)}{\overline{A(w)}}\right)\right] d\tau\right\} - 1 =: \mathscr{G}(f, \Omega). \tag{4.4}$$

We first claim that the expression $\mathscr{G}(f, \Omega)$ on the right hand side of (4.4) defines a continuous mapping $\mathscr{G} : \mathscr{S}^{k+\beta} \to C^{k+\beta}(\mathbb{T})$. Note that Theorem 2.3 guarantees that $\mathscr{A}$ is continuous $U^{k+\beta} \to C^{k+\beta}(\mathbb{T})$; the quotient $\mathscr{A}/\overline{\mathscr{A}}$ is continuous between the same spaces thanks to the restriction $|\mathscr{A}| > 0$ embedded in the definition of $U^{k+\beta}$, and the same is then true for the argument $\arg(\mathscr{A}/\overline{\mathscr{A}})$ thanks to the fact that $\mathscr{A}/\overline{\mathscr{A}}$ has winding number 0 by Lemma 4.1. The Cauchy integral operator appearing in (4.4) is a bounded linear operator from $C^{k+\beta}(\mathbb{T})$ to itself; this is for instance a very special case of Theorem 2.3. Composing with the exponential, we therefore obtain the desired continuity of $\mathscr{G}$ and the claim is proved.

Setting $k = 1$ and $\beta = \alpha$, we see that any $(f, \Omega) \in \mathscr{S}^{1+\alpha}$ has $f' = \mathscr{G}(f, \Omega) \in C^{1+\alpha}(\mathbb{T})$ and hence $f \in C^{2+\alpha}(\mathbb{T})$. Moreover, the continuity of $\mathscr{G} : \mathscr{S}^{1+\alpha} \to C^{1+\alpha}(\mathbb{T})$ implies that the inclusion $\mathscr{S} \hookrightarrow C^{2+\alpha}(\mathbb{T}) \times \mathbb{R}$ is continuous. Iterating this argument with $k = 2$ and so on, we discover that $f \in C^{k+\alpha}(\mathbb{T})$ for all k and that the inclusions $\mathscr{S}^{1+\alpha} \hookrightarrow C^{k+\alpha}(\mathbb{T}) \times \mathbb{R}$ are all continuous.

By Lemma 4.3, $E_\delta^{1+\alpha}$ is a closed and bounded subset of $U^{1+\alpha}$, and hence a compact subset of $U^{1+\alpha/2}$. Since $\mathscr{S}^{1+\alpha/2} \subset U^{1+\alpha/2}$ is closed, $E_\delta^{1+\alpha} \cap \mathscr{S}^{1+\alpha/2}$ is also a compact subset of $\mathscr{S}^{1+\alpha/2}$. And since $\mathscr{S}^{1+\alpha/2}$ includes continuously into $C^k(\mathbb{T}) \times \mathbb{R}$, we conclude that $E_\delta^{1+\alpha} \cap \mathscr{S}^{1+\alpha}$ is a compact subset of $C^k(\mathbb{T}) \times \mathbb{R}$ as desired. In particular, it is bounded in $C^k(\mathbb{T}) \times \mathbb{R}$, which implies (4.3). □

We are now in a position to apply the following version of Theorem 9.1.1 in [BT03] as modified in [CSV16]. In the abstract setting of Theorem 3.1, let

$$\mathcal{C}_{\text{loc}} = \big\{(\tilde{x}(s), \tilde{\lambda}(s)) : 0 < s < \varepsilon\big\}$$

be the portion of the local bifurcation curve with $s > 0$.

Theorem 4.5 (Analytic global bifurcation)**.** *In the setting of Theorem 3.1, suppose in addition that*

(d) $\mathcal{F}_x(x,\lambda)$ *is a Fredholm operator of index zero whenever* $\mathcal{F}(x,\lambda)=0$*; and*

(e) *for some sequence* $(Q_j)_{j=1}^\infty$ *of bounded closed subsets of* U *with* $U=\cup_j Q_j$*, the set* $\{(x,\lambda)\in U : \mathcal{F}(x,\lambda)=0\}\cap Q_j$ *is compact for each* j*.*

Then there exists a continuous curve $\mathcal{C}$ *of solutions, with*

$$\mathcal{C}_{\mathrm{loc}}\subset\mathcal{C}=\{(\tilde{x}(s),\tilde{\lambda}(s)) : 0<s<\infty\}\subset\mathcal{F}^{-1}(0)$$

where $(\tilde{x},\tilde{\lambda})\colon(0,\infty)\to X\times\mathbb{R}$ *is continuous, and such that one of the following occurs*

(i) *there exists* $T>0$ *such that, after a reparametrization,* $(\tilde{x}(s+T),\tilde{\lambda}(s+T))=(\tilde{x}(s),\tilde{\lambda}(s))$*;*

(ii) *for every* $j\in\mathbb{N}$ *there exists* $s_j>0$ *such that* $(\tilde{x}(s),\tilde{\lambda}(s))\notin Q_j$ *for* $s>s_j$*.*

Moreover, the curve $\mathcal{C}$ *has a real-analytic parametrization locally around each of its points, and is unique (up to reparametrization).*

Applying Theorem 4.5 to our problem we obtain the following intermediate result.

Theorem 4.6 (Global bifurcation of vortex patches)**.** *There exists a continuous curve* $\mathscr{C}$ *of solutions which extends* $\mathscr{C}_{\mathrm{loc}}$*,*

$$\mathscr{C}_{\mathrm{loc}}\subset\mathscr{C}=\{(\tilde{f}(s),\tilde{\Omega}(s)) : 0<s<\infty\}\subset\mathscr{S}$$

and such that either

(i) *there exists* $T>0$ *such that, after a reparametrization,* $(\tilde{f}(s+T),\tilde{\Omega}(s+T))=(\tilde{f}(s),\tilde{\Omega}(s))$*; or*

(ii) *as* $s\to\infty$*,*

$$\min\left\{\min_{\mathbb{T}}\left|\Omega\overline{\tilde{\phi}}+\tfrac{1}{2}\mathcal{C}(\phi)\overline{\tilde{\phi}}\right|,\ \frac{1}{1+|\Omega|+\|\tilde{\phi}\|_{C^{1+\alpha}}},\ \frac{\pi}{2}-\max_{\mathbb{T}}\left|\arg\frac{w\tilde{\phi}'}{\tilde{\phi}}\right|,\ \min_{\mathbb{T}}|\tilde{\phi}'|,\ \min_{\mathbb{T}}|\tilde{\phi}|\right\}\longrightarrow 0, \tag{4.5}$$

where here $\tilde{\phi}(s)=w+\tilde{f}(s)$*.*

Moreover, the curve $\mathscr{C}$ *has a real-analytic parametrization locally around each of its points, and is unique (up to reparametrization).*

Proof. We apply Theorem 4.5. By Theorem 3.3, $\mathscr{F}\colon U\to Y$ satisfies the hypotheses of Theorem 3.1, and by Lemma 4.2 $\mathscr{F}$ satisfies the Fredholm index zero assumption (d). Setting $Q_j=E^{3+\alpha}_{1/j}$, Lemmas 4.3 and 4.4 guarantee that the remaining assumption (e) is satisfied. With this choice of Q_j, alternative (ii) above is a restatement of alternative (ii) in Theorem 4.5, except that the Hölder exponent has been reduced from $3+\alpha$ to $1+\alpha$ thanks to Lemma 4.4. ☐

5 Streamlines

In this section we study the level sets of the relative stream function Ψ, including of course the boundary ∂D of the vortex patch. These curves, called streamlines, represent fluid particle trajectories in the rotating frame. In addition to their independent interest, the results in this section will allow us to eliminate the alternative (i) in Theorem 4.6 that the global curve $\mathscr{C}$ of vortex patches forms a closed loop.

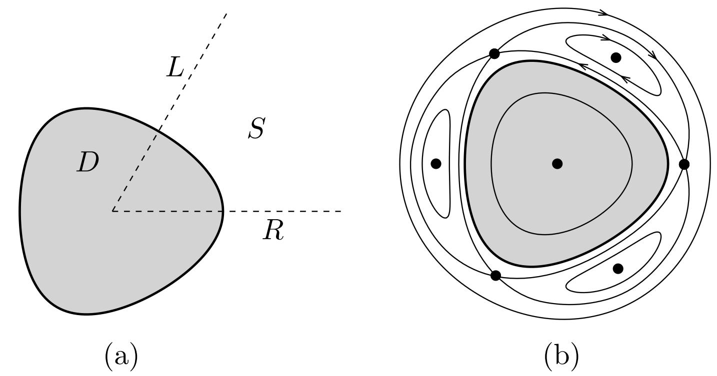

Figure 5: (a) The sector S and rays L, R in (5.1). (b) Level curves and critical points of Ψ.

For a solution $(\phi - w, \Omega) \in \mathscr{S}$, the symmetries $\phi(e^{2\pi i/m}w) = e^{2\pi i/m}\phi(w)$ and $\phi(\overline{w}) = \overline{\phi(w)}$ imply that the solution Ψ of (1.2a)–(1.2c) is $2\pi/m$-periodic and even in the polar coordinate θ. Thus it is enough to describe Ψ on the fundamental sector S with left and right boundary portions L, R given in polar coordinates by

$$S = \{re^{i\theta} : r > 0,\, 0 < \theta < \pi/m\}, \quad L = \{r > 0,\, \theta = \pi/m\}, \quad R = \{r > 0,\, \theta = 0\}. \tag{5.1}$$

See Figure 5a for an illustration. We will show in Section 5.3 that every vortex patch in $\mathscr{C}$ satisfies

$$\Psi_r > 0 \text{ on } \partial D, \tag{5.2a}$$
$$\Psi_\theta > 0 \text{ on } S, \tag{5.2b}$$
$$\Psi_{\theta\theta} > 0 \text{ on } R, \tag{5.2c}$$
$$\Psi_{\theta\theta} < 0 \text{ on } L. \tag{5.2d}$$

With $r = \tilde{r}(\theta)$ a polar parametrization of the boundary ∂D of the vortex patch, (5.2) and the identity $\Psi(\tilde{r}(\theta), \theta) = 0$ together imply the inequalities

$$\tilde{r}'(\theta) < 0 \text{ for } 0 < \theta < \frac{\pi}{m}, \quad \tilde{r}''(0) < 0, \quad \tilde{r}''\Big(\frac{\pi}{m}\Big) > 0$$

claimed in Theorem 1.1(c).

Definition 5.1 (Nodal set)**.** We define the "nodal set" $\mathscr{N}$ to be the subset of $\mathscr{S} \setminus \mathscr{T}$ where (5.2) holds.

In Section 5.2, we will show that the vortex patches along the local curve $\mathscr{C}_{\text{loc}}$ not only satisfy (5.2) but also satisfy the additional inequalities

$$\Psi^-_{r\theta} < 0 \text{ on } S \setminus D, \tag{5.3a}$$
$$(r\partial_r)^2\Psi^- < 0 \text{ on } \mathbb{C} \setminus D, \tag{5.3b}$$
$$\Psi^-_{r\theta\theta} < 0 \text{ on } R \setminus D, \tag{5.3c}$$
$$\Psi^-_{r\theta\theta} > 0 \text{ on } L \setminus D, \tag{5.3d}$$

which imply that the contour plot of Ψ on $\mathbb{C}\setminus D$ looks qualitatively like Figure 5b; see Theorem 1.2.

5.1 Preliminary lemmas

First we prove two simple lemmas which will be useful for both the local and global arguments to follow.

Lemma 5.2 (Robustness of simple roots)**.** *Consider the Banach space*

$$Z = \{g \in C^1([0,1],\mathbb{R}) : g(0) = 0\}$$

of C^1 functions vanishing at 0, and the subset

$$V = \{g \in Z : g(t) > 0 \text{ for } t > 0,\ g'(0) > 0\}$$

of functions which are positive away from 0 and have a simple root. Then

(a) *$V \subset Z$ is open.*

(b) *If $G\colon (-1,1) \to Z$ is a C^1 map with $G(0) = 0$ and $G'(0) \in V$, then $G(s) \in V$ for $s > 0$ sufficiently small.*

Proof. First we prove (a). Let $g_0 \in V$ and let $g \in Z$ have $\|g - g_0\|_{C^1} < \varepsilon$ for some $\varepsilon > 0$ to be determined. From $g_0 \in V$ we deduce that $\partial_t g_0 > 0$ on $[0,\delta]$ and $g_0 > 0$ on $[\delta,1]$ for some $\delta > 0$. As the above intervals are compact, we can choose ε small enough that the same strict inequalities hold for g. The remaining inequality $g(t) > 0$ for $t \in (0,\delta)$ then follows from the mean value theorem. From the differentiability of G we have $\|s^{-1}G(s) - G'(0)\|_Z \to 0$ as $s \searrow 0$, and so $s^{-1}G(s) \in V$ for s sufficiently small. Since V is invariant under multiplication with positive scalars, this proves (b). ☐

Lemma 5.3 (Weaker streamline conditions)**.** *For a solution in $\mathscr{S} \setminus \mathscr{T}$, the conditions* (5.2) *are equivalent to the weaker conditions*

$$\Psi_r > 0 \text{ on } \partial D, \tag{5.4a}$$
$$\Psi_\theta > 0 \text{ on } \partial D, \tag{5.4b}$$
$$\Psi_{\theta\theta} > 0 \text{ on } R \cap \partial D, \tag{5.4c}$$
$$\Psi_{\theta\theta} < 0 \text{ on } L \cap \partial D. \tag{5.4d}$$

Proof. Suppose that (5.4) holds. Then (5.2a) holds since it is just (5.4a), and so it suffices to prove (5.2b)–(5.2d). To show (5.2b), we use the maximum principle. Differentiating (1.2a) shows that Ψ_θ is harmonic on both $S \cap D$ and $S \setminus D$, is continuous across ∂D, and vanishes at infinity. By symmetry, we also have $\Psi_\theta = 0$ on ∂S. Thus if $\inf_S \Psi_\theta < 0$, by the maximum principle this infimum would have to be achieved on $S \cap \partial D$, a contradiction. Therefore $\inf_S \Psi_\theta = 0$, and so the strong maximum principle implies $\Psi_\theta > 0$ on S. Applying the Hopf lemma on R and L, we see that (5.2c) and (5.2d) hold except potentially on $R \cap \partial D$ and $L \cap \partial D$. But there the inequalities follow from (5.4c) and (5.4d), and so (5.2) holds as desired. ☐

5.2 Streamlines of small solutions

In this section we prove that both (5.2) and (5.3) hold along $\mathscr{C}_{\mathrm{loc}}$.

We begin by using Lemmas 2.1 and 2.2 to translate the expansion for the conformal mapping ϕ in Theorem 3.3(ii) into expansions for the derivatives of Ψ restricted to ∂D.

Lemma 5.4 (Expansions for derivatives of Ψ)**.** *Along the local curve $\mathscr{C}_{\mathrm{loc}}$ of vortex patches $(\tilde f(s), \tilde\Omega(s))$, the partial derivatives of the corresponding relative stream functions $\Psi^- = \Psi|_{\mathbb{C}\setminus D}$ have the following expansions:*

$$\begin{aligned}
\partial_\theta \Psi^-(\phi(w);s) &= s\tfrac{1}{2}\operatorname{Im} w^m + O(s^2),\\
r\partial_r \Psi^-(\phi(w);s) &= \tfrac{1}{2m} + O(s),\\
\partial_\theta r\partial_r \Psi^-(\phi(w);s) &= -s\tfrac{m}{2}\operatorname{Im} w^m + O(s^2),\\
(r\partial_r)^2 \Psi^-(\phi(w);s) &= -\tfrac{m-1}{m} + O(s),\\
\partial_\theta^2 \Psi^-(\phi(w);s) &= s\tfrac{m}{2}\operatorname{Re} w^m + O(s^2),\\
\partial_\theta^2 r\partial_r \Psi^-(\phi(w);s) &= -s\tfrac{m^2}{2}\operatorname{Re} w^m + O(s^2).
\end{aligned} \tag{5.5}$$

Proof. By Theorem 3.3(ii), we have the asymptotic expansions

$$\phi(w;s) := w + \tilde{f}(s)(w) = w + \frac{s}{w^{m-1}} + O(s^2), \qquad \tilde{\Omega}(s) = \frac{m-1}{2m} + O(s^2) \tag{5.6}$$

in $C^{3+\alpha}(\mathbb{T})$ and $\mathbb{R}$ respectively. From Lemmas 2.1 and 2.2 and the analyticity of the Cauchy operator, we know that the compositions $\partial_z^k \Psi^- \circ \phi$ depend analytically on s as elements of $C^{4-k+\alpha}(\mathbb{T})$ for $k = 1, 2, 3$. Inserting the expansion (5.6) into (2.3) and using the calculus of residues to evaluate the integrals, a relatively short calculation gives

$$\partial_z \Psi^-(\phi(w;s);s) = \frac{1}{4w} - \frac{m-1}{4m}\overline{w} - s\frac{m-1}{4m}\frac{1}{\overline{w}^{m-1}} + O(s^2). \tag{5.7}$$

Increasingly tedious calculations of the same type yield

$$\begin{aligned} \partial_z^2 \Psi^-(\phi(w;s);s) &= -\frac{1}{4w^2} - s\frac{m-1}{4}\frac{1}{w^{m+2}} + O(s^2), \\ \partial_z^3 \Psi^-(\phi(w;s);s) &= \frac{1}{2w^3} + s\frac{(m-1)(m+4)}{4}\frac{1}{w^{m+3}} + O(s^2). \end{aligned} \tag{5.8}$$

Alternatively, one can derive (5.8) directly from (5.7) using the identities

$$\partial_z^2 \Psi^-(\phi(w;s);s) = \frac{\partial_w[\partial_z \Psi^-(\phi(w;s);s)]}{\phi'(w;s)}, \qquad \partial_z^3 \Psi^-(\phi(w;s);s) = \frac{\partial_w[\partial_z^2 \Psi^-(\phi(w;s);s)]}{\phi'(w;s)}.$$

Using the convenient formulas $r\partial_r = z\partial_z + \overline{z}\partial_{\overline{z}}$ and $\partial_\theta = i(z\partial_z - \overline{z}\partial_{\overline{z}})$, the partials of Ψ^- can then be easily calculated in terms of $\partial_z \Psi^-, \partial_z^2 \Psi^-, \partial_z^3 \Psi^-$:

$$\begin{aligned} \partial_\theta \Psi^-(z) &= -2\,\mathrm{Im}[z\partial_z \Psi^-], \\ r\partial_r \Psi^-(z) &= 2\,\mathrm{Re}\left[z\partial_z \Psi^-\right], \\ \partial_\theta r\partial_r \Psi^-(z) &= -2\,\mathrm{Im}\left[z\partial_z \Psi^- + z^2\partial_z^2 \Psi^-\right], \\ (r\partial_r)^2 \Psi^-(z) &= 2\,\mathrm{Re}\left[z\partial_z \Psi^- + z^2\partial_z^2 \Psi^-\right] - \Omega|z|^2, \\ \partial_\theta^2 \Psi^-(z) &= -2\,\mathrm{Re}\left[z\partial_z \Psi^- + z^2\partial_z^2 \Psi^-\right] - \Omega|z|^2, \\ \partial_\theta^2 r\partial_r \Psi^-(z) &= -2\,\mathrm{Re}\left[z\partial_z \Psi^- + 3z^2\partial_z^2 \Psi^- + z^3\partial_z^3 \Psi^-\right] - \Omega|z|^2, \end{aligned} \tag{5.9}$$

where here we repeatedly have used the identity $\partial_z\partial_{\overline{z}}\Psi^- = \partial_{\overline{z}}\partial_z\Psi^- = \frac{1}{4}\Delta\Psi^- = -\frac{\Omega}{2}$. Plugging (5.7)–(5.8) into (5.9), and using the identity $\overline{w} = 1/w$ for $w \in \mathbb{T}$, we finally obtain (5.5) as desired. ☐

We can now establish (5.2) and (5.3) by using the expansions in Lemma 5.4, Lemma 5.2, and several non-obvious maximum principle arguments.

Proposition 5.5 (Signs of derivatives of Ψ)**.** *The inequalities* (5.2) *and* (5.3) *hold along* $\mathscr{C}_{\mathrm{loc}}$*, after possibly reducing* ε *in Definition 3.4.*

Proof. By Lemma 5.3, to prove (5.2) it suffices to prove (5.4). The first inequality (5.4a) as well as the last two (5.4c)–(5.4d) follow immediately from the expansion (5.5) in Lemma 5.4.

Thus in order to prove (5.2) it remains only to show that (5.4b) holds. For this, we use the polar coordinate representation

$$\phi(e^{it}) = \rho(t)e^{i\vartheta(t)}$$

introduced in Lemma 2.5, where we are temporarily suppressing dependence on s. The symmetries (2.9) guarantee that ρ is even and $2\pi/m$-periodic in t, and hence in particular that $\rho'(t) = 0$ for $t = 0, \pi/m$. Therefore, if $g \in C^1(\mathbb{C} \setminus D)$, we have

$$\begin{aligned} \frac{d}{dt} g(\phi(e^{it})) &= \partial_r g(\phi(e^{it}))\rho'(t) + \partial_\theta g(\phi(e^{it}))\vartheta'(t) \\ &= \mathrm{Re}\left(\frac{e^{it}\phi'(e^{it})}{\phi(e^{it})}\right)\partial_\theta g(\phi(e^{it})) \qquad \text{at } t = 0, \pi/m. \end{aligned}$$

Setting $w = e^{it}$, we abbreviate this as

$$\frac{d}{dt} = \operatorname{Re}\left(\frac{w\phi'}{\phi}\right)\frac{\partial}{\partial\theta} \qquad \text{at } t = 0, \pi/m. \tag{5.10}$$

Reintroducing the dependence on s, (5.4b) is equivalent to

$$g(t;s) := \partial_\theta \Psi(\phi(e^{it};s);s) > 0 \quad \text{ for } 0 < t < \pi/m. \tag{5.11}$$

The symmetries (2.9) guarantee that g is an odd and $2\pi/m$-periodic function of t. Moreover, as in the proof of Lemma 5.4, g depends analytically on s as a $C^{3+\alpha}$ function, and the expansion (5.5) gives $g(t;0) \equiv 0$ and

$$g_s(t;0) = \tfrac{1}{2}\sin mt > 0 \quad \text{ for } 0 < t < \pi/m.$$

Finally, at $t = 0$ the expansions (5.6) and (5.5) yield

$$g_t(0;s) = \operatorname{Re}\left(\frac{\phi'}{\phi}\right)\partial_{\theta\theta}\Psi(\phi(1;s);s) = s\tfrac{m}{2} + O(s^2),$$

while at $t = \pi/m$ we similarly obtain

$$g_t(\pi/m;s) = -s\tfrac{m}{2} + O(s^2).$$

Applying Lemma 5.2(b), we deduce that (5.11) and hence (5.4b) hold for $s > 0$ sufficiently small.

The proof of (5.3a), (5.3c), and (5.3d) is quite similar. Setting

$$g(t;s) = \partial_\theta r\partial_r \Psi^-(\phi(e^{it};s);s),$$

g is again an odd and $2\pi/m$-periodic function of t, this time with the expansions

$$g(t;s) = -s\tfrac{m}{2}\sin mt + O(s^2), \qquad g_t(0;s) = -s\tfrac{m^2}{2} + O(s^2), \qquad g_t(\pi/m;s) = +s\tfrac{m^2}{2} + O(s^2).$$

Lemma 5.2(b) then guarantees that $g < 0$ for $0 < t < \pi/m$ and $s > 0$ sufficiently small, and hence that $\partial_\theta r\partial_r\Psi < 0$ on $S \cap \partial D$. Now $\partial_\theta r\partial_r\Psi$ is harmonic on $S \setminus D$, vanishes at infinity, and vanishes along ∂S. Thus, as in the proof of Lemma 5.3, the maximum principle forces $\partial_\theta r\partial_r\Psi < 0$ on $S \setminus D$. The inequalities (5.3c) and (5.3d) hold on $R \cap \partial D$ and $L \cap \partial D$ by the expansion (5.5), and on $R \setminus \partial D$ and $L \setminus \partial D$ by the Hopf lemma applied to $\partial_\theta r\partial_r\Psi$.

The only remaining inequality is (5.3b). A direct calculation shows that the function

$$\varphi(r,\theta) = (r\partial_r)^2\Psi^-(r,\theta) + 2\Omega r^2$$

is harmonic. As with our other harmonic functions, it vanishes at infinity. From the expansions (5.5) and (5.6), $\varphi(\phi(w;s);s) = O(s)$, and so by the maximum principle

$$\|\varphi\|_{L^\infty(\mathbb{C}\setminus D)} = O(s).$$

We also of course have

$$\min_{z\in\partial D(s)} \Omega(s)|z|^2 = \min_{w\in\mathbb{T}} \Omega(s)|\phi(w;s)|^2 = \frac{m-1}{2m} - O(s).$$

Combining these two facts, we find that on $\mathbb{C} \setminus D$,

$$\begin{aligned}(r\partial_r)^2\Psi(z;s) &= \varphi(z;s) - 2\Omega|z|^2 \\ &< \|\varphi\|_{L^\infty(\mathbb{C}\setminus D)} - 2\min_{z\in\partial D}\Omega(s)|z|^2 \\ &= -\tfrac{m-1}{m} + O(s) < 0\end{aligned}$$

for s sufficiently small. □

Combining Proposition 5.5 with simpler arguments for the other small solutions described by Theorem 3.3, we arrive at the following characterization of $\mathscr{N}$ near $\mathscr{T}$.

Lemma 5.6 (Nodal properties of small solutions)**.**

(a) *If $\Omega \neq \Omega_m$, then there exists a neighborhood $V \subset U$ of $(0, \Omega)$ such that $V \cap \mathscr{N} = \varnothing$.*

(b) *There exists a neighborhood $V \subset U$ of $(0, \Omega_m)$ such that $V \cap \mathscr{N} = \mathscr{C}_{\mathrm{loc}}$.*

Proof. We apply Theorem 3.3. If Ω is not one of the Ω_{nm}, then we can simply choose V by part (i) of that theorem to get $V \cap \mathscr{N} \subset \mathscr{N} \cap \mathscr{T} = \varnothing$. If, on the other hand, $\Omega = \Omega_{nm}$ for some $n > 1$, then by part (ii) of Theorem 3.3, we know that V can be chosen so that $V \cap (\mathscr{S} \setminus \mathscr{T})$ consists of solutions

$$f = \tilde{f}(s) = \frac{s}{w^{nm-1}} + o(s),$$

where $s \neq 0$ is the parameter along the bifurcation curve, and the remainder refers to a term which is $o(s)$ in X. Arguing as in Lemma 5.4, we discover that

$$\partial_\theta \Psi(\phi(w; s); s) = -s\tfrac{1}{2} \operatorname{Im} w^{-nm} + O(s^2).$$

In particular, $\partial_\theta \Psi$ changes sign on $S \cap \partial D$ for $s \neq 0$ sufficiently small, contradicting (5.4b). By Proposition 5.5, solutions with $n = 1$ and $s > 0$ is sufficiently small, i.e. solutions on $\mathscr{C}_{\mathrm{loc}}$, do lie in $\mathscr{N}$. On the other hand for $s < 0$ small we have

$$\partial_\theta \Psi(\phi(e^{\pi i/2m}; s); s) = -s\tfrac{1}{2} + O(s^2) < 0,$$

again contradicting (5.4b). □

With Proposition 5.5 in hand, we can also complete the proof of Theorem 1.2.

Proof of Theorem 1.2. First we claim that there exists a unique polar curve $r = r_c(\theta)$ in $\overline{S} \setminus D$ such that $\Psi_r > 0$ for $r < r_c$ and $\Psi_r < 0$ for $r > r_c$. Since $\Psi_r > 0$ on ∂D by (5.2a) and $\Psi_r \to -\infty$ as $r \to \infty$ by (1.2b), for each θ there exists a radius $r_c(\theta)$ such that $(r, \theta) \notin D$ and $\Psi_r(r_c(\theta), \theta) = 0$. From (5.3b) we see that $\Psi_{rr} = (r\partial r)^2 \Psi / r^2 < 0$ along $r = r_c$, which gives the uniqueness and also $r_c \in C^{2+\alpha}$. Indeed, by standard elliptic theory Ψ is analytic away from ∂D, and so by the analytic implicit function theorem r_c is analytic. We note that implicit differentiation yields $r_c' = -\Psi_{r\theta}/\Psi_{rr} < 0$.

From (5.2b) and the above arguments, the unique equilibria in $\overline{S} \setminus D$ are $P := (r_c(0), 0)$ and $Q := (r_c(\pi/m), \pi/m)$. At P, (5.3b) and (5.2c) give

$$\Psi_{\theta\theta}\Psi_{rr} - 2\Psi_{\theta r}^2 = \Psi_{\theta\theta}\Psi_{rr} < 0$$

so that P is a saddle. Similarly (5.3b) and (5.2d) guarantee that Q is a center. The remaining statements now follow from a straightforward nullcline analysis (in polar coordinates). □

5.3 Streamlines for the global curve

We now show that the inequalities (5.2) hold not just along the local curve $\mathscr{C}_{\mathrm{loc}}$ but also along the global curve $\mathscr{C}$. This will be sufficient to eliminate the alternative (i) in Theorem 4.6 that $\mathscr{C}$ forms a closed loop. Unlike in Section 5.2, we will not be able to rely on asymptotic expansions, and will instead have to use more subtle arguments involving the structure of the equations. Conjecture 1.4 is that the inequalities (5.3) also hold along $\mathscr{C}$, but we only have numerical evidence of this fact; see Section 7.

Proposition 5.7 (Robustness of the nodal set)**.** *The nodal set $\mathscr{N}$ is both relatively open and relatively closed in $\mathscr{S} \setminus \mathscr{T}$.*

Proof. First we claim that $\mathscr{N}$ is relatively open. To be concrete, fix $(f^0, \Omega^0) \in \mathscr{N}$ and consider $(f^1, \Omega^1) \in \mathscr{S} \setminus \mathscr{T}$ with $\|(f^1, \Omega^1) - (f^0, \Omega^0)\|_{X\times\mathbb{R}} < \varepsilon$ where $\varepsilon > 0$ is to be determined. Let Ψ^0, Ψ^1 be the corresponding relative stream functions. Then Ψ^0 satisfies (5.2), and by Lemma 5.3 we will have $(f^1, \Omega^1) \in \mathscr{N}$ as soon as Ψ^1 satisfies (5.4). From Lemma 2.1 and (5.9), we know that the composition $\partial_r \Psi \circ \phi$ depends continuously on $(f, \Omega) \in \mathscr{S}$ as an element of $C^{3+\alpha}(\mathbb{T})$. In particular, we can choose $\varepsilon > 0$ small enough that Ψ^1 satisfies (5.4a). For the remaining inequalities, we consider the composition

$$g(t) := \partial_\theta \Psi(\phi(e^{it}))$$

so that (5.4b) is equivalent to $g > 0$ for $0 < t < \pi/m$. Again thanks to Lemma 2.1 and (5.9), we see that g depends continuously on $(f, \Omega) \in \mathscr{S} \setminus \mathscr{T}$ as an element of $C^{3+\alpha}[0, 2\pi]$. Moreover, by symmetry, g is an odd and $2\pi/m$-periodic function of t. Differentiating using (5.10) yields

$$g'(t) = \operatorname{Re}\left(\frac{e^{it}\phi'(e^{it})}{\phi(e^{it})}\right) \partial_{\theta\theta}\Psi(\phi(e^{it})) \qquad \text{at } t = 0, \pi/m.$$

For $(f, \Omega) \in \mathscr{S}$, the factor multiplying $\partial_{\theta\theta}\Psi$ above is strictly positive, and indeed we can choose $\varepsilon > 0$ so that there is a uniform lower bound for the (f^1, Ω^1) under consideration. Thus, in this neighborhood, $g'(0) > 0$ is equivalent to (5.4d) and $g'(\pi/m) < 0$ is equivalent to (5.4c). Applying Lemma 5.2(a), we conclude that there exists $\delta > 0$ so that $\|g^1 - g^0\|_{C^1} < \delta$ implies that $g^1 > 0$ for $0 < t < \pi/m$, $(g^1)'(0) > 0$, and $(g^1)'(\pi/m) < 0$. Thanks to the continuous dependence of the compositions $\Psi_\theta \circ \phi$, $\Psi_{\theta\theta} \circ \phi$ on $(f, \Omega) \in \mathscr{S}$ as elements of $C^{2+\alpha}(\mathbb{T})$, we can therefore choose $\varepsilon > 0$ in terms of δ so that Ψ^1 satisfies (5.4b)–(5.4d) as desired.

We now come to the core of the proof: showing that $\mathscr{N}$, which is defined in terms of strict inequalities, is nevertheless relatively closed. Suppose that a sequence of patches $(f^n, \Omega^n) \in \mathscr{N}$ converges to some patch $(f, \Omega) \in \mathscr{S} \setminus \mathscr{T}$, and let Ψ^n, Ψ be the associated relative stream functions. Taking limits in (5.4), we see that

$$\Psi_r \geq 0 \text{ on } \partial D, \tag{5.12a}$$
$$\Psi_\theta \geq 0 \text{ on } \partial D, \tag{5.12b}$$
$$\Psi_{\theta\theta} \geq 0 \text{ on } R \cap \partial D, \tag{5.12c}$$
$$\Psi_{\theta\theta} \leq 0 \text{ on } L \cap \partial D. \tag{5.12d}$$

Now $(f, \Omega) \in U_2$ implies that $\Omega\overline{\phi} + \frac{1}{2}\mathcal{C}(\phi)\overline{\phi} \neq 0$, and so by (2.8)

$$\tfrac{1}{4}\left(\Psi_r^2 + r^{-2}\Psi_\theta^2\right) = |\partial_z \Psi|^2 = |\Omega\overline{\phi} + \tfrac{1}{2}\mathcal{C}(\phi)\overline{\phi}| \neq 0. \tag{5.13}$$

Suppose for the sake of contradiction that $\Psi_r(z_0) = 0$ at some point $z_0 = \phi(e^{it_0}) \in \partial D$, in which case (5.13) forces $\Psi_\theta(z_0) \neq 0$. Writing $\phi = \rho e^{i\vartheta}$ as in Lemma 2.5, $\Psi|_{\partial D} \equiv 0$ gives

$$0 = \frac{d}{dt}\Psi(\rho e^{i\vartheta}) = \Psi_r \rho' + \Psi_\theta \vartheta' = \Psi_\theta \vartheta' \tag{5.14}$$

at z_0, and hence that $\vartheta' = 0$ there. But $(f, \Omega) \in U_3$ forces

$$\vartheta'(t) = \operatorname{Re}\left(\frac{e^{it}\phi'(e^{it})}{\phi(e^{it})}\right) > 0,$$

so this is a contradiction.

We now turn to (5.12b)–(5.12d). Since $\Psi_r > 0$ on ∂D, we know that ∂D is a $C^{3+\alpha}$ curve which can be written in polar coordinates as $r = R(\theta)$. Thus the restrictions $\Psi^+ := \Psi|_D$ and $\Psi^- := \Psi|_{\mathbb{C}\setminus\overline{D}}$ are also $C^{3+\alpha}$ up to their respective boundaries. As in the proof of Lemma 5.3, applying the strong maximum principle to $\Psi_\theta^\pm$ shows that $\Psi_\theta > 0$ on $S \setminus \partial D$.

We now study the values Ψ_θ on $S \cap \partial D$ using the Hopf lemma. Since $\Psi|_{\partial D} \equiv 0$ and $\Psi_r > 0$, an outward-pointing normal vector along ∂D is

$$\frac{\partial}{\partial n} := \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\Psi_\theta}{\Psi_r}\frac{\partial}{\partial \theta}.$$

Suppose for the sake of contradiction that $\Psi_\theta = 0$ at some point $z_0 \in S \cap \partial D$. Applying the Hopf lemma separately to $\Psi_\theta^\pm$, we find

$$\Psi_{\theta r}^+ = \frac{\partial}{\partial n}\Psi_\theta^+ < 0 < \frac{\partial}{\partial n}\Psi_\theta^- = \Psi_{\theta r}^- \tag{5.15}$$

at z_0. On the other hand, differentiating $\nabla\Psi^+ = \nabla\Psi^-$ along ∂D yields

$$\Psi_{\theta\theta}^+ - \frac{\Psi_\theta}{\Psi_r}\Psi_{\theta r}^+ = \Psi_{\theta\theta}^- - \frac{\Psi_\theta}{\Psi_r}\Psi_{\theta r}^-, \tag{5.16}$$

$$\Psi_{r\theta}^+ - \frac{\Psi_\theta}{\Psi_r}\Psi_{rr}^+ = \Psi_{r\theta}^- - \frac{\Psi_\theta}{\Psi_r}\Psi_{rr}^-. \tag{5.17}$$

At z_0, $\Psi_\theta = 0$ and so (5.17) reduces to $\Psi_{r\theta}^+ = \Psi_{r\theta}^-$, contradicting (5.15). This completes the proof of (5.4b).

Finally, we treat (5.4c)–(5.4d) using the Serrin edge-point lemma [Fra00, Appendix E]. Suppose for the sake of contradiction that $\Psi_{\theta\theta} = 0$ at the singleton $z_0 \in \partial D \cap R$. Here we have dropped the superscript $\pm$ since $\Psi \in C^1(\mathbb{C})$ and symmetry force $\Psi_{\theta\theta}^+ = \Psi_{\theta\theta}^-$ at z_0. Symmetry also forces $\Psi_{\theta r} = \Psi_{\theta rr} = \Psi_{\theta\theta\theta} = 0$ along R. Considering $R \cap D$ as a lateral boundary of $S \cap D$, the Hopf lemma implies that $\Psi_{\theta\theta}^+ > 0$ there. Similarly we find that $\Psi_{\theta\theta}^- > 0$ on $R \setminus \overline{D}$. Thus we must have $\Psi_{\theta\theta r}^+ \le 0$ and $\Psi_{\theta\theta r}^- \ge 0$ at z_0. On the other hand, differentiating (5.17) along the boundary once more and plugging in $\Psi_{\theta\theta} = \Psi_{\theta r} = \Psi_{\theta rr} = \Psi_{\theta\theta\theta} = 0$ we eventually discover that $\Psi_{\theta\theta r}^+ = \Psi_{\theta\theta r}^-$ at z_0, and so we must have $\Psi_{\theta\theta r}^+ = \Psi_{\theta\theta r}^- = 0$ there. But now we have shown that all of the first and second partials of Ψ_θ^+ vanish at z_0, violating the Serrin edge-point lemma. The argument at $\partial D \cap L$ is similar. □

We can now rule out the possibility of a loop by combining the previous two lemmas.

Proposition 5.8 (No loop)**.** *The nodal properties* (5.2) *hold for all elements of* $\mathscr{C}$*, i.e.* $\mathscr{C} \subset \mathscr{N}$*. Moreover, alternative* (i) *in Theorem 4.6 does not occur and so* (ii) *must occur.*

Proof. With $\tilde f, \tilde\Omega$ as in Theorem 4.6, set

$$s^* = \sup\{\tilde s > 0 : (\tilde f(s), \tilde\Omega(s)) \in \mathscr{N} \text{ for } 0 < s < \tilde s\}.$$

By Lemma 5.6, the set on the right hand side is nonempty. Assume for the sake of contradiction that $s^* < \infty$. By Lemma 5.7, if $(\tilde f(s^*), \tilde\Omega(s^*)) \notin \mathscr{T}$, then there is a neighborhood of $(\tilde f(s^*), \tilde\Omega(s^*))$ in $\mathscr{S} \setminus \mathscr{T}$ which is contained in $\mathscr{N}$, a contradiction. So $(\tilde f(s^*), \tilde\Omega(s^*)) \in \mathscr{T}$. Applying Lemma 5.6, the only possibility is then $(\tilde f(s^*), \tilde\Omega(s^*)) = (0, \Omega_m)$. Appealing to Theorem 3.3(ii) and Lemma 5.6(b), this means that for $0 < s < s^*$ the global curve $\mathscr{C}$ has revisited portions of the local curve $\mathscr{C}_{\text{loc}}$ twice (once in either direction) without first revisiting the bifurcation point $(0, \Omega_m)$. This contradicts the analytic construction of $\mathscr{C}$ in [BT03, Theorem 9.1.1] or alternatively the fact that $\mathscr{C}$ has a local real-analytic reparametrization; see [CSV16, Proof of Theorem 5]. □

5.4 Analyticity of the patch boundary

In Lemma 4.4 we showed that ϕ and hence also ∂D are smooth for every vortex patch in $\mathscr{S}$. For patches close to the unit disk, Castro, Córdoba, and Gómez-Serrano [CCGS16b] have shown that ∂D and hence also ϕ are in fact analytic. (Their argument also applies near ellipses, and when the Euler equation is replaced by the generalized Surface Quasi-Geostrophic equation.) In this section, we observe that *every* solution in $\mathscr{S}$ is analytic. The proof relies on a theorem of Kinderlehrer, Nirenberg, and Spruck [KNS78] for elliptic free-boundary problems.

A consequence of Proposition 5.8 is that every solution along the global curve $\mathscr{C}$ satisfies $\partial\Psi/\partial n > 0$ on ∂D, where n is a normal vector pointing out of D. In the following lemma, we prove this more generally.

Lemma 5.9. *Let Ψ, D, Ω solve* (1.2) *with $D \in C^1$ and $\Omega < 1/2$. Then*

$$\frac{\partial\Psi}{\partial n} > 0 \text{ on } \partial D.$$

If $\Omega > 1/2$, then the reverse inequality holds.

Proof. First assume that $\Omega < 1/2$. From (1.2), Ψ satisfies

$$\Delta\Psi = 1 - 2\Omega > 0 \text{ in } D, \qquad \Psi = 0 \text{ on } \partial D.$$

By the strong maximum principle, Ψ therefore achieves its maximum over $\overline{D}$ on ∂D, where it is constant. Since ∂D is C^1, by the Hopf lemma either $\partial\Psi/\partial n > 0$ at every point of ∂D or Ψ is constant in D. Since $\Delta\Psi > 0$ in D, Ψ cannot be constant, and so the proof is complete. The argument for $\Omega > 1/2$ is identical except that all of the inequalities are reversed. □

Theorem 5.10 (Analyticity of ∂D)**.** *Let Ψ, D, Ω solve* (1.2) *with $D \in C^1$. If $\Psi \in C^2(\mathbb{C} \setminus D) \cap C^2(\overline{D})$, then ∂D is analytic.*

Proof. If $\Omega = 1/2$, then D is a disk by [Hmi15], and hence ∂D is certainly analytic. So assume that $\Omega \neq 1/2$. Introducing the notation $\Omega^+ = D$, $\Omega^- = \mathbb{C} \setminus D$, and $\Gamma = \partial D$, we have that $\Psi \in C^1(\Omega^+ \cup \Omega^- \cup \Gamma) \cap C^2(\Omega^+ \cup \Gamma) \cap C^2(\Omega^- \cup \Gamma)$ satisfies the inhomogeneous linear elliptic equations

$$\begin{aligned} F(z, \Psi, D\Psi, D^2\Psi) &:= \Delta\Psi + 2\Omega = 0 && \text{in } \Omega^+, \\ G(z, \Psi, D\Psi, D^2\Psi) &:= \Delta\Psi + 2\Omega - 1 = 0 && \text{in } \Omega^-. \end{aligned}$$

Moreover, Lemma 5.9 implies

$$\Psi = 0, \ \frac{\partial\Psi}{\partial n} \neq 0 \qquad \text{on } \Gamma.$$

Thus, by [KNS78, Theorem 3.1′], $\Gamma = \partial D$ is analytic. □

We note that Theorem 5.10 and Lemma 5.9 have local versions where only part of ∂D is assumed to be C^1.

Corollary 5.11. *Let $(f, \Omega) \in \mathscr{S}$ and let $D \subset \mathbb{C}$ be the associated vortex patch. Then ∂D and f are both real-analytic.*

Proof. From the regularity of f we know that $D \in C^{3+\alpha}$, and so standard elliptic theory gives $\Psi \in C^2(\overline{D}) \cap C^2(\mathbb{C} \setminus D)$. Thus Theorem 5.10 applies and ∂D is analytic. The conformal mapping Φ therefore extends to an holomorphic (and one-to-one) mapping on $\{|w| > 1 - \varepsilon\}$ for some $\varepsilon > 0$ [Pom92, Proposition 3.1], and so in particular $t \mapsto f(e^{it}) = \Phi(e^{it}) - e^{it}$ is a real-analytic function of t as desired. □

6 Uniform bounds

We now turn our attention to the remaining alternative (ii) in Theorem 4.6. We will show that the various quantities appearing in (4.5) can all be controlled by the first and third terms, i.e. by the relative fluid speed and the tangent angle of the interface.

6.1 Uniform regularity

In this subsection we establish control over the Hölder norm $\|\phi\|_{C^{1+\alpha}}$ appearing in (4.5). The first step is the following estimate from the theory of conformal mappings.

Lemma 6.1 (Koebe 1/4 theorem)**.** *Any $(\phi - w, \Omega) \in U_1$ satisfies the bound $\|\phi\|_{L^\infty} \leq 4$.*

Proof. From Section 2.2, we know that ϕ extends to a conformal mapping $\Phi\colon \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C}$. Consider the related conformal mapping $g\colon \overline{\mathbb{D}} \to \mathbb{C}$ defined by $g(\zeta) = 1/\Phi(1/\zeta)$. We easily check that $g(0) = 0$ and $g'(0) = 1$. Thus by the Koebe 1/4 theorem (see for instance [Pom92, Theorem 1.3]), we obtain $|g(\zeta)| \geq 1/4$ for $\zeta \in \mathbb{T}$. Rewriting in terms of Φ we obtain $|\Phi(w)| \leq 4$ for $w \in \mathbb{T}$ as desired. □

We will also want to use the geometric information contained in the condition $(\phi - w, \Omega) \in U_3$. For this we introduce the notation

$$\gamma(w) = \arg \frac{w\phi'(w)}{\phi(w)}. \tag{6.1}$$

This is the quantity which appears in the definition (4.2) of $E^{1+\alpha}_\delta$, and represents (up to a sign) the tangent angle between ∂D and a circle. The next lemma states that a bound $\|\gamma\|_{L^\infty} < \pi/2$ implies a bound on $\|\phi'\|_{L^p}$ for some $p > 1$.

Lemma 6.2 ([Gai62])**.** *For $(\phi - w, \Omega) \in U_3$ and γ defined in* (6.1) *we have*

$$\int_0^{2\pi} |\phi'(e^{it})|^p \, dt \leq \frac{2\pi \cdot 4^p}{\cos(p\|\gamma\|_{L^\infty})} \qquad \text{for } 0 \leq p < \frac{\pi/2}{\|\gamma\|_{L^\infty}}. \tag{6.2}$$

Proof. Recall from Section 2.2 that $U_3 \subset U_1$, and that $(\phi - w, \Omega) \in U_1$ implies that ϕ extends to a conformal mapping $\Phi\colon \mathbb{C} \setminus \mathbb{D} \to \mathbb{C}$. Using Φ, we define the holomorphic function

$$F(w) = \log \frac{w\Phi'(w)}{\Phi(w)} = u(w) + i\gamma(w)$$

with real part u and imaginary part γ. Since $F(\infty) = 0$, the calculus of residues yields

$$1 = e^{pF(\infty)} = \frac{1}{2\pi i} \int_{\mathbb{T}} e^{pF(w)} \frac{dw}{w} = \frac{1}{2\pi} \int_0^{2\pi} e^{pu(e^{it})} \cos[p\gamma(e^{it})] \, dt \tag{6.3}$$

for any p. Assuming that $p\|\gamma\|_{L^\infty} < \pi/2$, we have $\cos(p\gamma(e^{it})) \geq \cos(p\|\gamma\|_{L^\infty})$ so that (6.3) implies

$$\int_0^{2\pi} \frac{|\Phi'(e^{it})|^p}{|\Phi(e^{it})|^p} \, dt = \int_0^{2\pi} e^{pu(e^{it})} \, dt \leq \frac{2\pi}{\cos(p\|\gamma\|_{L^\infty})}. \tag{6.4}$$

Estimating $|\Phi(e^{it})| \leq 4$ using Lemma 6.1 we are left with (6.2) as desired. □

Next we need to obtain bounds for $\Omega\overline{\phi} + \frac{1}{2}\mathcal{C}(\phi)\overline{\phi}$ which, unlike those used in Lemma 4.4, do not depend on $C^{1+\alpha}$ bounds for ϕ. We will accomplish this by using Lemma 2.1 and the following elliptic estimate for the relative stream function Ψ.

Lemma 6.3 (Basic elliptic estimate)**.** *Let $(\phi - w, \Omega) \in \mathscr{S}$ and fix $\beta \in (0, 1)$. Then there exists a constant C depending only β so that the corresponding relative stream function Ψ and vortex patch D satisfy*

$$\|\partial_z \Psi\|_{C^\beta(\overline{D})} < C|1 - 2\Omega|.$$

Proof. By Lemma 6.1, we know that $\|\phi\|_{L^\infty(\mathbb{T})} \le 4$ and hence that $D \subset B_4$. By (1.2), we know that Ψ satisfies

$$\Delta\Psi = 1_D - 2\Omega \text{ in } B_{10}$$

in the sense of distributions, and that $\|1_D - 2\Omega\|_{L^\infty(B_{10})} = |1-2\Omega|$. Thus, for instance by [GT01, Exercise 4.8], we have

$$\|\partial_z\Psi\|_{C^\beta(B_5)} \le C(\beta)|1-2\Omega|.$$

Since $\overline{D} \subset B_5$, this then implies the desired bound on $\|\partial_z\Psi\|_{C^\beta(\overline{D})}$. □

With the above lemmas in place, we can now establish the desired bound for $\|\phi\|_{C^{1+\alpha}}$. Our hypothesis will be that the first and third terms in (4.5) are controlled, i.e.

$$\left\|\arg \frac{w\phi'}{\phi}\right\|_{L^\infty} < \frac{\pi}{2} - \delta, \qquad \inf_{\mathbb{T}} |\Omega\overline{\phi} + \tfrac{1}{2}\mathcal{C}(\phi)\overline{\phi}| > \delta. \tag{6.5}$$

Lemma 6.4 (Control of $\|\phi\|_{C^{1+\alpha}}$)**.** *Let $(\phi - w, \Omega) \in \mathscr{S}$ with $|\Omega| \le 10$ and suppose* (6.5) *holds for some $\delta > 0$. Then there exists C depending only on δ so that $\|\phi\|_{C^{1+\alpha}} < C$.*

Proof. In what follows we use C to denote any positive constant depending only on δ. From Lemma 6.2 we know that there exists $p > 1$ and depending only on δ such that $\|\phi'\|_{L^p} < C$. From Lemma 6.1 we have $\|\phi\|_{L^\infty} < 4$, so Sobolev embedding gives $\|\phi\|_{C^\sigma} < C$ for some σ depending on p. With $\beta \in (0,1)$ arbitrary but fixed, we also know by Lemma 6.3 that $\|\partial_z\Psi\|_{C^\beta(\overline{D})} < C$.

We now write $\mathscr{F}(\phi - w, \Omega) = 0$ as the Riemann–Hilbert problem $\operatorname{Im}(A\phi') = 0$ where

$$A := (\Omega\overline{\phi} + \tfrac{1}{2}\mathcal{C}(\phi)\overline{\phi})w = 2\partial_z\Psi(\phi(w))w.$$

Thanks to (6.5), we have

$$|A| = |\Omega\overline{\phi} + \tfrac{1}{2}\mathcal{C}(\phi)\overline{\phi}| = 2|\partial_z\Psi(\phi(w))| > \delta.$$

Lemmas 2.4 and 4.1 yield the identity

$$\phi'(w) = \exp\left\{\frac{w}{2\pi}\int_{\mathbb{T}} \frac{1}{\tau - w}\left[\frac{1}{\tau}\arg\left(\frac{\partial_z\Psi(\phi(\tau))\tau}{\overline{\partial_z\Psi(\phi(\tau))\tau}}\right) - \frac{1}{w}\arg\left(\frac{\partial_z\Psi(\phi(w))w}{\overline{\partial_z\Psi(\phi(w))w}}\right)\right] d\tau\right\}. \tag{6.6}$$

From $\|\phi\|_{C^\sigma} < C$ and $\|\partial_z\Psi\|_{C^\beta(\overline{D})} < C$, we get $\|\partial_z\Psi \circ \phi\|_{C^{\sigma\beta}(\mathbb{T})} < C$. Since $|\partial_z\Psi| > \delta/2$, it is then straightforward to show that

$$\left\|\frac{1}{\tau}\arg\left(\frac{\partial_z\Psi(\phi(\tau))\tau}{\overline{\partial_z\Psi(\phi(\tau))\tau}}\right)\right\|_{C^{\sigma\beta}} < C.$$

The Cauchy integral is a bounded operator from $C^{\sigma\beta} \to C^{\sigma\beta}$, and so after composing with the exponential we obtain $\|\phi'\|_{C^{\sigma\beta}} < C$.

In particular, we now know that $\|\phi\|_{C^{\sqrt{\alpha}}} < C$, and so we can repeat the above argument with $\sigma = \beta = \sqrt{\alpha}$ to obtain $\|\phi'\|_{C^\alpha} < C$ as desired. □

6.2 Other bounds

We next turn to the other terms in (4.5). First we establish control over Ω by using the nonexistence results of Hmidi [Hmi15] and Fraenkel [Fra00] together with our result Proposition 5.8 on nodal properties:

Lemma 6.5 (Control of Ω)**.** *Along $\mathscr{C}$, $0 < \Omega < 1/2$.*

Proof. Suppose for the sake of contradiction that there exists a solution $(f,\Omega) \in \mathscr{C}$ with $\Omega = 1/2$ or $\Omega = 0$. Then by [Hmi15] or Fraenkel [Fra00] (as cited in [Hmi15]), we must have $f \equiv 0$, i.e. $(f,\Omega) \in \mathscr{T}$. But then $(f,\Omega) \notin \mathscr{N}$, contradicting Proposition 5.8. $\square$

Finally, we bound the two remaining quantities in (4.5) in terms of the first and third.

Lemma 6.6 (Remaining bounds on ϕ)**.** *Let $(\phi - w, \Omega) \in \mathscr{S}$, and suppose that* (6.5) *holds for some $\delta > 0$. Then there exists a constant $C > 0$ depending only δ such that*

$$|\phi'|,\, |\phi| \geq \frac{1}{C}.$$

Proof. By Lemmas 6.4 and 6.5, we have $\|\phi\|_{C^{1+\alpha}} < C$, where here and in what follows C is a positive constant whose value may change from line to line but which depends only on δ. To get the lower bound on $|\phi'|$, we take the multiplicative inverse of (6.6) and use our bounds on $\|\phi\|_{C^\alpha}$. Arguing as in the proof of Lemma 6.4 we find that $\|1/\phi'\|_{C^\alpha} < C$ and hence that $\min_{\mathbb{T}}|\phi'| > 1/C$.

To get the lower bound on $|\phi|$, we first get a lower bound on $\|\phi\|_{L^\infty}$ using the Schwarz lemma: Since the function $g(w) = \Phi(w)/w$ is holomorphic at infinity with $g(\infty) = 1$, we have by the maximum modulus principle that

$$\|\phi\|_{L^\infty(\mathbb{T})} = \|g\|_{L^\infty(\mathbb{C}\setminus D)} > 1. \tag{6.7}$$

Next we note that

$$\log \frac{\min_{\mathbb{T}}|\phi|}{\max_{\mathbb{T}}|\phi|} = \log \left| \frac{\phi(\pi/m)}{\phi(0)} \right| = -\operatorname{Re} \int_0^{\pi/m} \frac{d}{dt} \log \phi(e^{it})\, dt = \int_0^{\pi/m} \operatorname{Im} \frac{e^{it}\phi'(e^{it})}{\phi(e^{it})}\, dt.$$

Estimating this integral as in (6.4), we obtain

$$\log \frac{\min_{\mathbb{T}}|\phi|}{\max_{\mathbb{T}}|\phi|} \leq \int_0^{\pi/m} \frac{|\phi'(e^{it})|}{|\phi(e^{it})|}\, dt \leq \frac{2\pi}{\cos(\pi/2 - \delta)}.$$

Taking exponentials gives

$$\min_{\mathbb{T}}|\phi| \geq \max_{\mathbb{T}}|\phi| \exp\left(-\frac{2\pi}{\cos(\pi/2-\delta)} \right) > \exp\left(-\frac{2\pi}{\cos(\pi/2-\delta)} \right) > 1/C,$$

where in the second-to-last inequality we have used (6.7). $\square$

6.3 Proof of Theorem 1.1

We are now ready to prove our main result, Theorem 1.1.

Proof of Theorem 1.1. Conclusion (a) of Theorem 1.1 is immediate from the construction thus far: Theorem 4.6 constructed $\mathscr{C}$ as an extension of $\mathscr{C}_{\text{loc}}$, which indeed starts at the circular patch $(0, \Omega_m)$; see Theorem 3.3(ii) and Definition 3.4. As mentioned at the start of Section 5, the conclusion (c) of the theorem is implied by (5.2), which holds by Proposition 5.8. Since (d) follows from Corollary 5.11, it therefore remains to show (b).

By Proposition 5.8, alternative (i) in Theorem 4.6 does not occur. Therefore alternative (ii) occurs, that is

$$\min\left\{ \min_{\mathbb{T}} \left|\tilde\Omega\overline{\tilde\phi} + \tfrac{1}{2}\mathcal{C}(\tilde\phi)\overline{\tilde\phi}\right|,\, \frac{1}{1+|\tilde\Omega| + \|\tilde\phi\|_{C^{1+\alpha}}},\, \frac{\pi}{2} - \max_{\mathbb{T}} \left| \arg \frac{w\tilde\phi'}{\tilde\phi} \right|,\, \min_{\mathbb{T}}|\tilde\phi'|,\, \min_{\mathbb{T}}|\tilde\phi| \right\} \longrightarrow 0$$

as $s \to \infty$. Applying Lemmas 6.4, 6.5, and 6.6, we see that this implies the simpler condition

$$\min\left\{ \min_{\mathbb{T}} \left|\tilde\Omega\overline{\tilde\phi} + \tfrac{1}{2}\mathcal{C}(\tilde\phi)\overline{\tilde\phi}\right|,\, \frac{\pi}{2} - \max_{\mathbb{T}} \left| \arg \frac{w\tilde\phi'}{\tilde\phi} \right| \right\} \longrightarrow 0 \tag{6.8}$$

as $s \to \infty$. Letting $\tilde\Psi(s)$ be the relative stream function and $\tilde D(s)$ the vortex patch associated to $(\tilde f(s), \tilde\Omega(s))$, we claim that

$$0 < \min_{\partial \tilde D} \tilde\Psi_r \to 0$$

as $s \to \infty$. The left inequality is just a restatement of the nodal property (5.2a). Suppose for the sake of contradiction that $\tilde\Psi_r > \varepsilon$ on $\partial\tilde D$ along some subsequence $s_n \to \infty$ for some $\varepsilon > 0$. Then the first term in (6.8) is $|\nabla\tilde\Psi|^2/4 \ge \varepsilon^2/4$ (see (2.8) or (5.13)), and so

$$\max_{\mathbb{T}} \left| \arg \frac{w\tilde\phi'}{\tilde\phi} \right| \to \frac{\pi}{2} \text{ as } n \to \infty. \tag{6.9}$$

Differentiating $\tilde\Psi \circ \tilde\phi \equiv 0$ as in (5.14) and using Lemma 2.5, we see that

$$|\tilde\phi| \tan\left(\arg \frac{w\tilde\phi'}{\tilde\phi} \right) = \frac{|\phi| \operatorname{Im}(w\tilde\phi'/\tilde\phi)}{\operatorname{Re}(w\tilde\phi'/\tilde\phi)} = -\frac{\tilde\rho'}{\tilde\vartheta'} = \frac{\tilde\Psi_\theta}{\tilde\Psi_r},$$

and hence that

$$\left\| \arg \frac{w\tilde\phi'}{\tilde\phi} \right\|_{L^\infty(\mathbb{T})} = \tan^{-1} \left\| \frac{\tilde\Psi_\theta/r}{\tilde\Psi_r} \right\|_{L^\infty(\partial\tilde D)} \le \tan^{-1}\left(\varepsilon^{-1} \|\tilde\Psi_\theta/r\|_{L^\infty(\partial\tilde D)} \right)$$

for all n. Therefore the only way for (6.9) to occur is if $\|\tilde\Psi_\theta/r\|_{L^\infty(\partial\tilde D)} \to \infty$. But by Lemma 6.3 $\|\partial_z\tilde\Psi\|_{C^{1/2}(\tilde D)}$, say, is bounded along $\mathscr{C}$, and hence this is impossible. □

7 Numerical streamline patterns

In this section we numerically calculate global branches of rotating vortex patches with $m = 3, 4, 5, 6$. As mentioned in the introduction, similar branches have previously been calculated in [DZ78, WOZ84, Ove86], with the striking conclusion that there are limiting solutions with sharp 90° corners. Our contribution is that we additionally calculate the full stream function Ψ; the results suggest that the qualitative features in Theorem 1.2 persist along the whole branch (Conjecture 1.4) and indeed are related to the formation of sharp corners.

7.1 Numerical method

We approximate the trace ϕ of the conformal mapping Φ by a Fourier series with M modes:

$$\phi(e^{it}) \approx e^{it} + \sum_{n=1}^{M} a_n e^{-in(m-1)t},$$

where $a_1, \dots, a_M$ are real. This is in line with the normalization $\Phi'(\infty) = 0$ in Section 2.2. With an (inverse) fast Fourier transform, the values of $\phi(e^{it})$ and $\phi'(e^{it})$ are then approximated at $N > mM$ evenly spaced values $t_1, \dots, t_N$. These physical grid points $z_n = \phi(e^{it_n})$ become denser in the regions where ∂D has the high curvature. The integral appearing in (1.5) is then approximating with the trapezoid rule,

$$\int_{\mathbb{T}} \frac{\overline{\phi(\tau)} - \overline{\phi(e^{it_n})}}{\phi(\tau) - \phi(e^{it_n})} \phi'(\tau)\, d\tau \approx \overline{ie^{it}\phi'(e^{it})} \frac{2\pi}{N} + \sum_{k \neq n} \frac{\overline{\phi(e^{it_k})} - \overline{\phi(e^{it_n})}}{\phi(e^{it_k}) - \phi(e^{it_n})} \phi'(e^{it_k}) \frac{2\pi i e^{it_k}}{N},$$

where we have evaluated the integrand at $\tau = e^{it_n}$ by calculating the limit

$$\lim_{t \to s} \frac{\overline{\phi(e^{is})} - \overline{\phi(e^{it})}}{\phi(e^{is}) - \phi(e^{it_n})} \phi'(e^{is}) = -\overline{\phi'(e^{is})} e^{2is}.$$

Symmetry class m	3	4	5	6
Fourier modes M	1023	1023	511	255
Physical gridpoints N	6144	8192	5120	3072

Table 1: Number of Fourier modes and gridpoints used for different values of m.

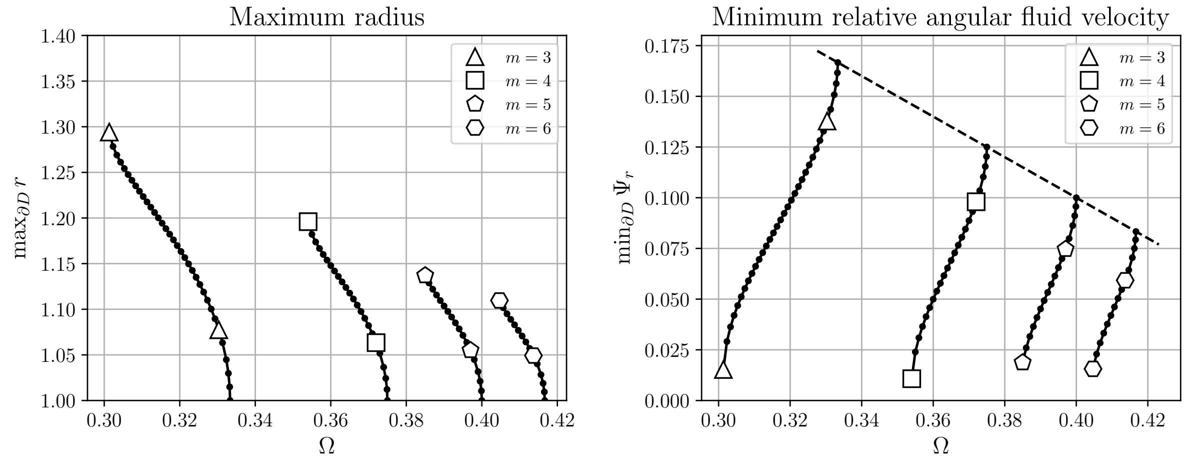

Figure 6: Maximum radius and minimum (relative) angular velocity along ∂D for the numerically computed branches. The solutions with markers appear in Figures 7 and 8, and the dashed line is the analytical formula for trivial solutions.

Substituting these approximations into (2.14) yields an approximation of $\mathscr{F}(\phi - w, \Omega)(e^{it})$ for $t = t_1, \dots, t_N$. Taking a fast Fourier transform, we obtain

$$\mathscr{F}(\phi - w, \Omega)(e^{it}) \approx \operatorname{Im} \sum_{n=1}^{M} b_n e^{inmt}$$

for real coefficients $b_1, \dots, b_M$. This process defines a finite-dimensional mapping

$$\mathscr{F}^M : \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^M, \qquad \mathscr{F}^M(a_1, \dots, a_M; \Omega) = (b_1, \dots, b_M),$$

whose roots correspond to rotating vortex patches.

We find roots of $\mathscr{F}^M$ by using a standard Newton–Krylov scheme. This is a Newton-type method in which the action of the Jacobian matrix $\mathscr{F}_a^M(a;\Omega)$ of $\mathscr{F}^M$ is approximated by

$$\mathscr{F}_a^M(a;\Omega)\alpha \approx \frac{\mathscr{F}^M(a + \varepsilon\alpha; \Omega) - \mathscr{F}^M(a;\Omega)}{\varepsilon}$$

so that the action of the inverse matrix can in turn be approximated using an iterative LGMRES method. Fixing $\delta > 0$, we seek solutions at the discrete frequencies $\Omega^k = \Omega_m - k\delta$. We begin tracing out the branch by computing a solution a^1 of $\mathscr{F}^M(a^1; \Omega^1) = 0$ using $a^1 = a^0 := (\sqrt{\delta}, 0, \dots, 0)$ as our initial guess. We then solve $\mathscr{F}^M(a^2; \Omega^2) = 0$ for a^2 using a^1 as an initial guess and so on. The process terminates when too many Newton iterations are needed or else when the solutions are no longer well-resolved by M Fourier modes.

7.2 Results

We applied the above method with $m = 3, 4, 5, 6$, a grid spacing of $\delta = 0.001$ in Ω, and the numbers M of Fourier modes and N of gridpoints given in Table 1. To better resolve the local bifurcation, we computed two additional solutions near the start of each branch.

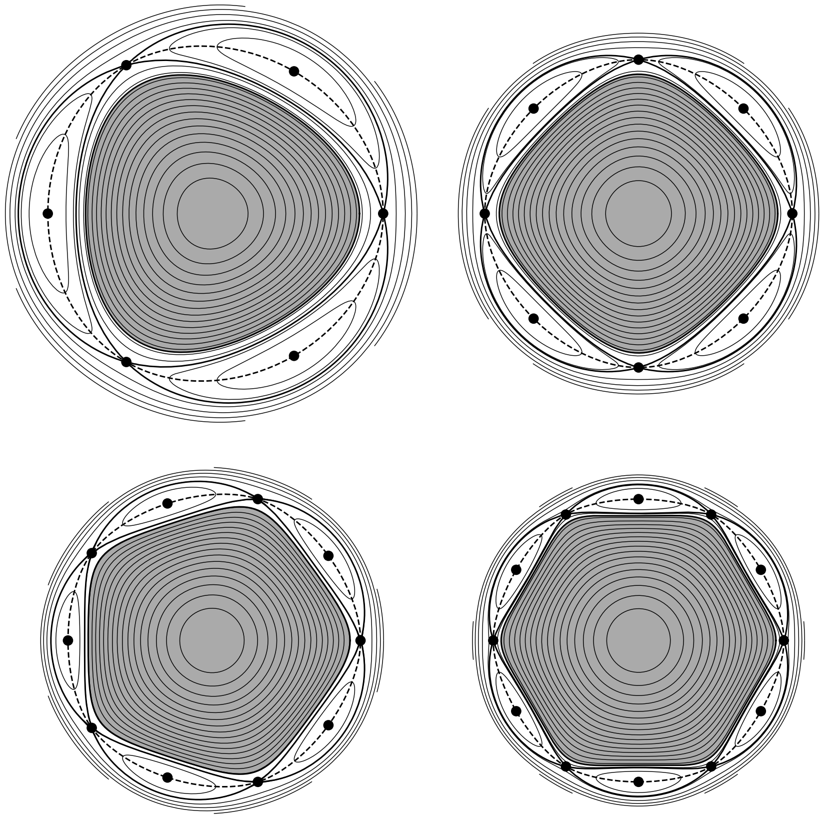

Figure 7: For each numerical branch in Figure 6, the level curves of the stream function Ψ for the solution with the fifth-largest value of Ω. The shaded region is the patch D, the dashed lines are the curves $\Psi_r = 0$ where there is no angular fluid velocity, and the markers are the critical points of Ψ (besides the origin).

The maximum radii $\max_{\partial D} r = \|\phi\|_{L^\infty}$ are shown in Figure 6. As seen in previous work, the angular frequency Ω appears to be decreasing along each branch while the radius increases. Theorem 1.1 predicts that $\min_{\partial D} \Psi_r$ should also limit to zero, and evidence of this can indeed be seen in Figure 6.

Now we turn to Conjecture 1.4 on the level curves of the relative stream function Ψ. Looking at the proof of Theorem 1.2, the conjecture is true provided the inequalities

$$\Psi^-_{r\theta} < 0 \text{ on } S \cap \partial D, \quad \Psi^-_{r\theta\theta} < 0 \text{ on } R \cap \partial D, \quad \Psi^-_{r\theta\theta} > 0 \text{ on } L \cap \partial D$$

and also

$$\max_{\partial D} \left((r\partial_r)^2 \Psi^- + 2\Omega r^2 \right) < \min_{\partial D} 2\Omega r^2$$

hold for all solutions. Thanks to Lemma 2.2, the above quantities are readily calculated in terms of ϕ, and the inequalities do seem to hold for all of the solutions we computed.

Applying Green's theoerem to (2.1), we can also compute and plot the full stream function Ψ for any of our numerical solutions. Level curves of Ψ for several solutions are shown in Figures 7 and 8. The solutions in Figure 7 are part of the way up our numerical branches, while the solutions in 8 are at the very end. As expected, all of these phase portraits have the qualitative features from Theorem 1.2. At the end of each branch, the saddle points approach the boundary ∂D; see Figures 9 and 10. It would seem that, in the limit, the saddle point coincides with the corner point on ∂D, which in turn is made up of heteroclinics from Theorem 1.2.

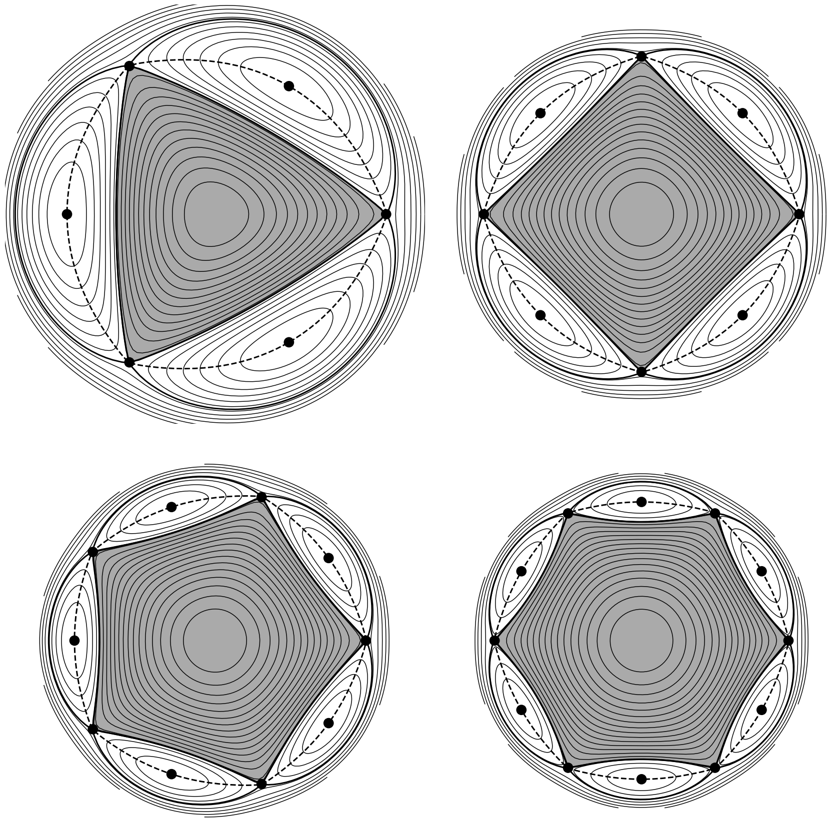

Figure 8: For each numerical branch in Figure 6, the level curves of the stream function Ψ for the solution with the smallest value of Ω. The shaded region is the patch D, the dashed lines are the curves $\Psi_r = 0$ where there is no angular fluid velocity, and the markers are the critical points of Ψ (besides the origin).

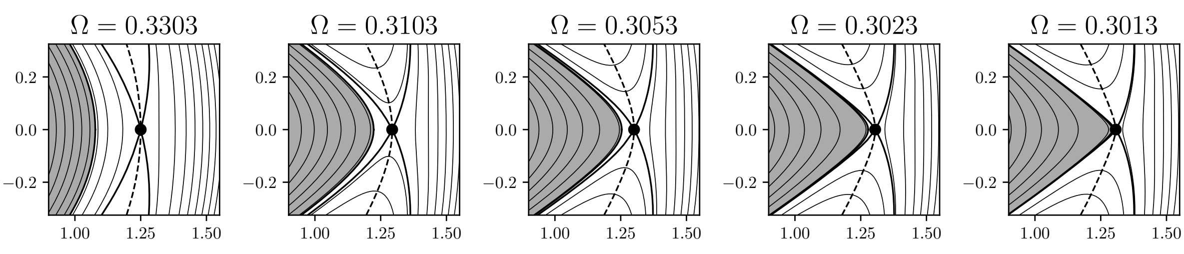

Figure 9: Streamline plots of several patches with $m = 3$, magnified to show the approach of the saddle point to ∂D. As in Figures 7 and 8, the shaded region is D, the dashed line is the curve $\Psi_r = 0$, and the marked point is the saddle.

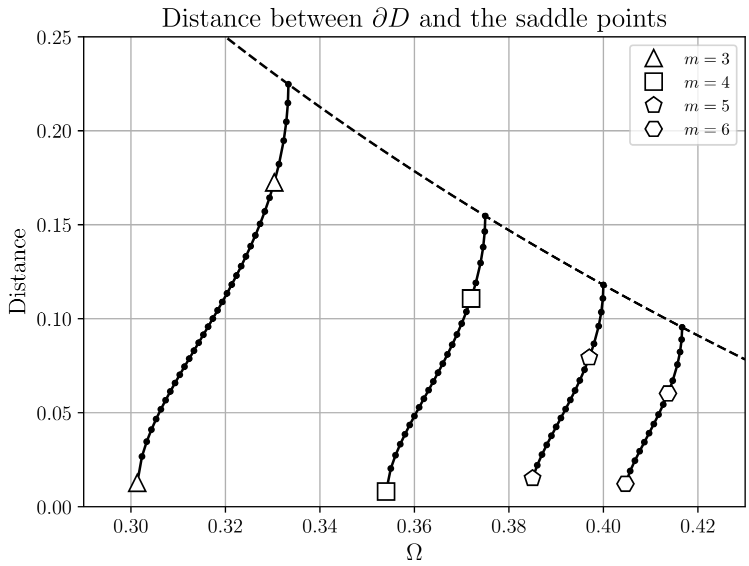

Figure 10: The distance between the closest saddle point and ∂D for the numerical branches from Figure 6. The dashed curve is the analytical formula for trivial solutions.

A Appendix

A.1 Linear Riemann–Hilbert problems

In this section we prove Lemma 2.4 by recalling some basic facts from the classical theory of Riemann–Hilbert problems. As mentioned in Section 2.2, the space $X^{k+\beta}$ in Lemma 2.4 consists of the traces $f = F|_{\mathbb{T}}$ of certain holomorphic functions F on the exterior of the unit disk $\mathbb{D}$. In this section it is more convenient to work with the full mappings F.

To cite the classical theory, we first consider holomorphic functions on the *interior* of the unit disk, introducing the (real) Banach spaces

$$\mathscr{Z}_1 = \big\{F_1 \in C^{k-1+\beta}(\overline{\mathbb{D}}, \mathbb{C}) : F_1 \text{ is holomorphic on } \mathbb{D}\big\}, \qquad \mathscr{Y} = C^{k+\beta}(\mathbb{T}, \mathbb{R}),$$

and fixing a $C^{k-1+\beta}(\mathbb{T}, \mathbb{C})$ coefficient function a_1 with winding number 0. From [Mus72, §40], we have the following results for the linear operator $\mathcal{L}_1 \colon \mathscr{Z}_1 \to \mathscr{Y}$ defined by $\mathcal{L}_1 F_1 = \operatorname{Re}\{a_1 F_1|_{\mathbb{T}}\}$.

Theorem A.1 (Properties of $\mathcal{L}_1$)**.**

(a) $\ker \mathcal{L}_1 = \operatorname{span}\{F_1^0\}$ *is one-dimensional, where for* $w \in \mathbb{D}$

$$F_1^0(w) = \exp\left\{-\frac{1}{4\pi}\int_{\mathbb{T}} \frac{\theta_1(\tau)}{\tau}\, d\tau + \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\theta_1(\tau)}{\tau - w}\, d\tau\right\}, \qquad \theta_1(w) := \arg\left(-\frac{\overline{a_1(w)}}{a_1(w)}\right).$$

(b) *For* $h \in \mathscr{Y}$*, the general solution of* $\mathcal{L}_1 F_1 = h$ *is given by*

$$\begin{aligned} F_1(w) = \frac{F_1^0(w)}{\pi i} &\int_{\mathbb{T}} \frac{1}{\tau - w} \frac{h(\tau)}{a_1(\tau) F_1^0(\tau)}\, d\tau \\ - \frac{F_1^0(w)}{2\pi i} &\int_{\mathbb{T}} \frac{h(\tau)}{a_1(\tau) F_1^0(\tau)\tau}\, d\tau + C F_1^0(w) \end{aligned} \qquad \text{for } w \in \mathbb{D},$$

where C *is an arbitrary real constant. In particular,* $\mathcal{L}_1$ *is Fredholm with index 1.*

Using a simple change of variables, we now obtain an analogue of Theorem A.1 with the interior of the unit disk replaced by the exterior and where imaginary instead of real parts are taken. The relevant analogue of $\mathcal{Z}_1$ is

$$\mathcal{Z}_2 = \Big\{F_2 \in C^{k-1+\beta}(\mathbb{C}\setminus\mathbb{D},\mathbb{C}) : F_2 \text{ is holomorphic on } \mathbb{C}\setminus\overline{\mathbb{D}}, \text{ bounded at } \infty\Big\},$$

and we define $\mathcal{L}_2\colon \mathcal{Z}_2 \to \mathcal{Y}$ by $\mathcal{L}_2 f = \operatorname{Im}\{a_2 F_2|_{\mathbb{T}}\}$. As before we assume that $a_2 \in C^{k-1+\beta}(\mathbb{T},\mathbb{C})$ and has zero winding number. With the identifications

$$a_1(w) = i\overline{a_2(w)}, \qquad F_1(w) = \overline{F_2(1/\overline{w})}, \tag{A.1}$$

one can easily check that $\mathcal{L}_1 F_1 = h$ if and only if $\mathcal{L}_2 F_2 = h$. Moreover the mapping $F_2 \mapsto F_1$ is linear and invertible $\mathcal{Z}_2 \to \mathcal{Z}_1$. This implies the following corollary of Theorem A.1:

Corollary A.2 (Properties of $\mathcal{L}_2$)**.**

(a) $\ker \mathcal{L}_2 = \operatorname{span}\{F_2^0\}$ *is one-dimensional, where for* $|w| > 1$

$$F_2^0(w) = \exp\left\{\frac{1}{4\pi}\int_{\mathbb{T}} \frac{\theta_2(\tau)}{\tau}\,d\tau + \frac{w}{2\pi}\int_{\mathbb{T}}\frac{\theta_2(\tau)}{\tau - w}\frac{d\tau}{\tau}\right\}, \qquad \theta_2(w) := \arg\left(\frac{a_2(w)}{\overline{a_2(w)}}\right). \tag{A.2}$$

(b) *For* $h \in \mathcal{Y}$*, the general solution of* $\mathcal{L}_2 F_2 = h$ *is given by, for* $|w| > 1$*,*

$$F_2(w) = -\frac{wF_2^0(w)}{\pi}\int_{\mathbb{T}} \frac{1}{\tau - w}\frac{h(\tau)}{a_2(\tau)F_2^0(\tau)}\frac{d\tau}{\tau} - \frac{F_2^0(w)}{2\pi}\int_{\mathbb{T}}\frac{h(\tau)}{a_2(\tau)F_2^0(\tau)}\frac{d\tau}{\tau} + CF_2^0(w), \tag{A.3}$$

where C is an arbitrary real constant. In particular, $\mathcal{L}_2$ is Fredholm with index 1.

Proof. Make the change of variables (A.1), and use the substitution $\tau \mapsto 1/\tau$ to rewrite the integrals in (A.2) and (A.3). □

Assuming that our coefficient a_2 has the symmetries $a_2(\overline{w}) = \overline{a_2(w)}$ and $a_2(e^{2\pi i/m}w) = a_2(w)$, it is easy to show that $\mathcal{L}_2$ restricts to a map $\mathcal{L}_3\colon \mathcal{Z}_3 \to Y$ with the same kernel and Fredholm index, where

$$\mathcal{Z}_3 = \Big\{F \in \mathcal{Z}_2 : F(\overline{w}) = \overline{F(w)},\ F(e^{2\pi i/m}w) = F(w)\Big\}.$$

These symmetries of a_2 also imply

$$\int_{\mathbb{T}} \frac{\theta_2}{\tau}d\tau = 0, \qquad \int_{\mathbb{T}} \frac{h}{a_2 F_2^0 \tau}\,d\tau = 0 \tag{A.4}$$

for all $h \in Y$, allowing for the formulas (A.2) and (A.3) to be simplified.

Finally we let

$$\mathcal{X} = \Big\{G \in C^{k+\beta}(\mathbb{C}\setminus\mathbb{D},\mathbb{C}) : G \text{ is holomorphic on } \mathbb{C}\setminus\overline{\mathbb{D}}, \text{ bounded at } \infty\Big\}$$

be a subspace of $\mathcal{Z}_2$ with additional regularity, and

$$\mathcal{X}_3 = \Big\{G \in \mathcal{X} : G(e^{2\pi i/m}w) = e^{2\pi i/m}G(w),\ G(\overline{w}) = \overline{G(w)},\ G(\infty) = 0\Big\}$$

be a further subspace with additional symmetries, to be compared with those inherent in the definition of $X^{k+\beta}$. The derivative operator $\frac{d}{dw}$ is invertible $\mathcal{X}_3 \to \mathcal{Z}_3$. Indeed, the only potential complication is the uniqueness of inverses, and this holds thanks to the constraint $G(\infty) = 0$ in the definition of $\mathcal{X}_3$ (equivalently, the Laurent series of a function in $\mathcal{X}_3$ has no constant term).

Proof of Lemma 2.4. Let $\tilde{X}^{k+\beta} = \operatorname{span}\{w\} + X^{k+\beta}$, and let $\mathcal{T} : \mathcal{X}_3 \to \tilde{X}^{k+\beta}$ be the trace operator, $\mathcal{T}g = g|_{\mathbb{T}}$. We easily check that $\mathcal{T}$ is invertible, and consider the composite mapping

$$\mathcal{S} = \mathcal{L}_3 \frac{d}{dw} \mathcal{T}^{-1} : \tilde{X}^{k+\beta} \longrightarrow Y^{k-1+\beta}.$$

Our above arguments show that $\mathcal{S}$ is Fredholm with index 1 and that its kernel is spanned by the function $g_0 \in \tilde{X}^{k+\beta}$ characterized by $g_0' = F_2^0|_{\mathbb{T}}$. Using the usual formulas for the limiting values of Cauchy integrals (e.g. [Mus72, Equation 16.4]), and remembering the cancellation (A.4), we obtain

$$g_0'(w) = \exp\left\{ \frac{w}{2\pi} \int_{\mathbb{T}} \frac{\tau^{-1}\theta_2(\tau) - w^{-1}\theta_2(w)}{\tau - w} \, d\tau \right\}.$$

Furthermore, $F_2^0(\infty) = 1$ implies that the coefficient of w in the Fourier series of $g_0(w)$ is also 1, and hence that $g_0 - w \in X^{k+\beta}$. Thus g_0 is a solution to (2.10), including the requirement that $g_0 - w \in X^{k+\beta}$. Moreover, any solution $g \in \tilde{X}^{k+\beta}$ of $\operatorname{Im}\{a_2 g\} = 0$ is of the form $g = Cg_0$ for some real constant C. Since C is also the coefficient of w in the Fourier series for g, we conclude that $g - w \in X^{k+\beta}$ if and only if $C = 1$, and so g_0 is indeed the unique solution of (2.10). This completes the proof of (a).

To prove (b), we note that $L : X^{k+\beta} \to Y^{k-1+\beta}$ is simply the restriction $\mathcal{S}|_{X^{k+\beta}}$. Since $X^{k+\beta} \subset \tilde{X}^{k+\beta}$ has codimension 1 and $X^{k+\beta} \cap \ker \mathcal{S} = \varnothing$, the above Fredholm properties of $\mathcal{S}$ imply that L is invertible. Taking the limit of (A.3) as w approaches $\mathbb{T}$ and using the cancellation (A.4) to drop one of the terms, we finally recover (2.11). □

A.2 Derivatives of the stream function

In this section we complete the proof of Lemma 2.2 expressing the partial derivatives of Ψ^- in terms of the trace ϕ of the conformal mapping Φ. Recall that $\Psi^- = \Psi^{D,\Omega}|_{\mathbb{C}\setminus D}$ is a restriction of $\Psi^{D,\Omega}$, which was explicitly defined in (2.1).

In the proof of Lemma 2.1, we have shown that

$$\partial_z \Psi^-(z) = -\frac{1}{8\pi i} \int_{\partial D} \frac{\overline{z} - \overline{\zeta}}{z - \zeta} \, d\zeta - \frac{\Omega}{2} \overline{z} \tag{A.5}$$

holds for $z \notin \overline{D}$. Differentiating, we obtain

$$\partial_z^2 \Psi(z) = +\frac{1}{8\pi i} \int_{\partial D} \frac{\overline{z} - \overline{\zeta}}{(z - \zeta)^2} \, d\zeta, \qquad \partial_z^3 \Psi(z) = -\frac{1}{4\pi i} \int_{\partial D} \frac{\overline{z} - \overline{\zeta}}{(z - \zeta)^3} \, d\zeta.$$

Now we make the change of variable $z = \Phi(w)$ and $\zeta = \phi(\tau)$ and integrate by parts using the rule

$$\int_{\mathbb{T}} F(\tau, \overline{\tau}) G'(\tau) \, d\tau = -\int_{\mathbb{T}} \left(\partial_\tau F(\tau, \overline{\tau}) - \frac{1}{\tau^2} \partial_{\overline{\tau}} F(\tau, \overline{\tau}) \right) G(\tau) \, d\tau.$$

For $\partial_z^2 \Psi^-$ this gives

$$\begin{aligned}
\partial_z^2 \Psi^-(\Phi(w)) &= \frac{1}{8\pi i} \int_{\mathbb{T}} \frac{\overline{\Phi(w)} - \overline{\phi(\tau)}}{(\Phi(w) - \phi(\tau))^2} \phi'(\tau) \, d\tau \\
&= \frac{1}{8\pi i} \int_{\mathbb{T}} (\overline{\Phi(w)} - \overline{\phi(\tau)}) \, \partial_\tau \left(\frac{1}{\Phi(w) - \phi(\tau)} \right) d\tau \\
&= \frac{1}{8\pi i} \int_{\mathbb{T}} \frac{1}{\tau^2} \partial_{\overline{\tau}} (\overline{\Phi(w)} - \overline{\phi(\tau)}) \frac{1}{\Phi(w) - \phi(\tau)} d\tau \\
&= -\frac{1}{8\pi i} \int_{\mathbb{T}} \frac{\overline{\phi'(\tau)}}{\tau^2 \phi'(\tau)} \frac{1}{\Phi(w) - \phi(\tau)} \phi'(\tau) \, d\tau \\
&= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{F_2(\Phi)(w) - F_2(\phi)(\tau)}{\Phi(w) - \phi(\tau)} \phi'(\tau) \, d\tau
\end{aligned}$$

where here

$$F_2(\phi) = \frac{1}{4}\frac{\overline{\phi'}}{w^2\phi'},$$

and the term added in the last step vanishes by the Cauchy integral formula. Taking the limit as w approaches $\mathbb{T}$ yields the equality in (2.8). Integrating by parts twice for $\partial_z^3\Psi^-$ we similarly obtain

$$\begin{aligned}
\partial_z^3\Psi(\Phi(w)) &= -\frac{1}{4\pi i}\int_{\mathbb{T}} \frac{\overline{\Phi(w)}-\overline{\phi(\tau)}}{(\Phi(w)-\phi(\tau))^3}\,\phi'(\tau)\,d\tau \\
&= -\frac{1}{8\pi i}\int_{\mathbb{T}} \big(\overline{\Phi(w)}-\overline{\phi(\tau)}\big)\partial_\tau\left(\frac{1}{(\Phi(w)-\phi(\tau))^2}\right)d\tau \\
&= +\frac{1}{8\pi i}\int_{\mathbb{T}} \frac{\overline{\phi'(\tau)}}{\tau^2}\frac{1}{(\Phi(w)-\phi(\tau))^2}\,d\tau \\
&= -\frac{1}{8\pi i}\int_{\mathbb{T}} \frac{\overline{\phi'(\tau)}}{\tau^2\phi'(\tau)}\partial_\tau\left(\frac{1}{\Phi(w)-\phi(\tau)}\right)d\tau \\
&= +\frac{1}{8\pi i}\int_{\mathbb{T}} \left[\partial_\tau\left(\frac{\overline{\phi'(\tau)}}{\tau^2\phi'(\tau)}\right)-\frac{1}{\tau^2}\partial_{\overline{\tau}}\left(\frac{\overline{\phi'(\tau)}}{\tau^2\phi'(\tau)}\right)\right]\frac{1}{\Phi(w)-\phi(\tau)}\,d\tau \\
&= +\frac{1}{8\pi i}\int_{\mathbb{T}} \left[-\frac{2\overline{\phi'(\tau)}}{\tau^3\phi'(\tau)}-\frac{\overline{\phi'(\tau)}\phi''(\tau)}{\tau^2\phi'(\tau)^2}-\frac{\overline{\phi''(\tau)}}{\tau^4\phi'(\tau)}\right]\frac{1}{\Phi(w)-\phi(\tau)}\,d\tau \\
&= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{F_3(\Phi)(w)-F_3(\phi)(\tau)}{\Phi(w)-\phi(\tau)}\phi'(\tau)\,d\tau,
\end{aligned}$$

where

$$F_3(\phi) = -\frac{\overline{\phi'(w)}}{2w^3\phi'(w)}-\frac{\overline{\phi'(w)}\phi''(w)}{4w^2\phi'(w)^2}-\frac{\overline{\phi''(w)}}{4w^4\phi'(w)}.$$

Taking the limit as w approaches $\mathbb{T}$ yields the remaining equality in (2.8), and the proof of Lemma 2.2 is complete.

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