ON CONFORMAL GAUSS MAPS

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Abstract. We characterise the maps into the space of 2-spheres in S^n that are the conformal Gauss maps of conformal immersions of a surface into S^n. In particular, we give an invariant formulation and efficient proof of a characterisation, due to Dorfmeister–Wang [4, 5], of the harmonic maps that are conformal Gauss maps of Willmore surfaces.

Introduction

A useful tool in conformal surface geometry is the *central sphere congruence* [1, §67; 10] or *conformal Gauss map* [2]. Geometrically, the central sphere congruence of a surface in the conformal n-sphere attaches to each point of the surface a 2-sphere, tangent to the surface at that point and having the same mean curvature vector as the surface at that point. The space of 2-spheres in S^n may be identified with the Grassmannian of $(3,1)$-planes in $\mathbb{R}^{n+1,1}$ and so the central sphere congruence may be viewed as a map, the conformal Gauss map, to this Grassmannian.

The utility of this construction is that it links the (parabolic) conformal geometry of the sphere to the (reductive) pseudo-Riemannian geometry of the Grassmannian. For example, a surface is Willmore if and only if the conformal Gauss map is harmonic [1, §81; 2; 6; 9]. In another direction, away from umbilic points, the metric induced by the conformal Gauss map, which is in the conformal class of the surface, is invariant by conformal diffeomorphisms of S^n and even arbitrary rescalings of the ambient metric [7, 11].

The purpose of this short note is to characterise those maps into the Grassmannian which are the conformal Gauss map of a conformal immersion. In so doing, we build on a result of Dorfmeister–Wang [4, 5] which treats the case where the map is harmonic. As a by-product of our analysis, we give an invariant formulation and efficient proof of their result.

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1. The conformal Gauss map

We view the conformal n-sphere S^n as the projective lightcone $\mathbb{P}(\mathcal{L})$ of $\mathbb{R}^{n+1,1}$ [3, Livre II, Chapitre VI; 8, Chapter 1]. Here $\mathcal{L} = \{v \in \mathbb{R}^{n+1,1}_{\times} \mid (v,v) = 0\}$ and $(\,,\,)$ is the signature $(n+1,1)$ inner product.

Let $f : \Sigma \to S^n = \mathbb{P}(\mathcal{L})$ be a conformal immersion of a Riemann surface into the conformal n-sphere. Equivalently, f is a null line subbundle of the trivial bundle $\underline{\mathbb{R}}^{n+1,1} := \Sigma \times \mathbb{R}^{n+1,1}$.

Define $f^{1,0} \leq \underline{\mathbb{C}}^{n+2}$ by

$$f^{1,0} = \operatorname{span}\{\sigma, \mathrm{d}_Z \sigma \mid \sigma \in \Gamma f, Z \in T^{1.0}\Sigma\}.$$

Here the notation $U \leq V$ means U is a subbundle of V. That f is a conformal immersion is equivalent to $f^{1,0}$ being a rank 2 isotropic subbundle of $\underline{\mathbb{C}}^{n+2}$. Set $f^{0,1} := \overline{f^{1,0}}$ and note that[1] $f^{1,0} \cap f^{0,1} = f$.

The *conformal Gauss map of* f is the bundle of $(3,1)$-planes $V \leq \underline{\mathbb{R}}^{n+1,1}$ given by

$$V = \operatorname{span}\{\sigma, \mathrm{d}_Z \sigma, \mathrm{d}_{\bar{Z}} \sigma, \mathrm{d}_Z \mathrm{d}_{\bar{Z}} \sigma \mid \sigma \in \Gamma f, Z \in \Gamma T^{1,0}\Sigma\}.$$

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[1] We make no notational distinction between a real bundle and its complexification

We have a decomposition $\underline{\mathbb{R}}^{n+1,1} = V \oplus V^\perp$ which induces a decomposition of the flat connection d:

$$\mathrm{d} = \mathcal{D} + \mathcal{N},$$

where $\mathcal{D}$ is a metric connection preserving V and $V^\perp$ while $\mathcal{N}$ is a 1-form taking values in skew-endomorphisms of $\underline{\mathbb{R}}^{n+1,1}$ which *permute* V and $V^\perp$.

Remark. We may view V as a map from Σ into the Grassmannian of $(3,1)$-planes in $\mathbb{R}^{n+1,1}$ and then $\mathcal{N}$ can be identified with its differential.

The flatness of d yields the structure equations of the situation:

$$0 = R^{\mathcal{D}} + \tfrac{1}{2}[\mathcal{N} \wedge \mathcal{N}] \tag{1.1a}$$

$$0 = \mathrm{d}^{\mathcal{D}} \mathcal{N}. \tag{1.1b}$$

Here $\mathrm{d}^{\mathcal{D}}$ is the exterior derivative on bundle-valued forms with $\mathcal{D}$ used to differentiate coefficients.

The conformal Gauss map V is defined by the following properties:

1. $f^{0,1} \leq V$;
2. $f^{0,1} \leq \ker \mathcal{N}^{1,0}$.

Now, for Z a local section of $T^{1,0}\Sigma$, $\mathcal{N}_Z$ is skew while $f^{0,1}$ is maximal isotropic in V so that

$$\mathcal{N}_Z V^\perp = (\ker \mathcal{N}_{Z|V})^\perp \subseteq (f^{0,1})^\perp \cap V = f^{0,1} \leq \ker \mathcal{N}_{Z|V} \tag{1.2}$$

and we conclude:

Lemma 1.1 (c.f. [4, Proposition 2.2])**.** *If V is the conformal Gauss map of a conformal immersion then $(\mathcal{N}^{1,0})^2{}_{|V^\perp} = 0$.*

Following [4], we say that V with the property of Lemma 1.1 is *strongly conformal*.

The conformal Gauss map also satisfies a second order condition. First note that (1.2) tells us that

$$\mathcal{N}_Z V^\perp \subseteq f^{0,1}. \tag{1.3}$$

Moreover, $f^{0,1}$ is $\mathcal{D}^{0,1}$-stable thanks to the following lemma which will see further use in Section 2:

Lemma 1.2. *Let $W \leq V$ be maximal isotropic in V with a never-vanishing section w such that $\mathcal{D}_{\bar{Z}} w \in W$, for $\bar{Z} \in T^{0,1}\Sigma$. Then W is $\mathcal{D}^{0,1}$-stable.*

Proof. Let $u \in \Gamma W$ be a local section so that u, w locally frame W. It suffices to show that $\mathcal{D}_{\bar{Z}} u \in W$. However,

$$(\mathcal{D}_{\bar{Z}} u, w) = -(u, \mathcal{D}_{\bar{Z}} w) = 0;$$
$$(\mathcal{D}_{\bar{Z}} u, u) = \tfrac{1}{2}\bar{Z}(u,u) = 0,$$

since W is isotropic. Thus $\mathcal{D}_{\bar{Z}} u \in W^\perp \cap V = W$ since W is maximal isotropic in V. □

In the case at hand, for $\sigma \in \Gamma f$, we have $\mathcal{D}_{\bar{Z}} \sigma \in f^{0,1}$ by definition so Lemma 1.2 applies to show that $f^{0,1}$ is $\mathcal{D}^{0,1}$-stable.

Now contemplate the *tension field* $\tau_V := * \mathrm{d}^{\mathcal{D}} * \mathcal{N}$ of V. Since $*\mathcal{N} = i(\mathcal{N}^{1,0} - \mathcal{N}^{0,1})$, (1.1b) yields

$$\tau_V = 2i * \mathrm{d}^{\mathcal{D}} \mathcal{N}^{1,0} = -2i * \mathrm{d}^{\mathcal{D}} \mathcal{N}^{0,1}.$$

In view of the last paragraph, $\tau_V V^\perp$ takes values in $f^{0,1}$ since $* \mathrm{d}^{\mathcal{D}} \mathcal{N}^{1,0} V^\perp$ does. However, τ_V is real so that $\tau_V V^\perp$ takes values in $f^{0,1} \cap f^{1,0} = f$:

$$\tau_V V^\perp \subseteq f. \tag{1.4}$$

In particular, since f is a null line subbundle on which $\mathcal{N}$ vanishes, we conclude:

Proposition 1.3. *If V is the conformal Gauss map of a conformal immersion with tension field τ_V. Then:*

$$\mathcal{N} \circ \tau_{V|V^{\perp}} = 0 \tag{1.5a}$$

$$(\tau_V)^2{}_{|V^{\perp}} = 0. \tag{1.5b}$$

This line of argument additionally give us some control on the rank of $\mathcal{N} : T\Sigma \otimes V^{\perp} \to V$:

Lemma 1.4. *Let V be the conformal Gauss map of a conformal immersion f, then the set*

$$A := \{p \in \Sigma \mid \mathcal{N}_Z V^{\perp} = \mathcal{N}_{\bar{Z}} V^{\perp} \neq \{0\}, Z \in T_p\Sigma\}$$

is nowhere dense.

Proof. Any open set in the closure of A must contain an open set where $\mathcal{N}_Z V^{\perp} = \mathcal{N}_{\bar{Z}} V^{\perp} \neq \{0\}$. On this latter set, we immediately see from (1.3) that $\mathcal{N}_Z V^{\perp} = f$. Since $\tau_V V^{\perp} \leq f$ also, by (1.4), we rapidly conclude (c.f. Lemma 2.2 below) that f is $\mathcal{D}^{0,1}$-stable and so $\mathcal{D}$-stable. Since $\mathcal{N} f = 0$ also, f is constant: a contradiction. □

In the next section, we will establish a generic converse to these results.

2. Reconstruction of f from V

Suppose now that we have a bundle $V \leq \underline{\mathbb{R}}^{n+1,1}$ of $(3,1)$-planes and ask whether V is the conformal Gauss map of some conformal immersion f. Our task is then to construct $f \leq V$ but, in fact, it will be more convenient to construct $f^{0,1}$:

Proposition 2.1. *Let $W \leq V$ be a maximal isotropic subbundle of V such that:*

1. *W is $\mathcal{D}^{0,1}$-stable;*
2. *$\mathcal{N}^{1,0} W = 0$, or, equivalently (c.f. (1.2)), $\mathcal{N}_Z V^{\perp} \subseteq W$, for all $Z \in T^{1,0}\Sigma$.*

Then $f := W \cap \overline{W}$ is a real, null, line subbundle which, on the open set where it immerses, is conformal with $W = f^{0,1}$ and conformal Gauss map V.

Proof. Since V has signature $(3,1)$, W and $\overline{W}$ must intersect in a line bundle, necessarily null and real. Since f is real, $f \leq \ker \mathcal{N}^{1,0} \cap \ker \mathcal{N}^{0,1}$ so that, for $\sigma \in \Gamma f$, $\bar{Z} \in T^{0,1}\Sigma$,

$$\mathrm{d}_{\bar{Z}}\, \sigma = \mathcal{D}_{\bar{Z}} \sigma + \mathcal{N}_{\bar{Z}} \sigma = \mathcal{D}_{\bar{Z}} \sigma \in W,$$

since $f \leq W$ and W is $\mathcal{D}^{0,1}$-stable. Thus $W = f^{0,1}$ on the set where f immerses. We conclude that, on that set, f is conformal, since $f^{0,1}$ is isotropic and V is the conformal Gauss map of f since $f^{0,1} \leq \ker \mathcal{N}^{1,0}$. □

For our main result, we need the following simple observation:

Lemma 2.2. *Let $V \leq \underline{\mathbb{R}}^{n+1,1}$ be a bundle of $(3,1)$-planes with tension field τ_V. Let $w = \mathcal{N}_Z \nu$, for $\nu \in \Gamma V^{\perp}$ and $Z \in T\Sigma$. Then $\mathcal{D}_{\bar{Z}} w \in \mathcal{N}_Z V^{\perp} + \tau_V V^{\perp}$.*

Proof. For suitable $Z \in T^{1,0}\Sigma$, $\tau_V = \mathcal{D}_{\bar{Z}} \mathcal{N}_Z - \mathcal{N}^{1,0}_{[\bar{Z},Z]}$ so that

$$\begin{aligned}\mathcal{D}_{\bar{Z}} w = \mathcal{D}_{\bar{Z}}(\mathcal{N}_Z \nu) &= (\mathcal{D}_{\bar{Z}} \mathcal{N}_Z)\nu + \mathcal{N}_Z(\mathcal{D}_{\bar{Z}} \nu) \\ &= \tau_V \nu + \mathcal{N}^{1,0}_{[\bar{Z},Z]} \nu + \mathcal{N}_Z(\mathcal{D}_{\bar{Z}} \nu) \in \mathcal{N}_Z V^{\perp} + \tau_V V^{\perp}.\end{aligned}$$

□

With all this in hand, we have:

Theorem 2.3. *Let $V \leq \underline{\mathbb{R}}^{n+1,1}$ be a bundle of $(3,1)$-planes with tension field τ_V. Suppose that:*

1. *V is strongly conformal.*
2. *Equations (1.5) hold.*

3. $\{p \in \Sigma \mid \mathcal{N}_Z V^\perp = \mathcal{N}_{\bar{Z}} V^\perp \neq \{0\}, Z \in T_p\Sigma\}$ *is empty.*

Set $U := \mathcal{N}_Z V^\perp + \tau_V V^\perp$ *and restrict attention to the open dense subset of* Σ *where* U *has fibres of locally constant dimension and so is a vector bundle.*

Then $\operatorname{rank} U \leq 2$ *and we have:*

(a) *Where* $\operatorname{rank} U = 2$, *there is a unique real, null line subbundle* $f \leq V$ *which, where it immerses, is a conformal immersion with conformal Gauss map* V.

(b) *Where* $\operatorname{rank} U = 1$, *there are exactly two real, null line subbundles* $f, \hat{f} \leq V$, *which, where they immerse, are conformal immersions with conformal Gauss map* V. *In this case,* V *is harmonic and* $f, \hat{f}$ *are a dual pair of Willmore, thus* S*-Willmore* [6], *surfaces.*

(c) *Where* $\operatorname{rank} U = 0$, V *is constant and there are infinitely many real, null line subbundles* $f \leq V$ *defining conformal immersions with conformal Gauss map* V.

Proof. First note that hypotheses 1 and 2 amount to the assertion that $U \leq V$ is isotropic so that $\operatorname{rank} U \leq 2$.

We now consider each possibility for $\operatorname{rank} U$ in turn.

First suppose that $\operatorname{rank} U = 2$. Then U is maximal isotropic in V and is $\mathcal{D}^{0,1}$-stable by Lemma 1.2 in view of Lemma 2.2. By construction $\mathcal{N}_Z V^\perp \subseteq U$ so that we may take $U = W$ in Proposition 2.1 to learn that V is the conformal Gauss map of $f = U \cap \overline{U}$ where the latter immerses.

Now suppose that $\operatorname{rank} U = 1$. We claim that $U = \mathcal{N}_Z V^\perp$: first this holds on a dense open set Ω, (if $\mathcal{N}_Z V^\perp$ vanishes on an open set, so does τ_V) so that, by hypothesis 3, we have $U \cap \overline{U} = \{0\}$ on Ω. Since τ_V is real, we must have $\tau_V = 0$ on Ω and hence everywhere so that the claim follows and V is a harmonic map. It is now immediate that U is $\mathcal{D}^{0,1}$-stable. By hypothesis 3, we have that $U \cap \overline{U} = \{0\}$ everywhere so that there are exactly two real, null line subbundles $f_1, f_2 \leq V$ orthogonal to $U \oplus \overline{U}$ and we set $W_i = f_i \oplus U$, $i = 1, 2$. Lemma 1.2, applied to a section w of U assures us that each W_i is $\mathcal{D}^{0,1}$-stable so that Proposition 2.1 gives that each f_i is conformal where it immerses with conformal Gauss map V. In this case, the f_i are dual Willmore surfaces.

Finally, if $\mathcal{N}^{1,0} = 0$ then $\mathcal{N}$ vanishes also so that V is d-stable and so constant. Thus $S^2 := \mathbb{P}(\mathcal{L} \cap V)$ is a conformal 2-sphere and any conformal immersion $f : \Sigma \to S^2$ (in particular, any meromorphic function on Σ, off its branch locus) has V as conformal Gauss map. $\square$

Remarks.

1. The caveat that f immerse is not vacuous: one can readily construct V satisfying the hypotheses of Theorem 2.3 for which f we find is constant. Indeed, given constant $f \in \mathbb{P}(\mathcal{L})$, let $W \leq \underline{\mathbb{C}}^{n+2}$ be a non-constant rank 2 isotropic subbundle containing f with $\overline{W}$ holomorphic with respect to the trivial holomorphic structure of $\underline{\mathbb{C}}^{n+2}$ and choose $V^\perp$ to be a complement to $W + \overline{W}$ in $f^\perp$. Then it is not difficult to show that W is $\mathcal{D}^{0,1}$-stable and $\mathcal{N}^{1,0} W = \{0\}$.
2. For strongly conformal V, equations (1.5) are not independent. Indeed, when $\operatorname{rank} \mathcal{N}^{1,0}{}_{|V^\perp} = 2$, $\mathcal{N}_Z V^\perp$ is maximal isotropic in V so that (1.5a) forces $\tau_V V^\perp \leq \mathcal{N}_Z V^\perp$. Thus $\tau_V V^\perp$ is isotropic and (1.5b) holds. Again, when $\operatorname{rank} \mathcal{N}^{1,0}{}_{|V^\perp} = 1$, it is easy to see that (1.5a) holds automatically.

In the interesting case of harmonic V (so that $\tau_V = 0$), matters simplify considerably. Here, of course, hypothesis 2 of Theorem 2.3 is vacuous. Moreover, $\mathcal{N}^{1,0}$ is a holomorphic 1-form with respect to the Koszul–Malgrange holomorphic structure of $\underline{\mathbb{C}}^{n+2}$ with $\bar{\partial}$-operator $\mathcal{D}^{0,1}$. It follows that $\mathcal{N}_{Z|V^\perp}$ has constant rank off a divisor and, moreover, that there is a $\mathcal{D}^{0,1}$-holomorphic subbundle of $\underline{\mathbb{C}}^{n+2}$ that coincides with $\mathcal{N}_Z V^\perp$ away from that divisor. In this setting, we conclude with Dorfmester–Wang:

Corollary 2.4 (c.f. [4, Theorem 3.11; 5, Theorem 3.11])**.** *Let* $V \leq \underline{\mathbb{R}}^{n+1,1}$ *be a strongly conformal harmonic bundle of* $(3,1)$*-planes.*

Let $U \leq V$ *be the* $\mathcal{D}^{0,1}$*-holomorphic, isotropic bundle that coincides with* $\mathcal{N}_Z V^\perp$ *off a divisor.*

(a) *if* $\operatorname{rank} U = 2$*, there is a unique real, null line subbundle* $f \leq V$ *which, where it immerses, is a Willmore, non* S*-Willmore, surface with conformal Gauss map* V*.*

(b) *if* $\operatorname{rank} U = 1$ *and* $U \cap \overline{U} = \{0\}$*, there are exactly two real, null line subbundles* $f, \hat{f} \leq V$*, which, where they immerse, are a dual pair of* S*-Willmore surfaces.*

Remark. In the notation of Dorfmeister–Wang [4, 5], after a gauge transformation that renders $V, V^\perp$ constant, $\mathcal{N}^{1,0}$ is represented by the matrix B_1.

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