BLOW-UP FOR THE 3-DIMENSIONAL AXIALLY SYMMETRIC HARMONIC MAP FLOW INTO S^2

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Abstract. We construct finite time blow-up solutions to the 3-dimensional harmonic map flow into the sphere S^2,

$$u_t = \Delta u + |\nabla u|^2 u \quad \text{in } \Omega \times (0,T)$$
$$u = u_b \quad \text{on } \partial\Omega \times (0,T)$$
$$u(\cdot,0) = u_0 \quad \text{in } \Omega,$$

with $u(x,t) : \bar\Omega \times [0,T) \to S^2$. Here Ω is a bounded, smooth axially symmetric domain in $\mathbb{R}^3$. We prove that for any circle $\Gamma \subset \Omega$ with the same axial symmetry, and any sufficiently small $T > 0$ there exist initial and boundary conditions such that $u(x,t)$ blows-up exactly at time T and precisely on the curve Γ, in fact

$$|\nabla u(\cdot,t)|^2 \rightharpoonup |\nabla u_*|^2 + 8\pi\delta_\Gamma \quad \text{as} \quad t \to T.$$

for a regular function $u_*(x)$, where δ_Γ denotes the Dirac measure supported on the curve. This the first example of a blow-up solution with a space-codimension 2 singular set, the maximal dimension predicted in the partial regularity theory by Chen-Struwe and Cheng [5, 6].

1. Introduction and main result

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. We denote by S^2 the standard 2-sphere. We consider the *harmonic map flow* for maps from Ω into S^2, given by the semilinear parabolic equation

$$\begin{cases} u_t = \Delta u + |\nabla u|^2 u & \text{in } \Omega \times (0,T) \\ u = u_b & \text{on } \partial\Omega \times (0,T) \\ u(\cdot,0) = u_0 & \text{in } \Omega \end{cases} \tag{1.1}$$

for a function $u : \Omega \times [0,T) \to S^2$. Here $u_0 : \bar\Omega \to S^2$ is a given smooth map and $\varphi = u_0\big|_{\partial\Omega}$. Local existence and uniqueness of a classical solution follows from the pioneering work by Eells and Sampson [11] and K.C. Chang [3]. Equation (1.1) formally corresponds to the negative L^2-gradient flow for the Dirichlet energy $\int_\Omega |\nabla u|^2 dx$. This energy is decreasing along smooth solutions $u(x,t)$:

$$\frac{\partial}{\partial t}\int_\Omega |\nabla u(\cdot,t)|^2 = -\int_\Omega |u_t(\cdot,t)|^2.$$

Chen-Struwe [5] found a global H^1-weak solution in any dimension. In the two-dimensional case $\Omega \subset \mathbb{R}^2 \mapsto S^2$ this solution can only become singular at a finite number of points in space-time [18].

If $T > 0$ designates the first instant at which smoothness of (1.1) is lost, standard parabolic regularity leads to the fact that

$$\|\nabla u(\cdot,t)\|_\infty \to +\infty \quad \text{as} \quad t \uparrow T.$$

In the two-dimensional case, substantial knowledge on the possible blow-up structure has been obtained in [10, 12, 15, 16, 18, 19]. Blow-up takes place only about a finite number of points $q_1, \ldots, q_k$, around which the approximate form $u(x,t) \approx U\left(\frac{x-\xi(t)}{\lambda(t)}\right)$ with $\lambda(t) \to 0$ where U is a finite-energy harmonic map, namely a solution of

$$\Delta U + |\nabla U|^2 U = 0, \quad |U| \equiv 1 \quad \text{in } \mathbb{R}^2, \quad , \quad \int_{\mathbb{R}^2} |\nabla U|^2 < +\infty$$

and $\lambda(t) \to 0$ as $t \to T$. Moreover (up to subsequences), we have

$$|\nabla u(\cdot,t)|^2 \rightharpoonup |\nabla u_*|^2 + \sum_{i=1}^k 4\pi m_i \, \delta_{q_i} \quad \text{as} \quad t \to T, \tag{1.2}$$

for some positive integers m_i where δ_q denotes the unit Dirac mass at q.

Less is known in the higher dimensional case $\Omega \subset \mathbb{R}^n \mapsto S^2$ in problem (1.1). Chen-Struwe and Cheng [5, 6] have proven that the blow-up set in Ω is at most $(n-2)$-dimensional in the Hausdorff sense. More refined information on the singular set has been derived by Lin and Wang in [14], see also [13].

While various important blow-up classification results are available, finding solutions explicitly exhibiting blow-up behavior has been rather difficult. In fact, in the two-dimensional case they were even believed not to exist, see [4]. The first example of a blowing-up solution in the case $\Omega = B_2 \subset \mathbb{R}^2$, the unit two-dimensional ball was found by Chang-Ding-Ye [4] in the *1-corrotational symmetry class*,

$$u(x,t) = \begin{pmatrix} e^{i\theta} \sin v(r,t) \\ \cos v(r,t) \end{pmatrix}, \quad x = re^{i\theta}.$$

where $v(r,t)$ is a scalar function. System (1.1) reduces to the radial scalar equation

$$v_t = v_{rr} + \frac{v_r}{r} - \frac{\sin v \cos v}{r^2}, \quad v(0,t) = 0, \quad r \in (0,1).$$

Suitable initial and boundary conditions and the use of barriers lead to finite-time blow-up at some $T > 0$ in the form $v(r,t) \approx w(\frac{r}{\lambda(t)})$ with

$$w(\rho) = \pi - 2\arctan(\rho).$$

Van den Berg, Hulshof and King [1] formally found that generically,

$$\lambda(t) \approx \kappa \frac{T-t}{|\log(T-t)|^2} \quad \text{as} \quad t \to T.$$

for some $\kappa > 0$. Raphael and Schweyer [17] rigorously constructed an entire 1-corrotational solution with this blow-up rate. At the level of u, the solutions mentioned above have the form

$$u(x,t) \approx W\left(\frac{x}{\lambda(t)}\right)$$

where $W(y)$ is the canonical 1-corrotational harmonic map

$$W(y) = \frac{1}{1+|y|^2} \begin{pmatrix} 2y \\ |y|^2 - 1 \end{pmatrix}, \quad y \in \mathbb{R}^2. \tag{1.3}$$

which satisfies

$$\int_{\mathbb{R}^2} |\nabla W|^2 = 4\pi, \quad W(\infty) = \mathbf{e}_3,$$

where

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.4}$$

We achieved in [7] the first construction of a blow-up solution without symmetries in (1.1) in the case $\Omega \subset \mathbb{R}^2 \mapsto S^2$: for an arbitrary $\Omega \subset \mathbb{R}^2$, given points $q_1, \ldots, q_k \in \Omega$ and $u_b = \mathbf{e}_3$ there is for any sufficiently small $T > 0$ a solution $u(x,t)$ with precisely these k blow-up points which, consistently with (1.2), satisfies

$$|\nabla u(\cdot,t)|^2 \rightharpoonup |\nabla u_*|^2 + \sum_{i=1}^k 4\pi \, \delta_{q_i} \quad \text{as} \quad t \to T, \tag{1.5}$$

which near each q_j and after a rigid constant rotation has the approximate form

$$u(x,t) \approx W\left(\frac{x-q_j}{\lambda_j(t)}\right), \quad \lambda_j(t) = \kappa_j \frac{T-t}{|\log(T-t)|^2} \quad \text{as} \quad t \to T.$$

Part of the difficulty in the construction is due to the *instability* of the blow-up phenomenon here described once the 1-corrotational symmetry is violated, see [7]. This instability had been numerically conjectured in [2].

In the case $\Omega \subset \mathbb{R}^3 \to S^2$ only one example has been know, again in the 1-corrotational class and $\Omega = B_3$, the unit ball in $\mathbb{R}^3$. In this case the ansatz takes the form

$$u(x,t) = \begin{pmatrix} e^{i\theta} \sin v(r,z,t) \\ \cos v(r,z,t) \end{pmatrix}, \quad x = \begin{pmatrix} re^{i\theta} \\ z \end{pmatrix}$$

System (1.1) reduces to the scalar equation

$$v_t = v_{rr} + \frac{v_r}{r} + v_{zz} - \frac{\sin v \cos v}{r^2}, \quad v(0,z,t) = 0, \quad r \in (0,1). \tag{1.6}$$

Adapting the barrier method in [4], Grotowski [8] found boundary and initial conditions and a solution to (1.6) that blows up on a subset of the z-axis $r = 0$. (See a related result in [9].) No information on the structure (or dimension) of this set or on the blow-up rate is provided.

In this paper we construct the first example of a solution with a 1-dimensional blow-up set in an arbitrary axisymmetric bounded domain $\Omega \subset \mathbb{R}^3$. We observe that this example saturates the estimate for the dimension $n-2$ of the singular set found in [6] (for $n = 3$).

Before stating our main result we introduce the setting we will consider. We say that $\Omega \subset \mathbb{R}^3$ is an axisymmetric domain if it can be expressed in the form

$$\Omega = \{(re^{i\vartheta}, z) \ / \ (r,z) \in \mathcal{D}, \quad \vartheta \in [0, 2\pi]\}. \tag{1.7}$$

where $\mathcal{D} \subset \{(r,z) / \ r \geq 0\} \subset \mathbb{R}^2$. When Ω is axisymmetric, it is natural to look for solutions of (1.1) with the same axial symmetry, namely

$$u(x,t) = \tilde{u}(r,z,t), \quad x = (re^{i\vartheta}, z), \quad (r,z) \in \mathcal{D},$$

for a function $\tilde{u} : \mathcal{D} \times (0,T) \to S^2$.

We fix in what follows and axisymmetric, smooth and bounded domain Ω of the form (1.7) Let us consider a point $(r_0, z_0) \in \mathcal{D}$ with $r_0 > 0$ and let Γ be the curve inside Ω given by the copy of S^1,

$$\Gamma := \{(r_0 e^{i\vartheta}, z_0) \ / \ \vartheta \in [0, 2\pi)\} \subset \Omega \tag{1.8}$$

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be an axisymmetric domain and consider problem* (1.1) *with boundary condition $u_b \equiv \mathbf{e}_3$. Then for all sufficiently small $T > 0$ there exists an initial condition and a solution $u(x,t)$ that blows-up exactly on the curve Γ in* (1.8)*, with a profile of the form*

$$u(x,t) = W\left(\frac{(r,z) - \xi(t)}{\lambda(t)} \right) + u_*(x), \quad x = (re^{i\vartheta}, z) \quad \text{as} \quad t \to T.$$

where $W(y)$ is the standard two-dimensional 1-corrotational map (1.3)*, $u_* \in H^1(\Omega)$, $\lambda(t) \to 0$ and $\xi(t) \to (r_0, z_0)$.*

The proof provides much finer information on the asymptotic profile. In particular we have, analogously to (1.5),

$$|\nabla u(\cdot,t)|^2 \ \rightharpoonup \ |\nabla u_*|^2 + 4\pi \, \delta_\Gamma \quad \text{as} \quad t \to T,$$

with δ_Γ the uniform Dirac measure on the curve Γ. Moreover, writing $\xi(t) = (\xi_1(t), \xi_2(t))$ we have the asymptotic expressions

$$\begin{cases} \xi_1(t) = \sqrt{r_0^2 + 2(T-t)} + O((T-t)^{1+\sigma}), \\ \xi_2(t) = z_0 + O((T-t)^{1+\sigma}), \\ \lambda(t) = |\kappa| \dfrac{T-t}{|\log(T-t)|^2}(1+o(1)), \end{cases}$$

as $t \uparrow T$, for some $\kappa \in \mathbb{C}$, $\sigma > 0$.

The proof of this result takes strong advantage of the symmetry of revolution of the domain. In fact, restricting the problem to the class of axisymmetric functions, Problem (1.1) reduces to a problem only involving the variables (r,z) and the two-dimensional domain $\mathcal{D}$. We will closely follow the steps of the main result in [7] and make reference to intermediate technical results there.

With a very similar proof we can construct simultaneous blow-up in any finite number of disjoint circles Γ. It would be a very interesting issue to consider the case $r_0 = 0$ case in which the singularity would asymptotically collapse onto a point in the z-axis. Lifting the revolution symmetry assumption potentially obtaining other blow-up sets is a very interesting and difficult issue.

2. The axially symmetric problem

In the setting of Theorem 1 it is natural to look for solutions which are axially symmetric. More precisely, we look of a solution of (1.1) with boundary condition $u_b = \mathbf{e}_3$ of the form

$$u(x,t) := \tilde u(r,z), \quad x = (re^{i\vartheta}, z), \quad (r,z) \quad \text{in } \mathcal{D}.$$

where $\tilde u : \mathcal{D} \subset \mathbb{R}^2 \to S^2$. We directly check that in this situation our problem becomes

$$\begin{cases} \tilde u_t = \tilde u_{rr} + \dfrac{1}{r}\tilde u_r + \tilde u_{zz} + |\nabla \tilde u|^2 \tilde u & \text{in } \mathcal{D} \times (0,T) \\ \tilde u_r = 0 \quad \text{on } \{r=0\} \cap \mathcal{D} \times (0,T) \\ \tilde u = \mathbf{e}_3 \quad \text{on } (\partial\mathcal{D} \setminus \{r=0\}) \times (0,T) \\ \tilde u(\cdot,0) = \tilde u_0, \end{cases} \tag{2.1}$$

where $\nabla \tilde u = (\tilde u_r, \tilde u_z)$. We want to find a solution $\tilde u(x,z)$ that blows up exactly at the point $q = (r_0, z_0)$ as $t \to T$ in the form

$$\tilde u(r,z) \approx W\left(\frac{(r,z)-\xi(t)}{\lambda(t)}\right), \quad \lambda(t) \to 0, \quad \xi(t) \to q_0.$$

To make a precise ansatz, we consider the family of two-dimensional 1-corrotational harmonic maps

$$U_{\lambda,\xi,\omega}(r,z) := Q_\omega\, W(y), \quad y = \frac{(r,z)-\xi}{\lambda}, \quad \xi \in \mathbb{R}^2\ \omega \in \mathbb{R},\ \lambda > 0,$$

where $W(y)$ is the canonical 1-corrotational harmonic map (1.3) and Q_ω is the ω-rotation matrix

$$Q_\omega := \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

All these functions satisfy the elliptic equation

$$U_{rr} + U_{zz} + |\nabla U|^2 U = 0 \quad \text{in } \mathbb{R}^2, \quad |U| = 1. \tag{2.2}$$

For any sufficiently small number $T > 0$ we look for an initial datum u_0 such that the solution $\tilde u(r,z,t)$ of problem (2.1) looks at main order like

$$U_{\lambda(t),\xi(t),\omega(t)}(r,z) = Q_{\omega(t)}\, W(y), \quad y = \frac{(r,z)-\xi(t)}{\lambda(t)},$$

for certain functions $\xi(t)$, $\lambda(t)$ and $\omega(t)$ of class $C^1([0,T])$ such that

$$\xi(T) = q, \quad \lambda(T) = 0, \quad \omega(T) = 0.$$

We consider a first approximation $U(r,z,t)$ which smoothly interpolates $U_{\lambda(t),\xi(t),\omega(t)}(r,z,t)$ with $(r,z) \approx q$ and the constant vector $\mathbf{e}_3$. Let $\eta(\zeta)$ be a smooth cut-off function so that

$$\eta(\zeta) = \begin{cases} 1 & \text{for } \zeta < 1, \\ 0 & \text{for } \zeta > 2. \end{cases}$$

For a fixed small number $\delta > 0$ we let

$$\eta^\delta(r,z) := \eta\left(\frac{|(r,z)-q|}{\delta}\right)$$

and set

$$U(r,z,t) \;:=\; \eta^\delta(r,z)\, U_{\lambda(t),\xi(t),\omega(t)}(r,z,t) \;+\; (1-\eta^\delta(r,z))\,\mathbf{e}_3. \tag{2.3}$$

We shall find values for these functions so that for a small remainder $v(x,t)$ we have that $\tilde u = U + v$ solves (2.1). The condition $|U+v| = 1$ tells us that u can be written as

$$u(x,t) = U + \Pi_{U^\perp}\varphi + a(\Pi_{U^\perp}\varphi)U, \tag{2.4}$$

where φ is a small function with values into $\mathbb{R}^3$ and we denote

$$\Pi_{U^\perp}\varphi := \varphi - (\varphi\cdot U)U, \quad a(\zeta) := \sqrt{1-|\zeta|^2} - 1.$$

The term $a(\Pi_{U^\perp}\varphi)$ has a quadratic size in φ. We choose to decompose the remainder $\varphi(r,z,t)$ in (2.4) as the addition of an "outer" part, better expressed in the original variables (r,z), and an "inner" part which is supported near the singularity and it is naturally expressed as function of the slow variable y. More precisely, we let

$$\varphi(r,z,t) \;=\; \varphi^{out}(r,z,t) + \varphi^{in}(y,t), \quad y = \frac{(r,z)-\xi(t)}{\lambda(t)} \tag{2.5}$$

where

$$\varphi^{in}(y,t) \;= \eta_{R(t)}\,(y)\, Q_{\omega(t)}\phi(y,t), \quad \phi(y,t)\cdot W(y) \equiv 0$$

and $\eta_R(y) := \eta\left(\frac{|y|}{R}\right)$ The function $\phi(y,t)$ is defined for $|y| < 3R(t)$ where $R(t)\to +\infty$ and $\lambda(t)R(t)\to 0$ as $t\to T$. With these definitions we see that $\Pi_{U^\perp}\varphi^{in} = \varphi^{in}$.

We choose to the decompose the outer part $\varphi^{out}(x,t)$ in (2.5) as

$$\varphi^{out}(x,t) = \Phi^0[\omega,\lambda,\xi] + Z^*(x,t) \;+\; \psi(x,t),$$

where Φ^0 and $+\ Z^*(x,t)$ are explicit functions chosen as follows: $\Phi^0[\omega,\lambda,\xi]$ is a function (which will be precisely described in the next section) that at main order eliminates the largest slow-decaying part of the error of approximation $E(r,z,t)$ in (2.1), namely $E = S(U)$, where

$$S(\tilde u) := -\tilde u_t + \tilde u_{rr} + \frac{\tilde u_r}{r} + \tilde u_{zz} + |\nabla\tilde u|^2\tilde u.$$

Writing $p(t) := \lambda(t)e^{i\omega(t)}$ and using polar coordinates

$$(r,z) = \xi(t) + se^{i\theta},$$

we require

$$\Phi^0_t - \Phi^0_{rr} - \Phi^0_{zz} \approx \frac{2}{s}\begin{bmatrix}\dot p(t)e^{i\theta}\\ 0\end{bmatrix} \approx E(r,z,t).$$

With the aid of Duhamel's formula for the standard heat equation, we find that the following function is a good approximate solution:

$$\Phi^0[\omega,\lambda,\xi](s,\theta,t) := \begin{bmatrix}\varphi^0(s,t)e^{i\theta}\\ 0\end{bmatrix} \tag{2.6}$$

$$\varphi^0(s,t) = -\int_{-T}^t \dot p(\tau)sk(z(s),t-\tau)\,d\tau$$

$$z(s) = \sqrt{s^2+\lambda^2}, \quad k(z,t) = 2\frac{1-e^{-\frac{z^2}{4t}}}{z^2},$$

where for technical reasons $p(t)$ is assumed to be defined in $[-T,T]$, that is, also for some negative values of t. On the other hand, we let $Z^*: \Omega\times(0,\infty)\to\mathbb{R}^3$ satisfy

$$\begin{cases} Z^*_t = \Delta_x Z^* \quad \text{in } \Omega\times(0,\infty),\\ Z^*(\cdot,t) = 0 \quad \text{in } \partial\Omega\times(0,\infty),\\ Z^*(\cdot,0) = Z^*_0 \quad \text{in } \Omega,\end{cases} \tag{2.7}$$

where $Z_0^*(x)$ is is a small, sufficiently regular, axially symmetric function, more precisely

$$Z_0^*(x) = \tilde Z_0^*(r,z) = \begin{bmatrix} \tilde z_0^*(r,z) \\ \tilde z_{03}^*(r,z) \end{bmatrix}, \quad \tilde z_0^*(r,z) = \tilde z_{01}^*(r,z) + i\tilde z_{02}^*(r,z), \quad x = (re^{i\vartheta}, z). \tag{2.8}$$

function essentially satisfying

$$\tilde Z_0^*(q) = 0, \quad \operatorname{div} \tilde z_0^*(q) + i \operatorname{curl} \tilde z_0^*(q) \neq 0.$$

where we denote

$$\operatorname{div} \tilde z_0^*(r,z) = \partial_r \tilde z_{01}^*(r,z) + \partial_z \tilde z_{02}^*(r,z), \quad \operatorname{curl} \tilde z_0^*(r,z) = \partial_r \tilde z_{02}^*(r,z) - \partial_z \tilde z_{01}^*(r,z). \tag{2.9}$$

Of course we have $Z^*(x,t) = \tilde Z^*(r,z,t)$. Then for $(r,z,t) \in \mathcal{D} \times (0,T)$ we make the ansatz

$$\begin{cases} \tilde u(r,z,t) &= U(r,z,t) \,+\, v(r,z,t), \\ v(r,z,t) &= \Pi_{U^\perp}\left(\eta^\delta\, \Phi^0[\omega,\lambda,\xi] + \tilde Z^* + \psi\right) \,+\, \eta_R Q_\omega \phi + aU \end{cases} \tag{2.10}$$

for a blowing-up solution $\tilde u(r,z,t)$ of (2.1), where ϕ and ψ are lower order corrections. Our task is to find functions $\omega(t), \lambda(t), \xi(t)$, $\psi(x,t)$ and $\phi(y,t)$ as described above, such that the remainder v remains uniformly small.

We will define a system of equations that we call the *inner-outer gluing system*, essentially of the form

$$\begin{cases} \lambda^2 \phi_t \;=\; L_W[\phi] \;+\; H[p,\xi,\psi,\phi], \quad \phi\cdot W = 0 \quad \text{in } \mathbb{R}^2 \times (0,T) \\ \quad \psi_t \;=\; \psi_{rr} + \dfrac{\psi_r}{r} + \psi_{zz} \;+\; G[p,\xi,\psi,\phi] \qquad\qquad \text{in } \mathcal{D}\times(0,T) \end{cases}$$

where

$$L_W[\phi] = \Delta_y \phi + |\nabla_y W|^2 \phi + 2(\nabla_y \phi \cdot \nabla_y W) W, \quad \phi\cdot W = 0 \tag{2.11}$$

is the linearized operator for equation (2.2) around $U = W$, so that if the pair of functions $(\phi(y,t), \psi(x,t))$ solves it then $\tilde u$ given by (2.10) is a solution of (2.1). The point is to adjust the parameter functions ω, λ, ξ such that the inner problem can be solved for $\phi(y,t)$ which decays as $|y| \to \infty$. To fix the idea, let us consider the approximate elliptic equation, where time is regarded just as a parameter,

$$L_W[\phi] \;+\; H[p,\xi,0,0] = 0 \quad \text{in } \mathbb{R}^2$$

As we will discuss, a space-decaying solution $\phi(y,t)$ to this problem exists if a set of orthogonality conditions of the form

$$\int_{\mathbb{R}^2} H[p,\xi,0,0](y,t)\, Z(y)\, dy = 0 \quad \text{for all} \quad Z \in \mathcal{Z}$$

where $\mathcal{Z}$ is a 4-dimensional space constituted by decaying functions $Z(y)$ with $L_W[Z] = 0$. These solvability conditions lead to an essentially explicit system of equations for the parameter functions which will tell us in particular that for some small $\sigma > 0$

$$\begin{aligned} p(t) &= -(\operatorname{div} \tilde z_0^*(q) + i \operatorname{curl} \tilde z_0^*(q)) \frac{|\log T|}{\log^2(T-t)} (1 + O(|\log T|^{-1+\sigma})), \\ \xi_1(t) &= \sqrt{r_0^2 + 2(T-t)} + O((T-t)^{1+\sigma}) \\ \xi_2(t) &= z_0 + O((T-t)^{1+\sigma}), \end{aligned}$$

and we recall that we are consistently asking $\operatorname{div} \tilde z_0^*(q) + i\operatorname{curl} \tilde z_0^*(q) \neq 0$.

In the next sections we will carry out in detail the program for the construction sketched above.

3. The linearized operator around the bubble

We can represent $W(y)$ in polar coordinates,

$$W(y)=\begin{pmatrix} e^{i\theta}\sin w(\rho)\\ \cos w(\rho)\end{pmatrix},\quad w(\rho)=\pi-2\arctan(\rho),\quad y=\rho e^{i\theta}.$$

We notice that

$$w_\rho=-\frac{2}{1+\rho^2},\quad \sin w=-\rho w_\rho=\frac{2\rho}{1+\rho^2},\quad \cos w=\frac{\rho^2-1}{1+\rho^2}.$$

For the linearized operator L_W in (2.11) we have that $L_W[Z_{lj}]=0$ where

$$\begin{cases} Z_{01}(y)=\rho w_\rho(\rho)\,E_1(y) & Z_{02}(y)=\rho w_\rho(\rho)\,E_2(y)\\ Z_{11}(y)=w_\rho(\rho)\,[\cos\theta\,E_1(y)+\sin\theta\,E_2(y)] & Z_{12}(y)=w_\rho(\rho)\,[\sin\theta\,E_1(y)-\cos\theta\,E_2(y)]\\ Z_{-1,1}(y)=\rho^2 w_\rho(\rho)[\cos\theta E_1(y)-\sin\theta E_2(y)] & Z_{-1,2}(y)=\rho^2 w_\rho(\rho)[\sin\theta E_1(y)+\cos\theta E_2(y)].\end{cases}\tag{3.1}$$

and

$$E_1(y)=\begin{pmatrix} e^{i\theta}\cos w(\rho)\\ -\sin w(\rho)\end{pmatrix},\quad E_2(y)=\begin{pmatrix} ie^{i\theta}\\ 0\end{pmatrix}.$$

These vectors from an orthonormal basis of the tangent space to S^2 at the point $W(y)$.

The linearized operator at functions orthogonal to U. We consider the linearized operator L_U analogous to L_W but taken around our basic approximation U, that is,

$$L_U[\varphi]=\varphi_{rr}+\varphi_{zz}+|\nabla U|^2\varphi+2(\nabla\varphi\cdot\nabla U)U.$$

It will be especially significant to compute the action of L_U on functions with values pointwise orthogonal to U. In what remains of this section we will derive various formulas that will be very useful later on.

For an arbitrary function $\Phi(r,z)$ with values in $\mathbb{R}^3$ we denote the projection

$$\Pi_{U^\perp}\Phi:=\Phi-(\Phi\cdot U)U.$$

A direct computation shows the validity of the following:

$$L_U[\Pi_{U^\perp}\Phi]=\Pi_{U^\perp}(\Phi_{rr}+\Phi_{zz})+\tilde L_U[\Phi]$$

where

$$\tilde L_U[\Phi]:=|\nabla U|^2\Pi_{U^\perp}\Phi-2\nabla(\Phi\cdot U)\nabla U,$$

with $\nabla=(\partial_r,\partial_z)$ and

$$\nabla(\Phi\cdot U)\nabla U=\partial_r(\Phi\cdot U)\,\partial_r U+\partial_z(\Phi\cdot U)\,\partial_z U.$$

A very convenient expression for $\tilde L_U[\Phi]$ is obtained if we use polar coordinates. Writing in complex notation

$$\Phi(r,z)=\Phi(s,\theta),\quad (r,z)=\xi+se^{i\theta},$$

we find

$$\tilde L_U[\Phi]=-\frac{2}{\lambda}w_\rho(\rho)\,[(\Phi_s\cdot U)Q_\omega E_1-\frac{1}{s}(\Phi_\theta\cdot U)Q_\omega E_2],\quad \rho=\frac{s}{\lambda}.\tag{3.2}$$

We mention two consequences of formula (3.2). Let us assume that $\Phi(x)$ is a C^1 function $\Phi:\mathcal{D}\to\mathbb{C}\times\mathbb{R}$, which we express in the form

$$\Phi(r,z)\ =\ \begin{pmatrix}\varphi_1(r,z)+i\varphi_2(r,z)\\ \varphi_3(r,z)\end{pmatrix}.\tag{3.3}$$

We also denote

$$\varphi=\varphi_1+i\varphi_2,\quad \bar\varphi=\varphi_1-i\varphi_2$$

and define the operators

$$\operatorname{div}\varphi=\partial_r\varphi_1+\partial_z\varphi_2,\quad \operatorname{curl}\varphi=\partial_r\varphi_2-\partial_z\varphi_1.$$

Then the following formula holds:

$$\tilde L_U[\Phi]=\tilde L_U[\Phi]_0+\tilde L_U[\Phi]_1+\tilde L_U[\Phi]_2\ ,\tag{3.4}$$

where

$$\begin{cases} \tilde L_U[\Phi]_0 = \lambda^{-1}\rho w_\rho^2 \left[\, \operatorname{div}(e^{-i\omega}\varphi)\, Q_\omega E_1 + \operatorname{curl}(e^{-i\omega}\varphi)\, Q_\omega E_2 \,\right] \\ \tilde L_U[\Phi]_1 = -\, 2\lambda^{-1} w_\rho \cos w \left[\, (\partial_r\varphi_3)\cos\theta + (\partial_z\varphi_3)\sin\theta \,\right] Q_\omega E_1 \\ \qquad\qquad - 2\lambda^{-1} w_\rho \cos w \left[\, (\partial_r\varphi_3)\sin\theta - (\partial_z\varphi_3)\cos\theta \,\right] Q_\omega E_2 \ , \\ \tilde L_U[\Phi]_2 = \quad \lambda^{-1}\rho w_\rho^2 \left[\, \operatorname{div}(e^{i\omega}\bar\varphi)\, \cos 2\theta - \operatorname{curl}(e^{i\omega}\bar\varphi)\, \sin 2\theta \,\right] Q_\omega E_1 \\ \qquad\qquad + \lambda^{-1}\rho w_\rho^2 \left[\, \operatorname{div}(e^{i\omega}\bar\varphi)\, \sin 2\theta + \operatorname{curl}(e^{i\omega}\bar\varphi)\, \cos 2\theta \,\right] Q_\omega E_2. \end{cases} \tag{3.5}$$

Another corollary of formula (3.2) that we single out is the following: assume that

$$\Phi(r,z) = \begin{pmatrix} \phi(s)e^{i\theta} \\ 0 \end{pmatrix}, \quad x = \xi + s e^{i\theta}, \quad \rho = \frac{s}{\lambda}$$

where $\phi(s)$ is complex valued. Then

$$\tilde L_U[\Phi] = \frac{2}{\lambda} w_\rho(\rho)^2 \left[\operatorname{Re}\,(e^{-i\omega}\partial_s\phi(s)) Q_\omega E_1 + \frac{1}{s}\operatorname{Im}\,(e^{-i\omega}\phi(s)) Q_\omega E_2 \right]. \tag{3.6}$$

For the proof of the formulas above see [7], section 2.

4. The ansatz and the inner-outer gluing system

The equation we want to solve is $S(\tilde u) = 0$, with $\tilde u = U + v$. A useful observation that we make is that as long as the constraint $|\tilde u| = 1$ is kept at all times and $\tilde u = U + v$ with $|v| \le \frac12$ uniformly, then for $\tilde u$ to solve equation (2.1) it suffices that

$$S(U + v) = b(r,z,t)U \tag{4.1}$$

for some scalar function b. Indeed, we observe that since $|\tilde u| \equiv 1$ we have

$$b\,(U\cdot \tilde u) = S(\tilde u)\cdot \tilde u = -\frac12 \frac{d}{dt}|\tilde u|^2 + \frac12(\partial_r^2 + \partial_z^2)|\tilde u|^2 + \frac{1}{2r}\partial_r|\tilde u|^2 = 0,$$

and since $U \cdot u \ge \frac12$, we find that $b \equiv 0$.

We find the following expansion for $S(U + v)$ with $v = \Pi_{U^\perp}\varphi + a(\Pi_{U^\perp}\varphi)U$:

$$S(U + \Pi_{U^\perp}\varphi + aU) = S(U) - \partial_t \Pi_{U^\perp}\varphi + L_U(\Pi_{U^\perp}\varphi) + \frac1r \partial_r(\Pi_{U^\perp}\varphi) + N_U(\Pi_{U^\perp}\varphi) + c(\Pi_{U^\perp}\varphi)U$$

where for $\zeta = \Pi_{U^\perp}\varphi$, $a = a(\zeta)$,

$$\begin{aligned} L_U(\zeta) &= \zeta_{rr} + \zeta_{zz} + |\nabla U|^2\zeta + 2(\nabla U\cdot\zeta)U \\ N_U(\zeta) &= \left[2\nabla(aU)\cdot\nabla(U+\zeta) + 2\nabla U\cdot\nabla\zeta + |\nabla\zeta|^2 + |\nabla(aU)|^2\,\right]\zeta - aU_t + \frac{a}{r}\partial_r U \\ &\quad + 2\nabla a \nabla U, \\ c(\zeta) &= a_{rr} + a_{zz} - a_t + (|\nabla(U + \zeta + aU)|^2 - |\nabla U|^2)(1+a) - 2\nabla U\cdot\nabla\zeta + \frac1r(\partial_r a). \end{aligned}$$

Since we just need to have an equation of the form (4.1) satisfied, we find that

$$\tilde u = U + \Pi_{U^\perp}\varphi + a(\Pi_{U^\perp}\varphi)U$$

solves (2.1) if and only if φ satisfies

$$0 = S(U) - \partial_t \Pi_{U^\perp}\varphi + L_U(\Pi_{U^\perp}\varphi) + \frac1r \partial_r(\Pi_{U^\perp}\varphi) + N_U(\Pi_{U^\perp}\varphi) + b(r,z,t)U,$$

for some scalar function b. We use the ansatz (2.10) for $\tilde u$, namely

$$\tilde u(r,z,t) \ = U + \Pi_{U^\perp}\varphi \ + \ a(\Pi_{U^\perp}\varphi)U, \quad \varphi := \Pi_{U^\perp}\left(\eta^\delta\, \Phi^0[\omega,\lambda,\xi] + \Psi^*\right) + \eta_R Q_\omega\phi, \tag{4.2}$$

where we will later decompose $\Psi^* = \tilde{Z}^* + \psi$ for a suitable $\tilde{Z}^*$. Equation $S(\tilde{u}) = 0$ then becomes

$$\begin{aligned}
0 = {} & \lambda^{-2}\eta Q_\omega[-\lambda^2\phi_t + L_W[\phi] + \lambda^2 Q_{-\omega}\tilde{L}_U[\Psi^*]] \\
& + \eta Q_\omega(\lambda^{-1}\dot{\lambda}y\cdot\nabla_y\phi + \lambda^{-1}\dot{\xi}\cdot\nabla_y\phi - \dot{\omega}J\phi) \\
& + \eta^\delta\tilde{L}_U[\Phi^0] + \eta^\delta\Pi_{U^\perp}[-\partial_t\Phi^0 + (\partial_r^2+\partial_z^2)\Phi^0 + S(U)] + \mathcal{E}^{out,0} \\
& - \partial_t\Psi^* + \Delta\Psi^* + (1-\eta)\tilde{L}_U[\Psi^*] + Q_\omega[((\partial_r^2+\partial_z^2)\eta)\phi + 2\nabla\eta\nabla\phi - \eta_t\phi] \\
& + \frac{1}{r}\partial_r\left(\Pi_{U^\perp}\left(\eta^\delta\,\Phi^0[\omega,\lambda,\xi] + \Psi^*\right) + \eta_R Q_\omega\phi\right) \\
& + N_U(\eta Q_\omega\phi + \Pi_{U^\perp}(\Phi^0+\Psi^*)) + ((\Psi^*+\Phi^0)\cdot U)U_t + bU,
\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}
\mathcal{E}^{out,0} = {} & \tilde{L}_U[\eta^\delta\Phi^0] + \Pi_{U^\perp}[(-\partial_t+\partial_r^2+\partial_z^2)(\delta^\eta\Phi^0)] \\
& - \eta^\delta\tilde{L}_U[\Phi^0] - \eta^\delta\Pi_{U^\perp}[-\partial_t\Phi^0 + (\partial_r^2+\partial_z^2)\Phi^0] + (1-\eta^\delta)S(U).
\end{aligned}$$

We note that from the definition (2.3) and the fact that $U_{\lambda(t),\xi(t),\omega(t)}$ satisfies the harmonic map equation (2.2), we have

$$S(U) = -U_t + \frac{1}{r}\partial_r U + \mathcal{E}^{out,1}, \quad |\mathcal{E}^{out,1}| + |\nabla\mathcal{E}^{out,1}| \le C\lambda.$$

Invoking formulas (3.1) to compute U_t we get

$$U_t = \dot{\lambda}\partial_\lambda U_{\lambda,\xi,\omega} + \dot{\omega}\partial_\omega U_{\lambda,\xi,\omega} + \partial_\xi U_{\lambda,\xi,\omega}\cdot\dot{\xi} = \mathcal{E}_0 + \mathcal{E}_1,$$

where, setting $y = \frac{(r,z)-\xi}{\lambda} = \rho e^{i\theta}$, we have

$$\begin{aligned}
\mathcal{E}_0(r,z,t) &= -Q_\omega[\frac{\dot{\lambda}}{\lambda}\rho w_\rho(\rho)\,E_1(y) + \dot{\omega}\rho w_\rho(\rho)\,E_2(y)\,] \\
\mathcal{E}_1(r,z,t) &= -\frac{\dot{\xi}_1}{\lambda}\,w_\rho(\rho)\,Q_\omega[\,\cos\theta\,E_1(y) + \sin\theta\,E_2(y)] \\
&\qquad - \frac{\dot{\xi}_2}{\lambda}\,w_\rho(\rho)\,Q_\omega[\sin\theta\,E_1(y) - \cos\theta\,E_2(y)\,].
\end{aligned}$$

The choice (2.6) of Φ^0 is so that it cancels $\mathcal{E}_0$ at main order. The other terms in $S(U)$ behave better, since $\mathcal{E}_1$ has faster space decay in ρ and the other terms in $S(U)$ are smaller. We note that

$$\mathcal{E}_0(r,z,t) \approx \tilde{\mathcal{E}}_0(r,z,t) := -\frac{2s}{s^2+\lambda^2}\begin{bmatrix}\dot{p}(t)e^{i\theta}\\ 0\end{bmatrix},$$

and a direct computation yields

$$\Phi_t^0 + (\partial_r^2+\partial_z^2)\Phi^0 + \tilde{\mathcal{E}}_0 = \tilde{\mathcal{R}}_0 + \tilde{\mathcal{R}}_1, \quad \tilde{\mathcal{R}}_0 = \begin{pmatrix}\mathcal{R}_0\\ 0\end{pmatrix}, \quad \tilde{\mathcal{R}}_1 = \begin{pmatrix}\mathcal{R}_1\\ 0\end{pmatrix}$$

where

$$\begin{aligned}
\mathcal{R}_0 &:= -re^{i\theta}\frac{\lambda^2}{z^4}\int_{-T}^t \dot{p}(\tau)(zk_z - z^2k_{zz})(z(s),t-\tau)\,d\tau \\
\mathcal{R}_1 &:= -e^{i\theta}\mathrm{Re}\,(e^{-i\theta}\dot{\xi}(t))\int_{-T}^t \dot{p}(\tau)\,k(z(s),t-\tau)\,d\tau \\
&\qquad + \frac{r}{z^2}e^{i\theta}\,(\lambda\dot{\lambda}(t) - \mathrm{Re}\,(re^{i\theta}\dot{\xi}(t)))\int_{-T}^t \dot{p}(\tau)\ zk_z(z(s),t-\tau)\,d\tau.
\end{aligned}$$

We observe that $\mathcal{R}_1$ is actually a term of smaller order. Using formulas (3.4), (3.6) and the facts

$$\frac{\lambda^2 r}{z^4} = \frac{1}{4\lambda}\rho w_\rho^2, \quad \frac{r}{z^2}(1-\cos w) = \frac{1}{2\lambda}\rho w_\rho^2,$$

we derive an expression for the quantity:

$$\begin{aligned}
&\tilde L_U[\Phi^0] + \Pi_{U^\perp}[-\partial_t\Phi^0 + (\partial_r^2+\partial_z^2)\Phi^0 + S(U)]\\
&= \tilde L_U[\Phi^0] - \mathcal{E}_1 + \Pi_{U^\perp}[\tilde{\mathcal{E}}_0] - \mathcal{E}_0 + \Pi_{U^\perp}[\tilde{\mathcal{R}}_0] + \Pi_{U^\perp}[\tilde{\mathcal{R}}_1]\\
&= \mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi] + \Pi_{U^\perp}[\tilde{\mathcal{R}}_1] + +\Pi_{U^\perp}\left[\frac{1}{r}U + \mathcal{E}^{out,1}\right]
\end{aligned}$$

where

$$\mathcal{K}_0[p,\xi] = \mathcal{K}_{01}[p,\xi] + \mathcal{K}_{02}[p,\xi]$$

with

$$\begin{aligned}
\mathcal{K}_{01}[p,\xi] &:= -\frac{2}{\lambda}\rho w_\rho^2 \int_{-T}^t \Big[\mathrm{Re}\,(\dot p(\tau)e^{-i\omega(t)})Q_\omega E_1 + \mathrm{Im}\,(\dot p(\tau)e^{-i\omega(t)})Q_\omega E_2\Big]\\
&\qquad\qquad \cdot k(z,t-\tau)\,d\tau \qquad (4.4)\\
\mathcal{K}_{02}[p,\xi] &:= \frac{1}{\lambda}\rho w_\rho^2\left[\dot\lambda - \int_{-T}^t \mathrm{Re}\,(\dot p(\tau)e^{-i\omega(t)}) r k_z(z,t-\tau)z_r\,d\tau\right]Q_\omega E_1\\
&\quad - \frac{1}{4\lambda}\rho w_\rho^2\cos w\left[\int_{-T}^t \mathrm{Re}\,(\dot p(\tau)e^{-i\omega(t)})\,(zk_z - z^2k_{zz})(z,t-\tau)\,d\tau\right]Q_\omega E_1\\
&\quad - \frac{1}{4\lambda}\rho w_\rho^2\left[\int_{-T}^t \mathrm{Im}\,(\dot p(\tau)e^{-i\omega(t)})\,(zk_z - z^2k_{zz})(z,t-\tau)\,d\tau\right]Q_\omega E_2, \qquad (4.5)\\
\mathcal{K}_1[p,\xi] &:= \frac{1}{\lambda}w_\rho\left[\,\mathrm{Re}\,\big((\dot\xi_1 - i\dot\xi_2)e^{i\theta}\big)Q_\omega E_1 + \mathrm{Im}\,\big((\dot\xi_1 - i\dot\xi_2)e^{i\theta}\big)Q_\omega E_2\right]. \qquad (4.6)
\end{aligned}$$

We insert this decomposition in equation (4.3) and see that we will have a solution to the equation if the pair (ϕ,Ψ^*) solves the *inner-outer gluing system*

$$\begin{cases}
\lambda^2\phi_t = L_W[\phi] + \lambda^2 Q_{-\omega}\Big[\tilde L_U[\Psi^*] + \mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi]\Big] + \lambda^2\chi_{D_{2R}}\frac{1}{r}Q_{-\omega}\partial_r U \quad \text{in } D_{2R}\\
\phi\cdot W = 0 \quad \text{in } D_{2R}\\
\phi(\cdot,0) = 0 = \phi(\cdot,T),
\end{cases} \tag{4.7}$$

$$\partial_t\Psi^* = (\partial_r^2+\partial_z^2)\Psi^* + g[p,\xi,\Psi^*,\phi] \quad \text{in } \mathcal{D}\times(0,T), \tag{4.8}$$

where χ_A is characteristic function of a set A,

$$\begin{aligned}
g[p,\xi,\Psi^*,\phi] &:= (1-\eta)\tilde L_U[\Psi^*] + (\Psi^*\cdot U)U_t \qquad (4.9)\\
&+ Q_\omega\big(((\partial_r^2+\partial_z^2)\eta)\phi + 2\nabla\eta\nabla\phi - \eta_t\phi\big)\\
&+ \eta Q_\omega\big(-\dot\omega J\phi + \lambda^{-1}\dot\lambda y\cdot\nabla_y\phi + \lambda^{-1}\dot\xi\cdot\nabla_y\phi\big)\\
&+ (1-\eta)[\mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi]] + \Pi_{U^\perp}[\tilde{\mathcal{R}}_1] + (\Phi^0\cdot U)U_t\\
&+ \frac{1}{r}\partial_r\left(\Pi_{U^\perp}\big(\eta^\delta\,\Phi^0[\omega,\lambda,\xi] + \Psi^*\big) + \eta_R Q_\omega\phi\right) + (1-\eta)\frac{1}{r}\partial_r U + \eta^\delta\mathcal{E}^{out,1} + \mathcal{E}^{out,0}\\
&+ N_U(\eta Q_\omega\phi + \Pi_{U^\perp}(\Phi^0 + \Psi^*)),
\end{aligned}$$

and we denote

$$D_{\gamma R} = \{(y,t)\in\mathbb{R}^2\times(0,T)\ /\ |y| < \gamma R(t)\}.$$

Indeed if (ϕ,Ψ^*) solves this system, then $\tilde u$ given by (4.2) solves equation (2.1). The boundary condition $\tilde u = \mathbf{e}_3$ on $(\partial\mathcal{D}\setminus\{r=0\})\times(0,T)$ amounts to

$$\Pi_{U^\perp}[\Phi^0+\Psi^*] + a(\Pi_{U^\perp}[U+\Phi^0+\Psi^*])U = (\mathbf{e}_3 - U)$$

and then it suffices that we take the boundary condition for (4.8):

$$\Psi^* = \mathbf{e}_3 - U - \Phi^0 \quad \text{on } (\partial\mathcal{D}\setminus\{r=0\})\times(0,T). \tag{4.10}$$

We also impose

$$\partial_r \Psi^* = 0 \quad \text{on } \{r=0\} \cap \mathcal{D} \times (0,T). \tag{4.11}$$

Since we want $\tilde{u}(r,z,t)$ to be a small perturbation of $U(x,t)$ when we stand close to (r_0, z_0, T), it is natural to require that Ψ^* satisfies the final condition

$$\Psi^*(r_0, z_0, T) = 0.$$

This constraint amounts to three Lagrange multipliers when we solve the problem, which we choose to put in the initial condition. Then we assume

$$\Psi^*\big(r,z,0\big) = Z_0^*(x) + c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3,$$

where c_1, c_2, c_3 are undetermined constants and $Z_0^*(x)$ is a small function for which specific assumptions will later be made.

5. The reduced equations

In this section we will informally discuss the procedure to achieve our purpose in particular deriving the order of vanishing of the scaling parameter $\lambda(t)$ as $t \to T$.

The main term that couples equations (4.7) and (4.8) inside the second equation is the linear expression

$$Q_\omega[((\partial_r^2 + \partial_z^2)\eta)\phi + 2\nabla\eta\nabla\phi + \eta_t\phi],$$

which is supported in $|y| = O(R)$. This motivates the fact that we want ϕ to exhibit some type of space decay in $|y|$ since in that way Ψ^* will eventually be smaller and in turn that would make the two equations at main order *uncoupled*. Equation (4.7) has the form

$$\begin{aligned}
&\lambda^2\phi_t = L_W[\phi] + h[p,\xi,\Psi^*](y,t) \quad \text{in } D_{2R}\\
&\phi \cdot W = 0 \quad \text{in } D_{2R}\\
&\phi(\cdot,0) = 0 \quad \text{in } B_{2R(0)},
\end{aligned}$$

where, for convenience we assume that $h(y,t)$ is defined for all $y \in \mathbb{R}^2$ extending it outside D_{2R} as

$$h[p,\xi,\Psi^*] = \lambda^2 Q_{-\omega}\mathcal{K}_0[p,\xi] + \lambda^2 Q_{-\omega}\left[\tilde{L}_U[\Psi^*] + \mathcal{K}_1[p,\xi] + \frac{1}{r}\partial_r U\right]\chi_{D_{2R}}, \tag{5.1}$$

where $\mathcal{K}_0$ is defined in (4.4), (4.5) and $\mathcal{K}_1$ in (4.6). If $\lambda(t)$ has a relatively smooth vanishing as $t \to T$ it seems natural that the term $\lambda^2\phi_t$ be of smaller order and then the equation is approximately represented by the elliptic problem

$$L_W[\phi] + h[p,\xi,\Psi^*] = 0, \quad \phi \cdot W = 0 \quad \text{in } \mathbb{R}^2. \tag{5.2}$$

Let us consider the decaying functions $Z_{lj}(y)$ defined in formula (3.1), which satisfy $L_W[Z_{lj}] = 0$. If $\phi(y,t)$ is a solution of (5.2) with sufficient decay, then necessarily

$$\int_{\mathbb{R}^2} h[p,\xi,\Psi^*](y,t) \cdot Z_{lj}(y)\,dy = 0 \qquad \text{for all} \quad t \in (0,T), \tag{5.3}$$

for $l = 0,1$, $j = 1,2$. These relations amount to an integro-differential system of equations for $p(t)$, $\xi(t)$, which, as a matter of fact, *detemine* the correct values of the parameters so that the solution (ϕ, Ψ^*) with appropriate asymptotics exists.

We derive next useful expressions for relations (5.3). Let us first define

$$\mathcal{B}_{0j}[p](t) := \frac{\lambda}{2\pi}\int_{\mathbb{R}^2} Q_{-\omega}[\mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi]] \cdot Z_{0j}(y)\,dy. \tag{5.4}$$

$$\tilde{\mathcal{B}}_{0j}[p,\xi] := \frac{\lambda}{2\pi}\int_{B_{2R}} Q_{-\omega}(\frac{1}{r}\partial_r U) \cdot Z_{0j}(y)\,dy$$

Using (4.4), (4.5) the following expressions for $\mathcal{B}_{01}$, $\mathcal{B}_{02}$ are readily obtained:

$$\mathcal{B}_{01}[p](t) = \int_{-T}^{t} \mathrm{Re}\,(\dot p(\tau)e^{-i\omega(t)})\,\Gamma_1\left(\frac{\lambda(t)^2}{t-\tau}\right)\frac{d\tau}{t-\tau} - 2\dot\lambda(t)$$
$$\mathcal{B}_{02}[p](t) = \int_{-T}^{t} \mathrm{Im}\,(\dot p(\tau)e^{-i\omega(t)})\,\Gamma_2\left(\frac{\lambda(t)^2}{t-\tau}\right)\frac{d\tau}{t-\tau}$$

where $\Gamma_j(\tau)$, $j=1,2$ are the smooth functions defined as follows:

$$\Gamma_1(\tau) = -\int_0^\infty \rho^3 w_\rho^3\left[K(\zeta) + 2\zeta K_\zeta(\zeta)\frac{\rho^2}{1+\rho^2} - 4\cos(w)\zeta^2 K_{\zeta\zeta}(\zeta)\right]_{\zeta=\tau(1+\rho^2)} d\rho$$
$$\Gamma_2(\tau) = -\int_0^\infty \rho^3 w_\rho^3\left[K(\zeta) - \zeta^2 K_{\zeta\zeta}(\zeta)\right]_{\zeta=\tau(1+\rho^2)} d\rho$$

where

$$K(\zeta) = 2\frac{1-e^{-\frac{\zeta}{4}}}{\zeta},$$

and we have used that $\int_0^\infty \rho^3 w_\rho^3 d\rho = -2$. Using these expressions we find that

$$|\Gamma_l(\tau) - 1| \le C\tau(1+|\log\tau|) \qquad \text{for } \tau < 1,$$
$$|\Gamma_l(\tau)| \le \frac{C}{\tau} \qquad \text{for } \tau > 1, l = 1,2.$$

Let us define

$$\mathcal{B}_0[p] := \frac{1}{2}e^{i\omega(t)}\left(\mathcal{B}_{01}[p] + i\mathcal{B}_{02}[p]\right), \quad \tilde{\mathcal{B}}_0[p] := \frac{1}{2}e^{i\omega(t)}\left(\tilde{\mathcal{B}}_{01}[p] + i\tilde{\mathcal{B}}_{02}[p]\right) \tag{5.5}$$

and

$$a_{0j}[p,\xi,\Psi^*] := -\frac{\lambda}{2\pi}\int_{B_{2R}} Q_{-\omega}\tilde L_U[\Psi^*]\cdot Z_{0j}(y)\,dy, j = 1,2,$$
$$a_0[p,\xi,\Psi^*] := \frac{1}{2}e^{i\omega(t)}\left(a_{01}[p,\xi,\Psi^*] + i a_{02}[p,\xi,\Psi^*]\right). \tag{5.6}$$

Similarly, we let

$$\mathcal{B}_{1j}[p,\xi](t) := \frac{\lambda}{2\pi}\int_{\mathbb{R}^2} Q_{-\omega}\left[\mathcal{K}_0[p,\xi] + \mathcal{K}_1[p,\xi] + \chi_{D_{2R}}\frac{1}{r}\partial_r U\right]\cdot Z_{1j}(y)\,dy, j = 1,2,$$
$$\mathcal{B}_1[p,\xi](t) := \mathcal{B}_{11}[p,\xi](t) + i\mathcal{B}_{12}[p,\xi](t).$$

At last, we set

$$a_{1j}[p,\xi,\Psi^*] := \frac{\lambda}{2\pi}\int_{B_{2R}} Q_{-\omega}\tilde L_U[\Psi^*]\cdot Z_{1j}(y)\,dy, \quad j = 1,2,$$
$$a_1[p,\xi,\Psi^*] := -e^{i\omega(t)}(a_{11}[p,\xi,\Psi^*] + i a_{12}[p,\xi,\Psi^*]).$$

We get that the four conditions (5.3) reduce to the system of two complex equations

$$\mathcal{B}_0[p] = a_0[p,\xi,\Psi^*] - \tilde{\mathcal{B}}_0[p,\xi], \tag{5.7}$$
$$\mathcal{B}_1[\xi] = a_1[p,\xi,\Psi^*]. \tag{5.8}$$

At this point we will make some preliminary considerations on this system that will allow us to find a first guess of the parameters $p(t)$ and $\xi(t)$. First, we observe that

$$\mathcal{B}_0[p] = \int_{-T}^{t-\lambda^2}\frac{\dot p(\tau)}{t-\tau}d\tau + O(\|\dot p\|_\infty), \quad \tilde{\mathcal{B}}_0[p,\xi] = O(\lambda^{1-\sigma}),$$

for any $\sigma > 0$.

To get an approximation for a_0, let us write

$$\Psi^* = \begin{bmatrix}\psi^*\\ \psi_3^*\end{bmatrix}, \quad \psi^* = \psi_1^* + i\psi_2^*.$$

From formula (3.4) we find that

$$\tilde{L}_U[\Psi^*](y) = [\tilde{L}_U]_0[\Psi^*] + [\tilde{L}_U]_1[\Psi^*] + [\tilde{L}_U]_2[\Psi^*],$$

where

$$\begin{aligned}
\lambda Q_{-\omega}[\tilde{L}_U]_0[\Psi^*] = &\quad \rho w_\rho^2 \left[\operatorname{div}(e^{-i\omega}\psi^*)\, E_1 + \operatorname{curl}(e^{-i\omega}\psi^*)\, E_2 \right] \\
\lambda Q_{-\omega}[\tilde{L}_U]_1[\Psi^*] = & -2w_\rho \cos w \left[(\partial_r \psi_3^*)\cos\theta + (\partial_z \psi_3^*)\sin\theta \right] E_1 \\
& - 2w_\rho \cos w \left[(\partial_r \psi_3^*)\sin\theta - (\partial_z \psi_3^*)\cos\theta \right] E_2 \ , \\
\lambda Q_{-\omega}[\tilde{L}_U]_2[\Psi^*] = &\quad \rho w_\rho^2 \left[\operatorname{div}(e^{i\omega}\bar{\psi}^*)\, \cos 2\theta - \operatorname{curl}(e^{i\omega}\bar{\psi}^*)\, \sin 2\theta \right] E_1 \\
& + \rho w_\rho^2 \left[\operatorname{div}(e^{i\omega}\bar{\psi}^*)\, \sin 2\theta + \operatorname{curl}(e^{i\omega}\bar{\psi}^*)\, \cos 2\theta \right] E_2,
\end{aligned}$$

and the differential operators in Ψ^* on the right hand sides are evaluated at (r, z, t) with $(r, z) = \xi(t) + \lambda(t)y$, $y = \rho e^{i\theta}$ while $E_l = E_l(y)$, $l = 1, 2$. From the above decomposition, assuming that Ψ^* is of class C^1 in space variable, we find that

$$a_0[p, \xi, \Psi^*] = [\operatorname{div}\psi^* + i \operatorname{curl}\psi^*](\xi, t) + o(1),$$

where $o(1) \to 0$ as $t \to T$.

Similarly, we have that

$$a_1(p, \xi) = 2(\partial_r \psi_3^* + i\partial_z \psi_3^*)(\xi, t) \int_0^\infty \cos w\, w_\rho^2 \rho\, d\rho = o(1) \quad \text{as} \quad t \to T,$$

since $\int_0^\infty w_\rho^2 \cos w \rho\, d\rho = 0$.

Using (4.6), (3.1) and the fact that $\int_0^\infty \rho w_\rho^2 d\rho = 2$ we get

$$\mathcal{B}_1[\xi](t) \;=\; 2[\,\dot{\xi}_1(t) + i\dot{\xi}_2(t)\,] + \frac{2}{\xi_1(t)} + O(\lambda^\sigma),$$

for some $\sigma > 0$ (actually $\sigma = 2\beta$ where $R \approx \lambda^\beta$.)

Let us discuss informally how to handle (5.7)-(5.8). For this we simplify this system in the form

$$\int_{-T}^{t-\lambda^2} \frac{\dot{p}(\tau)}{t-\tau} d\tau = [\operatorname{div}\psi^* + i \operatorname{curl}\psi^*](\xi(t), t) + o(1) + O(\|\dot{p}\|_\infty)$$

$$\dot{\xi}_1(t) = -\frac{1}{\xi_1(t)} + o(1) \quad \text{as} \quad t \to T. \tag{5.9}$$

$$\dot{\xi}_2(t) = o(1) \quad \text{as} \quad t \to T. \tag{5.10}$$

We assume for the moment that the function $\Psi^*(x, t)$ is fixed, sufficiently regular, and we regard T as a parameter that will always be taken smaller if necessary. Recall that we want $\xi(T) = (r_0, z_0)$ where $(r_0, z_0) \in \mathcal{D}$, $r_0 \neq 0$ is given, and $\lambda(T) = 0$. Equation (5.10) suggests us to take $\xi_2(t) \equiv z_0$ as a first approximation, while (5.9) suggest that ξ_1 is given at main order by

$$\xi_1(t) = \sqrt{r_0^2 + 2(T-t)}.$$

Neglecting lower order terms, we arrive at the "clean" equation for $p(t) = \lambda(t)e^{i\omega(t)}$,

$$\int_{-T}^{t-\lambda(t)^2} \frac{\dot{p}(s)}{t-s} ds = a_0^* \tag{5.11}$$

where $a_0^* = \operatorname{div}\psi^*(q, 0) + i \operatorname{curl}\psi^*(q, 0)$. At this point we make the following assumption:

$$\operatorname{div}\psi^*(q, 0) + i \operatorname{curl}\psi^*(q, 0) \neq 0. \tag{5.12}$$

We claim that a good approximate solution of (5.11) as $t \to T$ is given by

$$\dot{p}(t) = -\frac{\kappa}{\log^2(T-t)}$$

for a suitable $\kappa \in \mathbb{C}$. In fact, substituting, we have

$$
\begin{aligned}
\int_{-T}^{t-\lambda(t)^2} \frac{\dot p(s)}{t-s}\, ds &= \int_{-T}^{t-(T-t)} \frac{\dot p(s)}{t-s}\, ds + \dot p(t)\left[\log(T-t) - 2\log(\lambda(t))\right] + \int_{t-(T-t)}^{t-\lambda(t)^2} \frac{\dot p(s) - \dot p(t)}{t-s} ds \\
&\approx \int_{-T}^{t} \frac{\dot p(s)}{T-s}\, ds - \dot p(t)\log(T-t) \qquad (5.13)
\end{aligned}
$$

as $t \to T$. We see that by the explicit form of p,

$$
\frac{d}{dt}\left[\int_{-T}^{t} \frac{\dot p(s)}{T-s}\, ds - \dot p(t)\log(T-t)\right] = 0,
$$

and hence the right hand side of (5.13) is constant. As a conclusion, equation (5.11) is approximately satisfied if κ is such that

$$
\kappa \int_{-T}^{T} \frac{\dot p(s)}{T-s} = a_0^*.
$$

Imposing $p(T) = 0$ we gives us the approximate expression

$$
p(t)(t) = a_0^* \frac{|\log T|(T-t)}{\log^2(T-t)}(T-t)\,(1+o(1)) \quad \text{as} \quad t \to T.
$$

6. Solving the inner-outer gluing system

Our purpose is to determine, for a given $(r_0, z_0) \in \mathcal{D}$ and a sufficiently small $T > 0$, a solution (ϕ, Ψ^*) of system (4.7)-(4.8) with a boundary condition of the form (4.10), (4.11) such that $\tilde u(r,z,t)$ given by (4.2) blows up with $U(x,t)$ as its main order profile. This will only be possible for adequate choices of the parameter functions $\xi(t)$ and $p(t) = \lambda(t)e^{i\omega(t)}$. These functions will eventually be found by fixed point arguments, but a priori we need to make some assumptions regarding their behavior.

First, we define

$$
\lambda_*(t) = \frac{|\log T|(T-t)}{|\log(T-t)|^2}.
$$

We will assume that for some positive numbers a_1, a_2, σ independent of T the following hold:

$$
\begin{aligned}
a_1|\dot\lambda_*(t)| \le |\dot p(t)| \le a_2|\dot\lambda_*(t)| \quad &\text{for all} \quad t \in (0,T), \\
|\dot\xi(t)| \le \lambda_*(t)^\sigma \qquad &\text{for all} \quad t \in (0,T).
\end{aligned}
$$

We also take

$$
R(t) = \lambda_*(t)^{-\beta},
$$

where $\beta \in (0, \frac{1}{2})$.

To solve the outer equation (4.8) we will decompose Ψ^* in the form $\Psi^* = \tilde Z^* + \psi$ where we let $Z^* : \Omega \times (0,\infty) \to \mathbb{R}^3$ satisfy (2.7) with $Z_0^*(x)$ a function satisfying certain conditions to be described below. Since we would like that $\tilde u(r,z,t)$ given by (4.2) has a blow-up behavior given at main order by that of $U(x,t)$, we will require

$$
\Psi^*(r_0, z_0, T) = 0.
$$

This constraint has three parameters. Therefore we need three "Lagrange multipliers" which we include in the initial datum.

6.1. **Assumptions on Z_0^*.** Let us recall that Z^* solves the heat equation (2.7) with initial condition Z_0^*. We assume first that Z_0^* is axially symmetric so that $Z_0^*(x) = \tilde{Z}_0^*(r,z)$ and use the notation (2.8) and (2.9). A first condition that we require, consistent with (5.12), is $\operatorname{div} \tilde{z}_0^*(q) + i \operatorname{curl} \tilde{z}_0^*(q) \neq 0$. In addition we require that $\tilde{Z}_0^*(r_0,z_0) \approx 0$ in a non-degenerate way. We want also Z^* to be sufficiently small, but independently of T, so that the heat equation (2.7) is a good approximation of the linearized harmonic map flow far from the singularity. More precisely, we assume that for some $\alpha_0 > 0$ small and some $\alpha_1 > 0$, all independent of T, we have

$$\begin{cases} \|Z_0^*\|_{C^3(\overline{\Omega})} \leq \alpha_0, \\ |\tilde{Z}_0^*(r_0,z_0)| \leq 5T, \\ |(D\tilde{z}_0^*(r_0,z_0))^{-1}| \leq \alpha_1, \\ \alpha_0 \leq |\operatorname{div} \tilde{z}_0^*(r_0,z_0) + i \operatorname{curl} \tilde{z}_0^*(r_0,z_0)|. \end{cases} \tag{6.1}$$

(The notation here is analogous to (2.8) and (2.9).)

6.2. **Linear theory for the inner problem.** The inner problem (4.7) is written as

$$\begin{cases} \lambda^2 \partial_t \phi = L_W[\phi] + h[p,\xi,\Psi^*] \quad \text{in } D_{2R} \\ \phi \cdot W = 0 \quad \text{in } D_{2R} \\ \phi(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases}$$

where $h[p,\xi,\Psi^*]$ is given by (5.1). To find a good solution to this problem we would like that $h[p,\xi,\Psi^*]$ satisfies the orthogonality conditions (5.3).

We split the right hand side $h[p,\xi,\Psi^*]$ and the inner solution into components with different roles regarding these orthogonality conditions.

Recall that

$$h[p,\xi,\Psi^*] = \lambda^2 Q_{-\omega} \tilde{L}_U[\Psi^*]\chi_{D_{2R}} + \lambda^2 Q_{-\omega}\mathcal{K}_0[p,\xi] + \lambda^2 Q_{-\omega}\mathcal{K}_1[p,\xi]\chi_{D_{2R}},$$

the decomposition of $\tilde{L}_U$ given in (3.4):

$$\tilde{L}_U[\Psi^*] = \tilde{L}_U[\Psi^*]_0 + \tilde{L}_U[\Psi^*]_1 + \tilde{L}_U[\Psi^*]_2\ ,$$

with $\tilde{L}_U[\Phi]_j$ defined in (3.5). Using the notation (3.3), we then define

$$\begin{aligned} \tilde{L}_U[\Phi]_1^{(0)} = &- 2\lambda^{-1} w_\rho \cos w \left[(\partial_{x_1}\varphi_3(\xi(t),t)) \cos\theta + (\partial_{x_2}\varphi_3(\xi(t),t))) \sin\theta \right] Q_\omega E_1 \\ &- 2\lambda^{-1} w_\rho \cos w \left[(\partial_{x_1}\varphi_3(\xi(t),t))) \sin\theta - (\partial_{x_2}\varphi_3(\xi(t),t))) \cos\theta \right] Q_\omega E_2\ . \end{aligned}$$

We then decompose the function h defined in (5.1)

$$h = h_1 + h_2 + h_3$$

where

$$\begin{aligned} h_1[p,\xi,\Psi^*] &= \lambda^2 Q_{-\omega}(\tilde{L}_U[\Psi^*]_0 + \tilde{L}_U[\Psi^*]_2)\chi_{D_{2R}} + \lambda^2 Q_{-\omega}\mathcal{K}_0[p,\xi], \\ h_2[p,\xi,\Psi^*] &= \lambda^2 Q_{-\omega}\tilde{L}_U[\Psi^*]_1^{(0)}\chi_{D_{2R}} + \lambda^2 Q_{-\omega}\mathcal{K}_1[p,\xi]\chi_{D_{2R}}, \\ h_3[p,\xi,\Psi^*] &= \lambda^2 Q_{-\omega}(\tilde{L}_U[\Psi^*]_1 - \tilde{L}_U[\Psi^*]_1^{(0)})\chi_{D_{2R}}. \end{aligned} \tag{6.2}$$

Next we decompose $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$. The function ϕ_1 will solve the inner problem with right hand side $h_1[p,\xi,\Psi^*]$ projected so that it satisfies essentially (5.3). The advantage of doing this is that h_1 has faster spatial decay, which gives better bounds for the solution. For this we let, for any function $h(y,t)$ defined in $\mathbb{R}^2 \times (0,T)$ with sufficient decay,

$$c_{lj}[h](t) := \frac{1}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{lj}|^2} \int_{\mathbb{R}^2} h(y,t) \cdot Z_{lj}(y)\, dy. \tag{6.3}$$

Note that $h[p,\xi,\Psi^*]$ is defined in $\mathbb{R}^2 \times (0,T)$, and for simplicity we will assume that the right hand sides appearing in the different linear equations are always defined in $\mathbb{R}^2 \times (0,T)$.

We would like that ϕ_1 solves

$$\lambda^2 \partial_t \phi_1 = L_W[\phi_1] + h_1[p, \xi, \Psi^*] - \sum_{l=-1}^{1} \sum_{j=1}^{2} c_{lj}[h_1(p, \xi, \Psi^*)] w_\rho^2 Z_{lj} \quad \text{in } D_{2R},$$

but the estimates for ϕ_1 are better if the projections $c_{0j}[h(p, \xi, \Psi^*)]$ are modified slightly.

Here is the precise result that we will use later. We define the norms

$$\|h\|_{\nu,a} = \sup_{\mathbb{R}^2 \times (0,T)} \frac{|h(y,t)|}{\lambda_*^\nu (1+|y|)^{-a}}, \tag{6.4}$$

and

$$\|\phi\|_{*,\nu,a,\delta} = \sup_{D_{2R}} \frac{|\phi(y,t)| + (1+|y|)|\nabla_y \phi(y,t)|}{\lambda_*^\nu \max(\frac{R^{\delta(5-a)}}{(1+|y|)^3}, \frac{1}{(1+|y|)^{a-2}})}. \tag{6.5}$$

Proposition 6.1. *Let $a \in (2,3)$, $\delta \in (0,1)$, $\nu > 0$. Assume $\|h\|_{\nu,a} < \infty$. Then there is a solution $\phi = \mathcal{T}_{\lambda,1}[h]$, $\tilde{c}_{0j}[h]$ of*

$$\begin{cases} \lambda^2 \partial_t \phi = L_W[\phi] + h - \sum_{j=1,2} \tilde{c}_{0j}[h] Z_{0j} \chi_{B_1} - \sum_{\substack{l=-1,1 \\ j=1,2}} c_{lj}[h] Z_{lj} \chi_{B_1} & \text{in } D_{2R} \\ \phi \cdot W = 0 & \text{in } D_{2R} \\ \phi(\cdot, 0) = 0 & \text{in } B_{2R(0)} \end{cases}$$

where c_{lj} is defined in (6.3)*, which is linear in h, such that*

$$\|\phi\|_{*,\nu,a,\delta} \le C \|h\|_{\nu,a}$$

and such that

$$|c_{0j}[h] - \tilde{c}_{0j}[h]| \le C \lambda_*^\nu R^{-\frac{1}{2}\delta(a-2)} \|h\|_{\nu,a}.$$

The function ϕ_2 solves the equation with right hand side $h_2[p, \xi, \Psi^*]$, which is in *mode 1*, a notion that we define next. Let $h(y,t) \in \mathbb{R}^3$, be defined in $\mathbb{R}^2 \times (0,T)$ or D_{2R} with $h \cdot W = 0$. We say that h is a mode $k \in \mathbb{Z}$ if h has the form

$$h(y,t) = \text{Re}(\tilde{h}_k(|y|,t) e^{ik\theta}) E_1 + \text{Re}(\tilde{h}_k(|y|,t) e^{ik\theta}) E_2,$$

for some complex valued function $\tilde{h}_k(\rho, t)$. Consider

$$\begin{cases} \lambda^2 \partial_t \phi = L_W[\phi] + h - \sum_{j=1,2} c_{1j}[h] w_\rho^2 Z_{1j} & \text{in } D_{2R} \\ \phi \cdot W = 0 & \text{in } D_{2R} \\ \phi(\cdot, 0) = 0 & \text{in } B_{2R(0)} \end{cases} \tag{6.6}$$

Proposition 6.2. *Let $a \in (2,3)$, $\delta \in (0,1)$, $\nu > 0$. Assume that h is in mode 1 and $\|h\|_{\nu,a} < \infty$. Then there is a solution $\phi = \mathcal{T}_{\lambda,2}[h]$ of* (6.6)*, which is linear in h, such that*

$$\|\phi\|_{\nu,a-2} \le C \|h\|_{\nu,a}.$$

In the above statemen the norm $\|\phi\|_{\nu,a-2}$ analogous to the one in (6.4), but the supremum is taken in D_{2R}.

Another piece of the inner solution, ϕ_3, will handle $h_3[p, \xi, \Psi^*]$, which does not satisfy orthogonality conditions in mode 0. We will still project it to satisfy the orthogonality condition in mode 1. Let us consider then (6.6) without any orthogonality conditions on h in mode 0. We define

$$\|\phi\|_{**,\nu} = \sup_{D_{2R}} \frac{|\phi(y,t)| + (1+|y|)\,|\nabla_y \phi(y,t)|}{\lambda_*(t)^\nu R(t)^2 (1+|y|)^{-1}}. \tag{6.7}$$

Proposition 6.3. *Let $1 < a < 3$ and $\nu > 0$. There exists a $C > 0$ such that if $\|h\|_{a,\nu} < +\infty$ there is a solution $\phi = \mathcal{T}_{\lambda,3}[h]$ of* (6.6)*, which is linear in h and satisfies the estimate*

$$\|\phi\|_{**,\nu} \ \leq \ C\|h\|_{a,\nu}.$$

Note that we allow a to be less than 2 in the previous proposition.

Next we have a variant of Proposition 6.3 when h is in mode -1.

Proposition 6.4. *Let $2 < a < 3$ and $\nu > 0$. There exists a $C > 0$ such that for any h in mode -1 with $\|h\|_{a,\nu} < +\infty$, there is a solution $\phi = \mathcal{T}_{\lambda,4}[h]$ of problem* (6.6)*, which is linear in h and satisfies the estimate*

$$\|\phi\|_{***,\nu} \leq C\|h\|_{a,\nu},$$

where

$$\|\phi\|_{***,\nu} = \sup_{D_{2R}} \frac{|\phi(y,t)| + (1+|y|)\,|\nabla_y \phi(y,t)|}{\lambda_*(t)^\nu \log(R(t))}.$$

Propositions 6.1–6.4 are proved in [7], section 6.

6.3. **The equations for** $p = \lambda e^{i\omega}$**.** We need to choose the free parameters p, ξ so that $c_{lj}[h(p,\xi,\Psi^*)] = 0$ for $l = -1, 0, 1$, $j = 1, 2$. This will be easy to do for $l = 1$ (mode 1), but mode $l = 0$ is more complicated.

To handle c_{0j} we note that by definitions (5.1), (5.4), (5.6)

$$c_{0,j}[h(p,\xi,\Psi^*)] = \frac{2\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{0j}|^2} \left(\mathcal{B}_{0j}[p] - a_{0j}[p,\xi,\Psi^*]\right)$$

where B_0, a_0 are defined in (5.5), (5.6) and we recall that $p = \lambda e^{i\omega}$.

So to achieve $c_{0j}[h(p,\xi,\Psi^*)] = 0$ we should solve

$$\mathcal{B}_0[p](t) = a_0[p,\xi,\Psi^*](t), \quad t \in [0,T], \tag{6.8}$$

adjusting the parameters $\lambda(t)$ and $\omega(t)$. We define the following norms. Let I denote either the interval $[0,T]$ or $[-T,T]$. For $\Theta \in (0,1)$, $l \in \mathbb{R}$ and a continuous function $g : I \to \mathbb{C}$ we let

$$\|g\|_{\Theta,l} = \sup_{t\in I} (T-t)^{-\Theta} |\log(T-t)|^l |g(t)|,$$

and for $\gamma \in (0,1)$, $m \in (0,\infty)$, and $l \in \mathbb{R}$ we let

$$[g]_{\gamma,m,l} = \sup\, (T-t)^{-m} |\log(T-t)|^l \frac{|g(t)-g(s)|}{(t-s)^\gamma},$$

where the supremum is taken over $s \leq t$ in I such that $t - s \leq \frac{1}{10}(T-t)$.

We have then the following result, whose proof is in [7], section 13.

Proposition 6.5. *Let $\alpha, \gamma \in (0, \frac{1}{2})$, $l \in \mathbb{R}$, $C_1 > 1$. There is $\alpha_0 > 0$ such that if $\Theta \in (0, \alpha_0)$ and $m \leq \Theta - \gamma$, then for $a : [0,T] \to \mathbb{C}$ is such that*

$$\begin{cases} \dfrac{1}{C_1} \leq |a(T)| \leq C_1, \\ T^\Theta |\log T|^{1+\sigma-l} \|a(\cdot) - a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1} \leq C_1, \end{cases} \tag{6.9}$$

for some $\sigma > 0$, then, for $T > 0$ small enough there are two operators $\mathcal{P}$ and $\mathcal{R}_0$ so that $p = \mathcal{P}[a] : [-T,T] \to \mathbb{C}$ satisfies

$$\mathcal{B}_0[p](t) = a(t) + \mathcal{R}_0[a](t), \quad t \in [0,T], \tag{6.10}$$

with

$$\begin{aligned} &|\mathcal{R}_0[a](t)| \\ &\leq C\Big(T^\sigma + T^\Theta \frac{\log|\log T|}{|\log T|} \|a(\cdot) - a(T)\|_{\Theta,l-1} + [a]_{\gamma,m,l-1}\Big) \frac{(T-t)^{m+(1+\alpha)\gamma}}{|\log(T-t)|^l}, \end{aligned}$$

for some $\sigma > 0$.

The idea of the proof og Proposition 6.5 is to notice that

$$\mathcal{B}_0[p] \approx \int_{-T}^{t-\lambda_*(t)^2} \frac{\dot{p}(s)}{t-s} ds.$$

and decompose

$$\begin{aligned} S_\alpha[g] &:= g(t)[-2\log\lambda_*(t) + (1+\alpha)\log(T-t)] + \int_{-T}^{t-(T-t)^{1+\alpha}} \frac{g(s)}{t-s} ds, \\ R_\alpha[g] &:= -\int_{t-(T-t)^{1+\alpha}}^{t-\lambda_*^2} \frac{g(t)-g(s)}{t-s} ds. \end{aligned} \tag{6.11}$$

where $\alpha > 0$ is fixed. We solve a modified equation where in (6.10) we drop $R_\alpha[\dot{p}]$, and so the remainder $\mathcal{R}_0$ is essentially $R_\alpha[\dot{p}]$.

Another modification to equations (6.8) that we introduce is to replace $a_0[p,\xi,\Psi^*]$ by its main term. To do this we write

$$a_0[p,\xi,\Psi] = a_0^{(0)}[p,\xi,\Psi] + a_0^{(1)}[p,\xi,\Psi] + a_0^{(2)}[p,\xi,\Psi]$$

where

$$a_0^{(l)}[p,\xi,\Psi] = -\frac{\lambda}{4\pi} e^{i\omega} \int_{B_{2R}} \Big(Q_{-\omega}\tilde{L}_U[\Psi]_l \cdot Z_{01} + iQ_{-\omega}\tilde{L}_U[\Psi]_l \cdot Z_{02} \Big)\, dy$$

for $l = 0,1,2$.

We define

$$\begin{aligned} c_0^*[p,\xi,\Psi^*](t) := \frac{4\pi\lambda}{\int_{\mathbb{R}^2} w_\rho^2 |Z_{01}|^2} e^{-i\omega} \Big(&\mathcal{R}_0 \left[a_0^{(0)}[p,\xi,\Psi^*] \right](t) + a_0^{(1)}[p,\xi,\Psi^*](t) \\ &+ a_0^{(2)}[p,\xi,\Psi^*](t) \Big) - (c_0[h[p,\xi,\Psi^*]] - \tilde{c}_0[h_1[p,\xi,\Psi^*]]) - \tilde{\mathcal{B}}_0[p,\xi], \end{aligned}$$

and

$$c_{01}^* := \mathrm{Re}(c_0^*), \quad c_{02}^* := \mathrm{Im}(c_0^*),$$

where $\mathcal{R}_0$ is the operator given Proposition 6.5 and $\tilde{c}_0 = \tilde{c}_{01} + i\tilde{c}_{02}$ are the operators defined in Proposition 6.1.

6.4. **The system of equations.** We transform the system (4.7)-(4.8) in the problem of finding functions $\psi(r,z,t)$, $\phi_1(y,t),\dots,\phi_4(y,t)$, parameters $p(t) = \lambda(t)e^{i\omega(t)}$, $\xi(t)$ and constants c_1, c_2, c_3 such that the following system is satisfied:

$$\begin{cases} \psi_t = (\partial_r^2 + \partial_z^2)\psi + g(p,\xi,Z^*+\psi,\phi_1+\phi_2+\phi_3+\phi_4) & \text{in } \mathcal{D}\times(0,T) \\ \psi = (\mathbf{e}_3 - U) - \Phi^0 & \text{on } (\partial\mathcal{D}\setminus\{r=0\})\times(0,T) \\ \partial_r\psi = 0 & \text{on } (\{r=0\}\cap\mathcal{D})\times(0,T) \\ \psi(\cdot,0) = (c_1\,\mathbf{e_1} + c_2\,\mathbf{e_2} + c_3\,\mathbf{e_3})\chi + (1-\chi)(\mathbf{e}_3 - U - \Phi^0) & \text{in } \mathcal{D} \\ \psi(r_0,z_0,T) = -Z^*(r_0,z_0,T) \end{cases} \tag{6.12}$$

$$\begin{cases} \lambda^2\partial_t\phi_1 = L_W[\phi_1] + h_1[p,\xi,\Psi^*] - \sum_{j=1,2} \tilde{c}_{0j}[h_1[p,\xi,\Psi^*]] w_\rho^2 Z_{0j} \\ \qquad - \sum_{\substack{l=-1,1 \\ j=1,2}} c_{lj}[h_1[p,\xi,\Psi^*]] w_\rho^2 Z_{lj} \quad \text{in } D_{2R} \\ \phi_1 \cdot W = 0 \quad \text{in } D_{2R} \\ \phi_1(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{6.13}$$

$$\begin{cases} \lambda^2\partial_t\phi_2 = L_W[\phi_2] + h_2[p,\xi,\Psi^*] - \sum_{j=1,2} c_{1j}[h_2[p,\xi,\Psi^*]]w_\rho^2 Z_{1j} & \text{in } D_{2R} \\ \phi_2\cdot W = 0 \quad \text{in } D_{2R} \\ \phi_2(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{6.14}$$

$$\begin{cases} \lambda^2\partial_t\phi_3 = L_W[\phi_3] + h_3 - \sum_{j=1,2} c_{1j}[h_3[p,\xi,\Psi^*]]w_\rho^2 Z_{1j} \\ \qquad\qquad + \sum_{j=1,2} c_{0j}^*[p,\xi,\Psi^*]w_\rho^2 Z_{0j} \quad \text{in } D_{2R} \\ \phi_3\cdot W = 0 \quad \text{in } D_{2R} \\ \phi_3(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{6.15}$$

$$\begin{cases} \lambda^2\partial_t\phi_4 = L_W[\phi_4] + \sum_{j=1,2} c_{-1,j}[h_1[p,\xi,\Psi^*]]w_\rho^2 Z_{-1j} \\ \phi_4\cdot W = 0 \quad \text{in } D_{2R} \\ \phi_4(\cdot,t) = 0 \quad \text{on } \partial B_{2R(t)} \\ \phi_4(\cdot,0) = 0 \quad \text{in } B_{2R(0)} \end{cases} \tag{6.16}$$

$$c_{0j}[h(p,\xi,\Psi^*)](t) - \tilde c_{0j}[p,\xi,\Psi^*](t) = 0 \quad \text{for all} \quad t\in(0,T), \quad j=1,2, \tag{6.17}$$

$$c_{1j}[h(p,\xi,\Psi^*)](t) = 0 \quad \text{for all} \quad t\in(0,T), \quad j=1,2. \tag{6.18}$$

In (6.12) χ is a smooth cut-off function with compact support in $\mathcal{D}$ which is identically 1 on a fixed neighborhood of (r_0,z_0) independent of T and the function $g(p,\xi,\Psi^*,\phi)$ is given by (4.9).

We see that if $(\phi_1,\phi_2,\phi_3,\phi_4,\psi,p,\xi)$ satisfies system (6.12)–(6.18) then the functions

$$\phi = \phi_1+\phi_2+\phi_3+\phi_4, \quad \Psi^* = \tilde Z^* + \psi$$

solve the outer-inner gluing system (4.7)–(4.8).

6.5. **The fixed point formulation.** We consider the inner-outer system including the equations for the parameters p and ξ (6.12)–(6.18) as a fixed point problem for certain operators that we describe below. First we define the functional spaces we will use for the functions $\psi,\phi_1,\dots,\phi_4,p,\xi$.

For the outer problem (6.12) we define, given $\Theta>0$, $\gamma\in(0,\frac12)$ the norm

$$\begin{aligned} \|\psi\|_{\sharp,\Theta,\gamma} &:= \lambda_*(0)^{-\Theta}\frac{1}{|\log T|\lambda_*(0)R(0)}\|\psi\|_{L^\infty(\Omega\times(0,T))} + \lambda_*(0)^{-\Theta}\|\nabla\psi\|_{L^\infty(\Omega\times(0,T))} \\ &\quad + \sup_{\Omega\times(0,T)} \lambda_*(t)^{-\Theta-1}R(t)^{-1}\frac{1}{|\log(T-t)|}|\psi(x,t)-\psi(x,T)| \\ &\quad + \sup_{\Omega\times(0,T)} \lambda_*(t)^{-\Theta}|\nabla\psi(x,t)-\nabla\psi(x,T)| \\ &\quad + \sup \lambda_*(t)^{-\Theta}(\lambda_*(t)R(t))^{2\gamma}\frac{|\nabla\psi(x,t)-\nabla\psi(x',t')|}{(|x-x'|^2+|t-t'|)^\gamma}, \end{aligned} \tag{6.19}$$

where the last supremum is taken in the region

$$x,x'\in\Omega, \quad t,t'\in(0,T), \quad |x-x'|\le 2\lambda_*R(t), \quad |t-t'|<\frac14(T-t).$$

Then we define

$$\begin{aligned} F = \{\psi\in L^\infty(\mathcal{D}\times(0,T)) : \psi &\text{ is Lipschitz continuous with respecto to } (r,z) \text{ in } \mathcal{D}\times(0,T) \\ &\text{and } \|\psi\|_{\sharp,\Theta,\gamma}<\infty\} \end{aligned} \tag{6.20}$$

with the norm $\| \ \|_{\sharp,\Theta,\gamma}$.

For the functions ϕ_i in the inner equations (6.13)–(6.16) we consider the spaces

$$
\begin{aligned}
E_1 &= \{\phi_1 \in L^\infty(D_{2R}) : \nabla_y \phi_1 \in L^\infty(D_{2R}),\ \|\phi_1\|_{*,\nu_1,a_1,\delta} < \infty\}\\
E_2 &= \{\phi_2 \in L^\infty(D_{2R}) : \nabla_y \phi_2 \in L^\infty(D_{2R}),\ \|\phi_2\|_{\nu_2,a_2} < \infty\}\\
E_3 &= \{\phi_3 \in L^\infty(D_{2R}) : \nabla_y \phi_3 \in L^\infty(D_{2R}),\ \|\phi_3\|_{**,\nu_3} < \infty\}\\
E_4 &= \{\phi_4 \in L^\infty(D_{2R}) : \nabla_y \phi_4 \in L^\infty(D_{2R}),\ \|\phi_4\|_{***,\nu_4} < \infty\}
\end{aligned}
$$

and use the notation

$$
\begin{aligned}
E &= E_1 \times E_2 \times E_3 \times E_4,\\
\Phi &= (\phi_1, \phi_2, \phi_3, \phi_4) \in E\\
\|\Phi\|_E &= \|\phi_1\|_{*,\nu_1,a_1,\delta} + \|\phi_2\|_{\nu_2,a_2-2} + \|\phi_3\|_{**,\nu_3} + \|\phi_4\|_{***,\nu_4}.
\end{aligned}
$$

To introduce the space for the parameter p, we recall the integral operator $\mathcal{B}_0$ defined in (5.5), which has the approximate form

$$
\mathcal{B}_0[p] = \int_{-T}^{t-\lambda^2} \frac{\dot p(s)}{t-s} ds \ + O\big(\|\dot p\|_\infty\big).
$$

Proposition 6.5 gives an approximate inverse $\mathcal{P}$ of the operator $\mathcal{B}_0$, so that given a satisfying (6.9), $p := \mathcal{P}\,[a]$, satisfies the equation

$$
\mathcal{B}_0[p] = a + \mathcal{R}_0[a], \quad \text{in } [0,T],
$$

for a small remainder $\mathcal{R}_0[a]$. The proof of that proposition in [7] gives a decomposition

$$
\mathcal{P}[a] = p_{0,\kappa} + \mathcal{P}_1[a], \tag{6.21}
$$

where $p_{0,\kappa}$ is defined by

$$
p_{0,\kappa}(t) = \kappa |\log T| \int_t^T \frac{1}{|\log(T-s)|^2}\, ds, \quad t \le T,
$$

$\kappa = \kappa[a] \in \mathbb{C}$, and the function $p_1 = \mathcal{P}_1[a]$ has the estimate

$$
\|p_1\|_{*,3-\sigma} \le C |\log T|^{1-\sigma} \log^2(|\log T|),
$$

where $\| \ \|_{*,3-\sigma}$ is defined by

$$
\|g\|_{*,k} = \sup_{t\in[-T,T]} |\log(T-t)|^k |\dot g(t)|,
$$

and $\sigma \in (0,1)$. This leads us to define the space

$$
X_1 := \{p_1 \in C([-T,T;\mathbb{C}]) \cap C^1([-T,T;\mathbb{C}]) \mid p_1(T) = 0,\ \|p_1\|_{*,3-\sigma} < \infty\},
$$

with the norm $\|p_1\|_{*,3-\sigma}$ and represent p by the pair (κ, p_1) in the form $p = p_{0,\kappa} + p_1$.

Finally, for the parameter ξ we denote by ξ^0 the explicit function

$$
\xi^0(t) = (\sqrt{r_0^2 + 2(T-t)}, z_0), \quad t \in [0,T],
$$

and represent $\xi = \xi^0 + \xi^1$ with ξ^1 in the space

$$
X_2 = \{\xi \in C^1([0,T];\mathbb{R}^2) \ : \ \dot\xi(T) = 0\}
$$

with the norm

$$
\|\xi\|_{X_2} = \|\xi\|_{L^\infty(0,T)} + \sup_{t\in(0,T)} \lambda_*(t)^{-\sigma} |\dot\xi(t)|
$$

where $\sigma \in (0,1)$ is fixed.

Let $\mathcal{B}$ denote the closed subset of $F \times E \times \mathbb{C} \times X_1 \times X_2$ defined by $(\psi, \Phi, \kappa, p_1, \xi^1) \in \mathcal{B}$ if:

$$\begin{cases} \|\psi\|_F + \|\Phi\|_E \leq 1 \\ \quad |\kappa - \kappa_0| \leq \dfrac{1}{|\log T|^{1/2}} \\ \|p_1\|_{*,3-\sigma} \leq C_0 |\log T|^{1-\sigma} \log^2(|\log T|) \\ \quad \|\xi^1\|_{X_2} \leq 1, \end{cases} \tag{6.22}$$

where $\kappa_0 = \operatorname{div} \tilde{z}_0^*(r_0, z_0) + i \operatorname{curl} \tilde{z}_0^*(r_0, z_0)$ and C_0 is a large fixed constant.

Next we define an operator $\mathcal{A} : \mathcal{B} \to F \times E \times \mathbb{C} \times X_1 \times X_2$ so that a fixed point of it will give a solution to the full system (6.12)–(6.18). This operator is defined by

$$\mathcal{A} = (\mathcal{A}_0, \mathcal{F}, \mathcal{K}, \tilde{\mathcal{P}}_1, \mathcal{X}_1),$$

where

$$\begin{aligned} &\mathcal{A}_0 : \mathcal{B} \to F, \quad \mathcal{F} : \mathcal{B} \to E, \\ &\mathcal{K} : \mathcal{B} \to \mathbb{C} \quad \tilde{\mathcal{P}}_1 : \mathcal{B} \to X_1, \quad \mathcal{X}_1 : \mathcal{B} \to X_2, \end{aligned}$$

and where $\mathcal{A}_0$ will hande (6.12), $\mathcal{F}$ is related to (6.13)–(6.16) and $\mathcal{K}$, $\tilde{\mathcal{P}}_1$, $\mathcal{X}_1$ deal with the equations for p and ξ, (6.17), (6.18).

To define $\mathcal{A}_0$, we need first a linear result about the exterior problem (6.12). Thus we consider the inhomogeneous linear heat equation

$$\begin{cases} \quad \psi_t = (\partial_r^2 + \partial_z^2)\psi + \dfrac{1}{r}\partial_r \psi + f(r,z,t) \quad \text{in } \mathcal{D} \times (0,T) \\ \quad \psi = 0 \quad \text{on } (\partial\mathcal{D} \setminus \{r=0\}) \times (0,T) \\ \quad \partial_r \psi = 0 \quad \text{on } (\mathcal{D} \cap \{r=0\}) \times (0,T) \\ \psi(r_0, z_0, T) = 0 \\ \quad \psi(r,z,0) = (c_1 \mathbf{e_1} + c_2 \mathbf{e_2} + c_3 \mathbf{e_3})\eta_1 \quad \text{in } \mathcal{D}, \end{cases} \tag{6.23}$$

for suitable constants c_1, c_2, c_3, where $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ are defined in (1.4), $(r_0, z_0) \in \mathcal{D}$, $r_0 > 0$, and $T > 0$ is sufficiently small. The fixed smooth cut-off η_1 has compact support in $\mathcal{D}$ and is such that $\eta_1 \equiv 1$ in a small neighborhood of (r_0, z_0). The right hand side is assumed to satisfy $\|f\|_{**} < \infty$ where

$$\|f\|_{**} := \sup_{\mathcal{D} \times (0,T)} \Big(1 + \sum_{i=1}^{3} \varrho_i(r,z,t)\Big)^{-1} |f(r,z,t)|,$$

and the weights are defined by

$$\varrho_1 := \lambda_*^{\Theta} (\lambda_* R)^{-1} \chi_{\{s \leq 3R\lambda_*\}}, \quad \varrho_2 := T^{-\sigma_0} \frac{\lambda_*^{1-\sigma_0}}{s^2} \chi_{\{s \geq R\lambda_*\}}, \quad \varrho_3 := T^{-\sigma_0},$$

where $s = |(r,z) - (r_0, z_0)|$, $\Theta > 0$ and $\sigma_0 > 0$ is small. (The factor T^{σ_0} in front of ϱ_2 and ϱ_3 is a simple way to have parts of the error small in the outer problem.) These weights naturally adapt to the form of the outer error g in (4.9). The next lemma gives a solution to (6.23) as a linear operator of f.

Lemma 6.1. *Assume $\beta \in (0, \frac{1}{2})$, $\Theta \in (0, \beta)$. For $T > 0$ small there is a linear operator that maps a function $f : \mathcal{D} \times (0,T) \to \mathbb{R}^3$ with $\|f\|_{**} < \infty$ into ψ, c_1, c_2, c_3 so that (6.23) is satisfied. Moreover the following estimate holds*

$$\|\psi\|_{\sharp,\Theta,\gamma} + \frac{\lambda_*(0)^{-\Theta} (\lambda_*(0) R(0))^{-1}}{|\log T|} (|c_1| + |c_2| + |c_3|) \leq C \|f\|_{**}, \tag{6.24}$$

where $\gamma \in (0, \frac{1}{2})$.

Proof. We use Lemmas 8.1–8.3 in [7], which put together, can be summarized as follows.

Assume $\beta \in (0, \frac{1}{2})$, $\Theta \in (0, \beta)$. *If* $f : \mathbb{R}^2 \times (0,T) \to \mathbb{R}^3$ *satisfies* $\|f\|_{**} < \infty$, *the solution* ψ *to*

$$\partial_t \psi = (\partial_r^2 + \partial_z^2)\psi + f(r,z) \quad \text{in } \mathbb{R}^2 \tag{6.25}$$

given by Duhamel's formula satisfies

$$\|\psi\|_{\sharp,\Theta,\gamma} \le C\|f\|_{**},$$

where $\gamma \in (0, \frac{1}{2})$.

To obtain Lemma 6.1 from the above statement we first show that the solution to

$$\begin{cases} \psi_t = (\partial_r^2 + \partial_z^2)\psi + \dfrac{1}{r}\partial_r \psi + f(r,z,t) \quad \text{in } \mathcal{D} \times (0,T) \\ \psi = 0 \quad \text{on } (\partial\mathcal{D} \setminus \{r=0\}) \times (0,T) \\ \partial_r \psi = 0 \quad \text{on } (\mathcal{D} \cap \{r=0\}) \times (0,T) \\ \psi(r,z,0) = 0 \quad \text{in } \mathcal{D}, \end{cases} \tag{6.26}$$

satisfies

$$\|\psi\|_{\sharp,\Theta,\gamma} \le C\|f\|_{**}. \tag{6.27}$$

Indeed, let $\psi[f]$ be the solution of (6.25) given by Duhamel's formula. We then rewrite the solution ψ of (6.26) as $\psi = \eta\psi[f] + \tilde{\psi}_1$ where η_1 is as before. Then $\tilde{\psi}_1$ satisfies

$$\begin{cases} \partial_t \tilde{\psi}_1 = (\partial_r^2 + \partial_z^2)\tilde{\psi}_1 + \dfrac{1}{r}\partial_r \tilde{\psi}_1 + \dfrac{1}{r}\partial_r(\eta_1\psi[f]) + (\partial_r^2 + \partial_z^2)\eta_1\psi[f] + 2\nabla\eta_1\nabla\psi[f] \quad \text{in } \mathcal{D} \times (0,T) \\ \tilde{\psi}_1 = -\psi[f] \quad \text{on } (\partial\mathcal{D} \setminus \{r=0\}) \times (0,T) \\ \partial_r \tilde{\psi}_1 = 0 \quad \text{on } (\mathcal{D} \cap \{r=0\}) \times (0,T) \\ \tilde{\psi}_1(r,z,0) = 0 \quad \text{in } \mathcal{D}, \end{cases} \tag{6.28}$$

The function $\tilde{\psi}_1$ can be regarded as $\tilde{\psi}_1(r,z,t) = \psi_1(x,t)$ where ψ_1 solves a non-homogeneous problem in the three dimensional axially symmetric domain Ω. The estimate $\|\psi[f]\|_{\sharp,\Theta,\gamma} \le \|f\|_{**}$ gives sufficient control of the terms involving $\psi[f]$ in (6.28) so that for $\tilde{\psi}_1$ we also obtain $\|\tilde{\psi}_1\|_{\sharp,\Theta,\gamma} \le \|f\|_{**}$. This proves (6.27). Finally, using (6.27) one can show that for the problem (6.23) there are choices of c_i so that $\psi(r_0, z_0, T) = 0$, and these constants satisfy (6.24). □

Let $\psi = \mathcal{U}(f)$ be the operator constructed in Lemma 6.1 and set

$$\tilde{g}[p,\xi,\Psi^*,\phi] := g[p,\xi,\Psi^*,\phi] - \frac{1}{r}\partial_r\psi$$

with g defined in (4.9). We then define

$$\mathcal{A}_0(\psi,\Phi,\kappa,p_1,\xi^1) = \mathcal{U}(\tilde{g}[p_{0,\kappa} + p_1, \xi^0 + \xi^1, Z^* + \psi, \phi]) \tag{6.29}$$

where $\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$ and $\Phi = (\phi_1, \ldots, \phi_4)$.

Next we define

$$\mathcal{F}(\psi,\Phi,\kappa,p_1,\xi^1) = (\mathcal{F}_1(\psi,\Phi,\kappa,p_1,\xi^1), \mathcal{F}_2(\psi,\Phi,\kappa,p_1,\xi^1), \mathcal{F}_3(\psi,\Phi,\kappa,p_1,\xi^1), \mathcal{F}_4(\psi,\Phi,\kappa,p_1,\xi^1)) \tag{6.30}$$

where

$$\mathcal{F}_1(\psi,\Phi,\kappa,p_1,\xi^1)=\mathcal{T}_{\lambda,1}(h_1[p,\xi,\Psi^*])$$
$$\mathcal{F}_2(\psi,\Phi,\kappa,p_1,\xi^1)=\mathcal{T}_{\lambda,2}(h_2[p,\xi,\Psi^*])$$
$$\mathcal{F}_3(\psi,\Phi,\kappa,p_1,\xi^1)=\mathcal{T}_{\lambda,3}\Big(h_3[p,\xi,\Psi^*]+\sum_{j=1}^2 c_{0j}^*[p,\xi,\Psi^*]w_\rho^2 Z_{0j}\Big)$$
$$\mathcal{F}_4(\psi,\Phi,\kappa,p_1,\xi^1)=\mathcal{T}_{\lambda,4}\Big(\sum_{j=1}^2 c_{-1,j}[h_1[p,\xi,\Psi^*]]w_\rho^2 Z_{-1,j}\Big),$$

where $p=p_{0,\kappa}+p_1$, $\xi=\xi^0+\xi^1$, $\Psi^*=Z^*+\psi$.

To define the operators $\mathcal{K}$ and $\tilde{\mathcal{P}}_1$, we recall that Proposition 6.5 gives the decomposition (6.21) where $\kappa=\kappa[a]$ and $p_1=\mathcal{P}_1[a]$. We define

$$\mathcal{K}(\psi,\Phi,\kappa,p_1,\xi^1)=\kappa\left[a_0^{(0)}[p,\xi,\Psi^*]\right] \tag{6.31}$$
$$\tilde{\mathcal{P}}_1(\psi,\Phi,\kappa,p_1,\xi^1)=\mathcal{P}_1\left[a_0^{(0)}[p,\xi,\Psi^*]\right], \tag{6.32}$$

where, again, $p=p_{0,\kappa}+p_1$, $\xi=\xi^0+\xi^1$, $\Psi^*=Z^*+\psi$.

Finally, we introduce the operator $\mathcal{X}_1$. By (6.3), (6.18) is equivalent to

$$\int_{\mathbb{R}^2} h[p,\xi,\Psi^*]\cdot Z_{1j}(y)\,dy=0,\quad t\in(0,T),\ j=1,2,$$

and recalling (5.1), this is equivalent to

$$\dot{\xi}_j=-\frac{1}{4\pi}(1+(2R)^{-2})\int_{B_{2R}} Q_{-\omega}\Big(\tilde{L}_U[\Psi^*]+\frac{1}{r}\partial_r U\Big)\cdot Z_{1j},\quad j=1,2.$$

Then we define

$$\mathcal{X}_1(\psi,\Phi,\kappa,p_1,\xi^1)=(r_0,z_0)+\int_t^T b(\psi,\Phi,\kappa,p_1,\xi^1)(s)\,ds \tag{6.33}$$

with

$$b_{11}(\psi,\Phi,\kappa,p_1,\xi^1)(t)=\frac{1}{4\pi}(1+(2R)^{-2})\int_{B_{2R}} Q_{-\omega}\Big(\tilde{L}_U[\Psi^*]+\frac{1}{r}\partial_r U\Big)\cdot Z_{1j}-\frac{1}{\xi_1^0(t)}$$
$$b_{12}(\psi,\Phi,\kappa,p_1,\xi^1)(t)=\frac{1}{4\pi}(1+(2R)^{-2})\int_{B_{2R}} Q_{-\omega}\Big(\tilde{L}_U[\Psi^*]+\frac{1}{r}\partial_r U\Big)\cdot Z_{1j}.$$

6.6. **Choice of constants.** We state here the constraints we impose in the parameters involved in the different norms. The values assumed will be sufficient for the inner-outer gluing scheme to work.

- $\beta\in(0,\frac{1}{2})$ is so that $R(t)=\lambda_*(t)^{-\beta}$.
- $\alpha\in(0,\frac{1}{2})$ appears in Proposition 6.5. It is the parameter used to define the remainder $\mathcal{R}_\alpha$ in (6.11).
- We use the norm $\|\ \|_{*,\nu_1,a_1,\delta}$ (6.5) to measure the solution ϕ_1 in (6.13). Here we will ask that $\nu_1\in(0,1)$, $a_1\in(2,3)$, and $\delta>0$ small and fixed.
- We use the norm $\|\ \|_{\nu_2,a_2-2}$ (6.4) to measure the solution ϕ_2 in (6.14), with $\nu_2\in(0,1)$, $a_2\in(2,3)$.
- We use the norm $\|\ \|_{**,\nu_3}$ (6.7) for the solution ϕ_3 of (6.15), with $\nu_3>0$.
- We use the norm $\|\ \|_{***,\nu_4}$ for the solution ϕ_4 of (6.16), with $\nu_4>0$.
- We are going to use the norm $\|\ \|_{\sharp,\Theta,\gamma}$ with a parameters Θ, γ satisfying some restrictions given below.
- We have parameters m, l in Proposition 6.5. We work with m given by

  $$m=\Theta-2\gamma(1-\beta).$$

  and l satisfying $l<1+2m$.

We will assume that

$$\alpha - 1 + 2\beta > 0$$

which ensures that $m + (1+\alpha)\gamma > \Theta$.

To get the estimates for the outer problem (6.12), we need $\beta \in (0, \frac{1}{2})$, $\Theta \in (0, \beta)$ and

$$\Theta < \min\Big(\beta, \frac{1}{2} - \beta, \nu_1 - 1 + \beta(a_1 - 1), \nu_2 - 1 + \beta(a_2 - 1), \nu_3 - 1, \nu_4 - 1 + \beta\Big)$$

$$\Theta < \min\Big(\nu_1 - \delta\beta(5 - a_1) - \beta, \nu_2 - \beta, \nu_3 - 3\beta, \nu_4 - \beta\Big).$$

Also to control the nonlinear terms in (6.12) we need $\delta > 0$ in $\| \ \|_{*,\nu_1,a_1,\delta}$ to be small. To find Θ in the range above we need

$$\nu_1 > \max\big(1 - \beta(a_1 - 1), \delta\beta(5 - a_1) - \beta\big), \qquad \nu_2 > \max\big(1 - \beta(a_2 - 1), \beta\big),$$
$$\nu_3 > \max(1, 3\beta), \qquad \nu_4 > \max(1 - \beta, \beta).$$

To solve the inner system given by equations (6.13), (6.14), (6.15), and (6.16) we will need

$$\nu_1 < 1, \qquad \nu_2 < 1 - \beta(a_2 - 2),$$
$$\nu_3 < \min\big(1 + \Theta + \sigma_1, 1 + \Theta + 2\gamma\beta, \nu_1 + \frac{1}{2}\delta\beta(a_1 - 2)\big), \qquad \nu_4 < 1,$$

where $\sigma_1 \in (0, \gamma(\alpha - 1 + 2\beta))$.

6.7. **The proof of Theorem 1.** Let us consider the operator

$$\mathcal{A} = (\mathcal{A}_0, \mathcal{F}, \mathcal{K}, \tilde{\mathcal{P}}_1, \mathcal{X}_1) \tag{6.34}$$

where $\mathcal{A}_0$, $\mathcal{F}$, $\mathcal{K}$, $\tilde{\mathcal{P}}_1$, $\mathcal{X}_1$ are given in (6.29), (6.30), (6.31), (6.32), (6.33).

The proof of Theorem 1 consists in showing that $\mathcal{A} : \mathcal{B} \subset F \times E \times \mathbb{C} \times X_1 \times X_2 \to F \times E \times \mathbb{C} \times X_1 \times X_2$ has a fixed point, where $\mathcal{B}$ is defined by (6.22). We do this using the Schauder fixed point theorem. The estimates needed to show that $\mathcal{A}$ maps $\mathcal{B}$ into itself and the compactness are obtained in a similar way. They are based on the following estimates for the operators e $\mathcal{A}_0$, $\mathcal{F}$, $\mathcal{K}$, $\tilde{\mathcal{P}}_1$, $\mathcal{X}_1$. We claim that if $(\psi, \Phi, \kappa, p_1, \xi^1) \in \mathcal{B}$ then

$$\begin{cases} \|\mathcal{A}_0(\psi, \Phi, \kappa, p_1, \xi^1)\|_{\sharp,\Theta,\gamma} \le CT^\sigma \\ \|\mathcal{F}(\psi, \Phi, \kappa, p_1, \xi^1)\|_E \le CT^\sigma \\ |\mathcal{K}(\psi, \Phi, \kappa, p_1, \xi^1) - \kappa_0| \le \dfrac{C}{|\log T|} \\ \|\tilde{\mathcal{P}}_1(\psi, \Phi, \kappa, p_1, \xi^1)\|_{*,3-\sigma} \le C|\log T|^{1-\sigma} \log^2(|\log T|) \\ \|\mathcal{X}_1(\psi, \Phi, \kappa, p_1, \xi^1)\|_{X_2} \le CT^\sigma. \end{cases} \tag{6.35}$$

We give below the proof of some of the estimates stated above. We first show that for $(\psi, \Phi, \kappa, p_1, \xi^1) \in \mathcal{B}$,

$$\|\mathcal{A}_0(\psi, \Phi, \kappa, p_1, \xi^1)\|_{\sharp,\Theta,\gamma} \le CT^\sigma.$$

For the proof let us write $\tilde{g} = g_1 + g_2 + g_3 + g_4 + g_5$ where

$$\begin{aligned} g_1 &= Q_\omega\big(((\partial_r^2 + \partial_z^2)\eta)\phi + 2\nabla\eta\nabla\phi - \eta_t\phi\big) \\ &\quad + \eta Q_\omega\big(-\dot{\omega}J\phi + \lambda^{-1}\dot{\lambda}y \cdot \nabla_y\phi + \lambda^{-1}\dot{\xi} \cdot \nabla_y\phi\big) \\ g_2 &= (1 - \eta)\tilde{L}_U[\Psi^*] + (\Psi^* \cdot U)U_t \\ g_3 &= (1 - \eta)[\mathcal{K}_0[p, \xi] + \mathcal{K}_1[p, \xi]] + \Pi_{U^\perp}[\tilde{\mathcal{R}}_1] + (\Phi^0 \cdot U)U_t, \\ g_4 &= N_U(\eta Q_\omega\phi + \Pi_{U^\perp}(\Phi^0 + \Psi)^*) \\ g_5 &= \frac{1}{r}\partial_r\left(\Pi_{U^\perp}\big(\eta^\delta\, \Phi^0[\omega, \lambda, \xi] + Z^*\big) + \eta_R Q_\omega\phi\right) + (1 - \eta)\frac{1}{r}\partial_r U + \eta^\delta \mathcal{E}^{out,1} + \mathcal{E}^{out,0}. \end{aligned}$$

We claim that

$$\|g_1\|_{**} \le CT^\sigma \|\Phi\|_E,$$

for some $\sigma > 0$. Indeed, we have

$$\begin{aligned}
|(\partial_r^2+\partial_z^2)\eta\phi_1| &\le C\lambda_*^{\nu_1-2}R^{-a_1}\chi_{[|x-q|\le 3\lambda_* R]}\|\phi_1\|_{*,\nu_1,a_1,\delta}\\
|(\partial_r^2+\partial_z^2)\eta\phi_2| &\le C\lambda_*^{\nu_2-2}R^{-a_2}\chi_{[|x-q|\le 3\lambda_* R]}\|\phi_2\|_{\nu_2,a_2-2}\\
|(\partial_r^2+\partial_z^2)\eta\phi_3| &\le C\lambda_*^{\nu_3-2}R^{-1}\chi_{[|x-q|\le 3\lambda_* R]}\|\phi_3\|_{**,\nu_3}\\
|(\partial_r^2+\partial_z^2)\eta\phi_4| &\le C\lambda_*^{\nu_4-2}R^{-2}\log R\chi_{[|x-q|\le 3\lambda_* R]}\|\phi_4\|_{***,\nu_4}.
\end{aligned}$$

If

$$\Theta < \min(\nu_1-1+\beta(a_1-1), \nu_2-1+\beta(a_2-1), \nu_3-1, \nu_4-1+\beta),$$

we find that for any $j=1,2,3,4$:

$$|\phi_j(\partial_r^2+\partial_z^2)\eta| \le CT^\sigma\lambda_*^{\Theta-1+\beta}\chi_{[|x-q|\le 3\lambda_* R]}\|\Phi\|_E,$$

for some $\sigma > 0$. Then we have

$$\|Q_\omega((\partial_r^2+\partial_z^2)\eta)\phi\|_{**} \le CT^\sigma\|\Phi\|_E$$

and similarly

$$\|(\partial_t\eta)Q_\omega\phi\|_{**} + \|Q_\omega\lambda^{-1}\nabla\eta\nabla_y\phi\|_{**} \le CT^\sigma\|\Phi\|_E.$$

The other terms g_2, g_3, g_4, g_5 can be estimated in the same way. In the estimate for g_2 it is important to have the property that $\Psi^* = Z^* + \psi$ vanishes at (r_0, z_0, T).

Next we estimate the operator $\mathcal{F}_1$. The other operators $\mathcal{F}_2, \dots, \mathcal{F}_4$ are handled similarly. We claim that for $(\psi, \Phi, \kappa, p_1, \xi^1) \in \mathcal{B}$, we have

$$\|\mathcal{F}_1(\psi,\Phi,\kappa,p_1,\xi^1)\|_{*,a_1,\nu_1} \le C\lambda_*(0)^\sigma(\|\psi\|_{\sharp,\Theta,\gamma} + \|\dot p\|_{L^\infty(-T,T)} + \|Z_0\|_{C^2}). \tag{6.36}$$

Indeed, by Proposition 6.1 we have

$$\|\mathcal{F}_1(\Phi)\|_{*,\nu_1,a_1,\delta} \le C\|h_1[p,\xi,\Psi^*]\|_{\nu_1,a_1}. \tag{6.37}$$

From the definition of h_1 (6.2) and recalling that $\Psi^* = Z^* + \psi$ we get

$$\begin{aligned}
&\|h_1[p,\xi,\Psi^*]\|_{\nu_1,a_1}\\
&\quad\le \|\lambda^2 Q_{-\omega}(\tilde L_U[\psi]_0 + \tilde L_U[\psi]_2)\chi_{D_{2R}}\|_{\nu_1,a_1} + \|\lambda^2 Q_{-\omega}(\tilde L_U[Z^*]_0 + \tilde L_U[Z^*]_2)\chi_{D_{2R}}\|_{\nu_1,a_1}\\
&\qquad+ \|\lambda^2 Q_{-\omega}\mathcal{K}_0[p,\xi]\|_{\nu_1,a_1}.
\end{aligned}$$

We claim that for $j=0$ and $j=2$:

$$\|\lambda^2 Q_{-\omega}\tilde L_U[\psi]_j\,\chi_{D_{2R}}\|_{\nu_1,a_1} \le CT^\sigma\lambda_*(0)^\Theta\|\psi\|_{\sharp,\Theta,\gamma} \tag{6.38}$$

Indeed, from (3.5) we get, for $j=0$ and $j=2$:

$$|\lambda^2 Q_{-\omega}\tilde L_U[\psi]_j| \le C\frac{\lambda_*}{(1+|y|)^3}\|\nabla\psi\|_{L^\infty}.$$

We use $\nu_1 < 1$ and $a_1 < 3$ to estimate for $|y| \le 2R$

$$\frac{\lambda_*}{(1+|y|)^3} \le \frac{\lambda_*^{\nu_1}}{(1+|y|)^{a_1}}\lambda_*(0)^{1-\nu_1}.$$

Then for $|y| \le 2R$ and $j=0,2$:

$$|\lambda^2 Q_{-\omega}\tilde L_U[\psi]_j| \le C\frac{\lambda_*^{\nu_1}}{(1+|y|)^{a_1}}\lambda_*(0)^{1-\nu_1}\|\nabla\psi\|_{L^\infty} \le C\frac{\lambda_*^{\nu_1}}{(1+|y|)^{a_1}}\lambda_*(0)^{1-\nu_1}\lambda_*(0)^\Theta\|\psi\|_{\sharp,\Theta,\gamma},$$

and (6.38) follwos. Next we claim that

$$\|\lambda^2 Q_{-\omega}\tilde L_U[Z^*]_j\chi_{D_{2R}}\|_{\nu_1,a_1} \le CT^\sigma\|Z_0\|_{C^2}, \tag{6.39}$$

for $j = 0, 2$ and some $\sigma > 0$. Indeed, we use the assumption (6.1) and standard estimates for the heat equation to obtain for $j = 0, 2$:

$$|\lambda^2 Q_{-\omega} \tilde{L}_U [Z^*]_j \, \chi_{D_{2R}}| \leq C \frac{\lambda_*}{(1+\rho)^3} \|Z_0\|_{C^2(\overline{\Omega})},$$

Since $\nu_1 < 1$, we get

$$\|\lambda^2 Q_{-\omega} \tilde{L}_U [Z^*]_j \, \chi_{D_{2R}}\|_{\nu_1, a_1} \leq C \lambda_*(0)^{1-\nu_1} \|Z_0\|_{C^2(\overline{\Omega})}.$$

This implies (6.39). Next we estimate $\lambda^2 Q_{-\omega} \mathcal{K}_0[p, \xi]$. We claim that

$$\|\lambda^2 Q_{-\omega} \mathcal{K}_0[p, \xi]\|_{\nu_1, a_1} \leq C T^\sigma \|\dot{p}\|_{L^\infty(-T,T)}. \tag{6.40}$$

Indeed, consider $\mathcal{K}_{01}$ given in (4.4). We have

$$|\lambda^2 Q_{-\omega} \mathcal{K}_{01}[p, \xi]| \leq C \frac{\lambda_*}{(1+\rho)^3} \int_{-T}^{t} |\dot{p}(s) k(z, t-s)| \, ds.$$

A direct computation shows that

$$\begin{aligned} \|\lambda^2 Q_{-\omega} \tilde{L}_U [\mathcal{K}_{01}[p, \xi]] \chi_{D_{2R}}\|_{\nu_1, a_1} &\leq C \lambda_*(0)^{1-\nu_1} \|\dot{p}\|_{L^\infty(-T,T)} \\ &\leq C T^\sigma \|\dot{p}\|_{L^\infty(-T,T)}, \end{aligned}$$

for some $\sigma > 0$. The estimate for $\mathcal{K}_{02}$ is similar, and we obtain (6.40). Combining (6.38), (6.39), and (6.40) we finally obtain

$$\|h_1[p, \xi, \Psi^*]\|_{\nu_1, a_1} \leq C T^\sigma (\|\psi\|_{\sharp, \Theta, \gamma} + \|\dot{p}\|_{L^\infty(-T,T)} + \|Z_0^*\|_{C^2}),$$

and combining with (6.37) we get (6.36).

Compactness of the operator $\mathcal{A}$ in (6.34) is proved using suitable variants of (6.35). Indeed, the previous computations show that if we vary the parameters $\Theta, \gamma, \nu_j, a_j, \delta, \sigma$ of the norms slightly, so that the restrictions in §6.6 are kept, then we still obtain (6.35) where the norms in the left hand side are defined with the new parameters while $\mathcal{B}$ is defined with the old parameters. More precisely, one can show, for example, that if Θ', γ' are fixed close to Θ, γ, then for $(\psi, \Phi, \kappa, p_1, \xi^1) \in \mathcal{B}$ (this set defined still with $\Theta, \gamma, \ldots$) we get

$$\|\mathcal{A}_0(\psi, \Phi, \kappa, p_1, \xi^1)\|_{\sharp, \Theta', \gamma'} \leq C T^\sigma,$$

(for a possibly different $\sigma > 0$). Then one proves that if $\gamma < \gamma'$, $\Theta' - \Theta > 2(\gamma' - \gamma)$ one has a compact embedding in the sense that if $(\psi_n)_n$ is a bounded sequence in the norm $\| \, \|_{\sharp, \Theta', \gamma'}$, then for a subsequence it converges in the norm $\| \, \|_{\sharp, \Theta, \gamma}$. This compact embedding is a direct consequence of a standard diagonal argument using Ascoli's theorem, and examining the estimates for a uniform smallness control of its values near time T. Similar statements hold for the other components Φ, κ, p_1, ξ^1. The proof is concluded. $\square$

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