Symplectic alternating nil-algebras

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Abstract
In this paper we continue developing the theory of symplectic alternating algebras that was started in [3]. We focus on nilpotency, solubility and nil-algebras. We show in particular that symplectic alternating nil-2 algebras are always nilpotent and classify all nil-algebras of dimension up to 8.

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1 Introduction
Symplectic alternating algebras have arisen in the study of 2-Engel groups (see [1], [2]) but seem also to be of interest in their own right, with many beautiful properties. Some general theory was developed in [3].

Definition. Let $F$ be a field. A symplectic alternating algebra over $F$ is a triple $L = (V, (\cdot, \cdot), \cdot)$ where $V$ is a symplectic vector space over $F$ with respect to a non-degenerate alternating form $(\cdot, \cdot)$ and $\cdot$ is a bilinear and alternating binary operation on $V$ such that

$$(u \cdot v, w) = (v \cdot w, u)$$

for all $u, v, w \in V$. 

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Notice that \((u \cdot x, v) = (x \cdot u, v) = - (v \cdot x, u) = (u, v \cdot x)\). The multiplication by \(x\) from the right is therefore a self-adjoint linear operation with respect to the alternating form. We know that the dimension of a symplectic alternating algebra must be even and we will refer to a basis \(x_1, y_1, ..., x_r, y_r\) with the property that \((x_i, x_j) = (y_i, y_j) = 0\) and \((x_i, y_j) = \delta_{ij}\) as a standard basis. We will also adopt the left-normed convention for multiple products. Thus \(x_1 x_2 \cdots x_n\) stands for \((\cdots (x_1 x_2) \cdots) x_n\). If \(x_1, x_2, \ldots, x_{2r}\) is a basis for the symplectic vector space, then the alternating product is determined from the values of all triples \((x_i x_j, x_k) = (x_j x_k, x_i) = (x_k x_i, x_j)\) for \(1 \leq i < j < k \leq 2r\).

Given a standard basis \(x_1, y_1, \ldots, x_r, y_r\) for a symplectic alternating algebra \(L\), we can describe \(L\), as follows. Consider the two isotropic subspaces \(F x_1 + \cdots + F x_r\) and \(F y_1 + \cdots + F y_r\). It suffices then to write only down the products of \(x_i x_j, y_i y_j, 1 \leq i < j \leq r\). The reason for this is that having determined these products we have determined \((uv, w)\) for all triples \(u, v, w\) of basis vectors, since two of those are either some \(x_i, x_j\) or some \(y_i, y_j\) in which case the triple is determined from \(x_i x_j\) or \(y_i y_j\). The only restraints on the products \(x_i x_j\) and \(y_i y_j\) come from \((x_i x_j, x_k) = (x_j x_k, x_i) = (x_k x_i, x_j)\) and \((y_i y_j, y_k) = (y_j y_k, y_i) = (y_k y_i, y_j)\).

It is clear that the only symplectic alternating algebra of dimension 2 is the abelian one. Furthermore, it is easily seen that up to isomorphism there are two symplectic alternating algebras of dimension 4: one is abelian whereas the other one has the following multiplication table (see [3]).

\[
\begin{align*}
x_1 x_2 &= 0 \\
y_1 y_2 &= -y_1 \\
x_1 y_1 &= x_2 \\
x_1 y_2 &= -x_1 \\
x_2 y_1 &= 0 \\
x_2 y_2 &= 0.
\end{align*}
\]

Of course, the presentation is determined by \(x_1 x_2 = 0\) and \(y_1 y_2 = -y_1\) as the other products are consequences of these two. The symplectic alternating algebras of dimension 6 have been classified in [3], when the field has three elements: there are 31 such algebras of which 15 are simple.

As we said before, some general theory was developed in [3]. In particular it was shown that a symplectic alternating algebra is either semi simple or has an abelian ideal. In this paper we continue developing a structure theory for symplectic alternating algebras and we are motivated by the following question that was posed in [3]:

**Question.** What can one say about the structure of symplectic alternating nil-algebras? In particular, does a symplectic alternating nil-algebra have to be nilpotent?
If \( k \) is a positive integer, we say that a symplectic alternating algebra \( L \) is \( \text{nil-}k \) if \( xy^k = 0 \) for all \( x, y \in L \). More generally, a symplectic alternating \( \text{nil-algebra} \) is a symplectic alternating \( \text{nil-k} \) algebra for some positive integer \( k \). Also, we define \( a \in L \) to be a \( \text{right nil-}k \) element if \( ax^k = 0 \) for all \( x \in L \) and to be a \( \text{right nil-element} \) if it is right nil-k for some \( k \). Similarly, \( a \in L \) is a \( \text{left nil-}k \) element when \( xa^k = 0 \) for all \( x \in L \) and a \( \text{left nil-element} \) if it is left nil-k for some \( k \).

Furthermore, we say that a symplectic alternating algebra is \( \text{nilpotent} \) if \( x_1 x_2 \cdots x_n = 0 \) for all \( x_1, x_2, \ldots, x_n \in L \) and for some integer \( n \geq 1 \). As usual, the \( \text{nilpotency class} \) of \( L \) is the smallest \( c \geq 0 \) such that \( x_1 x_2 \cdots x_{c+1} = 0 \) for all \( x_1, x_2, \ldots, x_{c+1} \in L \).

In the following, we first discuss connections between nilpotency and solubility of a symplectic alternating algebra. We will see in particular that every symplectic alternating algebra that is abelian-by-nilpotent is nilpotent. We then move to \( \text{nil-k} \) elements and to symplectic alternating \( \text{nil-k} \) algebras. We get a positive answer to the question above for \( k = 2 \) and, when the dimension is \( \leq 8 \), also for \( k = 3 \). We finish with the classification of all \( \text{nil-algebras} \) of dimension up to 8.

\section{Nilpotency and solubility}

For subspaces \( U, V \) of a symplectic alternating algebra \( L \), we define \( UV \) in the usual way as the subspace consisting of all linear spans of elements of the form \( uv \) where \( u \in U \) and \( v \in V \). We define the \textit{lower central series} \((L^i)_{i \geq 1}\) inductively by \( L_1 = L \) and \( L^{i+1} = L^i \cdot L \). Clearly
\[ L^1 \geq L^2 \geq \ldots \]
which implies in particular that every \( L^i \) is an ideal. We can also define the \textit{upper central series} \((Z^i(L))_{i \geq 0}\) naturally by \( Z^0(L) = \{0\} \), \( Z^1(L) = Z(L) = \{a \in L : ax = 0 \text{ for all } x \in L\} \) and \( Z^{i+1}(L) = \{a \in L : ax \in Z^i(L) \text{ for all } x \in L\} \). In [3], Lemma 2.2, the author proves that the lower and the upper central series are related as follows:
\[ Z^i(L) = (L^{i+1})^\perp. \]
It follows that \( Z^i(L) \) is an ideal since, in a symplectic alternating algebra, \( I^\perp \) is an ideal whenever \( I \) is an ideal (see [3], Lemma 2.1); but this also follows directly from \( Z^{i+1}(L) \cdot L \leq Z^i(L) \). Notice also that the \( \text{dim}(Z^i(L)) + \text{dim}(L^{i+1}) = \text{dim}(L) \). We then have that \( L \) is nilpotent of class \( c \geq 0 \) if and only if \( c \) is the smallest integer such that \( Z^c(L) = L \) or, equivalently, \( L^{c+1} = \{0\} \). One more way to characterize the nilpotency in terms of the lower central series is given by the following result.
Proposition 2.1. Let $L$ be a symplectic alternating algebra. Then $L$ is nilpotent if and only if there exists $i \geq 1$ such that $L^i$ is isotropic.

Proof. Let $L$ be nilpotent and denote by $c$ its nilpotency class. Then $L = Z_c(L) = (L^{c+1})^\perp$ and hence $L^{c+1}$ is isotropic. Conversely, let $L^i$ be isotropic for some $i \geq 1$. Then
\[(u_1 \cdots u_i, v_1 \cdots v_i) = 0\]
whenever $u_1, \ldots, u_i, v_1, \ldots, v_i$ belong to $L$. It follows
\[(u_1, v_1 \cdots v_i u_i \cdots u_2) = 0\]
and thus $L$ is nilpotent of class at most $2i - 2$ since the symplectic form is non-degenerate.

As usual, the derived series $(L^{(i)})_{i \geq 0}$ is defined inductively by $L^{(0)} = L$, $L^{(1)} = L \cdot L = L^2$ and $L^{(i+1)} = L^{(i)} \cdot L^{(i)}$. Then
\[L^{(0)} = L^{(1)} = \cdots\]
and we say that a symplectic alternating algebra $L$ is soluble if there exists an integer $n \geq 0$ such that $L^{(n)} = \{0\}$. The smallest $n$ enjoying this property is then referred to as the derived length of $L$. Thus $L$ has derived length 0 if and only if it has order one. Also, the symplectic alternating algebras with derived length at most 1 are just the abelian ones. A symplectic alternating algebra which is soluble of derived length at most 2 is said to be metabelian.

Lemma 2.2. If $L$ is a symplectic alternating algebra then $L^{(i)} \subseteq L^{i+1}$. In particular, if $L$ is nilpotent of class $i$ then $L$ is soluble of derived length at most $i$.

Proof. We argue by induction on $i$. The claim is obviously true when $i = 0$ being $L^{(0)} = L = L^1$. Assuming $i > 0$ and $L^{(i)} \subseteq L^{i+1}$, we get $L^{(i+1)} = L^{(i)} \cdot L^{(i)} \subseteq L^{i+1} \cdot L = L^{i+2}$, as required.

Next result is rather odd and shows that all metabelian symplectic alternating algebras are nilpotent. It also shows that the inclusion in last lemma is not optimal.

Proposition 2.3. Let $L$ be a symplectic alternating algebra. Then $L$ is metabelian if and only if it is nilpotent of class at most 3.

Proof. We have that $L$ is metabelian if and only if $xy(zw) = 0$ for all $x, y, z, w \in L$, that is $(xy(zw), t) = 0$ for all $t \in L$. This means $0 = (xy, zw) = (x, zw)$ and $L$ is nilpotent of class at most 3.
Not all soluble symplectic alternating algebras are however nilpotent as the following example shows.

**Example 2.4.** Consider

\[ L : \begin{align*}
  x_1x_2 &= 0 \\
  y_1y_2 &= -y_1,
\end{align*} \]

the only nonabelian symplectic alternating algebra of dimension 4 over a field \( F \). We have

\[ Z(L) = Fx_2 \quad \text{and} \quad L^2 = Z(L)^\perp = Fx_1 + Fx_2 + Fy_1. \]

Here \( L^{(3)} = L^{(2)} \cdot L^{(2)} = Fx_2 \cdot Fx_2 = \{0\} \) and \( L \) is soluble of derived length 3 but it is not nilpotent. In fact \( y_1y_2^n = (-1)^ny_1 \) for any integer \( n \geq 1 \).

However, we have the following strong generalisation of Proposition 2.3.

**Proposition 2.5.** Let \( L \) be a symplectic alternating algebra. If \( L \) is abelian-by-(nilpotent of class \( \leq c \)) then it is nilpotent of class at most \( 2c + 1 \).

**Proof.** Let \( I \) be an abelian ideal of \( L \) such that \( L/I \) is nilpotent of class at most \( c \). Then \( L^{c+1} \subseteq I \) and

\[ (x_1 \cdots x_{c+1} \cdot (y_1 \cdots y_{c+1}), z) = 0 \]

for all \( x_1, \ldots, x_{c+1}, y_1, \ldots, y_{c+1}, z \in L \). Thus

\[ (x_1, y_1 \cdots y_{c+1}z x_{c+1} \cdots x_2) = 0 \]

and \( L \) is nilpotent of class at most \( 2c + 1 \). \( \square \)

This result fails if we assume that our algebra is nilpotent-by-abelian. The example above still provides a counterexample, for \( L^2 \) is nilpotent and \( L/L^2 \) is abelian.

### 3 Nil-elements

Let \( L \) be a symplectic alternating algebra and \( x \) be a left nil-element of \( L \). We say that an element \( a \in L \) has *nil-\( x \) degree* \( m \) if \( m \) is the smallest positive integer such that \( ax^m = 0 \). Pick \( a \in L \) of maximal nil-\( x \) degree \( k \) and let

\[ V(a) = \langle a, ax, ax^2, \ldots, ax^{k-1} \rangle. \]
We know that this is an isotropic subspace in $L$ (see [3], Lemma 2.10). Then there exists $b \in L$ such that 

$$(a, b) = (ax, b) = \ldots = (ax^{k-2}, b) = 0 \text{ and } (ax^{k-1}, b) = 1.$$ 

Since $(a, bx^{k-1}) = (ax^{k-1}, b) = 1$, we have that the nil-$x$ degree of $b$ is $k$. Notice also that 

$$(ax^r, bx^s) = (ax^{r+s}, b)$$

which is 1 if $r + s = k - 1$ but 0 otherwise. So that the subspace 

$$V(a) + V(b) = V(a) \oplus V(b) = \langle a, bx^{k-1} \rangle \oplus \langle ax, bx^{k-2} \rangle \oplus \ldots \oplus \langle ax^{k-1}, b \rangle$$

is a perpendicular direct sum of hyperbolic subspaces.

Let $W = W(a, b) = V(a) + V(b)$. The multiplication by $x$ from the right gives us a linear map on $L$. Then $W$ is invariant under the right multiplication by $x$ and the same is then true for the orthogonal complement $W^\perp$: in fact, for all $y \in W^\perp$ and $z \in W$ we have $(yx, z) = -(y, zx) = 0$ as $zx \in W$. Now, we can take $c \in W^\perp$ of maximal nil-$x$ degree, say $m$. Then, as before, we get $d \in L$ of nil-$x$ degree $m$ and $W(c, d) = V(c) + V(d)$ is a perpendicular direct sum. Thus we inductively see that $L$ splits up into a perpendicular direct sum 

$$L = W(a_1, b_1) \oplus \cdots \oplus W(a_n, b_n). \quad (1)$$

We will refer to such a decomposition as a primary decomposition of $L$ with respect to multiplication by $x$ from the right. We will also use the notation 

$$
\begin{pmatrix}
    a & bx^{k-1} \\
    ax & bx^{k-2} \\
    \vdots & \vdots \\
    ax^{k-1} & b
\end{pmatrix}
$$

for the subspace $W(a, b)$.

**Proposition 3.1.** Let $L$ be a symplectic alternating algebra. If $x \in L$ is a left nil-element, then $C_L(x)$ is even dimensional.

**Proof.** Consider a decomposition as above with respect to right multiplication by $x$. We have seen that the cyclic subspaces come in pairs, say that 

$$L = V(a_1) \oplus V(b_1) \oplus \cdots \oplus V(a_n) \oplus V(b_n).$$

The kernel of each of these is one dimensional, hence $C_L(x)$ has dimension $2n$. \qed
For the remainder of this section we focus on right nil-2 elements. In general, a left nil-2 element needs not to be a right nil-2 element. In Example 2.4, $y_1$ is a left nil-2 element that is not a right nil-element. However, the converse is always true.

**Lemma 3.2.** Let $L$ be a symplectic alternating algebra. If $a$ is a right nil-2 element of $L$, then:

(i) $ayz = -azy$ for all $y, z \in L$;

(ii) $a$ is left nil-2;

(iii) $C_L(a)$ is an ideal;

(iv) $La$ and $Fa + La$ are abelian ideals and the latter is the smallest ideal containing $a$.

**Proof.** (i) We have
\[0 = a(y + z)(y + z) = (ay + az)(y + z) = ayz + azy\]
and $ayz = -azy$.

(ii) For all $x \in L$, we have $0 = -a(a + x)^2 = xa(a + x) = xa^2$.

(iii) Let $x, y \in L$ and $b \in C_L(a)$. Then $0 = a(x + b)^2 = ax(x + b) = axb$ which implies $0 = (axb, y) = (a(by), x)$. Thus $a(by) = 0$ and $by \in C_L(a)$.

(iv) That $La$ is an ideal follows immediately from $uax = -uxa$ and of course it follows then that $Fa + La$ is an ideal, the smallest ideal containing $a$. As $a$ is left nil-2 and since $ax(ya) = -a(ya)x = 0$, it is clear that both the ideals are abelian. \qed

**Theorem 3.3.** Let $X$ be a set of right nil-2 elements in a symplectic alternating algebra $L$ and denote by $I(X)$ the smallest ideal of $L$ containing $X$. Then
\[I(X) = \sum_{a \in X} Fa + La.\]

Furthermore, if $|X| = c$ then $I(X)$ is nilpotent of class at most $c$.

**Proof.** Let $a \in X$. By Lemma 3.2 (iv) we know that $I(a) = Fa + La$ is the smallest ideal containing $a$ and that $I(a)$ is abelian. It follows that $I(X) = \sum_{a \in X} I(a)$. Since each of these ideals is abelian it is clear that $I(X)^{c+1} = \{0\}$, here $c = |X|$. \qed
It follows in particular that the ideal generated by all the right nil-2 elements is always a nilpotent ideal.

4 Nil-2 algebras

The results concerning right nil-2 elements lead to the following characterization of symplectic alternating nil-2 algebras.

**Theorem 4.1.** Let \( L \) be a symplectic alternating algebra. Then the following are equivalent:

(i) \( L \) is nil-2;

(ii) \( C_L(x) \) is an ideal for any \( x \in L \);

(iii) \( I(x) \) is abelian for any \( x \in L \);

(iv) the identity \( xyz = -xzy \) holds in \( L \);

(v) the identity \( x(yz) = xzy \) holds in \( L \).

**Proof.** First we show that (i) \( \iff \) (ii) \( \iff \) (iii). From Lemma 3.2, we know that (i) implies (ii) and (iii). To see that (iii) implies (i), take any \( a, x \in L \). As \( I(x) \) is abelian and \( ax, x \in I(x) \), it follows that \( ax^2 = 0 \). Finally to show that (ii) implies (i), notice that \( x \in C_L(x) \) and as \( C_L(x) \) is an ideal we also have \( ax \in C_L(x) \). The latter gives \( ax^2 = 0 \).

We finish the proof by showing that (i) \( \Rightarrow \) (iv) \( \Rightarrow \) (v) \( \Rightarrow \) (i). The fact that (i) implies (iv) follows from Lemma 3.2. If (iv) holds, then \( x(yz) = -yzx = yxz = -xyz = xzy \) that gives us (v). Finally (i) follows from (v) by taking \( y = z \).

It follows from Theorem 3.3 that all symplectic alternating nil-2 algebras are nilpotent. We next analyse this in more details.

**Theorem 4.2.** Let \( L \) be a symplectic alternating algebra over a field \( F \) of characteristic \( \neq 2 \). If \( L \) is nil-2, then \( L \) is nilpotent of class at most 3.

**Proof.** Let \( x, y, z, t \in L \). By Theorem 4.1, \( xy(tz) = xytz \) and \( xy(tz) = -x(tz)y = -xzyt = xzyt = -xyzt \). It follows that \( 2xyzt = 0 \) and, since \( \text{char} \ F \neq 2 \), we conclude that \( xytz = 0 \).

Moreover, the bound provided is optimal as there exists a nil-2 algebra which is nilpotent of class 3.

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Example 4.3. Let $F$ be any field and $L$ be the linear span of

\[
\begin{align*}
    x_1 &= a & y_1 &= tcb \\
    x_2 &= b & y_2 &= tac \\
    x_3 &= c & y_3 &= tba \\
    x_4 &= ab & y_4 &= tc \\
    x_5 &= ca & y_5 &= tb \\
    x_6 &= bc & y_6 &= ta \\
    x_7 &= abc & y_7 &= t.
\end{align*}
\]

As a symplectic vector space we let $L = (Fx_1 + Fy_1) \oplus \cdots \oplus (Fx_7 + Fy_7)$, a perpendicular direct sum of hyperbolic subspaces (where $(x_i, y_i) = 1$ for $i = 1, \ldots, 7$). We turn this into a symplectic alternating nil-2 algebra by adding an alternating product satisfying condition (iv) of Theorem 4.1. As the identity (iv) is multilinear it suffices that $xyz = -xzy$ whenever $x, y, z$ are generators. The condition implies that the only non-trivial triples $(uv, w) = (vw, u) = (wu, v)$ are

\[
\begin{align*}
    (x_1x_2, y_4) &= 1 \\
    (x_3x_1, y_5) &= 1 \\
    (x_2x_3, y_6) &= 1 \\
    (x_4x_3, y_7) &= 1 \\
    (x_5x_2, y_7) &= 1 \\
    (x_6x_1, y_7) &= 1.
\end{align*}
\]

Conversely one can easily check that this alternating product turns $L$ into a symplectic alternating nil-2 algebra that is nilpotent of class 3.

Theorem 4.4. Let $F$ be a field of characteristic 2 and let $L$ be a symplectic alternating algebra of dimension $n = 2m$. If $L$ is nil-2, then $L$ is nilpotent of class at most $\lfloor \log_2 (m + 1) \rfloor$.

Proof. Let $\{x_1, \ldots, x_n\}$ be a basis of $L$. If char $F = 2$, then $L$ is commutative and, by Theorem 4.1, it is also associative. It follows that

\[ u_1 \cdots u_n = 0 \text{ for all } u_1, \ldots, u_n \in L \text{ if and only if } x_1 \cdots x_n = 0. \]

But $(x_1 \cdots x_n, x_i) = 0$ for any $i \in \{1, \ldots, n\}$. Hence $x_1 \cdots x_n = 0$ and $L$ is nilpotent of class at most $n - 1$. So, if we denote by $c$ the nilpotency class of $L$, then $c < n$. Since the class is $c$ there is a non-zero product $x_{i_1} \cdots x_{i_c}$ and without loss of generality we can suppose that $x_1 \cdots x_c \neq 0$. Now, let

\[ x_I = x_{i_1} \cdots x_{i_r}, \]
for any $I = \{i_1, \ldots, i_r\} \subseteq \{1, \ldots, c\}$ and let

$$X = \{x_I : \emptyset \subset I \subseteq \{1, \ldots, c\}\}.$$  

We prove that $X$ is a linearly independent subset of $L$. Assume

$$\alpha_1 x_{I_1} + \ldots + \alpha_m x_{I_m} = 0$$

where $m \leq 2^c - 1$ and $|I_1| \leq \ldots \leq |I_m|$. Let $\alpha_j$ be the least non zero coefficient and $J = \{1, \ldots, c\}\setminus I_j$. Then, multiplying by $\prod_{k \in J} x_k$, we get

$$\alpha_j x_1 \cdots x_c = 0$$

and thus $x_1 \cdots x_c = 0$ which is a contradiction. Thus $X$ is linearly independent and $|X| = 2^c - 1$. Hence $2^c - 1 \leq 2m$ and $2^c < 2m + 2$. Then $c < \log_2(2(m + 1)) = 1 + \log_2(m + 1)$ and so $c \leq \log_2(m + 1)$, as we claimed.

Indeed, the bound we have just got is the best possible, as shown in the following construction:

**Example 4.5.** Let $F$ be the field with 2 elements and let $r > 3$. There exists a symplectic alternating nil-2 algebra $L$ over $F$ of dimension $2(2^{r-1} - 1)$ which is nilpotent of class $r - 1$. In fact, define $L$ to be the linear span of all monomials in $x_1, \ldots, x_r$ with no repeated entries and of weight less than $r$. Then $L$ has dimension $2^r - 2$ over $F$. Let

$$(x_{i_1} \ldots x_{i_n}, x_{j_1} \ldots x_{j_m}) = 0$$

except if $n + m = r$ and $\{i_1, \ldots, i_n, j_1, \ldots, j_m\} = \{1, \ldots, r\}$, and 1 otherwise. This gives a symplectic vector space. Let

$$x_{i_1} \ldots x_{i_n} \cdot x_{j_1} \ldots x_{j_m} = x_{i_1} \ldots x_{i_n} x_{j_1} \ldots x_{j_m}$$

if $i_1, \ldots, i_n, j_1, \ldots, j_m$ are distinct and $\{i_1, \ldots, i_n, j_1, \ldots, j_m\} \subset \{1, \ldots, r\}$, and 0 otherwise. Then $L$ is a symplectic alternating algebra that is nilpotent of class $r - 1$. Since $L$ is commutative and associative, it is also nil-2.

## 5 Nil-3 algebras

In this section we describe some general properties of a symplectic alternating nil-3 algebra $L$.

**Lemma 5.1.** For any $x, y, z \in L$ the following identities hold:
(i) \( \sum_{\sigma \in S_3} x y_{\sigma(1)} y_{\sigma(2)} y_{\sigma(3)} = 0; \)

(ii) \( \sum_{\sigma \in S_2} x y_{\sigma(1)} y_{\sigma(2)} z + x y_{\sigma(1)} (z y_{\sigma(2)}) + x (z y_{\sigma(1)} y_{\sigma(2)}) = 0. \)

**Proof.** The proof of (i) is straightforward. To see why (ii) holds notice that, for any \( u \in L \), from (i) we have

\[
0 = \left( \sum_{\sigma \in S_2} x y_{\sigma(1)} y_{\sigma(2)} u + x y_{\sigma(1)} u y_{\sigma(2)} + x u y_{\sigma(1)} y_{\sigma(2)}, z \right)
\]

\[
= \sum_{\sigma \in S_2} (x y_{\sigma(1)} y_{\sigma(2)} , z u ) + (x y_{\sigma(1)} , z y_{\sigma(2)} u ) + (x , z y_{\sigma(2)} y_{\sigma(1)} u )
\]

\[
= -\left( \sum_{\sigma \in S_2} x y_{\sigma(1)} y_{\sigma(2)} z + x y_{\sigma(1)} (z y_{\sigma(2)}) + x (y_{\sigma(2)} y_{\sigma(1)} , u ) \right).
\]

\[\square\]

In the following we will use the notation \( x \{ y_1, y_2, y_3 \} \) for the first sum in Lemma 5.1 and similarly \( x \{ y_1, y_2 \} = xy_1y_2 + xy_2y_1. \)

**Lemma 5.2.** For any \( x, y, z \in L \) the following hold:

(i) \( yx^2y = -yxyx \in Lx; \)

(ii) if \( zx^2y = 0 \) then \( yx^2z \in Lx; \)

(iii) \( yx^2(zx^2) \in Lx \cap C_L(x); \)

(iv) if \( yx^2(zx^2) = 0 \) then \( yx^2(zx) \in Lx \cap C_L(x). \)

**Proof.** (i) First we have

\[
0 = y(x + y)^3 = yx(x + y)^2 = (yx^2 + yxy)(x + y) = yx^2y + yxyx.
\]

(ii) Assume \( zx^2y = 0 \). Then we get

\[
0 = x \{ x, y, z \} = xy \{ x, z \} + xz \{ x, y \} = xyxz + xyzx + xzyx.
\]
that gives \(yx^2z \in Lx\).

(iii) We see that

\[
0 = -x\{x, yx, zx^2\} = yx^2\{x, zx^2\} = yx^2(zx^2)x.
\]

Then also

\[
0 = x\{x, y, zx^2\} = xy\{x, zx^2\} = xy(zx^2) + xy(zx^2)x
\]

that implies \(yx^2(zx^2) \in Lx \cap C_L(x)\).

(iv) Let \(yx^2(zx^2) = 0\). Since

\[
0 = x\{x, yx^2, z\} = xz(yx^2)x,
\]

it follows

\[
yx^2(zx)x = 0.
\]

Notice also

\[
0 = x\{x, y, zx\} = xy\{x, zx\} + x(zx)\{x, y\} = xy(zx) + xy(zx)x + x(zx)yx.
\]

Thus \(yx^2(zx) \in Lx \cap C_L(x)\).

\[\square\]

6 Classification of nil-algebras of dimension \(\leq 8\)

Before embarking on the classification of the symplectic alternating nil-algebras of dimension \(\leq 8\), we prove the following result.

Proposition 6.1. If \(L\) is a symplectic alternating nil-\(k\) algebra, then \(\dim(L) \geq 2(k + 1)\).

Proof. Suppose by contradiction \(\dim(L) = 2k\) and take \(x \in L\) which is not left nil-(\(k - 1\)). By (1), there is only one possible primary decomposition for the multiplication by \(x\) from the right. This is

\[
\begin{pmatrix}
a & bx^{k-1} \\
ax & bx^{k-2} \\
\vdots & \vdots \\
ax^{k-1} & b
\end{pmatrix}.
\]

It is easy to see that \(x = cx^{k-1}\) for some \(c \in L\). Then \(0 = x(-cx^{k-2})^k = x\), which is impossible. \[\square\]

As a consequence, all the nonabelian nil-algebras of dimension \(\leq 8\) are the nil-2 algebras of dimension either 6 or 8 and the nil-3 of dimension 8.
6.1 Nil-2 algebras of dimension 6

Let $L$ be a symplectic alternating nil-2 algebra of dimension 6 over a field $F$. Assume that $L$ is not abelian and let $x \in L \setminus Z(L)$. Because of (1), we have that the only primary decomposition of $L$ with respect to multiplication by $x$ from the right is
\[
\begin{pmatrix}
  a & bx \\
  ax & b
\end{pmatrix}
\oplus
\begin{pmatrix}
  c & d
\end{pmatrix}
\]
where $cx = dx = 0$.

By Theorem 4.1, $axc = -xac = xca = 0$ and similarly $ax$ commutes with $d, a, ax, bx$. As $C_L(ax)$ is even dimensional, it follows that $ax$ commutes also with $b$ and thus $ax \in Z(L)$. Similarly $bx \in Z(L)$ and $Lx \subseteq Z(L)$. Of course this is also true if $x \in Z(L)$. We have thus shown that $Ly \subseteq Z(L)$ for all $y \in L$ and thus $L$ is nilpotent of class 2.

Now we have
\[x = \alpha ax + \beta bx + u\]
for some $\alpha, \beta \in F$ and $u \in Fc + Fd$. As $x \notin Lx$ we must have that $u$ is nontrivial. Also $au = ax$ and $bu = bx$. We can thus, without loss of generality, replace $x$ by $u$ and suppose that $x$ is orthogonal to $a, ax, b, bx$.

Next we turn to $ab$. Notice that $ab$ is orthogonal to $a, b, ax, bx$ and $(x, ab) = (-bx, a) = (a, bx) = 1$. Hence we have the primary decomposition
\[
\begin{pmatrix}
  a & bx \\
  ax & b
\end{pmatrix}
\oplus
\begin{pmatrix}
  x & ab
\end{pmatrix}
\]
with respect to multiplication by $x$ from the right. The structure is now completely determined. So there is just one nonabelian nil-2 algebra of dimension 6.

6.2 Nil-2 algebras of dimension 8

Let $L$ be a symplectic alternating nil-2 algebra of dimension 8 over a field $F$. Assume that $L$ is not abelian and let $x \in L \setminus Z(L)$. We cannot have $x \in Lx$ as this would imply that $x = xz$ for some $z \in L$ and then $x = xz^2 = 0$. By (1), this implies that there is only one possible primary decomposition of $L$ with respect to multiplication by $x$ from the right. This is
\[
\begin{pmatrix}
  a & bx \\
  ax & b
\end{pmatrix}
\oplus
\begin{pmatrix}
  c & d \\
  e & f
\end{pmatrix}
\]
where $cx = dx = ex = fx = 0$.

By Theorem 4.1, $axc = -xac = xca = 0$ and similarly we see that $ax$ commutes with $d, e, f, bx$ as well as, of course, with $a$ and $ax$. Since $C_L(ax)$ is even dimensional, it follows that $ax$ commutes also with $b$ and $ax \in Z(L)$. The same argument shows that $bx \in Z(L)$. So $Lx \subseteq Z(L)$ and obviously
this is also true if \( x \in \mathbb{Z}(L) \). We have thus shown that \( Ly \subseteq \mathbb{Z}(L) \) for all \( y \in L \) and \( L \) is nilpotent of class 2. Now we have that

\[
x = \alpha ax + \beta bx + u
\]

for some \( \alpha, \beta \in F \) and for \( u \in Fc + Fd + Fe + Ff \). As \( x \) cannot be in \( Lx \) we must have that \( u \) is nontrivial. Now \( au = ax \) and \( bu = bx \) so we can, without loss of generality, replace \( x \) by \( u \) and so we can suppose that \( x \) is orthogonal to \( a, b, ax, bx \). Next consider the element \( ab \). We have that \( ab \) is orthogonal to \( a, b \) and as \( ab \in \mathbb{Z}(L) \), we also have that \( ab \) is orthogonal to \( ax \) and \( bx \). Furthermore \((x, ab) = (−bx, a) = (a, bx) = 1\). So we have a primary decomposition

\[
\left( \begin{array}{cc}
  a & bx \\
  ax & b \\
\end{array} \right) \oplus \left( \begin{array}{c}
  x \\
  ab \\
\end{array} \right) \oplus \left( \begin{array}{cc}
  c & d \\
\end{array} \right)
\]

with \( cx = dx = 0 \). But now \( Fa+Fax+Fbx+Fb+Fx+Fab \) is invariant under multiplication by \( a \) and \( b \). It follows that its orthogonal complement, \( Fc + Fd \), is also invariant under multiplication by \( a \) and \( b \). The only possibility then is that \( ca = da = cb = db = 0 \). Notice, finally, that \( cd \) is orthogonal to \( a, ax, b, bx, x, ab \) as well as to \( c, d \) and thus \( cd = 0 \). The structure of \( L \) is thus determined. All triples \((uv, w)\) involving \( ax, bx, ab, c, d \) are trivial and \((ax, b) = (xb, a) = (ba, x) = 1\). So there is only one nonabelian nil-2 algebra of dimension 8.

### 6.3 Nil-3 algebras of dimension 8

Let \( L \) be a symplectic alternating nil-3 algebra of dimension 8 over a field \( F \). Suppose that \( x \in L \) is not left nil-2. By (1), there is only one possible primary decomposition for the multiplication by \( x \) from the right. This is

\[
L = \left( \begin{array}{ccc}
  a & bx^2 & b \\
  ax & bx & b \\
  ax^2 & b & b \\
\end{array} \right) \oplus \left( \begin{array}{c}
  u \\
  t \\
\end{array} \right)
\]

where \( ux = tx = 0 \).

**Lemma 6.2.** The following properties hold:

(i) \( Lx^2 \) is abelian;
(ii) \( Lx^2(Lx) \subseteq Lx^2 \);
(iii) \( ax^2(ax) = −ax^2ax \) and \( bx^2(bx) = −bx^2bx \);
(iv) if \( bx^2(ax) = 0 \) then \( ax^2(ax) = r bx^2 \) for some \( r \in F \);
(v) if \( ax^2(bx) = 0 \) then \( bx^2(bx) = s ax^2 \) for some \( s \in F \).
Proof. (i) As $Lx \cap C_L(x) = Lx^2$, it follows from Lemma 5.2 (iii) that $ax^2(bx^2) \in Lx^2 = Fx^2 \oplus Fbx^2$. Suppose 

$$ax^2(bx^2) = ax^2 + \beta bx^2$$

for some $\alpha, \beta \in F$. Then 

$$0 = ax^2(bx^2)^3 = ax^2 + \alpha^3 bx^2$$

implies $\alpha = 0$ and 

$$0 = bx^2(ax^2)^3 = -\beta^3 bx^2$$

gives $\beta = 0$. Thus $ax^2(bx^2) = 0$ and $Lx^2$ is abelian.

(ii) This follows by (i) and Lemma 5.2 (iv), since $Lx \cap C_L(x) = Lx^2$.

(iii) We have 

$$0 = -x\{a, x, ax\} = ax\{x, ax\} + ax^2\{a, x\} = ax^2(ax) + ax^2ax$$

and similarly $0 = bx^2(bx) + bx^2bx$.

(iv) By (ii), we know that 

$$ax^2(ax) = sax^2 + rbx^2$$

for some $r, s \in F$. Then 

$$0 = -x(ax)^3 = ax^2(ax)^2 = s^2 ax^2 + sbx^2$$

implies $s = 0$ and hence $ax^2(ax) = rbx^2$.

We get (v) in the same manner. 

Notice that the following result holds with the roles of $a$ and $b$ interchanged.

**Lemma 6.3.** If $ax^2(ax) = rbx^2$ for some $r \in F$, then $ax^2(bx) = 0$. Furthermore, $ax^2 \in Z(L)$ when $r = 0$.

*Proof.* By (i) of Lemma 5.2, $ax^2a \in Lx$. As $(ax^2a, a) = 0$ and 

$$(ax^2a, ax) = -(ax^2(ax), a) = r,$$

we have 

$$ax^2a = ax + \beta ax^2 - rbx$$

for some $\alpha, \beta \in F$. Then 

$$ax^2ax = ax + \beta ax^2 - rbx.$$ 

But $ax^2ax = -ax^2(ax) = -rbx^2$ by Lemma 6.2 (iii), thus $\alpha ax^2 = 0$. It follows that $\alpha = 0$ and 

$$ax^2a = \beta ax^2 - rbx,$$
so that $ax^2a$ is orthogonal to $bx$ and thus $ax^2(bx)$ is orthogonal to $a$. However, $ax^2(bx) \in Lx^2$ by (ii) of Lemma 6.2, hence

$$ax^2(bx) = \gamma ax^2$$

for some $\gamma \in F$. Moreover $0 = ax^2(bx)^3 = \gamma^3 ax^2$, hence $\gamma = 0$ and $ax^2(bx) = 0$.

Now assume $r = 0$. Then

$$ax^2a = \beta ax^2$$

and we have

$$0 = ax^2a^3 = \beta^3 ax^2$$

which gives $\beta = 0$ and $ax^2a = 0$.

We now turn to $ax^2u$ and $ax^2t$. They both lie in $Lx$ by (ii) of Lemma 5.2 and are orthogonal to $a, ax, bx$. If $\beta = (ax^2u, b)$ and $\gamma = (ax^2t, b)$, we have

$$ax^2u = \beta ax^2 \quad \text{and} \quad ax^2t = \gamma ax^2.$$ 

Then, as before, we get $\beta = \gamma = 0$. We have thus seen that $ax^2$ commutes with $a, ax, ax^2, bx, bx^2, u, t$ and, as the dimension of $C_L(ax^2)$ is even, it follows that $ax^2b = 0$ and $ax^2 \in Z(L)$.

**Corollary 6.4.** Let $y, z \in L$. If $yz^2(yz) = 0$ then $yz^2 \in Z(L)$.

**Proof.** If $yz^2 = 0$, this is obvious. Otherwise this follows from Lemma 6.3 with $y$ in the role of $a$ and $z$ in the role of $x$. \qed

**Remark 6.5.** In particular if $yz^2(yz) = 0$ for all $y, z \in L$, then $Lz^2 \subseteq Z(L)$.

Furthermore, we have:

**Lemma 6.6.** $Z(L) \cap Lx^2 \neq \{0\}$.

**Proof.** If $ax^2(ax) = 0$, then $ax^2 \in Z(L)$ by the previous lemma. So we may assume $ax^2(ax) \neq 0$. By Lemma 6.2 (ii), the multiplication by $ax$ from the right gives us a linear operator on $Lx^2$ that is a nil operator and so with a nontrivial kernel. This means that we have

$$(b + \alpha a)x^2(ax) = 0$$

for some $\alpha \in F$. Without loss of generality we can replace $b$ by $b + \alpha a$ and thus assume that

$$bx^2(ax) = 0.$$
By Lemma 6.2 (iv) we have \( ax^2(ax) = rbx^2 \) for some \( r \in F \setminus \{0\} \) and hence \( ax^2(bx) = 0 \) by Lemma 6.3. Then (v) of Lemma 6.2 gives that there exists \( s \in F \) such that \( bx^2(bx) = sax^2 \). This implies
\[
0 = bx^2(ax + bx)^3 = rs^2ax^2
\]
and we get \( s = 0 \). It follows \( bx^2(bx) = 0 \) and \( bx^2 \in Z(L) \) again applying Lemma 6.3.

We now turn to the structure of \( L \). This is determined by the value of all triples \((vz, w) = (zw, v) = (wv, z)\) where \( v, z, w \) are pairwise distinct basis vectors. As any such triples has either two vectors from \( \{a, ax, ax^2, b, bx, bx^2\} \) or two vectors from \( \{u, t\} \), we only need to determine \( ut \) and the products of any two elements from \( \{a, ax, ax^2, b, bx, bx^2\} \).

According with Lemma 6.6, we will assume
\[
bx^2 \in Z(L). \tag{3}
\]
Then we also have
\[
ax^2(ax) = rbx^2 \quad \text{and} \quad ax^2(bx) = 0 \tag{4}
\]
by Lemma 6.2 (iv) and Lemma 6.3, respectively.

**Step 1.** We can assume that \( ax^2b = 0 \) and \( ax^2a = -rbx \).

*Proof.* By Lemma 5.2, (ii) and (i), \( ax^2b \) and \( ax^2a \) are in \( Lx \). Also \( ax^2b \) is orthogonal to \( ax, b, bx \) and
\[
ax^2b = \alpha bx^2
\]
for \( \alpha = -(ax^2b, a) \). If \( r = 0 \), then Lemma 6.3 implies \( ax^2 \in Z(L) \) and so \( ax^2b = 0 \). Let \( r \neq 0 \), then \( ax^2(b - \frac{\alpha}{r}ax) = 0 \). Replacing \( b \) by \( b - \frac{\alpha}{r}ax \), we can assume that \( ax^2b = 0 \). One can check that (3) and (4) still hold.

Next, we have that \( ax^2a \) is orthogonal to \( a, b, bx \) and
\[
(ax^2a, ax) = -(ax^2(ax), a) = -r(bx^2, a) = r.
\]
Thus \( ax^2a = -rbx \). \( \square \)

Suppose now that \( x = y + z \) with \( y \in \langle a, ax, ax^2, b, bx, bx^2 \rangle \) and \( z \in \langle u, t \rangle \). Then \( 0 = yx \) and thus \( y \in Lx^2 \). Notice that \( z \neq 0 \) since otherwise \( x = y = cx^2 \) for some \( c \in L \) and \( 0 = x(-cx)^2 = x \). Without loss of generality, we can suppose that \( z = u \). Hence
\[
x = u + \alpha ax^2 + \beta bx^2
\]
for some \( \alpha, \beta \in F \).
Let us calculate the effect of multiplying with 

\[ u = x - \alpha ax^2 - \beta bx^2. \]

Firstly, we have

\[ ut = xt - \alpha ax^2 t. \]

However, \( ax^2 t \in Lx \) by Lemma 5.2 (ii) and is orthogonal to \( a, ax, bx \). Thus \( ax^2 t = 0 \) and

\[ ut = xt. \]

Recall that \( bx^2 \in Z(L) \) and that \( ax^2 b = ax^2 (bx) = 0 \), whereas \( ax^2 a = -r bx \) and \( ax^2 (ax) = r bx^2 \). Using this, we see that

\[ au = ax + \alpha ax^2 a = ax - \alpha rbx \]

and

\[ au^2 = (ax - \alpha rbx)(x - \alpha ax^2 - \beta bx^2) \]
\[ = ax^2 + \alpha ax^2 (ax) - \alpha rbx^2 \]
\[ = ax^2 + \alpha rbx^2 - \alpha rbx^2 \]
\[ = ax^2. \]

One also sees that \( bu = bx \) and \( bu^2 = bx^2 \). Replacing \( x \) by \( u \) and \( a, ax, ax^2, b, bx, bx^2 \) by \( a, au, au^2, b, bu, bu^2 \), we still have a decomposition into hyperbolic subspaces. One can now check that (3), (4) and Step 1 are still valid with \( x \) replaced by \( u \). So without loss of generality we can assume that \( u = x \). We thus have a primary decomposition

\[ L = \begin{pmatrix} a & bx^2 \\ ax & bx \\ ax^2 & b \end{pmatrix} \oplus \begin{pmatrix} x & t \end{pmatrix} \]

where

\[ xt = 0. \] (5)

**Step 2.** \( ax(bx) = 0 \).

*Proof.* From \( ax^2 b = 0 \), we get

\[ 0 = -x\{a,b,x\} = ax\{b,x\} + bx\{a,x\} = axbx + bxax. \] (6)

Since the values

\[ (axb,b), (axb,ax), (axb,ax^2), (axb,bx^2) \]

and

\[ (bxa,a), (bxa,bx), (bxa,ax^2), (bxa,bx^2) \]

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are all trivial, we have

\[ axb = \alpha ax + y, \quad y \in Fbx^2 + Fx + Ft \]  

(7)

and

\[ bxa = \beta bx + z, \quad z \in Fbx^2 + Fx + Ft, \]  

(8)

respectively. By (6), (7) and (8), it follows that

\[ \alpha ax^2 = axbx = -bxax = -\beta bx^2 \]

which implies \( \alpha = \beta = 0 \). Hence \((axb, bx) = (bxa, ax) = 0\) and thus

\[ (ax(bx), a) = (ax(bx), b) = 0. \]

Clearly, \( ax(bx) \) is also orthogonal to \( ax, bx, ax^2, bx^2, x \) and thus

\[ ax(bx) = \alpha x \]

for some \( \alpha \in F \). But we have

\[
\begin{align*}
0 &= -x\{a, ax, bx\} \\
&= ax\{ax, bx\} + ax^2\{a, bx\} + bx^2\{a, ax\} \\
&= ax(bx)(ax) + ax^2a(bx) \\
&= ax(bx)(ax) - r(bx)^2 \\
&= ax(bx)(ax).
\end{align*}
\]

Then

\[ 0 = ax(bx)(ax) = \alpha x(ax) = -\alpha ax^2 \]

and \( \alpha = 0. \)

**Step 3.** We can assume that \( bxb = 0 \) and \( axa = rb \).

*Proof.* Let us first consider \( bxb \). It is orthogonal to \( ax, ax^2, b, bx, bx^2, x \). We then have

\[ bxb = \alpha bx^2 + \beta x \]

where \( \alpha = -(bxb, a) \) and \( \beta = (bxb, t) \). Since

\[ 0 = xb^3 = -\beta xb, \]

we get \( \beta = 0 \). It follows that

\[ 0 = bx(b - ax). \]

Replacing \( b \) by \( b - \alpha x \) and \( t \) by \( t - \alpha ax^2 \) respectively, (3), (4), (5) and the previous steps still hold. Thus we can assume \( bxb = 0. \)
We turn to $axa$. It is clear that $axa$ is orthogonal to $a, ax, bx, bx^2, x$ and that
\[(axa, ax^2) = (ax^2, a(ax)) = (ax^2(ax), a) = r(bx^2, a) = -r.\]
Suppose $(axa, b) = \alpha$ and $(axa, t) = \beta$. Then
\[axa = \alpha ax^2 + rb + \beta x. \quad (9)\]
We next show that $axa(bx) \in Lx$ and in order to do this we prove that $a(bx)x = 0$. That this is sufficient follows from
\[0 = a\{a, x, bx\} = ax\{a, bx\} + a(bx)\{a, x\} = axa(bx) + a(bx)ax + a(bx)xa.\]
As $ax(bx) = 0$, by (8) we know that $a(bx) \in Fax^2 + Fx + Ft$. But
\[(a(bx), b) = 0 \quad \text{and} \quad (a(bx), x) = -1,\]
and thus
\[a(bx) = \gamma x + t \quad \text{and} \quad a(bx)x = 0. \quad (10)\]
Let $axa(bx) = \alpha_1 ax + \alpha_2 ax^2 + \beta_1 bx + \beta_2 bx^2$. Since
\[(axa(bx), a) = (axa(bx), b) = (axa(bx), ax) = (axa(bx), bx) = 0,\]
$axa(bx)$ is trivial and, by (9), we get
\[0 = axa(bx) = -\beta bx^2.\]
Thus $\beta = 0$ and $ax(a - \alpha x) = rb$. If we replace $a$ by $a - \alpha x$ and $t$ by $t + \alpha bx^2$, then (3), (4), (5) and all the previous steps hold. So we can assume that $axa = rb$. \qed

**Step 4.** $AXB = t$ and $bxa = -t$.

**Proof.** We first consider $axt$ which is clearly orthogonal to $x$ and $t$. As the product of $ax$ with $a, ax, ax^2, bx, bx^2$ is orthogonal to $t$, $axt$ is also orthogonal to $a, ax, ax^2, bx, bx^2$. Hence, for some $\alpha \in F$,
\[axt = \alpha ax^2 \quad \text{and} \quad ax(t - \alpha x) = 0.\]
Replacing $t$ by $t - \alpha x$ we can assume that
\[axt = 0.\]
It follows that $(axb, t) = 0$, thus $axb$ is orthogonal to $t$. As the products of $ax$ with $a, ax, bx, ax^2, bx^2$ are orthogonal to $b$, we have that $axb$ is orthogonal to $t, a, ax, bx, ax^2, bx^2, b$. Also $(axb, x) = -1$ and so
\[axb = t.\]
We now turn to \(bxa\). By (10), we know that
\[
bxa = -t - \gamma x.
\]
Since
\[
0 = -x(a + b)^3 = (ax + bx)(a + b)^2 = (axa + axb + bxa)(a + b) = (rb + t - t - \gamma x)(a + b) = -rab + \gamma ax + \gamma bx,
\]
we get
\[
0 = (-rab + \gamma ax + \gamma bx, bx) = \gamma.
\]
Thus \(bxa = -t\).

**Step 5.** We can assume that \(ab = 0\).

**Proof.** Clearly, \(ab\) is orthogonal to \(a, b\) and, since \(ax^2, bx, bx^2\) commute with \(b\), we have that \(ab\) is also orthogonal to \(ax^2, bx, bx^2\). As \(bx\) is orthogonal to \(a\) we also have \(ab\) orthogonal to \(ax^2, bx, bx^2\). Then
\[
(ab, ax) = -(b, axa) = -(b, rb) = 0
\]
and the only generator left is \(t\). Hence
\[
ab = \alpha x
\]
for some \(\alpha \in F\).

We consider two cases. Suppose first that \(yz^2(yz) = 0\) for all \(y, z \in L\). Then \(r = 0\) and by Remark 6.5
\[
axb = ab^2 \in Z(L)
\]
which is absurd except if \(\alpha = 0\). Hence \(ab = 0\) in this case.

If the identity \(yz^2(yz) = 0\) does not hold for all \(y, z \in L\), without loss of generality we can assume \(ax^2(ax) = rbx^2\) with \(r \neq 0\). Thus
\[
0 = ba^3 = axaxa = \alpha rb
\]
implies \(\alpha = 0\) and hence \(ab = 0\) also in this case.

As candidates for our examples we thus have a one parameter family of symplectic alternating algebras
\[
L(r) = \begin{pmatrix} a & bx^2 \\ ax & bx \\ ax^2 & b \end{pmatrix} \oplus \begin{pmatrix} x & t \end{pmatrix}.
\]
Notice that $t \in Z(L(r))$ since $vt$ is orthogonal to $x, t$ and $(vt, w) = (vw, t) = 0$ for all $v, w \in \{a, ax, ax^2, b, bx, bx^2\}$: the only nontrivial products not involving $x$ are

\[
\begin{align*}
ax &= rb \\
ax^2 a &= -rbx \\
ax^2(ax) &= rbx^2 \\
axb &= t \\
bbxa &= -t.
\end{align*}
\]

It remains to check that $L(r)$ is nil-$3$.

**Proposition 6.7.** $L(r)$ is a nil-$3$ algebra for all $r \in F$.

**Proof.** Let $z = a_1a + a_2ax + a_3ax^2 + \beta_1b + \beta_2bx + \gamma x$. It suffices to show that $yz^3 = 0$ for the basis elements $a, ax, ax^2, b, bx, x$. Using the description of $L(r)$, we have $bxz^2 = (-\alpha_1t + \gamma bx^2)z = 0$ and then:

\[
\begin{align*}
ax^3 &= (-\alpha_2rb + \alpha_3rbx + \beta_2t + \gamma ax)z^2 \\
&= (-\alpha_2rb + \gamma ax)z^2 \\
&= (\alpha_2^2rt - \alpha_2\gamma rbx + \gamma a_1rb - \gamma a_3rbx^2 + \gamma \beta_1t + \gamma^2ax^2)z \\
&= (-\alpha_2\gamma rbx + \alpha_1rb + \gamma^2ax^2)z \\
&= \alpha_2\gamma a_1rt - \alpha_2\gamma^2rbx^2 - \gamma a_1a_2rt + \\
&\quad + \gamma^2a_1rbx - \gamma^2a_1rbx + \gamma^2\alpha_2rbx^2 \\
&= 0; \\
ax^2z &= (a_1rb - \alpha_3rbx^2 + \beta_1t + \gamma ax^2)z^2 \\
&= (a_1rb + \gamma ax^2)z^2 \\
&= (-\alpha_1a_2rt + \alpha_1\gamma rbx - \gamma a_1rbx + \gamma a_2rbx^2)z \\
&= 0; \\
ax^2z^2 &= (-\alpha_1rbx + \alpha_2rbx^2)z^2 = 0;
\end{align*}
\]

\[
\begin{align*}
x^2z^3 &= (-\alpha_1rt + \gamma bx^3)z^2 = 0; \\
xz^3 &= (-\alpha_1t + \gamma bx^3)z^2 = 0;
\end{align*}
\]

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\[ xz^3 = (-\alpha_1 ax - \alpha_2 ax^2 - \beta_1 bx - \beta_2 bx^2)z^2 \]
\[ = (-\alpha_1 ax - \alpha_2 ax^2)z^2 \]
\[ = (-\alpha_1^2 r + \alpha_1 \alpha_3 r bx^2 - \alpha_1 \beta_1 t + \alpha_1 \gamma ax^2 + \alpha_2 \alpha_1 r bx - \alpha_2^2 rbx^2)z \]
\[ = (-\alpha_1^2 r - \alpha_1 \gamma ax^2 + \alpha_2 \alpha_1 r bx)z \]
\[ = \alpha_1^2 \alpha_2 r t - \alpha_1 \gamma rbx + \alpha_1^2 \gamma rbx \]
\[ - \alpha_1 \gamma \alpha_2 r bx^2 - \alpha_2 \alpha_1^2 rt + \alpha_2 \alpha_1 \gamma rbx^2 \]
\[ = 0. \]

We finally prove the nilpotency of \( L(r) \).

**Theorem 6.8.** \( L(r) \) is nilpotent of class 3 if \( r = 0 \) and of class 5 if \( r \neq 0 \).

**Proof.** Let \( r = 0 \). Then \( Z(L) = F ax^2 + F bx^2 + Ft \) by Lemma 6.3. Moreover
\[ L^2 = Lx + Ft \quad \text{and} \quad L^3 = Lx^2 + Ft = Z(L), \]
so that \( L(0) \) is nilpotent of class 3.

Assume \( r \neq 0 \). Then
\[ L^2 = \langle b, ax, bx, ax^2, bx^2, t \rangle, \quad L^3 = \langle b, bx, ax^2, bx^2, t \rangle \]
\[ L^4 = \langle bx, bx^2, t \rangle, \quad L^5 = \langle bx^2, t \rangle, \quad L^6 = \{0\}. \]

This proves that \( L(r) \) is nilpotent of class 5. \( \square \)

The parameter \( r \in F \) is not unique. Recall that \( r = (a, ax^2(ax)) \). Now \( Z_3(L) = (L^4)^{-1} = \langle b, bx, ax^2, bx^2, t \rangle \). Let
\[ \bar{a} = \alpha_1 a + \beta_1 ax + \gamma x + u \quad \text{and} \quad \bar{x} = \alpha_2 a + \beta_2 ax + \delta x + v \]
with \( u, v \in Z_3(L) \). Tediou but direct calculations show that
\[ (\bar{a}, \bar{a}x^2(\bar{a}x)) = (\alpha_1 \delta - \alpha_2 \gamma)^3 r. \]

This implies that for \( r, s \neq 0 \) we have that \( L(r) \cong L(s) \) if and only if \( r \) and \( s \)
are in the same coset of the abelian group \( F^*/(F^*)^3 \) (where \( F^* = F \setminus \{0\} \)).
Adding \( L(0) \), we see that there are up to isomorphism exactly \( |F^*/(F^*)^3| + 1 \)
symplectic alternating algebras of dimension 8 that are nil-3 but not nil-2.
If \( F \) is algebraically closed then this number is 2. As \((\mathbb{R}^*)^3 = \mathbb{R}\), this is also true when the underlying field is the field of real numbers. On the other hand, \( \mathbb{Q}^*/(\mathbb{Q}^*)^3 \) is infinite so over the rational field we have an infinite number of examples. If \( F \) is finite then \( F^* \) is cyclic and thus \( |F^*/(F^*)^3| \) is 1 or 3 depending on whether 3 divides \( |F| - 1 \) or not.
References

