Model order reduction by balanced proper orthogonal decomposition and by rational interpolation

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Abstract

We show that model order reduction by rational interpolation (also known as moment matching or rational Krylov) can be seen as the special case of balanced Proper Orthogonal Decomposition where all the snapshots are retained and particular numerical procedures are used to obtain the snapshots.

1 Introduction

Three popular methods for model order reduction are

1. Balanced truncation and its variations,
2. (Balanced) proper orthogonal decomposition (POD, also known as Karhunen–Loéve decomposition),
3. Rational interpolation (also known as moment matching, [rational] Krylov and Padé approximation).

It is well-known that balanced POD is an approximate version of balanced truncation (this is in fact already contained in the article [9] that introduced balanced truncation). Other approximate balanced truncation algorithms (such as the ADI method, low-rank Smith methods and approximating the Gramians by numerical quadrature of their frequency domain integral representation) have been connected to Krylov subspaces (see e.g. [8], [5], [10]). It is the objective of this note to point out a direct connection between balanced POD and model reduction by rational interpolation.

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For the convenience of the reader and to establish notation, in Sections 2.1 and 2.2 we very briefly review model order reduction by balanced POD and by rational interpolation. Section 3 contains the main results. We first (Section 3.1) consider the simplest case of interpolation at infinity and link this to balanced POD with snapshots obtained by forward Euler. We next (Section 3.2) consider the case of interpolation at a finite positive real point s_0 and connect this to balanced POD with snapshots obtained by backward Euler with stepsize $\frac{1}{s_0}$. We subsequently (Section 3.3) comment on interpolation at several distinct points. In principle the backward Euler connection extends to complex interpolation points if we allow for complex stepsizes. However, a slightly more satisfactory connection is obtained for complex interpolation points by considering multi-stage implicit numerical methods instead of backward Euler to generate the POD snapshots (Section 3.4). For simplicity of exposition, Sections 2 and 3 deal only with SISO systems. We comment on MIMO systems in Section 3.5. Finally, in Section 4 we consider two very simple examples that illustrate the connections made.

2 A very short review of model order reduction

Given the dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x^0, \quad y(t) = Cx(t) + Du(t),$$

with state space $\mathbb{R}^n$, input space $\mathbb{R}$ and output space $\mathbb{R}$ and a pair of operators $S : \mathbb{R}^n \to \mathbb{R}^k$ and $T : \mathbb{R}^k \to \mathbb{R}^n$ with $ST = I$ a reduced order system

$$\dot{x}_r(t) = A_r x_r(t) + B_r u_r(t), \quad x_r(0) = x_r^0, \quad y_r(t) = C_r x_r(t) + D_r u_r(t),$$

with state space $\mathbb{R}^k$, input space $\mathbb{R}$ and output space $\mathbb{R}$ is obtained by setting

$$\begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} := \begin{bmatrix} SAT & SB \\ CT & D \end{bmatrix}.$$

Thus the reduced order system is obtained by a Petrov–Galerkin projection. Note that the condition $ST = I$ implies that $(TS)^2 = T(ST)S = TS$ so that $TS : \mathbb{R}^n \to \mathbb{R}^n$ is indeed a projection.

Remark 1. For future reference note that if $Q : \mathbb{R}^k \to \mathbb{R}^k$ is a similarity transformation and if we define $\tilde{S} = QS$ and $\tilde{T} = TQ^{-1}$, then the pair $\tilde{S}$, $\tilde{T}$ is another Petrov–Galerkin pair whose reduced order system is related to the one obtained from the pair S, T by the similarity transformation Q. In particular, these reduced order systems have the same transfer function.

2.1 Model order reduction by balanced POD

Model order reduction by balanced POD is a Petrov–Galerkin projection method where the Petrov–Galerkin operators are defined as follows ([11], [12]):

1. Approximate the solution of

$$\dot{w}(t) = Aw(t), \quad w(0) = B, \tag{1}$$

   at time instances $\{t_i\}_{i=1}^N$, which gives the vector $\mathcal{B}_N := [\hat{w}(t_i)]_{i=1}^N \in \mathbb{R}^N$.

2. Approximate the solution of

$$\dot{z}(t) = A^* z(t), \quad z(0) = C^*, \tag{2}$$

   at time instances $\{t_i\}_{i=1}^N$, which gives the vector $\mathcal{C}_N := [\hat{z}(t_i)]_{i=1}^N \in \mathbb{R}^N$.

3. Form the empirical Hankel operator $\mathcal{H}_N : \mathbb{R}^N \to \mathbb{R}^N$ as

$$\mathcal{H}_N := \mathcal{C}_N^* \mathcal{B}_N.$$

4. Compute the Singular Value Decomposition of the empirical Hankel operator

$$\mathcal{H}_N = U \Sigma V^*,$$

   where without loss of generality we can assume that the diagonal elements of Σ are ordered in a decreasing manner. We have $U, \Sigma, V : \mathbb{R}^N \to \mathbb{R}^N$.

5. Decompose

$$\Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & \Sigma_e \end{bmatrix}, \quad U = [U_r,\ U_e], \quad V = [V_r,\ V_e],$$

   where $\Sigma_r : \mathbb{R}^k \to \mathbb{R}^k$, $V_r : \mathbb{R}^k \to \mathbb{R}^N$ and $U_r : \mathbb{R}^k \to \mathbb{R}^N$. We may assume without loss of generality that $\Sigma_r > 0$.

6. Form the operators

$$S := \Sigma_r^{-1/2} U_r^* \mathcal{C}_N^*, \quad T := \mathcal{B}_N V_r \Sigma_r^{-1/2}.$$

7. Form the reduced order system

$$\begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} := \begin{bmatrix} SAT & SB \\ CT & D \end{bmatrix}.$$

Remark 2. Note that S and T indeed form a Petrov–Galerkin pair since

$$\begin{aligned} ST &= \Sigma_r^{-1/2} U_r \mathcal{C}_N^* \mathcal{B}_N V_r^* \Sigma_r^{-1/2} \\ &= \Sigma_r^{-1/2} U_r^* \mathcal{H}_N V_r \Sigma_r^{-1/2} \\ &= \Sigma_r^{-1/2} U_r^* U \Sigma V^* V_r \Sigma_r^{-1/2} = I, \end{aligned}$$

where we have used that

$$\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = V^* V = \begin{bmatrix} V_r^* \\ V_e^* \end{bmatrix} [V_r,\ V_e] = \begin{bmatrix} V_r^* V_r & V_r^* V_e \\ V_e^* V_r & V_e^* V_e \end{bmatrix},$$

which shows that $V^* V_r = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$ and we have used that similarly $U_r^* U = [I, 0]$.

Remark 3. Note that the time instances $\{t_i\}_{i=1}^N$ at which snapshots are taken are user specified. The user also has to choose numerical methods to simulate the ODEs in steps 1 and 2. In the literature usually not much information is provided about these choices, especially the latter choice (of numerical method). When we consider the connection with model order reduction by rational interpolation we see that these choices are in fact crucial (mainly of course when relatively few snapshots are generated).

2.2 Model order reduction by rational interpolation

Model order reduction by rational interpolation is a Petrov–Galerkin projection method where the Petrov–Galerkin operators are defined as follows ([2, Chapter 11.3]):

1. Form the partial generalized controllability operators

$$\begin{aligned} \mathcal{R}_N(s) &:= [(sI-A)^{-1}B, \ldots, (sI-A)^{-N}B], \\ \mathcal{R}_N(\infty) &:= [B, AB, \ldots, A^{N-1}B], \end{aligned}$$

   and the partial generalized observability operators

$$\mathcal{O}_N(s) := \begin{bmatrix} C(sI-A)^{-1} \\ \vdots \\ C(sI-A)^{-N} \end{bmatrix}, \qquad \mathcal{O}_N(\infty) := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix}.$$

2. Form the operators

$$V := [\mathcal{R}_{k_1}(s_1), \ldots, \mathcal{R}_{k_m}(s_m)], \tag{3}$$

$$W := \begin{bmatrix} \mathcal{O}_{k_1}(s_{m+1}) \\ \cdots \\ \mathcal{O}_{k_m}(s_{2m}) \end{bmatrix}, \tag{4}$$

   where $s_i \in \mathbb{C} \cup \{\infty\}$ $(i = 1, \ldots, 2m)$ are not eigenvalues of A and $k_i \in \mathbb{N}$ $(i = 1, \ldots, m)$.

3. Assuming that WV is invertible define $Z := (WV)^{-1}W$.

4. Form the reduced order system

$$\begin{bmatrix} A_r & B_r \\ C_r & D_r \end{bmatrix} := \begin{bmatrix} ZAV & ZB \\ CV & D \end{bmatrix}.$$

The reduced order system has the following property (G is the transfer function of the full order system and G_r the transfer function of the reduced order system):

$$G_r^{(n)}(s_i) = G^{(n)}(s_i), \quad n = 0, \ldots, k_i - 1,$$

where $G^{(n)}(\infty)$ must be interpreted as the n-th derivative at zero of $\tilde{G}(z) := G(1/z)$.

If $s_i = s_{m+i}$ and $k_i = 1$ or if $m = 1$ and $s_1 = s_2 = \infty$, then the reduced order system in fact has the stronger property:

$$G_r^{(n)}(s_i) = G^{(n)}(s_i), \quad n = 0, \ldots, 2k_i - 1,$$

see [2, Section 11.3.1].

Remark 4. Note that the interpolation points $\{s_i\}_{i=1}^{2m}$ and their multiplicities k_i are user specified. It is known that generically any reduced order system can be obtained by the above procedure for appropriate choices of interpolation points [4], [2, Chapter 11.3.2]. Therefore, these choices are crucial.

3 Rational interpolation as balanced POD

We will show that model order reduction by rational interpolation is in fact model order reduction by balanced POD where the number of snapshots taken is equal to the dimension of the reduced order system, the numerical method (and the stepsize) used to obtain the snapshots is connected to the choice of interpolation points and actually no singular value decomposition is performed (all the snapshots are retained).

3.1 Interpolation at infinity

If we approximate the solution of the primal problem (1) using forward Euler with stepsize h then

$$\hat{w}(hi) = (I + hA)^i B.$$

So the resulting vector of snapshots is

$$\mathcal{B}_N = [B, (I + hA)B, \ldots, (I + hA)^{N-1}B].$$

Similarly applying forward Euler with stepsize h to the dual problem (2) results in the vector of snapshots

$$\mathcal{C}_N = [C^*, (I + hA^*)C^*, \ldots, (I + hA^*)^{N-1}C^*].$$

Using the upper triangular matrix M defined by

$$M_{ij} = \binom{j-1}{i-1} h^{i-1} \quad i = 1 \ldots, N, \ j = i, \ldots, N,$$

these vectors of snapshots can be written in terms of the partial controllability and observability operators as

$$\mathcal{B}_N = \mathcal{R}_N(\infty)M, \quad \mathcal{C}_N = \mathcal{O}_N(\infty)^* M.$$

It follows that the empirical Hankel operator satisfies

$$\mathcal{H}_N = M^* \mathcal{O}_N(\infty) \mathcal{R}_N(\infty) M. \tag{5}$$

Using the above we have that (since no reduction is performed) the POD Petrov–Galerkin operators are given by

$$S = \Sigma^{-1/2} U^* M^* \mathcal{O}_N(\infty), \quad T = \mathcal{R}_N(\infty) M V \Sigma^{-1/2}.$$

We now use Remark 1 with $Q = MV\Sigma^{-1/2}$ to obtain the equivalent Petrov–Galerkin pair (using (5) in the third equality)

$$\begin{aligned}
\tilde{S} &= MV\Sigma^{-1}U^* M^* \mathcal{O}_N(\infty) = M\mathcal{H}_N^{-1} M^* \mathcal{O}_N(\infty) = \left(\mathcal{O}_N(\infty)\mathcal{R}_N(\infty)\right)^{-1} \mathcal{O}_N(\infty),\\
\tilde{T} &= \mathcal{R}_N(\infty).
\end{aligned}$$

This we recognize as the Petrov–Galerkin pair corresponding to interpolation at infinity.

Recapitulating: taking snapshots at $\{hi\}_{i=0}^{N-1}$ using forward Euler with stepsize h and forming the POD reduced order system on the basis of all of these snapshots (so no reduction based on the singular value decomposition) gives the same reduced order system as interpolating the transfer function and its first $2N-1$ derivatives at infinity using the method described in Section 2.2.

3.2 Interpolation at a finite point

If we approximate the solution of the primal problem (1) using backward Euler with stepsize h then

$$\hat{w}(hi) = (I - hA)^{-i} B.$$

If we ignore the zero-th iterate $\hat{w}(0)$, then the resulting vector of snapshots is

$$\mathcal{B}_N = [(I - hA)^{-1}B, \ldots, (I - hA)^{-N}B].$$

Similarly applying backward Euler with stepsize h to the dual problem (2) results in the vector of snapshots

$$\mathcal{C}_N = [(I - hA^*)^{-1}C^*, \ldots, (I - hA^*)^{-N}C^*].$$

Using the diagonal matrix $\tilde{M}$ defined by

$$\tilde{M}_{ii} = h^{-i}, \quad i = 1 \ldots, N,$$

these vectors of snapshots can be written in terms of the partial generalized controllability and observability operators as

$$\mathcal{B}_N = \mathcal{R}_N(1/h)\tilde{M}, \quad \mathcal{C}_N = \mathcal{O}_N(1/h)^* \tilde{M}.$$

It follows that the empirical Hankel operator satisfies

$$\mathcal{H}_N = \tilde{M}^* \mathcal{O}_N(1/h) \mathcal{R}_N(1/h) \tilde{M}. \tag{6}$$

Completely analogously to the forward Euler case we then have

$$\tilde{S} = \left(\mathcal{O}_N(1/h)\mathcal{R}_N(1/h)\right)^{-1} \mathcal{O}_N(1/h),$$
$$\tilde{T} = \mathcal{R}_N(1/h).$$

This we recognize as the Petrov–Galerkin pair corresponding to interpolation at $s_0 = \frac{1}{h}$.

Recapitulating: taking snapshots at $\{hi\}_{i=1}^N$ using backward Euler with stepsize h and forming the balanced POD reduced order system on the basis of all of these snapshots (so no reduction based on the singular value decomposition) gives the same reduced order system as interpolating the transfer function and its first $N-1$ derivatives at $s_0 = \frac{1}{h}$ using the method described in Section 2.2.

3.3 Interpolation at multiple points

If we approximate the solution of the primal problem (1) using forward Euler with stepsize h_1 to obtain $\{\hat{w}_1(h_1 i)\}_{i=0}^{N_1-1}$ and using backward Euler with stepsizes h_j $(j = 2, \ldots, m)$ to obtain $\{\hat{w}_j(h_j i)\}_{i=1}^{N_j}$ and collect these in a vector $\mathcal{B}_N$, then similarly as before

$$\mathcal{B}_N = V\hat{M}_B,$$

where V is the matrix formed from partial generalized controllability matrices as in (3) and $\hat{M}_B$ is an invertible operator depending on the stepsizes h_j $(j = 1, \ldots, m)$.

If we approximate the solution of the dual problem (2) using forward Euler with stepsize h_{m+1} to obtain $\{\hat{z}_1(h_{m+1} i)\}_{i=0}^{N_{m+1}-1}$ and using backward Euler with stepsizes h_j $(j = m+2, \ldots, 2m)$ to obtain $\{\hat{z}_j(h_j i)\}_{i=1}^{N_j}$ and collect these in a vector $\mathcal{C}_N$, then similarly as before

$$\mathcal{C}_N = W^*\hat{M}_C,$$

where W is the matrix formed from partial generalized observability matrices as in (4) and $\hat{M}_C$ is an invertible operator depending on the stepsizes h_j $(j = m+1, \ldots, 2m)$.

Entirely analogously to before (noting that $Q := \hat{M}_B V \Sigma^{-1/2}$), it follows that the POD reduced order system formed on the basis of all of these snapshots (so no reduction based on the singular value decomposition) is the same reduced order system as the one obtained by interpolation of the transfer function at $s_1 = s_{m+1} = \infty$ and the points $s_j = \frac{1}{h_j}$ $(j = 2, \ldots, m,\ j = m+1, \ldots, 2m)$ with the relevant multiplicities using the method described in Section 2.2.

3.4 Interpolation at complex points

If we allow for complex stepsizes, then an interpretation of model reduction by interpolation at complex points in terms of backward Euler balanced POD can be obtained exactly as above. However, a more convincing (though still not very satisfactory) correspondence can be obtained by considering multi-stage implicit methods. We indicate below how the choice of interpolation points is related to the poles of what in the numerical ODE literature is called the *stability function* of the numerical method. We consider a typical example that can easily be adapted to similar situations. We first note that the stability function for forward Euler is $\phi(z) = 1 + z$, which has a single pole at infinity, and the stability function for backward Euler is $\phi(z) = \frac{1}{1-z}$, which has a single pole at a real point ($z = 1$ in fact). The connection with the positioning of the interpolation points in the complex plane should be apparent.

The typical example that we consider is Hammer–Hollingsworth [7] (see also [3]). With stepsize h this results in the approximate solution

$$\hat{w}(hi) = (12I + 6hA + h^2A^2)^i(12I - 6hA + h^2A^2)^{-i}B,$$

and similarly for the dual system. This is not quite enough to make the connection with rational interpolation, so we also consider an adapted version of Hammer–Hollingsworth with stepsize h (we note that this adapted version is not a very good numerical method), which results in

$$\hat{w}(hi) = (-12I + 6hA + h^2A^2)^i(12I - 6hA + h^2A^2)^{-i}B,$$

and similarly for the dual system. We now form the POD snapshot vectors by taking the zero-th iterate and first iterate of Hammer–Hollingsworth and the first iterate of the adapted Hammer–Hollingsworth:

$$\mathcal{B} = [B, (12I+6hA+h^2A^2)(12I-6hA+h^2A^2)^{-1}B, (-12I+6hA+h^2A^2)^i(12I-6hA+h^2A^2)^{-i}B].$$

It is easy to see that the range of $\mathcal{B}$ is the same as that of

$$\tilde{\mathcal{B}} = [B, (I - hz_0A)^{-1}B, (I - h\bar{z}_0A)^{-1}B],$$

where $z_0 = 3 + i\sqrt{3}$ and similarly for the dual system. It follows that balanced POD with these snapshots corresponds to the following multi-point rational interpolation: interpolation of $G'(\infty)$ and of $G(s_0)$ and $G(\bar{s}_0)$ with $s_0 = \frac{1}{h(3+i\sqrt{3})}$. We note that the stability function of Hammer–Hollingsworth is

$$\phi(z) = \frac{-12 + 6z + z^2}{12 - 6z + z^2},$$

which has simple poles at $3 \pm i\sqrt{3}$, i.e. the reciprocals of $\frac{1}{3\pm i\sqrt{3}}$, which explains the placing of the resulting interpolation points.

Interpolation in other pairs of complex conjugate points than those where one is on the line through $\frac{1}{(3+i\sqrt{3})}$ can be achieved by considering other two-stage implicit Runge–Kutta methods (we note that again these will generally not be very good methods). Interpolation at more than one pair of complex points can be achieved by combining the above with the idea in Section 3.3.

Remark 5. As remarked above, the position of the interpolation points in the complex plane is related to the stability function of the numerical method.

Implicit methods have stability functions with finite poles, explicit methods have stability functions with poles at infinity. A popular implicit Runge–Kutta method is Crank-Nicolson, which has stability function

$$\phi(z) = \frac{1 + z/2}{1 - z/2} = -1 + \frac{2}{1 - z/2}.$$

It is easy to see that balanced POD with snapshots obtained by Crank-Nicolson with stepsize $2h$ and balanced POD with snapshots obtained by backward Euler with stepsize h gives the same result provided that the zero-th iterate is included in both cases. Note that we *excluded* the zero-th iterate in the backward Euler case when making the connection with interpolation at $\frac{1}{h}$. Including the zero-th iterate means that we not only interpolate moments at $\frac{1}{h}$, but also one at infinity (this follows as in Section 3.3).

3.5 MIMO systems

Balanced POD for MIMO systems simply uses all of the columns of the B matrix subsequently as $w(0)$ for the primal system (and similarly for C^* for the dual system) and collects all the resulting snapshots for the different columns into the rows of a matrix $\mathcal{B}_N$ (respectively $\mathcal{C}_N$). The procedure than carries on as in the SISO case with forming the empirical Hankel operator $\mathcal{H}_N := \mathcal{C}_N^* \mathcal{B}_N$ and so on ([11], [12]). Note that if the number of inputs is not equal to the number of outputs, then $\mathcal{H}_N$ is not square.

Model order reduction by rational interpolation in the MIMO case is typically done by tangential interpolation ([6], [1]), i.e. for each interpolation point s_i ($i = 1, \dots, m$) there is a vector in the input space u_i such that the interpolation condition is $G(s_i)u_i = G_r(s_i)u_i$ and for each interpolation point s_i ($i = m+1, \dots, 2m$) there is a vector in the output space y_i such that the interpolation condition is $G(s_i)^* y_i = G_r(s_i)^* y_i$. Similar interpolation conditions can be put on the derivatives. The vectors u_i and y_i are user specified. To obtain such a reduced order model, in the algorithm described in Section 2.2 one simply replaces B when it occurs in combination with s_i by Bu_i and C^* when it occurs in combination with s_i by $C^* y_i$ (see [1]). Note that in this formulation, the resulting matrix WV (which is the interpolation equivalent of the empirical Hankel operator $\mathcal{H}_N$ in balanced POD) is forced to be square.

It is now easy to see that tangential interpolation corresponds to balanced POD (with the choice of step-size and numerical method as explained in the SISO case) where not all columns of B and C^* are used for obtaining the snapshots, but only some linear combinations of them (namely Bu_i and $C^* y_i$). Of course when, for a fixed s_i, $\{u_i\}$ forms a basis for the input space and $\{y_i\}$ forms a basis for the output space (this corresponds to matrix interpolation rather than 'pure' tangential interpolation), then in fact all columns are used (but note that this choice of vectors is only allowed in the interpolation framework if the number of inputs equals the number of outputs).

4 Examples

We illustrate the results by two very simple examples. They are not chosen to illustrate the model reduction procedures themselves (which are each amply illustrated in the literature), but only their connection. We first consider a SISO system and then a MISO system.

4.1 A SISO example

As the full-order model, we consider a standard (piecewise linear) finite element approximation of the following heat equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2}, \quad t > 0, \quad x \in (0,1),$$

$$w(0,x) = 0, \quad \frac{\partial w}{\partial x}(t,0) = u(t), \quad w(t,1) = 0, \quad y(t) = -w(t,0).$$

The four-dimensional finite element approximation that we will take as our full-order system has transfer function

$$G(s) = \frac{13.86s^3 + 2565s^2 + 109452s + 875615}{s^4 + 281.1s^3 + 21092s^2 + 401324s + 875615},$$

and realization

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{rrrr|r} -70.27 & 89.07 & -23.75 & 5.94 & -13.86 \\ 44.54 & -82.14 & 47.51 & -11.88 & 3.71 \\ -11.88 & 47.51 & -70.27 & 41.57 & -0.99 \\ 2.97 & -11.88 & 41.57 & -58.39 & 0.25 \\ \hline -1 & 0 & 0 & 0 & 0 \end{array}\right].$$

The two-dimensional approximation obtained from this by forward Euler has transfer function

$$G_{FE}(s) = \frac{13.86s + 1292}{s^2 + 189.2s + 4379},$$

and it can be checked that this indeed matches the first four moments at infinity. The realization obtain using forward Euler (in this case with $h = 2$) is

$$\left[\begin{array}{rr|r} -158.8 & -21.2 & -3.04 \\ -21.2 & -30.4 & 2.15 \\ \hline -3.04 & 2.15 & 0 \end{array}\right],$$

and the realization obtained by interpolation at infinity is

$$\left[\begin{array}{rr|r} 0 & -4379 & 1 \\ 1 & -189.2 & 0 \\ \hline 13.86 & -1329 & 0 \end{array}\right],$$

and it can be seen that these are indeed related by the similarity transformation

$$Q = MV\Sigma^{-1/2} = \begin{bmatrix} 0.0011 & 0.4664 \\ 0.0023 & 0.0032 \end{bmatrix}.$$

The two-dimensional approximation obtained from the full-order model by backward Euler with step size one has transfer function

$$G_{BE}(s) = \frac{9.228s + 101.4}{s^2 + 43.03s + 101.4},$$

and it can be checked that indeed $G^{k)}(1) = G_{BE}^{(k)}(1)$ for $k = 0, 1, 2, 3$. The realization obtain using backward Euler is

$$\left[\begin{array}{cc|c} -3.30 & -5.46 & -1.81 \\ -5.46 & -39.73 & -2.44 \\ \hline -1.81 & -2.44 & 0 \end{array}\right],$$

and the realization obtained by interpolation at one is

$$\left[\begin{array}{cc|c} -44 & -1 & 45 \\ 145 & 1 & -145 \\ \hline 0.76 & 0.17 & 0 \end{array}\right],$$

and it can be seen that these are indeed related by the similarity transformation

$$\tilde{Q} = \tilde{M}V\Sigma^{-1/2} = \begin{bmatrix} -2.24 & -16.79 \\ -0.63 & 60.05 \end{bmatrix}.$$

The three-dimensional approximation obtained from the full-order model by adapted Hammer-Hollingsworth as described above (which must have an odd order) with step size one has transfer function

$$G_{HH}(s) = \frac{13.86s^2 + 1138s + 10525}{s^3 + 178s^2 + 4645s + 10526},$$

and it can be checked that indeed

$$G\left(\frac{1}{3 + i\sqrt{3}}\right) = G_{HH}\left(\frac{1}{3 + i\sqrt{3}}\right) \text{ and } G\left(\frac{1}{3 - i\sqrt{3}}\right) = G_{HH}\left(\frac{1}{3 - i\sqrt{3}}\right),$$

and that the first two moments at infinity are matched. Similarly as above, the respective realizations (adapted Hammer-Hollingsworth and interpolation) are seen to be similar.

We now consider an illustration of Remark 5. The two-dimensional approximation obtained from the full-order model by backward Euler with step size one where also the zero-th iterate is used in balanced POD has transfer function

$$G_{BE0}(s) = \frac{13.86s + 335.7}{s^2 + 102.2s + 338.4}, \tag{7}$$

and it can be checked that indeed $G^{k)}(1) = G_1^{(k)}(1)$ for $k = 0, 1$ and that the first two moments at infinity are matched. The two-dimensional approximation obtained from the full-order model by Crank-Nicolson with step size two where also the zero-th iterate is used in balanced POD has transfer function (7). The two-dimensional approximation obtained from the full-order model by Crank-Nicolson with step size two where the zero-th iterate is not used in balanced POD however has transfer function

$$G_{CN}(s) = \frac{13.84s + 430.5}{s^2 + 124.9s + 510.8},$$

and this does not have any obvious interpolation properties.

For comparison with what from a numerical ODE perspective seems a more sensible thing to do, the two-dimensional approximation obtained from the full-order model by Hammer-Hollingsworth (not the adapted version) with step size one where the zero-th iterate is not used in balanced POD has transfer function

$$G_{HH1} = \frac{12.97s + 902.2}{s^2 + 162.8s + 3128},$$

and if we do include the zero-th iterate, then the transfer function is

$$G_{HH0} = \frac{13.86s + 407.6}{s^2 + 125.3s + 452.1}.$$

Again, these functions do not have obvious interpolation properties.

4.2 A MISO example

As the full-order model, we consider a standard (piecewise linear) finite element approximation of the following heat equation with two inputs (in comparison to the previous section, there is an additional point control) and one output

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + u_2(t)\delta_{2/3}(x), \quad t > 0, \quad x \in (0,1),$$

$$w(0,x) = 0, \quad \frac{\partial w}{\partial x}(t,0) = u_1(t), \quad w(t,1) = 0, \quad y(t) = -w(t,0).$$

The finite element approximation with a finite-element space of dimension four that we will take as our full-order system has transfer function $G = [G_1, G_2]$ with G_1 the full order transfer function from Section 4.1 and

$$G_2(s) = \frac{0.9897s^3 - 142.5s^2 + 437808}{s^4 + 281.1s^3 + 21092s^2 + 401324s + 875615}.$$

Balanced POD with snapshots obtained by backward Euler with stepsize $h = 1$ gives the four-dimensional approximation (here –contrary to the rest of this note– a singular value decomposition is performed, but it just omits the zero singular values which arise because the empirical Hankel matrix is not square)

$$G_{BE}(s) = \left[\frac{8.115s + 81.64}{s^2 + 35.36s + 81.64}, \frac{-1.046s + 40.82}{s^2 + 35.36s + 81.64}\right].$$

This cannot be obtained by tangential interpolation as described above, since the number of inputs (two) is not equal to the number of outputs (one). However, as can be checked $G(1) = G_{BE}(1)$ and $G'(1) = G'_{BE}(1)$.

If we consider tangential interpolation with $s_1 = s_2 = 1$, $u_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $u_2 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, $s_3 = 1$, $y_1 = 1$, $s_4 = 10$, $y_2 = 1$ then we obtain

$$G_{TI}(s) = \left[\frac{8.526s + 83.38}{s^2 + 36.53s + 83.27}, \frac{-0.9746s + 41.70}{s^2 + 36.53s + 83.27}\right],$$

and we see that $G(1) = G_{TI}(1)$, $G'(1) = G'_{TI}(1)$ and $G(10) = G_{TI}(10)$. The realization obtained is

$$\left[\begin{array}{rr|rr} -23.36 & 7.56 & 24.36 & -7.56 \\ 29.68 & -13.17 & -29.68 & 14.17 \\ \hline 0.76 & 0.34 & 0 & 0 \end{array}\right].$$

The same transfer function is obtained by applying backward Euler balanced POD (as always in this note: without reduction based on the singular value decomposition) with the following choices of snapshots. Stepsize $h = 1$ with the initial conditions Bu_1 and Bu_2 for the primal system and initial condition C^* for the dual system. Stepsize $h = \frac{1}{10}$ with initial condition C^* for the dual system. This gives the realization

$$\left[\begin{array}{rr|rr} -3.83 & 15.72 & -3.98 & -0.84 \\ 2.67 & -32.7 & 3.37 & -1.42 \\ \hline -1.04 & 1.3 & 0 & 0 \end{array}\right].$$

It can be checked that these two realizations are indeed related by the similarity transformation

$$\hat{Q} = \hat{M}_B V \Sigma^{-1/2} = \begin{bmatrix} -1.07 & 5.96 \\ -0.66 & -9.60 \end{bmatrix}.$$

5 Conclusion

As is known, model order reduction by balanced POD with a large number of snapshots approximates model order reduction by balanced truncation. In this paper we have shown that on the other hand model order reduction by balanced POD with the number of snapshots equal to the dimension of the reduced order model and assuming that the snapshots are obtained by certain numerical methods is the same as model order reduction by rational interpolation (the interpolation points depending on the numerical method used to generate the snapshots).

It is interesting to note that interpolation at infinity corresponds to an explicit method whereas interpolation at finite points corresponds to implicit methods. It is well-known that implicit methods are better for stiff equations (e.g. those arising from spatial discretization of partial differential equations) and this –together with the connection made here between balanced POD and rational interpolation– seems to explain the observation often made that interpolation at finite points is better for certain problems than interpolation at infinity.

References

[1] A. C. Antoulas, C.A. Beattie, and S. Gugercin. Interpolatory model reduction of large-scale dynamical systems. In *Efficient Modeling and Control of Large-Scale Systems*. Springer-Verlag, 2010.

[2] Athanasios C. Antoulas. *Approximation of large-scale dynamical systems*, volume 6 of *Advances in Design and Control*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2005.

[3] J. C. Butcher. Implicit Runge-Kutta processes. *Math. Comp.*, 18:50–64, 1964.

[4] K. Gallivan, A. Vandendorpe, and P. Van Dooren. Model reduction via truncation: an interpolation point of view. *Linear Algebra Appl.*, 375:115–134, 2003.

[5] K. Gallivan, A. Vandendorpe, and P. Van Dooren. Sylvester equations and projection-based model reduction. *J. Comput. Appl. Math.*, 162(1):213–229, 2004.

[6] K. Gallivan, A. Vandendorpe, and P. Van Dooren. Model reduction of MIMO systems via tangential interpolation. *SIAM J. Matrix Anal. Appl.*, 26(2):328–349 (electronic), 2004/05.

[7] Preston C. Hammer and Jack W. Hollingsworth. Trapezoidal methods of approximating solutions of differential equations. *Math. Tables Aids Comput.*, 9:92–96, 1955.

[8] Jing-Rebecca Li. *Model reduction of large linear systems via low rank system gramians*. PhD thesis, Massachusetts Institute of Technology, 2000.

[9] Bruce C. Moore. Principal component analysis in linear systems: controllability, observability, and model reduction. *IEEE Trans. Automat. Control*, 26(1):17–32, 1981.

[10] J.R. Phillips and L.M. Silveira. Poor mans tbr: a simple model reduction scheme. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 24:43–55, 2005.

[11] C. W. Rowley. Model reduction for fluids, using balanced proper orthogonal decomposition. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 15(3):997–1013, 2005.

[12] John R. Singler and Belinda A. Batten. A proper orthogonal decomposition approach to approximate balanced truncation of infinite dimensional linear systems. *Int. J. Comput. Math.*, 86(2):355–371, 2009.