Lexicographic Probabilities and Robustness*

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Abstract

We characterize lexicographic conditional probability systems (LCPSs). Our aim is to address an issue left open in an important contribution by Blume, Brandenburger, and Dekel (1991a). They provide a characterization of LCPSs, but one of their axioms quite explicitly imposes disjointness of the supports in an LCPS. The main new axiom is robustness, which is a weak continuity requirement on preferences. It requires preferences between acts x and y to be robust to (unchanged by) small perturbations in payoffs x_ω and y_ω for *some* state ω where x and y are different. LCPSs are characterized as the subclass of lexicographic probability systems satisfying robustness. As a corollary we provide an axiomatization of LCPSs in terms of standard properties satisfied by subjective expected utility preferences (as in Anscombe and Aumann (1963)) and robustness.

Keywords: Lexicographic probability systems, non-Archimedean preferences, subjective expected utility, epistemic game theory.

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1 Introduction

In the Anscombe and Aumann (1963) - framework, of decision making under risk and uncertainty, expected utility preferences satisfy the Archimedean axiom (Archimedeaneness). Roughly speaking, it requires there to be no act deemed infinitely better than any other act. An important implication of Archimedeaneness is hence that it precludes some events to be infinitely more likely than other events of *strictly positive* probability. In some applications, most notably in epistemic game theory, the Archimedean axiom is for this reason considered a demanding property.[1]

This paper is an attempt to address some issues involved in obtaining subjective expected utility (SEU) representations of non archimedean preferences. Blume, Brandenburger, and Dekel (1991a) gives an extensive treatment of non archimedean SEU preferences.[2] They essentially propose two different representations of non archimedean preferences: Lexicographic Probability Systems (LPSs) and Lexicographic Conditional Probability Systems (LCPSs). In an LPS a decision maker is assumed to have an ordered sequence of beliefs over acts. An act x is preferred to y if and only if the first belief that makes a difference (between acts) makes x look better than y. An LCPS is a slightly more restrictive representation requiring the beliefs to have disjoint supports. However, this later property is desirable both because of its use in epistemic game theory and also because imposing it avoids some of the unintuitive behavior consistent with general LPSs (see Blume et al. (1991a, p.70) for an elaboration on this issue).

Blume et al. (1991a) provide an axiomatization of LPSs using well known properties satisfied by SEU preferences and a weak version of the Archimedean axiom. However, and as argued in a recent contribution by Dekel, Friedenberg, and Siniscalchi (2016), their characterization of LCPSs lack a compelling behavioral foundation. They write: "First, while Blume, Brandenburger, and Dekel (1991a) provide an axiom that characterizes LCPSs within the class of LPSs, their axiom has a flavour of reverse-engineering: it says no more than the probabilities in the LPS have disjoint support; it offers no further normative or other appeal." In this paper we try to address this problem by providing an axiomatization of LCPSs such that disjointness of the probabilities arises as a consequence of the axioms, rather than being part of their formulation.

Our main new axiom is a robustness property of preferences. Intuitively, robustness requires preferences to be robust to small changes in state contingent payoffs. To illustrate, suppose that a DM strictly prefers an act x to y, then robustness demands that there is *some* state $\omega \in \Omega$, among states favouring x or y, such that perturbing the state contingent pay-

[1]See for instance Blume, Brandenburger, and Dekel (1991b), Brandenburger (1992), Mailath, Samuelson, and Swinkels (1997), Lee (2013, 2016), Catonini and De Vito (2014),Yang (2015).

[2]Non Archimedean preferences are also treated in Hausner and Wendel (1952), Fishburn (1971), Blume, Brandenburger, and Dekel (1989), LaValle and Fishburn (1991), Borie (2016).

offs x_ω and y_ω, of x and y, by a sufficiently small amount leaves the DM's overall preference unchanged. In other words, the DM's preference is *robust* to small changes in x_ω and y_ω. Technically, robustness can be seen as a weak continuity requirement since every continuous preference relation satisfies robustness. Roughly speaking, continuous preferences require robustness to small perturbations in coordinates $(x_\omega)_{\omega\in A}$ for *any* subset A of states, whereas robust preferences only require robustness to small perturbations in x_ω for *some* state ω.

So why does robustness imply the existence of an LPS with disjoint supports? The next example[3] illustrates some of our main ideas further:

Example 1.1. Consider a state space of cardinality two $\Omega = \{1,2\}$ and assume that acts are functions $x : \Omega \to [0,1]$. Let $(\mu^1,\dots,\mu^K)$ be a sequence of probability measures on Ω. Assume that $\succsim$ is a preference relation defined by $x \succsim y$ if and only if $\big(x(1)\mu^i(1) + x(2)\mu^i(2)\big) \geq_L \big(y(1)\mu^i(1) + y(2)\mu^i(2)\big)$.[4] In this setting robustness requires that if $x \succ y$ then there is an $\varepsilon > 0$ and a state ω with $x_\omega \neq y_\omega$ such that $(x'_\omega, x_{-\omega}) \succ (y'_\omega, y_{-\omega})$ for all $x'_\omega, y'_\omega \in [0,1]$ with $|x'_\omega - x_\omega| < \varepsilon$ and $|y'_\omega - y_\omega| < \varepsilon$.

Let us now see why imposing robustness on $\succsim$ implies that $\succsim$ is representable by an LCPS. Suppose first that neither state is infinitely more likely than the other (thus μ^1 is a full support probability measure). We will show that every act x has a certainty equivalent.[5] W.l.o.g. assume that $x(1) > x(2)$. Let $\hat\alpha$ satisfy $x(1)\mu^1(1) + x(2)\mu^1(2) = \hat\alpha(\mu^1(1) + \mu^1(2))$. Since $x(1) > x(2)$ it follows that $x(1) > \hat\alpha$ and $\hat\alpha > x(2)$. We show that robustness implies that $(x(1), x(2)) \sim (\hat\alpha, \hat\alpha)$. Suppose not, then $(x(1), x(2)) \succ (\hat\alpha, \hat\alpha)$ or $(\hat\alpha, \hat\alpha) \succ (x(1), x(2))$. We consider the case where $(x(1), x(2)) \succ (\hat\alpha, \hat\alpha)$. The other case follows using similar reasoning. By robustness there is a (small) $\varepsilon > 0$ such that either preference (1) or preference (2) holds:

$$(x(1), x(2)) \succ (\hat\alpha + \varepsilon, \hat\alpha) \tag{1}$$

$$(x(1), x(2)) \succ (\hat\alpha, \hat\alpha + \varepsilon) \tag{2}$$

But the above preferences imply that either of the following inequalities hold:

$$x(1)\mu^1(1) + x(2)\mu^1(2) \geq (\hat\alpha + \varepsilon)\mu^1(1) + \hat\alpha\mu^1(2)$$
$$x(1)\mu^1(1) + x(2)\mu^1(2) \geq \hat\alpha\mu^1(1) + (\hat\alpha + \varepsilon)\mu^1(2)$$

In both cases we have a contradiction, since in either case it follows that $x(1)\mu^1(1) + x(2)\mu^1(2) > \hat\alpha(\mu^1(1) + \mu^1(2))$. This shows that every act has a certainty equivalent and hence preferences are expected utility.

[3] Example 1.1 was suggested to me by an anonymous referee.

[4] $\geq_L$ denotes the standard lexicographic ordering on $\mathbb{R}^K$ defined by $x \geq_L y$ if and only if whenever $y_i > x_i$ there is a $j < i$ with $x_j > y_j$.

[5] α is a certainty equivalent of x if $x \sim (\alpha, \alpha)$.

Next suppose that one state is infinitely more likely than the other state. In this case robustness has no bite (as it is trivially satisfied) and it is readily verified that preferences $\succsim$ are represented by standard lexicographic preferences and hence are LCPS. $\lhd$

The preceding example suggests the following explanation of the robustness axiom (in the two-state case): when one state is infinitely more likely than the other state (case 2 above) then robustness has no bite. When both states are equally likely the robustness axiom reduces to the Archimedean axiom via assertion of certainty equivalents. Thus, in the two state case preferences are either lexicographic (and hence LCPS) or standard expected utility preferences.

Although instructive, the previous example does not give a complete picture of the implications of the robustness axiom. There are some further issues involved in extending the result to arbitrary state spaces. These issues are related to the restriction of robustness to states ω where x and y are different. In the two state case this restriction is unnecessary since LCPSs are characterized by a weaker form of robustness only requiring robustness to some state $\omega \in \Omega$, rather than to a state ω in the possibly smaller set $\{\omega : x(\omega) \neq y(\omega)\} \subseteq \Omega$.[6] Considering state spaces with more than two states this restriction becomes crucial. In section 3, following the definition of robustness, we argue that there are LPSs satisfying this weaker form of robustness that are not LCPSs.

The remainder of the paper investigates to what extent the logic of the previous examples carries over to a general setting. By suitably modifying the robustness axiom, our main result characterizes LCPSs as the subclass of LPSs satisfying robustness for general state spaces. We then move on to discuss an axiomatic foundation of LCPSs for finite state spaces and to compare our findings with those of Blume et al. (1991a).

The organization of the paper is as follows. Section 2 introduces notation and defines LCPSs. In section 3 we state and discuss the robustness axiom. Section 4 states our main result and also gives a brief proof outline. An axiomatization of LCPSs for finite state spaces is in section 5. A short discussion of related literature is in section 6. The proof of the main result is in appendix A. Finally, other versions of robustness are discussed in appendix B.

2 Setup

Let Ω be a state space equipped with a σ-algebra $\mathscr{B}$. The elements of $\mathscr{B}$ are called *measurable sets* or *events*. Let C denote a finite set of consequences and let $\Delta(C) \subseteq \mathbb{R}^d$ denote the set of lotteries on C. Equip $\Delta(C)$ with the subspace topology from $\mathbb{R}^d$. An *act* x is a

[6] This can be seen by inspection of example 1.1 above. Nowhere did we use that the robust state ω is such that $x(\omega) \neq y(\omega)$. We provide a direct proof of the equivalence of these two notions of robustness in proposition B.3 in appendix B.

measurable function[7] $x:\Omega\to\Delta(C)$ taking states to $\Delta(C)$. Let $x(\omega)$ denote the outcome in state ω. For every measurable set E and any acts x, y let (x_E, y_{-E}) denote the act defined by $(x_E, y_{-E})(\omega) = x(\omega)$ for all $\omega\in E$ and $(x_E, y_{-E})(\omega) = y(\omega)$ for all $\omega\in\Omega\backslash E$. An act x is a *constant act* if there is a $p\in\Delta(C)$ such that $x(\omega) = p$ for all $\omega\in\Omega$. Fix a lottery $x_0\in\Delta(C)$ and denote by $\mathbf{0}$ the constant act equal to x_0 for all $\omega\in\Omega$. We will abuse notation and let x_E denote the act $(x_E, \mathbf{0}_{-E})$. The set of all acts is denoted by X. A decision maker has preferences $\succsim$ on the set of acts X.

Definition 2.1. A Lexicographic Probability System (LPS) on Ω is a sequence of probability measures $(\mu^1,...,\mu^K)$ where each μ^i is a probability measure on Ω. An LPS $(\mu^1,...,\mu^K)$ is a Lexicographic Conditional Probability System (LCPS) if there are measurable sets $U^1,...,U^K$ such that $\mu^i(U^i) = 1$ for all $i\in\{1,...,K\}$ and such that $\mu^i(U^j) = 0$ for all distinct $i, j\in\{1,...,K\}$. $\triangleleft$

If Ω is a finite state space let $\operatorname{supp}(\mu) = \{\omega\in\Omega:\mu(\omega) > 0\}$ denote the support of a measure μ on Ω. For finite state spaces it follows that a sequence of measures $(\mu^1,...,\mu^K)$ is an LCPS if and only if the supports of the measures in $(\mu^1,...,\mu^K)$ are disjoint. We now come to the main definition studied in this paper:

Definition 2.2. A preference relation $\succsim$ on X is represented by an LPS (LCPS) if there is an LPS (LCPS) $(\mu^1,...,\mu^K)$ on Ω and a continuous function $u:\Delta(C)\to\mathbb{R}$ such that for all $x, y\in X$:

$$x\succsim y\Leftrightarrow\left(\int_\Omega u(x)d\mu^l\right)_{l=1}^K\geq_L\left(\int_\Omega u(y)d\mu^l\right)_{l=1}^K. \tag{3}$$

$\triangleleft$

An LPS $(\mu^1,...,\mu^K)$ representing $\succsim$ has minimal length K, if there is no LPS $(\nu^1,...,\nu^S)$ with $S<K$ and such that equation (3) holds.

3 Robustness

In this section we introduce and discuss the robustness axiom. Say that a measurable set E has *positive measure* if for all constant acts $x, y\in X$: $x\succ y$ implies $(x_E, z_{-E})\succ(y_E, z_{-E})$ for all $z\in X$. Let $D(x, y) = \{\omega\in\Omega: x(\omega)\neq y(\omega)\}$, i.e. $D(x, y)$ is the set of states where x and y are different.

> **Robustness** (ROB): For all $x, y\in X$, if $x\succ y$ then there is a measurable $E\subseteq D(x, y)$ with positive measure and an $\varepsilon>0$ such that $(x'_E, x_{-E})\succ(y'_E, y_{-E})$ for all $x'_E, y'_E\in X$ with $||x'_E - x_E|| < \varepsilon$ and $||y'_E - y_E|| < \varepsilon$.[8]

[7]A function $x:\Omega\to\Delta(C)$ is measurable if the inverse image of every open set $O\subseteq\Delta(C)$ is in $\mathscr{B}$, i.e. $x^{-1}(O)\in\mathscr{B}$ for all open sets $O\subseteq\Delta(C)$.

[8]The topology on $\Delta(C)$ is generated by the Euclidean metric $||(.)||_d$ on $\mathbb{R}^d$. For all $x, y\in X$ define $||x-y|| = \sup_{\omega\in\Omega}||x(\omega)-y(\omega)||_d$. Note that this function is well-defined as we have $||x(\omega)-y(\omega)||_d\leq\sqrt{2}$ for all $\omega\in\Omega$.

The definition above is well founded since if $x \succ y$ then $D(x, y)$ itself has positive measure.[9] To gain some intuition for this property consider the following formulation of robustness which is equivalent in the case of finite state spaces Ω.

> **Robustness**: For all $x, y \in X$, if $x \succ y$ then there is an $\omega \in D(x, y)$ and an $\varepsilon > 0$ such that $(x'_\omega, x_{-\omega}) \succ (y'_\omega, y_{-\omega})$ for all $x'_\omega, y'_\omega \in \Delta(C)$ with $||x'_\omega - x_\omega|| < \varepsilon$ and $||y'_\omega - y_\omega|| < \varepsilon$.

The "for some" quantifier in "there is an $\omega \in D(x, y)$..." and the restriction to the set $D(x, y) \subseteq \Omega$ in the formulation of robustness is crucial. Requiring an LPS to be robust with respect to some $\omega \in \Omega$, is not sufficient to characterize LCPSs. There are LPSs satisfying this weaker robustness property that are not LCPSs. Roughly speaking, this is because if we only require robustness w.r.t. some $\omega \in \Omega$ then if $x \succ y$ we can take the robust state $\omega \in \Omega$ to be in the support of a higher order belief with support outside $D(x, y)$. This is illustrated by example 3.1 below. The LPS in example 3.1 satisfies the weaker form of robustness but is not an LCPS.[10]

Suppose instead that we require robustness with respect to every state $\omega \in \Omega$ then, as shown in corollary B.2 in appendix B, it follows that the decision maker uses a subjective expected utility preference as in Anscombe and Aumann (1963). For similar reasons requiring robustness w.r.t. every $\omega \in D(x, y)$ is a too strong condition (see corollary B.1 in appendix B).

The following example shows that not every $\succsim$ represented by an LPS satisfies robustness. Thus robustness is logically independent from other properties characterizing preferences representable by an LPS.

Example 3.1. Assume that $\Omega = \{1, 2, 3\}$, $C = \{c_0, c_1\}$ and $u : \Delta(C) \to \mathbb{R}$ is an affine function (hence continuous) with $u(c_0) = 0$ and $u(c_1) = 1$. Also assume that $\mu^1 = (\frac{1}{2}, \frac{1}{2}, 0)$ and $\mu^2 = (0, \frac{1}{2}, \frac{1}{2})$. Let $\succsim$ be a preference relation defined on the set of all acts $x : \Omega \to \Delta(C)$ such that equation (3) holds. Since $u(c_i) = i$ for all i, each act x is representable as a vector $x = (x_1, x_2, x_3)$[11] where $x_i \in [0, 1]$ is the probability put on consequence c_1. We claim that $\succsim$ violates robustness. Let $x = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ and $y = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3})$; then $x \succ y$. But for all $\frac{1}{3} \geq \varepsilon > 0$ we have $(\frac{2}{3}, \frac{1}{3} + \varepsilon, \frac{2}{3}) \succ x$ and $y \succ (\frac{1}{3} - \varepsilon, \frac{2}{3}, \frac{2}{3})$. A contradiction to robustness. $\triangleleft$

The following lemma shows that every $\succsim$ representable by an LCPS satisfies robustness.

Lemma 3.1. *If a preference relation $\succsim$ is representable by an LCPS $(\mu^1, \ldots, \mu^K)$ then $\succsim$ satisfies robustness.*

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[9] To be precise, the other axioms characterizing an LPS assures that $D(x, y)$ has positive measure.

[10] To see this note that it is possible to perturb x and y in state 3 without changing overall preferences.

[11] Thus we use $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ as a shorthand notation for $((\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{1}{3}))$ and similarly for other acts in this example.

Proof. Let $U^1, \ldots, U^K$ be measurable sets such that $\mu^i(U^i) = 1$ for all $i \in \{1, \ldots, K\}$ and such that $\mu^i(U^j) = 0$ for all distinct $i, j \in \{1, \ldots, K\}$. Let $x, y \in X$ and $x \succ y$ and let $l \in \{1, \ldots, K\}$ be the smallest number such that $\int_\Omega u(x) d\mu^l > \int_\Omega u(y) d\mu^l$. Then $\mu^l(D(x,y)) > 0$. Set $E = D(x,y) \cap U^l$, then $\mu^l(E) = \mu^l(D(x,y) \cap U^l) + \mu^l(D(x,y) \cap (\Omega \setminus U^l)) = \mu^l(D(x,y)) > 0$. Let $c = \int_E u(x) d\mu^l - \int_E u(y) d\mu^l > 0$. Choose $\varepsilon_0 > 0$ such that $2\varepsilon_0 < c$. Then there is an $\varepsilon > 0$ such that $||x'_E - x_E|| < \varepsilon$ and $||y'_E - y_E|| < \varepsilon$ implies that $|u(x(\omega)) - u(x'(\omega))| < \varepsilon_0$ and $|u(y(\omega)) - u(y'(\omega))| < \varepsilon_0$ for all $\omega \in \Omega$ (follows since u is uniformly continuous on $\Delta(C)$). Note that $\int_{\Omega \setminus E} u(x) d\mu^l = \int_{(\Omega \setminus D(x,y)) \cup (\Omega \setminus U^l)} u(x) d\mu^l = \int_{\Omega \setminus D(x,y)} u(x) d\mu^l = \int_{\Omega \setminus D(x,y)} u(y) d\mu^l = \int_{\Omega \setminus E} u(y) d\mu^l$. Thus it follows that:

$$\int_E u(x') d\mu^l + \int_{\Omega \setminus E} u(x) d\mu^l > \int_E (u(x) - \varepsilon_0) d\mu^l + \int_{\Omega \setminus E} u(x) d\mu^l =$$

$$\int_E u(x) d\mu^l + \int_{\Omega \setminus E} u(x) d\mu^l - \varepsilon_0 \mu^l(E) > \int_E u(y) d\mu^l + \int_{\Omega \setminus E} u(y) d\mu^l + c - \varepsilon_0 \mu^l(E) >$$

$$\int_E (u(y') - \varepsilon_0) d\mu^l + \int_{\Omega \setminus E} u(y) d\mu^l + c - \varepsilon_0 \mu^l(E) =$$

$$\int_E u(y') d\mu^l + \int_{\Omega \setminus E} u(y) d\mu^l + c - 2\varepsilon_0 \mu^l(E) > \int_E u(y') d\mu^l + \int_{\Omega \setminus E} u(y) d\mu^l.$$

Further, since $E \subseteq U^l$ and $\mu^k(U^l) = 0$ for all $k < l$ it follows that $\int_E u(x') d\mu^k + \int_{\Omega \setminus E} u(x) d\mu^k = \int_E u(y') d\mu^k + \int_{\Omega \setminus E} u(y) d\mu^k$ for all $k < l$. Hence $(x'_E, x_{-E}) \succ (y'_E, y_{-E})$ whenever $||x'_E - x_E|| < \varepsilon$ and $||y'_E - y_E|| < \varepsilon$. □

4 The main result

We next present and discuss the main result of the paper. We also give a brief proof outline.

Theorem 4.1. *For a preference relation $\succsim$ on X the following statements are equivalent:*

a) There is a continuous function $u : \Delta(C) \to \mathbb{R}$ and an LCPS $(\mu^1, \ldots, \mu^K)$ such that equation (3) holds.

b) There is a continuous function $u : \Delta(C) \to \mathbb{R}$ and an LPS $(\mu^1, \ldots, \mu^K)$ such that equation (3) holds and $\succsim$ satisfies robustness ROB.

Remark: By inspecting the proof of theorem 4.1 it follows that if a preference relation satisfies robustness and equation (3) with utility function $u : \Delta(C) \to \mathbb{R}$ and LPS $(\mu^1, \ldots, \mu^K)$ then $\succsim$ satisfies equation (3) with a possibly different LCPS $(\nu^1, \ldots, \nu^K)$ but with *the same* utility function $u : \Delta(C) \to \mathbb{R}$.

The proof of theorem 4.1 is in appendix A. The basic idea of proof is the following. To ease the exposition, let us consider a finite state space Ω. Assume that $\succsim$ is represented by an

LPS $(\mu^1,...,\mu^K)$ such that definition 2.2 holds. If each μ^i was unique then it would suffice to show that the supports of $\mu^1,...,\mu^K$ are pairwise disjoint. However, as explained in Blume et al. (1991a), each μ^k for $k \in \{1,...,K\}$ is only unique up to linear combinations of the form $\mu^k = \sum_{i=1}^k \alpha^i \mu^i$ where the α^i are such that $\sum_{i=1}^k \alpha^i \mu^i$ is a probability distribution and $\alpha^k > 0$. Thus given an LPS $(\mu^1,...,\mu^K)$ representing $\succsim$, we successively, by an induction argument, transform the measures in $(\mu^1,...,\mu^K)$ to obtain a new sequence of measures with disjoint supports. Note that the supports of the measures of an LPS satisfying robustness may fail to be disjoint, although $\succsim$ could be represented by such a set of measures. The following example illustrates this point further:

Example 4.1. Assume that $\Omega = \{1,2,3\}$, $C = \{c_0, c_1\}$ and $u : \Delta(C) \to \mathbb{R}$ is affine with $u(c_i) = i$. Let $\mu^1 = (1,0,0)$, $\mu^2 = (\frac{1}{2},\frac{1}{2},0)$ and $\mu^3 = (\frac{1}{2},0,\frac{1}{2})$, so the supports of the measures are not disjoint. Also, let $\succsim$ be a preference relation satisfying equation (3). Then (μ^1,μ^2,μ^3) is not an LCPS. However, the LCPS with $\mu'^1 = (1,0,0)$, $\mu'^2 = (0,1,0)$ and $\mu'^3 = (0,0,1)$ represents $\succsim$. $\triangleleft$

So if imposing robustness on an LPS does not give disjointness of the supports of the measures in an LPS, how are the measures related? Roughly speaking, robustness implies that if the supports of a lower order and a higher order belief in a lexicographic system are overlapping then the support of the lower order belief must be contained in the support of the higher order belief. Furthermore, the restriction of the higher order belief to the support of the lower order belief must be identical to the lower order belief. Robustness implies that the system of probability measures $(\mu^1,...,\mu^K)$ forms a "martingale sequence" with respect to their supports $\Pi^1,...,\Pi^K$: I.e., robustness implies that $\mu^k(\omega|\Pi^j) = \mu^j(\omega)$ for all $k > j$.

5 An axiomatization of LCPSs

For the remainder of this paper (section 5 and section 6) we will restrict attention to finite state spaces Ω. This restriction allows us to obtain an axiomatization of LCPSs in terms of robustness and other properties satisfied by SEU preferences. Further, it allows a comparison to Blume et al. (1991a) who provides an axiomatization of LCPSs for finite state spaces. The following axioms will be used in our axiomatization of LCPSs:

A1 *Weak order*: $\succsim$ is a weak order[12] on X.

A2 *Independence*: for all $x,y,z \in X$ and $\alpha \in (0,1)$, if $x \succ y$ $(x \sim y)$ then $\alpha x + (1-\alpha)z \succ \alpha y + (1-\alpha)z$ $(\alpha x + (1-\alpha)z \sim \alpha y + (1-\alpha)z)$.

A3 *Nontriviality*: There are acts $x, y \in X$ such that $x \succ y$.

[12]A weak order $\succsim$ on X is a complete and transitive binary relation. A binary relation $\succsim$ is *complete* if for all $x,y \in X$: $x \succsim y$ or $y \succsim x$, and it is *transitive* if for all $x,y,z \in X$: $x \succsim y$ and $y \succsim z$ imply $x \succsim z$.

For each $A \subseteq \Omega$ define a conditional relation $\succsim_A$ by $x \succsim_A y$ if and only if $(x_A, z_{-A}) \succsim (y_A, z_{-A})$ for some $z \in X$. It follows by axiom A1 and A2 that $\succsim_A$ is independent of the choice of z.[13]

A4 *State independence:* For any two states $\omega, \omega' \in \Omega$ and any two constant acts $x, y \in X$ it holds that $x \succsim_\omega y$ if and only if $x \succsim_{\omega'} y$.

Axiom A1 - A3 are from Blume et al. (1991a) and they are standard in a literature on decision making under uncertainty. The state independence axiom A4 is imposed in Blume et al. (1991a) in order for their characterization of LCPSs to make sense. Essentially, axiom A4 rules out savage null sets.

In Blume et al. (1991a) preferences representable by an LPS are characterized by assuming the Conditional Archimedean axiom A5' in addition to axiom A1 - A4. Technically, axiom A5' is a weakening of the classical Archimedean axiom, only requiring it to hold conditional on states ω, i.e. for $\succsim$ restricted to $\succsim_\omega$:

A5' *Conditional Archimedean property*: for all $x, y, z \in X$ and for each $\omega \in \Omega$, if $x \succ_\omega y \succ_\omega z$ then there are $0 < \alpha < \beta < 1$ such that $\beta x + (1-\beta) z \succ_\omega y \succ_\omega \alpha x + (1-\alpha) z$.

The following lemma shows that robustness is a stronger axiom than (implies) axiom A5'. Thus, by theorem 3.1 in Blume et al. (1991a, p. 66), axiom A1 - A4 together with robustness imply that $\succsim$ is representable by an LPS.

Lemma 5.1. *Robustness ROB implies the Conditional Archimedean property A5'.*

Proof. Let $x, y, z \in X$ and $\omega \in \Omega$ be such that $x \succ_\omega y \succ_\omega z$. Since $x \succ_\omega y$ we have $(x_\omega, u_{-\omega}) \succ (y_\omega, u_{-\omega})$ for all $u \in X$. Fix $u \in X$, by robustness there is an $\varepsilon > 0$ such that $(x'_\omega, u_{-\omega}) \succ (y'_\omega, u_{-\omega})$ for all $x'_\omega, y'_\omega \in \Delta(C)$ with $||x'_\omega - x_\omega|| < \varepsilon$ and $||y'_\omega - y_\omega|| < \varepsilon$. Note that $\varepsilon > 0$ above is independent of $u \in X$ since by the independence axiom we have for all $x, y, u, v \in X$: $(x_\omega, u_{-\omega}) \succ (y_\omega, u_{-\omega})$ if and only if $(x_\omega, v_{-\omega}) \succ (y_\omega, v_{-\omega})$. Let $\beta \in (0,1)$ be such that $(1-\beta)||z_\omega - x_\omega|| < \varepsilon$ then it follows that $||\beta x_\omega + (1-\beta) z_\omega - x_\omega|| = (1-\beta)||z_\omega - x_\omega|| < \varepsilon$. Hence $\beta x + (1-\beta) z \succ_\omega y$ by previous reasoning.

Since $y \succ_\omega z$ we have $(y_\omega, u_{-\omega}) \succ (z_\omega, u_{-\omega})$ for all $u \in X$. By robustness there is an $\varepsilon > 0$ such that $(y'_\omega, u_{-\omega}) \succ (z'_\omega, u_{-\omega})$ for all $u \in X$ and for all $y'_\omega, z'_\omega \in \Delta(C)$ with $||y'_\omega - y_\omega|| < \varepsilon$ and $||z'_\omega - z_\omega|| < \varepsilon$. Let $\alpha \in (0, \beta)$ be such that $\alpha ||z_\omega - x_\omega|| < \varepsilon$ then it follows that $||\alpha x_\omega + (1-\alpha) z_\omega - z_\omega|| = \alpha ||x_\omega - z_\omega|| < \varepsilon$. Hence $y \succ_\omega \alpha x + (1-\alpha) z$ and the lemma follows. □

Using lemma 5.1 we obtain the following axiomatization of LCPSs (for finite state spaces) as a corollary to theorem 3.1 in Blume et al. (1991a, p. 66) and theorem 4.1.

[13] To see this, suppose that $(x_A, z_{-A}) \succsim (y_A, z_{-A})$ but $(y_A, w_{-A}) \succ (x_A, w_{-A})$. Then independence gives $\frac{1}{2}(x_A, w_{-A}) + \frac{1}{2}(x_A, z_{-A}) \succsim \frac{1}{2}(x_A, w_{-A}) + \frac{1}{2}(y_A, z_{-A})$ and $\frac{1}{2}(x_A, z_{-A}) + \frac{1}{2}(y_A, w_{-A}) \succ \frac{1}{2}(x_A, z_{-A}) + \frac{1}{2}(x_A, w_{-A})$. But transitivity then implies that $\frac{1}{2}(x_A, z_{-A}) + \frac{1}{2}(x_A, w_{-A}) \succ \frac{1}{2}(x_A, z_{-A}) + \frac{1}{2}(x_A, w_{-A})$. This contradicts completeness (reflexivity) of $\succsim$.

Theorem 5.2. *For a preference relation $\succsim$ on X the following statements are equivalent:*

a) There is an affine function $u : \Delta(C) \to \mathbb{R}$ and an LCPS $(\mu^1, ..., \mu^K)$ on Ω such that for all $x, y \in X$:

$$x \succsim y \Leftrightarrow \left(\sum_{\omega\in\Omega} u(x_\omega)\mu^l(\omega)\right)_{l=1}^K \geq_L \left(\sum_{\omega\in\Omega} u(y_\omega)\mu^l(\omega)\right)_{l=1}^K. \tag{4}$$

b) $\succsim$ satisfies axiom A1 - A4 and ROB.

Furthermore $u : \Delta(C) \to \mathbb{R}$ is unique up to positive affine transformation and, among LCPSs $(\mu_1, ..., \mu_K)$ of minimal length K, $(\mu_1, ..., \mu_K)$ is unique. For each $\omega \in \Omega$ there is a $k \in \{1, ..., K\}$ such that $\mu^k(\omega) > 0$.

Proof. Let $\succsim$ satisfy axiom A1 - A4 and ROB. Then it follows by lemma 5.1 that $\succsim$ satisfies axiom A5'. Hence by theorem 3.1 in Blume et al. (1991a, p. 66) it follows that $\succsim$ is representable by an LPS $(\mu^1, ..., \mu^K)$ with an affine $u : \Delta(C) \to \mathbb{R}$. But every affine function is continuous and hence it follows by theorem 4.1 and the remark following theorem 4.1 that $\succsim$ is representable by an LCPS with affine $u : \Delta(C) \to \mathbb{R}$. The converse is straightforward. Finally, note that the uniqueness properties of the LCPS $(\mu^1, ..., \mu^K)$ and u follow by (Blume et al., 1991a, Thm. 5.3, p. 71). □

6 Relation to the literature

The only characterization of LCPSs that we are aware of is (Blume et al., 1991a, Thm 5.3). However, as already mentioned, they assume an axiom that quite explicitly imposes disjointness of the supports in the LPS. They define a relation $\gg$ on subsets of events, where $S \gg T$ is interpreted as "S is infinitely more likely than T". Formally, for disjoint events $S, T \subseteq \Omega$ and $S \neq \emptyset$,$\gg$ is defined by $S \gg T$ if for all $x, y, u, v \in X$: $x \succ_S y$ implies that $(x_{-T}, u_T) \succ_{S\cup T} (y_{-T}, v_T)$. Instead of robustness they impose the following condition:

A5” There is a partition $\{\Pi^1, ..., \Pi^K\}$ of Ω such that: (a) for each k, if $x \succ_{\Pi^k} y \succ_{\Pi^k} z$ then there exists $0 < \alpha < \beta < 1$ such that $\beta x + (1-\beta)z \succ_{\Pi^k} y \succ_{\Pi^k} \alpha x + (1-\alpha)z$; (b) $\Pi^1 \gg ... \gg \Pi^K$.

Theorem 5.3 in Blume et al. (1991a) shows that axiom A1 - A4 and A5” implies the existence of an LCPS $(\mu^1, ..., \mu^K)$ where each μ^i has support Π^i. Since the partition $\Pi^1, ..., \Pi^K$ is part of the formulation of axiom A5”, using axiom A5” gives a characterization of a subclass of LCPSs, namely the subclass with partition exactly equal to that given by $\Pi^1, ..., \Pi^K$. In this way $\Pi^1, ..., \Pi^K$ is exogenously given in Blume et al. (1991a). In contrast the partition $\Pi^1, ..., \Pi^K$ is determined endogenously, as a consequence of axiom A1 -A4 and ROB, in our model.

The robustness condition proposed here is related to, but distinct from, other robustness conditions proposed to model standard lexicographic preferences. Petri and Voorneveld (2016) characterize lexicographic preferences by assuming a condition they call robustness. Let $P(x, y) = \{\omega \in \Omega : x \succ_\omega y\}$. Their condition is as follows:

B1 For all $x, y \in X$, if $x \succ y$, $P(y, x) \neq \emptyset$, and $|P(x, y)|/|P(y, x)| \geq 2$,

then there is an $\omega \in P(x, y)$ with $(z_\omega, x_{-\omega}) \succsim (z_\omega, y_{-\omega})$ for all $z_\omega \in \Delta(C)$.

In a sense condition B1 is much stronger than robustness ROB. Condition B1 requires preferences to be robust to potentially large changes in state contingent payoffs x_ω and y_ω, whereas robustness ROB only allows for small changes in x_ω and y_ω. Moreover, B1 demands that preferences be robust to changes in $\omega \in P(x, y)$. In our setting changes in x_ω and y_ω are small and there is no reason why $\omega \in P(y, x)$ should be excluded.

Goswami, Mitra, and Sen (2018) characterize lexicographic preferences on $\mathbb{R}^n$. One of their axioms is called mild continuity and is is formulated as follows:

C1 For all $x, y \in \mathbb{R}^n$, and $S \subseteq \{i \in \{1, ..., n\} : x_i \neq y_i\}$ with $|S| = 2$. If $x \succ_S y$,

then there is an $\varepsilon > 0$ with $x' \succ_S y'$ for all $x' \in B(x, \varepsilon)$ and $y' \in B(y, \varepsilon)$.

Mild continuity is related to, but fundamentally different from, our robustness axiom. Note that mild continuity imposes restrictions on subsets S of $\{1, ..., n\}$ with $|S| = 2$. In particular, mild continuity does not imply the conditional Archimedean axiom. As a consequence, robustness cannot be replaced by mild continuity in an axiomatization of LCPSs. To see this, suppose that there is only one state ω and a set of consequences C. Define a relation $\succsim = \succsim_\omega$ by $p \succsim_\omega q$ if and only if $p \geq_L q$ for all $p, q \in \Delta(C)$. Then $\succsim$ satisfies axiom A1-A4. Further, note that $\succsim$ is mildly continuous as there is only one state ω. However, $\succsim$ does not satisfy the conditional Archimedean axiom: there is only one state ω and $\succsim_\omega$ is clearly not continuous.

A further advantage of robustness is that it generalizes fairly straightforwardly to uncountable state spaces Ω. Mild continuity is formulated for finite state spaces and due to it being restricted to binary subsets S (i.e. $|S| = 2$) it is not clear how to generalize this axiom further. A natural generalization is the following:

C2 For all $x, y \in X$, and $S \subseteq D(x, y)$ with positive measure. If $x \succ_S y$,

then there is an $\varepsilon > 0$ with $x' \succ_S y'$ for all $x' \in B(x, \varepsilon)$ and $y' \in B(y, \varepsilon)$.

But as the next example illustrates there are LCPSs that fail to satisfy this axiom.

Example 6.1. Assume that $\Omega = [0,1]$ and let $\succsim$ be an LCPS with uniform beliefs: μ_1 on $[0,\frac{1}{2}]$ and μ_2 on $[\frac{1}{2},1]$. Let acts x and y be defined by:

$$x(\omega) = \begin{cases} 0 & \text{if } \omega \in [0,\frac{1}{4}] \\ 1 & \text{if } \omega \in [\frac{1}{4},\frac{1}{2}] \\ \frac{1}{2} & \text{if } \omega \in [\frac{1}{2},1] \end{cases} \qquad y(\omega) = \begin{cases} 1 & \text{if } \omega \in [0,\frac{1}{4}] \\ 0 & \text{if } \omega \in [\frac{1}{4},\frac{1}{2}] \\ 0 & \text{if } \omega \in [\frac{1}{2},1] \end{cases}$$

Then $x \succ y$ and $D(x,y) = \Omega$. Axiom C2 then requires that there is an $\varepsilon > 0$ such that $x' \succ y'$ for all $x' \in B(x,\varepsilon)$ and $y' \in B(y,\varepsilon)$. But this is not the case as for every $\varepsilon > 0$ we have $y \succ x'$ where x' is defined by:

$$x'(\omega) = \begin{cases} 0 & \text{if } \omega \in [0,\frac{1}{4}] \\ 1-\varepsilon & \text{if } \omega \in [\frac{1}{4},\frac{1}{2}] \\ \frac{1}{2} & \text{if } \omega \in [\frac{1}{2},1] \end{cases}$$

◁

A Appendix: Proof of theorem 4.1

Proof. Suppose that there is an LPS $(\mu^1,\dots,\mu^K)$ on Ω and a continuous function $u : \Delta(C) \to \mathbb{R}$ such that for all $x, y \in X$:

$$x \succsim y \Leftrightarrow \left(\int_\Omega u(x)d\mu^l\right)_{l=1}^K \geq_L \left(\int_\Omega u(y)d\mu^l\right)_{l=1}^K. \tag{5}$$

We assume that $(\mu^1,\dots,\mu^K)$ is of *minimal length*. I.e. there is no LPS $(\mu^1,\dots,\mu^S)$ on Ω with $S < K$ such that equation (5) holds. Our aim is to show that there is an LPS $(\mu^1,\dots,\mu^K)$ representing $\succsim$ and there are measurable sets $U^1,\dots,U^K$ such that $\mu^i(U^i) = 1$ for all i and $\mu^i(U^j) = 0$ for all distinct $i, j \in \{1,\dots,K\}$. We prove the claim by an induction argument. More precisely, it is clear that the proof of the main theorem follows from the following claim:

Claim A.1. *For every $m \in \mathbb{N}$ there is an LPS $(\mu^1,\dots,\mu^K)$ such that equation (5) holds and there are measurable sets $U^1,\dots,U^K$ such that $\mu^i(U^i) = 1$ for all $i \in \{1,\dots,K\}$ and $\mu^i(U^j) = \mu^j(U^i) = 0$ for all $i, j \in \{1,\dots,K\}$ with $i < m$ and $j \neq i$.*

The base case $m = 1$ follows by taking $U^i = \Omega$ for all $i \in \{1,\dots,n\}$ and by noting that $\mu^i(U^j) = \mu^j(U^i) = 0$ for all $i, j \in \{1,\dots,n\}$ with $i < 1$ vacuously holds. Let $m \in \mathbb{N}$. As induction hypothesis assume:

There is an LPS $(\mu^1,\dots,\mu^K)$ s.t. equation (5) holds and are measurable sets $U^1,\dots,U^K$ (IH) with $\mu^i(U^i) = 1$ and $\mu^i(U^j) = \mu^j(U^i) = 0$ for all $i, j \in \{1, ..., K\}$ with $i < m$ and $j \neq i$.

The remainder of the proof is devoted to proving the induction step, i.e. that claim A.1 holds for $m+1$. The proof is broken up into several lemmas.

The following lemmas give the existence of lotteries in $\Delta(C)$ with desirable properties. We will use these results frequently below.

Lemma A.2. *There are $p_*, p^* \in \Delta(C)$ with $u(p_*) < u(p^*)$ and such that $u(p_*) \leq u(p) \leq u(p^*)$ for all $p \in \Delta(C)$. Moreover, if $r \in [u(p_*), u(p^*)]$ then there is a $p \in \Delta(C)$ with $u(p) = r$.*

Proof. Since $u : \Delta(C) \to \mathbb{R}$ is a continuous non-constant function on $\Delta(C)$ there are $p_*, p^* \in \Delta(C)$ with $u(p_*) < u(p^*)$ and such that $u(p_*) \leq u(p) \leq u(p^*)$ for all $p \in \Delta(C)$. Let $r \in [u(p_*), u(p^*)]$. Define a function $v(\alpha) = u(\alpha p_* + (1-\alpha)p^*)$. Note that v is a continuous function of α. It follows by the intermediate value theorem that there is an $\alpha' \in [0,1]$ with $v(\alpha') = r$. Since $\alpha' p_* + (1-\alpha')p^* \in \Delta(C)$ the claim follows. □

For the remainder of the proof fix two lotteries $p_*, p^* \in \Delta(C)$ as given by lemma A.2.

Lemma A.3. *Let $\alpha \in (0,1)$ and $\beta \neq \alpha$. Then there are lotteries $a, b, c \in \Delta(C)$ with $u(p^*) > u(a) > u(b) > u(c) > u(p_*)$ such that*

$$\alpha u(a) + (1-\alpha)u(c) = u(b) \tag{6}$$

$$\beta u(a) + (1-\beta)u(c) \neq u(b). \tag{7}$$

Proof. Let $r \in \mathbb{R}$ be such that $u(p^*) > r > u(p_*)$. By lemma A.2 there is a lottery $b \in \Delta(C)$ with $u(b) = r$. Let $\varepsilon > 0$ be small enough such that $u(p^*) > u(b) + \varepsilon\frac{1-\alpha}{\alpha} > u(b) - \varepsilon > u(p_*)$. Another application of lemma A.2 gives a lottery $a \in \Delta(C)$ such that $u(a) = r + \varepsilon\frac{1-\alpha}{\alpha}$. Similarly, there is a $c \in \Delta(C)$ with $u(c) = r - \varepsilon$. Hence $\alpha u(a) + (1-\alpha)u(c) = \alpha(r + \varepsilon\frac{1-\alpha}{\alpha}) + (1-\alpha)(r-\varepsilon) = r = u(b)$. This proves equation (6). Equation (7) follows since $\alpha \neq \beta$ and $u(b) \neq u(c)$. To see this, note that $u(a) = \frac{u(b)-(1-\alpha)u(c)}{\alpha}$. Hence $\alpha[\beta u(a) + (1-\beta)u(c)] = \alpha[\beta\left(\frac{u(b)-(1-\alpha)u(c)}{\alpha}\right) + (1-\beta)u(c)] = \beta u(b) + (\alpha-\beta)u(c) \neq \alpha u(b)$. Divide the last inequality by α and the lemma follows. □

To prove our next lemma we will invoke a version of the Lebesgue decomposition theorem. The version below follows as a direct corollary to Rudin (1966, Theorem 6.10, p.121).[14] To state it some further notation is introduced. Let μ, ν be measures on Ω. We write $u \perp \nu$ if there is a measurable set A such that $\mu(A) = \nu(\Omega \setminus A) = 0$ and $u \ll \nu$ if for all measurable sets $E \subseteq \Omega$: $\nu(E) = 0$ implies $\mu(E) = 0$. A measure μ on Ω is called finite if $\mu(\Omega) < +\infty$.

Lemma A.4. *Let μ and ν be positive and finite measures on (X, Ω) then there is a pair of positive and finite measures μ_0 and μ_1 such that:*

1. *$\mu = \mu_0 + \mu_1$ and*

[14] A short and self-contained proof of the Lebesgue decomposition theorem is in Titkos (2015).

2. $\mu_0 \perp \nu$ *and* $\mu_1 \ll \nu$.

Roughly speaking, our next lemma formalises the intuition (from the finite case) that the support of a lower order belief μ^m must be contained in the support of a higher order belief μ^i for all $i > m$, and that the restriction of the higher order belief μ^i to the support of the lower order belief μ^m must be identical to the lower order belief.

Lemma A.5. *For every $i \in \{m+1,...,K\}$ there is a probability measure μ_0^i and an $\alpha^i \in (0,1]$ such that*

$$\mu^i = \alpha^i \mu_0^i + (1-\alpha^i)\mu^m \quad \textit{and} \quad \mu_0^i \perp \mu^m.$$

Proof. By lemma A.4 there are measures ν_0^i, ν_1^i such that $\mu^i = \nu_0^i + \nu_1^i$ and $\nu_0^i \perp \mu^m$ and $\nu_1^i \ll \mu^m$. If $\nu_1^i(\Omega) = 0$ then $\mu^i = \nu_0^i$ and by setting $\alpha^i = 1$ we are done. If $\nu_0^i(\Omega) = 0$ then $\mu^i = \nu_1^i$ and we set $\alpha^i = 0$. If $\nu_0^i(\Omega) > 0$ and $\nu_1^i(\Omega) > 0$ define $\mu_0^i = \frac{\nu_0^i}{\nu_0^i(\Omega)}$ and $\mu_1^i = \frac{\nu_1^i}{\nu_1^i(\Omega)}$ and set $\alpha^i = \nu_0^i(\Omega)$. Then we have $\mu^i = \alpha^i \mu_0^i + (1-\alpha^i)\mu_1^i$. Our aim is to show that $\mu_1^i = \mu^m$ for all $i > m$ with $\alpha^i \in [0,1)$.

Since $\nu_0^i \perp \mu^m$ for all $i > m$ there are sets $(A^i)_{i>m}$ such that $\nu_0^i(A^i) = \mu^m(\Omega \setminus A^i) = 0$ for all $i > m$. Set

$$A = \bigcap_{i>m} A^i \cap U^m.$$

We first note that $\mu^m(A) = 1$ since $\mu^m(A^i) = 1$ for all $i > m$ and $\mu^m(U^m) = 1$.

Assume by contradiction that $\mu_1^i \neq \mu^m$ for some $i > m$. Then there is a measurable set $E \subseteq \Omega$ with $1 > \mu^m(E \cap A) > 0$ and $\mu^m(E \cap A) > \mu_1^i(E \cap A)$. To see this, first observe that since $\mu_1^i \neq \mu^m$ and $\mu_1^i \ll \mu^m$ there is a measurable set $E \subseteq \Omega$ with $1 > \mu^m(E) > 0$ and $\mu^m(E) \neq \mu_1^i(E)$. Note further that $\mu^m(E \cap A) = \mu^m(E) > 0$. If $\mu^m(E) > \mu_1^i(E)$ then $\mu^m(E \cap A) = \mu^m(E) > \mu_1^i(E) \geq \mu_1^i(E \cap A)$ and we are done. If, on the other hand, $\mu_1^i(E) > \mu^m(E)$ then $\mu^m(E^c) > \mu_1^i(E^c)$ and hence $\mu^m(E^c \cap A) = \mu^m(E^c) > \mu_1^i(E^c) \geq \mu_1^i(E^c \cap A)$.

Set $\alpha = \mu^m(E \cap A)$ and $\beta = \mu_1^i(E \cap A)$.[15] Then $\alpha \in (0,1)$ and $\alpha \neq \beta$ and it follows by lemma A.3 that there are lotteries $a, b, c \in \Delta(C)$ such that $u(p^*) > u(a) > u(b) > u(c) > u(p_*)$ and

$$u(a)\mu^m(E \cap A) + u(c)\mu^m(E^c \cap A) = u(b) \tag{8}$$

$$u(a)\mu_1^i(E \cap A) + u(c)\mu_1^i(E^c \cap A) < u(b) \tag{9}$$

[15] Note that $1 = \mu^m(A)$ and $\mu_1^i \ll \mu^m$ implies $1 = \mu_1^i(A)$ so $\mu_1^i(E \cap A) + \mu_1^i(E^c \cap A) = 1$.

Define an act $x \in X$ as follows:

$$x(\omega) = \begin{cases} c & \text{if } \omega \in E \cap A \\ a & \text{if } \omega \in E^c \cap A \\ b & \text{otherwise.} \end{cases}$$

Further let $y \in X$ be a constant act equal to b everywhere, i.e. $y(\omega) = b$ for all $\omega \in \Omega$.

By construction of x and y it follows that $x \succ y$ or $y \succ x$. To see this observe that $\int_\Omega u(x) d\mu^m = u(a)\mu^m(E^c \cap A) + u(c)\mu^m(E \cap A) = u(b) = \int_\Omega u(y) d\mu^m$. We also have $\int_\Omega u(x) d\mu^j = \int_\Omega u(y) d\mu^j$ for all $j < m$. This follows since $\mu^j(A) \leq \mu^j(U^m) = 0$ for all $j < m$ implies that $\int_\Omega u(x) d\mu^j = \int_A u(x) d\mu^j + \int_{\Omega\setminus A} u(x) d\mu^j = \int_A u(y) d\mu^j + \int_{\Omega\setminus A} u(y) d\mu^j = \int_\Omega u(y) d\mu^j$ for all $j < m$.

Further, we have $\int_A u(x) d\mu^i = u(a)\mu^i(E^c \cap A) + u(c)\mu^i(E \cap A) = u(a)\nu_1^i(E^c \cap A) + u(c)\nu_1^i(E \cap A) \neq \nu_i^1(A) u(b) = \int_A u(y) d\mu^i$. Since $\int_{\Omega\setminus A} u(x) d\mu^i = \int_{\Omega\setminus A} u(y) d\mu^i$ it follows that $\int_\Omega u(x) d\mu^i \neq \int_\Omega u(y) d\mu^i$. Thus $x \succ y$ or $y \succ x$.

We will next show that $y \succ x$ and robustness ROB lead to a contradiction (a similar contradiction obtains if $x \succ y$). First observe that $D(x, y) = (E \cap A) \cup (E^c \cap A) = A$ (and since $\mu^m(A) = 1$ it follows that $D(x, y)$ has positive measure). Let $\varepsilon > 0$ and $F \subseteq D(x, y) = A$ be a set of positive measure. We show that $\mu^m(F) > 0$. Since F has positive measure it follows that $\mu^i(F) > 0$ for some $i \geq m$. But since $F \subseteq A$ we have $\nu_0^i(F) \leq \nu_0^i(A) = 0$. Hence $0 < \mu^i(F) = \nu_1^i(F) + \nu_0^i(F) = \nu_1^i(F)$. But since $\mu^m \gg \nu_1^i$ we must also have $\mu^m(F) > 0$.

Let $p, q \in \Delta(C)$ be such that $u(a) < u(p)$, $u(c) < u(q)$ and $||p - a|| < \varepsilon$ and $||q - c|| < \varepsilon$.[16] Define x' on Ω by :

$$x'(\omega) = \begin{cases} q & \text{if } \omega \in F \cap E \\ p & \text{if } \omega \in F \cap E^c \\ x(\omega) & \text{otherwise} \end{cases}$$

It is clear that $||x_F - x'_F|| < \varepsilon$. By construction of x, y and x' it then follows that:

$$\int_\Omega u(x'_F, x_{-F}) d\mu^m = \int_F u(x'_F, x_{-F}) d\mu^m + \int_{F^c} u(x'_F, x_{-F}) d\mu^m =$$

$$u(p)\mu^m(F \cap E^c) + u(q)\mu^m(F \cap E) + \int_{F^c} u(x'_F, x_{-F}) d\mu^m >$$

$$u(a)\mu^m(F \cap E^c) + u(c)\mu^m(F \cap E) + \int_{F^c} u(x_F, x_{-F}) d\mu^m =$$

[16] Such lotteries $p, q \in \Delta(C)$ exist by lemma A.2 and since $u(a) < u(p^*)$ and $u(c) < u(p^*)$.

$$\int_F u(x)d\mu^m + \int_{F^c} u(x)d\mu^m = \int_\Omega u(x)d\mu^m = \int_\Omega u(y)d\mu^m.$$

The strict inequality above follows since $\mu^m(F) > 0$ (so $\mu^m(F\cap E) > 0$ or $\mu^m(F\cap E^c) > 0$) and since $u(p) > u(a)$ and $u(q) > u(c)$. We also have that $\int_\Omega u(x'_F, x_{-F})d\mu^j = \int_\Omega u(x)d\mu^j = \int_\Omega u(y)d\mu^j$ for all $j < m$ since $F \subseteq U^m$ so it follows by induction hypothesis (IH) that $\mu^j(F) \leq \mu^j(U^m) = 0$ for all $j < m$. Hence $(x'_F, x_{-F}) \succ y$ and $||x_F - x'_F|| < \varepsilon$ lead to a contradiction to robustness.

To summarize, we have shown that there for every $i \in \{m+1,...,K\}$ is an $\alpha^i \in [0,1]$ and a probability measure μ_0^i with $\mu_0^i \perp \mu^m$ such that $\mu^i = \alpha^i\mu_0^i + (1-\alpha^i)\mu^m$. But if $\alpha^i = 0$ then $\mu^i = \mu^m$, a contradiction to minimality of $(\mu^1,...,\mu^K)$. Thus $\alpha^i \in (0,1]$ for all $i > m$. □

The next lemma extends lemma A.5 to show that it holds for all measurable acts (not just indicator functions as in lemma A.5). The proof follows by a straightforward application of the dominated convergence theorem.

Lemma A.6. *For every $i \in \{m+1,...,K\}$ the following formula holds:*

$$\int_\Omega u(x)d\mu^i = \alpha^i \int_\Omega u(x)d\mu_0^i + (1-\alpha^i)\int_\Omega u(x)d\mu^m \tag{10}$$

for all acts $x \in X$.

Proof. First note that since $u \circ x$ is a measurable function it is the pointwise limit of a sequence of simple functions $(f_n(\omega))_{n=1}^\infty$ (Aliprantis and Border, 2007, Corollary 4.37, p.145). Moreover, this sequence of simple functions may be chosen such that $f_n(\omega) \in [u(p_*), u(p^*)]$ for all $\omega \in \Omega$. Let f be a simple function with $f(\omega) \in [u(p_*), u(p^*)]$ for all $\omega \in \Omega$ then $f(\omega) = \sum_{k=1}^n r_k \mathbf{1}_{E_k}(\omega)$ where E_k are measurable and $r_k \in [u(p_*), u(p^*)]$ for all $k \geq 1$. Since $r_1,...,r_n \in [u(p_*), u(p^*)]$ there are by lemma A.2 lotteries $a_1,...,a_n \in \Delta(C)$ with $u(a_i) = r_i$ for all $i \geq 1$. Thus $f(\omega) = \sum_{k=1}^n u(a_k)\mathbf{1}_{E_k}(\omega)$. By the dominated convergence theorem (Tao, 2011, Theorem 1.4.49, p.111) and the reasoning above it suffices to show formula (10) for simple acts, i.e assume $x \in X$ is such that $x = \sum_{k=1}^n a_k \mathbf{1}_{E_k}(\omega)$ where E_k are measurable and $a_k \in \Delta(C)$ for all $k \geq 1$.

$$\int_\Omega u(x)d\mu^i = \sum_{k=1}^n u(a_k)\mu^i(E_k) = \tag{11}$$

$$\sum_{k=1}^n u(a_k)[\alpha^i\mu_0^i(E_k) + (1-\alpha^i)\mu^m(E_k)] = \tag{12}$$

$$\alpha^i \sum_{k=1}^n u(a_k)\mu_0^i(E_k) + (1-\alpha^i)\sum_{k=1}^n u(a_k)\mu^m(E_k) = \tag{13}$$

$$\alpha^i \int_\Omega u(x)d\mu_0^i + (1-\alpha^i)\int_\Omega u(x)d\mu^m. \tag{14}$$

The second equality in (11) follows by lemma A.5. By rearranging, we obtain the equality in (12). The equality in (13) follows by definition of the integral of a simple function. □

We next define measurable sets $(V^1,...,V^K)$ and an LPS $(\nu^1,...,\nu^K)$ by:

$$V^i = \begin{cases} U^i & \text{if } i < m \\ \bigcap_{i>m} A^i \cap U^m & \text{if } i = m \\ U^i \cap (\Omega \setminus A^i) & \text{if } i > m \end{cases} \qquad \nu^i = \begin{cases} \mu^i & \text{if } i < m \\ \mu^m & \text{if } i = m \\ \mu_0^i & \text{if } i > m \end{cases}$$

where the sets $A^{m+1},...,A^K$ and the measures $\mu_0^{m+1},...,\mu_0^K$ are as constructed in the proof of lemma A.5. To finish the proof of claim A.1 we need to show that $\nu^i(V^i) = 1$ for all $i \in \{1,...,K\}$, $\nu^i(V^j) = \nu^j(V^i) = 0$ for all $i < m+1$ and $i \neq j$ and that equation (5) with LPS $(\nu^1,...,\nu^K)$ holds. We first show that equation (5) with LPS $(\nu^1,...,\nu^K)$ holds.

Lemma A.7. *Equation (5) with LPS $(\nu^1,...,\nu^K)$ holds.*

Proof. Let $\succsim'$ on X be a preference relation defined by equation (5) and using LPS $(\nu^1,...,\nu^K)$. Recall that $\succsim$ is the preference relation defined by equation (5) and using LPS $(\mu^1,...,\mu^K)$. To show this it suffices to show that:

$$x \succ' y \implies x \succ y$$
$$x \sim' y \implies x \sim y$$

Let $x, y \in X$ and $x \succ' y$ then there is an $l \in \{1,...,K\}$ such that $\int_\Omega u(x)d\nu^l > \int_\Omega u(y)d\nu^l$ and $\int_\Omega u(x)d\nu^k = \int_\Omega u(y)d\nu^k$ for all $k < l$. If $l \leq m$ then clearly $x \succ y$ since $\nu^i = \mu^i$ for all $i \leq m$. Next, consider the case with $l > m$. Then $\nu^l = \mu_0^l$ and by using equation (10) it follows that:

$$\int_\Omega u(x)d\mu^l = \alpha^l \int_\Omega u(x)d\mu_0^l + (1-\alpha^l)\int_\Omega u(x)d\mu^m > \tag{15}$$
$$\alpha^l \int_\Omega u(y)d\mu_0^l + (1-\alpha^l)\int_\Omega u(x)d\mu^m = \tag{16}$$
$$\alpha^l \int_\Omega u(y)d\mu_0^l + (1-\alpha^l)\int_\Omega u(y)d\mu^m = \int_\Omega u(y)d\mu^l. \tag{17}$$

where the strict inequality in equation (15) follows since $\alpha^l > 0$. For all $m < i < l$ we have that $\int_\Omega u(x)d\mu^i = \int_\Omega u(y)d\mu^i$, which follows using similar reasoning as above, replacing the strict inequality in equation (15) with an equality. Since $\mu^i = \nu^i$ for all $i \leq m$ it follows that $x \succ y$.

If $x \sim' y$ then $\int_\Omega u(x)d\nu^i = \int_\Omega u(y)d\nu^i$ for all $i \in \{1,...,K\}$. To show that $x \sim y$ it hence suffices to show that $\int_\Omega u(x)d\mu^i = \int_\Omega u(y)d\mu^i$ for all $i > m$. This again, follows using similar reasoning as in equation (15) - (17), replacing the strict inequality in (15) with an equality. □

The following lemma concludes the proof of claim A.1 (and hence the theorem).

Lemma A.8. *We have that* $\nu^i(V^i) = 1$ *for all* $i \in \{1,...,K\}$, $\nu^i(V^j) = \nu^j(V^i) = 0$ *for all* $i, j \in \{1,...,K\}$ *with* $i < m+1$ *and* $i \neq j$.

Proof. We check that $\nu^i(V^i) = 1$ for all $i \in \{1,...,K\}$. Note first that $\nu^i(V^i) = \mu^i(U^i) = 1$ for all $i < m$. Since $\mu^m(U^m) = 1$ and $\mu^m(A^i) = 1$ for all $i > m$ it follows that $\nu^m(V^m) = \mu^m(\bigcap_{i>m} A^i \cap U^m) = 1$. Since $\mu^i(U^i) = 1$ for all $i > m$ it follows that $1 = \mu^i(U^i) = \alpha^i \mu_0^i(U^i) + (1-\alpha^i)\mu^m(U^i)$. But this implies that $\mu_0^i(U^i) = 1$ for all $i > m$. Hence, since $\nu_0^i(\Omega \setminus A^i) = \nu_0^i(\Omega)$, we have

$$\nu^i(V^i) = \mu_0^i(U^i \cap (\Omega \setminus A^i)) = \mu_0^i(\Omega \setminus A^i) = \frac{\nu_0^i(\Omega \setminus A^i)}{\nu_0^i(\Omega)} = 1.$$

We next show that $\nu^i(V^m) = \nu^m(V^i) = 0$ for all $i \neq m$. If $i < m$ then $\nu^i(V^m) = \mu^i(V^m) \leq \mu^i(U^m) = 0$ and $\nu^m(V^i) = \mu^m(U^i) = 0$ by induction hypothesis (IH). If $m < i$ then we have:

$$\nu^i(V^m) = \mu_0^i\left(\bigcap_{i>m} A^i \cap U^m\right) = \frac{\nu_0^i(\bigcap_{i>m} A^i \cap U^m)}{\nu_0^i(\Omega)} \leq \frac{\nu_0^i(A^i)}{\nu_0^i(\Omega)} = 0.$$

We further note that $\nu^m(V^i) = \mu^m(U^i \cap (\Omega \setminus A^i)) \leq \mu^m(\Omega \setminus A^i) = 0$.

It remains to show that $\nu^i(V^j) = \nu^j(V^i) = 0$ for all $i, j \in \{1,...,K\}$ with $i < m$ and $i \neq j$. Clearly, $\nu^i(V^j) = \mu^i(U^j) = 0$ and $\nu^j(V^i) = \mu^j(U^i) = 0$ for all $i < m$ and $j < m$ by induction hypothesis (IH). The case $i < m$ and $j = m$ has already been shown. So suppose that $i < m$ and $m < j$. Then $\nu^i(V^j) = \mu^i(V^j) = \mu^i(U^j \cap (\Omega \setminus A^j)) \leq \mu^i(U^j) = 0$ and since $\nu_0^j(U^i) \leq \mu^j(U^i) = 0$ it follows that

$$\nu^j(V^i) = \mu_0^j(U^i) = \frac{\nu_0^j(U^i)}{\nu_0^j(\Omega)} = 0.$$

□

B Appendix: Other versions of Robustness

In our formulation of the robustness axiom we required robustness w.r.t. some event $E \subseteq D(x, y)$ of positive measure. In this section we consider stronger and weaker versions of our main robustness axiom. The subsequent discussion is intended to facilitate the understanding of the main robustness axiom. Our aim is to illustrate why other natural formulations of robustness are not characterizing properties of LCPSs.

B.1 Strong Robustness

By assuming robustness w.r.t. every measurable set $E \subseteq D(x,y)$ we obtain a characterization of a subclass of LCPS of which both lexicographic and SEU preferences are special cases.

> **Strong Robustness** (SROB): For all $x, y \in X$ and for all $E \subseteq D(x,y)$ of positive measure, if $x \succ y$ then there is an $\varepsilon > 0$ such that $(x'_E, x_{-E}) \succ (y'_E, y_{-E})$ for all $x'_E, y'_E \in X$ with $||x'_E - x_E|| < \varepsilon$ and $||y'_E - y_E|| < \varepsilon$.

Strong robustness requires robustness w.r.t. to changes in payoffs of *any* measurable subset $E \subseteq D(x,y)$ of positive measure, whereas robustness ROB only requires preferences to be robust to changes in *some* subset $E \subseteq D(x,y)$ of positive measure.

Corollary B.1. *Let $\succsim$ be a binary relation on X. Then the following statements are equivalent:*

a) There is a continuous function $u : \Delta(C) \to \mathbb{R}$ and an LCPS $(\mu^1, ..., \mu^K)$ such that equation (3) holds and such that U^i is an atom of μ^i for all $1 \leq i < K$.

b) There is a continuous function $u : \Delta(C) \to \mathbb{R}$ and an LPS $(\mu^1, ..., \mu^K)$ such that equation (3) holds and such that $\succsim$ satisfies strong robustness SROB.

Proof. Let $\succsim$ be such that b) holds. Then theorem 5.2 gives a minimal LCPS $(\mu^1, ..., \mu^K)$ such that equation (3) holds. If $K = 1$ we are done. So suppose that $K > 1$. Let $i < K$. We first show that U^i is an atom of the measure μ^i, i.e. if $E \subset U^i$ and $\mu^i(E) < \mu^i(U^i)$ then $\mu^i(E) = 0$. Suppose not, then there is a subset E of U^i with $0 < \mu^i(E) < 1$. This implies that $0 < \mu^i(E^c) < 1$. Further let $F = U^K$.

Set $\alpha = \mu^i(E)$ and $\beta = 1$. By lemma A.3 there are lotteries $a, b, c \in \Delta(C)$ such that $u(p^*) > u(a) > u(b) > u(c) > u(p_*)$ and

$$u(a)\mu^i(E) + u(c)\mu^i(E^c) = u(b) \tag{18}$$

$$u(a) > u(b) \tag{19}$$

Define an act $x \in X$ as follows:

$$x(\omega) = \begin{cases} a & \text{if } \omega \in E \cap U^i \\ c & \text{if } \omega \in E^c \cap U^i \\ a & \text{if } \omega \in U^K \setminus U^i \\ b & \text{otherwise.} \end{cases}$$

Further, let $y \in X$ be a constant act equal to b everywhere, i.e. $y(\omega) = b$ for all $\omega \in \Omega$. Then $\int_\Omega u(x) d\mu^i = u(a)\mu^i(E \cap U^i) + u(c)\mu^i(E^c \cap U^i) + u(a)\mu^i(U^K \setminus U^i) = u(a)\mu^i(E) + u(c)\mu^i(E^c) =$

$u(b) = \int_\Omega u(y)d\mu^i$ and since $x(\omega) = b$ for all $\omega \notin U^i \cup U^K$ it follows that $\int_\Omega u(x)d\mu^l = \int_\Omega u(y)d\mu^l$ for all $l \in \{1,....,n\}\setminus\{i,K\}$. Further, $\int_\Omega u(x)d\mu^K = u(a)\mu^K(U^K \setminus U^i) = u(a) > u(b) = \int_\Omega u(y)d\mu^K$ and hence $x \succ y$. Strong robustness SROB then implies that there is an $\varepsilon > 0$ such that $(x'_E, x_{-E}) \succ y$ for all $x'_E \in X$ with $||x'_E - x_E|| < \varepsilon$. But let $p \in \Delta(C)$ be a simple lottery such that $u(a) > u(p)$ and $||a - p|| < \varepsilon$. Define x' on Ω by :

$$x'(\omega) = \begin{cases} p & \text{if } \omega \in E \cap U^i \\ x(\omega) & \text{otherwise} \end{cases}$$

Then since $x' = (x'_E, x_{-E})$ and $\int_\Omega u(x')d\mu^l = \int_\Omega u(x)d\mu^l = \int_\Omega u(y)d\mu^l$ for all $l \in \{1,....,n\} \setminus \{i,K\}$ and $\int_\Omega u(x')d\mu^i < \int_\Omega u(x)d\mu^i = \int_\Omega u(y)d\mu^i$ it follows that $y \succ (x'_E, x_{-E})$. A contradiction. The proof that an LCPS with U^i an atom of μ^i for all $1 \le i < K$ satisfies SROB is straightforward. □

A corollary to the above result is that if each μ^i is atomless then a preference relation $\succsim$ represented by an LPS satisfies strong robustness if and only if $\succsim$ has an SEU representation.

B.2 Stronger robustness

Assuming an even stronger robustness condition we obtain a characterization of SEU preferences under risk and uncertainty as in Anscombe and Aumann (1963).

> **Stronger Robustness** (SSROB): For all $x, y \in X$ and for all $E \subseteq \Omega$ of positive measure, if $x \succ y$ then there is an $\varepsilon > 0$ such that $(x'_E, x_{-E}) \succ (y'_E, y_{-E})$ for all $x'_E, y'_E \in X$ with $||x'_E - x_E|| < \varepsilon$ and $||y'_E - y_E|| < \varepsilon$.

Stronger robustness requires robustness w.r.t. to changes in payoffs of any set $E \subseteq \Omega$ of positive measure, whereas strong robustness SROB requires preferences to be robust to changes in any measurable subset E of the possibly smaller set $D(x, y) \subseteq \Omega$. It is clear that SSROB $\Rightarrow$ SROB $\Rightarrow$ ROB.

Corollary B.2. *Let $\succsim$ be a binary relation on X. Then the following statements are equivalent:*

a) There is a probability measure μ on Ω and a continuous function $u : \Delta(C) \to \mathbb{R}$ such that for all $x, y \in X$:

$$x \succsim y \Leftrightarrow \int_\Omega u(x)d\mu \ge \int_\Omega u(y)d\mu.$$

b) There is a continuous function $u : \Delta(C) \to \mathbb{R}$ and an LPS $(\mu^1, ..., \mu^K)$ such that equation (3) holds and such that $\succsim$ satisfies stronger robustness SSROB.

Proof. Let $\succsim$ be such that b) holds. Then corollary B.1 gives a minimal LCPS $(\mu^1, ..., \mu^K)$ such that equation (3) holds and such that U^i is an atom of μ^i for all $1 \le i < K$. We claim that

$K = 1$. Suppose not, then $K > 1$. Let $a, b \in \Delta(C)$ be lotteries such that $u(a) > u(b) > u(p_*)$. Define an act x by

$$x(\omega) = \begin{cases} b & \text{if } \omega \in U^1 \\ a & \text{otherwise} \end{cases}$$

Further, let $y \in X$ be a constant act equal to b everywhere, i.e. $y(\omega) = b$ for all $\omega \in \Omega$. Then it is clear that $x \succ y$. Stronger robustness SSROB then implies that there is an $\varepsilon > 0$ such that $(x'_{U^1}, x_{-U^1}) \succ y$ for all $x'_{U^1} \in X$ with $||x'_{U^1} - x_{U^1}|| < \varepsilon$. But let $p \in \Delta(C)$ be a simple lottery such that $u(b) > u(p)$ and $||b - p|| < \varepsilon$. Define x' on Ω by:

$$x'(\omega) = \begin{cases} p & \text{if } \omega \in U^1 \\ x(\omega) & \text{otherwise} \end{cases}$$

Then it follows that $y \succ (x'_{U^1}, x_{-U^1})$. A contradiction. The proof that SEU preferences satisfy SSROB follows by continuity of SEU preferences. □

B.3 Weak Robustness

Finally, we discuss a weakening of our main robustness axiom, only requiring robustness w.r.t. to some measurable subset $E \subseteq \Omega$.

> **Weak Robustness** (WROB): For all $x, y \in X$, if $x \succ y$ then there is a measurable $E \subseteq \Omega$ with positive measure and an $\varepsilon > 0$ such that $(x'_E, x_{-E}) \succ (y'_E, y_{-E})$ for all $x'_E, y'_E \in X$ with $||x'_E - x_E|| < \varepsilon$ and $||y'_E - y_E|| < \varepsilon$.

Weak robustness is not strong enough to characterize the class of LCPSs. Unfortunately, we are not able to fully characterize the class of LPSs satisfying weak robustness. The following example shows that the class of LPSs satisfying weak robustness is a strict subset of the class of LPSs.

Example B.1. Assume that $\Omega = \{1, 2\}$, $C = \{c_0, c_1\}$ and $u : \Delta(C) \to \mathbb{R}$ is affine with $u(c_0) = 0$ and $u(c_1) = 1$. Let $\mu^1 = (\frac{1}{2}, \frac{1}{2})$, $\mu^2 = (1, 0)$. Also, let $\succsim$ be a preference relation satisfying equation (3). We show that $\succsim$ fails to satisfy weak robustness. Let $x = (1, 0)$ and $y = (\frac{1}{2}, \frac{1}{2})$. Then $x \succ y$. Suppose that the robust state is 1, then weak robustness implies that $(1, 0) \succ (\frac{1}{2} + \varepsilon, \frac{1}{2})$. A contradiction. Similarly, if state 2 is the robust state then $(1, 0) \succ (\frac{1}{2}, \frac{1}{2} + \varepsilon)$. A contradiction again. ◁

Since every LCPS satisfies robustness, and hence weak robustness, example 3.1 of the main text shows that the class of LCPSs is a strict subset of the class of LPSs satisfying weak robustness.

It was mentioned in the introduction that weak robustness and robustness are equivalent for binary state spaces Ω. This result follows from example 1.1. For completeness we provide a direct proof of this observation below.

Proposition B.3. *Let $\Omega = \{1, 2\}$. A preference relation represented by an LPS satisfies weak robustness WROB if and only if it satisfies robustness ROB.*

Proof. Clearly robustness implies weak robustness so we prove the converse. Suppose that $\succsim$ satisfies weak robustness WROB and is represented by an LCPS $(\mu^1, ..., \mu^K)$ on $\Omega = \{1, 2\}$. We show that robustness ROB holds. Let $x, y \in X$ with $x = (x(1), x(2)) \succ (y(1), y(2)) = y$. If $D(x, y) = \{1, 2\}$ then since $\succsim$ satisfies weak robustness the result follows. Otherwise, assume w.l.o.g. that $D(x, y) = \{1\}$. Then since $\succsim$ is an LPS and $x \succ y$ it follows that $u(x(1)) > u(y(1))$ and hence it follows that $\succsim$ is robust with respect to coordinate one. □

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