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FLUCTUATION THEORY AND EXIT SYSTEMS FOR POSITIVE SELF-SIMILAR MARKOV PROCESSES[1]

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For a positive self-similar Markov process, X, we construct a local time for the random set, Θ, of times where the process reaches its past supremum. Using this local time we describe an exit system for the excursions of X out of its past supremum. Next, we define and study the *ladder process* (R, H) associated to a positive self-similar Markov process X, namely a bivariate Markov process with a scaling property whose coordinates are the right inverse of the local time of the random set Θ and the process X sampled on the local time scale. The process (R, H) is described in terms of a ladder process linked to the Lévy process associated to X via Lamperti's transformation. In the case where X never hits 0, and the upward ladder height process is not arithmetic and has finite mean, we prove the finite-dimensional convergence of (R, H) as the starting point of X tends to 0. Finally, we use these results to provide an alternative proof to the weak convergence of X as the starting point tends to 0. Our approach allows us to address two issues that remained open in Caballero and Chaumont [*Ann. Probab.* **34** (2006) 1012–1034], namely, how to remove a redundant hypothesis and how to provide a formula for the entrance law of X in the case where the underlying Lévy process oscillates.

1. Introduction. In recent years there has been a growing interest in the theory of positive self-similar Markov processes (pssMp). Recall that a pssMp $X = (X_t, t \geq 0)$ is a right-continuous left-limited positive-valued strong Markov process with the following scaling property. There exists an $\alpha \neq 0$ such that for any $0 < c < \infty$,

$$\{(cX_{tc^{-1/\alpha}}, t \geq 0), \mathbb{P}_x\} \stackrel{(\mathrm{d})}{=} \{(X_t, t \geq 0), \mathbb{P}_{cx}\}, \qquad x > 0.$$

This class of processes has been introduced by Lamperti [21] in a seminal paper where, among other interesting results, he established a one-to-one relation between pssMp killed at 0 and real-valued Lévy processes. (Here we allow in the definition of a Lévy process the additional possibility of being sent to a cemetery state after an independent and exponentially distributed time).

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Lamperti proved that any pssMp killed at 0 is the exponential of a Lévy process time changed by the right-continuous inverse of an additive functional. We will refer to this relation as Lamperti's transformation, and we will describe it in more detail in Section 2. This relation allows one to embed the theory of Lévy processes into that of pssMp. This embedding has proved to be a powerful tool in unravelling the asymptotic behavior of pssMp (e.g., [13]), in establishing various interesting identities (e.g., [12]) and in linking these processes with other areas of applied probability such as mathematical finance [30], continuous state branching processes [20] and fragmentation theory [3], to name but a few.

Nevertheless implementing the theory of Lévy processes for such purposes has never been as simple as one might hope for on account of the time change, which relates both classes together and destroys many of the convenient homogeneities that are to be found in the theory of Lévy processes. For example, it is known that a nondecreasing Lévy process with finite mean grows linearly, owing to the law of large numbers, while a pssMp associated via Lamperti's transformation to such a Lévy process, growths with a polynomial rate whose order is given by the index of self-similarity (see, e.g., [4] and [26]).

Our main objective in this paper is to shed light on fine properties for the paths of pssMp, namely to establish a fluctuation theory, built from the fluctuation theory of Lévy processes as well as classical excursion theory (see, e.g., [2] and [19] for background). We will provide several new identities for pssMp and present an alternative approach to that proposed by [4, 5] and [8] for the existence of entrance laws for pssMp. The latter allows us to address the problem of establishing an identity for the entrance law at 0+ for pssMp associated via Lamperti's transformation to an oscillating Lévy process. To present our results in more detail we need to introduce further notation and preliminary results.

2. Preliminaries and main results. Let $\mathbb{D}$ be the space of càdlàg paths defined on $[0,\infty)$, with values in $\mathbb{R}\cup\Delta$, where Δ is a cemetery state. Each path $\omega\in\mathbb{D}$ is such that $\omega_t=\Delta$, for any $t\geq\inf\{s\geq 0:\omega_s=\Delta\}:=\zeta(\omega)$. As usual we extend any function $f:\mathbb{R}\to\mathbb{R}$ to $\mathbb{R}\cup\Delta$ by $f(\Delta)=0$. For each Borel set A we also define $\varsigma_A=\inf\{s>0:\omega_s\in A\}$, writing, in particular, for convenience ς_0 instead of $\varsigma_{\{0\}}$. The space $\mathbb{D}$ is endowed with the Skohorod topology and its Borel σ-field. We will denote by X the canonical process of the coordinates. Moreover, let $\mathbf{P}$ be a reference probability measure on $\mathbb{D}$ under which the process ξ is a Lévy process; we will denote by $(\mathcal{G}_t, t\geq 0)$ the complete filtration generated by ξ. We will *assume that the Lévy process* $(\xi,\mathbf{P})$ *has an infinite lifetime*. Although this assumption remains in place throughout the paper, it is not necessary and we indicate below how it may be removed.

Fix $\alpha\in\mathbb{R}\setminus\{0\}$, and let $(\mathbb{P}_x, x>0)$ be the laws of the $1/\alpha$-pssMp associated to $(\xi,\mathbf{P})$ via the Lamperti representation. Formally, define

$$A_t=\int_0^t \exp\{\alpha\xi_s\}\,\mathrm{d}s,\qquad t\geq 0, \tag{2.1}$$

and let $\tau(t)$ be its inverse,

$$\tau(t) = \inf\{s > 0 : A_s > t\}$$

with the usual convention, $\inf\{\varnothing\} = \infty$. For $x > 0$, we denote by $\mathbb{P}_x$ the law of the process

$$x \exp\{\xi_{\tau(tx^{-\alpha})}\}, \qquad t > 0.$$

The Lamperti representation ensures that the laws $(\mathbb{P}_x, x > 0)$ are those of a pssMp in the filtration $\{\mathcal{F}_t := \mathcal{G}_{\tau(t)}, t \geq 0\}$ where index of self-similarity $1/\alpha$. It follows that $T_0 = \inf\{t > 0 : X_t = 0\}$ has the same law under $\mathbb{P}_x$ as $x^\alpha A_\infty$ under $\mathbf{P}$ with

$$A_\infty = \int_0^\infty \exp\{\alpha \xi_s\} \, \mathrm{d}s.$$

Our assumption that $(\xi, \mathbf{P})$ has infinite lifetime implies that the random variable A_∞ is finite a.s. or infinite a.s. according as $\lim_{t\to\infty} \alpha\xi_t = -\infty$, a.s. or $\limsup_{t\to\infty} \alpha\xi_t = \infty$, a.s. (see, e.g., Theorem 1 in [6]). As a consequence, either $(X, \mathbb{P}_x)$ never hits 0 a.s. or continuously hits 0 in a finite time a.s., independently of the starting point $x > 0$. Specifically in the latter case, the process $(X, \mathbb{P}_x)$ does not jump to 0. Lamperti [21] proved that all pssMp that do not jump to 0 can be constructed this way.

The assumption that $(\xi, \mathbf{P})$ has infinite lifetime can be removed by killing the Lévy process ξ at an independent and exponentially distributed time with some parameter $q > 0$ and then applying the above explained transformation (Lamperti's transformation) to the resulting killed Lévy process. Equivalently, one may kill the pssMp $(X, \mathbb{P})$, associated via Lamperti's transformation to a Lévy process $(\xi, \mathbf{P})$, with infinite lifetime, by means of the multiplicative functional

$$\exp\left\{-q \int_0^t X_s^{-\alpha} \, \mathrm{d}s\right\}, \qquad t \geq 0.$$

Using the Feymman–Kac formula to describe the infinitesimal generator of the latter process it is readily seen that both procedures lead to equivalent processes. Therefore, we do not lose generality by making the assumption that $(\xi, \mathbf{P})$ has infinite lifetime, as all our results concern local properties of the process $(X, \mathbb{P})$ or only make sense for pssMp that never hit 0.

Our main purpose is to study the paths of a pssMp by decomposing them into the instants where it reaches its past-supremum. To this end, let $(M_t, t \geq 0)$ be the past supremum of X, $M_t = \sup\{X_s, 0 \leq s \leq t\}$, for $0 \leq t \leq T_0$, and define the process X *reflected at its past supremum*,

$$\frac{M}{X} := \left(\frac{M_t}{X_t}, 0 \leq t < T_0\right).$$

It is easy to verify, using either the scaling and Markov properties or Lamperti's transformation, that for $t, s \geq 0$,

$$M_{t+s} = X_t \left(\frac{M_t}{X_t} \vee \widetilde{M}_{sX_t^{-\alpha}}\right), \qquad X_{t+s} = X_t \widetilde{X}_{sX_t^{-\alpha}},$$

where $\widetilde{M}$ (resp., $\widetilde{X}$) is a copy of M (resp., of X) which is independent of $\mathcal{F}_t$. It follows that the process X reflected at its past supremum is not Markovian. Nevertheless, using standard arguments it is easily established that the process Z defined by

$$Z_t = \left(\frac{M_t}{X_t}, M_t\right), \qquad 0 \le t < T_0,$$

is a strong Markov process. Hence the random set of times Θ

$$\Theta = \left\{0 \le t < T_0 : Z_t = \left(\frac{M_t}{X_t}, M_t\right) \in \{1\} \times \mathbb{R}_+\right\}$$

is a homogeneous random set of X in the sense of [23], but it is not regenerative in general because of its dependence on the values of M. In [18] it has been proved, as a consequence of the main result therein, that in a rather general framework (X, M) is a strong Markov process, and several functionals related to it have been studied.

Our first aim is to describe the process X at the instants of time in Θ by means of the introduction of a *local time*. To do so we observe that because of Lamperti's transformation any element in Θ is the image under the time change τ of some instant where the underlying Lévy process reaches its past supremum. This suggests that the random set Θ can be described by means of the local time at 0 of the underlying Lévy process reflected at its past supremum. To make precise this idea we need to introduce further notions related to the fluctuation theory of Lévy processes.

We recall that the process ξ reflected at its past supremum, $\overline{\xi} - \xi = (\sup_{s \le t} \xi_s - \xi_t, t \ge 0)$, is a strong Markov process. Although all our results are true in general, for brevity we will hereafter assume that:

0 *is regular for* $\overline{\xi} - \xi$ *or*, *equivalently*, *that* 0 *is regular for* $[0, \infty)$ *for the process* ξ.

Under these assumptions it is known that there is a continuous local time at 0 for $\overline{\xi} - \xi$, that as usual we will denote by $L = (L_s, s \ge 0)$ (see, e.g., Chapter IV in [2]). The instants where ξ reaches its past supremum and the position of ξ at such times is described by the so-called upward ladder time and height processes for ξ, (L_t^{-1}, h_t), $t \ge 0$, which are, respectively, defined by

$$L_t^{-1} = \begin{cases} \inf\{s > 0 : L_s > t\}, & t < L_\infty, \\ \infty, & t \ge L_\infty, \end{cases} \quad \text{and} \quad \mathrm{h}_t = \begin{cases} \xi_{L_t^{-1}}, & t < L_\infty, \\ \infty, & t \ge L_\infty. \end{cases}$$

It is known that the upward ladder process (L^{-1}, h) is a bivariate Lévy process with increasing coordinates, and its characteristics can be described in terms of $(\xi, \mathbf{P})$ (see, e.g., [2], Chapter VI). The downward ladder time and height processes are defined analogously, replacing ξ by its dual $(\widehat{\xi}, \widehat{\mathbf{P}}) := (-\xi, \mathbf{P})$. We will assume that the downward and upward ladder time subordinators are normalized such that their respective Laplace exponents $\phi, \widehat{\phi}$, satisfy $\phi(1) = 1 = \widehat{\phi}(1)$.

We recall that there exists a constant $a \geq 0$ such that the inverse of the local time is given by

$$L_t^{-1} = at + \sum_{s \leq t} \Delta L_s^{-1}, \qquad t \geq 0, \tag{2.2}$$

and

$$\int_0^t \mathbf{1}_{\{\overline{\xi}_s - \xi_s = 0\}} \, \mathrm{d}s = aL_t, \qquad t \geq 0.$$

It is known that $a > 0$ if and only if 0 is irregular for $(-\infty, 0)$ for ξ. In that case the downward ladder time and height processes are compound Poisson processes with inter-arrival rate $1/a$. To see this recall, from the many statements that make up the Wiener–Hopf factorization, that

$$\phi(\lambda)\widehat{\phi}(\lambda) = \lambda, \qquad \lambda \geq 0$$

(see, e.g., [2], Section VI.2). It follows that

$$a = \lim_{\lambda \to \infty} \frac{\phi(\lambda)}{\lambda} = \lim_{\lambda \to \infty} \frac{1}{\widehat{\phi}(\lambda)} = \frac{1}{\Pi_{\hat{L}^{-1}}(0, \infty)},$$

where $\Pi_{\hat{L}^{-1}}$ denotes the Lévy measure of $\widehat{L}^{-1}$. Given that the downward ladder height subordinator $\widehat{\mathrm{h}}$ stays constant in the same intervals where $\widehat{L}^{-1}$ does, the claim about the inter-arrival rate of $\widehat{\mathrm{h}}$ follows.

We denote by $\{\epsilon_t, t \geq 0\}$ the process of excursions of ξ from $\overline{\xi}$, namely

$$\epsilon_t(s) = \begin{cases} \xi_{L_{t-}^{-1}} - \xi_{L_{t-}^{-1}+s}, 0 \leq s \leq L_t^{-1} - L_{t-}^{-1}, & \text{if } L_t^{-1} - L_{t-}^{-1} > 0, \\ \Delta, & \text{if } L_t^{-1} - L_{t-}^{-1} = 0. \end{cases}$$

It is well known that this process forms a Poisson point process on the space of real valued càdlàg paths with lifetime ζ and whose intensity measure will be denoted by $\overline{n}$. This measure is the so-called excursion measure from 0 for $\overline{\xi} - \xi$. We will denote by ϵ the coordinate process under $\overline{n}$. Under $\overline{n}$ the coordinate process has the strong Markov property and its semigroup is that of $(\widehat{\xi}, \widehat{\mathbf{P}})$ killed at $\varsigma_{(-\infty,0]}$, the first hitting time of $(-\infty, 0]$. Recall that for $A \subseteq \mathbb{R}$, we denote by ς_A the first hitting time of the set A for the Lévy process ξ. Next, let $\widehat{\mathcal{V}}$ be the measure over $\mathbb{R}_+$ defined by

$$\widehat{\mathcal{V}}(\mathrm{d}y) = a\delta_0(\mathrm{d}y) + \overline{n}\left(\int_0^\zeta \mathbf{1}_{\{\epsilon(s) \in \mathrm{d}y\}} \, \mathrm{d}s\right), \qquad y \geq 0, \tag{2.3}$$

and $\widehat{\mathcal{V}}(x) = a + \overline{n}(\int_0^\zeta \mathbf{1}_{\{\epsilon(s) < x\}} \, \mathrm{d}s)$ for $x > 0$. We recall that $\widehat{\mathcal{V}}(\mathrm{d}x)$ equals the potential measure $\mathbf{E}(\int_0^\infty \mathbf{1}_{\{\widehat{\mathrm{h}}_s \in \mathrm{d}x\}} \, \mathrm{d}s)$, of the downward ladder height subordinator $\widehat{\mathrm{h}}$. This is due to the fact that

$$\mathbf{E}\left(\int_0^\infty \mathbf{1}_{\{\widehat{\mathrm{h}}_s \in \mathrm{d}x\}} \, \mathrm{d}s\right) = \overline{n}\left(\int_0^\zeta \mathbf{1}_{\{\epsilon(s) \in \mathrm{d}x\}} \, \mathrm{d}s\right) \qquad \text{on } x > 0$$

(see, e.g., [2], exercise VI.6.5). Moreover, the potential measure $\mathbf{E}(\int_0^\infty \mathbf{1}_{\{\widehat{\mathrm{h}}_s \in \mathrm{d}x\}}\,\mathrm{d}s)$ has an atom at 0 if and only if $\widehat{\mathrm{h}}$ is a compound Poisson process. The latter happens if and only if 0 is irregular for $(-\infty, 0)$ and regular for $(0, \infty)$, for ξ, and then, in this case, the aforementioned atom is of size a.

We introduce a new process

$$Y_t := \begin{cases} at + \sum_{s \le t} \int_{L_{s-}^{-1}}^{L_s^{-1}} \exp\{-\alpha(\xi_{L_{s-}^{-1}} - \xi_u)\}\,\mathrm{d}s, & 0 \le t < L_\infty, \\ \infty, & t \ge L_\infty. \end{cases} \tag{2.4}$$

It will be proved in Lemma 2 that the process Y is well defined. Using the fact that the process (L^{-1}, h, Y) is defined in terms of functionals of the excursions from 0 of the process ξ reflected at it past supremum, standard arguments allow us to ensure that the former process is a three-dimensional Lévy process whose coordinates are subordinators, and they are not independent because they have simultaneous jumps.

We will denote by $(\xi^\uparrow, \mathbf{P}^\uparrow)$ the process obtained by conditioning ξ to stay positive. We refer to [11] and the references therein for further details on the construction of $(\xi^\uparrow, \mathbf{P}^\uparrow)$. We will denote by $\mathbf{P}^\dagger$ the law of $\xi^\uparrow$ reflected in its future infimum, $(\xi_t^\uparrow - \inf_{s \ge t} \xi_s^\uparrow, t \ge 0)$.

Given that the local time for $\overline{\xi} - \xi$ is an additive functional in the filtration $\mathcal{G}_t$, the process

$$\widetilde{L}_t := L_{\tau(t)}, \qquad t \ge 0,$$

is an additive functional in the filtration $(\mathcal{F}_t, t \ge 0)$ where $\mathcal{F}_t = \mathcal{G}_{\tau(t)}$ for $t \ge 0$. Moreover, $\widetilde{L}$ grows only on the instants where the process X reaches its past supremum, and thus a natural choice for the local time of Θ would be $\widetilde{L}$. However, this additive functional cannot be obtained as a limit of an occupation measure and does not keep track of the values visited by the supremum. It is for these reasons that we will instead consider the process $L^\Theta = (L_t^\Theta, t \ge 0)$, defined by

$$L_t^\Theta = \int_{(0,t]} X_s^\alpha \,\mathrm{d}\widetilde{L}_s, \qquad t \ge 0. \tag{2.5}$$

The latter is an additive functional of $Z = (M/X, M)$, and it is carried by the set Θ. Unusually, however, the right-continuous inverse of L^Θ is not a subordinator on account of the fact that Θ does not have the regenerative property as it depends on the position of the past supremum. Nevertheless, L^Θ has several properties common to local times of regenerative sets as is demonstrated by the following result.

PROPOSITION 1 (Local time). *The constant a, defined in (2.2), is such that*

$$\int_0^t \mathbf{1}_{\{M_s = X_s\}}\,\mathrm{d}s = aL_t^\Theta, \qquad t \ge 0.$$

Assume that the underlying Lévy process ξ is such that 0 *is regular for* $(0,\infty)$. *The function $\widehat{\mathcal{V}}$ normalizes the occupation measure in the following way*:

$$\lim_{\varepsilon\to 0}\frac{1}{\widehat{\mathcal{V}}(\log(1+\varepsilon))}\int_0^t \mathbf{1}_{\{M_s/X_s\in[1,1+\varepsilon)\}}\,\mathrm{d}s = L_t^{\Theta},$$

uniformly over bounded intervals in t where convergence is taken in $\mathbb{P}_x$-probability, for any starting point $x>0$. We will therefore refer to the process L^{Θ} as the local time of the set Θ.

An elementary but interesting remark is the following. By making a change of variables we obtain that under $\mathbb{P}_1$

$$L_{T_0}^{\Theta} = \int_0^{L_\infty} e^{\alpha \mathrm{h}_s}\,\mathrm{d}s,$$

where we recall that $T_0=\inf\{t>0: X_t=0\}$. Note that $L_{T_0}^{\Theta}$ is finite if and only if $L_\infty<\infty$ or $L_\infty=\infty$ and $\alpha<0$. The terminal value of the local time L^{Θ} has the same law as the exponential functional of a subordinator stopped at an independent exponential time with some parameter $q\geq 0$, where the value $q=0$ is allowed to include the case where this random time is a.s. infinite.

In the following result we establish that there is a kernel associated to L^{Θ}, which we denote by N^x, with $x>0$ being the initial value of the process of coordinates, so that $(L^{\Theta}, N^{\cdot})$ form an exit system for Θ, which is an example of the one introduced in [23].

THEOREM 1. *Suppose that $F:\mathbb{R}_+^2\times\mathbb{D}\to\mathbb{R}_+$ is a measurable function. Then for each $x>0$, define the kernel N^x by*

$$N^x(F) = x^{-\alpha}\overline{n}\left(F\left(x, xe^{-\epsilon(\zeta)}, xe^{\epsilon(\tau(t/x^{\alpha}))}, 0\leq t\leq x^{\alpha}\int_0^{\zeta} e^{\alpha\epsilon(s)}\,\mathrm{d}s\right)\right).$$

Let G *be the set of left extrema of the excursion intervals complementary to the random set* Θ, $D_s=\inf\{t>s: Z_t\in\Theta\}$ *for* $s\geq 0$, *and* $k_{\cdot}$ *the killing operator. The exit formula*

$$\mathbb{E}_x\left(\sum_{G\in\mathsf{G}} V_G F\left(M_G, M_{D_G}, \left(\frac{M}{X}\circ\theta_G\right)\circ k_{D_G}\right)\right) = \mathbb{E}_x\left(\int_0^{\infty}\mathrm{d}L_s^{\Theta} V_s N^{X_s}(F)\right)$$

holds for every positive, left-continuous and $(\mathcal{F}_t, t\geq 0)$*-adapted process* $(V_t, t\geq 0)$.

REMARK 1. Observe that the kernel $N^{\cdot}(\cdot)$ has the following scaling property: for $c>0$, the image of $N^{\cdot}(\cdot)$, under the dilation $(cX_{tc^{-\alpha}}, t\geq 0)$, equals $c^{\alpha}N^{c\cdot}(\cdot)$.

We are now ready to define the ascending ladder process associated to the pssMp X. The name arises from the analogous process appearing in the fluctuation theory of Lévy processes.

THEOREM 2 (Ascending ladder process). *Let* $\{R_t, t \geq 0\}$ *be the right-continuous inverse of* L^Θ, *that is*,

$$R_t = \inf\{s > 0 : L_s^\Theta > t\}, \qquad t \geq 0,$$

$H_t := X_{R_t}$, $t \geq 0$, *and* $K_t := \int_0^{R_t} X_s^{-\alpha}\,\mathrm{d}s$, $t \geq 0$. *The following properties hold*:

(i) *The process* (K, R, H) *has the same law under* $\mathbb{P}_x$ *as the process*

$$\left\{\left(L_t^{-1}, x^\alpha \int_{(0,t]} e^{\alpha \mathrm{h}_{s-}}\,\mathrm{d}Y_s, x e^{\mathrm{h}_t}\right), t \geq 0\right\},$$

time changed by the inverse of the additive functional $(x^\alpha \int_0^t e^{\alpha \mathrm{h}_s}\,\mathrm{d}s, t \geq 0)$, *under* $\mathbf{P}$.

(ii) *The process* (R, H) *is a Feller process in* $[0, \infty) \times (0, \infty)$ *and has the following scaling property*: *for every* $c > 0$, $((c^\alpha R_{tc^{-\alpha}}, cH_{tc^{-\alpha}}), t \geq 0)$ *issued from* $(x_1, x_2) \in \mathbb{R}_+^2$ *has the same law as* (R, H) *issued from* $(c^\alpha x_1, cx_2)$.

We observe that Theorem 2 implies directly the following corollary.

COROLLARY 1. *Let* (R, H) *be as in the previous theorem.*

(i) *The process* R *is an increasing self-similar process with index* 1. *It is not a Markov process. Although, if* X *has no positive jumps, then* R *has independent increments.*

(ii) *The process* H *is the* $1/\alpha$*-increasing self-similar Markov process which is obtained as the Lamperti transform of the upward ladder height subordinator* $\{\mathrm{h}_t, t \geq 0\}$ *associated to the Lévy process* ξ.

REMARK 2. It is important to mention that it is possible to state the analogues of Proposition 1 and Theorems 1 and 2 for the past infimum and X reflected in its infumum, $\mathrm{I}_t = \inf_{0 \leq s \leq t} X_s$, $t \geq 0$, $(\mathrm{I}_t, X_t/\mathrm{I}_t)$, $t \geq 0$. These are easily deduced from our results using the elementary fact that X has the same law as $1/\widehat{X}^{(-\alpha)}$, where $\widehat{X}^{(-\alpha)}$ denotes the pssMp with self-similarity index $1/(-\alpha)$, which is obtained by applying Lamperti's transformation to $\widehat{\xi} := -\xi$. We omit the details.

Similar to fluctuation theory of Lévy processes, the process (R, H) is in general a simpler mathematical object to manipulate since its coordinates are increasing processes and provide information about X at its running supremum.

Next, we will explain how the process (R, H) can be used to provide an alternative approach to those methods proposed by [4, 5] and [8] with regard to establishing the existence of entrance laws for pssMp. Working with the process (R, H) has the advantage of allowing us to give an explanation of the necessary and sufficient conditions for the existence of entrance laws, and in doing so we are able to remove an extra assumption in the main theorem of [8] as well as establish a general formula for the entrance law at 0 for X, thereby answering an open question from [8].

Assume hereafter that $\alpha > 0$. In [21] Lamperti remarked that the Feller property at 0 may fail for some pssMp and raised the question of providing necessary and sufficient conditions for the process X to be a Feller process in $[0, \infty)$. For pssMp that hit 0 in a finite time this problem has been solved in complete generality by Rivero [27, 28] and Fitzsimmons [17]. It is known that for pssMp that never hit the state 0, the latter question is equivalent to studying the existence of entrance laws. This task, as well as the proof of the weak convergence of $(X, \mathbb{P}_x)$ as x tends to zero, in the sense of finite-dimensional distributions or in the sense of the weak convergence with respect to the Skorohod topology, has been carried by Bertoin and Caballero in [4], Bertoin and Yor in [5] and Chaumont and Caballero in [8]. To explain their results and our results, we will assume hereafter that the pssMp X is such that $\limsup_{t\to\infty} X_t = \infty$ a.s. which is equivalent to assuming that the underlying Lévy process ξ is such that $\limsup_{t\to\infty} \xi_t = \infty$. In the case that the process $\{\mathrm{h}_s, s \geq 0\}$ is not arithmetic, they provide necessary and sufficient conditions for the process X to be Feller on $[0, \infty)$ and to have weak convergence with respect to the Skorohod topology as the starting point tends to 0. These conditions are that $\mathbf{E}(\mathrm{h}_1) < \infty$ and $\mathbf{E}(\log^+(\int_0^{\varsigma(1,\infty)} e^{\alpha\xi_s}\,\mathrm{d}s)) < \infty$. One of the main contributions of this paper is that we will prove that in fact the sole condition $\mathbf{E}(\mathrm{h}_1) < \infty$ is necessary and sufficient for the latter convergence to hold.

Our first key observation in this direction is to remark that Lamperti's question regarding the Feller nature of pssMp on $[0, \infty)$ is equally applicable to the process (R, H). The purpose of the next theorem is to provide an answer to this question. The result can be seen as an extension for the ladder process (R, H) of the main result in [4].

THEOREM 3. *If* h *is not arithmetic and* $\mu_+ = \mathbf{E}(\mathrm{h}_1) < \infty$, *then for every* $t > 0$ *the bivariate measure* $\mathbb{P}_x(R_t \in \mathrm{d}s, H_t \in \mathrm{d}y)$ *converges weakly as* $x \to 0+$ *to a measure that we will denote by* $\mathbb{P}_{0+}^{R,H}(R_t \in \mathrm{d}s, H_t \in \mathrm{d}y)$ *in* $\mathbb{R}_+^2$, *which is such that for any measurable function* $F : \mathbb{R}_+^2 \to \mathbb{R}_+$,

$$\mathbb{E}_{0+}^{R,H}(F(R_t, H_t)) = \frac{1}{\alpha\mu_+}\mathbf{E}\left(F\left(\frac{t\widetilde{I}}{I_\mathrm{h}}, \frac{t^{1/\alpha}}{I_\mathrm{h}^{1/\alpha}}\right)\frac{1}{I_\mathrm{h}}\right), \qquad t \geq 0,$$

where $I_\mathrm{h} = \int_0^\infty e^{-\alpha \mathrm{h}_s}\,\mathrm{d}s$, *and* $\widetilde{I}$ *is the weak limit of* $e^{-\alpha\mathrm{h}_t}\int_{(0,t]} e^{\alpha\mathrm{h}_{s-}}\,\mathrm{d}Y_s$, *as* $t \to \infty$. *Furthermore,* $\widetilde{I}$ *has the same law as* $\int_0^\infty \exp\{-\alpha\xi_s^\uparrow\}\,\mathrm{d}s$. *Finally, the process* $\{(R_t, H_t), t \geq 0\}$ *converges in the sense of finite-dimensional distributions as the starting point of* X *tends to* 0.

It is implicit in Theorem 3 that $\widetilde{I}$ is a nondegenerate random variable.

In Corollary 2 we will represent the resolvent of X in terms of (R, H) and then use Theorem 3 to prove the finite-dimensional convergence of the process X as the starting point tends to $0+$, and to obtain the formula for the entrance law at $0+$ for X in the following result. Weak convergence with respect to the Skorohod topology will be then obtained by a tightness argument.

THEOREM 4. *Assume that ξ is not arithmetic and that $\mu_+ = \mathbf{E}(\mathrm{h}_1) < \infty$. Then $\mathbb{P}_x$ converges weakly with respect to the Skorohod topology, as the starting point x tends to $0+$, toward a probability measure $\mathbb{P}_{0+}$. The process $((X, \mathbb{P}_x), x \geq 0)$ is a strong Markov process and the one-dimensional law of X under $\mathbb{P}_{0+}$ is determined by*

$$\mathbb{E}_{0+}(f(X_t)) = \int_0^\infty f\left(\frac{t^{1/\alpha}}{x^{1/\alpha}}\right)\frac{1}{x}\eta(\mathrm{d}x), \tag{2.6}$$

where η is the measure defined by

$$\eta(f) = \frac{1}{\alpha\mu_+}\int_{\mathbb{R}_+^3} \mathbf{P}(\widetilde{I} \in \mathrm{d}t)\widehat{\mathcal{V}}(\mathrm{d}x)\mathbf{P}_x^{\dagger}\left(\int_0^{\varsigma_0} e^{-\alpha\xi_u}\,\mathrm{d}u \in \mathrm{d}s\right) f\left(e^{\alpha x}(t+s)\right)$$

and $\int_0^\infty x^{-1}\eta(\mathrm{d}x) = 1$.

Roughly speaking, our approach to proving Theorem 4 relies on the idea that we need first to prove the convergence of its ladder height process.

REMARK 3. In [8], it has been proved that if $\mathbb{E}(\log^+ \int_0^{\varsigma(1,\infty)} \exp\xi_s\,\mathrm{d}s) = \infty$, then $\mathbb{P}_x$ converges weakly toward the degenerated process $X_\cdot \equiv 0$. Under conditions of Theorem 4, the weak limit is not degenerated; therefore, a simple argument by contradiction shows that, within the context of Theorem 4, it necessarily holds that $\mathbb{E}(\log^+ \int_0^{\varsigma(1,\infty)} \exp\xi_s\,\mathrm{d}s) < \infty$. Note also that the conditions of Theorem 4, that is, that ξ is not arithmetic, and $\mathbf{E}(\mathrm{h}_1) < \infty$ are equivalent to the following:

$$\xi \text{ is not arithmetic} \quad \text{and} \quad \begin{cases} \text{either } 0 < \mathbf{E}(\xi_1) \leq \mathbf{E}(|\xi_1|) < \infty, \\ \text{or } \mathbf{E}(|\xi_1|) < \infty, \mathbf{E}(\xi_1) = 0 \text{ and } J < \infty, \end{cases}$$

where $J = \int_{[1,\infty)} \frac{x\pi(x,\infty)\,\mathrm{d}x}{1+\int_0^x \mathrm{d}y \int_y^\infty \pi(-\infty,-z)\,\mathrm{d}z}$ and π is the Lévy measure of ξ (see [8], Section 2.1).

REMARK 4. When $\mathbf{E}(\mathrm{h}_1) < \infty$ and ξ drifts to $+\infty$, that is, $\mathbf{E}(\xi_1) > 0$, an expression for the entrance law under $\mathbb{P}_{0+}$ has already been obtained by Bertoin and Yor [5]. More precisely, for every $t > 0$ and measurable function $f : \mathbb{R}^+ \to \mathbb{R}^+$, we have

$$\mathbb{E}_{0+}(f(X_t)) = \frac{1}{\alpha\mathbf{E}(\xi_1)}\mathbf{E}\left(I^{-1} f\left((t/I)^{1/\alpha}\right)\right). \tag{2.7}$$

In this case and under some mild technical conditions on ξ which can be found in [10], formula (2.6) can be recovered using pathwise arguments as follows: Let m be the unique time when ξ reaches its overall minimum. It is well known (cf. [10, 24]) that $(\xi_t, 0 \le t < m)$ and $(\xi_{t+m} - \xi_m, t \ge 0)$ are independent, and the latter process has the same law as ξ conditioned to stay positive. Moreover, from Lemma 8 of [14], the law of the pre-minimum process is characterized by

$$\mathbf{E}\big(H(\xi_t - \xi_m, 0 \le t < m)\big) = \widehat{\kappa} \int_0^\infty \widehat{\mathcal{V}}(\mathrm{d}x)\mathbf{P}_x^{\downarrow}\big(H(\xi_t, 0 \le t < \zeta)\big),$$

where $\widehat{\kappa}$ is the killing rate of the subordinator $\widehat{\mathrm{h}}$, and $\mathbf{P}_x^{\downarrow}$ is the law of the Lévy process ξ starting from x and conditioned to hit 0 continuously. Then we write

$$I = \int_0^\infty e^{-\alpha\xi_s}\,\mathrm{d}s = e^{-\alpha\xi_m}\bigg(\int_0^m e^{-\alpha(\xi_s - \xi_m)}\,\mathrm{d}s + \int_0^\infty e^{-\alpha(\xi_{m+s} - \xi_m)}\,\mathrm{d}s\bigg).$$

From this identity we obtain that for any measurable function $g : \mathbb{R}^+ \to \mathbb{R}^+$,

$$\mathbf{E}(g(I)) = \widehat{\kappa} \int_0^\infty \mathbf{P}(I^\uparrow \in \mathrm{d}r) \int_0^\infty \widehat{\mathcal{V}}(\mathrm{d}x)\mathbf{P}_x^{\downarrow}\bigg(\int_0^\zeta e^{-\alpha\xi_s}\,\mathrm{d}s \in \mathrm{d}u\bigg) g\big(e^{\alpha x}(r+u)\big),$$

where $I^\uparrow = \int_0^\infty e^{-\alpha(\xi_{m+s} - \xi_m)}\,\mathrm{d}s \overset{(\mathrm{d})}{=} \int_0^\infty e^{-\alpha\xi_s^\uparrow}\,\mathrm{d}s$. Taking $g(x) = (\alpha\mathbf{E}(\xi_1))^{-1} \times f(x^{-1/\alpha})x^{-1}$, it gives

$$\begin{aligned}
&\frac{1}{\alpha\mathbf{E}(\xi_1)}\mathbf{E}\big(I^{-1} f\big((1/I)^{1/\alpha}\big)\big) \\
&\quad = \frac{\widehat{\kappa}}{\alpha\mathbf{E}(\xi_1)} \int_0^\infty \mathbf{P}(I^\uparrow \in \mathrm{d}r) \int_0^\infty \widehat{\mathcal{V}}(\mathrm{d}x)\mathbf{P}_x^{\downarrow}\bigg(\int_0^\zeta e^{-\alpha\xi_s}\,\mathrm{d}s \in \mathrm{d}u\bigg) \\
&\qquad \times f\big(e^{-x}(r+u)^{-1/\alpha}\big)\frac{1}{e^{\alpha x}(r+u)}.
\end{aligned}$$

It follows from Theorem 5 in [10] that the law of the canonical process killed at its first hitting time of 0, ς_0, under $\mathbf{P}_x^\dagger$ equals that of the process ξ issued from x and conditioned to hit 0 continuously under $\mathbf{P}_x^{\downarrow}$. Finally, we use the fact that $\mathbf{E}(\xi_1) = \widehat{\kappa}\mathbf{E}(\mathrm{h}_1)$, which is a simple consequence of the Wiener–Hopf factorization (see, e.g., [15], Corollary 4.4) to recover formula (2.6) in Theorem 4.

The rest of this paper is organized as follows. Section 3 is mainly devoted to prove Proposition 1 and Theorem 1. In proving Proposition 1 we will use that the local time at 0 of a Lévy process reflected in its past supremum can be approximated by an occupation time functional. This result is of interest in itself and, to the best of our knowledge this cannot be found in the literature, so we have included a proof. Next we use the excursion theory for Lévy processes reflected in its past supremum to establish Theorem 1. The main purpose of Section 4 is to prove Theorem 2. To this end we will establish an elementary but key lemma that allows us to describe the time change appearing in Lamperti's transformation in terms of the excursions of the process ξ out of its past supremum. In Section 4.1 we describe the q-resolvent of X in terms of the ladder process (R, H), which will be useful in the proof of the convergence results. As an application of these results, in Section 5 we will prove Theorems 3 and 4.

3. Proofs of Proposition 1 and Theorem 1. The proof of Proposition 1 needs the following analogous result for Lévy processes which is of interest in itself.

LEMMA 1. *Let ξ be a Lévy process for which* 0 *is regular for* $(0,\infty)$. *Then*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\widehat{\mathcal{V}}(\varepsilon)}\int_0^{L_t^{-1}}\mathbf{1}_{\{\overline{\xi}_s-\xi_s<\varepsilon\}}\,\mathrm{d}s=t\wedge L_\infty,$$

uniformly over bounded intervals of t in the L^2-norm. Furthermore,

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\widehat{\mathcal{V}}(\varepsilon)}\int_0^{t}\mathbf{1}_{\{\overline{\xi}_s-\xi_s<\varepsilon\}}\,\mathrm{d}s=L_t,$$

uniformly over bounded intervals in probability.

PROOF. In the case where $\widehat{\mathcal{V}}(0)=a>0$, we know that

$$\int_0^t\mathbf{1}_{\{\overline{\xi}_s-\xi_s=0\}}\,\mathrm{d}s=aL_t,\qquad t>0.$$

Using this fact it is readily seen that the claims are true, so we can restrict ourselves to the case $a=0$. The proof of the first assertion in this lemma follows the basic steps of the proof of this result for Lévy processes with no negative jumps due to Duquesne and Le Gall [16].

Let $\mathcal{N}$ be a Poisson random measure on $\mathbb{R}_+\times\mathbb{D}(\mathbb{R}_+,\mathbb{R})$ with intensity $\mathrm{d}t\times\overline{n}(\mathrm{d}\epsilon)$. For every $t>0$, put

$$J_\varepsilon(t):=\frac{1}{\widehat{\mathcal{V}}(\varepsilon)}\int\mathcal{N}(\mathrm{d}u,\mathrm{d}\epsilon)\cdot\mathbf{1}_{\{u\le t\}}\int_0^\zeta\mathbf{1}_{\{[0,\varepsilon)\}}(\epsilon(s))\,\mathrm{d}s.$$

It follows that

$$\mathbf{E}(J_\varepsilon(t))=\frac{t}{\widehat{\mathcal{V}}(\varepsilon)}\overline{n}\biggl(\int_0^\zeta\mathbf{1}_{\{[0,\varepsilon)\}}(\epsilon(s))\,\mathrm{d}s\biggr)=t,$$

where the second equality follows from the definition of $\widehat{\mathcal{V}}(\varepsilon)$. Furthermore,

$$\mathbf{E}((J_\varepsilon(t))^2)=(\mathbf{E}(J_\varepsilon(t)))^2+\frac{t}{(\widehat{\mathcal{V}}(\varepsilon))^2}\overline{n}\biggl(\biggl(\int_0^\zeta\mathbf{1}_{\{\epsilon(s)\in[0,\varepsilon)\}}\,\mathrm{d}s\biggr)^2\biggr),$$

the latter equality can be verified using that for each $\varepsilon>0$, the process $(J_\varepsilon(t), t\ge 0)$ is a subordinator whose Lévy measure is the image measure of $\overline{n}$ under the mapping $\epsilon \mapsto \widehat{\mathcal{V}}(\varepsilon)^{-1} \int_0^\zeta \mathbf{1}_{\{[0,\varepsilon)\}}(\epsilon(s))\,\mathrm{d}s$, and thus the second moment is obtained by differentiating twice the Laplace transform and using the Lévy–Khintchine formula. The right-most term in the latter identity can be estimated as follows:

$$\begin{aligned}
&\overline{n}\left(\left(\int_0^\zeta \mathbf{1}_{\{\epsilon(s)\in[0,\varepsilon)\}}\,\mathrm{d}s\right)^2\right)\\
&\quad = 2\overline{n}\left(\int_{0\le s\le t\le\zeta} \mathbf{1}_{\{\epsilon(s)\in[0,\varepsilon)\}}\mathbf{1}_{\{\epsilon(t)\in[0,\varepsilon)\}}\,\mathrm{d}s\,\mathrm{d}t\right)\\
&\quad = 2\overline{n}\left(\int_0^\zeta \mathbf{1}_{\{\epsilon(s)\in[0,\varepsilon)\}}\mathbf{E}_{-\epsilon(s)}\left(\int_0^{\varsigma(0,\infty)} \mathbf{1}_{\{\xi(t)\in(-\varepsilon,0]\}}\,\mathrm{d}t\right)\mathrm{d}s\right)\\
&\quad \le 2\widehat{\mathcal{V}}(\varepsilon)\sup_{y\in[0,\varepsilon)}\mathbf{E}_{-y}\left(\int_0^{\varsigma(0,\infty)} \mathbf{1}_{\{\xi(t)\in(-\varepsilon,0]\}}\,\mathrm{d}t\right),
\end{aligned}$$

where we have used the Markov property for excursions in the second equality.

By Theorem VI.20 in [2] it is known that there exists a constant, say k, such that for $y < \varepsilon$

$$\begin{aligned}
\mathbf{E}_{-y}\left(\int_0^{\varsigma(0,\infty)} \mathbf{1}_{\{\xi(t)\in(-\varepsilon,0]\}}\,\mathrm{d}t\right) &= k\int_{[0,y)} \mathcal{V}(\mathrm{d}z)\int_{[0,\infty)} \widehat{\mathcal{V}}(\mathrm{d}x)\mathbf{1}_{(-\varepsilon,0]}(-y+z-x)\\
&= k\int_{[0,y)} \mathcal{V}(\mathrm{d}z)\int_{[0,\infty)} \widehat{\mathcal{V}}(\mathrm{d}x)\mathbf{1}_{[0,\varepsilon)}(x+y-z)\\
&\le k\mathcal{V}[0,y)\widehat{\mathcal{V}}[0,\varepsilon)\\
&\le k\widehat{\mathcal{V}}(\varepsilon)\mathcal{V}[0,\varepsilon)\\
&= o(\widehat{\mathcal{V}}(\varepsilon)),
\end{aligned}$$

where the second inequality follows from the fact that $\widehat{\mathcal{V}}(z) \le \widehat{\mathcal{V}}(\varepsilon)$, $0 < z \le \varepsilon$, and the fifth from the fact that $\mathcal{V}[0,\epsilon) \to 0$ as $\epsilon \to 0+$, because 0 is regular for $(0,\infty)$ (which implies that the upward subordinator is not a compound Poisson process and so that its renewal measure $\mathcal{V}$ does not have an atom at 0). It therefore follows that

$$\overline{n}\left(\left(\int_0^\zeta \mathbf{1}_{\{\epsilon(s)\in[0,\varepsilon)\}}\,\mathrm{d}s\right)^2\right) = o((\widehat{\mathcal{V}}(\varepsilon))^2). \tag{3.1}$$

These estimates allow us to ensure that

$$\lim_{\varepsilon\to 0}\mathbf{E}\big((J_\varepsilon(t)-t)^2\big) = 0.$$

Moreover, thanks to the fact that $(J_\varepsilon(t)-t, t\ge 0)$ is a martingale, we can apply Doob's inequality to deduce that

$$\lim_{\varepsilon\to 0}\mathbf{E}\Big(\sup_{s\le t}\big(J_\varepsilon(s)-s\big)^2\Big) = 0.$$

The first assertion of the theorem follows since the pair

$$\left(\frac{1}{\widehat{\mathcal{V}}(\varepsilon)}\int_0^{L_t^{-1}} \mathbf{1}_{\{\overline{\xi}_s-\xi_s<\epsilon\}}\,\mathrm{d}s, L_\infty\right)$$

has the same law as $(J_\varepsilon(t\wedge\nu),\nu)$ where $\nu=\inf\{t>0:\mathcal{N}(0,t]\times\{\zeta=\infty\}\geq 1\}$.

Now to prove the second assertion we fix $t>0$, let $\varepsilon_1,\delta>0$ and take $T>0$ large enough such that $\mathbb{P}(L_t>T)<\varepsilon_1/3$. It follows using the inequalities $L_{L_s}^{-1}\geq s\geq L_{L_s-}^{-1}$ that

$$\begin{aligned}
&\mathbf{P}\left(\sup_{s\leq t}\left|\frac{1}{\widehat{\mathcal{V}}(\varepsilon)}\int_0^s \mathbf{1}_{\{\overline{\xi}_u-\xi_u<\varepsilon\}}\,\mathrm{d}u-L_s\right|>\delta\right)\\
&\quad\leq \mathbf{P}\left(\sup_{s\leq t}\left|\frac{1}{\widehat{\mathcal{V}}(\varepsilon)}\int_0^{L_{L_s}^{-1}} \mathbf{1}_{\{\overline{\xi}_u-\xi_u<\varepsilon\}}\,\mathrm{d}u-L_s\right|>\delta/2, L_t<T\right)\\
&\qquad+\mathbf{P}\left(\sup_{s\leq t}\frac{1}{\widehat{\mathcal{V}}(\varepsilon)}\int_{L_{L_s-}^{-1}}^{L_{L_s}^{-1}} \mathbf{1}_{\{\overline{\xi}_u-\xi_u<\varepsilon\}}\,\mathrm{d}u>\delta/2, L_t<T\right)+\varepsilon_1/3\\
&\quad\leq \mathbf{P}\left(\sup_{s\leq T}\left|\frac{1}{\widehat{\mathcal{V}}(\varepsilon)}\int_0^{L_s^{-1}} \mathbf{1}_{\{\overline{\xi}_u-\xi_u<\varepsilon\}}\,\mathrm{d}u-s\right|>\delta/2\right)\\
&\qquad+\mathbf{P}\left(\sup_{s\leq T}\int_{L_{s-}^{-1}}^{L_s^{-1}} \mathbf{1}_{\{\overline{\xi}_u-\xi_u<\varepsilon\}}\,\mathrm{d}u>\delta\widehat{\mathcal{V}}(\varepsilon)/2\right)+\varepsilon_1/3.
\end{aligned}\tag{3.2}$$

It follows from the first assertion in Lemma 1 that ε can be chosen so that the first term in the right-hand side in inequality (3.2) is smaller than $\varepsilon_1/3$. Moreover, as the random objects

$$\int_{L_{s-}^{-1}}^{L_s^{-1}} \mathbf{1}_{\{\overline{\xi}_u-\xi_u<\varepsilon\}}\,\mathrm{d}u,\qquad s\geq 0,$$

are the values of the points in a Poisson point process in $\mathbb{R}_+$ whose intensity measure is the image of $\overline{n}$ under the mapping $\epsilon\mapsto\int_0^\zeta \mathbf{1}_{\{\epsilon(u)<\varepsilon\}}\,\mathrm{d}u$, it follows that

$$\begin{aligned}
&\mathbf{P}\left(\sup_{s\leq T}\int_{L_{s-}^{-1}}^{L_s^{-1}} \mathbf{1}_{\{\overline{\xi}_u-\xi_u<\varepsilon\}}\,\mathrm{d}u>\delta\widehat{\mathcal{V}}(\varepsilon)/2\right)\\
&\qquad=1-\exp\left\{-T\overline{n}\left(\int_0^\zeta \mathbf{1}_{\{\epsilon(u)<\varepsilon\}}\,\mathrm{d}u>\delta\widehat{\mathcal{V}}(\varepsilon)/2\right)\right\}.
\end{aligned}\tag{3.3}$$

From the Markov inequality and (3.1) we have also that

$$\begin{aligned}
\overline{n}\left(\int_0^\zeta \mathbf{1}_{\{\epsilon(u)<\varepsilon\}}\,\mathrm{d}u>\delta\widehat{\mathcal{V}}(\varepsilon)/2\right)&\leq\frac{4}{\delta^2\widehat{\mathcal{V}}(\varepsilon)^2}\overline{n}\left(\left(\int_0^\zeta \mathbf{1}_{\{\epsilon(u)<\varepsilon\}}\,\mathrm{d}u\right)^2\right)\\
&=o(1)\qquad\text{as }\varepsilon\to 0.
\end{aligned}$$

It follows then that by taking ε small enough the right-hand term in (3.3) can be made smaller than $\varepsilon_1/3$, which finishes the proof of the second claim. □

PROOF OF PROPOSITION 1. To prove the first claim we recall that the constant a is such that

$$\int_0^t \mathbf{1}_{\{\overline{\xi}_s-\xi_s=0\}}\,\mathrm{d}s = aL_t, \qquad t\geq 0.$$

Hence, by making a time change we get that for every $t>0$,

$$\begin{aligned}\int_0^t \mathbf{1}_{\{M_s=X_s\}}\,\mathrm{d}s &= x^\alpha\int_0^{\tau(t/x^\alpha)}\mathbf{1}_{\{\overline{\xi}_u-\xi_u=0\}}e^{\alpha\xi_u}\,\mathrm{d}u = ax^\alpha\int_{(0,\tau(t/x^\alpha)]}e^{\alpha\xi_u}\,\mathrm{d}L_u\\ &= a\int_{(0,t]}\big(xe^{\xi_{\tau(s/x^\alpha)}}\big)^\alpha\,\mathrm{d}L_{\tau(s/x^\alpha)} = aL_t^\Theta\end{aligned}\tag{3.4}$$

under $\mathbb{P}_x$. Now to prove the second claim we observe first that, as in the proof of Lemma 1, we can restrict ourselves to the case where $a=0$.

For notational convenience, and without loss of generality thanks to the self-similarity of X, we will assume that X is issued from 1. By applying Lamperti's transformation and making a time change we obtain the elementary inequalities

$$\begin{aligned}&\frac{1}{\widehat{\mathcal{V}}(\log(1+\varepsilon))(1+\varepsilon)^\alpha}\int_0^{\tau(t)}\mathbf{1}_{\{\overline{\xi}_s-\xi_s\in[0,\log(1+\varepsilon))\}}e^{\alpha\overline{\xi}_s}\,\mathrm{d}s\\ &\quad\leq\frac{1}{\widehat{\mathcal{V}}(\log(1+\varepsilon))}\int_0^t\mathbf{1}_{\{M_s/X_s\in[1,1+\varepsilon)\}}\,\mathrm{d}s\\ &\quad=\frac{1}{\widehat{\mathcal{V}}(\log(1+\varepsilon))}\int_0^{\tau(t)}\mathbf{1}_{\{\overline{\xi}_s-\xi_s\in[0,\log(1+\varepsilon))\}}e^{-\alpha(\overline{\xi}_s-\xi_s)}e^{\alpha\overline{\xi}_s}\,\mathrm{d}s\\ &\quad\leq\frac{1}{\widehat{\mathcal{V}}(\log(1+\varepsilon))}\int_0^{\tau(t)}\mathbf{1}_{\{\overline{\xi}_s-\xi_s\in[0,\log(1+\varepsilon))\}}e^{\alpha\overline{\xi}_s}\,\mathrm{d}s.\end{aligned}\tag{3.5}$$

Let $\delta,t>0$ fixed. We infer the following inequalities:

$$\begin{aligned}&\mathbb{P}_1\bigg(\sup_{r\leq t}\bigg|\frac{1}{\widehat{\mathcal{V}}(\log(1+\varepsilon))}\int_0^r\mathbf{1}_{\{M_u/X_u\in[1,1+\varepsilon)\}}\,\mathrm{d}u-L_r^\Theta\bigg|>\delta\bigg)\\ &\quad\leq\mathbf{P}\bigg(\frac{1}{\widehat{\mathcal{V}}(\log(1+\varepsilon))}\int_0^{\tau(r)}\mathbf{1}_{\{\overline{\xi}_s-\xi_s\in[0,\log(1+\varepsilon))\}}e^{\alpha\overline{\xi}_s}\,\mathrm{d}s-L_r^\Theta>\delta,\\ &\qquad\qquad\qquad\text{for some } r\leq t\bigg)\\ &\qquad+\mathbf{P}\bigg(L_r^\Theta-\frac{1}{(1+\varepsilon)^\alpha\widehat{\mathcal{V}}(\log(1+\varepsilon))}\\ &\qquad\qquad\times\int_0^{\tau(r)}\mathbf{1}_{\{\overline{\xi}_s-\xi_s\in[0,\log(1+\varepsilon))\}}e^{\alpha\overline{\xi}_s}\,\mathrm{d}s>\delta,\text{ for some } r\leq t\bigg).\end{aligned}\tag{3.6}$$

Next we will prove that the probability in the first term on the right-hand side tends to 0 as $\varepsilon \to 0$. The arguments used to prove that the second one tends to 0 are similar, so we omit them. Consider the event

$$\mathcal{A} = \left\{ \sup_{s \le T} \left| \frac{1}{\widehat{\mathcal{V}}(\log(1+\varepsilon))} \int_0^s \mathbf{1}_{\{\overline{\xi}_u - \xi_u \in [0, \log(1+\varepsilon))\}} \, \mathrm{d}u - L_s \right| < \delta_1, \right.$$
$$\left. \tau(t) < T, (2e^{\alpha \overline{\xi}_T} - 1) < \delta/\delta_1 \right\}$$

for $\delta_1, \varepsilon, T > 0$. From Lemma 1 and standard arguments it follows that δ_1, T and ε can be chosen so that the probability of the event $\mathcal{A}$ is arbitrarily close to 1. By integrating by parts twice, we have that on $\mathcal{A}$

$$\begin{aligned}
&\frac{1}{\widehat{\mathcal{V}}(\log(1+\varepsilon))} \int_0^{\tau(r)} \mathbf{1}_{\{\overline{\xi}_s - \xi_s \in [0, \log(1+\varepsilon))\}} e^{\alpha \overline{\xi}_s} \, \mathrm{d}s \\
&\quad = e^{\alpha \overline{\xi}_{\tau(r)}} \frac{1}{\widehat{\mathcal{V}}(\log(1+\varepsilon))} \int_0^{\tau(r)} \mathbf{1}_{\{\overline{\xi}_s - \xi_s \in [0, \log(1+\varepsilon))\}} \, \mathrm{d}s \\
&\qquad - \int_{(0,\tau(r)]} \left(\frac{1}{\widehat{\mathcal{V}}(\log(1+\varepsilon))} \int_0^s \mathbf{1}_{\{\overline{\xi}_u - \xi_u \in [0, \log(1+\varepsilon))\}} \, \mathrm{d}u \right) \mathrm{d}e^{\alpha \overline{\xi}_s} \\
&\quad \le e^{\alpha \overline{\xi}_{\tau(r)}} \big(L_{\tau(r)} + \delta_1 \big) \\
&\qquad - \int_{(0,\tau(r)]} L_s \, \mathrm{d}e^{\alpha \overline{\xi}_s} + \delta_1 (e^{\alpha \overline{\xi}_T} - 1) \\
&\quad \le \int_{(0,\tau(r)]} e^{\alpha \overline{\xi}_u} \, \mathrm{d}L_u + \delta
\end{aligned}$$

for every $r \le t$. It follows that on $\mathcal{A}$,

$$\frac{1}{\widehat{\mathcal{V}}(\log(1+\varepsilon))} \int_0^r \mathbf{1}_{\{M_s/X_s \in [1, 1+\varepsilon)\}} \, \mathrm{d}s - L_r^{\Theta} \le \delta \qquad \text{for every } r \le t,$$

which finishes the proof of our claim. □

PROOF OF THEOREM 1. Thanks to self-similarity we can suppose without loss of generality that $x = 1$. Recall that ϵ denotes the typical excursion of the Markov process $\overline{\xi} - \xi$. For $s > 0$, let $d_s = \inf\{t > s : \xi_t = \overline{\xi}_t\}$. First of all, observe that the left extrema G (resp., right extrema, D_G) of the of the excursion intervals complementary to the homogeneous random set Θ, are related to left extrema g (resp., right extrema, d_g) of the excursion intervals complementary to the regenerative set $\{t \ge 0 : \overline{\xi}_t - \xi_t = 0\}$, by the relation $G = A_g$ for some g (resp., $D_G = A_{d_g}$ for the corresponding d_g). Using this fact and Lamperti's transformation we obtain

$$
\begin{aligned}
&\mathbb{E}_1\biggl(\sum_{G\in\mathsf{G}} V_G F\biggl(M_G, M_{D_G}, \biggl(\frac{M}{X}\circ\theta_G\biggr)\circ k_{D_G}\biggr)\biggr)\\
&\quad = \mathbf{E}\biggl(\sum_g V_{A_g} F\bigl(e^{\overline{\xi}_g}, e^{\overline{\xi}_g} e^{-(\overline{\xi}_g-\xi_{d_g})},\\
&\qquad\qquad\qquad \bigl(\exp\{(\overline{\xi}-\xi)_{\tau(A_g+u)}\}, 0\le u\le A_{d_g}-A_g\bigr)\bigr)\biggr)\\
&\quad = \mathbf{E}\biggl(\sum_g V_{A_g} F\bigl(e^{\overline{\xi}_g}, \bigl(e^{\overline{\xi}_g} e^{-\epsilon(d_g)}, \exp\{\epsilon\bigl(\tau_\epsilon(u/e^{\alpha\overline{\xi}_g})\bigr)\},\\
&\qquad\qquad\qquad 0\le u\le e^{\alpha\overline{\xi}_g} A^{\{\epsilon\}}_{d_g}\bigr)\bigr)\circ\theta_g\biggr),
\end{aligned}
$$

where we denote by $A^{\{\epsilon\}}_t = \int_0^t e^{\alpha\epsilon(u)}\,\mathrm{d}u$, $t\ge 0$, and τ_ϵ the inverse of $A^{\{\epsilon\}}$. By the compensation formula from the excursion theory of Markov processes we deduce that the right-hand side in the latter equation equals

$$
\mathbf{E}\biggl(\int_0^\infty \mathrm{d}L_s V_{A_s}\overline{n}\bigl(F\bigl(e^{\overline{\xi}_s}, e^{\overline{\xi}_s} e^{-\epsilon(\zeta)}, \bigl(\exp\{\epsilon\bigl(\tau_\epsilon(u/e^{\alpha\overline{\xi}_s})\bigr)\}, 0\le u\le e^{\alpha\overline{\xi}_s} A^{\{\epsilon\}}_\zeta\bigr)\bigr)\bigr)\biggr).
$$

Finally, using again Lamperti's transformation and the fact that L_τ has as support the homogeneous random set Θ, we get that the previous expectation is equal to

$$
\begin{aligned}
&\mathbf{E}\biggl(\int_0^\infty \mathrm{d}L_{\tau(s)} V_s\overline{n}\bigl(F\bigl(e^{\overline{\xi}_{\tau(s)}}, e^{\overline{\xi}_{\tau(s)}} e^{-\epsilon(\zeta)}, \bigl(\exp\{\epsilon\bigl(\tau_\epsilon(u/e^{\alpha\overline{\xi}_{\tau(s)}})\bigr)\},\\
&\qquad\qquad\qquad 0\le u\le e^{\alpha\overline{\xi}_{\tau(s)}} A^{\{\epsilon\}}_\zeta\bigr)\bigr)\bigr)\biggr)\\
&\quad = \mathbb{E}_1\biggl(\int_0^\infty \mathrm{d}L^\Theta_s V_s N^{M_s}(F)\biggr)\\
&\quad = \mathbb{E}_1\biggl(\int_0^\infty \mathrm{d}L^\Theta_s V_s N^{X_s}(F)\biggr),
\end{aligned}
$$

which finishes the proof. $\square$

4. Proof of Theorem 2. The following lemma will be helpful throughout the sequel. Recall that the process A was defined in (2.1).

LEMMA 2. *The following equality holds* **P**-*a.s.*:

$$
A_{L_t^{-1}} = \int_{(0,t]} \exp\{\alpha \mathrm{h}_{s-}\}\,\mathrm{d}Y_s, \qquad t\ge 0, \tag{4.1}
$$

where we recall that $Y_t := at + \sum_{u\le t}\int_0^{L_u^{-1}-L_{u-}^{-1}} \exp\{\alpha(\xi_{s+L_{u-}^{-1}} - \xi_{L_{u-}^{-1}})\}\,\mathrm{d}s$, $t \ge 0$. *The process* (L^{-1}, h, Y) *is a Lévy process with increasing coordinates and*, *when* $\alpha > 0$, $\mathbf{E}(Y_1) < \infty$.

PROOF. Indeed, it follows by decomposing the interval $[0, L_t^{-1}]$ into the excursion intervals that

$$
\begin{aligned}
A_{L_t^{-1}} &= \int_0^{L_t^{-1}} \exp\{\alpha\xi_s\}\,\mathrm{d}s \\
&= \int_0^{L_t^{-1}} \exp\{\alpha\xi_s\}\mathbf{1}_{\{\bar{\xi}_s-\xi_s=0\}}\,\mathrm{d}s + \sum_{u\le t}\int_{L_{u-}^{-1}}^{L_u^{-1}} \exp\{\alpha\xi_s\}\,\mathrm{d}s \\
&= a\int_0^{L_t^{-1}} \exp\{\alpha\xi_s\}\,\mathrm{d}L_s + \sum_{u\le t} e^{\alpha\xi_{L_{u-}^{-1}}}\int_0^{L_u^{-1}-L_{u-}^{-1}} \exp\{\alpha(\xi_{s+L_{u-}^{-1}} - \xi_{L_{u-}^{-1}})\}\,\mathrm{d}s \\
&= a\int_0^{t} \exp\{\alpha\xi_{L_u^{-1}}\}\,\mathrm{d}u + \sum_{u\le t} e^{\alpha\xi_{L_{u-}^{-1}}}\int_0^{L_u^{-1}-L_{u-}^{-1}} \exp\{\alpha(\xi_{s+L_{u-}^{-1}} - \xi_{L_{u-}^{-1}})\}\,\mathrm{d}s \\
&\ge a\int_0^{t} \exp\{\alpha\xi_{L_u^{-1}}\}\,\mathrm{d}u \\
&\quad + e^{(\alpha\vee 0)\xi_{L_{t-}^{-1}}}\sum_{u\le t}\int_0^{L_u^{-1}-L_{u-}^{-1}} \exp\{\alpha(\xi_{s+L_{u-}^{-1}} - \xi_{L_{u-}^{-1}})\}\,\mathrm{d}s.
\end{aligned}
$$

On account of the fact that for $t > 0$, $A_{L_t^{-1}} < \infty$ a.s. we infer that

$$
\sum_{u\le t}\int_0^{L_u^{-1}-L_{u-}^{-1}} \exp\{\alpha(\xi_{s+L_{u-}^{-1}} - \xi_{L_{u-}^{-1}})\}\,\mathrm{d}s < \infty, \qquad \text{a.s.}
$$

It follows that the subordinator Y is well defined. The former calculations and the fact that $\xi_{L^{-1}}$ has countably many discontinuities imply that

$$
\begin{aligned}
A_{L_t^{-1}} &= a\int_0^{t} \exp\{\alpha\xi_{L_{u-}^{-1}}\}\,\mathrm{d}u + \sum_{u\le t} e^{\alpha\xi_{L_{u-}^{-1}}}\int_0^{L_u^{-1}-L_{u-}^{-1}} \exp\{\alpha(\xi_{s+L_{u-}^{-1}} - \xi_{L_{u-}^{-1}})\}\,\mathrm{d}s \\
&= \int_{(0,t]} \exp\alpha\xi_{L_{s-}^{-1}}\,\mathrm{d}Y_s.
\end{aligned}
$$

The fact that (L^{-1}, h, Y) has independent and stationary increments is a consequence of the fact that these processes can be explicitly constructed in terms of the Poisson point process of excursions of ξ from its past supremum. To prove that when $\alpha > 0$, Y has finite mean we use the compensation formula for Poisson point processes

$$\begin{aligned}
\mathbf{E}(Y_t) &= at + \mathbf{E}\left(\sum_{u\leq t}\int_0^{L_u^{-1}-L_{u-}^{-1}} \exp\{\alpha(\xi_{s+L_{u-}^{-1}} - \xi_{L_{u-}^{-1}})\}\,\mathrm{d}s\right)\\
&= at + \mathbf{E}\left(\int_0^t \overline{n}\left(\int_0^{\zeta} \exp\{-\alpha\epsilon(s)\}\,\mathrm{d}s\right)\mathrm{d}u\right)\\
&= t\left(\int_0^{\infty} e^{-\alpha y}\widehat{\mathcal{V}}(\mathrm{d}y)\right) = \frac{t}{\phi_{\hat{\mathrm{h}}}(\alpha)} < \infty,
\end{aligned}$$

where the third and fourth identities follow from the definition of $\widehat{\mathcal{V}}$ in (2.3) and the fact that this is the potential measure of the downward ladder height subordinator $\widehat{\mathrm{h}}$, whose Laplace exponent is given by $\phi_{\hat{\mathrm{h}}}$. □

PROOF OF THEOREM 2. On account of self-similarity we may prove the result under the assumption that $X_0 = 1$. We denote by τ_{h} the time change induced by Lamperti's transformation when applied to the ladder height subordinator h,

$$\tau_{\mathrm{h}}(t) = \inf\left\{s > 0 : \int_0^s e^{\alpha \mathrm{h}_u}\,\mathrm{d}u > t\right\}, \qquad t \geq 0.$$

By making a change of variables in the definition of the process L^{Θ} we observe the relation

$$L_t^{\Theta} = \int_0^{L_{\tau(t)}} \exp\{\alpha \mathrm{h}_s\}\,\mathrm{d}s, \qquad t \geq 0.$$

So that its right-continuous inverse is given by $R_t = A_{L^{-1}_{\tau_{\mathrm{h}}(t)}}$, $t \geq 0$. From Lemma 2, it has the property that its increments are given by

$$\begin{aligned}
R_{t+s} - R_t &= \int_{(\tau_{\mathrm{h}}(t),\tau_{\mathrm{h}}((t+s))]} \exp\{\alpha \mathrm{h}_{u-}\}\,\mathrm{d}Y_u\\
&= z^{\alpha}\int_{(0,\widetilde{\tau}_{\mathrm{h}}(s/z^{\alpha})]} \exp\{\alpha\widetilde{\mathrm{h}}_{u-}\}\,\mathrm{d}\widetilde{Y}_u \qquad z = \exp\{\mathrm{h}_{\tau_{\mathrm{h}}(t)}\}\\
&= H_t^{\alpha}\widetilde{R}_{s/H_t^{\alpha}},
\end{aligned}$$

where $\widetilde{\mathrm{h}}_u = \mathrm{h}_{u+\tau_{\mathrm{h}}(t)} - \mathrm{h}_{\tau_{\mathrm{h}}(t)}$, $\widetilde{Y}_u = Y_{u+\tau_{\mathrm{h}}(t)} - Y_{\tau_{\mathrm{h}}(t)}$, and given that $\tau_{\mathrm{h}}(t)$ is a stopping time in the filtration $\mathcal{H}_v := \sigma\{(L_u^{-1}, \mathrm{h}_u, Y_u), 0 \leq u \leq v\}$, $v \geq 0$, it follows that the process $\widetilde{R}$ is a copy of R issued from 0 and independent of $\mathcal{H}_{\tau_{\mathrm{h}}(t)}$.

On account of the fact that H is obtained by time changing X, which is a strong Markov process, by the right-continuous inverse of the additive functional L^{Θ}, it follows by standard arguments that H is a strong Markov process. Hence the couple (R, H) is a Markov process and the process H is $1/\alpha$-self-similar.

Define $B_{H,t} = \int_0^t H_s^{-\alpha}\,\mathrm{d}s$. Then we have the following equalities:

$$B_{H,t} = \int_{(0,R_t]} X_s^{-\alpha}\,\mathrm{d}L_s^{\Theta} = \int_{(0,R_t]} \mathrm{d}\widetilde{L}_s = \widetilde{L}_{R_t}, \qquad t \geq 0,$$

where we recall that $\widetilde{L}_t = L_{\tau(t)}$. Denote by $C_{H,t}, t \geq 0$, the inverse of the functional $B_{H,\cdot}$. We obtain from our previous calculations that

$$B_{H,t} = L_{\tau(A_{L^{-1}_{\tau_{\mathrm{h}}(t)}})} = \tau_{\mathrm{h}}(t), \qquad C_{H,t} = \int_0^t e^{\alpha \mathrm{h}_s}\,\mathrm{d}s, \qquad R_{C_{H,t}} = A_{L_t^{-1}}, \qquad t \geq 0.$$

These relations allow us to ensure that the process obtained by time changing the process $\log(H_\cdot/H_0)$ by C_H is the process $\mathrm{h}_t = \xi_{L_t^{-1}}, t \geq 0$. We recall that because of Lamperti's transformation

$$\tau(t) = \int_0^t X_s^{-\alpha}\,\mathrm{d}s, \qquad K_t = \tau(R_t) = L^{-1}_{\tau_h(t)}, \qquad t \geq 0.$$

Hence the representation obtained in Lemma 2 allows us to ensure that (K, R, H) time changed by the inverse of $C_{H,\cdot}$ equals $\{(L_t^{-1}, \int_{(0,t]} e^{\alpha \mathrm{h}_{s-}}\,\mathrm{d}Y_s, e^{\mathrm{h}_t}), t \geq 0\}$.

To prove that the process (R, H) is a Feller process in $[0,\infty) \times (0,\infty)$, we observe first that for $t, s \geq 0$

$$e^{\mathrm{h}_{\tau_{\mathrm{h}}(t)}}\big(Y_{\tau_{\mathrm{h}}(t+s)} - Y_{\tau_{\mathrm{h}}(t)}\big) \leq R_{t+s} - R_t \leq e^{\mathrm{h}_{\tau_{\mathrm{h}}(t+s)}}\big(Y_{\tau_{\mathrm{h}}(t+s)} - Y_{\tau_{\mathrm{h}}(t)}\big),$$

$$H_{t+s} - H_t = e^{\mathrm{h}_{\tau_{\mathrm{h}}(t)}}\big(e^{\mathrm{h}_{\tau_{\mathrm{h}}(t+s)} - \mathrm{h}_{\tau_{\mathrm{h}}(t)}} - 1\big),$$

where $X_0 = x = H_0$. By construction τ_{h} is a continuous functional, and thus if $(t_n)_{n\geq 0}$ is a convergent sequence of positive reals with limit t, then the sequence of stopping times $(\tau_{\mathrm{h}}(t_n))_{n\geq 0}$ converges a.s. to $\tau_{\mathrm{h}}(t)$. So, the latter inequalities together with the right-continuity and quasi-left continuity of the Lévy process (h, Y) (see, e.g., Proposition I.7 in [2]) imply that if $t_n \uparrow t$ or $t_n \downarrow t$, then

$$(R_{t_n}, H_{t_n}) \xrightarrow[n\to\infty]{} (R_t, H_t), \qquad \mathbb{P}_1\text{-a.s.}$$

Hence, the Feller property is a simple consequence of these facts and the scaling property. $\square$

REMARK 5. For further generality we can chose to time change X with the right-continuous inverse of the additive functional

$$L_t^{\Theta,\beta} := \int_0^t X_s^{\beta}\,\mathrm{d}\widetilde{L}_s, \qquad t \geq 0,$$

for some $\beta \in \mathbb{R} \setminus \{0\}$, fixed. In that case, we have that the process $(K^{(\beta)}, R^{(\beta)}, H^{(\beta)})$, defined by

$$R_t^{(\beta)} := \inf\{s \geq 0 : L_s^{\Theta,\beta} > t\}, \qquad K_t^{(\beta)} := \int_0^{R_t^{(\beta)}} X_s^{-\alpha}\,\mathrm{d}s,$$

$$H_t^{(\beta)} := X_{R_t^{(\beta)}}, \qquad t \geq 0,$$

has the same law under $\mathbb{P}_x$ as the process

$$\left\{\left(L_t^{-1}, x^\alpha \int_{(0,t]} e^{\alpha \mathrm{h}_{s-}}\,\mathrm{d}Y_s, x e^{\mathrm{h}_t}\right), t \geq 0\right\},$$

time changed by the inverse of the additive functional $(x^\beta \int_0^t e^{\beta \mathrm{h}_s}\,\mathrm{d}s, t \geq 0)$, under $\mathbf{P}$. In this case, the process $(R^{(\beta)}, H^{(\beta)})$ has the following scaling property. For every $c > 0$,

$$((c^\alpha R^{(\beta)}_{tc^{-\beta}}, cH^{(\beta)}_{tc^{-\beta}}), t \geq 0)$$

issued from $(x_1, x_2) \in \mathbb{R}^2_+$ has the same law as $(R^{(\beta)}, H^{(\beta)})$ issued from $(c^\alpha x_1, cx_2)$. The proof of these facts follows along the same lines of the proof of Theorem 2.

4.1. *The resolvent of X.* The main purpose of this subsection is to establish a formula for the resolvent of X in terms of the ladder process (R, H). This result will be very helpful in the proof of Theorem 4.

COROLLARY 2. *Let $\kappa_q(x, \mathrm{d}y)$ be the kernel defined by*

$$\kappa_q(z, \mathrm{d}y) = az^\alpha \delta_z(\mathrm{d}y) + \overline{n}\left(\int_0^\zeta z^\alpha e^{-\alpha\epsilon(s)} e^{-qz^\alpha \int_0^s e^{-\alpha\epsilon(u)}\,\mathrm{d}u} \mathbf{1}_{\{ze^{-\epsilon(s)} \in \mathrm{d}y\}}\,\mathrm{d}s\right),$$
$$y > 0, z > 0.$$

Then the q-resolvent of X, V_q, satisfies

$$V_q f(x) = \mathbb{E}_x\left(\int_0^{L^\Theta_{T_0}} H_t^{-\alpha} e^{-qR_t} \kappa_q(H_t, f)\,\mathrm{d}t\right)$$

for any measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ and $x > 0$.

PROOF. The proof is a consequence of Proposition 1, Theorem 1, the identity

$$\begin{aligned} V_q f(x) &:= \mathbb{E}_x\left(\int_0^\infty e^{-qt} f(X_t)\,\mathrm{d}t\right) \\ &= \mathbb{E}_x\left(\int_0^\infty e^{-qs} \mathbf{1}_{\{X_s = M_s\}} f(X_s)\,\mathrm{d}s\right) \\ &\quad + \mathbb{E}_x\left(\sum_{G \in \mathsf{G}} e^{-qG}\left(\int_0^{D-G} e^{-qs} f\left(M_G\left(\frac{M_G}{X_{s+G}}\right)^{-1}\right)\right)\mathrm{d}s\right) \end{aligned} \tag{4.2}$$

(with the notation of Theorem 1) and that $L^\Theta_{T_0} = \int_0^{L_\infty} e^{\alpha \mathrm{h}_s}\,\mathrm{d}s$ under $\mathbf{P}$, since L_∞ is the lifetime of h. $\square$

Observe that for $q=0$, the kernel κ_0 becomes

$$\kappa_0(z,f) = az^\alpha f(z) + \overline{n}\left(\int_0^\zeta z^\alpha e^{-\alpha\epsilon(s)} f\big(ze^{-\epsilon(s)}\big)\,\mathrm{d}s\right)$$
$$= \int_{[0,\infty[} (ze^{-y})^\alpha f(ze^{-y})\widehat{\mathcal{V}}(\mathrm{d}y),$$

where the second identity is a consequence of (2.3). Recall that H time changed by the inverse of $\int_0^t H_s^{-\alpha}\,\mathrm{d}s$, $t\geq 0$, is equal to e^{h}. As a consequence we obtain that the 0-resolvent for X is given by the formula

$$V_0 f(x) = \iint_{[0,\infty)\times[0,\infty)} x^\alpha e^{\alpha(z-y)} f(xe^{z-y})\mathcal{V}(\mathrm{d}z)\widehat{\mathcal{V}}(\mathrm{d}y),$$

where $\mathcal{V}(\mathrm{d}y)$ denotes the renewal measure for h, that is to say

$$\mathcal{V}(\mathrm{d}y) = \mathbf{E}\left(\int_0^{L_\infty} \mathbf{1}_{\{\mathrm{h}(t)\in\mathrm{d}y\}}\,\mathrm{d}t\right), \qquad y\geq 0.$$

5. Applications to entrance laws and weak convergence. We will assume hereafter that $\alpha>0$ and X is such that $\limsup_{t\to\infty} X_t=\infty$ a.s. which is equivalent to assume that, for the underlying Lévy process ξ, it holds that $\limsup_{t\to\infty}\xi_t=\infty$, a.s.

LEMMA 3. *For every $t>0$, the following equality in law holds*:

$$\begin{aligned}(5.1)\qquad &\big(\big(\mathrm{h}_t-\mathrm{h}_{(t-s)-}, 0\leq s\leq t\big), e^{-\alpha\mathrm{h}_t}A_{L_t^{-1}}\big)\\ &\stackrel{(d)}{=}\left(\big(\mathrm{h}_s, 0\leq s\leq t\big), \int_{(0,t]} e^{-\alpha\mathrm{h}_{r-}}\,\mathrm{d}Y_r + \sum_{u\leq t}(e^{-\alpha\Delta\mathrm{h}_u}-1)e^{\alpha\mathrm{h}_{u-}}\Delta Y_u\right).\end{aligned}$$

Furthermore, if $\alpha>0$ and $\mathbf{E}(\mathrm{h}_1)<\infty$, then the stochastic process

$$W_t := \int_{(0,t]} e^{-\alpha\mathrm{h}_{r-}}\,\mathrm{d}Y_r + \sum_{u\leq t}(e^{-\alpha\Delta\mathrm{h}_u}-1)e^{-\alpha\mathrm{h}_{u-}}\Delta Y_u$$

converges a.s., as $t\to\infty$, to a random variable $\widetilde{I}$ satisfying

$$\widetilde{I} := \int_0^\infty \exp\{-\alpha\mathrm{h}_s\}\,\mathrm{d}Y_{s-} \stackrel{(d)}{=} \int_0^\infty \exp\{-\alpha\xi_s^\uparrow\}\,\mathrm{d}s.$$

PROOF. We start by proving the time reversal property described in (5.1). On the one hand the duality lemma for Lévy processes implies that for $t>0$

$$\big(\big(\mathrm{h}_t-\mathrm{h}_{(t-s)-}, Y_t-Y_{(t-s)-}\big), 0\leq s\leq t\big) \stackrel{(\mathrm{d})}{=} \big((\mathrm{h}_s, Y_s), 0\leq s\leq t\big).$$

It follows, making a change of variables of the form $t-u$ together with the above identity in law, that

$$
\begin{aligned}
e^{-\alpha \mathrm{h}_t}\int_{(0,t]} e^{\alpha \mathrm{h}_{s-}}\,\mathrm{d}Y_s &= \int_{(0,t]} e^{-\alpha(\mathrm{h}_t-\mathrm{h}_{(t-u)-})}\,\mathrm{d}(Y_t-Y_{t-u})\\
&\stackrel{(\mathrm{d})}{=} \int_{(0,t]} e^{-\alpha \mathrm{h}_s}\,\mathrm{d}Y_{s-}\\
&= \int_{(0,t]} e^{-\alpha \mathrm{h}_{r-}}\,\mathrm{d}Y_r + \sum_{u\le t}(e^{-\alpha\Delta \mathrm{h}_u}-1)e^{-\alpha \mathrm{h}_{u-}}\Delta Y_u.
\end{aligned}
$$

A similar identity for general Lévy processes can be found in [22]. Observe that the process T defined by $T_t = Y_t + \sum_{s\le t}(e^{-\alpha\Delta \mathrm{h}_s}-1)\Delta Y_s$, $t\ge 0$, is a Lévy process and for any $t>0$,

$$
\int_{(0,t]} e^{-\alpha \mathrm{h}_{r-}}\,\mathrm{d}Y_r + \sum_{u\le t}(e^{-\alpha\Delta \mathrm{h}_u}-1)e^{-\alpha \mathrm{h}_{u-}}\Delta Y_u = \int_{(0,t]} e^{-\alpha \mathrm{h}_{s-}}\,dT_s.
$$

To finish the proof of the convergence of W_t, we should verify that, when $\alpha>0$ and $\mathbf{E}(\mathrm{h}_1)<\infty$, the right-hand side of the above equality converges a.s. Thanks to the assumption that $\mathbf{E}(\mathrm{h}_1)<\infty$, according to Theorem 2.1 and Remark 2.2 in [22] it suffices to verify that $\mathbf{E}(\log^+(T_1))<\infty$. Indeed, given that $T_1\le Y_1$ a.s. it follows from Lemma 2 that under our assumptions $\mathbf{E}(Y_1)<\infty$, and thus $\mathbf{E}(\log^+(Y_1))<\infty$ which in turn implies that $\mathbf{E}(\log^+(T_1))<\infty$.

Finally the identity in law between $\widetilde{I}$ and $\int_0^\infty \exp\{-\alpha\xi_s^{\uparrow}\}\,\mathrm{d}s$ follows from the Doney–Tanaka path construction of the process conditioned to stay positive. Roughly speaking, the latter allows us to construct the process $\xi^{\uparrow}$ by pasting, at the level of the last supremum, the excursions of ξ from its past supremum, reflected and time reversed. More precisely, for $t\ge 0$, let

$$
g_t = \sup\{s<t : \overline{\xi}_s - \xi_s = 0\}, \qquad d_t = \inf\{s>t : \overline{\xi}_s - \xi_s = 0\},
$$

$$
\widetilde{\mathcal{R}}_t = \begin{cases} (\overline{\xi}-\xi)_{(d_t+g_t-t)-}, & \text{if } d_t-g_t>0,\\ 0, & \text{if } d_t-g_t=0,\end{cases} \qquad \mathcal{R}_t = \overline{\xi}_{d_t} + \widetilde{\mathcal{R}}_t.
$$

The process $\{\mathcal{R}_t, t\ge 0\}$, under $\mathbf{P}$ has the same law as $(\xi^{\uparrow},\mathbf{P}^{\uparrow})$. See [15], Section 8.5.1, for a proof of this result. Taking account of the Doney–Tanaka construction, we may proceed as in the proof of Lemma 2 and obtain that

$$
\begin{aligned}
&\int_0^\infty \exp\{-\alpha\xi_s^{\uparrow}\}\,\mathrm{d}s\\
&\quad\stackrel{(\mathrm{d})}{=} \int_0^\infty \exp\{-\alpha\overline{\xi}_{d_s}\}\mathbf{1}_{\{d_s=g_s\}}\,\mathrm{d}s\\
&\qquad + \sum_{t\in\widetilde{\mathsf{G}}}\int_{g_t}^{d_t} \exp\{-\alpha(\overline{\xi}_{d_s} + (\overline{\xi}-\xi)_{(d_s+g_s-s)-})\}\,\mathrm{d}s,
\end{aligned}
$$

where $\widetilde{\mathsf{G}}$ denotes the left extrema of the excursion intervals of ξ from $\overline{\xi}$. We recall that g_s, d_s, remain constant along the excursion intervals, that $d_s = L^{-1}_{L_s}$, $g_s = L^{-1}_{L_s-}$, and that $d_s = g_s$ if and only if s belongs to the interior of the random set of points where $\overline{\xi} - \xi$, takes the value 0. Using these facts we obtain the following identities:

$$
\begin{aligned}
&\int_0^\infty \exp\{-\alpha\xi^{\uparrow}_s\}\,\mathrm{d}s\\
&\quad\stackrel{\text{(d)}}{=} \int_0^\infty \exp\{-\alpha\overline{\xi}_s\}\mathbf{1}_{\{\xi_s=\overline{\xi}_s\}}\,\mathrm{d}s\\
&\qquad + \sum_{t\geq 0}\int_{L^{-1}_{t-}}^{L^{-1}_t} \exp\{-\alpha\bigl(\overline{\xi}_{L^{-1}_t} + (\overline{\xi}-\xi)_{L^{-1}_t+L^{-1}_{t-}-s}\bigr)\}\,\mathrm{d}s\\
&\quad= a\int_0^\infty \exp\{-\alpha\overline{\xi}_s\}\,\mathrm{d}L_s\\
&\qquad + \sum_{t\geq 0} e^{-\alpha\overline{\xi}_{L^{-1}_t}}\int_{L^{-1}_{t-}}^{L^{-1}_t} \exp\{-\alpha(\overline{\xi}_{L^{-1}_{t-}} - \xi_{L^{-1}_t+L^{-1}_{t-}-s})\}\,\mathrm{d}s\\
&\quad= a\int_0^\infty \exp\{-\alpha\overline{\xi}_s\}\,\mathrm{d}L_s\\
&\qquad + \sum_{t\geq 0} e^{-\alpha\overline{\xi}_{L^{-1}_t}}\int_0^{L^{-1}_t-L^{-1}_{t-}} \exp\{-\alpha(\overline{\xi}_{L^{-1}_{t-}} - \xi_{L^{-1}_{t-}+u})\}\,\mathrm{d}u\\
&\quad= \int_0^\infty \exp\{-\alpha\overline{\xi}_{L^{-1}_t}\}\,\mathrm{d}Y_t.
\end{aligned}
$$

To conclude we should justify that

$$
\int_0^\infty \exp\{-\alpha\overline{\xi}_{L^{-1}_t}\}\,\mathrm{d}Y_t = \int_0^\infty \exp\{-\alpha\overline{\xi}_{L^{-1}_t}\}\,\mathrm{d}Y_{t-}, \qquad \mathbf{P}\text{-a.s.}
$$

Indeed, we have that for every $s > 0$

$$
\int_{(0,s]} \exp\{-\alpha\overline{\xi}_{L^{-1}_t}\}\,\mathrm{d}Y_t = \int_{(0,s)} \exp\{-\alpha\overline{\xi}_{L^{-1}_t}\}\,\mathrm{d}Y_{t-} + (Y_s - Y_{s-})e^{-\alpha\overline{\xi}_{L^{-1}_s}}.
$$

Furthermore, the last term on the right-hand side above tends to 0 as s growths to infinity, $\mathbf{P}$-a.s. because Y_s and $\overline{\xi}_{L^{-1}_s}$ have linear growth, which in turn is thanks to the hypothesis $\mathbf{E}(\mathrm{h}_1) < \infty$, and the fact $\mathbf{E}(Y_1) < \infty$ (which was established in Lemma 2). The identity follows. $\square$

PROOF OF THEOREM 3. Our objective will be achieved in three main steps. First, we will prove that the resolvent of (R, H) under $\mathbb{P}_x$ has a nondegenerate limit as $x \to 0$. Second, we will deduce the finite-dimensional convergence, using an argument that follows along the lines of Bertoin and Yor's proof of the finite-dimensional convergence of a pssMp as the starting point tends to 0. (Our argument is essentially a rewording of those in the three paragraphs following the second display on page 396 of [5]. We include the arguments here for sake of completeness.) We recall that the identity in law between $\widetilde{I}$ and $\int_0^\infty \exp\{-\alpha\xi_s^{\uparrow}\}\,\mathrm{d}s$, has been proved in Lemma 3. So, to finish we will prove the formula in Theorem 3 for the limit law of R, H under $\mathbb{P}_x$ as $x \to 0$.

Step 1. Observe that by construction the process (R, H) issued from $(R_0 = y, H_0 = x)$ has the same law as $(R + y, H)$ issued from $(R_0 = 0, H_0 = x)$. We denote by $V_q^{R,H}$, $q \geq 0$, the q-resolvent of the process (R, H), namely for continuous and bounded $f : \mathbb{R}_+^2 \to \mathbb{R}_+$,

$$V_q^{R,H} f(y, x) := \mathbb{E}_x\biggl(\int_0^\infty e^{-qt} f(y + R_t, H_t)\,\mathrm{d}t\biggr), \tag{5.2}$$

where we recall that by construction $R_0 = 0$, $\mathbb{P}_x$-a.s. As this operator is clearly continuous in y we can assume without loss of generality that $y = 0$. We will prove that $V_q^{R,H} f(0, x)$ has a nondegenerate limit as $x \to 0+$. Recall that under $\mathbb{P}_x$ the process $((R_t, H_t)_{t\geq 0})$ is equal in law to the process

$$\bigl(\bigl(x^\alpha A_{L^{-1}_{\tau_{\mathrm{h}}(t/x^\alpha)}}, x e^{\mathrm{h}_{\tau_{\mathrm{h}}(t/x^\alpha)}}\bigr), t \geq 0\bigr) \qquad \text{under } \mathbf{P}.$$

We will need the following identity for $f : \mathbb{R}^2 \to \mathbb{R}_+$ measurable:

$$\begin{aligned}
&\mathbb{E}_x\biggl(\int_0^\infty e^{-qt} f(R_t, H_t)\,\mathrm{d}t\biggr)\\
&\quad= \mathbf{E}\biggl(\int_0^\infty e^{-qx^\alpha \int_0^t e^{\alpha \mathrm{h}_u}\,\mathrm{d}u} x^\alpha e^{\alpha\mathrm{h}_t} f(x^\alpha A_{L_t^{-1}}, x e^{\mathrm{h}_t})\,\mathrm{d}t\biggr)\\
&\quad= \int_0^\infty \mathbf{E}\biggl(\exp\biggl\{-q x^\alpha e^{\alpha\mathrm{h}_t}\int_0^t e^{-\alpha(\mathrm{h}_t - \mathrm{h}_u)}\,\mathrm{d}u\biggr\}\\
&\qquad\qquad \times x^\alpha e^{\alpha\mathrm{h}_t} f(x^\alpha e^{\alpha\mathrm{h}_t} e^{-\alpha\mathrm{h}_t} A_{L_t^{-1}}, x e^{\mathrm{h}_t})\,\mathrm{d}t\biggr)\\
&\quad= \int_0^\infty \mathbf{E}\biggl(\exp\biggl\{-q x^\alpha e^{\alpha\mathrm{h}_t}\int_0^t e^{-\alpha\mathrm{h}_{u-}}\,\mathrm{d}u\biggr\}\\
&\qquad\qquad \times x^\alpha e^{\alpha\mathrm{h}_t} f(x^\alpha e^{\alpha\mathrm{h}_t} W_t, x e^{\mathrm{h}_t})\,\mathrm{d}t\biggr)\\
&\quad= \mathbf{E}\biggl(\int_0^\infty \exp\biggl\{-q x^\alpha e^{\alpha\mathrm{h}_{\tau_{\mathrm{h}}(u/x^\alpha)}}\int_0^{\tau_{\mathrm{h}}(u/x^\alpha)} e^{-\alpha\mathrm{h}_{s-}}\,\mathrm{d}s\biggr\}\\
&\qquad\qquad \times f\bigl(x^\alpha e^{\alpha\mathrm{h}_{\tau_{\mathrm{h}}(u/x^\alpha)}} W_{\tau_{\mathrm{h}}(u/x^\alpha)}, x e^{\mathrm{h}_{\tau_{\mathrm{h}}(u/x^\alpha)}}\bigr)\,\mathrm{d}u\biggr),
\end{aligned} \tag{5.3}$$

where in the first equality we made the change of variables $u = \tau_h(t)$; in the second we applied Fubini's theorem; in the third we used the time inversion property obtained in Lemma 3 (note also that the process W was defined in Lemma 3); finally in the fourth we used Fubini's theorem again and applied the change of variables $t = \tau_{\mathrm{h}}(u)$.

Next we note that a consequence of Lemma 3 is that under our assumptions the pair $(W_{\tau_{\mathrm{h}}(s)}, \int_0^{\tau_{\mathrm{h}}(s)} e^{-\alpha \mathrm{h}_{t-}}\,\mathrm{d}t)$ converges a.s. to $(\widetilde{I}, I_{\mathrm{h}})$, as $s \to \infty$. The fact that h is nonarithmetic and $\mathbf{E}(\mathrm{h}_1) < \infty$ imply that $H_t t^{-1/\alpha}$ converges weakly to a nondegenerated random variable Z, by the main theorem in [4]. Using these results we deduce that

$$\begin{aligned}&\left(u^{-1/\alpha} e^{\mathrm{h}_{\tau(u)}}, u^{-1/\alpha} e^{\mathrm{h}_{\tau(u)}} \int_0^{\tau(u)} e^{-\alpha \mathrm{h}_{s-}}\,\mathrm{d}s, \left(u^{-1/\alpha} e^{\mathrm{h}_{\tau(u)}}\right)^{\alpha} W_{\tau_{\mathrm{h}}(u)}\right)\\ &\qquad \xrightarrow[u\to\infty]{(\mathrm{d})} \left(Z, Z\int_0^{\infty} e^{-\alpha \mathrm{h}_s}\,\mathrm{d}s, Z^{\alpha}\widetilde{I}\right)\end{aligned}$$

under $\mathbf{P}$. Thus if $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a continuous and bounded function then equation (5.3) and Fatou's lemma imply that

$$\liminf_{x\to 0+} \mathbb{E}_x\left(\int_0^{\infty} \mathrm{d}t\, e^{-qt} f(R_t, H_t)\right) \geq \int_0^{\infty} \mathbf{E}(\exp\{-qtZ^{\alpha} I_{\mathrm{h}}\} f(tZ^{\alpha}\widetilde{I}, t^{1/\alpha} Z))\,\mathrm{d}t.$$

Furthermore, let $M = \sup_{(z,y)\in\mathbb{R}_+^2} f(y,z)$ and $f^c(z,y) = M - f(z,y)$, and apply the latter estimate to f^c to get that

$$\begin{aligned}\frac{M}{q} &- \limsup_{x\to 0+} \mathbb{E}_x\left(\int_0^{\infty} e^{-qt} f(R_t, H_t)\,\mathrm{d}t\right)\\ &= \liminf_{x\to 0+} \mathbb{E}_x\left(\int_0^{\infty} e^{-qt} f^c(R_t, H_t)\,\mathrm{d}t\right)\\ &\geq \int_0^{\infty} \mathbf{E}(\exp\{-qtZ^{\alpha} I_{\mathrm{h}}\} f^c(tZ^{\alpha}\widetilde{I}, t^{1/\alpha} Z))\,\mathrm{d}t\\ &= M\int_0^{\infty} \mathbf{E}(\exp\{-qtZ^{\alpha} I_{\mathrm{h}}\})\,\mathrm{d}t\\ &\qquad - \int_0^{\infty} \mathbf{E}(\exp\{-qtZ^{\alpha} I_{\mathrm{h}}\} f(tZ^{\alpha}\widetilde{I}, t^{1/\alpha} Z))\,\mathrm{d}t.\end{aligned}$$

We have thus proved the inequalities

$$\begin{aligned}\int_0^{\infty} &\mathbf{E}(\exp\{-qtZ^{\alpha} I_{\mathrm{h}}\} f(tZ^{\alpha}\widetilde{I}, t^{1/\alpha} Z))\,\mathrm{d}t\\ &\leq \liminf_{x\to 0+} \mathbb{E}_x\left(\int_0^{\infty} e^{-qt} f(R_t, H_t)\,\mathrm{d}t\right)\\ &\leq \limsup_{x\to 0+} \mathbb{E}_x\left(\int_0^{\infty} e^{-qt} f(R_t, H_t)\,\mathrm{d}t\right)\end{aligned}$$

$$\leq \frac{M(1-\mathbf{E}((Z^\alpha I_{\mathrm{h}})^{-1}))}{q} + \int_0^\infty \mathbf{E}(\exp\{-qtZ^\alpha I_{\mathrm{h}}\} f(tZ^\alpha \widetilde{I}, t^{1/\alpha} Z))\,\mathrm{d}t.$$

Next, we need to verify that

$$\mathbf{E}((Z^\alpha I_{\mathrm{h}})^{-1}) = 1. \tag{5.4}$$

To this end, observe the following duality identity:

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\mathbf{E}\biggl(\exp\biggl\{-qx^\alpha e^{\alpha \mathrm{h}_{\tau_{\mathrm{h}}(t/x^\alpha)}} \int_0^{\tau_{\mathrm{h}}(t/x^\alpha)} e^{-\alpha \mathrm{h}_{s-}}\,\mathrm{d}s\biggr\}\biggr)\,\mathrm{d}t \\ = \int_0^\infty e^{-qt}\mathbf{E}\biggl(\exp\biggl\{-\lambda x^\alpha e^{\alpha \mathrm{h}_{\tau_{\mathrm{h}}(t/x^\alpha)}} \int_0^{\tau_{\mathrm{h}}(t/x^\alpha)} e^{-\alpha \mathrm{h}_{s-}}\,\mathrm{d}s\biggr\}\biggr)\,\mathrm{d}t,\end{aligned} \tag{5.5}$$

valid for $\lambda, q \geq 0$. The proof of this identity uses the same arguments as in (5.3). Taking $x \to 0$ and using the dominated convergence theorem we deduce from (5.5) that

$$\begin{aligned}\mathbf{E}\biggl(\frac{1}{\lambda + qZ^\alpha I_h}\biggr) &= \int_0^\infty e^{-\lambda t}\mathbf{E}(\exp\{-qZ^\alpha I_{\mathrm{h}} t\})\,\mathrm{d}t \\ &= \int_0^\infty e^{-qt}\mathbf{E}(\exp\{-\lambda Z^\alpha I_{\mathrm{h}} t\})\,\mathrm{d}t \\ &= \mathbf{E}\biggl(\frac{1}{q + \lambda Z^\alpha I_h}\biggr)\end{aligned} \tag{5.6}$$

for $q, \lambda > 0$. We may now let q tend to $0+$ and apply the monotone convergence theorem to obtain the claimed equality.

The latter facts and a change of variables lead to

$$\begin{aligned}\lim_{x\to 0+} \mathbb{E}_x\biggl(\int_0^\infty e^{-qt} f(R_t, H_t)\biggr) &= \int_0^\infty \mathbf{E}(\exp\{-qtZ^\alpha I_{\mathrm{h}}\} f(tZ^\alpha \widetilde{I}, t^{1/\alpha} Z))\,\mathrm{d}t \\ &= \int_0^\infty e^{-qt}\mathbf{E}\biggl(f\biggl(\frac{t\widetilde{I}}{I_{\mathrm{h}}}, \frac{t^{1/\alpha}}{I_{\mathrm{h}}^{1/\alpha}}\biggr)\frac{1}{Z^\alpha I_{\mathrm{h}}}\biggr)\,\mathrm{d}t.\end{aligned}$$

We have thus proved the convergence of the resolvent of (R, H) under $\mathbb{P}_x$ as $x \to 0$.

Step 2. We define for $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ measurable

$$\begin{aligned}V_q^{R,H} f(y, 0) &= \lim_{x\to 0+} \mathbb{E}_x\biggl(\int_0^\infty e^{-qt} f(y + R_t, H_t)\,\mathrm{d}t\biggr) \\ &= \int_0^\infty e^{-qt}\mathbf{E}\biggl(f\biggl(y + \frac{t\widetilde{I}}{I_{\mathrm{h}}}, \frac{t^{1/\alpha}}{I_{\mathrm{h}}^{1/\alpha}}\biggr)\frac{1}{Z^\alpha I_{\mathrm{h}}}\biggr)\,\mathrm{d}t, \qquad y \geq 0.\end{aligned} \tag{5.7}$$

Let C_0 be the space of continuous functions on $\mathbb{R}_+^2$ with limit 0 at infinity. Observe that for every $t > 0$, the stopping time $\tau_h(tx^{-\alpha})$ tends to ∞ (resp., to 0) as $x \to 0$ (resp., to ∞) and that $A_{L_s^{-1}}$ tends to ∞ (resp., to 0) as $s \to \infty$ (resp., $s \to 0$). Moreover, given that

$$A_u/u \xrightarrow[u\to 0]{} 1, \qquad \tau_h(u)/u \xrightarrow[u\to 0]{} 1 \quad \text{and} \quad L_u^{-1}/u \xrightarrow[u\to 0]{} a,$$

P-almost surely, it follows that

$$x^\alpha A_{L^{-1}_{\tau_h(x^{-\alpha})}} \xrightarrow[x\to\infty]{} a,$$

P-a.s. Using these facts and the results in Theorem 2(i), we deduce that for every function $f \in C_0$, the function $x, y \mapsto \mathbb{E}_x(f(y + R_t, H_t))$ has a limit 0 at infinity. Now, applying the result in Theorem 2(ii) it follows that the operator $V_q^{R,H}$ maps C_0 into C_0, and furthermore the resolvent equation holds on C_0. It follows from (5.4), (5.7) and the Feller property in Theorem 2 that for $f \in C_0$, $qV_q^{R,H}$ converges pointwise to f as $q \to \infty$. By the discussion on page 83 in [25] it follows that this implies the uniform convergence of $qV_q^{R,H} f$ toward f as $q \to \infty$ for $f \in C_0$. Now we invoke the Hille–Yoshida theory to deduce that associated to the resolvent family $qV_q^{R,H}$ there is a unique strongly continuous Markovian semi-group on C_0. The finite-dimensional convergence follows.

Step 3. We will next establish the formula for the entrance law for (R, H). On the one hand, by the scaling property of (R, H) the weak convergence of the one-dimensional law of (R, H) as the starting point of X tends to 0, is equivalent to the weak convergence of $(t^{-1}R_t, t^{-1/\alpha}H_t)$ as $t \to \infty$, under $\mathbb{P}_1$, as $t \to \infty$. Moreover, using the self-similarity and Feller properties of (R, H) and arguing as in Sections 5.2 to 5.4 of [9], one may show that the process

$$\mathrm{OU}_t := (e^{-t}R_{e^t-1}, e^{-t/\alpha}H_{e^t-1}), \qquad t \geq 0,$$

is an homogeneous Markov process which, under our assumptions, has a unique invariant measure. On the other hand, it has been proved in [29] that for any $c > 0$, the measure defined for $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ positive and measurable by

$$c\mathbf{E}\left(f\left(\frac{s\widetilde{I}}{I_{\mathrm{h}}}, \frac{s^{1/\alpha}}{I_{\mathrm{h}}^{1/\alpha}}\right)\frac{1}{I_{\mathrm{h}}}\right)$$

is an entrance law for the process (R, H). It readily follows that this measure, taking $s = 1$, is an invariant measure for the process OU. Furthermore, the constant c can be chosen so that the measure defined above is a probability measure. This follows as a consequence of the fact that under our assumptions, $\mathbf{E}(I_{\mathrm{h}}^{-1}) = \alpha\mu_+ < \infty$ (see, e.g., [4] and [5]). Therefore, we conclude that

$$\frac{1}{\alpha\mu_+}\mathbf{E}\left(f\left(\frac{s\widetilde{I}}{I_{\mathrm{h}}}, \frac{s^{1/\alpha}}{I_{\mathrm{h}}^{1/\alpha}}\right)\frac{1}{I_{\mathrm{h}}}\right) = \mathbf{E}\left(f\left(\frac{s\widetilde{I}}{I_{\mathrm{h}}}, \frac{s^{1/\alpha}}{I_{\mathrm{h}}^{1/\alpha}}\right)\frac{1}{Z^\alpha I_{\mathrm{h}}}\right)$$

for every positive and measurable function f. This implies in particular that

$$\mathbf{E}(Z^{-\alpha}|I_{\mathrm{h}}, \widetilde{I}) = \frac{1}{\alpha\mu_+}.$$

Which finishes the proof of Theorem 3. □

5.1. *Proof of Theorem 4: Finite-dimensional convergence.* We will prove this result in two main steps, first of all we will prove the convergence in the finite-dimensional sense and then in Section 5.2 we will prove that the convergence holds in the Skorohod's sense.

As in the proof of Theorem 3, the most important tool to prove convergence of finite-dimensional distributions is to establish the convergence of the resolvent of X as x tends to 0. One may then appeal to reasoning along the lines the proof of Theorem 1 in [5] (see also the second step in the proof of our Theorem 3). Note that, while Bertoin and Yor [5] require that the underlying Lévy process drifts to ∞, we may circumvent the use of this condition as we are able to write the resolvent of X in terms of the process (R, H). We omit the details. We will finish the proof by establishing the formula for the entrance law.

First of all we recall that in Corollary 2 we established that the q-resolvent of X is given by

$$V_q f(x) = \int_0^\infty \mathbb{E}_x(H_t^{-\alpha} e^{-qR_t}\kappa_q(H_t, f))\,\mathrm{d}t \tag{5.8}$$

for every $f:\mathbb{R}_+ \to \mathbb{R}_+$ measurable and bounded. It follows that for $q > 0$,

$$\mathbb{E}_x\left(\int_0^\infty e^{-qR_t} H_t^{-\alpha}\kappa_q(H_t, 1)\,\mathrm{d}t\right) = \frac{1}{q}, \qquad x > 0. \tag{5.9}$$

To prove the convergence of the resolvent V_q it will be useful to know that the mapping $x \mapsto x^{-\alpha}\kappa_q(x, 1)$, for $x > 0$ defines a decreasing, continuous and bounded function. Indeed for every $x > 0$

$$\begin{aligned} x^{-\alpha}\kappa_q(x, 1) &= a + (qx^\alpha)^{-1}\overline{n}\left(1 - \exp\left\{-qx^\alpha \int_0^\zeta e^{-\alpha\epsilon(s)}\,\mathrm{d}s\right\}\right) \\ &= a + \int_0^\infty \mathrm{d}y\,\overline{n}\left(\int_0^\zeta e^{-\alpha\epsilon(s)}\,\mathrm{d}s > y\right)e^{-qx^\alpha y}. \end{aligned}$$

Given that for $\alpha > 0$

$$\lim_{x\to 0+} \frac{\kappa_q(x, 1)}{x^\alpha} = a + \overline{n}\left(\int_0^\zeta e^{-\alpha\epsilon(s)}\,\mathrm{d}s\right) = \mathbf{E}(Y_1) < \infty,$$

the claim follows. The previous limit together with the dominated convergence theorem imply that if $f:\mathbb{R}_+ \to \mathbb{R}$ is a continuous and bounded function then so is

$x \mapsto x^{-\alpha}\kappa_q(x, f)$, for $x > 0$. So, Theorem 3 implies that if f is a continuous and bounded function, then

$$\lim_{x\to 0+} \mathbb{E}_x(e^{-qR_t} H_t^{-\alpha}\kappa_q(H_t, f)) = \mathbb{E}_{0+}^{R,H}(e^{-qR_t} H_t^{-\alpha}\kappa_q(H_t, f)).$$

This limit result together with Fatou's lemma imply that

$$\liminf_{x\to 0+} V_q f(x) \geq \int_0^\infty \mathbb{E}_{0+}^{R,H}(e^{-qR_t} H_t^{-\alpha}\kappa_q(H_t, f))\,\mathrm{d}t. \tag{5.10}$$

To determine the upper limit of $V_q f$ we first claim that

$$\int_0^\infty \mathbb{E}_{0+}^{R,H}(e^{-qR_t} H_t^{-\alpha}\kappa_q(H_t, 1))\,\mathrm{d}t = \frac{1}{q}. \tag{5.11}$$

To see why the above claim is true, note that $\mathbb{P}_{0+}(R_t \in \mathrm{d}s, H_t \in \mathrm{d}y)$ is an entrance law for the semigroup of the self-similar Markov process (R, H), and hence for any $\epsilon > 0$

$$\begin{aligned}
&\int_0^\infty \mathbb{E}_{0+}^{R,H}(e^{-qR_t} H_t^{-\alpha}\kappa_q(H_t, 1))\,\mathrm{d}t \\
&\quad= \int_0^\varepsilon \mathbb{E}_{0+}^{R,H}(e^{-qR_t} H_t^{-\alpha}\kappa_q(H_t, 1))\,\mathrm{d}t \\
&\qquad+ \int_{\mathbb{R}_+\times(0,\infty)} \mathbb{P}_{0+}^{R,H}(R_\varepsilon \in du, H_\varepsilon \in dx) \\
&\qquad\quad\times \int_0^\infty \mathbb{E}_x\big(e^{-q(u+R_t)} H_t^{-\alpha}\kappa_q(H_t, 1)\big)\,\mathrm{d}t \\
&\quad\leq \mathbf{E}(Y_1)\int_0^\varepsilon \mathbb{E}_{0+}^{R,H}(e^{-qtR_1})\,\mathrm{d}t + \frac{1}{q}\mathbb{E}_{0+}^{R,H}(e^{-qR_\varepsilon}) \\
&\quad\leq \mathbf{E}(Y_1)\varepsilon + \frac{1}{q}\mathbb{E}_{0+}^{R,H}(e^{-q\varepsilon R_1}),
\end{aligned} \tag{5.12}$$

where the first inequality comes from (5.9) and the fact that $x^{-\alpha}\kappa_q(x, 1) \leq \mathbf{E}(Y_1) < \infty$, $\forall x \geq 0$. Identity (5.11) is obtained by taking $\varepsilon \to 0+$ in (5.12) and combining the resulting inequality together with an analogous lower bound which can be obtained in a similar way.

By applying the inequality (5.10) to the function $f^c(x) = \sup_{z>0} f(z) - f(x)$, $x > 0$, and using the latter identity we obtain that

$$\limsup_{x\to 0+} V_q f(x) \leq \int_0^\infty \mathbb{E}_{0+}^{R,H}(e^{-qR_t} H_t^{-\alpha}\kappa_q(H_t, f))\,\mathrm{d}t. \tag{5.13}$$

We have therefore proved that for every continuous and bounded function $f : \mathbb{R}_+ \to \mathbb{R}_+$,

$$V_q f(0) := \lim_{x\to 0+} V_q f(x) = \int_0^\infty \mathbb{E}_{0+}^{R,H}(e^{-qR_t} H_t^{-\alpha}\kappa_q(H_t, f))\,\mathrm{d}t \tag{5.14}$$

and, in particular,

$$V_q 1(0) = \frac{1}{q}. \tag{5.15}$$

To finish this part of the proof we will now describe $V_q f(0)$. Using the self-similarity of (R, H), making a change of variables and from the identity for the law of (R_1, H_1) under $\mathbb{P}^{R,H}_{0+}$ obtained in Theorem 3, we obtain that

$$\begin{aligned} V_q f(0) &= \int_0^\infty \mathbb{E}^{R,H}_{0+}(e^{-qtR_1}(t^{1/\alpha}H_1)^{-\alpha}\kappa_q(t^{1/\alpha}H_1, f))\,\mathrm{d}t \\ &= \alpha \int_0^\infty \mathbb{E}^{R,H}_{0+}(\exp\{-qt^\alpha R_1 H_1^{-\alpha}\}H_1^{-\alpha}\kappa_q(t, f))\,\frac{\mathrm{d}t}{t} \\ &= \frac{\alpha}{\alpha\mu_+}\int_0^\infty \mathbf{E}(\exp\{-qt^\alpha \widetilde{I}\}\kappa_q(t, f))\,\frac{\mathrm{d}t}{t}. \end{aligned} \tag{5.16}$$

Next, we observe that the kernel κ_q can be represented as

$$\kappa_q(z, f) = \int_0^\infty f(ze^{-x})(ze^{-x})^\alpha \mathbf{E}^\dagger_x\biggl(\exp\biggl\{-qz^\alpha \int_0^{\varsigma_0} e^{-\alpha\xi_u}\,\mathrm{d}u\biggr\}\biggr)\widehat{\mathcal{V}}(\mathrm{d}x), \tag{5.17}$$

where we recall $\mathbf{E}^\dagger$ is the law of the process ξ conditioned to stay positive, reflected at its future infimum, and ς_0 denotes its first hitting time of 0. Indeed, this is an easy consequence of the fact that the image under time reversal of $\overline{n}$ is the excursion measure, say $n^\uparrow$, of the process of excursions of $\xi^\uparrow$ from its future infimum (see, e.g., Lemma 4 of [1]), that under $n^\uparrow$ the coordinate process has the Markov property with semi-group $P^\dagger_t f(x) = \mathbf{E}^\dagger_x(f(X_t), t < \varsigma_0)$, $t \geq 0$, and the formula (2.3).

Hence, using identity (5.17), the former limit and making some elementary manipulations we obtain that

$$\begin{aligned} V_q f(0) &= \frac{\alpha}{\alpha\mu_+}\mathbf{E}\biggl(\int_0^\infty e^{-qv^\alpha \widetilde{I}}\kappa_q(v, f)\,\frac{\mathrm{d}v}{v}\biggr) \\ &= \frac{\alpha}{\alpha\mu_+}\mathbf{E}\biggl(\int_0^\infty e^{-qv^\alpha \widetilde{I}}\int_0^\infty f(ve^{-x})(ve^{-x})^\alpha \\ &\qquad\qquad \times \mathbf{E}^\dagger_x\biggl(\exp\biggl\{-qv^\alpha\int_0^{\varsigma_0} e^{-\alpha\xi_u}\,\mathrm{d}u\biggr\}\biggr)\widehat{\mathcal{V}}(\mathrm{d}x)\,\frac{\mathrm{d}v}{v}\biggr) \\ &= \frac{\alpha}{\alpha\mu_+}\int_0^\infty l^{\alpha-1}f(l)\int_0^\infty \mathbf{E}(e^{-ql^\alpha e^{\alpha x}\widetilde{I}}) \\ &\qquad\qquad \times \mathbf{E}^\dagger_x\biggl(\exp\biggl\{-ql^\alpha e^{\alpha x}\int_0^{\varsigma_0} e^{-\alpha\xi_u}\,\mathrm{d}u\biggr\}\biggr)\widehat{\mathcal{V}}(\mathrm{d}x)\,\mathrm{d}l \\ &= \alpha\int_0^\infty l^{\alpha-1}f(l)\biggl(\int_0^\infty e^{-ql^\alpha x}\eta(\mathrm{d}x)\biggr)\mathrm{d}l, \end{aligned} \tag{5.18}$$

where η is the sigma finite measure defined by

$$\eta(f) = \frac{1}{\alpha\mu_+}\int_{\mathbb{R}^3_+} \mathbf{P}^\dagger_x\left(\int_0^{\varsigma_0} e^{-\alpha\xi_u}\,\mathrm{d}u \in \mathrm{d}s\right) f\left(e^{\alpha x}(t+s)\right)\mathbf{P}(\widetilde{I} \in \mathrm{d}t)\widehat{\mathcal{V}}(\mathrm{d}x).$$

Note that on account of (5.15) we also have that $\int_{\mathbb{R}_+} x^{-1}\eta(\mathrm{d}x) = 1$. To complete the proof we invert the Laplace transform in (5.18) and recover that the entrance law of X under $\mathbb{P}_{0+}$ is given by

$$\mathbb{E}_{0+}(f(X_t)) = \int_{\mathbb{R}_+} f(t^{1/\alpha}x^{-1/\alpha})x^{-1}\eta(\mathrm{d}x).$$

5.2. *Proof of the weak convergence in Theorem* 4. In Theorems 3 and 4, we proved the convergence in the sense of finite-dimensional distributions for (R, H) and X under $\mathbb{P}_x$, as $x \downarrow 0$. As a consequence, we deduce the following corollary which corresponds to the last part of the statement of Theorem 4 and which was already obtained in [8] under the additional hypothesis $\mathbf{E}(\log^+ \times \int_0^{\varsigma(1,\infty)} \exp\xi_s\,\mathrm{d}s) < \infty$.

COROLLARY 3. *Under the conditions of Theorem* 4, *the family of probability measures* $(\mathbb{P}_x)$ *converges weakly toward* $\mathbb{P}_{0+}$ *as* x *tends to* 0.

PROOF. Fix a sequence $(x_n)_{n\geq 1}$ of positive real numbers which converges to 0. Recall from Theorem 4 that the sequence of probability measures $(\mathbb{P}_{x_n})$ converges to $\mathbb{P}_{0+}$ in the sense of finite-dimensional distributions as $n \to \infty$. We will first prove that the sequence $(\mathbb{P}_{x_n})$ actually converges weakly on $\mathbb{D}([0, 1])$.

To this end, we apply Theorem 15.4 of [7]. First since X has $\mathbb{P}_{0+}$-a.s. no fixed discontinuities, the condition $\mathbb{P}_{0+}(X_{1-} = X_1) = 1$ is satisfied. Then for $0 < \delta < 1$, define

$$W(\delta) = \sup\min\{|X_t - X_{t_1}|, |X_{t_2} - X_t|\},$$

where the supremum extends over t_1, t and t_2 in [0, 1] satisfying

$$t_1 \leq t \leq t_2 \quad \text{and} \quad t_2 - t_1 \leq \delta.$$

From Theorem 15.4 of [7], it remains to prove that for all $\varepsilon > 0$ and $\chi > 0$, there exist $0 < \delta < 1$ and an integer n_0 such that for all $n \geq n_0$

$$\mathbb{P}_{x_n}\big(W(\delta) > \varepsilon\big) \leq \chi. \tag{5.19}$$

Let $\gamma, \delta \in (0, 1)$ such that $\delta < \gamma$ and note that for t_1, t and t_2 in [0, 1] satisfying $t_1 \leq t \leq t_2$ and $t_2 - t_1 \leq \delta$, if $t_1 \in [0, \gamma)$, then $\min\{|X_t - X_{t_1}|, |X_{t_2} - X_t|\} \leq \sup_{0\leq t\leq 2\gamma} X_t$. Hence

$$W(\delta) \leq \sup_{0\leq t\leq 2\gamma} X_t + \sup\min\{|X_t - X_{t_1}|, |X_{t_2} - X_t|\},$$

where the supremum extends over t_1, t and t_2 in $[\gamma, 1]$ satisfying $t_1 \le t \le t_2$ and $t_2 - t_1 \le \delta$. The second term on the right-hand side of the above inequality is smaller than $W(\delta) \circ \theta_\gamma$, so that

$$W(\delta) \le \sup_{0\le t\le 2\gamma} X_t + W(\delta) \circ \theta_\gamma. \tag{5.20}$$

From (5.20) and the Markov property one has for all $n \ge 1$ and for all δ, γ as above,

$$\mathbb{P}_{x_n}\big(W(\delta) > \varepsilon\big) \le \mathbb{P}_{x_n}\Big(\sup_{0\le t\le 2\gamma} X_t > \varepsilon/2\Big) + \mathbb{E}_{x_n}\big(\mathbb{P}_{X_\gamma}\big(W(\delta) > \varepsilon/2\big)\big). \tag{5.21}$$

To deal with the first term in (5.21), we pick $u > 0$ and we write

$$\begin{aligned}
\mathbb{P}_{x_n}\Big(\sup_{0\le t\le 2\gamma} X_t > \varepsilon/2\Big) &= \mathbb{P}_{x_n}\Big(\sup_{0\le t\le 2\gamma} X_t > \varepsilon/2, R_u < 2\gamma\Big) \\
&\quad + \mathbb{P}_{x_n}\Big(\sup_{0\le t\le 2\gamma} X_t > \varepsilon/2, R_u \ge 2\gamma\Big) \\
&\le \mathbb{P}_{x_n}(R_u < 2\gamma) + \mathbb{P}_{x_n}(H_u > \varepsilon/2).
\end{aligned}$$

From Theorem 3, $\mathbb{P}_{x_n}(R_u \in \mathrm{d}s, H_u \in \mathrm{d}y)$ converges weakly to $\mathbb{P}_{0+}(R_u \in \mathrm{d}s, H_u \in \mathrm{d}y)$, as n tends to ∞, hence $\lim_n \mathbb{P}_{x_n}(R_u < 2\gamma) = \mathbb{P}_{0+}(R_u < 2\gamma)$ and $\lim_n \mathbb{P}_{x_n}(H_u > \varepsilon/2) = \mathbb{P}_{0+}(H_u > \varepsilon/2)$ (without loss of generality we can make sure that 2γ and $\varepsilon/2$ are points of continuity of the distribution functions of H_u and R_u, resp., under $\mathbb{P}_{0+}$). Moreover since $H_0 = 0$, $\mathbb{P}_{0+}$ a.s., we have $\lim_{u\downarrow 0} \mathbb{P}_{0+}(H_u > \varepsilon/2) = 0$, so we may find $u > 0$ and an integer n_1 such that for all $n \ge n_1$, $\mathbb{P}_{x_n}(H_u > \varepsilon/2) < \chi/4$. Then since $R_u > 0$, $\mathbb{P}_{0+}$-a.s., we may find $\gamma \in (0, 1)$ and an integer n_2 such that for all $n \ge n_2$, $\mathbb{P}_{x_n}(R_u < 2\gamma) \le \chi/4$.

Next we deal with the second term in (5.21). From Theorem 4, $\mathbb{P}_{x_n}(X_\gamma \in dz)$ converges weakly toward $\mathbb{P}_{0+}(X_\gamma \in dz)$ as n tends to ∞. Moreover, we may easily check, using the Lamperti representation and general properties of Lévy processes, that $x \mapsto \mathbb{P}_x(W(\delta) > \varepsilon/2)$ is continuous on $(0,\infty)$. Since $\mathbb{P}_{0+}(X_\gamma = 0) = 0$, we have $\lim_{n\to\infty} \mathbb{E}_{x_n}(\mathbb{P}_{X_\gamma}(W(\delta) > \varepsilon/2)) = \mathbb{E}_{0+}(\mathbb{P}_{X_\gamma}(W(\delta) > \varepsilon/2))$. For all $x \ge 0$, $\lim_{\delta\downarrow 0} W(\delta) = 0$, $\mathbb{P}_x$-a.s. (see pages 110 and 119 of [7]) so that using dominated convergence, $\lim_{\delta\downarrow 0} \mathbb{E}_{0+}(\mathbb{P}_{X_\gamma}(W(\delta) > \varepsilon/2)) = 0$. Then, we may find n_3 and δ such that for all $n \ge n_3$, $\mathbb{E}_{x_n}(\mathbb{P}_{X_\gamma}(W(\delta) > \varepsilon/2)) \le \chi/2$.

We conclude that (5.19) is satisfied with δ and $n_0 = \max(n_1, n_2, n_3)$, so that the sequence $(\mathbb{P}_{x_n})$ restricted to $\mathbb{D}([0, 1])$ converges weakly to $\mathbb{P}_{0+}$. Then it follows from the same arguments that the sequence $(\mathbb{P}_{x_n})$ restricted to $\mathbb{D}([0, t])$ converges weakly to $\mathbb{P}_{0+}$, for each $t > 0$. Finally it remains to apply Theorem 16.7 of [7] to conclude that $(\mathbb{P}_{x_n})$ converges weakly to $\mathbb{P}_{0+}$ on $\mathbb{D}([0,\infty))$. $\square$

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