A reformulation of the biharmonic map equation

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Abstract

The known Euler-Lagrange equation for (intrinsic) biharmonic maps is unsuitable for the study of some of the critical points of the corresponding functional, as it requires too much regularity. We derive and discuss a variant of the equation that does not have this shortcoming.

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1 Introduction

Biharmonic maps between two Riemannian manifolds are the critical points of a certain functional involving derivatives up to second order. As usual, the critical points are characterised by an Euler-Lagrange equation. Many questions in the theory are therefore reduced to questions on a specific partial differential equation. On the other hand, a closer examination of the underlying variational problem shows that the study of the Euler-Lagrange equation in its usual form gives an incomplete picture of the set of critical points or even the set of global minimisers. This is because some of these points are so irregular that some of the terms in the equation appear meaningless.

In a recent work [2], we have rewritten the Euler-Lagrange equation in the special case of biharmonic maps into a homogeneous space. The new system of equations does not have the shortcomings described; that is, it can be tested for any map that may conceivably be a critical point of the functional, although we did assume additional regularity to prove that the system is equivalent to criticality. In this paper, we first point out a connection between this reformulation of the Euler-Lagrange equation and the notion of Jacobi fields along a map. We then use generalisations of Jacobi fields to give an alternative to the usual Euler-Lagrange equation for general target manifolds. Again the condition that we formulate is meaningful under minimal assumptions, and in the presence of sufficient regularity, it is equivalent to the known Euler-Lagrange equation.

Rather than proving new properties of biharmonic maps (which we do not), the purpose of this paper is to propose a new point of view, especially for studying the energy landscape of the underlying functional rather than smooth

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solutions of the equation. Our formulation of the problem involves another partial differential equation, describing Jacobi fields, that is interesting in its own right but has not received wide attention from analysts (although Jacobi fields along harmonic maps have been studied [3]). Thus the paper also makes a case for further work on this equation.

2 Biharmonic maps

Let (M, g) and (N, h) denote two smooth Riemannian manifolds, and suppose that N is without boundary. A map $u : M \to N$ is called harmonic if it is a critical point of the Dirichlet energy

$$E_1(u) = \frac{1}{2} \int_M |du|^2 \, d\mu_g.$$

Here $d\mu_g$ denotes the volume form of (M, g). The quantity du is a section of the vector bundle $T^*M \otimes u^{-1}TN$. We use the notation $\langle \cdot, \cdot \rangle$ for the metric and $|\cdot|$ for the norm on this and similar vector bundles.

Let ∇ denote the covariant derivative on $u^{-1}TN$ (and similar bundles) induced by the Levi-Civita connection on N. Then harmonic maps satisfy the Euler-Lagrange equation

$$\operatorname{trace} \nabla du = 0. \tag{1}$$

We use the abbreviation $\tau(u) = \operatorname{trace} \nabla du$. This section of $u^{-1}TN$ is called the tension field of u.

In this paper, we study the functional

$$E_2(u) = \frac{1}{2} \int_M |\tau(u)|^2 \, d\mu_g$$

and the Euler-Lagrange equation for its critical points. This equation has been calculated by Jiang [4]. We write Δ for the Laplacian on sections of $u^{-1}TN$ coming from ∇, with a sign convention that makes the operator negative semidefinite. If we write R for the Riemann curvature tensor on N, then the equation derived by Jiang is

$$\Delta\tau(u) + \operatorname{trace} R(\tau(u), du)du = 0. \tag{2}$$

Once the functional E_2 is studied on the natural Sobolev space, a notion of weak solutions of the Euler-Lagrange equation is required. It is not obvious from (2) how to interpret the equation in this context. For this reason, we derived a different version of the equation (equivalent to (2) if u is sufficiently regular) under the assumption that N is a compact homogeneous space in a previous paper [2]. We now describe a somewhat different approach, relying on similar ideas and leading to an equivalent formulation, which we will later generalise to target manifolds that are not necessarily homogeneous.

Consider a Killing vector field Ξ on N and define $X = \Xi \circ u$. This is a section of $u^{-1}TN$ and automatically satisfies the equation

$$\Delta X + \operatorname{trace} R(X, du)du = D_{\tau(u)}\Xi(u), \tag{3}$$

where D is the covariant derivative on N. We regard this as a variant of the equation

$$\Delta X + \operatorname{trace} R(X, du)du = 0. \tag{4}$$

Solutions of (4) are called Jacobi fields along u, and we note that u is a solution of (2) if, and only if, the tension field $\tau(u)$ is a Jacobi field along u. This, however, is not the point that we wish to make.

Let δ be the L^2-adjoint of the exterior derivative d. We also use the symbol Δ for the (negative semidefinite) Laplace-Beltrami operator $\Delta = -\delta d$ on M. Then we calculate

$$\begin{aligned}\Delta \langle \tau(u), X\rangle + 2\delta \langle \tau(u), \nabla X\rangle &= \langle \Delta\tau(u), X\rangle - \langle \tau(u), \Delta X\rangle \\ &= \langle \Delta\tau(u), X\rangle + \langle \tau(u), \operatorname{trace} R(X, du)du\rangle \\ &= \langle \Delta\tau(u) + \operatorname{trace} R(\tau(u), du)du, X\rangle\end{aligned}$$

if $X = \Xi \circ u$ for a Killing vector field Ξ by (3). When we assume that N is a compact homogeneous space, then it follows from a construction of Hélein [1] that there exists a finite set of Killing vector fields that span every tangent space of N. It then follows that u is biharmonic if, and only if,

$$\Delta \langle \tau(u), X\rangle + 2\delta \langle \tau(u), \nabla X\rangle = 0$$

for $X = \Xi(u)$ whenever Ξ is a Killing vector field on N.

In general, we do not insist on working with equation (3), but rather define

$$J(X) = \Delta X + \operatorname{trace} R(X, du)du \tag{5}$$

for sections X of $u^{-1}TN$. For biharmonic maps, we then compute

$$\Delta \langle \tau(u), X\rangle + 2\delta \langle \tau(u), \nabla X\rangle + \langle \tau(u), J(X)\rangle = 0. \tag{6}$$

If u is smooth, then it is not difficult to see that equation (2) is equivalent to the condition that (6) is satisfied for all smooth sections X of $u^{-1}TN$. We will show below that a similar statement holds for weak solutions of the biharmonic map equation. As long as we are in a situation where we can work with equation (2), then this observation may not be very useful. But the new criterion allows to work in spaces of maps with much less regularity, in particular in the Sobolev spaces that are natural when we study the functional E_2 with variational methods.

3 Sobolev spaces

In the calculus of variations, when we consider a functional such as the Dirichlet energy E_1, it is natural to work in an appropriate Sobolev space. In order to define this space, the target manifold N is typically embedded in a Euclidean space by virtue of the Nash embedding theorem. Such an approach would be possible here, too, but some of the points that we wish to make may be clearer if we avoid an ambient space and work entirely in N.

Let $\Lambda(N)$ be the space of all smooth functions $\phi : N \to \mathbb{R}$ with gradient $\operatorname{grad}\phi$ satisfying $h(\operatorname{grad}\phi, \operatorname{grad}\phi) \leq 1$ on N. Furthermore, we define $\tilde{H}^1(M;N)$ to be the space of all measurable functions $u : M \to N$ such that for all $\phi \in$

$\Lambda(N)$, the composition $\phi \circ u$ belongs to the homogeneous Sobolev space $\dot{H}^1(M)$, which is defined as usual. We obtain the space $H^1(M;N)$ from $\tilde{H}^1(M;N)$ by identifying functions that coincide almost everywhere, as usual. If M is compact with a smooth boundary and N is a submanifold of a Euclidean space, then it easy to see that $H^1(M;N)$ coincides with the corresponding Sobolev space defined elsewhere (e.g., by Schoen and Uhlenbeck [8]), because the coordinate functions in the ambient space give rise to functions in $\Lambda(N)$. If $u \in H^1(M;N)$, then in every coordinate chart of M and at almost every point of the chart, the quantities $\frac{\partial}{\partial x^\alpha}(\phi \circ u)$ exist for all $\phi \in \Lambda(N)$ and $\alpha = 1, \ldots, m$, and thus du is well-defined almost everywhere. In particular, we can extend the Dirichlet energy E_1 to $H^1(M;N)$.

When we study biharmonic maps and the energy E_2, then we need to consider second (and eventually higher order) derivatives as well. To this end, suppose that $u \in H^1(M;N)$ and consider the set $L^2(u^{-1}TN)$ of square integrable sections of $u^{-1}TN$ (where again, two sections are identified if they coincide almost everywhere). We use similar notation for other vector bundles such as $T^*M \otimes u^{-1}TN$ over M. Let $\Lambda(TN)$ be the set of all smooth sections Φ of TN with a covariant derivative $D\Phi$ satisfying $h(D\Phi, D\Phi) \leq 1$ everywhere. Then for $\Phi \in \Lambda(TN)$, the section $\Phi \circ u$ of $u^{-1}TN$ has a covariant derivative $\nabla(\Phi \circ u)$ in $L^2(T^*M \otimes u^{-1}TN)$, in local coordinates (on M) expressed by

$$\nabla(\Phi \circ u) = dx^\alpha \otimes D_{\partial u/\partial x^\alpha} \Phi(u).$$

We use these compositions as 'test vector fields' to define a weak covariant derivative for other sections of $u^{-1}TN$.

Let $X \in L^2(u^{-1}TN)$. We say that $Y \in L^2(T^*M \otimes u^{-1}TN)$ is the weak covariant derivative of X if for all $\Phi \in \Lambda(TN)$, the equation

$$d \langle \Phi \circ u, X \rangle = \langle \nabla(\Phi \circ u), X \rangle + \langle \Phi \circ u, Y \rangle$$

holds in the distribution sense. In this case, we write $\nabla X = Y$. The space $H^1(u^{-1}TN)$ consists of all $X \in L^2(u^{-1}TN)$ such that $\nabla X \in L^2(T^*M \otimes u^{-1}TN)$. Moreover, we define $H^2(M;N)$ as the space of all $u \in H^1(M;N)$ with $du \in H^1(T^*M \otimes u^{-1}TN)$.

In general, if N is a submanifold of a Euclidean space, then $H^2(M;N)$ is *not* the same as the second order Sobolev space used in the majority of papers on variational aspects of biharmonic maps, but it is readily seen that it coincides with a space defined by the second author [5].

In a similar way, we can define a weak tension field of u, even without assuming that $u \in H^2(M;N)$: let $u \in H^1(M,N)$ and suppose that there exists a section $T \in L^2(u^{-1}TN)$ such that for all $\Phi \in \Lambda(TN)$,

$$\delta \langle du, \Phi \circ u \rangle + \langle T, \Phi \circ u \rangle + \langle du, \nabla(\Phi \circ u) \rangle = 0$$

distributionally. Then we write $\tau(u) = T$ and we define

$$E_2(u) = \frac{1}{2} \int_M |\tau(u)|^2 \, d\mu_g.$$

Otherwise, we define $E_2(u) = \infty$.

We can now interpret the harmonic map equation (1) for $u \in H^1(M;N)$. Indeed, there exist solutions that are not classically differentiable and do not belong to $H^2(M;N)$, for example the solutions constructed by Rivière [7]. Before we study the biharmonic map equation (2), we need a weak version of the Laplacian Δ on $u^{-1}TN$.

Let $X \in H^1(u^{-1}TN)$. Suppose that there exists a $Y \in L^2(u^{-1}TN)$ such that for all $\Phi \in \Lambda(TN)$,

$$\delta \langle \nabla X, \Phi \circ u \rangle + \langle Y, \Phi \circ u \rangle + \langle \nabla X, \nabla(\Phi \circ u) \rangle = 0.$$

Then we write $\Delta X = Y$ (and when we use the notation, we tacitly assume that such a Y exists). It is easy to see that ΔX is unique if it exists. If $u \in H^2(M;N)$, then we can define ΔX even for $X \in L^2(u^{-1}TN)$ by the condition

$$\Delta \langle X, \Phi \circ u \rangle + 2\delta \langle X, \nabla(\Phi \circ u) \rangle + \langle X, \Delta(\Phi \circ u) \rangle = \langle \Delta X, \Phi \circ u \rangle \tag{7}$$

for all $\Phi \in \Lambda(TN)$—provided, of course, that a section $\Delta X \in L^2(u^{-1}TN)$ with this property exists. With this characterisation, we can define what it means for u to be a weak solution of (2), although we will require a certain amount of regularity of u.

If all we know is that $u \in H^1(M;N)$ with $\tau(u) \in L^2(u^{-1}TN)$, then we are not able to interpret either of the two terms in (2). For the weakly harmonic maps with low regularity mentioned above, however, more cannot be deduced, as even the assumption $u \in H^2(M;N)$ would imply some degree of higher regularity. On the other hand, for a harmonic map, we have $E_2(u) = 0$, so this is a global minimiser of the energy. It should be a solution of the Euler-Lagrange equation.

Using the auxiliary vector fields in equation (6) has the advantage that we may impose additional conditions on X that we cannot expect for $\tau(u)$. We can use this equation under the assumption that $X \in H^1(u^{-1}TN)$ with $J(X) \in L^2(u^{-1}TN)$ and

$$\operatorname*{ess\,sup}_M |X| < \infty,$$

i.e, with $X \in L^\infty(u^{-1}TN)$ for the obvious definition of this space. If N is compact and Ξ is a Killing vector field on N, then the condition is met in particular by $X = \Xi \circ u$.

Definition 3.1. *Suppose that $u \in H^1(M;N)$. A section $X \in H^1(u^{-1}TN)$ is called an* almost Jacobi field *along u if $X \in L^\infty(u^{-1}TN)$ and $J(X) \in L^2(u^{-1}TN)$. We say that u is a* very weakly biharmonic map *if $E_2(u) < \infty$ and for every almost Jacobi field $X \in H^1(u^{-1}TN)$ along u, equation (6) holds true.*

In contrast, the usual notion of weak solutions of (2) requires at least that $u \in H^2(M;N) \cap W^{1,4}(M;N)$ (for the obvious definition of this space), and even then there may be difficulties for unbounded or incomplete target manifolds. But if N is compact and $u \in H^2(M;N) \cap W^{1,4}(M;N)$, then we can define weakly biharmonic maps through identity (7) for $X = \tau(u)$. In such a situation, we expect only $\Delta\tau(u) \in L^1(u^{-1}TN)$, but this is sufficient, as any $\Phi \in \Lambda(TN)$ is then bounded. The above concept is consistent with this definition of weak solutions.

Theorem 3.1. *Suppose that N is compact. Then a map in $H^2(M;N) \cap W^{1,4}(M;N)$ is weakly biharmonic if, and only if, it is very weakly biharmonic.*

In the case of homogeneous target spaces, we made use of the divergence structure of the equations in our previous paper [2] to prove a conditional regularity result. The same would be possible here if we knew that it suffices to test (6) with Jacobi fields along u. But this would require more information about solutions of equation (4). There are many other open questions. For example, can very weak solutions always be interpreted as critical points of E_2 in a well-defined way? Do minimisers of E_2 (say, for fixed boundary data if M has a boundary) necessarily satisfy (6) if they have only minimal regularity? But at least these questions can now be formulated!

4 Proof of Theorem 3.1

Suppose that N is compact and $u \in H^2(M;N) \cap W^{1,4}(M;N)$. We first assume that u is very weakly biharmonic. Then for $\Phi \in \Lambda(TN)$, we note that $J(\Phi \circ u) \in L^2(u^{-1}TN)$. Hence $X = \Phi \circ u$ is an almost Jacobi field, and by the hypothesis, equation (6) holds. Combining it with (5), we obtain

$$\Delta \langle \tau(u), \Phi \circ u \rangle + 2\delta \langle \tau(u), \nabla(\Phi \circ u) \rangle + \langle \tau(u), \Delta(\Phi \circ u) \rangle \\ + \langle \Phi \circ u, \operatorname{trace} R(\tau(u), du)du \rangle = 0.$$

By (7), this means exactly that (2) is satisfied weakly.

The proof of the reverse implication is somewhat more delicate. It relies mostly on well-known results from the theory of partial differential equations, but their application requires some care, because in the Sobolev spaces that we use, we cannot work with local coordinates in N.

Let $X \in H^1(u^{-1}TN)$ be an almost Jacobi field along u. From (5), it follows that $\Delta X \in L^2(u^{-1}TN)$ and there exists a constant C_1, depending only on m and the curvature of N, such that

$$\|\Delta X\|_{L^2(M)} \leq C_1 \|X\|_{L^\infty(M)} \|du\|_{L^4(M)}^2 + \|J(X)\|_{L^2(M)}. \tag{8}$$

We first want to show that every point in the interior of M has a neighbourhood Ω such that $X \in H^2((u|_\Omega)^{-1}TN) \cap W^{1,4}((u|_\Omega)^{-1}TN)$.

Consider a smooth partition of unity

$$1 = \sum_{k=1}^{K} \chi_k$$

on N with the property that for every k, there exist smooth tangent vector fields $e_1^k, \dots, e_n^k$ on N which form an orthonormal basis of the tangent space at every point of the support of χ_k.

Let $\xi \in M \backslash \partial M$. For $r > 0$, let $B_r(\xi)$ denote the open ball in M about ξ with radius r. Fix $r_0 > 0$ such that $B_{2r_0}(\xi) \cap \partial M = \emptyset$. Let $r \in (0, r_0]$ and let $\eta : M \to \mathbb{R}$ be a smooth function with $0 \leq \eta \leq 1$ and with support $\operatorname{supp} \eta \subset B_r(\xi)$. Consider the functions $f_i^k = \eta \left\langle X, e_i^k \circ u \right\rangle$. We calculate

$$\Delta f_i^k = \eta \left\langle e_i^k \circ u, \Delta X \right\rangle + 2 \left\langle \nabla(e_i^k \circ u), \nabla(\eta X) \right\rangle + \eta \left\langle \Delta(e_i^k \circ u), X \right\rangle \\ + 2g\left(d\eta, \left\langle e_i^k \circ u, \nabla X \right\rangle\right) + \Delta\eta \left\langle e_i^k \circ u, X \right\rangle. \tag{9}$$

Most of the terms on the right-hand side belong to $L^2(M)$, except possibly for the term $2\left\langle\nabla(e_i^k\circ u),\nabla(\eta X)\right\rangle$. At the moment, we only know that it is in $L^{4/3}(M)$ by the Hölder inequality. From standard elliptic estimates, we still obtain $f_i^k\in W^{2,4/3}(M)$. Indeed, using (8), we find a constant C_2, depending only on the geometries of $B_{r_0}(\xi)$ and N, such that

$$\begin{aligned}\|f_i^k\|_{W^{2,4/3}(M)} \le{}& C_2\|X\|_{L^\infty(M)}\left(\|du\|_{L^4(M)}^2+\|\nabla du\|_{L^2(M)}+\|\Delta\eta\|_{L^\infty(M)}\right)\\ &+C_2\|d\eta\|_{L^\infty(M)}\|\nabla X\|_{L^2(M)}+C_2\|du\|_{L^4(B_r(\xi))}\|\nabla(\eta X)\|_{L^2(M)}\\ &+C_2\|J(X)\|_{L^2(M)}.\end{aligned}$$

A Gagliardo-Nirenberg inequality [6] then gives constants C_3 and C_4 with the same dependence and with

$$\begin{aligned}\|df_i^k\|_{L^{8/3}(M)} &\le C_3\|f_i^k\|_{L^\infty(M)}\|f_i^k\|_{W^{2,4/3}(M)}\\ &\le C_4\|X\|_{L^\infty(M)}^2\left(\|du\|_{L^4(M)}^2+\|\nabla du\|_{L^2(M)}+\|\Delta\eta\|_{L^\infty(M)}\right)\\ &\quad+C_4\|d\eta\|_{L^\infty(M)}\|X\|_{L^\infty(M)}\|\nabla X\|_{L^2(M)}\\ &\quad+C_4\|du\|_{L^4(B_r(\xi))}\|X\|_{L^\infty(M)}\|\nabla(\eta X)\|_{L^2(M)}\\ &\quad+C_4\|X\|_{L^\infty(M)}\|J(X)\|_{L^2(M)}.\end{aligned}$$

Define

$$p=\sup\left\{q\in[2,4]:\eta X\in W^{1,q}(u^{-1}TN)\right\}.$$

Then $p\ge 8/3$. If $q\in[2,p)$, then the Hölder inequality gives

$$\left\langle\nabla(e_i^k\circ u),\nabla(\eta X)\right\rangle\in L^{\frac{4q}{q+4}}(M).$$

With the same arguments as above, we then find an estimate for df_i^k in $L^{\frac{8q}{q+4}}(M)$. Eventually, using the formula

$$\eta X=\sum_{k=1}^K\sum_{i=1}^n\chi_k f_i^k e_i^k\circ u,$$

we obtain

$$\begin{aligned}\|\nabla(\eta X)\|_{L^{\frac{8q}{q+4}}(M)} \le{}& C_5\|X\|_{L^\infty(M)}^2\left(\|du\|_{L^4(M)}^2+\|\nabla du\|_{L^2(M)}+\|\Delta\eta\|_{L^\infty(M)}\right)\\ &+C_5\|d\eta\|_{L^\infty(M)}\|X\|_{L^\infty(M)}\|\nabla X\|_{L^2(M)}\\ &+C_5\|du\|_{L^4(B_r(\xi))}\|X\|_{L^\infty(M)}\|\nabla(\eta X)\|_{L^q(M)}\\ &+C_5\|X\|_{L^\infty(M)}\|J(X)\|_{L^2(M)}+C_5\|X\|_{L^\infty(M)}+C_5\end{aligned}$$

for a constant C_5 that depends only on the geometries of $B_{r_0}(\xi)$ and N. We can make $\|du\|_{L^4(B_r(\xi))}$ arbitrarily small by choosing r small. Note that $\frac{8q}{q+4}\ge q$ for $q\le 4$. Thus for a suitable choice of r, we obtain estimates for

$$\|\nabla(\eta X)\|_{L^{\frac{8q}{q+4}}(M)}$$

that are uniform in $q\in[2,p)$. In particular, it follows that $\eta X\in W^{1,p}(M)$ and that $p=4$. Using (9) again, we infer $\eta X\in H^2(u^{-1}TN)$.

We conclude that there exists a neighbourhood Ω of x_0 such that $X \in H^2((u|_\Omega)^{-1}TN) \cap W^{1,4}((u|_\Omega)^{-1}TN)$. Next we assume that u is a weak solution of (2) and we claim that

$$\Delta \langle \tau(u), X \rangle + 2\delta \langle \tau(u), \nabla X \rangle + \langle \tau(u), \Delta X \rangle + \langle \operatorname{trace} R(\tau(u), du)du, X \rangle = 0 \quad (10)$$

in Ω. In order to prove this, we apply (7) to the vector field $\tau(u)$ instead of X. For any $\phi \in C_0^\infty(\Omega)$ and any $\Phi \in \Lambda(TN)$, we obtain

$$\int_\Omega \left(\Delta\phi \langle \tau(u), \Phi \circ u \rangle + 2g\left(d\phi, \langle \tau(u), \nabla(\Phi \circ u) \rangle\right) + \phi \langle \tau(u), \Delta(\Phi \circ u) \rangle \right) d\mu_g$$
$$+ \int_\Omega \phi \langle \operatorname{trace} R(\tau(u), du)du, \Phi \circ u \rangle \, d\mu_g = 0. \quad (11)$$

By approximation, we prove the same for $\phi \in H_0^2(\Omega) \cap W_0^{1,4}(\Omega) \cap L^\infty(\Omega)$. Let $\psi \in C_0^\infty(\Omega)$ and set

$$\phi_i^k = \psi \left\langle X, e_i^k \circ u \right\rangle$$

and

$$\Phi_i^k = \chi_k e_i^k.$$

Then we have

$$\psi X = \sum_{k=1}^K \sum_{i=1}^n \phi_i^k \Phi_i^k \circ u.$$

If we use identity (11) for ψ_i^k and Φ_i^k, sum over k and i, and use the Leibniz rule, then we obtain

$$\int_\Omega \left(\Delta\psi \langle \tau(u), X \rangle + 2g\left(d\psi, \langle \tau(u), \nabla X \rangle\right) + \psi \langle \tau(u), \Delta X \rangle \right) d\mu_g$$
$$+ \int_\Omega \psi \langle \operatorname{trace} R(\tau(u), du)du, X \rangle \, d\mu_g = 0,$$

which is (10). Finally, this formula is another representation of (6).

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