ON THE BOX DIMENSIONS OF GRAPHS OF TYPICAL CONTINUOUS FUNCTIONS

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Abstract. Let $X \subseteq \mathbb{R}$ be a bounded set; we emphasize that we are not assuming that X is compact or Borel. We prove that for a typical (in the sense of Baire) uniformly continuous function f on X, the lower box dimension of the graph of f is as small as possible and the upper box dimension of the graph of f is as big as possible. We also prove a local version of this result. Namely, we prove that for a typical uniformly continuous function f on X, the lower local box dimension of the graph of f at all points $x \in X$ is as small as possible and the upper local box dimension of the graph of f at all points $x \in X$ is as big as possible.

1. Statements of the Main Results.

For a bounded subset X of $\mathbb{R}$, we investigate the set $C_{\mathsf{u}}(X)$ of uniformly continuous functions on X equipped with the uniform norm $\|\cdot\|_\infty$; we emphasize that the set X is completely arbitrary - for example, we are not assuming that X is compact or Borel. In particular, we find the lower box dimension and the upper box dimension of the graph of a typical (in the sense of Baire) function $f \in C_{\mathsf{u}}(X)$ (see Theorem 1). We also obtain a local version of this result, namely, for all $x \in X$, we find the lower local box dimension and upper local box dimension of the graph of a typical function

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$f \in C_{\mathsf{u}}(X)$ at x (see Theorem 2); recall that in a metric space $\mathcal{X}$, a set E is called co-meagre if its complement is meagre, and we say that a typical element $x \in \mathcal{X}$ has property P if the set $E = \{x \in \mathcal{X} \,|\, x \text{ has property } \mathsf{P}\}$ is co-meagre. We refer the reader to Oxtoby [Ox] for more details.

We start by recalling the definition of the lower and upper box dimensions of subsets of $\mathbb{R}^d$; we note that (with the exception of Theorem 4.1) we will only be interested in the cases $d = 1$ and $d = 2$ – however, for the benefit of the reader, we present the definitions for an arbitrary positive integer $d \in \mathbb{N}$. For $\delta > 0$, let

$$\mathcal{Q}^d_\delta = \left\{ \prod_{i=1}^{d} [n_i\delta, (n_i + 1)\delta] \,\middle|\, n_1, \ldots, n_d \in \mathbb{Z} \right\} \tag{1.1}$$

denote the standard δ-grid in $\mathbb{R}^d$, and for a subset E of $\mathbb{R}^d$ write

$$N_\delta(E) = \left| \left\{ Q \in \mathcal{Q}^d_\delta \,\middle|\, Q \cap E \neq \varnothing \right\} \right| \tag{1.2}$$

for the number of cubes in $\mathcal{Q}^d_\delta$ that intersect E. The lower and upper box dimensions of E are now defined by

$$\underline{\dim}_{\mathsf{B}}(E) = \liminf_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}\,, \tag{1.3}$$

and

$$\overline{\dim}_{\mathsf{B}}(E) = \limsup_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta}\,, \tag{1.4}$$

respectively. The reader referred to Falconer [Fa] for a thorough discussion of the properties of the box dimensions.

For $f \in C_{\mathsf{u}}(X)$, we will write graph(f) to denote the graph of f, ie.

$$\operatorname{graph}(f) = \{(x, f(x)) \,|\, x \in X\}\,.$$

The purpose of this paper is to investigate the box dimensions of the graphs of typical functions in $C_{\mathsf{u}}(X)$, and we therefore make the following definitions. For a set $X \subseteq \mathbb{R}$, we define the lower graph box dimension of X by

$$\underline{\dim}_{\mathsf{gr,B}}(X) = \inf_{g \in C_{\mathsf{u}}(X)} \underline{\dim}_{\mathsf{B}}(\operatorname{graph}(g))\,,$$

i.e. $\underline{\dim}_{\mathsf{gr,B}}(X)$ is the infimum of lower box dimensions of the graphs of functions in $C_{\mathsf{u}}(X)$. Similarly, we define the upper graph box dimension of X by

$$\overline{\dim}_{\mathsf{gr,B}}(X) = \sup_{g \in C_{\mathsf{u}}(X)} \overline{\dim}_{\mathsf{B}}(\operatorname{graph}(g))\,,$$

i.e. $\overline{\dim}_{\mathsf{gr,B}}(X)$ is the infimum of lower box dimensions of the graphs of functions in $C_{\mathsf{u}}(X)$.

We can now state the first of our main results giving a complete description of the global behaviour of a typical function $f \in C_{\mathsf{u}}(X)$.

Theorem 1 (Global results). *Let X be a bounded subset of $\mathbb{R}$.*

(1) *A typical function $f \in C_{\mathsf{u}}(X)$ satisfies*

$$\underline{\dim}_{\mathsf{B}}(\operatorname{graph}(f)) = \underline{\dim}_{\mathsf{gr,B}}(X)\,.$$

(2) *We have*

$$\dim_{\mathsf{gr,B}}(X) = \underline{\dim}_{\mathsf{B}}(X)\,.$$

(3) *A typical function $f \in C_{\mathsf{u}}(X)$ satisfies*

$$\overline{\dim}_{\mathsf{B}}(\operatorname{graph}(f)) = \overline{\dim}_{\mathsf{gr,B}}(X)\,.$$

(4) *If X has only finitely many isolated points, then we have*

$$\overline{\dim}_{\mathsf{gr,B}}(X) = \overline{\dim}_{\mathsf{B}}(X) + 1\,.$$

The proof of Theorem 1 is given in Sections 2–4. Theorem 1 says that for a typical $f \in C_{\mathsf{u}}(X)$, the lower and upper box dimensions are as big and as small as they can be, respectively.

We note that since the lower box dimension is an upper bound for Hausdorff dimension, Theorem 1.(1) strengthens a result by Maudlin & Williams [MW] saying the Hausdorff dimension of the graph of a typical function $f \in C([0,1])$ equals 1. We also note that Humke & Petruska [HP] proved that the packing dimension of a typical continous function $f \in C([0,1])$ is 2.

Part (4) of Theorem 1 finds the upper graph box dimension $\overline{\dim}_{\mathsf{gr,B}}(X)$ of a bounded subset X of $\mathbb{R}$ with only finitely many isolated points, and shows that in this case we have $\overline{\dim}_{\mathsf{gr,B}}(X) = \overline{\dim}_{\mathsf{B}}(X)+1$. While it is clear that

$$\overline{\dim}_{\mathsf{gr,B}}(X) \le \overline{\dim}_{\mathsf{B}}(X) + 1 \tag{1.5}$$

for any bounded subset X of $\mathbb{R}$, we note that if X has infinitely many isolated points, then the inequality in (1.5) may be strict; indeed, below we present an example of a bounded subset X of $\mathbb{R}$ with countably many isolated points for which

$$\overline{\dim}_{\mathsf{gr,B}}(X) < \overline{\dim}_{\mathsf{B}}(X) + 1\,.$$

In particular, this shows that Theorem 1.(4) cannot be extended to bounded sets with infinitely many isolated points.

Example. Let $X = \{0\} \cup \{\frac{1}{n} \,|\, n \in \mathbb{N}\}$. Then X is a bounded set with infinitely many isolated points. It is well-known that $\underline{\dim}_{\mathsf{B}}(X) = \overline{\dim}_{\mathsf{B}}(X) = \frac{1}{2}$ (see, for example, [Fa, p. 35]), and below we prove that $\overline{\dim}_{\mathsf{gr,B}}(X) \le 1$. It therefore follows that

$$\overline{\dim}_{\mathsf{gr,B}}(X) \le 1 < \frac{3}{2} = \overline{\dim}_{\mathsf{B}}(X) + 1\,.$$

We will now prove that $\overline{\dim}_{\mathsf{gr,B}}(X) \le 1$. To prove this, fix a function $f \in C_{\mathsf{u}}(X)$. Since f is uniformly continuous and X is bounded, we conclude that f is bounded, i.e. there is a real number $M > 0$ such that $|f(x)| \le M$ for all $x \in X$. Next, fix $\delta > 0$ and note that $N_\delta(\mathrm{graph}(f)) \le N_\delta(\mathrm{graph}(f|_{X\cap[0,\delta]})) + N_\delta(\mathrm{graph}(f|_{X\cap(\delta,1]})) \le 2\lceil M\rceil\lceil\frac{1}{\delta}\rceil + N_\delta(\{(\frac{1}{n}, f(\frac{1}{n}))\,|\, n = 1,\ldots,\lceil\frac{1}{\delta}\rceil\}) \le 2\lceil M\rceil\lceil\frac{1}{\delta}\rceil + \lceil\frac{1}{\delta}\rceil = c\lceil\frac{1}{\delta}\rceil$ where $c = 2\lceil M\rceil + 1$, whence $\overline{\dim}_{\mathsf{B}}(\mathrm{graph}(f)) = \limsup_{\delta\to 0}\frac{\log N_\delta(\mathrm{graph}(f))}{-\log\delta} \le \limsup_{\delta\to 0}\frac{\log(c\lceil\frac{1}{\delta}\rceil)}{-\log\delta} = 1$. Since $f \in C_{\mathsf{u}}(X)$ was arbitrary, we conclude from this that $\overline{\dim}_{\mathsf{gr,B}}(X) \le 1$.

Motivated by the above discussion we ask the following question.

Question. *If the set X is an infinite bounded subset of $\mathbb{R}$, is it true that the upper graph box dimension of X is either 1 or $\overline{\dim}_{\mathsf{B}}(X) + 1$?*

We now turn towards our second main result. Our second result gives a complete description of the local behaviour of the graph of a typical function $f \in C_{\mathsf{u}}(X)$. We begin by introducing the following definitions. For $E \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^d$, we define the lower local box dimension of x at E

$$\underline{\dim}_{\mathsf{loc,B}}(x;E) = \lim_{r\to 0}\underline{\dim}_{\mathsf{B}}(B(x,r)\cap E)\,,$$

and we define the upper local box dimension of x at E

$$\overline{\dim}_{\mathsf{loc,B}}(x;E) = \lim_{r\to 0}\overline{\dim}_{\mathsf{B}}(B(x,r)\cap E)\,,$$

The lower local and upper local box dimensions represent how erratically the set E behaves around the particular point x.

Next, we introduce some notation that is particularly useful when analyzing the local box dimensions of the graph of a function. Namely, for $X \subseteq \mathbb{R}$ and $f \in C_{\mathsf{u}}(X)$, we define the lower local graph box dimension of a point x in X by

$$\underline{\dim}_{\mathsf{gr,loc,B}}(x;X) = \inf_{g\in C_{\mathsf{u}}(X)}\underline{\dim}_{\mathsf{loc,B}}\Big(\,(x,g(x))\,;\,\mathrm{graph}(g)\,\Big)\,,$$

and we define the upper local graph box dimension of a point x in X by

$$\overline{\dim}_{\mathsf{gr,loc,B}}(x;X)=\sup_{g\in C_{\mathsf{u}}(X)}\overline{\dim}_{\mathsf{loc,B}}\Big(\,(x,g(x))\,;\,\mathrm{graph}(g)\,\Big)\,.$$

We can now present our second main result computing the local graph box dimensions of a typical function $f\in C_{\mathsf{u}}(X)$. Below we use the following notation, namely, if $E\subseteq\mathbb{R}$, then $1_E:\mathbb{R}\to\mathbb{R}$ denotes the indicator function on E.

Theorem 2. Local Results. *Let X be a bounded subset of $\mathbb{R}$.*

(1) *A typical function $f\in C_{\mathsf{u}}(X)$ satisfies*

$$\underline{\dim}_{\mathsf{loc,B}}\Big(\,(x,f(x))\,;\,\mathrm{graph}(f)\,\Big)=\underline{\dim}_{\mathsf{gr,loc,B}}(x;X)$$

for all $x\in X$.

(2) *We have*

$$\underline{\dim}_{\mathsf{gr,loc,B}}(x;X)=\underline{\dim}_{\mathsf{loc,B}}(x;X)$$

for all $x\in X$.

(3) *A typical function $f\in C_{\mathsf{u}}(X)$ satisfies*

$$\overline{\dim}_{\mathsf{loc,B}}\Big(\,(x,f(x))\,;\,\mathrm{graph}(f)\,\Big)=\overline{\dim}_{\mathsf{gr,loc,B}}(x;X)$$

for all $x\in X$.

(4) *Let $\mathcal{I}(X)$ denote the set of isolated points of X. If X has only finitely many isolated points, then we have*

$$\overline{\dim}_{\mathsf{gr,loc,B}}(x;X)=\overline{\dim}_{\mathsf{loc,B}}(x;X)+1_{\mathcal{I}(X)}(x)$$

for all $x\in X$, i.e.

$$\overline{\dim}_{\mathsf{gr,loc,B}}(x;X)=\begin{cases}\overline{\dim}_{\mathsf{loc,B}}(x;X)+1 & \textit{if } x\in X \textit{ is not an isolated point of } X;\\ 0 & \textit{if } x\in X \textit{ is an isolated point of } X.\end{cases}$$

The proof of Theorem 2 is given in Section 5. Theorem 2 says that for a typical $f\in C_{\mathsf{u}}(X)$ and for any point $(x,f(x))$ on the graph of f, the upper local box dimension of $(x,f(x))$ at graph(f) is as big as possible, and that the lower local box dimension of $(x,f(x))$ at graph(f) is as small as possible. This strengthens the statements in Theorem 1.(1) and Theorem 1.(3), in the sense that it shows that a typical uniformly continuous function is as irregular as possible not only globally, but also locally.

Observe that if the set X is compact, then continuity and uniform continuity are the same, and Theorem 1 and Theorem 2 therefore hold for the set $C(X)$ of continuous functions on X.

2. Proofs of Theorem 1.(1) and Theorem 1.(2).

The purpose of this section is to prove Theorem 1.(1) and Theorem 1.(2). We begin by proving three auxiliary lemmas.

Lemma 2.1. *Fix a bounded subset X of $\mathbb{R}$ and real numbers a and b with $X\subseteq[a,b]$. Let $f\in C_{\mathsf{u}}(X)$. Then there is a continuous function $F\in C([a,b])$ such that*

$$F|_X=f\,.$$

Proof.
Since f is uniformly continuous on X, it follows from [Si, p. 78] that there is a continuous function $\Phi:\overline{X}\to\mathbb{R}$ such that $\Phi|_X=f$; here $\overline{X}$ denotes the closure of X in $\mathbb{R}$. Finally, since $\overline{X}$ is closed, it now follows from Tietze's Extension Theorem that there a continuous function $F:[a,b]\to\mathbb{R}$ such that $F|_{\overline{X}}=\Phi$. □

Lemma 2.2. *Fix a bounded subset X of $\mathbb{R}$. Let $f \in C_{\mathsf{u}}(X)$ and let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial. Let $\lambda \in \mathbb{R}$ with $\lambda \neq 0$.*

(1) *We have*

$$\underline{\dim}_{\mathsf{B}}\Big(\operatorname{graph}\big(p|_X + \lambda f\big)\Big) = \underline{\dim}_{\mathsf{B}}\big(\operatorname{graph}(f)\big)$$

and

$$\overline{\dim}_{\mathsf{B}}\Big(\operatorname{graph}\big(p|_X + \lambda f\big)\Big) = \overline{\dim}_{\mathsf{B}}\big(\operatorname{graph}(f)\big)\,.$$

(2) *We have*

$$\underline{\dim}_{\mathsf{B}}\big(\operatorname{graph}(p|_X)\big) = \underline{\dim}_{\mathsf{B}}(X)$$

and

$$\overline{\dim}_{\mathsf{B}}\big(\operatorname{graph}(p|_X)\big) = \overline{\dim}_{\mathsf{B}}(X)\,.$$

Proof.
(1) Define $F : \operatorname{graph}(f) \to \operatorname{graph}(p|_X + \lambda f)$ by $F(x, f(x)) = (x, p(x) + \lambda f(x))$ and note that F is bijective with $F^{-1}(x, p(x) + \lambda f(x)) = (x, f(x))$. Also, an easy calculation shows that both F and F^{-1} are Lipschitz maps, whence $\underline{\dim}_{\mathsf{B}}(F(\operatorname{graph}(f))) = \underline{\dim}_{\mathsf{B}}(\operatorname{graph}(f))$ and $\overline{\dim}_{\mathsf{B}}(F(\operatorname{graph}(f))) = \overline{\dim}_{\mathsf{B}}(\operatorname{graph}(f))$. Since clearly $F(\operatorname{graph}(f)) = \operatorname{graph}(p|_X + \lambda f)$, we therefore immediately conclude that $\underline{\dim}_{\mathsf{B}}(\operatorname{graph}(p|_X + \lambda f)) = \underline{\dim}_{\mathsf{B}}(F(\operatorname{graph}(f))) = \underline{\dim}_{\mathsf{B}}(\operatorname{graph}(f))$ and $\overline{\dim}_{\mathsf{B}}(\operatorname{graph}(p|_X + \lambda f)) = \overline{\dim}_{\mathsf{B}}(F(\operatorname{graph}(f))) = \overline{\dim}_{\mathsf{B}}(\operatorname{graph}(f))$.
(2) This statement follows from (1) by putting $f = 0$. □

Lemma 2.3. *Fix a bounded subset X of $\mathbb{R}$. For a typical $f \in C_{\mathsf{u}}(X)$, we have*

$$\underline{\dim}_{\mathsf{B}}(\operatorname{graph}(f)) = \underline{\dim}_{\mathsf{B}}(X)\,.$$

Proof.
Let

$$L = \{f \in C_{\mathsf{u}}(X) \,|\, \underline{\dim}_{\mathsf{B}}(\operatorname{graph}(f)) = \underline{\dim}_{\mathsf{B}}(X)\}\,.$$

We must now prove that L is co-meagre. Since $C_{\mathsf{u}}(X)$ is a complete metric space when equipped with the uniform norm, it suffices to show that there is a countable family $(L_n)_n$ of open and dense subsets of $C_{\mathsf{u}}(X)$ such that

$$L = \bigcap_n L_n\,.$$

For $n \in \mathbb{N}$, define

$$L_n = \left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \begin{array}{l} \text{there is } \delta > 0 \text{ with } \delta < \dfrac{1}{n} \text{ such that} \\ \qquad\qquad\qquad \dfrac{\log N_\delta\big(\overline{\operatorname{graph}(f)}\big)}{-\log\delta} \leq \underline{\dim}_{\mathsf{B}}(X) + \dfrac{1}{n} \end{array} \right\}$$

(here $\overline{\operatorname{graph}(f)}$ denotes the closure of $\operatorname{graph}(f)$).

We first note that it follows from the definition of the lower box dimension that

$$L = \bigcap_n L_n\,.$$

Next, we prove that the set L_n is open and dense.

Claim 1. The set L_n is open in $C_{\mathsf{u}}(X)$.

Proof of Claim 1. Let $f \in L_n$. Since $f \in L_n$, we can choose $\delta > 0$ with $\delta < \frac{1}{n}$ such that

$$\frac{\log N_\delta\big(\,\overline{\operatorname{graph}(f)}\,\big)}{-\log \delta} \leq \underline{\dim}_{\mathsf{B}}(X) + \frac{1}{n}\,.$$

Let

$$r = \frac{1}{2} \inf_{\substack{Q' \in \mathcal{Q}_\delta^2 \\ Q' \cap \overline{\operatorname{graph}(f)} \neq \varnothing}} \inf_{\substack{Q'' \in \mathcal{Q}_\delta^2 \\ Q'' \cap \overline{\operatorname{graph}(f)} = \varnothing}} \operatorname{dist}(\,Q' \cap \overline{\operatorname{graph}(f)}\,,\, Q''\,)\,.$$

First observe that since $\overline{\operatorname{graph}(f)}$ is compact, we conclude that $r > 0$.

Next, we claim that

$$B(f,r) \subseteq L_n\,. \tag{2.1}$$

We now prove (2.1). Therefore fix $g \in B(f,r)$. Since $\|f-g\|_\infty < r$, the definition of r implies that

$$\Big\{Q \in \mathcal{Q}_\delta^2 \,\Big|\, Q \cap \overline{\operatorname{graph}(g)} \neq \varnothing\Big\} \subseteq \Big\{Q \in \mathcal{Q}_\delta^2 \,\Big|\, Q \cap \overline{\operatorname{graph}(f)} \neq \varnothing\Big\}\,.$$

This clearly implies that $N_\delta\big(\,\overline{\operatorname{graph}(g)}\,\big) \leq N_\delta\big(\,\overline{\operatorname{graph}(f)}\,\big)$, whence

$$\frac{\log N_\delta\big(\,\overline{\operatorname{graph}(g)}\,\big)}{-\log \delta} \leq \frac{\log N_\delta\big(\,\overline{\operatorname{graph}(f)}\,\big)}{-\log \delta} \leq \underline{\dim}_{\mathsf{B}}(X) + \frac{1}{n}\,.$$

We conclude from the above inequality that $g \in L_n$. This completes the proof of Claim 1

Claim 2. The set L_n is dense in $C_{\mathsf{u}}(X)$.

Proof of Claim 2. Let $f \in C_{\mathsf{u}}(X)$ and let $r > 0$. We must now find $g \in L_n$ such that $\|g-f\|_\infty < r$. Since X is bounded, there exists an interval $[a,b]$ such that $X \subseteq [a,b]$. It now follows from Lemma 2.1 that the function f can be extended to a continuous function F on $[a,b]$, and by Weierstrass' Approximation Theorem there exists a polynomial p such that $\sup_{x\in[a,b]} |F(x) - p(x)| < r$. Put $g = p|_X$. It is clear that g is uniformly continuous and Lemma 2.2 shows that $\underline{\dim}_{\mathsf{B}}\big(\,\overline{\operatorname{graph}(g)}\,\big) = \underline{\dim}_{\mathsf{B}}(\operatorname{graph}(g)) = \underline{\dim}_{\mathsf{B}}(\operatorname{graph}(p|_X)) = \underline{\dim}_{\mathsf{B}}(X)$. We conclude from this that $g \in L_n$. Also $\|p|_X - f\|_\infty = \|p|_X - F|_X\|_\infty \leq \sup_{x\in[a,b]} |p(x) - F(x)| < r$. This completes the proof of Claim 2.

Claim 1 and Claim 2 show that L is the intersection of a countable family $(L_n)_n$ of open and dense sets, and we therefore conclude that L is co-meagre. □

We can now prove Theorem 1.(1) and Theorem 1.(2).

Proof of Theorem 1.(1).

Since it is easily seen that $\underline{\dim}_{\mathsf{gr},\mathsf{B}}(X) = \underline{\dim}_{\mathsf{B}}(X)$, it follows from Lemma 2.3 that a typical function $f \in C_{\mathsf{u}}(X)$ satisfies

$$\begin{aligned}\underline{\dim}_{\mathsf{B}}(\operatorname{graph}(f)) &= \underline{\dim}_{\mathsf{B}}(X) \\ &= \underline{\dim}_{\mathsf{gr},\mathsf{B}}(X)\,.\end{aligned}$$

This completes the proof. □

Proof of Theorem 1.(2).

Ii is easily seen that $\underline{\dim}_{\mathsf{gr},\mathsf{B}}(X) = \underline{\dim}_{\mathsf{B}}(X)$. □

3. Proof of Theorem 1.(3).

The purpose of this section is to prove Theorem 1.(3). We start by providing an alternative characterization of the box dimension (see Lemma 3.1) based on open cubes (as opposed to the usual definition (1.1)-(1.4) based on closed cubes). The motivation for introducing this characterization is the following. Namely, the proof of Theorem 1.(3) requires a lower bound for the upper box dimension of the graph of a typical function, and methods for establishing good lower bounds for the box dimension of subsets E of $\mathbb{R}^d$ are often sensitive to the number of cubes from the grid $\mathcal{Q}_\delta^d$ who only intersect E by their boundaries. It is to overcome this problem that we provide an alternative characterization of the box dimension based on open cubes. We first introduce some notation. For $\delta > 0$ and $\mathbf{u} \in \mathbb{R}^d$ write

$$\mathcal{Q}_{\mathbf{u},\delta}^{\circ,d} = \left\{ \prod_{i=1}^{d} (n_i\delta, (n_i+1)\delta) \,\middle|\, (n_1, \ldots, n_d) \in \mathbf{u} + \mathbb{Z}^d \right\}.$$

Also, for a subset E of $\mathbb{R}^d$, we will write $N_{\mathbf{u},\delta}^\circ(E)$ for the number of open boxes from $\mathcal{Q}_{\mathbf{u},\delta}^{\circ,d}$ that intersect E, i.e.

$$N_{\mathbf{u},\delta}^\circ(E) = \left| \left\{ Q \in \mathcal{Q}_{\mathbf{u},\delta}^{\circ,d} \,\middle|\, Q \cap E \neq \varnothing \right\} \right|.$$

Finally, we write

$$U_d = \left\{ (u_1, \ldots, u_d) \,\middle|\, u_i = 0, \tfrac{1}{2} \right\}.$$

and put

$$N_\delta^\circ(E) = \sum_{\mathbf{u} \in U_d} N_{\mathbf{u},\delta}^\circ(E)\,.$$

Lemma 3.1. *For a bounded subset E of $\mathbb{R}^d$, we have*

$$\underline{\dim}_{\mathsf{B}}(E) = \liminf_{\delta \to 0} \frac{\log N_\delta^\circ(E)}{-\log \delta}$$

and

$$\overline{\dim}_{\mathsf{B}}(E) = \limsup_{\delta \to 0} \frac{\log N_\delta^\circ(E)}{-\log \delta}\,.$$

Proof.
This follows from standard arguments and the proof is therefore omitted. □

We can now prove Theorem 1.(3).

Proof of Theorem 1.(3).
We must prove that for a typical $f \in C_{\mathsf{u}}(X)$ we have

$$\overline{\dim}_{\mathsf{B}}(\mathrm{graph}(f)) = \overline{\dim}_{\mathsf{gr},\mathsf{B}}(X)\,.$$

Let

$$L = \left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \overline{\dim}_{\mathsf{B}}\,(\mathrm{graph}(f)) = \overline{\dim}_{\mathsf{gr},\mathsf{B}}(X) \right\}\,.$$

We must now prove that L is co-meagre. Since $C_{\mathsf{u}}(X)$ is a complete metric space when equipped with the uniform norm, it suffices to show that there is a countable family $(L_n)_n$ of open and dense subsets of $C_{\mathsf{u}}(X)$ such that

$$L = \bigcap_n L_n\,.$$

For $n \in \mathbb{N}$, define the set L_n by

$$L_n = \left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \begin{array}{l} \text{there is } \delta > 0 \text{ with } \delta < \dfrac{1}{n} \text{ such that} \\ \qquad\qquad \dfrac{\log N^{\circ}_{\delta}(\operatorname{graph}(f))}{-\log \delta} + \dfrac{1}{n} \geq \overline{\dim}_{\mathsf{gr},\mathsf{B}}(X) \end{array} \right\}.$$

We first note that it follows from the definition of the upper box dimension that

$$L = \bigcap_n L_n \,.$$

Next, we prove that the set L_n is open and dense.

Claim 1. The set L_n is open in $C_{\mathsf{u}}(X)$.
Proof of Claim 1. Let $f \in L_n$. Since $f \in L_n$, we can choose $\delta > 0$ with $\delta < \frac{1}{n}$ such that

$$\frac{\log N^{\circ}_{\delta}(\operatorname{graph}(f))}{-\log \delta} + \frac{1}{n} \geq \overline{\dim}_{\mathsf{gr},\mathsf{B}}(X) \,.$$

For each $\mathbf{u} = (u_1, u_2) \in U_2$, write

$$E_{\mathbf{u},\delta} = \bigcup_{m \in u_2 + \mathbb{Z}} \Big(\mathbb{R} \times \{m\delta\} \Big) \,,$$

i.e. $E_{\mathbf{u},\delta}$ denotes the horizontal lines that outline the grid $\mathcal{Q}^{\circ,2}_{\mathbf{u},\delta}$. For each $Q \in \mathcal{Q}^{\circ,2}_{\mathbf{u},\delta}$ with $Q \cap \operatorname{graph}(f) \neq \varnothing$, choose

$$x_Q \in Q \cap \operatorname{graph}(f) \,.$$

Next, put

$$r = \frac{1}{2} \min_{\mathbf{u} \in U_2} \min_{\substack{Q \in \mathcal{Q}^{\circ,2}_{\mathbf{u},\delta} \\ Q \cap \operatorname{graph}(f) \neq \varnothing}} \operatorname{dist}(x_Q, E_{\mathbf{u},\delta})$$

We claim that

$$r > 0 \,.$$

Indeed, for all $\mathbf{u} \in U_2$ and $Q \in \mathcal{Q}^{\circ,2}_{\mathbf{u},\delta}$ with $Q \cap \operatorname{graph}(f) \neq \varnothing$ we have $x_Q \in Q \cap \operatorname{graph}(f) \subseteq Q$, whence $x_Q \notin E_{\mathbf{u},\delta}$. We conclude from this that $\operatorname{dist}(x_Q, E_{\mathbf{u},\delta}) > 0$, and so $r > 0$.

Next we claim that

$$B(f, r) \subseteq L_n \,. \tag{3.1}$$

We now prove (3.1). Therefore fix $g \in B(f, r)$. Since $\|f - g\|_\infty < r$, the definition of r implies that if $\mathbf{u} \in U$, then

$$\left\{ Q \in \mathcal{Q}^{\circ,2}_{\mathbf{u},\delta} \,\middle|\, Q \cap \operatorname{graph}(f) \neq \varnothing \right\} \subseteq \left\{ Q \in \mathcal{Q}^{\circ,2}_{\mathbf{u},\delta} \,\middle|\, Q \cap \operatorname{graph}(g) \neq \varnothing \right\}.$$

This clearly implies that $N^{\circ}_{\mathbf{u},\delta}(\operatorname{graph}(f)) \leq N^{\circ}_{\mathbf{u},\delta}(\operatorname{graph}(g))$, and so $N^{\circ}_{\delta}(\operatorname{graph}(f)) \leq N^{\circ}_{\delta}(\operatorname{graph}(g))$, whence

$$\frac{\log N^{\circ}_{\delta}(\operatorname{graph}(g))}{-\log \delta} + \frac{1}{n} \geq \frac{\log N^{\circ}_{\delta}(\operatorname{graph}(f))}{-\log \delta} + \frac{1}{n} \geq \overline{\dim}_{\mathsf{gr},\mathsf{B}}(X) \,.$$

We conclude from the above inequality that $g \in L_n$. This completes the proof of Claim 1.

Claim 2. The set L_n is dense in $C_{\mathsf{u}}(X)$.
Proof of Claim 2. Let $f \in C_{\mathsf{u}}(X)$ and let $r > 0$. We must now find $g \in L_n$ such that $\|g - f\|_\infty < r$.

Without loss of generality, we may assume $\frac{r}{2} \leq \frac{1}{n}$. Since X is bounded, we can find real numbers a and b with $X \subseteq [a, b]$. It follows from Lemma 2.1 that there is a continuous function $F : [a, b] \to \mathbb{R}$

such that $F|_X = f$. Next, it follows from Weierstrass' Approximation Theorem that we can find a polynomial p satisfying $\sup_{x\in[a,b]} |p(x) - F(x)| < \frac{r}{4}$.

Note, that the definition of $\overline{\dim}_{\mathsf{gr},\mathsf{B}}(X)$ implies that there is a function $\varphi \in C_{\mathsf{u}}(X)$ such that

$$\overline{\dim}_{\mathsf{B}}(\mathrm{graph}(\varphi)) \geq \overline{\dim}_{\mathsf{gr},\mathsf{B}}(X) - \frac{r}{4}\,. \tag{3.2}$$

Also, since φ is bounded (because X is bounded and $\varphi : X \to \mathbb{R}$ is uniformly continuous), we can find a positive real number $c > 0$ such that

$$c \leq \frac{r}{4(\|\varphi\|_\infty + 1)}$$

Now define $g : X \to \mathbb{R}$ by

$$g = p|_X + c\varphi\,.$$

Clearly $g \in C_{\mathsf{u}}(X)$.

We now claim that $g \in \cap L_n$ and $\|f - g\|_\infty < r$.

We first show that $\|f - g\|_\infty < r$. Indeed, we have

$$\begin{aligned}
\|f - g\|_\infty &= \|f - p|_X - c\varphi\|_\infty \\
&\leq \|f - p|_X\|_\infty + c\|\varphi\|_\infty \\
&= \|F|_X - p|_X\|_\infty + c\|\varphi\|_\infty \\
&\leq \sup_{x\in[a,b]} |F(x) - p(x)| + c\|\varphi\|_\infty \\
&\leq \frac{r}{4} + \frac{r}{4(\|\varphi\|_\infty + 1)}\|\varphi\|_\infty \\
&< r\,.
\end{aligned}$$

This shows that $g \in B(f, r)$.

Next, we show that $g \in L_n$. By the definition of upper box dimension we can find $\delta < \frac{1}{n}$, such that

$$\frac{\log N_\delta(\mathrm{graph}(g))}{-\log\delta} + \frac{r}{4} \geq \overline{\dim}_{\mathsf{B}}(\mathrm{graph}(g))\,. \tag{3.3}$$

Since $\frac{r}{4} + \frac{r}{4} = \frac{r}{2} \leq \frac{1}{n}$, we conclude from (3.3) that

$$\begin{aligned}
\frac{\log N_\delta(\mathrm{graph}(g))}{-\log\delta} + \frac{1}{n} &\geq \frac{\log N_\delta(\mathrm{graph}(g))}{-\log\delta} + \frac{r}{4} + \frac{r}{4} \\
&\geq \overline{\dim}_{\mathsf{B}}(\mathrm{graph}(g)) + \frac{r}{4} \\
&= \overline{\dim}_{\mathsf{B}}(\mathrm{graph}(p|_X + c\varphi)) + \frac{r}{4}\,.
\end{aligned} \tag{3.4}$$

Also, observe that it follows from Lemma 2.2 that $\overline{\dim}_{\mathsf{B}}(\mathrm{graph}(p|_X + c\varphi)) = \overline{\dim}_{\mathsf{B}}(\mathrm{graph}(\varphi))$, and we therefore conclude from (3.4) that

$$\frac{\log N_\delta(\mathrm{graph}(g))}{-\log\delta} + \frac{1}{n} \geq \overline{\dim}_{\mathsf{B}}(\mathrm{graph}(\varphi)) + \frac{r}{4}\,. \tag{3.5}$$

Finally, combining (3.2) and (3.5) yields

$$\frac{\log N_\delta(\mathrm{graph}(g))}{-\log\delta} + \frac{1}{n} \geq \overline{\dim}_{\mathsf{gr},\mathsf{B}}(X)\,.$$

This shows that $g \in L_n$, and completes the proof of Claim 2.

Claim 1 and Claim 2 show that L is the intersection of a countable family $(L_n)_n$ of open and dense sets, and we therefore conclude that L is co-meagre. $\square$

4. Proof of Theorem 1.(4).

The purpose of this section is to prove Theorem 1.(4). However, we first prove a slightly more general result.

Theorem 4.1. *Let X be a bounded subset of $\mathbb{R}^d$ with only finitely many isolated points.*

(1) *We have*
$$\sup_{f\in C_{\mathsf{u}}(X)} \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(f)) = \underline{\dim}_{\mathsf{B}}(X)+1\,.$$

(2) *We have*
$$\sup_{f\in C_{\mathsf{u}}(X)} \overline{\dim}_{\mathsf{B}}(\mathrm{graph}(f)) = \overline{\dim}_{\mathsf{B}}(X)+1\,.$$

Proof.
Observe that if a set has finitely many isolated points, we may remove these without changing the lower and the upper box dimensions of the set. Hence we may suppose that X has no isolated points.

Let $\varepsilon>0$. We must now show that there is a uniformly continuous function $f:X\to\mathbb{R}$ such that $\underline{\dim}_{\mathsf{B}}(\mathrm{graph}(f))\geq\underline{\dim}_{\mathsf{B}}(X)+1-\varepsilon$ and $\overline{\dim}_{\mathsf{B}}(\mathrm{graph}(f))\geq\overline{\dim}_{\mathsf{B}}(X)+1-\varepsilon$.

Fix a positive integer n and write
$$\mathcal{V}_n=\Big\{Q\in\mathcal{Q}^d_{2^{-n}}\,\Big|\,Q\cap X\neq\varnothing\Big\}$$
(recall, that for $\delta>0$, the family $\mathcal{Q}^d_\delta$ of δ-cubes in $\mathbb{R}^d$ is defined in (1.1)). Since X does not have isolated points there is a subfamily $\mathcal{W}_n$ of $\mathcal{V}_n$ with $|\mathcal{W}_n|\geq\frac{1}{2^d}|\mathcal{V}_n|$ such that if $Q\in\mathcal{W}_n$, then none of the points in the set $X\cap Q$ are isolated in $X\cap Q$.

For each integer n with $n\geq 0$, we will now define a uniformly continuous function $f_n:X\to[0,\infty)$ and a finite set
$$E_n=\Big\{x_{Q,n}\,\Big|\,Q\in\mathcal{W}_n\Big\}\;\cup\;\Big\{y_{Q,n,i}\,\Big|\,Q\in\mathcal{W}_n\,,\,i=1,\dots,\lceil 2^{n(1-\varepsilon)}\rceil\Big\}$$
such that the following properties are satisfied
$$x_{Q,n},y_{Q,n,i}\in X\cap Q\,, \tag{4.1}$$
$$\left|\sum_{j=0}^{n-1}f_j(x_{Q,n})-\sum_{j=0}^{n-1}f_j(y_{Q,n,i})\right|\leq 2^{-n}\,, \tag{4.2}$$
$$\|f_n\|_\infty\leq 5\,\lceil 2^{n(1-\varepsilon)}\rceil\,2^{-n}\,, \tag{4.3}$$
$$f_n(x_{Q,n})=0\,, \tag{4.4}$$
$$f_n(y_{Q,n,i})=5i2^{-n}\,, \tag{4.5}$$
$$f_k(y_{Q,n,i})=0\text{ for }k<n\,. \tag{4.6}$$

Below we construct the functions f_n and the sets E_n inductively as follows.

First we put $f_0=0$ and $E_0=\varnothing$. Next assume that the functions $f_0,f_1,\dots,f_{n-1}$ and the sets $E_0,E_1,\dots,E_{n-1}$ have been constructed such that properties (4.1)–(4.6) are satisfied. We will now construct f_n and E_n. Fix $Q\in\mathcal{W}_n$. It follows from the definition of $\mathcal{W}_n$ that we can choose $x_{Q,n}\in(Q\cap X)\setminus(E_0\cup E_1\cup\ldots\cup E_{n-1})$. It also follows from the definition of $\mathcal{W}_n$ and the fact that the functions $f_0,f_1,\dots,f_{n-1}$ are (uniformly) continuous that we can choose points $y_{Q,n,i}\in(Q\cap X)\setminus(E_0\cup E_1\cup\ldots\cup E_{n-1})$ with $i=1,\dots,\lceil 2^{n(1-\varepsilon)}\rceil$ such that the points $x_{Q,n},y_{Q,n,1},\dots,y_{Q,n,\lceil 2^{n(1-\varepsilon)}\rceil}$ are distinct and
$$\left|\sum_{j=0}^{n-1}f_j(x_{Q,n})-\sum_{j=0}^{n-1}f_j(y_{Q,n,i})\right|\leq 2^{-n}$$

Now define $g_n : E_0 \cup E_1 \cup \ldots \cup E_{n-1} \cup E_n \to \mathbb{R}$ by

$$g_n(x) = \begin{cases} 0 & \text{if } x \in E_0 \cup E_1 \cup \ldots \cup E_{n-1}; \\ 0 & \text{if } x = x_{Q,n}; \\ 5i2^{-n} & \text{if } x = y_{Q,n,i} \text{ for } i = 1, \ldots, \lceil 2^{n(1-\varepsilon)} \rceil. \end{cases}$$

Next, observe that since the set $E_0 \cup E_1 \cup \ldots \cup E_{n-1} \cup E_n$ is finite, we can find a uniformly continuous function $f_n : X \to [0, \infty)$ such that $f_n|_{E_0 \cup E_1 \cup \ldots \cup E_{n-1} \cup E_n} = g_n$ and $0 = \min_{x \in E_0 \cup E_1 \cup \ldots \cup E_{n-1} \cup E_n} g_n(x) \le f(x) \le \max_{x \in E_0 \cup E_1 \cup \ldots \cup E_{n-1} \cup E_n} g_n(x) = 5 \lceil 2^{n(1-\varepsilon)} \rceil 2^{-n}$ for all $x \in X$. It is clear that the function f_n and the set $E_n = \{x_{Q,n} \,|\, Q \in \mathcal{W}_n\} \cup \{y_{Q,n,i} \,|\, Q \in \mathcal{W}_n\,,\, i = 1, \ldots, \lceil 2^{n(1-\varepsilon)} \rceil\}$ satisfy the properties in (4.1)–(4.6). This completes the construction of the functions f_n and the sets E_n.

We now construct $f \in C_{\mathsf{u}}(K)$ as follows. Namely, note that it follows from (4.3) that

$$\begin{aligned} \sum_n \|f_n\|_\infty &\le \sum_n 5 \lceil 2^{n(1-\varepsilon)} \rceil 2^{-n} \\ &\le 5 \sum_n \left(2^{-n\varepsilon} + 2^{-n}\right) \\ &< \infty \,. \end{aligned} \tag{4.7}$$

We conclude from (4.7) that the function f define by

$$f = \sum_n f_n$$

is a well-defined real valued uniformly continuous function.

Below we prove that $\underline{\dim}_{\mathsf{B}}(\mathrm{graph}(f)) \ge \underline{\dim}_{\mathsf{B}}(X) + 1 - \varepsilon$ and $\overline{\dim}_{\mathsf{B}}(\mathrm{graph}(f)) \ge \overline{\dim}_{\mathsf{B}}(X) + 1 - \varepsilon$. In order to prove this we first prove the following 2 claims.

Claim 1. *If n is a positive integer and $Q \in \mathcal{W}_n$, then $N_{2^{-n}}(\mathrm{graph}(f|_{Q \cap X})) \ge 2^{n(1-\varepsilon)}$.*

Proof of Claim 1. We first show that if $i, j = 1, \ldots, \lceil 2^{n(1-\varepsilon)} \rceil$, then

$$|f(y_{Q,n,i}) - f(y_{Q,n,j})| > 2^{-n} \,. \tag{4.8}$$

Indeed, we have

$$\begin{aligned} &|f_n(y_{Q,n,i}) - f_n(y_{Q,n,j})| \\ &\quad = \left| \left(\sum_{k=0}^{n} f_k(y_{Q,n,i}) - \sum_{k=0}^{n} f_k(y_{Q,n,j}) \right) - \left(\sum_{k=0}^{n-1} f_k(y_{Q,n,i}) - \sum_{k=0}^{n-1} f_k(y_{Q,n,j}) \right) \right| \\ &\quad \le \left| \sum_{k=0}^{n} f_k(y_{Q,n,i}) - \sum_{k=0}^{n} f_k(y_{Q,n,j}) \right| \\ &\qquad + \left| \sum_{k=0}^{n-1} f_k(y_{Q,n,i}) - \sum_{k=0}^{n-1} f_k(x_{Q,n}) \right| + \left| \sum_{k=0}^{n-1} f_k(x_{Q,n}) - \sum_{k=0}^{n-1} f_k(y_{Q,n,j}) \right|, \end{aligned}$$

whence

$$\begin{aligned} &|f(y_{Q,n,i}) - f(y_{Q,n,j})| \\ &\quad = \left| \sum_{k=0}^{n} f_k(y_{Q,n,i}) - \sum_{k=0}^{n} f_k(y_{Q,n,j}) \right| && \text{[by (4.6)]} \\ &\quad \ge |f_n(y_{Q,n,i}) - f_n(y_{Q,n,j})| \\ &\qquad - \left| \sum_{k=0}^{n-1} f_k(y_{Q,n,i}) - \sum_{k=0}^{n-1} f_k(x_{Q,n}) \right| - \left| \sum_{k=0}^{n-1} f_k(x_{Q,n}) - \sum_{k=0}^{n-1} f_k(y_{Q,n,j}) \right| \\ &\quad \ge |5i2^{-n} - 5j2^{-n}| - 2^{-n} - 2^{-n} && \text{[by (4.2) and (4.5)]} \\ &\quad = 5|i-j|2^{-n} - 2^{-n} - 2^{-n} \\ &\quad \ge 52^{-n} - 2^{-n} - 2^{-n} \\ &\quad > 2^{-n} \,. \end{aligned}$$

This completes the proof of (4.8).

It follows from (4.8) that distinct points in the set $\{(y_{Q,n,i}, f(y_{Q,n,i}))\,|\,i = 1,\dots,\lceil 2^{n(1-\varepsilon)}\rceil\}$ are at most 2^{-n} close, whence

$$N_{2^{-n}}(\operatorname{graph}(f)) \geq \Big|\{(y_{Q,n,i}, f(y_{Q,n,i}))\,|\,i = 1,\dots,\lceil 2^{n(1-\varepsilon)}\rceil\}\Big| = \lceil 2^{n(1-\varepsilon)}\rceil \geq 2^{n(1-\varepsilon)}\,.$$

This completes the proof of Claim 1.

Claim 2. *If n is a positive integer, then $N_{2^{-n}}(\operatorname{graph}(f)) \geq \frac{1}{2^d}\,N_{2^{-n}}(X)\,2^{n(1-\varepsilon)}$.*

Proof of Claim 2. It follows from Claim 1 that

$$\begin{aligned} N_{2^{-n}}(\operatorname{graph}(f)) &= \sum_{Q\in\mathcal{V}_n} N_{2^{-n}}(\operatorname{graph}(f|_{Q\cap X}))\\ &\geq \sum_{Q\in\mathcal{W}_n} N_{2^{-n}}(\operatorname{graph}(f|_{Q\cap X}))\\ &\geq \sum_{Q\in\mathcal{W}_n} 2^{n(1-\varepsilon)}\\ &= |\mathcal{W}_n|\,2^{n(1-\varepsilon)}\\ &\geq \frac{1}{2^d}\,|\mathcal{V}_n|\,2^{n(1-\varepsilon)}\\ &= \frac{1}{2^d}\,N_{2^{-n}}(X)\,2^{n(1-\varepsilon)}\,. \end{aligned}$$

This completes the proof of Claim 2.

We conclude from Claim 2 that

$$\frac{\log N_{2^{-n}}(\operatorname{graph}(f))}{-\log 2^{-n}} \geq \frac{\log N_{2^{-n}}(X)}{-\log 2^{-n}} + 1 - \varepsilon - \frac{d}{n} \tag{4.9}$$

for all positive integers n. The desired result follows immediately from (4.9). □

We can now prove Theorem 1.(4).

Proof of Theorem 1.(4).
The statement in Theorem 1.(4) follows immediately from Theorem 4.1.(1). □

5. Proof of Theorem 2.

The purpose of this section is to prove Theorem 2. We begin by proving two auxiliary lemmas.

Lemma 5.1. *Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces and let $\Phi : \mathcal{X} \to \mathcal{Y}$ be a map. Assume that*

$$\Phi(B(x,r)) = B(\Phi(x), r)$$

for all $x \in \mathcal{X}$ and all $r > 0$. Then the following hold.

1. *If N is a nowhere dense subset of $\mathcal{Y}$, then $\Phi^{-1}(N)$ is a nowhere dense subset of $\mathcal{X}$.*
2. *If M is a meagre subset of $\mathcal{Y}$, then $\Phi^{-1}(M)$ is a meagre subset of $\mathcal{X}$.*
3. *If E is a co-meagre subset of $\mathcal{Y}$, then $\Phi^{-1}(E)$ is a co-meagre subset of $\mathcal{X}$.*

Proof.
(1) Let $x \in \mathcal{X}$ and $r > 0$. We must now find $y \in \mathcal{X}$ and $s > 0$ such that $B(y,s) \subseteq B(x,r) \setminus \Phi^{-1}(N)$.

Since N is nowhere dense, we can find $u \in \mathcal{Y}$ and $\delta > 0$ such that

$$B(u,\delta) \subseteq B(\Phi(x),r) \setminus N\,.$$

Next, observe that since $u \in B(u,\delta) \subseteq B(\Phi(x),r) \setminus N \subseteq B(\Phi(x),r)$, we conclude that $d_{\mathcal{Y}}(u,\Phi(x)) < r$, and we can therefore choose $\rho > 0$ with

$$d_{\mathcal{Y}}(u,\Phi(x)) < \rho < r\,.$$

Now note that $u \in B(\Phi(x),\rho) = \Phi(B(x,\rho))$. We conclude from this that there is a point

$$y \in B(x,\rho)$$

with

$$\Phi(y) = u\,.$$

Finally, put

$$s = \min(\, r - d_{\mathcal{X}}(x,y)\,,\, \delta\,)\,.$$

We will now prove that $s > 0$ and $B(y,s) \subseteq B(x,r) \setminus \Phi^{-1}(N)$.

We first show that $s > 0$. Indeed, since $y \in B(x,\rho)$, we deduce that $d_{\mathcal{X}}(x,y) < \rho < r$, whence $r - d_{\mathcal{X}}(x,y) > 0$. This shows that $s > 0$.

Next, we show that $B(y,s) \subseteq B(x,r)$. Indeed, if $z \in B(y,s)$, then $d_{\mathcal{X}}(x,z) \le d_{\mathcal{X}}(x,y) + d_{\mathcal{X}}(y,z) < d_{\mathcal{X}}(x,y) + s \le d_{\mathcal{X}}(x,y) + r - d_{\mathcal{X}}(x,y) = r$. This shows that $z \in B(x,r)$.

Finally, we show that $B(y,s) \cap \Phi^{-1}(N) = \varnothing$. Assume, in order to reach a contradiction, that there is a point $z \in B(y,s) \cap \Phi^{-1}(N)$. Since $z \in \Phi^{-1}(N)$, we conclude that

$$\Phi(z) \in N\,. \tag{5.1}$$

On the other hand, since $z \in B(y,s)$, we also conclude that

$$\begin{aligned}\Phi(z) &\in \Phi(B(y,s))\\ &= B(\Phi(y),s)\\ &= B(u,s)\\ &\subseteq B(u,\delta)\\ &\subseteq B(\Phi(x),r) \setminus N\,.\end{aligned} \tag{5.2}$$

The desired contradiction follows immediately from (5.1) and (5.2).
(2) Let M be a meagre subset of $\mathcal{Y}$. We can therefore choose nowhere dense subsets $N_1, N_2, \dots$ of $\mathcal{Y}$ with $M = \cup_n N_n$. Since $\Phi^{-1}(N_n)$ is nowhere dense for all n (by Part (1)), we conclude that $\Phi^{-1}(M) = \Phi^{-1}(\cup_n N_n) = \cup_n \Phi^{-1}(N_n)$ is meagre.
(3) This follows easily from (2). $\square$

Lemma 5.2. *Fix $X \subseteq \mathbb{R}$. Let $f \in C_{\mathsf{u}}(X)$. and $x \in X$. Let $p_n, q_n \in \mathbb{R}$ for $n \in \mathbb{N}$. Assume that*

(i) $[p_1, q_n] \supseteq [p_2, q_2] \supseteq \dots$;
(ii) $x \in (p_n, q_n)$ *for all* n;
(iii) $q_n - p_n \to 0$.

Then

$$\underline{\dim}_{\mathsf{loc},\mathsf{B}}\Big(\,(x,f(x))\,;\,\mathrm{graph}(f)\,\Big) = \lim_n \underline{\dim}_{\mathsf{B}}\Big(\,\mathrm{graph}(f|_{X\cap[p_n,q_n]})\,\Big)$$

and

$$\overline{\dim}_{\mathsf{loc},\mathsf{B}}\Big(\,(x,f(x))\,;\,\mathrm{graph}(f)\,\Big) = \lim_n \overline{\dim}_{\mathsf{B}}\Big(\,\mathrm{graph}(f|_{X\cap[p_n,q_n]})\,\Big)\,.$$

Proof.
This is easily seen and the proof is therefore omitted. □

We can now prove Theorem 2.

Proof of Theorem 2.
(1) We must prove that for a typical $f \in C_{\mathsf{u}}(X)$ we have

$$\underline{\dim}_{\mathsf{loc},\mathsf{B}}\Big(\,(x,f(x))\,;\,\mathrm{graph}(f)\Big) = \underline{\dim}_{\mathsf{gr},\mathsf{loc},\mathsf{B}}\big(x;X\big)$$

for all $x \in X$.

For $p,q \in \mathbb{Q}$ with $(p,q)\cap X \neq \varnothing$, let

$$L_{p,q} = \left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \underline{\dim}_{\mathsf{B}}\Big(\ \mathrm{graph}(f|_{X\cap[p,q]})\Big) = \underline{\dim}_{\mathsf{gr},\mathsf{B}}\Big(\ X\cap[p,q]\ \Big) \right\},$$

and put

$$L = \bigcap_{\substack{p,q\in\mathbb{Q}\\(p,q)\cap X\neq\varnothing}} L_{p,q}\,.$$

We now prove the following two claims.

Claim 1. The set L is co-meagre in $C_{\mathsf{u}}(X)$.
Proof of Claim 1. It clearly suffices to show that if $p,q \in \mathbb{Q}$ with $(p,q)\cap X\neq\varnothing$, then the set $L_{p,q}$ is co-meagre in $C_{\mathsf{u}}(X)$.

We therefore fix $p,q\in\mathbb{Q}$ with $(p,q)\cap X\neq\varnothing$. Now put

$$E_{p,q} = \left\{ g \in C_{\mathsf{u}}(X\cap[p,q]) \,\middle|\, \underline{\dim}_{\mathsf{B}}\Big(\ \mathrm{graph}(g)\ \Big) = \underline{\dim}_{\mathsf{gr},\mathsf{B}}\Big(\ X\cap[p,q]\ \Big) \right\}.$$

It follows from Theorem 1.(1) that the set $E_{p,q}$ is co-meagre in $C_{\mathsf{u}}(X\cap[p,q])$. Next, applying Lemma 5.1 with $\mathcal{X} = C_{\mathsf{u}}(X)$, $\mathcal{Y} = C_{\mathsf{u}}(X\cap[p,q])$ and $\Phi : C_{\mathsf{u}}(X) \to C_{\mathsf{u}}(X\cap[p,q])$ defined by $\Phi(f) = f|_{X\cap[p,q]}$ shows that $\Phi^{-1}(E_{p,q})$ is meagre in $C_{\mathsf{u}}(X)$. Finally, since clearly

$$L_{p,q} = \Phi^{-1}(E_{p,q})\,,$$

we therefore conclude that $L_{p,q}$ is meagre in $C_{\mathsf{u}}(X)$. This completes the proof of Claim 1.

Claim 2. We have

$$L \subseteq \left\{ f \in C_{\mathsf{u}}(X) \,\middle|\, \forall x\in X\ :\ \underline{\dim}_{\mathsf{loc},\mathsf{B}}\Big(\,(x,f(x))\,;\,\mathrm{graph}(f)\Big) = \underline{\dim}_{\mathsf{gr},\mathsf{loc},\mathsf{B}}\big(x;X\big) \right\},$$

Proof of Claim 2. Let $f\in L$ and $x\in X$. We must now prove that

$$\underline{\dim}_{\mathsf{loc},\mathsf{B}}\Big(\,(x,f(x))\,;\,\mathrm{graph}(f)\Big) = \underline{\dim}_{\mathsf{gr},\mathsf{loc},\mathsf{B}}\big(x;X\big)\,.$$

Indeed, it is clear that

$$\underline{\dim}_{\mathsf{loc},\mathsf{B}}\Big(\,(x,f(x))\,;\,\mathrm{graph}(f)\Big) \geq \underline{\dim}_{\mathsf{gr},\mathsf{loc},\mathsf{B}}\big(x;X\big)\,,$$

and it therefore suffices to show that

$$\underline{\dim}_{\mathsf{loc},\mathsf{B}}\Big(\,(x,f(x))\,;\,\mathrm{graph}(f)\Big) \leq \underline{\dim}_{\mathsf{gr},\mathsf{loc},\mathsf{B}}\big(x;X\big)\,. \tag{5.3}$$

Below we prove (5.3). First, note that we can choose sequences $(p_n)_n$ and $(q_n)_n$ from $\mathbb{Q}$ with $[p_1, q_1] \supseteq [p_2, q_2] \supseteq \dots$, such that $x \in (p_n, q_n)$ for all n and $q_n - p_n \to 0$. Next, we have

$$\begin{aligned}\underline{\dim}_{\mathsf{loc,B}}\Big((x, f(x))\, ;\, \mathrm{graph}(f)\Big) &= \lim_n \underline{\dim}_{\mathsf{B}}\Big(\;\mathrm{graph}(f|_{X\cap[p_n,q_n]})\Big) && \text{[by Lemma 5.2]}\\ &= \lim_n \underline{\dim}_{\mathsf{gr,B}}\Big(X \cap [p_n, q_n]\Big) && \text{[since } f \in L \subseteq L_{p_n,q_n}\text{]}\\ &= \lim_n \inf_{g\in C_{\mathsf{u}}(X\cap[p_n,q_n])} \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(g))\,. && (5.4)\end{aligned}$$

However, for $\varphi \in C_{\mathsf{u}}(X)$, we have $\varphi|_{X\cap[p_n,q_n]} \in C_{\mathsf{u}}(X \cap [p_n, q_n])$, and from this we deduce that $\inf_{g\in C_{\mathsf{u}}(X\cap[p_n,q_n])} \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(g)) \le \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(\varphi|_{X\cap[p_n,q_n]}))$. This inequality clearly implies that $\inf_{g\in C_{\mathsf{u}}(X\cap[p_n,q_n])} \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(g)) \le \inf_{\varphi\in C_{\mathsf{u}}(X)} \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(\varphi|_{X\cap[p_n,q_n]}))$, whence

$$\lim_n \inf_{g\in C_{\mathsf{u}}(X\cap[p_n,q_n])} \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(g)) \le \lim_n \inf_{\varphi\in C_{\mathsf{u}}(X)} \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(\varphi|_{X\cap[p_n,q_n]}))\,. \tag{5.5}$$

Combining (5.4) and (5.5) now shows that

$$\underline{\dim}_{\mathsf{loc,B}}\Big((x, f(x))\, ;\, \mathrm{graph}(f)\Big) \le \lim_n \inf_{\varphi\in C_{\mathsf{u}}(X)} \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(\varphi|_{X\cap[p_n,q_n]}))\,. \tag{5.6}$$

Next, observe that for $\psi \in C_{\mathsf{u}}(X)$, we have

$$\inf_{\varphi\in C_{\mathsf{u}}(X)} \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(\varphi|_{X\cap[p_n,q_n]})) \le \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(\psi|_{X\cap[p_n,q_n]}))\,,$$

and so

$$\lim_n \inf_{\varphi\in C_{\mathsf{u}}(X)} \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(\varphi|_{X\cap[p_n,q_n]})) \le \lim_n \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(\psi|_{X\cap[p_n,q_n]}))\,.$$

Taking infimum over all $\psi \in C_{\mathsf{u}}(X)$ now gives

$$\lim_n \inf_{\varphi\in C_{\mathsf{u}}(X)} \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(\varphi|_{X\cap[p_n,q_n]})) \le \inf_{\psi\in C_{\mathsf{u}}(X)} \lim_n \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(\psi|_{X\cap[p_n,q_n]}))\,. \tag{5.7}$$

Finally, combining (5.6) and (5.7) shows that

$$\begin{aligned}\underline{\dim}_{\mathsf{loc,B}}\Big((x, f(x))\, ;\, \mathrm{graph}(f)\Big) &\le \lim_n \inf_{\varphi\in C_{\mathsf{u}}(X)} \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(\varphi|_{X\cap[p_n,q_n]}))\\ &\le \inf_{\psi\in C_{\mathsf{u}}(X)} \lim_n \underline{\dim}_{\mathsf{B}}(\mathrm{graph}(\psi|_{X\cap[p_n,q_n]}))\\ &= \inf_{\psi\in C_{\mathsf{u}}(X)} \underline{\dim}_{\mathsf{loc,B}}\Big((x, \psi(x))\, ;\, \mathrm{graph}(\psi)\Big) \qquad \text{[by Lemma 5.2]}\\ &= \underline{\dim}_{\mathsf{gr,loc,B}}(x; X)\,.\end{aligned}$$

This completes the proof of Claim 2.

Theorem 2.(1) follows immediately from Clam 1 and Claim 2.
(2) Let $x \in X$.

We first prove that $\underline{\dim}_{\mathsf{gr,loc,B}}(x; X) \le \underline{\dim}_{\mathsf{loc,B}}(x; X)$. Indeed, if $\mathcal{O}$ denotes the zero-function on X, then clearly

$$\begin{aligned}\underline{\dim}_{\mathsf{gr,loc,B}}(x; X) &\le \underline{\dim}_{\mathsf{loc,B}}\Big((x, \mathcal{O}(x))\, ;\, \mathrm{graph}(\mathcal{O})\Big)\\ &= \underline{\dim}_{\mathsf{loc,B}}\Big((x, 0)\, ;\, X \times \{0\}\Big)\\ &= \underline{\dim}_{\mathsf{loc,B}}(x; X)\,.\end{aligned}$$

Next, we prove that $\underline{\dim}_{\mathsf{gr,loc,B}}(x; X) \ge \underline{\dim}_{\mathsf{loc,B}}(x; X)$. It follows from (1) that there is a function $f \in C_{\mathsf{u}}(X)$ such that

$$\underline{\dim}_{\mathsf{gr,loc,B}}(x; X) = \underline{\dim}_{\mathsf{loc,B}}\Big((x, f(x))\, ;\, \mathrm{graph}(f)\Big)\,. \tag{5.8}$$

Note that we can choose sequences $(p_n)_n$ and $(q_n)_n$ from $\mathbb{Q}$ with $[p_1,q_1] \supseteq [p_2,q_2] \supseteq \ldots$, such that $x \in (p_n,q_n)$ for all n and $q_n - p_n \to 0$. It now follows from (5.8) that

$$\begin{aligned}
\underline{\dim}_{\mathsf{gr,loc,B}}(x;X) &= \underline{\dim}_{\mathsf{loc,B}}\Big(\,(x,f(x))\,;\,\mathrm{graph}(f)\Big)\\
&= \lim_n \underline{\dim}_{\mathsf{B}}\Big(\;\mathrm{graph}(f|_{X\cap[p_n,q_n]})\Big) && \text{[by Lemma 5.2]}\\
&\geq \lim_n \underline{\dim}_{\mathsf{B}}(X\cap[p_n,q_n])\\
&= \underline{\dim}_{\mathsf{loc,B}}(x;X)\,. && \text{[by Lemma 5.2 applied to the zero-function on } X]
\end{aligned}$$

(3) The proof of (3) is very similar to the proof of (1) and is therefore omitted.
(4) Let $x \in X$. If x is not an isolated point of X, then we must prove that

$$\overline{\dim}_{\mathsf{gr,loc,B}}(x;X) = \overline{\dim}_{\mathsf{loc,B}}(x;X) + 1\,. \tag{5.9}$$

However, the proof of (5.9) is very similar to the proof of (2) and is therefore omitted.

On the other hand, if x is an isolated point of X, then clearly

$$\overline{\dim}_{\mathsf{gr,loc,B}}(x;X) = 0\,.$$

This completes the proof. □

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