Random Walks in Changing Environments

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Declaration of any previous Submission of the Work

The material presented here for examination for the award of a higher degree by research has not been incorporated into a submission for another degree.

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Andrea Lelli

Declaration of Authorship

I am the author of this thesis, and the work described herein was carried out by myself personally in collaboration with my supervisor Dr. Alexandre Stauffer.

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Abstract

This thesis deals with Random Walks on graphs that change over time in a random manner, more precisely we analyse Random Walks on Dynamical Percolation. In this model, the edges of a graph G are either open or closed and refresh their status at rate μ independently from all other edges, while at the same time a random walker moves on G at rate 1 but only along edges which are open. In Chapter 3 we present the known results about mixing time for random walks in dynamical percolation and we give a sketch of the proof of the upper bound for the mixing time of Random Walk on Dynamical Percolation when $G_n = \mathbb{T}_n^d$ for all $p < p_c$ discussed in detail in [PSS15]. Later, we show a bound on the mixing time of the Random-Cluster model for lattices with polynomial growth. Finally, we introduce Random Walks on Dynamical Random Cluster. This model is similar to the Random Walks on Dynamical Percolation with the only difference that the refresh of the edges depends on the configuration of the open edges in the graph at the time of the update. We prove that on the d-dimensional torus with side length n, in the subcritical regime, the mixing time for the full system is bounded above by $\frac{n^2}{\mu}$ up to constants.

Acknowledgements

Usually, the acknowledgements section in a thesis is for the guests to have something to read during the celebration. However, with the coronavirus pandemic happening worldwide as I am writing this, I do not believe this thesis will have such a lucky faith. Nevertheless, I still want to mention and spend a few words for the people who have helped me in the writing of this thesis and the people who have had an impact on my life during my experience in Bath, as a testament for my future nostalgic leafing through.

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Questa tesi è dedicata a mia madre e mio padre.

Contents

Chapter 1

Introduction

"**Random Walk in Random Environment**" (RWRE) has become a classical model for random motion in random media. Following the "Lecture Notes on Random Walk in Random Environment" [Zei04] by Ofer Zeitouni: "*The definition of a RWRE involves two components: first, the environment, which is randomly chosen but kept fixed throughout the time evolution, and second, the random walk*".
The interest in RWRE began in the early 70's, and since then its theory has been developed by several researchers, including W. Bryc, F. Comets, A. Dembo, N. Gantert, Y. Peres, and S.R.S. Varadhan. Many fundamental results regarding **Ergodicity**, **law of large numbers,** 0-1 **law, large deviations** and others [Ali99, BD96, CZ+04] have been proved, and it is fair to say that RWRE is now a well understood model.

More recently, people started to look at random walks that move inside an environment which is, at the same time, changing itself.
The study of random walks in this particular setting is motivated by the fact that *real-world networks are not static, but evolve over time.*

Several challenges arise when we allow the environment to change while the walker moves:

- **Lack of stationarity**.*Random walks on a static and finite graph* (that is, a graph with a finite number of vertices that does not change over time) *always have a stationary distribution*, which is quite easy to state as it is proportional to the degrees of the vertices. On the other hand, *random walks on a changing graph* may not have a *stationary distribution*; for example consider an environment which switches between graphs with different stationary distributions;

- **Lack of reversibility**. Random walks on static graphs not only have a stationary distribution, but are also *time reversible*. On a changing environment, however, in cases where the random walk does have a stationary distribution, it may form a non-reversible process. See, for instance, Proposition 2.29 in Chapter 2;

- **Cover Time**. On static graphs of size n the so-called **cover time** (which is the expected time for the random walk to visit every vertex) is known to be at most Cn^3 for some universal constant $C > 0$ [Fei95]. However, if the graph changes over time the cover time could be even exponentially large in n [AKL08]. Moreover a similar example can be made to show that hitting times and the mixing time of random walks on changing graphs (when a stationary distribution exists and the walker converges to it) could be exponentially large in n.

The notion of "**Random Walk in Changing Environment**" (RWCE) has been introduced only very recently by Amir et al. [ABGGK15] as a random walk in which each step is performed in a different graph, on the same set of vertices.
The walker moves on a graph $G = (V, E)$ where the set of vertices is fixed, and each edge is equipped with a non-negative conductance $C_t(e)$ dependent on time. The walker decides which edge to traverse in each step with probability proportional to the conductance. A precise definition is given in Definition 2.15 in Chapter 2.

As we will discuss in Chapter 2, this notion of RWCE is very broad and gives rise to a large number of degenerate cases. For this reason, in this thesis we will focus on subclasses of RWCE, more specifically when the environment changes according to **dynamical percolation** (DP) and **dynamical random-cluster** (RC), namely **Random Walks in Dynamical Percolation** (RWDP) and **Random Walks in Dynamical Random-Cluster** (RWRC). In Chapter 2 we will define all these notions and in Chapter 3 we will present one of the few results known for the analysis of the mixing time of a random walk in a graph that changes over time, namely the upper bound for the mixing time of the random walk in dynamical percolation on the torus $\mathbb{T}_n^d$ when the probability of opening an edge is below the critical one, see Theorem 3.1 [PSS15]. In this paper the authors also provide a lower bound for the mixing time, and show it to be of the same order of the upper bound, thus finding the correct order of the mixing time.

The bulk of the thesis will be the analysis of RWRC, which is carried out in Chapter 4, this model is very similar to RWDP but the underlying graph evolves following the dynamics of the RC model. Here we prove an upper bound of the Mixing Time of RWRC on the d-dimensional torus, in the subcritical case, see Theorem 4.1.

In Chapter 5 we will refer to the work of A. Blanca and A. Sinclair [BS17], whose main result is an upper bound to the mixing time of the Random Cluster Glauber Dynamics on $\mathbb{Z}^2$, and borrowing some of their techniques we will extend their result to a larger class of graphs. This result was proved in parallel in [HS18] using different methods. This model can be seen as a generalisation of the independent percolation model, where an extra parameter on the number of the clusters is introduced in the formulation of the percolation probability distribution, and this modification makes the update of the edges depend on the configuration of the entire graph at the moment of the update. Because of this global dependence, to the present, still very little is known on the mixing time of the dynamics of the Random-Cluster model, namely the Glauber dynamics. Here we prove an upper bound for the mixing time of the Glauber dynamics for the random cluster model on vertex-transitive graphs of polynomial growth in subcritical setting, see Theorem 5.2.

Finally in Chapter 6 we identify future research directions to investigate.

We adopt the following convention regarding the constants that appear in the thesis. We use lower case letters c_i with $i = 0, 1, \dots$ for constants that we only use temporarily in a proof, the same letter may be used again in another proof but it will not refer to the previous one. Capital letters C_i with $i = 0, 1 \dots$ will be used to denote constants that are global within a chapter, the same letter may appear in two different chapters but it will not represent the same constant. Finally, capital letters with words as subscript will represent constants that are global in the whole thesis.

Chapter 2

Background Work

2.1 Markov Chain Mixing

In this thesis we will only look at Markov chains on a finite state space, which we denote by Ω. Our main objective is to bound the **mixing time** of certain Markov chains. Essentially we want to be able to find a good approximation of how long said Markov chain has to run before we can say its distribution is "close enough" to its stationary distribution. To study this, we first need to be able to quantify the "distance" of the distribution of a Markov chain from the equilibrium (its stationary distribution).

Definition 2.1. *The **total variation distance** between two probability distributions ϕ_1 and ϕ_2 on the same state space Ω is defined by*

$$\|\phi_1 - \phi_2\|_{TV} := \max_{A \subset \Omega} |\phi_1(A) - \phi_2(A)| . \tag{2.1}$$

In words, the total variation distance is the largest possible difference between the probabilities that the two distributions can assign to the same event.

Remark 2.2. *The total variation distance is a distance. Indeed:*

1. $\|\phi_1 - \phi_2\|_{TV} \geqslant 0$ *and it is equal to* 0 *if and only if* $\phi_1 = \phi_2$*;*
2. $\|\phi_1 - \phi_2\|_{TV} = \|\phi_2 - \phi_1\|_{TV}$*, i.e. it is symmetric;*
3. *the triangle inequality holds:* $\|\phi_1 - \phi_2\|_{TV} \leqslant \|\phi_1 - \pi\|_{TV} + \|\pi - \phi_2\|_{TV}$.

Proposition 2.3. *For all ϕ_1, ϕ_2 distributions on Ω*

$$\|\phi_1 - \phi_2\|_{TV} = \frac{1}{2} \sum_{x \in \Omega} |\phi_1(x) - \phi_2(x)| . \tag{2.2}$$

For $x \in \Omega$, we denote by $\mathbb{P}_x$ the probability measure induced by the Markov chain started from x.

Definition 2.4. *The distance $d(t)$ of a Markov chain $X_t \in \Omega$ at time t from its stationary distribution π:*

$$d(t) := \max_{x \in \Omega} \|\mathbb{P}_x(X_t \in \cdot) - \pi\|_{TV}, \tag{2.3}$$

$$\bar{d}(t) := \max_{x,y \in \Omega} \|\mathbb{P}_x(X_t \in \cdot) - \mathbb{P}_y(X_t \in \cdot)\|_{TV}. \tag{2.4}$$

More precisely, it is the distance between the distribution of X_t starting from the worst initial configuration and its stationary distribution.

As one would expect, the function $d(t)$ is decreasing in time, as follows from the next proposition.

Proposition 2.5. *Let P be the transition matrix of a Markov chain on Ω with stationary distribution π. Then for any $0 \leqslant s \leqslant t$:*

$$d(t) \leqslant d(s). \tag{2.5}$$

Lemma 2.6. *[LPwcbELW17]* $d(t) \leqslant \bar{d}(t) \leqslant 2d(t)$.

Definition 2.7. *A **coupling** of two probability distributions ϕ_1 and ϕ_2 is a pair of random variables (X, Y) defined on a single probability space $(\Omega \times \Omega, \mathcal{F}, \mathbb{P})$ such that the marginal distribution of X is ϕ_1 and the marginal distribution of Y is ϕ_2.*
In other terms: $\mathbb{P}(X \in A) = \phi_1(A)$ and $\mathbb{P}(Y \in B) = \phi_2(B)$ for all $A, B \subset \Omega$.

There always exists a coupling between any two probability distributions, for example letting X and Y be independent. However very interesting properties come out when we force the two random variables to assume the same value, as much as possible.

Proposition 2.8. *[LPwcbELW17] Let ϕ_1 and ϕ_2 be two probability distributions on Ω.*

$$\|\phi_1 - \phi_2\|_{TV} = \inf\{\mathbb{P}(X \neq Y) \,|\, (X, Y) \text{ is a coupling of } \mu \text{ and } \nu\}, \tag{2.6}$$

where the infimum is over all couplings between ϕ_1 and ϕ_2.

This provides an effective method of obtaining upper bounds on the total variation distance. The more the coupling increases the probability to find the same value for both X and Y, the better is the bound. Thus in general we look for the coupling that maximizes the probability for the two random variables to be equal.

We can also see coupling not only as coupling of pairs of distributions but also as coupling of entire Markov chain trajectories: consider two Markov chains (X_t), (Y_t) on Ω with common transition matrix P and starting points x_0 and y_0 respectively. Their distribution at time t will be $P^t(x_0, \cdot)$ for X_t and $P^t(y_0, \cdot)$ for Y_t of course. Thus we can define a **coupling of Markov chains** with transition matrix P to be any process $(X_t, Y_t)_{t=0}^{\infty}$ such that for all $t \geqslant 0$ and all $x_0, y_0 \in \Omega$ the joint distribution satisfies

$$\mathbb{P}_{x_0,y_0}(X_t \in \cdot) = P^t(x_0, \cdot), \quad \mathbb{P}_{x_0,y_0}(Y_t \in \cdot) = P^t(y_0, \cdot).$$

Any coupling of Markov chains can be modified so that the two chains remain together at all times after their first simultaneous visit to a single state, or equivalently:

$$\text{if } X_s = Y_s, \text{ then } X_t = Y_t \text{ for any } t \geqslant s.$$

To construct a coupling satisfying the above equation, simply run the chains according to the original coupling until they meet, then let them run together. From now, if (X_t) and (Y_t) are coupled Markov chains with $X_0 = x$ and $Y_0 = y$, we will denote $\mathbb{P}_{x,y}$ for the probability space where both (X_t) and (Y_t) are defined.

Proposition 2.9. *[LPwcbELW17] Let $\{(X_t, Y_t)\}$ be a coupling such that the two chains remain together once they meet, with $X_0 = x$ and $Y_0 = y$. Let τ_{couple} be the first time the chains meet*

$$\tau_{couple} := \min\{t : X_t = Y_t\}.$$

Then

$$\|\mathbb{P}_x(X_t \in \cdot) - \mathbb{P}_y(Y_t \in \cdot)\|_{TV} \leqslant \mathbb{P}_{x,y}(\tau_{couple} > t).$$

Proof. Observe that $\mathbb{P}_x(X_t = z) = \mathbb{P}_{x,y}(X_t = z)$ and $\mathbb{P}_y(Y_t = z) = \mathbb{P}_{x,y}(Y_t = z)$ because $\mathbb{P}_{x,y}$ is a coupling of (X_t) and (Y_t) and by definition it must respect the right marginal distributions.
Using Proposition 2.8 we find that

$$\|\mathbb{P}_x(X_t \in \cdot) - \mathbb{P}_y(Y_t \in \cdot)\|_{TV} \leqslant \mathbb{P}_{x,y}(X_t \neq Y_t) = \mathbb{P}_{x,y}(\tau_{\text{couple}} > t).$$

Which concludes the proof of the Proposition. □

Using Proposition 2.9 and Lemma 4.11 in [LPwcbELW17] we obtain the useful result, which gives a method to bound the distance from stationarity

Corollary 2.10. *Suppose that for each state $x, y \in \Omega$ there is a coupling (X_t, Y_t) with $X_0 = x$ and $Y_0 = y$. For each such coupling, let τ_{couple} be the first time the chains meet. Then*

$$\bar{d}(t) \leqslant \max_{x,y \in \Omega} \mathbb{P}_{x,y}(\tau_{couple} > t).$$

Suppose we want to know how long we have to wait before some Markov chain X_t reaches a distance from its stationary distribution of at most ε. It would be useful to have a time-parameter that formalizes this concept.

Definition 2.11. *The ε-**mixing time** is defined by:*

$$T_{mix}(\varepsilon) := \min\{t : d(t) \leqslant \varepsilon\}. \tag{2.7}$$

Remark 2.12. *If $\varepsilon \geqslant \varepsilon'$, then $T_{mix}(\varepsilon) \leqslant T_{mix}(\varepsilon')$.*
This immediately follows from Proposition 2.5.

By convention if we take $\varepsilon = \frac{1}{4}$, we call $T_{\text{mix}}(\frac{1}{4})$ the **mixing time**:

$$T_{\text{mix}} := T_{\text{mix}}(\frac{1}{4}).$$

The study of T_{mix} is often sufficient, because of the next result:

Proposition 2.13. *[LPwcbELW17] Let P be the transition matrix of an irreducible, aperiodic Markov chain with $T_{mix}(c) \leqslant T$ for some $c < \frac{1}{2}$. Then, for this Markov chain,*

$$T_{mix}(\varepsilon) \leqslant \lceil \frac{\log \varepsilon}{\log(2c)} \rceil T.$$

And, in particular,

$$T_{mix}(\varepsilon) \leqslant \lceil \log_2 \varepsilon^{-1} \rceil T_{mix}.$$

2.2 Random Walks in Changing Environment

This section is based on the results presented by Amir et al. [ABGGK15]. In their work they introduce the notion of Random Walks in Changing Environments, the class of processes that we will analyse in this thesis. The notion of "**Random Walk in Changing Environment**" (RWCE) has been introduced only very recently by Amir et al. [ABGGK15] as a random walk in which each step is performed in a different graph, on the same set of vertices.

Definition 2.14. *A **graph** $G = (V, E)$ consists of a vertex set V and an edge set E, where the elements of E are disordered pairs of vertices: $E \subset \{xy : x, y \in V, x \neq y\}$.*

- *When $xy \in E$, we write $x \sim y$ and say that y is a **neighbor** of x (and, also, x is a neighbor of y);*
- *For any $x \in V$, define $\mathcal{N}(x) \subseteq V$ the set of neighbors of x;*
- *The **degree** $\deg(x)$ of a vertex x is the number of neighbors of x;*
- *For any $S \subset V$, a subset of the edges $E(S)$ is the set of all edges in E with endpoints in S;*
- *The **graph distance** between two distinct vertices x, $y \in V$ is denoted $\mathrm{dist}_G(x, y)$; and the diameter of the graph is denoted by $\mathrm{diam}(G)$.*

In this thesis the size of the graph will be finite. We will equip the edges of the graph with non-negative numbers, which we call conductances. In this way, a graph $G = (V, E, \zeta)$ is the graph (V, E) where each edge $e \in E$ has conductance $\zeta(e)$.
We are now able to define what a **random walk in changing environment** (RWCE) is, in its broadest sense.

Definition 2.15. *A discrete time RWCE on a graph $G = (V, E)$ is a stochastic process $\{\langle X_t, G_t\rangle\}_{t=0}^{\infty}$, where $G_t = (V, E, \zeta_t)$ are graphs over a fixed vertex set V and edge set E with conductance function $\zeta_t : E \to [0, \infty)$, and for all $t \in \mathbb{N}$, $X_t \in V$ denotes the position of the walker at time t.*
Moreover, for all $v \in V$,

$$\mathbb{P}\left(X_{t+1} = v \mid \langle X_0, G_0\rangle, \ldots, \langle X_t, G_t\rangle\right) = \frac{\zeta_t(X_t v)}{\sum_{\{e \in E \mid X_t \in e\}} \zeta_t(e)}, \tag{2.8}$$

where we use the convention that $\zeta(X_t v) = 0$ if $v \notin \mathcal{N}(X_t)$.
We call the sequence $\{X_t\}_{t=0}^{\infty}$ the random walk and the sequence $\{G_t\}_{t=0}^{\infty}$ the changing environment.

Less formally, the law of the process governs the changes in the environment (the conductances), while the distribution of X_t is determined by the conductances on the graph. In the following examples the function ζ_t will be deterministic, but it could as well be a stochastic process as in **Dynamical Percolation**.
Moreover, the case $\zeta_t(e) = \zeta_0(e)$ for all $e \in E$ (where conductances are independent of time) is exactly the case of a random particle that moves on a graph, maybe random, but remains unchanged throughout time. As mentioned in the Chapter 1 the theory for this kind of problems is ample and goes under the name of RWRE.
The definition of RWCE is very broad, as it can implement any kind of behavior. In particular, in a setting like this we can not even find a criteria for transience/recurrence.

Definition 2.16. *Let $G = (V, E)$ be a infinite graph, namely V is an infinite set of vertices, a RWCE on G is called*

- ***recurrent*** *if the walker visits every vertex in V infinitely many times almost surely;*
- ***transient*** *if the walker visits every vertex in V a finite number of times almost surely;*
- ***mixed type*** *otherwise.*

Due to Kolmogorov 0-1 law, a random walk in a static graph can be either recurrent or transient, while RWCE can be of mixed type, as the following example illustrates.

Example 2.1. *[ABGGK15] Consider the RWCE with $V = \mathbb{N}$, $X_0 = 0$, and denote with $\zeta_t(j, j+1)$ the conductance on the edge that connects the vertices j and $j+1$. Assume the conductances are as follows:*

- $\zeta_t(j, j+1) = 2^{-j}$ *for* $j = t$
- $\zeta_t(j, j+1) = 1$ *otherwise,*

With probability $p = \prod_{t=0}^{\infty} \frac{1}{1+2^{-t}} > 0$ the RW will always go to the right (transient), and with probability $1 - p$ it will eventually perform a simple RW on the graph with conductances 2^{-j} which is of course recurrent.

For this reason we need to restrict to a smaller class of RWCE.

Definition 2.17. *A RWCE is called*

- ***proper*** *if $0 < \zeta_t(e) < \infty$ for all $t \in \mathbb{N}$ and $e \in E$. It is improper otherwise;*
- ***elliptic*** *if $\zeta_t > c$, for all t, with c some positive constant;*
- ***bounded from above (below)*** *by $G = (V, E, \zeta)$ if $\zeta_t(e) \leqslant \zeta(e)$ $(\zeta_t(e) \geqslant \zeta(e))$ for all $t \in \mathbb{N}$ and $e \in E$ almost surely;*
- ***nonadaptive*** *if the distribution of G_{t+1} given $G_0, \dots, G_t$ is independent of $X_0, \dots, X_t$. It is adaptive otherwise;*
- ***monotone increasing (decreasing)*** *if $\zeta_{t+1} \geqslant \zeta_t$ $(\zeta_{t+1} \leqslant \zeta_t)$ almost surely.*

To illustrate the importance of the above properties, the example below shows that adaptivity can produce unexpected behaviors.

Example 2.2. *[ABGGK15] Let X_t be a RW on $(G_t = (V, E, \zeta_t))$ with $V = \mathbb{N}$ and denote with $\zeta_t(j, j+1)$ the conductance on the edge that connects the vertices j and $j+1$. Assume the conductances are*

- $\zeta_t(X_t, X_t + 1) = 2$
- $\zeta_t(j, j+1) = 1$ *for* $j \neq X_t$.

This RWCE is bounded above by the one with $\zeta_t(e) = 2$ for all $t \in \mathbb{N}$ and $e \in E$, and below by the one with $\zeta_t(e) = 1$ for all $t \in \mathbb{N}$ and $e \in E$, which are both symmetric simple random walks.
Thus, it is bounded above and below by a ***recurrent*** *RW, it is proper, elliptic, but adaptive and non-monotone. However, this RWCE is simply a RW with a right drift and therefore it is* ***transient****.*

We can get a similar behavior even in a nonadaptive setting.

Example 2.3. *[ABGGK15] Let X_t be a RW on $(G_t = (V, E, \zeta_t))$ with $V = \mathbb{N}$ and denote with $\zeta_t(j, j+1)$ the conductance on the edge that connects the vertices j and $j+1$. Assume the conductances are*

- $\zeta_t(j, j+1) = 100$ *when* $t \equiv j \pmod{100}$
- $\zeta_t(j, j+1) = 1$ *otherwise*

This RWCE is bounded above by the one with $\zeta_t(e) = 100$ for all $t \in \mathbb{N}$ and $e \in E$ and below by the one with $\zeta_t(e) = 1$ for all $t \in \mathbb{N}$ and $e \in E$.
Thus, it is bounded above and below by a recurrent RW, it is nonadaptive, proper and elliptic, but it is transient. In fact, this is simply a RW with a right drift for an expected number of 101 steps, after which it is simple (i.e., unbiased) until the next wave of 100 conductance arrives. Therefore, it is transient.

Similarly we can obtain a recurrent RWCE bounded from above and below by transient RWs: simply set conductances $\zeta_t(j, j+1) = 1000 \times 2^j$ when $t \equiv -j \pmod{100}$ and $\zeta_t(j, j+1) = 2^j$ otherwise.
When the conductances change monotonically, then it is possible to establish sufficient conditions for recurrence/transience when G is a tree.

Theorem 2.18. *[ABGGK15] Let X_t be a RW on $(G_t = (V, E, \zeta_t))$ with G a tree then*

- *if ζ_t are monotonically increasing, nonadaptive and bounded above by conductances ζ_∞ of a recurrent graph, then the walk is recurrent;*
- *if ζ_t are monotonically decreasing, nonadaptive and bounded below by conductances ζ_∞ of a transient graph, then the walk is transient;*

If $G = \mathbb{N}$ the analysis can be refined.

Theorem 2.19. *[ABGGK15] Let X_t be a RW on $(G_t = (V, E, C_t))$ with $G = \mathbb{N}$ then*

- *if the conductances C_t monotonically decrease, but are bounded below and above by conductances $C_\infty^- \leqslant C_t \leqslant C_\infty^+$ of recurrent graphs, then the walk is recurrent;*
- *if the conductances C_t monotonically increase, but are bounded below and above by conductances $C_\infty^- \leqslant C_t \leqslant C_\infty^+$ of transient graphs, then the walk is transient;*

Unfortunately these criteria do not hold when the dimension of the graph is higher. The difficulty arises when the conductances are adapted to the walk, in fact Amir et al. [ABGGK15] show *how increasing the conductances of the graph with a strategy adapted to the walk can produce a process transient on $\mathbb{Z}^2$*. Finally they conjecture that this particular behavior cannot happen when the conductances are non-adaptive, claiming that "*there is an essential difference between adaptive and non-adaptive walks*". Part of this conjecture has then been proved in [DHM+17] using the **Evolving Sets** technique, which was introduced by Morris-Peres [MP05] and that will be more carefully analyzed later in this thesis. More precisely [DHM+17] proved the transience of any **Uniformly Lazy Random Walk** (ULRW) on $\mathbb{Z}^d$, for $d \geqslant 3$, when $\mathbb{Z}^d$ is equipped with uniformly bounded above and below conductances, which are non-adaptive and whose sum around a vertex is non-decreasing in time.

More generally it is believed that when the environment

- changes quickly enough,
- and is well "behaved", e.g. **uniformly elliptic**, which means the conductances are bounded from above and below, and in particular the probability of traversing each edge when the walk is at one of its endpoints is bounded away from 0 so that the walk cannot get stuck in a region,

the random walk will behave in a way that is similar to a random walk on the underlying (non-changing) graph. This has been quantified, especially in the case of $\mathbb{Z}^d$, by the derivation of the law of large numbers and central limit theorems under some conditions related to the mixing time of the environment; see for example the works of Avena et al. [AHR09,AdHR+11]. We will further restrict to a class of random walk on changing environments, which will be called random walk on dynamical percolation.

2.3 Dynamical Percolation

Before we define dynamical percolation we need to define percolation on a graph. Here time will play no role.

Definition 2.20. *The **percolation graph** on G (or, more precisely, the* independent *percolation graph on G) with parameter $0 \leqslant p \leqslant 1$, is the random subgraph $G_p = (V, E_p)$ obtained as follows:*

- *for each $e \in E$, $\mathbb{P}(e \in E_p) = p$;*
- *for any distinct $e_1, \ldots, e_k \in E$, the events $\{e_1 \in E_p\}, \ldots, \{e_k \in E_p\}$ are mutually independent.*

In other words, we declare each edge to be "open" (i.e., belonging to E_p) with probability p and "closed" (i.e., not belonging to E_p) with probability $1-p$, independently of the other edges.

When G is infinite one could ask what the probability for a certain vertex $v \in V$ to lie in an infinite connected component of the percolation graph is.

Definition 2.21. *Denoting $\mathcal{C}_v$ the connected component containing v in the percolation graph G_p, let*

$$\theta_v(p) := \mathbb{P}\left(|\mathcal{C}_v| = \infty\right).$$

*We call $\theta_v(p)$ the **percolation probability** at v.*

An immediate nice result is the following. A proof can be found, for example, in [Gri13].

Theorem 2.22. *For all $u, v \in V$ and $p \in [0,1]$, if $\theta_v(p) > 0$ then $\theta_u(p) > 0$.*

Observe that by this result, either $\theta_v(p) = 0$ for all $v \in V$ or $\theta_v(p) > 0$ for all $v \in V$. And therefore, the next quantity is well-defined and does not depend on v.

Definition 2.23. *Given a connected graph $G = (V, E)$ and a vertex $v \in V$, the **critical probability** or **percolation threshold** $p_c = p_c(G)$ is*

$$p_c := \inf\{p : \theta_v(p) > 0\}.$$

Moreover let $\mathcal{C}_\infty$ be the event that

$$\mathcal{C}_\infty := \{\exists \text{ infinite connected component in } G\}.$$

Theorem 2.24 ($0-1$ law)**.** *[Gri13] For any infinite connected locally finite graph G and any $p \in [0,1]$,*

$$\mathbb{P}(\mathcal{C}_\infty) \in \{0,1\}\,.$$

Proof. Observe that if $\theta_v(p) = 0$ for some $v \in V$

$$\mathbb{P}(\mathcal{C}_\infty) = \mathbb{P}(\bigcup_{v\in V} \{|\mathcal{C}_v| = \infty\}) \leqslant \sum_{v\in V} \theta_v(p) = 0.$$

On the other hand, if $\theta_v(p) > 0$ for some $v \in V$, then

$$\mathbb{P}(\mathcal{C}_\infty) \geqslant \mathbb{P}(|\mathcal{C}_v| = \infty) = \theta_v(p) > 0$$

And then $\mathbb{P}(\mathcal{C}_\infty) = 1$ because of Theorem 2.24. □

In particular, by monotonicity

$$\mathbb{P}(\mathcal{C}_\infty) = \begin{cases} 0 & \text{for } p < p_c \\ 1 & \text{for } p > p_c. \end{cases}$$

Thus we have a **phase transition** at $p = p_c$, meaning that if $p > p_c$ then almost surely **percolation occurs**, whereas if $p < p_c$ a.s. percolation does not occur. If $p = p_c$ it is in general unknown whether percolation occurs.
The critical probability p_c is important also in a finite graph setting.
Consider $\mathbb{T}_n^d$ the $d-$dimensional discrete torus with vertex set $[0,\dots,n)^d$, this is a finite graph and is obtained by taking the finite box $\mathbb{Z}_n^d$ and equipping it with so-called **periodic boundary conditions**. If we do percolation on this particular graph, we have

Theorem 2.25. *[Gri13, Theorem 8.18]*
Let $\bar{0} = (0,\dots,0) \in \mathbb{Z}^d$. If $p \in [0,1]$, as $n \to \infty$, we have

$$n^{-d} \max_{\mathcal{C}\subseteq \mathbb{T}_{n,p}^d} |\mathcal{C}| \xrightarrow{P} \theta_{\bar{0}}(p),$$

where the maximum is over the connected components of the percolation graph $\mathbb{T}_{n,p}^d$, and $\theta_{\bar{0}}(p)$ is the percolation probability for $\mathbb{Z}^d$ in $\bar{0}$ (note that, by symmetry, the percolation probability is equal to $\theta_v(p)$ for all $v \in \mathbb{T}_n^d$).
Moreover, if $\mathcal{C}_1$ is the largest connected component of $\mathbb{T}_{n,p}^d$,

$$n^{-d} \max_{\mathcal{C}\nsubseteq \mathcal{C}_1} |\mathcal{C}| \xrightarrow{P} 0,$$

where the maximum is ranged on all connected components of $\mathbb{T}_{n,p}^d \setminus \mathcal{C}_1$.

In particular, if $p > p_c(\mathbb{Z}^d)$ then in the percolation graph $\mathbb{T}_{n,p}^d$ with probability going to 1 with n there will be a unique connected component (often called **giant component**) with size of order n^d, furthermore all the other connected components will have size of order at most $\log n$.
We are finally able to introduce Dynamical Percolation. This model was introduced independently in [HPS97] and by Itai Benjamini. We will consider from now on processes in continuous time as it is easier to present them.

Definition 2.26. *Consider a graph $G = (V, E)$ and parameters p and $\mu > 0$. The dynamical percolation is the continuous time Markov process $\{\eta_t\}_{t=0}^{\infty}$ on $\Omega = \{0,1\}^E$ in which,*

- *each edge is equipped with a random clock distributed as a Poisson process of intensity μ independent of the other clocks;*
- *as soon as a clock rings we refresh the status of its edge to 1 with probability p or 0 with probability $1-p$, independently of everything else.*

Its **stationary distribution** is a product of Bernoulli(p).

2.4 Random Walk on Dynamical Percolation.

Definition 2.27. *If $G = (V, E)$ is a graph, $p \in [0,1]$, $\mu > 0$, the **Random Walk on Dynamical Percolation** (RWDP) is the process of a particle that lives on V and that, according to the rings of a clock distributed as a Poisson process of rate 1, chooses a neighbor (in the original graph G) uniformly at random and moves there if and only if the connecting edge is open (i.e., it has status 1 in the dynamical percolation, see Definition 2.26) at that time.*

If we denote with $\{X_t\}_{t=0}^{\infty}$ the position of the random walker at time t (this is not a Markov process), we have that

$$\{M_t\}_{t=0}^{\infty} := \{(X_t, \eta_t)\}_{t=0}^{\infty} \tag{2.9}$$

is a continuous time Markov chain on the space $\Omega = V \times \{0,1\}^E$. Here η_t is the dynamical percolation process on the graph that we defined in Definition 2.26. Observe that this process agrees with the definition of non-adaptive RWCE in 2.15. Indeed, this is just a particular case of the model introduced before where the conductances can assume only the values 0 and 1.

Proposition 2.28. *Let $G = (V, E)$ be a finite graph, then the stationary distribution of the RWCE on G we just defined is $\pi = u \times \pi_p$, where u is the stationary distribution of the SRW on G and π_p is the product measure with parameter p.*

Proof. Let $\varepsilon = \frac{1}{1+|E|\mu}$ be the probability that the walker jumps from $v \in V$ before any edge refreshes. We only need to prove that $u \times \pi_p$ is reversible for the Markov Chain $\{M_t\}_{t=0}^{\infty}$.
In fact, let $M = (X, \eta)$, $M' = (X', \eta') \in V \times \Omega$ be two configurations such that

- either $X = X'$ and there exists $e' \in E$ such that $\eta(e) = \eta'(e)$ for all $e \in E \setminus \{e'\}$, and $\eta(e') = 0$, $\eta'(e') = 1$ (**case a**);
- or $X \in \mathcal{N}(X')$, $\eta \equiv \eta'$ and $\eta(e) = \eta'(e) = 1$ for $e = XX'$ (**case b**).

We do not consider other configurations because otherwise the probability of jumping from one to the other is 0. Moreover, we will consider T_j to be the time of the j-th jump of M_t. Then we just need to prove that

$$\pi(M)\mathbb{P}(M_{T_{j+1}} = M' | M_{T_j} = M) = \pi(M')\mathbb{P}(M_{T_{j+1}} = M | M_{T_j} = M').$$

If so, then π is reversible for $\{M_t\}_{t=0}^{\infty}$ and thus is its stationary distribution.
In the following, we let $E(\eta)$ stand for the set of open edges in η, and similarly for η'. We

analyze each of the two cases separately.
Case a:

$$u(X)p^{|E(\eta)|}(1-p)^{|E|-|E(\eta)|}(1-\varepsilon)\frac{1}{|E|}p \\ = u(X')p^{|E(\eta')|}(1-p)^{|E|-|E(\eta')|}(1-\varepsilon)\frac{1}{|E|}(1-p),$$

which is verified because $u(X) = u(X')$ and $|E(\eta)| + 1 = |E(\eta')|$.
Case b:

$$u(X)p^{|E(\eta)|}(1-p)^{|E|-|E(\eta)|}\varepsilon\frac{1}{\deg(X)} = u(X')p^{|E(\eta')|}(1-p)^{|E|-|E(\eta')|}\varepsilon\frac{1}{\deg(X')},$$

which is verified because $u(X) = \frac{\deg(X)}{\sum_{v\in V}\deg(v)}$, and $|E(\eta)| = |E(\eta')|$. □

It is of vital importance that the random walker chooses a neighbor uniformly at random from the set of neighbors in the original graph G, and then jumps there if and only if the edge is open.

Proposition 2.29. *There exists a graph G such that, for the random walk on dynamical percolation on G, if the random walker chooses a neighbor uniformly at random only from the set of open edges, the process is not reversible.*

Proof. We find that there exists a graph G such that, for the random walk on dynamical percolation on G, we can find a cycle of configurations $M^1, \dots, M^k, M^1$ such that

$$\mathbb{P}(M^1 \to M^2 \to \cdots \to M^k \to M^1) \neq \mathbb{P}(M^1 \to M^k \to \cdots \to M^2 \to M^1), \tag{2.10}$$

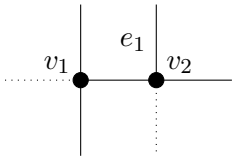
where with $M^1 \to M^2$ we indicate that the process jumps from the configuration M^1 to M^2. If we can find such a cycle then the process is not reversible. Indeed, by contradiction, if π were a reversible distribution, we would have for all t

$$\begin{aligned}
\mathbb{P}(M_{t+1} = M^2 | M_t = M^1) &= \frac{\pi(M^2)}{\pi(M^1)}\mathbb{P}(M_{t+1} = M^1 | M_t = M^2) \\
\mathbb{P}(M_{t+1} = M^3 | M_t = M^2) &= \frac{\pi(M^3)}{\pi(M^2)}\mathbb{P}(M_{t+1} = M^2 | M_t = M^3) \\
\vdots \quad &= \quad \vdots \\
\mathbb{P}(M_{t+1} = M^1 | M_t = M^k) &= \frac{\pi(M^1)}{\pi(M^k)}\mathbb{P}(M_{t+1} = M^k | M_t = M^1).
\end{aligned}$$

Multiplying the terms at the left hand side and the terms at the right hand side we would find

$$\mathbb{P}(M^1 \to M^2 \to \cdots \to M^k \to M^1) = \mathbb{P}(M^1 \to M^k \to \cdots \to M^2 \to M^1),$$

which is a contradiction.
Consider the RWDP on $\mathbb{T}_n^2$. Now we establish (2.10).



Imagine that we start from a configuration like the one in the figure above, where solid edges represent open edges and dashed edges represent closed edges. Consider the cycle below, formed by moves of the walker and updates of the status of the edge e_1 (with the status of all the other edges remaining unchanged),

$$v_1 \times e_1^1 \to v_2 \times e_1^1 \to v_2 \times e_1^0 \to v_1 \times e_1^0 \to v_1 \times e_1^1,$$

where, for the sake of notation, $v_1 \times e_1^i$ represents the configuration where the random walker is at v_1 and the edge e_1 has status i.
Let $\varepsilon = \frac{1}{1+|E|\mu}$ be the probability that the walker jumps from $v \in V$ before any edge refreshes. This cycle has probability of happening

$$c_1 = \varepsilon\frac{1}{3} \times (1-\varepsilon)\frac{1}{|E|}(1-p) \times \varepsilon\frac{1}{2} \times (1-\varepsilon)\frac{1}{|E|}p.$$

However if we reverse the cycle, the probability of this event would be

$$c_2 = (1-\varepsilon)\frac{1}{|E|}(1-p) \times \varepsilon\frac{1}{3} \times (1-\varepsilon)\frac{1}{|E|}p \times \varepsilon\frac{1}{3} \neq c_1.$$

Therefore this process is not reversible. □

2.5 The Random Cluster Model

The Random-Cluster model, often also denoted as the FK model, was introduced in the late 1960s by Kees Fortuin and Piet Kasteleyn in [KF69] as a unifying framework for studying percolation, Ising model and the Potts model. They initiated a study in stochastic geometry which has revealed to be a central tool in modelling and analysing the ferromagnet and its phase transition, one of the oldest challenges in classical statistical mechanics. Interest in the Random-Cluster model for applications on the Ising and Potts models was not shown until 1987, when Swendsen and Wang utilized the model in proposing an algorithm for the time evolution of Potts model [SW87].
Later, Aizenmann, Chayes, Chayes, and Newman used it to show discontinuity in long-range one-dimensional Ising and Potts models [ACCN88]. Edwards and Sokal showed how to do it with coupling [ES88].
To give a feeling of how much consideration is given to this model, Grimmett in [Gri06] says "*One of my main projects since 1992 has been to comprehend the (in)validity of the mantra "everything worth doing for Ising/Potts is best done via random cluster"*"; but then he also adds "*there is a lot to be said in favour of this assertion, but its weakness is its unconditionality*".

Definition 2.30. *The random-cluster model on a finite graph $G = (V, E)$ with parameter $p \in [0, 1]$ and $q > 0$ is the probability measure that assigns to each configuration $\eta \in \Omega := \{0, 1\}^E$ a probability*

$$\nu_{G,p,q}(\eta) := \frac{1}{Z_{RC}} p^{\#o(\eta)}(1-p)^{\#c(\eta)} q^{k(\eta)}. \tag{2.11}$$

Where $\#o(\eta)$ is the number of open edges in the configuration $\eta \in \Omega$, $\#c(\eta)$ is the number of closed edges, $k(\eta)$ is the number of connected components in η, and finally

$$Z_{RC} = \sum_{\eta\in\Omega} p^{\#o(\eta)}(1-p)^{\#c(\eta)} q^{k(\eta)},$$

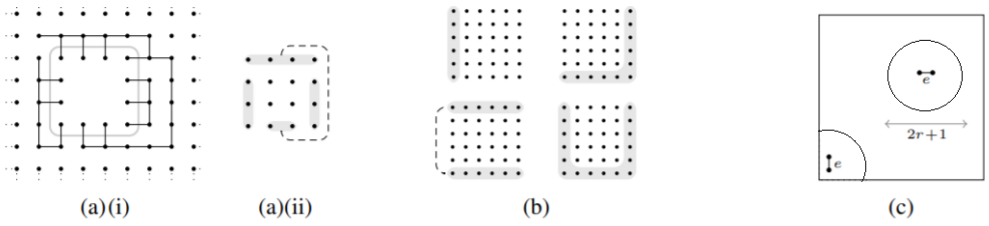


FIGURE 2.1: (a):(i) $\Lambda \subset \mathbb{Z}^2$ with a random-cluster configuration η^c in Λ^c, (ii) the boundary condition induced in Λ by η^c; (b) examples of homogeneous boundary conditions; (c) $B_\Lambda(e, r)$ for two edges e of Λ. [BS17].

is a normalising constant.

The Random-Cluster model can be seen as a generalisation of the bond percolation, as for $q = 1$ then $\nu_{G,p,1}$ is simply the product measure with density p. Moreover it is easy to see that this model can be coupled with Ising/Potts, in particular with Ising when $q = 2$, and $q > 2$ integer with Potts with q colours. For an in-depth description of the Ising model and the Potts model, the reder can refer to [Gri06]. If β is the inverse temperature, then by choosing $p = 1 - e^{-\beta}$, we can pass from random-cluster model to Ising/Potts just by sampling for any cluster one of the q colours uniformly at random and then assign to the vertices of the cluster the same colour chosen; the step in the opposite direction can be done by assigning a closed edge to neighbouring vertices with different colour, and for any other remaining pair decide to open the edge according to independent Bernoulli of parameter p [Gri06].
The case $q < 1$, $q \to 0$ is far less understood but it is related to the electrical-network theory of the graph G. Because the leading terms are the ones for $k(\eta) = 1$ (one cluster), the weak limit for $q \to 0$ converges to the product measure with density p conditioned on the resulting graph being connected; more limits arise if we allow both p and q to converge to 0: if $\frac{q}{p} \to 0$ the mass is concentrated on spanning trees; if p approaches 0 at the same rate as q does, the measure converges to the uniform measure on spanning forests [Gri06].

Definition 2.31. *A lattice is a locally finite vertex-transitive infinite connected graph* $\mathbb{G} = (\mathbb{V}, \mathbb{E})$. *We also distinguish a vertex* $0 \in V$ *and call it the origin.*

On a finite graph, all probabilities are polynomials in p and q, and are therefore smooth functions, whereas singularities and "phase transition" can occur when the graph is infinite. Just like percolation and Ising/Potts models, also the random-cluster model undergoes a phase transition at a certain critical probability p_c. Before going into detail, we need first to define random-cluster on infinite graphs. Let $V_\Lambda \subseteq V$ be some subset of vertices of $\mathbb{G}$, and let $\Lambda = (V_\Lambda, E_\Lambda)$, be the graph whose edge set E_Λ contains all edges in $\mathbb{E}$ with both endpoints in V_Λ. Call $\partial\Lambda$ the boundary of Λ, the set of vertices in V_Λ connected by an edge in $\mathbb{E}$ to $V_{\Lambda^c} = \mathbb{V} \setminus V_\Lambda$. We will later consider balls of radius $n > 0$ around $0 \in V$, but we retain extra-generality at this point.
For any fixed random-cluster configuration $\eta^c \in \Omega_{\Lambda^c} := \{0,1\}^{\mathbb{E}\setminus E_\Lambda}$, we may consider the conditional random-cluster measure induced in Λ by η^c. To this end it is convenient to introduce the standard concept of **boundary conditions**.

Definition 2.32. *A boundary condition for* Λ *is a partition* $\varsigma = (P_1, \dots, P_k)$ *of* $\partial\Lambda$ *which encodes how the vertices of* $\partial\Lambda$ *are connected in a fixed configuration* $\eta^c \in \Omega_{\Lambda^c}$. *Equivalently,*

for all $u, v \in \partial\Lambda$, $u, v \in P_i$ iff u, v are connected by a path in η^c; in this case we say that u and v are wired in ς.

Therefore given a partition ς of $\partial\Lambda$ we can define

$$\nu^{\varsigma}_{\Lambda,p,q}(\eta) := \frac{1}{Z^{\varsigma}_{\Lambda,p,q}} p^{\#o(\eta)}(1-p)^{\#c(\eta)} q^{k(\eta,\varsigma)}.$$

where now $k(\eta, \varsigma)$ is the number of components in η when the connectivities from the boundary condition ς are also considered: if $\mathcal{C}_1, \mathcal{C}_2$ are two connected components in η, and there exist $u \in \mathcal{C}_1 \cap \partial\Lambda$ and $v \in \mathcal{C}_2 \cap \partial\Lambda$ such that u and v are wired in ς, then $\mathcal{C}_1$ and $\mathcal{C}_2$ are identified as the same connected component in η and counted only once in $k(\eta, \varsigma)$.
The boundary condition ς influences $\nu^{\varsigma}_{\Lambda,p,q}(\eta)$ through the way $k(\eta, \varsigma)$ changes.

Definition 2.33. *An event $\mathcal{E} \subseteq \eta$ is said to be increasing if for all $\eta_1, \eta_2 \in \Omega$ such that $\eta_1 \in \mathcal{E}$ and $\eta_1 \leqslant \eta_2$ then $\eta_2 \in \mathcal{E}$. In other words, if the event is preserved by the addition of edges.*
Two probability measures ν_1, ν_2 on a space Ω, are in relation $\nu_1 \preceq \nu_2$, where $\preceq$ denotes stochastic domination, if $\nu_1(\mathcal{E}) \leqslant \nu_2(\mathcal{E})$, for all increasing events $\mathcal{E}$.

Definition 2.34. *For any pair of boundary conditions ς and ψ, denote by $\varsigma \leqslant \psi$ if the partition ς is a refinement of ψ (the connectivities induced by ς in $\partial\Lambda$ are induced also by ψ).*

Lemma 2.35. *[Gri06, Lemma 4.14] When $q \geqslant 1$, $\varsigma \leqslant \psi$ implies $\nu^{\varsigma}_{\Lambda,p,q} \preceq \nu^{\psi}_{\Lambda,p,q}$. When $q \leqslant 1$, $\varsigma \leqslant \psi$ implies $\nu^{\varsigma}_{\Lambda,p,q} \succeq \nu^{\psi}_{\Lambda,p,q}$.*

Definition 2.36. *Some boundary conditions will be of particular interest to us:*

1. *the minimal boundary condition, where no two vertices on $\partial\Lambda$ are connected to each other is called **free boundary condition** and we will denote it $\varsigma = 0$;*
2. *the maximal one, where all vertices are connected, is called the **wired boundary condition**, and we will denote it $\varsigma = 1$;*
3. *finally there is the **homogeneous boundary condition**, this is the class that contains all $\varsigma = (P_1, \ldots, P_k)$ such that $|P_i| > 1$ for at most one i.*

Both free and wired boundary conditions are homogeneous, see Figure 2.5. We can define the random cluster measure on lattices via weak limits of measures on finite subgraphs.

Definition 2.37. *[Gri06] Let $p \in [0, 1]$ and $q > 0$. Let $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ be a lattice, $\Omega = \{0, 1\}^{\mathbb{E}}$ be the state space of possible configurations and $\mathcal{F}$ the σ-field of all possible subsets of Ω. A probability measure ν^{ς} on $(\Omega, \mathcal{F})$ is called a limit-random-cluster measure with parameters p and q and boundary condition ς, if for some ς_n sequence of boundary conditions, ν^{ς} is an accumulation point of the family $\{\nu^{\varsigma_n}_{\Lambda_n,p,q} : \Lambda_n \subseteq \mathbb{V}\}$.*
Equivalently, there exists a sequence $\{(\Lambda_n)\}_{n=1}^{\infty}$ of balls satisfying $\Lambda_n \uparrow \mathbb{V}$ such that

$$\nu^{\varsigma_n}_{\Lambda_n,p,q} \to \nu^{\varsigma}.$$

In general it is not known whether the limit is independent of the choice of the sequences ς_n. It is however known that for any amenable vertex-transitive graphs, the limit is unique for $p \notin \mathcal{D}_q$ for some countable set $\mathcal{D}_q \subset [0, 1]$. See Theorem 4.63 on [Gri06].
An infinite graph $\mathbb{G}$ is **amenable** if $\inf_{\Lambda} \frac{|\partial\Lambda|}{|\Lambda|} = 0$, where the infimum is taken over all finite subsets of $\mathbb{V}$.

Definition 2.38. *Let*

$$\theta^{\varsigma}(p,\, q) := \nu^{\varsigma}_{\mathbb{G},p,q}(0 \leftrightarrow \infty)$$

where $0 \leftrightarrow \infty$ *means that vertex* 0 *is connected to* ∞ *(i.e. there is an infinite connected component that contains* 0*). And define*

$$p^{\varsigma}_c(q) := sup\{p :\, \theta^{\varsigma}(p,\, q) = 0\},$$

the critical value of p *separating the regime in which all open clusters are finite from that in which infinite clusters exist.*

We refer to the **subcritical phase** when $p < p^{\varsigma}_c(q)$, in which case we have $\theta^{\varsigma}(p,\, q) = 0$, and refer to the **supercritical phase** when $p > p^{\varsigma}_c(q)$, in which case we have $\theta^{\varsigma}(p,\, q) > 0$. Understanding the behaviour for $p = p^{\varsigma}_c$, in particular whether there is continuity of the phase transition, is very challenging and this is in fact another major open problem. It is known that on $\mathbb{Z}^2$ for $1 \leqslant q \leqslant 4$ there exists a unique Gibbs measure, i.e. $\theta^0(p^0_c(q), q) = \theta^1(p^1_c(q), q)$, or equivalently that the phase transition is continuous for $1 \leqslant q \leqslant 4$ [DCST17], and discontinuous for $q > 4$, $\theta^0(p^0_c(q), q) \neq \theta^1(p^1_c(q), q)$ [DCGH+16]. Because $\mathcal{D}_q$ is countable, we have $\nu^0_{p,q} = \nu^1_{p,q}$ for almost every $p \in [0, 1]$, which means $\theta^0(p,\, q) = \theta^1(p,\, q)$, for almost every p, and therefore

$$p^0_c(q) = p^1_c(q) =: p_c(q).$$

A major open problem is to quantify the critical point $p_c(q)$. In 2012 Beffara and Duminil-Copin [BDC12] proved the long-lasting conjecture that on $\mathbb{G} = \mathbb{Z}^2$ for $q \geqslant 1$

$$p_c(q) = \frac{\sqrt{q}}{\sqrt{q} + 1}.$$

Furthermore it is conjectured that there exists $Q(d) > 1$ such that

$$\begin{aligned} \mathcal{D}_q &= \emptyset, && \text{if } 1 \leqslant q \leqslant Q \\ \mathcal{D}_q &= \{p_c(q)\}, && \text{if } q > Q. \end{aligned}$$

Or equivalently that $\theta^0(p_c(q), q) = 0$ for all $q \geqslant 1$ whereas, $\theta^1(p_c(q), q)$ can be either 0 or 1, and, more precisely, $\theta^1(p_c(q), q) = 1$ for $q > Q$ (**first order phase transition**), and, for $1 \leqslant q \leqslant Q$, $\theta^1(p_c(q), q) = 0$ (**second order phase transition**). What is known so far, for what concerns the uniqueness of the infinite random-cluster measure, is that it holds if either $p \leqslant p_c(q)$ or $p > p\prime$, for some $p\prime = p\prime(q) > p_c(q)$. More precisely we have uniqueness throughout the subcritical phase, whereas establishing the same result for the supercritical regime is still an important open problem (see [Gri06] Theorem 5.33.)
Moreover we have another important result that is

Theorem 2.39. *[Gri06][0 − 1 infinite cluster law] If* $\mathbb{G}$ *is amenable, if* $\eta \in \Omega$ *is a configuration of random-cluster on* $\mathbb{G}$*, let* $I(\eta)$ *be the number of infinite open clusters of* η*. Then* $\nu_{p,q}(I(\eta) \in \{0, 1\}) = 1$.

The proof of the theorem relies on the uniqueness theorem of Burton-Keane [BK89], namely translation-invariance of the event $\{I(\eta) \in \{0, 1\}\}$ and the so called **finite energy property** of the measure $\nu_{p,q}$.

Definition 2.40. *A measure ν is said to have the finite energy property if*

$$0 < \nu(\eta(e) = 1 | \mathcal{T}_e) < 1, \quad \nu\text{-a.s. for any } e \in \mathbb{E},$$

where $\mathcal{T}_e$ is the σ-field of all possible subsets of $\{0,1\}^{\mathbb{E}\setminus\{e\}}$.

And this is true because of the following theorem, see Theorem 4.17 of [Gri06]

Theorem 2.41. *Let $\mathbb{G}$ be a lattice, and $e \in \mathbb{E}$; for any $p \in [0,1]$ and $q > 0$*

$$\min\left\{p, \frac{p}{p+q(1-p)}\right\} < \nu_{p,q}(\eta(e) = 1 |\, \mathcal{T}_e) < \max\left\{p, \frac{p}{p+q(1-p)}\right\}.$$

Many questions regard also what happens for $p \neq p_c(q)$. In particular, one expects the phase transition to be sharp, meaning that as soon as $p < p_c(q)$ there is **exponential decay of connectivities** or equivalently, the size of the open cluster containing a vertex of the graph is controlled by exponentially decaying probabilities. More formally, on a lattice $\mathbb{G}$ for $q \geqslant 1$ and any fixed $p < p_c(q)$, there exists a positive constant c_0 such that for all $u, v \in \mathbb{V}$:

$$\nu_{\mathbb{G},p,q}(u \leftrightarrow v) \leqslant e^{-c_0 \operatorname{dist}(u,v)}.$$

In [Ale01] Alexander proved that exponential decay of connectivities implies **exponential decay of finite volume connectivities** on $\mathbb{Z}^2$ uniformly over all boundary conditions:

$$\nu^{\varsigma}_{\Lambda,p,q}(u \overset{\Lambda}{\leftrightarrow} v) \leqslant e^{-c_0 \operatorname{dist}(u,v)},$$

for any boundary condition ς and all $u, v \in \Lambda$.
Beffara et al. proved **sharp phase transition** on $\mathbb{Z}^2$ in [BDC12], and then later Duminil-Copin et al. [DCRT17] proved it for general lattices: let Λ_n, from now on, be the ball of radius n around 0, then for all $p < p_c(q)$ there exists $C_{\text{spt}}(p) > 0$ such that

$$\nu^{\varsigma}_{\Lambda_n,p,q}(0 \leftrightarrow \partial\Lambda_n) \leqslant e^{-C_{\text{spt}} n}. \tag{2.12}$$

It is easy to see that there exist an absolute constant $C_{cd} > 0$ and $0 < p_0 < p_c$ such that for $p < p_0$ one has $C_{\text{spt}}(p) > -C_{cd} \log p$: consider for example the lattice $\mathbb{Z}^d$

$$\nu^{\varsigma}_{\Lambda_n,p,q}(0 \leftrightarrow \partial\Lambda_n) \leqslant \sum_{\text{paths from 0 to } \partial\Lambda_n} \left(\frac{p}{p+(1-p)q}\right)^n \leqslant (2d)^n p^n \leqslant e^{nC_{cd}\log p}, \tag{2.13}$$

for some constant $C_{cd} > 0$, here $p_0 = \frac{1}{2d}$.
However the power of sharp phase transition is that it extends this result to all $p < p_c(q)$.
The supercritical regime is less understood, but on $\mathbb{Z}^2$, its self-duality allows a fuller understanding because most of what we know for the subcritical regime can be transferred.

Definition 2.42. *Let $\mathbb{G}$ be a lattice and let $\Lambda_n = (V_n, E_n)$ be the ball of radius n around 0. Consider now $B_{\Lambda_n}(e,r) = (V(B_{\Lambda_n}(e,r)), E(B_{\Lambda_n}(e,r))) := B(e,r) \cap \Lambda_n \subseteq \Lambda_n$, the ball of radius r around $e \in E_n$ intersected with Λ_n itself, see Figure 2.5 (c); and let $\partial B_{\Lambda_n}(e,r)$ be the set of vertices $v \in B(e,r)$ connected by an edge in E_n to $V_n \setminus V(B(e,r))$. Then, if there exist c_{sm}, C_{sm} positive constants such that for all $n > 0$, for all $e \in E_n$, for all $r \geqslant c_{sm}$ and every pair of random cluster configurations A_1, A_2 in $\{0,1\}^{E_n \setminus E(B_{\Lambda_n}(e,r))}$, we have*

$$|\nu^{\varsigma}_{\Lambda_n,p,q}(\eta(e) = 1 |\, A_1) - \nu^{\varsigma}_{\Lambda_n,p,q}(\eta(e) = 1 |\, A_2)| \leqslant e^{-C_{sm} r}, \tag{2.14}$$

we say that the random-cluster measure $\nu^{\varsigma}_{\Lambda_n,p,q}$ has weak spatial mixing.

The **spatial mixing** was introduced by Martinelli and Olivieri in [MO94] for spin systems and by Alexander in [Ale01] for the FK model; it is a very important property that holds when the influence on e of the external configuration decays exponentially with r. It is generally known to hold for spin systems when the Glauber Dynamics has mixing time $O(\log n)$ [DVW04], and it is in general a useful ingredient to prove $O(\log n)$ mixing time. In [Ale01] Alexander showed that (2.12) implies (2.14) on $\mathbb{Z}^2$ for a certain class of boundary conditions ς and when q is integer; later Blanca et al. proved it in [BS17] on $\mathbb{Z}^2$ for all **side homogeneous boundary conditions** and any $q \geqslant 1$ not necessarily integer.

2.6 Glauber Dynamics of the Random Cluster model

The dynamics that we will consider is the discrete time **Heat-Bath Glauber Dynamics** for the random-cluster on a finite graph $G = (V, E)$, which at each step updates a uniformly random edge $e \in E$ by deciding its status according to the random-cluster measure conditioned on the current configuration of $E \setminus \{e\}$.

This Markov Chain is reversible w.r.t. $\nu_{G,p,q}$ and therefore converges to it.

Intuitively, at each step, we put an edge at the equilibrium conditioned on the configuration of the other edges.

The dynamics proceeds as follows. Start from an initial configuration $\eta \in \Omega$ by setting $X_0 = \eta$, then at each step of the chain proceed as follows

1. choose an edge $e \in E$ uniformly at random;
2. let η_1, η_2 be the configurations such that $X_n(e') = \eta_1(e') = \eta_2(e')$ for all $e' \neq e$ and $\eta_1(e) = 1$, $\eta_2(e) = 0$ and set $X_{n+1} = \eta_1$ with probability

$$\frac{\nu_{G,p,q}(\eta_1)}{\nu_{G,p,q}(\eta_1) + \nu_{G,p,q}(\eta_2)},$$

   otherwise set $X_{n+1} = \eta_2$.

This probability can be easily calculated, in fact this dynamics is equivalent to replacing the configuration X_n with η_1 with probability

$$\begin{cases} \frac{p}{p+q(1-p)} & \text{if } e \text{ is a } \textbf{cut edge} \text{ in } X_n, \\ p & \text{otherwise;} \end{cases}$$

where "cut edge" denotes an edge for which its status changes the number of connected components of the current configuration.

Notation 2.1. *With p_{ce} we will denote the probability to refresh an edge open when that edge is a cut-edge.*

$$p_{ce} := \frac{p}{p + q(1-p)}. \tag{2.15}$$

Remark 2.43. *Consider two Markov Chains X_n, Y_n of Glauber Dynamics for random-cluster measure and for simplicity let us denote by $X_n + e$ (respectively, $X_n - e$) the configuration that is identical to X_n for all $e' \neq e$ and $(X_n + e)(e) = 1$ (respectively, $(X_n - e)(e) = 0$). The **identity coupling** is the coupling of the two Markov Chains in which at each step we refresh*

the same edge, and the new status is chosen according to the realization of the same uniform in $[0,1]$ *random variable. More precisely, let* (n,e,U_n) *be an update of the random cluster configuration given by* X_n *at time* n *on edge* e*.* U_n *is a uniform random variable in* $[0,1]$ *and it decides the outcome of the update in the following manner: if* $U_n \leqslant \frac{\nu_{G,p,q}(X_n+e)}{\nu_{G,p,q}(X_n+e)+\nu_{G,p,q}(X_n+e)}$*, then we set the edge* e *open, otherwise we set it closed. It is easy to see from the definition that this is a Glauber Dynamics for the random cluster model. The identity coupling says that whenever an update* (n,e,U_n) *occurs on* X_n*, then we update* Y_n *according to the same tuple* (n,e,U_n)*, but the edge* e *in* Y_n *is updated open if* $U_n \leqslant \frac{\nu_{G,p,q}(Y_n+e)}{\nu_{G,p,q}(Y_n+e)+\nu_{G,p,q}(Y_n+e)}$*. If* $X_n = Y_n$ *then the update is exactly the same, however if the configurations differ it is easy to see that the updates may be different. In particular, if* e *is a cut edge in* X_n *and it is not in* Y_n*, or vice versa, the outcome will be different when* $U_n \in \left(\frac{p}{p+q(1-p)}, p\right)$*. We only need to check that this is indeed a coupling, or equivalently, that the two chains have the correct marginal distributions.* X_n *has the correct distribution by definition, and so* Y_n*: in fact, because* e *is uniformly chosen in* X_n*, it is in* Y_n*; moreover the update rule is the correct one.*

For $q \geqslant 1$ *the identity coupling is a* ***monotone coupling*** *in the sense that it preserves the partial order* $\preceq$*. This means that the coupling time starting from any pair of configurations is bounded by the coupling time starting from* $X_0 = \{0\}^E$*, and* $Y_0 = \{1\}^E$*, the configurations where all the edges are closed and open respectively.*

Glauber Dynamics for spin systems (Ising/Potts) have been extensively studied in the last two decades, the primary object has been to bound its mixing time, and identify the presence of **cutoff phenomenon** (i.e. a sharp transition in the total variation distance of a finite Markov Chain from equilibrium, dropping quickly from near 1, to near 0) [LPwcbELW17].

However, the Glauber Dynamics for random-cluster remain very poorly understood. This is due to the fact that while in Ising/Potts each vertex update depends on a set of vertices in a finite range (nearest-neighbours, second nearest neighbours, etc.), in the random cluster model instead, because connectivity is a global property each edge update has global dependence.

This has led to failure of existing techniques and lack of useful tools for the analysis of the Markov Chain behaviour.

The recent breakthrough [BDC12] has opened the doors to a complete understanding of the Glauber Dynamics for random-cluster measure on $\mathbb{Z}^2$, perfected by Blanca and Sinclair [BS17], where they prove that $O(n^2 \log n)$ is the correct order of the mixing time for the Random Cluster Glauber Dynamics in $\mathbb{T}_n^2$ in both subcritical and supercritical regimes.

They were able to use exponential decay of finite connectivities to prove the bound in subcritical regime and then extend the result to supercritical thanks to the planar duality of $\mathbb{Z}^2$.

Our work in Chapter 5 takes inspiration from theirs, in light of the fact that the sharp phase transition (the main ingredient they needed (2.12)) was recently proved for (vertex-transitive) lattices in [DCRT17]. In spin systems, a central idea is that, because updates depend on a set of vertices in a finite range, if two configurations agree everywhere except in some region A, then it takes many steps for a local Markov Chain under the identity coupling to propagate these disagreement to regions that are far from A.

The random-cluster model has long-range interactions, so disagreement could potentially propagate fast, however if we restrict our attention to pairs of configurations where one of the configurations is sampled from the stationary distribution (i.e. $\nu^{\zeta}_{\Lambda,p,q}$), then if $p \leqslant p_c(q)$, by (2.12), the probability of long paths decays exponentially with the length of the path.

2.7 Strassen's Monotone Coupling Theorem

Let $(\Omega, \leqslant)$ be a finite *poset* (partially ordered set), with the product topology and the corresponding Borel σ-algebra.

Definition 2.44. *For any function $f : \Omega \to \mathbb{R}$ we say f is increasing if $\omega_1 \leqslant \omega_2$ implies $f(\omega_1) \leqslant f(\omega_2)$ for any $\omega_1, \omega_2 \in \Omega$. Finally, for any two probability measures on Ω ν_1, and ν_2 we say that ν_2 stochastically dominates ν_1 and we denote it with $\nu_1 \preccurlyeq \nu_2$ if for all increasing functions f we have that*

$$\int f d\nu_1 \leqslant \int f d\nu_2$$

Theorem 2.45. *[S+65] Let $(\Omega, \leqslant)$ be a finite poset, and let ν_1, ν_2 be two probability distributions on Ω. If $\nu_1 \preccurlyeq \nu_2$ then there is a probability distribution ν_S on $\Omega \times \Omega$ satisfying*

1. *For all $\omega_1, \omega_2 \in \Omega$, $\nu_S(\omega_1, \omega_2) = 0$ unless $\omega_1 \leqslant \omega_2$;*
2. *$\sum_{\omega \in \Omega} \nu_S(\omega_1, \omega) = \nu_1(\omega_1)$ for all $\omega_1 \in \Omega$;*
3. *$\sum_{\omega \in \Omega} \nu_S(\omega, \omega_2) = \nu_2(\omega_1)$ for all $\omega_2 \in \Omega$.*

In words, Strassen's theorem says that if we have two random processes X_1, X_2 on Ω distributed with probability distributions ν_1, ν_2 such that $\nu_1 \preccurlyeq \nu_2$, then we can couple the two processes in such a way that $X_1 \leqslant X_2$ a.s. and the marginal distributions of the coupling are the correct ones.

Chapter 3

Mixing Time for Random Walks on Dynamical Percolation

In this chapter, we present the known results about mixing time for random walks in dynamical percolation and we give a sketch of the proof of the upper bound for the mixing time of Random Walk on Dynamical Percolation when $G_n = \mathbb{T}_n^d$ for all $p < p_c$ discussed in detail in [PSS15]. The strategies used in this proof will be important for the proof of the main result of this thesis. In their paper Peres et al. obtain the correct order for the mixing of the full system, and show that the order of the mixing of the walk is the same order as the mixing of the full system. They also conjecture the order of the mixing time in the supercritical case. Later Peres et al. [PSS17] prove the conjectured upper bound for the mixing time in the supercritical case up to a poly-logarithmic factor but only in the regime where $\theta(p) > \frac{1}{2}$, that the percolation probability at 0 is greater than $\frac{1}{2}$, see Definition 2.23. Hermon et al. find comparison results for the mixing time of random walk in dynamical percolation model with the mixing time of simple random walk on the underlying graph, for general graphs [HS19]. Sousi and Thomas study in [ST18] the supercritical phase when the graph is the complete graph, they prove that the mixing time of the full system exhibits cutoff and show that the *walker mixes faster than the environment*, in contrast to the case where the graph is the torus. Avena et al study in [AGvdHdH16, AGvdHdH18] the mixing time of the nonbacktracking random walk on a dynamical configuration model. The configuration model generates a graph uniformly at random among all graphs with a prescribed degree sequence, and the dynamics at every time step rewires uniformly at random a given proportion of the edges.

Theorem 3.1. *[PSS15] Let $G_n = \mathbb{T}_n^d$ with periodic boundary conditions.*
For any $p \in (0, p_c(\mathbb{Z}^d))$, there exists $C_0 > 0$, such that for all n, and for all $\mu \leqslant 1$, we have that

$$T_{mix}(G_n) \leqslant \frac{C_0 n^2}{\mu}, \tag{3.1}$$

and $T_{mix}(G_n)$ is the mixing time of $M_t = (X_t, \eta_t)$ the random walk in the dynamical percolation on G_n with parameter p.

Because $O(n^2)$ is the mixing time of a Lazy Simple Random Walk on the static graph $\mathbb{T}_n^d$ (with all open edges), this theorem tells us that we can relate the mixing time for a random walk on a graph which refreshes its edges at rate μ with the mixing time for a random walk on the underlying non changing graph. More precisely the dynamics is slowed down by a factor of $\frac{1}{\mu}$, that is the expected time to refresh a large portion of the edges of the graph.

[PSS15] proves also a lower bound for the process on $\mathbb{T}_n^d$ showing that it is the same as the upper bound up to a constant, and therefore yielding a precise estimate for the mixing time. Throughout this chapter we will work under the assumption that $\mu \leqslant 1$ as for large μ the analysis is easier and less interesting. In fact for $\mu \to \infty$ the graph refreshes so much faster than the walker moves that the dynamic is equivalent to a random walker that moves on a random graph. Now we give a relatively detailed overview on the proof of Theorem 3.1. We need to prove that for all $t \geqslant \frac{C_0 n^2}{\mu}$ and $M_0 = (X_0, \eta_0) \in V_n \times \{0,1\}^{E_n}$:

$$\|\mathbb{P}_{M_0}(M_t \in \cdot) - \pi\|_{TV} < \frac{1}{4}.$$

Let τ_{couple} be the first time the chains meet as seen in Proposition 2.9. We will construct a coupling for which we can bound the expected coupling time and then use Corollary 2.10 and Markov inequality. If $(M_t^1,\ M_t^2)$ is a coupling of the same process starting from $M_0^1 = (X_1, \eta_1)$ and $M_0^2 = (X_2, \eta_2)$ then we have that for all $t \geq 4 \max_{M_0^1, M_0^2} \mathbb{E}[\tau_{\text{couple}}]$

$$\max_{M_0} \|\mathbb{P}_{M_0}(M_t \in \cdot) - \pi\|_{TV} \leqslant \max_{M_0^1, M_0^2} \mathbb{P}(\tau_{\text{couple}} > t) \leqslant \frac{\max_{M_0^1, M_0^2} \mathbb{E}[\tau_{\text{couple}}]}{t} \leqslant \frac{1}{4}.$$

Therefore $T_{\text{mix}} \leqslant 4 \max_{M_0^1, M_0^2} \mathbb{E}[\tau_{\text{couple}}]$.
We only need to find a coupling for which $\max_{M_0^1, M_0^2} \mathbb{E}[\tau_{\text{couple}}] \leqslant O_{k,p}(1)\frac{n^2}{\mu}$.
The first issue in such a coupling is finding a way to deal with two processes that move on two different graphs (the underlying graph is the same but the open/closed state of the edges are different). For this purpose we resort to the notion of **regeneration time**. To define the regeneration time we first need to consider the increased state space

$$\Omega^{\text{ext}} := \left\{(v, \tilde{\eta}) \in V_n \times \{0, 1, \star\}^{E_n} \mid \tilde{\eta}(e) \in \{0,1\} \text{ for each } e \text{ adjacent to } v\right\}.$$

In other words the edges adjacent to the particle can be either 0 or 1, while the other edges could have a third state, denoted by $\star$, and which is meant to represent the edges that have been refreshed since the last time they were adjacent to the walker. Therefore, the state of the $\star$ edges will be will be 1 with probability p or 0 with probability $1-p$, independently of the state of everything else. In other words, from the point of view of the walker, edges with a $\star$ have unknown states. In the new Markov Chain $\tilde{M}_t = (X_t, \tilde{\eta}_t)$ the edges refresh at rate μ, and when an edge refreshes, if its adjacent to the random walker it refreshes to 1 with probability p or 0 with probability $1-p$, otherwise it refreshes to $\star$. When a transition of $\tilde{M}_t$ happens due to a jump of the walker the random walker will choose uniformly at random among the adjacent edges and move along it if the chosen one is 1, then the edges adjacent to the new position of the walker that are $\star$ will become 1 with probability p or 0 with probability $1-p$, independently of one another and of everything else. We want that conditioned on

1. the position of the walker,
2. the collection of the $\star$-edges,
3. the states of the non-$\star$-edges,

we have no information whatsoever about whether the $\star$-edges are open or closed, so that their state is still a Bernoulli(p) random variable.

Definition 3.2. *A **regeneration time** is the time at which the process* $\left\{\tilde{M}_t\right\}_{t\geqslant 0}$ *reaches the set*

$$\Omega_{REG} := \left\{(v,\tilde{\eta}) \in \Omega^{\text{ext}} \,|\, \tilde{\eta}(e) = 0 \text{ for all } e \text{ adjacent to } v \text{ and } \tilde{\eta}(e) = \star \text{ otherwise}\right\}.$$

When the process reaches Ω_{REG}, since all edges adjacent to the walker are closed, the walker is said to be stuck in its position, and the non adjacent edges, independently of everything else, are open with probability p and closed with probability $1-p$. We now explain the coupling. We will couple two instances of the process $(\tilde{M}_t)_t$ which we will denote by $\tilde{M}_t^1 = (X_t^1, \tilde{\eta}_t^1)$ and $\tilde{M}_t^2 = (X_t^2, \tilde{\eta}_t^2)$. The coupling will consist of two stages:

1. in the first one we let the two processes run independently until they simultaneously reach Ω_{REG}. We want them to simultaneously reach Ω_{REG} so that we can couple them in such a way that, from the point of view of the walkers, the graphs look the same; this is possible thanks to the property of vertex-transitivity of the torus;

2. after the first stage the two graphs look the same everywhere up to a translation: the translation given by the isomorphism of the graphs that maps X_t^1 onto X_t^2. Looking now at excursion from Ω_{REG} we try to reduce the displacement between the positions of the walkers $(\text{dist}(X_t^1, X_t^2))$ so that when at some time t the processes are both in Ω_{REG} and the walkers are in the same position $(X_t^1 = X_t^2)$ then we know t is a coupling time.

We get that $\tau_{\text{couple}} = T_1 + T_2$, where T_i is the time needed to complete stage i.

3.1 Analysis of the first stage.

Denoting

$$A_t := \{e : \tilde{\eta}_t(e) \in \{0,1\}\}$$

the set of edges without a $\star$ at time t, we shall prove that A_t decreases in size on a time scale of order $\frac{1}{\mu}$. Note that, if $\tilde{M}_t \in \Omega_{\text{REG}}$ then A_t only contains the $2d$ edges adjacent to the position of the walker.
The key in this stage is the exponential decay of the size of connected components when the percolation is done in a subcritical setting. From [AN84] we have

Theorem 3.3. *Let* $G_n = \mathbb{T}_n^d$*, for any* $p < p_c$*, there exists* $C_1 = C_1(p) > 0$ *such that for any* $v \in V$ *and any* $r > 0$ *we have*

$$\mathbb{P}(|\mathcal{C}(v)| \geqslant r) \leqslant \exp(-C_1 r).$$

With this in mind, it follows that, the random walker lives in a rather small and limited cluster and because of the subcritical setting has difficulty in moving outside this connected component. Intuitively then, while in an interval of size $\frac{\varepsilon}{\mu}$, for some small positive constant ε, a positive fraction of the edges of A_t get refreshed, the edges that get refreshed near the connected component where the walk is are still limited.
This causes A_t to decrease in size (in expectation) after an interval of order $\frac{1}{\mu}$, and in particular it yields the following result.

Proposition 3.4. *Let $\mathcal{F}_t^\star$ be the filtration induced by $\left\{\tilde{M}_t\right\}_{t\geqslant 0}$. There exist positive constants $C_2(d,p)$, $C_3(d,p)$ such that for all n, μ, and s,*

$$\mathbb{E}\left[|A_{s+\frac{C_2}{\mu}}| \mid \mathcal{F}_s^\star\right] \leqslant \frac{A_s}{4} + C_3 \log|A_s|.$$

Proof sketch. The key observation is that if $t_1 := s+\frac{r_1}{\mu}$, $t_2 := t_1+\frac{r_2}{\mu}$, with r_1, r_2 to be chosen, then

$$|A_{t_2}| \leqslant Q + |E(\mathcal{R}(t_1,t_2))|,$$

where Q is the number of edges in A_{t_1} which are not refreshed during $[t_1,t_2]$, whereas $E(\mathcal{R}(t_1,t_2))$ are the edges adjacent to the vertices in the range of the walker during the time interval $[t_1,t_2]$. Then, since $|E(\mathcal{R}(t_1,t_2))| \leqslant 2d|\mathcal{R}(t_1,t_2)|$, the proof would be completed provided we can choose t_1, t_2 such that

$$\mathbb{E}[Q \mid \mathcal{F}_s^\star] \leqslant \frac{|A_s|}{4}, \tag{3.2}$$

and

$$\mathbb{E}[|\mathcal{R}(t_1,t_2)| \mid \mathcal{F}_s^\star] \leqslant \frac{C_3}{2d}\log|A_s|. \tag{3.3}$$

The main challenge in establishing this is that if the random walker finds itself at time s in an atypically large component, the walker will visit several edges between refreshes of edges from A_s. This would tend to keep A_s large. So the idea is that we want to choose

- t_1 large enough, so that even if the walker is in an atypically large connected component at time s, the probability that this component remains large for the long time $t_1 - s$ is quite small;
- t_2 large enough, so that enough edges in A_{t_1} refresh during $[t_1,t_2]$.

Then we need to check that after we have chosen t_1 and t_2, because of the subcritical setting, the walker is stuck in his connected component and cannot go far away. Choose $\varepsilon = \varepsilon(d,p) < \frac{p_c-p}{4}$ such that $\frac{1}{\varepsilon} \in \mathbb{N}$. It can be verified [PSS15] that there exists c_0 such that for any $\ell \in \mathbb{N}$

$$\mathbb{E}[|A_{s+\frac{\ell\varepsilon}{\mu}}| \mid \mathcal{F}_s^\star] \leqslant c_0^\ell |A_s| \quad \text{almost surely.}$$

Recall that Q is the number of edges in A_{t_1} which are not refreshed during $[t_1,t_2]$ and that $t_1 = s+\frac{r_1}{\mu}$ and $t_2 = t_1+\frac{r_2}{\mu}$. Using this we obtain

$$\mathbb{E}[Q|\mathcal{F}_s^\star] = \mathbb{E}\left[\mathbb{E}\left[Q|\mathcal{F}_{t_1}^\star\right]|\mathcal{F}_s^\star\right] = \mathbb{E}[e^{-r_2}|A_{t_1}||\mathcal{F}_s^\star] \leqslant e^{-r_2}c_0^{\frac{r_1}{\varepsilon}}|A_s| \leqslant \frac{|A_s|}{4},$$

for $r_2(d,p,r_1)$ large enough. Thus (3.2) is satisfied. Now we see how having a large value of t_1 helps us to establish (3.3). Given $\mathcal{F}_s^\star$, the conditional probability that an edge e is open at time t_1 is at most the probability that it has not refreshed during $[s,t_1]$ (which is e^{-r_1}) plus the probability that it has refreshed and is open at time t_1 (which is at most p). Choosing $r_1 = r_1(d,p)$ large enough we ensure that this probability is at most

$$e^{-r_1} + p \leqslant \frac{p_c+p}{2} < p_c.$$

With our choice of r_1 and ε, it follows that for any $e \in E_n$

$$\mathbb{P}\left(e \text{ is open some time during } \left[t_1, t_1 + \frac{\varepsilon}{\mu}\right]\right) \leqslant \frac{p_c + p}{2} + \varepsilon p$$
$$=: p' < p_c.$$

This gives that the set of edges open between times t_1 and $t_1 + \frac{\varepsilon}{\mu}$ is stochastically dominated by independent percolation of parameter $p' < p_c$. It is important that this probability is less than p_c because we want to use Theorem 3.3 to prove that the connected component formed by the set of edges which are open *during* an interval of length $\frac{\varepsilon}{\mu}$ is small. So that the walker can barely move and its range is small. To use this piece of information we split our interval $[t_1, t_2]$ into $D = D(d, p)$ disjoint intervals of length $\frac{\varepsilon}{\mu}$ and prove (3.3) with $[t_1, t_2]$ replaced by $\mathcal{I} = [y, y + \frac{\varepsilon}{\mu}]$ for some $y = t_1 + \frac{\ell\varepsilon}{\mu}$, $\ell \geqslant 1$. Actually, since $\log|A_s| \leq 4d \log|V(A_s)|$, with $V(A_s)$ being the set of vertices adjacent to the edges in A_s, it is enough to prove that

$$\left\| \sum_{j \geqslant 1} \mathbb{P}\left(|\mathcal{R}[\mathcal{I}]| \geqslant c_1 j \log|V(A_s)| \,|\mathcal{F}_s^\star\right) \right\|_\infty < \infty,$$

for some constant $c_1 > 0$. Let $\bar{\eta}$ be the set of edges that are open some time during $\mathcal{I}$. By our choice of ε, conditioned on $\mathcal{F}_s^\star$, $\bar{\eta}$ is stochastically dominated by an i.i.d. process with density p'. Since $\mathcal{R}[\mathcal{I}]$ is necessarily contained inside of a $\bar{\eta}$-cluster, it can be proved that for each $j \geqslant 1$

$$\mathbb{P}\left(|\mathcal{R}[\mathcal{I}]| \geqslant c_1 j \log|V(A_s)| \,|\mathcal{F}_s^\star\right) \leqslant \mathbb{P}(\mathrm{dist}_{G_n}(X_y, X_s) \geqslant j\Gamma \,|\mathcal{F}_s^\star) +$$
$$\mathbb{P}\left(\bar{\eta} \text{ contains a cluster of size } \geqslant c_1 j \log|V(A_s)| \text{ intersecting } \mathbb{B}_{X_s}(j\Gamma) \,|\mathcal{F}_s^\star\right),$$

where $\Gamma = 4dc_2(d, p', D)|V(A_s)| \log|V(A_s)|$ and c_2 comes from Theorem 6.8 in [PSS15]. Moreover, $\mathbb{B}_{X_s}(j\Gamma)$ is the set of vertices within dist_{G_n}-distance $j\Gamma$ of X_s.
It is easy to check that Theorem 3.3 together with a union bound implies that the second terms are summable over j uniform in the conditioning, i.e.

$$\left\| \sum_{j \geqslant 1} \mathbb{P}\left(\bar{\eta} \text{ contains a cluster of size } \geqslant c_1 j \log|V(A_s)| \text{ intersecting } \mathbb{B}_{X_s}(j\Gamma) \,|\mathcal{F}_s^\star\right) \right\|_\infty$$

is a finite quantity. For the first term finally, apply Theorem 6.8 in [PSS15] to see that

$$\| \sum_{j \geqslant 1} \mathbb{P}(\mathrm{dist}_{G_n}(X_y, X_s) \geqslant j\Gamma \,|\mathcal{F}_s^\star) \|_\infty < \infty.$$

□

Looking at the process at integer multiples of $\frac{C_2}{\mu}$ we have a precise idea of how much $|A_t|$ is expected to decrease, moreover we have that if A_t^1 and A_t^2 are two independent copies of the process $\{A_t\}_{t \geqslant 0}$, each A_t^i corresponding to the process $\tilde{M}_t^i$, there exists $C_4(d, p) < \infty$ such that,

1. when $|A_t^1| + |A_t^2| > C_4$ then $\mathbb{E}[|A^1_{t+\frac{C_2}{\mu}}| + |A^2_{t+\frac{C_2}{\mu}}| \,|\, \mathcal{F}_t^{\star,1} \times \mathcal{F}_t^{\star,2}] \leqslant \frac{|A_t^1| + |A_t^2|}{3}$
2. when $|A_t^1| + |A_t^2| < C_4$ then there is a positive probability of both $\tilde{M}^1_{t+\frac{C_2}{\mu}}$ and $\tilde{M}^2_{t+\frac{C_2}{\mu}}$ being simultaneously in Ω_{REG}.

Part 1 above says that if T_3 is the first time that $|A_t^1| + |A_t^2| < C_4$, then

$$3^{\frac{\mu}{C_2}(t\wedge T_3)}\left(|A^1_{t\wedge T_3}| + |A^2_{t\wedge T_3}|\right)$$

is a super-martingale and since $|A_t^1|+|A_t^2| \geqslant 1$ for all t and $|A_0^1|+|A_0^2| \leqslant 2|E_n| \leqslant 2\frac{d|V_n|}{2} = dn^d$ it follows that

$$\mathbb{E}[T_3] \leqslant \frac{C_2}{\mu}\log_3 dn^d.$$

After time T_3, we have that $|A_t^1| + |A_t^2| < C_4$, so that the number of edges that are not equal to $\star$ is bounded above by the constant C_4. Therefore with positive probability, all these edges may refresh in such a way that A_t^1 and A_t^2 enter in a regeneration time after a time of order $\frac{1}{\mu}$. Then, using part 2 above we obtain that after a geometric number of trials the two process will enter Ω_{REG} simultaneously. Therefore there exists $C_5 > 0$ such that

$$\mathbb{E}[T_1] \leqslant \frac{C_2}{\mu}C_5(d,p)\log n = O(\frac{\log n}{\mu}).$$

3.2 Analysis of the second stage.

At this point both the processes are in Ω_{REG} and the graphs are the same up to a translation that maps X_t^1 onto X_t^2. We want to reduce the distance between the two walkers. We do this looking at excursions from Ω_{REG}, of course coupling the two processes in such a way that they keep entering Ω_{REG} simultaneously (in this stage they both start from Ω_{REG} and for this reason we will see the excursions are much shorter than the time of the first stage). We distinguish between two types of excursions. First, we look at those excursions where the walker reenters in Ω_{REG} within an interval of order $\frac{C_2}{\mu}$ and has jumped only at most once (so the walker ends either at the same position as in the beginning of the excursion or at one of the neighbors). We will call this a **simple regeneration time**. In fact, the simple regeneration times are defined carefully in [PSS15] so that conditioned on an excursion being a simple regeneration time, the motion of the walker during such excursion is distributed exactly as one jump of a lazy simple random walk on G_n. It is not difficult to check that *a regeneration time is a simple regeneration time with positive probability (depending on p and d, but not on n nor μ), as it depends only on a certain sequence of edge refreshes involving only a constant number of edges.* If the regeneration is not simple we couple the two processes in such a way that, with probability 1, they do not change their distance: we perform the same edge refreshes and the same jumps of the walkers in both processes. During simple regeneration times, we will use the fact that the displacement of the walker is distributed exactly like lazy simple random walk on G_n to couple X^1 and X^2 in the same way one would couple simple random walks on G_n. Consider $\{(\tau_j, U_j)\}_{j>0} \in \mathbb{N} \times G_n$ where τ_j is the j-th time the process reenters in Ω_{REG} and U_j is the displacement of the particle from τ_{j-1} to τ_j. The τ_j are integers because we look at the process at times multiples of $\frac{C_2}{\mu}$, so that at index τ_j we are looking at the process at time $\frac{C_2}{\mu}\tau_j$. It is easy to see that $\{(\tau_j - \tau_{j-1}, U_j)\}_{j\geqslant 1}$ are i.i.d. and

$$(\tau_j - \tau_{j-1}, U_j) \overset{\text{d}}{=} (\tau_1, U_1).$$

Of course U_j and $\tau_j - \tau_{j-1}$ are not independent of each other. If $\gamma > 0$ is the probability for a regeneration time to be simple, the joint distribution of (τ_1, U_1) is

$$\nu_{k,p,n,\mu} = \gamma(\delta_1 \times \nu_{\mathrm{LSRW}}) + (1-\gamma)m_{k,p,n,\mu}$$

where $m_{k,p,n,\mu}$ is some probability measure on $\mathbb{N} \times G_n$, and $\delta_1 \times \nu_{\mathrm{LSRW}}$ is the probability measure on $\mathbb{N} \times G_n$ with the first coordinate always 1 and the second one being one step of a lazy simple random walk on G_n. If $(\tau_j^1 - \tau_{j-1}^1, U_j^1)$ and $(\tau_j^2 - \tau_{j-1}^2, U_j^2)$ are the two copies of the same process we have to couple, we define their coupling to be

- with probability $1-\gamma$ one chooses an element from $\mathbb{N} \times G_n$ according to the distribution $m_{k,p,n,\mu}$ and uses it for both systems;
- with probability γ one takes the first coordinate to be 1 for both systems and does the coupled lazy simple random walk for the second coordinate of the two systems: equivalently, we let each walker choose one of the d coordinates uniformly at random, if they agree in this coordinate, then they both stay still, or both move "right" in this coordinate or both move "left" in this coordinate, where the probability of each such case is $\delta \geqslant \frac{1}{2}$ (which depends on how long the edge e remains open during the simple regeneration time, but is equivalent in both processes), $\frac{1-\delta}{2}$ and $\frac{1-\delta}{2}$, respectively. If the two walkers disagree in this coordinate, then with probability $(1-\delta)$ the first one jumps and the second one stays still, with probability $(1-\delta)$ the second one jumps and the first stays still, and both stay still with probability $2\delta - 1$. It is easy to check that this is a coupling of the two lazy simple random walks, see Theorem 5.5 of [LPwcbELW17].

Coupling the two processes in this way, we have that each time a simple regeneration time occurs, the walkers do a step of lazy simple random walk in the underlying graph. Therefore they need $O(n^2)$ instances of simple regeneration times to meet at the same vertex. Using the Proposition 3.4 and the same arguments used to bound the first stage, we find that if τ_1 is the first time the process reenters in Ω_{REG}, then

$$\mathbb{E}[\tau_1] \leqslant \frac{C_2}{\mu} C_5. \tag{3.4}$$

Let J be the number of steps in the above coupling until the walkers meet, whose expectation is at most $O(n^2)$, we have

$$T_2 = \frac{C_2}{\mu} \sum_{j=1}^{J} (\tau_j - \tau_{j-1})$$

and therefore by (3.4) and recalling $\mu \leqslant 1$

$$\mathbb{E}[T_2] = \frac{C_2}{\mu} \mathbb{E}[\sum_{j=1}^{J} (\tau_j - \tau_{j-1})] = \frac{C_2}{\mu} \mathbb{E}[J]\mathbb{E}[\tau_1] \leqslant O(\frac{n^2}{\mu}).$$

The proof of Theorem 3.1 is then completed because

$$\mathbb{E}[\tau_{\mathrm{couple}}] = \mathbb{E}[T_1] + \mathbb{E}[T_2] \leqslant O(\frac{\log n}{\mu}) + O(\frac{n^2}{\mu}) = O(\frac{n^2}{\mu}).$$

Chapter 4

Random Walks in Dynamical Random Cluster

In this chapter we will consider the dynamics of a random walker when the underlying graph changes according to a Glauber dynamics for the random cluster model. We will prove an upper bound of the mixing time when the underlying graph is the d-dimensional torus $\mathbb{T}_n^d$ when the probability of opening an edge is below a certain threshold smaller than the critical probability $p < p_0 < p_c(q)$. Although the result in [PSS15] is for any $p < p_c(1)$, our result recovers it for $q = 1$ and $p < p_0$, for a certain p_0 smaller than $p_c(1)$, and it shows that the mixing time is essentially the same as the random walk on the open fixed graph multiplied by a factor $\frac{1}{\mu}$. Throughout this chapter we will work under the assumption that $\mu \leqslant 1$ because for larger μ the graph refreshes faster and therefore mixes faster as well.

Theorem 4.1. *Let $q \geq 1$. For the Random Walk on the Dynamical Random-Cluster on the graph $\mathbb{T}_n^d$ there exists $p_0 < p_c(q, d)$ such that for all $p \leq p_0$ there exists a positive constant $C_0 = C_0(p, q) > 0$ for which*

$$T_{\text{mix}} \leq C_0(p, q)\frac{n^2}{\mu}, \tag{4.1}$$

for all $\mu \leqslant 1$, for all $n > 0$, where T_{mix} is the mixing time of the full system $\{M_t\}_{t \geqslant 0}$.

4.1 Probability space extension

In this section, we fix an arbitrary graph $G = (V, E)$ with constant degree and parameters $q \geqslant 1$, $\mu \leqslant 1$ and p. We consider the resulting random walk in dynamical random cluster which, as before, we denote by $\{M_t\}_{t \geq 0} = \{(X_t, \eta_t)\}_{t \geq 0}$. In the Random-Cluster Dynamics each edge is updated via a Poisson process of rate μ and every update is decided by the outcome of an independent uniform unit random variable U and the configuration of the graph at the moment of the update. When refreshed, if the edge is a *cut-edge* then it becomes open if $U < p_{ce}$ (recall the definition of p_{ce} in (2.15)) ; otherwise it becomes open if $U < p$. Note that $p > p_{ce}$. In particular if U turns out to be in the interval $(0, p_{ce}) \cup (p, 1)$ the outcome of the update is determined regardless of the configuration of the graph at the moment of the update. This suggests that with probability

$$p_\star := p_{ce} + 1 - p$$

the status of the edge is chosen independently of the status of the other edges. We will then make use of an auxiliary uniform unit random variable U' independent of everything else, and define a different updating procedure which will be equivalent to the original one. If $U' < p_\star$ the update is called a $\star$-*update* and, consequently, the edge will refresh open if $U < \frac{p_{ce}}{p_\star}$ and closed if $U > \frac{p_{ce}}{p_\star}$; otherwise, if $U' > p_\star$ we say the update is not a $\star$-update and a properly chosen rule for this situation, in which one has to inspect the configuration of the graph, extends the procedure to the Heat-Bath Glauber Dynamics for random-cluster. We therefore say that to each edge is associated a Poisson clock and when the clock of an edge rings, the edge is updated: an update is a triple (s, U', U), where $s > 0$ is the time at which the update occurs, $U' \in (0,1)$ says if it is a $\star$-update or not and finally $U \in (0,1)$ is the uniform random variable whose outcome dictates whether the edge is refreshed open or closed, subject to whether the update is $\star$ or not and, if not, whether e is a cut-edge.
In order to obtain upper bounds on the mixing time, it will be useful to take advantage of the concept of $\star$-update. We then introduce another Markov process which we denote by $\{\tilde{M}_t\}_{t\geq 0} = (X_t, \tilde{\eta}_t)_{t\geq 0}$. This process will retain more randomness than $\{M_t\}_{t\geq 0}$ and its state space will be

$$\Omega^{ext} := \{(v, \tilde{\eta}) \in V \times \{0,1,\star\}^E : \tilde{\eta}(e) \in \{0,1\} \text{ for each } e \text{ adjacent to } v\}.$$

We define $\{\tilde{M}_t\}_{t\geq 0}$ as follows. An edge e refreshes at rate μ

- if $U' < p_\star$ the update is a $\star$-update and the state of e will be $\star$ (we do not check the outcome of U until either the walker is on one of the endpoints of e or the outcome of a non-$\star$ update depends on the status of e).
- If the update is not $\star$ then its outcome, which will be 1 or 0, depends on whether the edge e is a cut-edge or not.

Remark 4.2. *Note that, when $q = 1$, we have that $p_{ce} = p$ and, hence, $p_\star = 1$. This means that all updates are $\star$-updates, and this representation is equivalent to the one described in Chapter 3.*

In the second item above, to decide whether e is a cut-edge we must check the status of the edges in the connected components of the two endpoints, say v_1, v_2, of e. Thus, when an update that is not $\star$ occurs on e we do what we call an *exploration of the edge*: we reveal the status of the edges adjacent to $\mathcal{C}_{v_1}$ or $\mathcal{C}_{v_2}$ which have a $\star$, by revealing the values of the variables U from the last update of each such edge; this amounts to revealing the status of the edges in $\mathcal{C}_{v_1} \cup \mathcal{C}_{v_2}$ and on the boundary of these components. Moreover, when the random walker moves along an edge, the $\star$'s are removed from all edges which become adjacent to the walker. Whenever we *remove the* $\star$ from an edge, the new status, that can be either 0 or 1, and that is determined by the random variable U from the last update of the edge, corresponds to a Bernoulli of parameter $\frac{p_{ce}}{p_\star}$, independently of everything else. The random walker will as before choose at rate 1 a uniformly random neighbor (in the original graph) and move along that edge if the edge is in state 1 otherwise the walkers do not jump. (Note that this edge can only be in state 1 or 0, but never in state $\star$, since it is adjacent to the walker.)
Clearly, dropping the $\star$'s at any time, using the random variable U from the last update of each edge with a $\star$ state, we recover the original process $\{M_t\}_{t\geq 0}$. Note that conditioned on

the position of the walker, the collection of $\star$-edges and the states of the other edges, we have no information concerning the states of the $\star$-edges.

4.2 Proof Overview

We start by recalling the proof in [PSS15] for the subcritical regime when $q = 1$, of which we gave a sketch in Chapter 3. There, they also define the $\star$-process (which they denote by $\tilde{M}_t$). Recall that, when $q = 1$, we have $p_\star = 1$, so all updates are $\star$-updates. With this, they define a stopping time τ_0 as the first time at which

$$\text{all edges adjacent to the walker are closed, and all remaining edges are in state } \star. \tag{4.2}$$

Then, one can define a sequence of times $\tau_1, \tau_2, \ldots$ so that τ_i is the first time after $\tau_{i-1} + \frac{C}{\mu}$, for some fixed constant $C > 0$, at which the event in (4.2) happens. These are regeneration times in the sense that the evolution of the full system from τ_i does not depend on what happened before τ_i. Once the full system is at a regeneration time τ_i, with positive probability the following sequence of events happen within time $\tau_i + \frac{C}{\mu}$:

1. an edge e adjacent to the walker opens;
2. when the walker jumps to the other endpoint of e, all the adjacent edges (which are in state $\star$) are sampled closed;
3. e remains open for some time of order $\frac{1}{\mu}$;
4. e closes before any of the other edges adjacent to e open, thereby locking the walker in one of e's endpoints;
5. the edges adjacent to the other endpoint of e (i.e., opposite to the location of the walker) refresh before the edges adjacent to the walker refresh.

When these events occur, the walker does nothing more than a jump to a uniformly random neighbor, and immediately gets back to a regeneration time (so $\tau_{i+1} = \tau_i + \frac{C}{\mu}$); such a regeneration time is then called a *simple random walk regeneration* since, at the end, what the walker did was just one step of a simple random walk in $\mathbb{T}_n^d$.

The proof in [PSS15] then goes by showing that the $\tau_{i+1} - \tau_i$ are of order $\frac{1}{\mu}$. Therefore, after time $\frac{n^2}{\mu}$, the walker underwent an order of n^2 regeneration times, a positive fraction of which being simple random walk regeneration. So it is possible to couple the full system with another copy of the full system so that, whenever the walker does a simple random walk regeneration, we employ one of the standard couplings of simple random walks on the torus. On the other hand, if the regeneration time is not a simple random walk regeneration, we couple the motion of the two walkers from one regeneration time to the next identically, so that the distance between the walkers does not change. Since an order of n^2 steps is necessary for two simple random walks on $\mathbb{T}_n^d$ to meet, we get that performing an order of n^2 simple random walk regenerations is enough for the two processes to meet, which translates to a mixing time of order $\frac{n^2}{\mu}$.

If we try to mimic the steps above for the case $q > 1$, we immediately run into the issue

that the event (4.2) now occurs very rarely. In fact, since non-$\star$-updates occur with positive probability, we will typically have a positive density of non-$\star$-edges. Therefore, it will take an exponential amount of time to reach a regeneration time as in (4.2), rendering this strategy useless.

We will devise a different strategy. We will, as before, construct a coupling between two copies of the full-system, where we see the edges "from the point of view of the walker" in the sense that whenever the edge $X_t + e$ updates at time t, where X_t is the position of the walker in the first copy, then in the second copy we will do the same update to the edge $\bar{X}_t + e$, where $\bar{X}_t$ is the location of the walker in the second copy. Note that to establish the mixing time of the full system we need to couple the environments and the walkers. For simplicity, we concentrate our discussion here on the coupling of the walkers (which is the most delicate bit), and assume for now that somehow we managed to couple the two environments: that is, the two copies are coupled modulo a translation of the walkers. Note that, from this moment, if we were to employ the *identity coupling* (that is, the second copy mimics all the edge updates and jumps of the walker from the first copy) we would get that the environments will remain coupled (from the point of view of the walkers) but the distance between the walkers will not change, thereby not allowing the walkers to couple.

With this information in mind, our idea is to observe a bit the environment and, whenever the environment looks "favorable enough", we attempt to do a coupling that could bring the walkers closer together, which will be a standard coupling of simple random walks. We will refer to such moments as *simple random walk moments*, as an allusion to the simple random walk regenerations described above, but with the fundamental difference that they will not be regeneration times. On the other hand, when the environment is not favorable enough, then doing a simple random walk moment is a bit too risky, so instead we resort to the identity coupling as a means to keeping the distance between the walkers unchanged and not spoiling the work done during the favorable regions of the environments.

But what does it mean for the environment to look favorable enough? In short terms, it will mean that the event (4.2) occurs *locally*. That is, at such times, all edges adjacent to the walkers will be closed and all edges in a small region around the walkers will be $\star$ (for example, all edges inside a ball of radius 3 around the walkers, excluding the edges adjacent to the walkers). At such a time, with positive probability, the sequence of events described above for the simple random walk regeneration occurs, and therefore we could attempt to perform one of the standard couplings of simple random walks. However, there are two important caveats.

The first caveat is that if we succeed in doing a simple random walk moment with a coupling of simple random walks, then the distance between the walkers will change. This means that the translation that maps the location of one walker to the location of the second walker will change, and this map is what we use to match the edges of the first copy to the edges of the second copy, when we view the edges from the point of view of the walkers. As a consequence, the environments will immediately *decouple*. Of course, if we only had *star*-edges (besides the ones adjacent to the walkers, as in the case $q = 1$), then the environments would not decouple since despite the change in the translation map, we would still match $\star$-edges in the first copy to $\star$-edges in the second copy, so we can easily maintain the environments coupled.

But, since $q \neq 1$ implies a density of non-$\star$ edges, the environments will necessarily decouple. Moreover, if we decide to just wait the environments to recouple completely, this would take a time of order $\frac{\log n}{\mu}$, which is just too long: it will lead to an upper bound on the mixing time of $\frac{n^2}{\mu} \log n$. So we will not recouple the environments completely, but will work with partially coupled environments.

The second caveat is that a simple random walk moment occurs with *positive* probability, so it is also possible that it turns out that a simple random walk moment does not take place. Then, what could happen in this case? If the environments were completely coupled, then we are guaranteed that we can perform identity coupling and keep the distance between the walkers unchanged. But we have just seen that the environments will typically not be fully coupled. Yet, if we knew that the environments are coupled in a neighborhood around the walkers and that the walkers will not exit this neighborhood, then identity coupling is still doable. That will be our strategy, but to implement it we will require a more delicate definition of what a favorable enough environment means.

We will use a multi-scale analysis to control the environment. This will reveal *future information* regarding the environment; that is, we will observe some information about the environment from time 0 to some time t, and then decide how to couple the walkers from time 0. Therefore, this construction will lead to a non-Markovian coupling.

A good picture to have in mind is that the environment is a process in space-time, where some regions are classified as favorable and others as unfavorable. We observe these regions from time 0 to time t, and then start observing the walkers which are paths in space-time that start growing from time 0. Whenever we see that the walkers are passing through a favorable part of the environment, where favorable will also imply that the walkers will not move outside some neighborhood around their current locations, we will try to do a simple random walk moment. If successful, the distance between the walkers may change and the environments may decouple, but still using (the yet-to-be-defined properties of) favorability we will be able to recouple the environments within a neighborhood around the walkers. If, instead, the simple random walk moment is not successful, then the walkers may move more than just one step of a simple random walk, but favorability will also imply that the walkers will not move too far away, in particular they will remain within a region where we know the environments were coupled. This will translate to a successful application of the identity coupling.

On the other hand, if we see that the walkers are approaching an unfavorable region of the environment, then we will want to do identity coupling but we will need to start preparing ourselves beforehand. The problem is that such an unfavorable region could be of an arbitrarily large scale, and the larger its size is, the earlier we need to start preparing for it. So when we see that in space-time the path of the walker is getting dangerously near an unfavorable region, we stop doing simple random walk moments even if in a smaller scale around the walkers the environment looks favorable. By switching off the simple random walk moments, we only apply identity coupling until the walkers reach the unfavorable region. We can show that such identity couplings will succeed and, since the translation map from one walker to the next will not change during this period, it will give enough time for the environments to couple in a region around the walkers that is as large as needed to contain the scale of the

unfavorable region that the walkers are approaching. Then, with the environments properly coupled, the walkers can enter the unfavorable region and move as wildly as the environment there allows, because we can perform identity coupling throughout the unfavorable region. So the walkers survive the traversal of the unfavorable region without changing their distance.

Then one can imagine that the proof ends by showing that n^2 instances of a simple random walk moment are enough to guarantee that we can couple the walkers. This is partially true. The fact is that, as mentioned above, we need to observe future information to carry out this coupling strategy. But in order to establish that the mixing time is at most t, we need to show that with a large enough probability the two copies of the full system are coupled at time t without revealing any information that goes beyond time t. So our strategy to finalize the proof is to choose an appropriate time $t' \in (0, t)$, reveal the information up to time t and do the coupling described above up to time t' , showing that within t' we have carried out an order of n^2 simple random walk moments, and that we coupled the walkers at time t' (the environments may, and typically will, be uncoupled except for a small region around the walkers). We will be able to show this first phase succeeds with positive probability.

Next, the goal is to try to do identity coupling from time t' to t in a similar manner as we were doing when approaching an unfavorable region. In this second phase, identity coupling can only fail due to information that we have not observed because we are limited to observe the environment up to time t. We will show that, with positive probability, identity coupling will indeed succeed from t' to t, leading to a coupling of the full system at time t. If any of these two phases fail, then we just restart from scratch. We only need to repeat the phases a constant number of times to guarantee that the whole coupling succeeds with probability at least 3/4.

4.3 Multi-scale setup

We start defining a multi-scale tessellation of $\mathbb{T}_d^n$, which will consist of partitioning $\mathbb{T}_d^n$ into boxes and defining the event that boxes are *good* or *bad*. Those events will be then used to define the favorable parts of the environment.

4.3.1 Multi-scale Tessellation

Let

$$\ell := p^{-\frac{1}{3d}}, \tag{4.3}$$

and m be a sufficiently large integer.
For each $k \geq 1$ we tessellate the graph $\mathbb{T}_n^d$ into cubes of length ℓ_k where

$$\ell_1 = \ell \quad \text{and} \quad \ell_{k+1} = mk^2\ell_k. \tag{4.4}$$

The reason why we choose k^2 is solely because later on we will have that $\sum \frac{1}{k^2}$ is summable. The cubes will be indexed by integer vectors $i \in \mathbb{Z}^d_{\frac{n}{\ell_k}}$, and denoted $S_k^{\text{core}}(i) \subset \mathbb{T}_n^d$ with

$$S_k^{\text{core}}(i) = \prod_{j=1}^{d} [i_j\ell_k,\, (i_j + 1)\ell_k)\,.$$

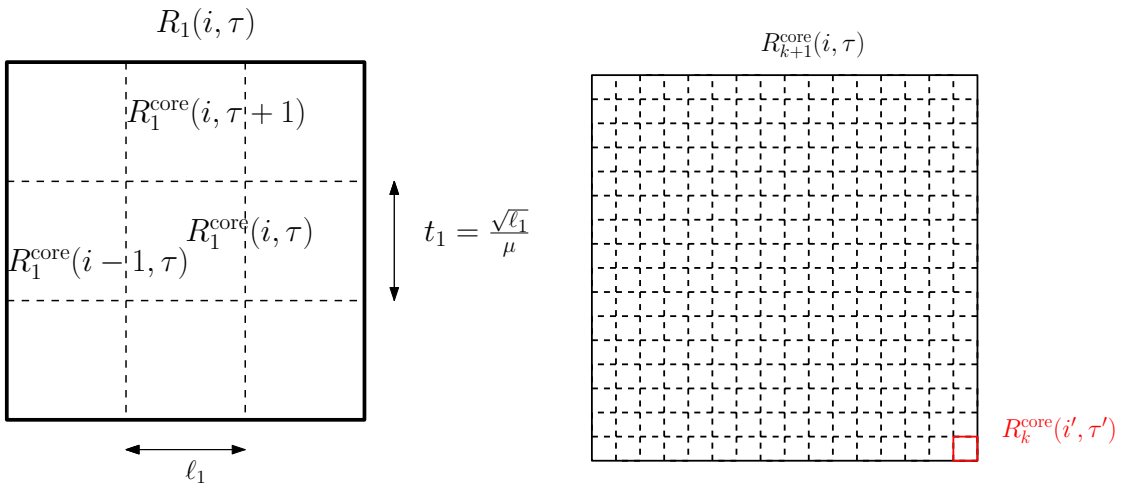


FIGURE 4.1: On the left a box of level 1 and its core (in the center) in $\mathbb{T}_n^d \times [0,T]$. On the right a box of level $k+1$ that contains boxes of level k.

We will consider a tiling of $\mathbb{T}_n^d$ with a hierarchy as each cube of level k is contained inside a unique cube of scale $k+1$. For simplicity we will assume ℓ_k divides n for all k we will consider; if that was not the case one could consider for each k some cubes to have length between ℓ_k and $2\ell_k$ to fully tessellate the torus. Moreover for any subset V_i of the vertices of $V(\mathbb{T}_n^d)$, we denote by

$$E(V_i) = \{(v_1, v_2) \in E(\mathbb{T}_n^d) : v_1, v_2 \in V_i\}$$

the set of all edges incident only to vertices in V_i. Now we define a multi-scale tessellation of time. At scale 1, we tessellate $\mathbb{R}_+$ into intervals of length $t_1 = \frac{\ell_2^1}{\mu}$ and then, for higher scales we define

$$t_{k+1} = mk^2 t_k, \quad k \geqslant 1.$$

We index the time intervals by $\tau \in \{0\} \cup \mathbb{N}$ and denote them by $T_k^{\text{core}}(\tau)$, where

$$T_k^{\text{core}}(\tau) = [\tau t_k,\, (\tau+1)t_k)\,.$$

Now for any $i \in \mathbb{Z}^d$, $k \geqslant 1$, and $\tau \in \mathbb{Z}$, we define the space-time k-box

$$R_k^{\text{core}}(i,\,\tau) := S_k^{\text{core}}(i) \times T_k^{\text{core}}(\tau). \tag{4.5}$$

Notice that a box of level $k+1$ contains $(mk^2)^{d+1}$ boxes of level k (see Figure 4.1).
We furthermore define the spatial boundary of $R_k^{\text{core}}(i,\tau)$ and denote it by

$$\partial_s R_k^{\text{core}}(i,\tau) = \bigcup_{j'=1}^{d} \left(\{(i_{j'})\ell_k, (i_{j'}+1)\ell_k\} \times \prod_{j \neq j'} [i_j \ell_k, (i_j+1)\ell_k] \right) \times T_k^{\text{core}}(\tau).$$

Define also the two time boundaries

$$\partial_t^+ R_k^{\text{core}}(i,\tau) = S_k^{\text{core}}(i) \times \{(\tau+1)t_k\} \quad \text{and} \quad \partial_t^- R_k^{\text{core}}(i,\tau) = S_k^{\text{core}}(i) \times \{\tau t_k\}\,.$$

For $k \geqslant 2$ and $\tau > 0$, each box $R_k^{\text{core}}(i,\tau)$, will be the central part of a larger box

$$R_k(i,\,\tau) := \bigcup_{(j_1,\ldots,j_d,\beta) \in \{-1,0,+1\}^{d+1}} R_k^{\text{core}}(i+j, \tau+\beta) = S_k(i) \times T_k(\tau),$$

where we let

$$S_k(i) := \bigcup_{(j_1,\dots,j_d)\in\{-1,0,+1\}^d} S_k^{\text{core}}(i+j), \quad \text{and} \quad T_k(\tau) := \bigcup_{\beta\in\{-1,0,+1\}} T_k^{\text{core}}(\tau+\beta),$$

and where with a slight abuse of notation we write $i_\iota + j_\iota$ instead of $(i_\iota + j_\iota) \mod \frac{n}{\ell_k}$. In words $R_k(i,\tau)$ is composed of a cube in space of side length $3\ell_k$ and a time interval of length $3t_k$, and it has $R_k^{\text{core}}(i,\tau)$ as its central part (see Figure 4.1).
Let $\{0\}\cup 2\mathbb{N}$ be the set of even natural numbers, for scale $k=1$ instead we will only consider boxes $R_1(i,\tau)$ with $\tau \in \{0\}\cup 2\mathbb{N}$ and

$$R_1(i,\tau) := \bigcup_{\substack{(j_1,\dots,j_d)\in\{-1,0,+1\}^d \\ \beta\in\{0,1\}}} R_1^{\text{core}}(i+j,\tau+\beta) = S_1(i)\times T_1(\tau),$$

where we let

$$S_1(i) := \bigcup_{(j_1,\dots,j_d)\in\{-1,0,+1\}^d} S_1^{\text{core}}(i+j), \quad \text{and} \quad T_1(\tau) := \bigcup_{\beta\in\{0,+1\}} T_1^{\text{core}}(\tau+\beta).$$

In words $R_1(i,\tau)$ is composed of a cube in space of side length $3\ell_1$ and a time interval of length $2t_1$. This will allow us to have boxes of scale 1 that intersect in the space-dimension but not in the time dimension, more precisely for any pair $\tau,\tau'\in\{0\}\cup 2\mathbb{N}$, $T_1(\tau)\cap T_1(\tau')=\emptyset$. This particular choice will allow us later on to define independent events which will play a key role in the final proof.
Similarly to the definition of space and time boundary of $R_k^{\text{core}}(i,\tau)$, for $k\geqslant 1$ we denote the space boundaries of $R_k(i,\tau)$ by

$$\partial_s R_k(i,\tau) = \bigcup_{j'=1}^{d}\left(\{(i_{j'}-1)\ell_k,(i_{j'}+2)\ell_k\}\times\prod_{j\neq j'}[(i_j-1)\ell_k,(i_j+2)\ell_k]\right)\times T_k(\tau),$$

whereas the time boundaries for $k\geqslant 2$ by

$$\partial_t^+ R_k(i,\tau) = S_k(i)\times\{(\tau+2)t_k\} \quad \text{and} \quad \partial_t^- R_k(i,\tau) = S_k(i)\times\{(\tau-1)t_k\},$$

and the time boundaries for $k=1$ by

$$\partial_t^+ R_1(i,\tau) = S_1(i)\times\{(\tau+2)t_1\} \quad \text{and} \quad \partial_t^- R_1(i,\tau) = S_1(i)\times\{\tau t_1\}.$$

Moreover, denote with $\gamma(p,q,d)$ the useful constant

$$\gamma(p,\,q,\,d)>0 \text{ such that } p+\gamma<\frac{1}{2d}. \tag{4.6}$$

Finally we denote by

$$\partial S_1(i) = \bigcup_{j'=1}^{d}\left(\{(i_{j'}-1)\ell,(i_{j'}+2)\ell\}\times\prod_{j\neq j'}[(i_j-1)\ell,(i_j+2)\ell]\right)$$

the boundary of $S_1(i)$, and by $S_1^{\text{inn}}(i)$ the *inner part* of $S_1(i)$ which is obtained by removing all the vertices that are at distance less than $\frac{\gamma}{6}\log^2\ell$ from the boundary of $S_1(i)$ (γ is the

constant that we have defined in (4.6)), more precisely

$$S_1^{\text{inn}}(i) := \{v \in S_1(i) : \|v - w\|_1 > \frac{\gamma}{6}\log^2 \ell, \text{ for all } w \in \partial S_1(i)\}.$$

4.3.2 Good Boxes at scale 1

Definition 4.3. *We say that an event A is restricted to a region $R \subset V(\mathbb{T}_n^d)$ and a time interval $[s_0, s_1]$ if it is measurable with respect to the σ-field generated by the updates of the edges that are inside R from time s_0 to s_1 and any random variables U, U' from updates of edges $e \in E(R)$ at times in (s_0, s_1).*

Denote by $\mathcal{C}_x(t)$ the connected component of open edges containing vertex $x \in V$ at time t, and with $\mathcal{C}_x(s, s')$ the connected component of $x \in V$ of all the edges which are open during some time in the interval $[s, s']$, equivalently $y \in \mathcal{C}_x(s, s')$ if there exists a path of edges $\mathcal{P}_{xy}(s, s') = \{(x, z_x) = e_0, e_1, \dots e_{n_{xy}} = (z_y, y)\}$, for some vertices z_x, z_y, such that e_i and e_{i+1} are adjacent and for all $e \in \mathcal{P}_{xy}(s, s')$ there exists an interval $(s_e, s'_e) \subset (s, s')$ and $\eta_t(e) = 1$ for all $t \in (s_e, s'_e)$. Now, we introduce the definition of **almost good boxes**. In words, we partition $T_1(\tau)$, the time interval of the box $R_1(i, \tau)$ into 6 consecutive (not necessarily of the same length) sub-intervals $T_1^j(\tau)$, then we say that $R_1(i, \tau)$ is almost good if

- (A_1) in the whole space-time interval of the box only oblivious updates ($\star$-updates) occur: this is to ensure that the graph behaves as dynamical percolation in the box;
- (A_2) in the first sub-interval $T_1^1(\tau)$ every edge in the box closes and does not open: this is to ensure that after the first sub-interval no information percolates from the past of the box, and to ensure that the walker, were it to be there, would be stuck on a vertex;
- (A_3) in every sub-interval after the first one each edge is refreshed enough times: this is to ensure that even if some information were to be revealed, it would be *forgotten* quickly;
- (A_4) lastly we require that in every sub-interval after the first one, the connected components are small enough: this is to ensure that, were the walker to be there, it would not be able to move much, in particular, we will see later (Lemma 4.25), if the walker is in the core of an almost good box then it is not able to exit the box before the end of $T_1(\tau)$.

Moreover we will define them in such a way that A_3, A_4 are independent from A_1 and A_2. More formally now, recall that $t_1 = \frac{\sqrt{\ell}}{\mu}$,

Definition 4.4 (Almost good boxes). *Consider a box $R_1(i, \tau)$. Partition the time interval into 6 consecutive sub-intervals*

$$T_1(\tau) = \bigcup_{j=1}^{6} T_1^j(\tau),$$

such that $T_1^1(\tau) = [\tau t_1, \tau t_1 + \frac{\log^2 \ell}{\mu})$ and $|T_1^j| = \frac{1}{5}(2t_1 - \frac{\log^2 \ell}{\mu})$ for $j = 2, \dots, 6$ are disjoint and ordered.
*The box $R_1(i, \tau)$ is said to be **almost good** if the following four conditions are satisfied:*

- *(A_1) For any $e \in E(S_1(i))$, all updates on e during $T_1(\tau)$ are $\star$-updates;*
- *(A_2) During the interval $T_1^1(\tau)$, every $e \in E(S_1(i))$ receives at least one update (s, U', U) and for every update it receives, U is such that the edge refreshes closed;*

(A_3) For each $e \in E(S_1(i))$, the number of $\star$-updates on edge e occurring during each one of the time intervals $T_1^j(\tau)$ for $j = 2, \dots, 6$ is at least $\frac{1}{60} p_\star \sqrt{\ell}$ (for the values of ℓ and p we will consider this will always be at least 1);

(A_4) Recall γ from (4.6), for all $j \in \{0, 1, \dots, \frac{2\ell^{\frac{1}{2}} - \log^2 \ell}{\gamma} - 1\}$, let

$$s_j(\tau) := \left(\tau\sqrt{\ell} + \log^2 \ell + j\gamma\right) \frac{1}{\mu}, \tag{4.7}$$

*and let $\tilde{\mathcal{C}}_x(j)$ be the connected component of x of all the edges open some time during $[s_j(\tau), s_{j+1}(\tau)]$ and that opened only after $s_0(\tau)$, then we require that, **if we only look at the $\star$-updates** during $T_1(\tau) \setminus T_1^1(\tau)$,*

$$\left|\tilde{\mathcal{C}}_x(j)\right| < \frac{\gamma}{6} \log^2(\ell), \text{ for all } x \in S_1^{\text{inn}}(i), \text{ for all } j \in \{0, 1, \dots, \frac{2\ell^{\frac{1}{2}} - \log^2 \ell}{\gamma} - 1\}.$$

Remark 4.5. *The event $\{R_1(i, \tau)$ is almost good$\}$ is restricted to the cube $S_1(i)$ and the interval $T_1(\tau)$.The decision of whether a box is almost good is completely independent of the walker's process, it only depends on the updates of the dynamical random cluster process. Moreover, by the way we define the events $A_1, \dots, A_4$ we have that A_3 and A_4 are independent from A_1 and A_2. We will need this later on when we will have to control the distribution of certain random variables conditioned on these events. Of course, in the definition of almost good boxes, because we take the intersection of all of them, this will not create inconsistencies with our process.*

Lemma 4.6. *There exists $p_0, C_1(d), C_2 > 0$ such that for all $p < p_0$ and for all (i, τ)*

$$\mathbb{P}\left(R_1(i, \tau) \text{ is almost good}\right) \geqslant \exp\left(-C_1 \ell^{d+\frac{1}{2}}(1 - p_\star)\right) \tag{4.8}$$

$$\mathbb{P}(A_3^c \cup A_4^c) \leqslant \exp\left(-C_2 \log\left(p^{-1}\right) \log^2(\ell)\right) \tag{4.9}$$

Proof. From the definition of almost good box and observing that A_3, A_4 are independent of A_1, A_2

$$\begin{aligned}
\mathbb{P}(R_1(i, \tau) \text{ is almost good}) =& \mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4) = \mathbb{P}(A_1)\mathbb{P}(A_2 \cap A_3 \cap A_4 \,|\, A_1) \\
=& \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)\mathbb{P}(A_3 \cap A_4|A_1, A_2) \\
\geqslant& \mathbb{P}(A_1)\mathbb{P}(A_2|A_1)(1 - \mathbb{P}(A_3^c) - \mathbb{P}(A_4^c)).
\end{aligned}$$

We will handle each term separately.
For a given edge, an update that is not $\star$ occurs at rate $(1 - p_\star)\mu$, then since in $E(S_1(i))$ there are at most $d3^d\ell^d$ edges

$$\mathbb{P}(A_1) \geq e^{-2d3^d\ell^{d+\frac{1}{2}}(1-p_\star)}.$$

Now we bound $\mathbb{P}(A_2|A_1)$ from below: we consider the process of updates occurring on a fixed edge. The updates occur at rate μ. Next, we divide this process into $\star$ updates (that occur at rate $\mu p_\star$) and non-$\star$ updates (that occur at rate $\mu(1 - p_\star)$), these are independent. Conditioned on A_1, non-$\star$ updates do not occur, so we divide the process of $\star$ updates into $\star$ updates that refresh the edge open (that occur at rate $\mu p_\star \frac{p_{ce}}{p_\star} = \mu p_{ce}$) and closed (at rate $\mu p_\star(1 - \frac{p_{ce}}{p_\star}) = \mu(1 - p)$). Now, we can bound $\mathbb{P}(A_2|A_1)$ by 1 minus the probability

that conditioned on A_1 there exists $e \in E(S_1(i))$ that is never updated closed during $T_1^1(\tau)$ (conditioned on A_1, each edge refreshes closed at rate $\mu(1-p)$), times the probability that conditioned on A_1 no edge $e \in E(S_1(i))$ is updated open during $T_1^1(\tau)$ (conditioned on A_1, each edge refreshes open at rate μp_{ce}). For p small (and ℓ large), by doing a union bound over the set of edges in the box ($d3^d\ell^d$ at most)

$$\begin{aligned}\mathbb{P}(A_2|A_1) &\geqslant (1 - d3^d\ell^d e^{-(1-p)\log^2 \ell})e^{-p_{ce}d3^d\ell^d \log^2 \ell} \\ &\geqslant \frac{1}{2}e^{-p_{ce}d3^d\ell^d \log^2 \ell} \\ &\geqslant e^{-2d3^d\ell^{d+\frac{1}{2}}(1-p_\star)},\end{aligned}$$

where the last line comes from the fact that for ℓ large, $\ell^{\frac{1}{2}}(1-p_\star) > p_{ce}\log^2 \ell$.
For A_3, denote $u := \frac{1}{60}p_\star\sqrt{\ell}$. We will bound the probability that A_3 does not occur by taking an union bound over the edges in the box $R_1(i,\tau)$ and calculate the probability that an edge e receives less than u $\star$-updates in each time interval $T_1^j(\tau)$ (there are 5 of them). For ℓ large, $\frac{\sqrt{\ell}-\log^2 \ell}{5} > \frac{\sqrt{\ell}}{6}$, therefore

$$\begin{aligned}\mathbb{P}(A_3^c) \leq& 5d3^d\ell^d \sum_{i=0}^{u} e^{-p_\star \frac{\sqrt{\ell}}{6}} \frac{(p_\star\sqrt{\ell})^i}{i!} \\ \leq& 5d3^d\ell^d(u+1)e^{-p_\star \frac{\sqrt{\ell}}{6}} \frac{(p_\star\sqrt{\ell})^u}{u!},\end{aligned}$$

where in the last inequality we used the fact that $\frac{p_\star\sqrt{\ell}}{u} > 1$. Using now $u! \geqslant (\frac{u}{e})^u$, from the definition of u we have that, for ℓ large (p small)

$$\mathbb{P}(A_3^c) \leq 5d3^d\ell^d(u+1)e^{-10u}\frac{(60)^u}{u!} \leqslant e^{-9u}(60u)^u\frac{e^u}{u^u} = e^{u(1+\log 60-9)} = e^{-c_0\sqrt{\ell}},$$

with $c_0 = \frac{1}{60}p_\star(8 - \log 60) > 0$.
For A_4, for any $s > 0$, assuming we start from all edges closed at time s_0,

$$\begin{aligned}&\mathbb{P}\left(e \text{ is open sometime in } \left(s, s+\frac{\gamma}{\mu}\right)\right) \\ \leq& \mathbb{P}\left(e \text{ is open at time } s\right) + \mathbb{P}\left(e \text{ refreshes in } \left(s, s+\frac{\gamma}{\mu}\right)\right) \\ \leq& \frac{p_{ce}}{p_\star} + 1 - e^{-\gamma} \leq p + \gamma < \frac{1}{2d}.\end{aligned} \tag{4.10}$$

We now want to bound the probability that there exists a $j \in \left\{0, \dots, \frac{2\ell^{\frac{1}{2}}}{\gamma} - 1\right\}$ and an $x \in S_1^{\text{inn}}(i)$ for which $|\mathcal{C}_x(s_j(\tau), s_{j+1}(\tau))| \geqslant \frac{\gamma}{6}\log^2 \ell$, with $s_j(\tau)$ as defined in (4.7). Because the process is translation invariant in space and time we take an union bound over the vertices in $S_1(i)$ and the intervals $(s_j(\tau), s_{j+1}(\tau))$ (there are less than $2t_1\frac{\mu}{\gamma}$ of them, and each one of them is equivalent to the interval $(s_0(\tau), s_1(\tau))$. By (4.10), now, for any j, $\mathcal{C}_x(s_j(\tau), s_{j+1}(\tau))$ is stochastically dominated by a subcritical independent percolation model and because the probability of an edge to be open is smaller than $\frac{1}{2d}$ we can use exponential

decay of connectivities from (2.13), therefore

$$\begin{aligned}\mathbb{P}(A_4^c) &\leq 3^d \frac{2\sqrt{\ell}}{\gamma} \mathbb{P}\left(\exists\, x \in S_1^i(i) \,:\, \left|\mathcal{C}_x\left(0, \frac{\gamma}{\mu}\right)\right| \geq \frac{\gamma}{6}\log^2(\ell)\right) \\ &\leq 3^{d+1}\ell^d \frac{\sqrt{\ell}}{\gamma} \mathbb{P}\left(\left|\mathcal{C}_x\left(0, \frac{\gamma}{\mu}\right)\right| \geq \frac{\gamma}{6}\log^2(\ell)\right) \\ &\leq \frac{3^{d+1}}{\gamma}\ell^{d+\frac{1}{2}} e^{-\frac{\gamma}{6}C_{cd}\log p^{-1}\log^2(\ell)} \leq e^{-\frac{\gamma}{10}C_{cd}\log p^{-1}\log^2(\ell)}.\end{aligned}$$

Putting everything together one has that for $C_1 = 5d3^d$

$$\begin{aligned}\mathbb{P}(R_1(i,\tau) \text{ is almost good}) &\geqslant e^{-4d3^d\ell^{d+\frac{1}{2}}(1-p_\star)}\left(1 - e^{-c_0\sqrt{\ell}} - e^{-C_{cd}\frac{\gamma}{10}\log p^{-1}\log^2(\ell)}\right) \\ &\geqslant e^{-C_1\ell^{d+\frac{1}{2}}(1-p_\star)}\end{aligned}$$

if p is small enough.
For the second inequality we simply notice that

$$\begin{aligned}\mathbb{P}(A_3^c \cup A_4^c) &\leqslant e^{-c_0\sqrt{\ell}} + e^{-C_{cd}\frac{\gamma}{10}\log p^{-1}\log^2(\ell)} \\ &\leqslant e^{-C_2\log p^{-1}\log^2(\ell)},\end{aligned}$$

for $C_2 = \frac{\gamma}{20}C_{cd}$. □

Remark 4.7. *By the choice we made for ℓ in (4.3), $\ell = p^{-\frac{1}{3d}}$, recalling moreover, that $p_\star = p_{ce} + 1 - p$, and that $p_{ce} = \frac{p}{p+(1-p)q}$, then it is easy to see that $\mathbb{P}(R_1(i,\tau)$ is almost good$)$ can be made arbitrarily close to 1 by choosing p small enough. Indeed, by Lemma 4.4*

$$\begin{aligned}\mathbb{P}\left(R_1(i,\tau) \text{ is almost good}\right) &\geqslant \exp\left(-C_1\ell^{d+\frac{1}{2}}(1-p_\star)\right) \\ &= \exp\left(-C_1 p^{\frac{4d-1}{6d}}\left(1 - \frac{1}{p+(1-p)q}\right)\right)\end{aligned}$$

Consider the quantity

$$p_\xi := 1 - e^{-\frac{C_2}{10^{d+1}}\log p^{-1}\log^2\ell}. \tag{4.11}$$

The reason for this choice for p_ξ is because we need it to be comparable to the bound we found for $\mathbb{P}(A_3^c \cup A_4^c)$ in Lemma 4.6. The reason for this choice will be clearer later in the proof of Lemma 4.9.

Definition 4.8. *We say that $R_1(i,\tau)$ is well refreshed if $A_3 \cap A_4$ happens in $R_1(i,\tau)$. Moreover we will denote the probability to be well refreshed with*

$$\nu_{wr}(i,\tau) := \mathbb{P}(R_1(i,\tau) \text{ is well refreshed}).$$

In particular, $R_1(i,\tau)$ is almost good if it is well refreshed and $\Theta_i^\tau = 1$. We can write $A_1 \cap A_2$ as intersection of independent events, however this cannot be done with $A_3 \cap A_4$. In fact, if $\tau \neq \tau'$ the event $A_3 \cap A_4$ in the box $R_1(i,\tau)$ is independent of $A_3 \cap A_4$ in $R_1(i',\tau')$ for all $i \neq i' \in \mathbb{T}_n^d$, but they are not independent if $\tau = \tau'$ and $R_1(i,\tau)$, $R_1(i',\tau)$ are neighboring boxes. We now want to use a result in [LSS97] to prove that for all $\tau \geqslant 0$ we can couple the probability distribution of the event $A_3 \cap A_4$ in the boxes $R_1(\cdot,\tau)$ with π_{p_ξ} the product measure of parameter p_ξ, in such a way that the first one stochastically dominates the second one.

Lemma 4.9. *Recall Definition 2.44 for the definition of stochastic domination. There exists $p_0 > 0$ such that for any $p < p_0$ and for any $\tau \geqslant 0$, the distribution $\nu_{wr}(\cdot, \tau)$ of well refreshed boxes identified by index τ can be bounded in the following way:*

$$\nu_{wr}(\cdot, \tau) \succcurlyeq \pi_{p_\xi}(\cdot).$$

Proof. We notice first of all that from the memoryless property of the Poisson process $A_3 \cap A_4$ in $R_1(i, \tau)$ have bounded dependence, meaning that if $R_1(i, \tau) \cap R_1(i', \tau) = \emptyset$ then they are independent. Let Δ be the number of boxes that a box $R_1(i, \tau)$ intersects. Then $\Delta < 5^{d+1} - 1$ (consider a $d+1$-dimensional hypercube centered in $R_1(i, \tau)$ and with side length 5ℓ, (5 times the side length of one core), this hypercube contains every box that intersects $R_1(i, \tau)$ and if we count the cores in the hypercube we get 5^{d+1} from which we subtract 1 that represents the box itself). We want to apply Theorem 1.3 in [LSS97] which states that if we can find constants α, $r \in (0, 1)$ such that for all $i \in \mathbb{N}_{\frac{n}{\ell}}$

$$(1 - \alpha)(1 - r)^{\Delta - 1} \geqslant 1 - \nu_{wr}(i, \tau) \tag{4.12}$$

$$(1 - \alpha)\alpha^{\Delta - 1} \geqslant 1 - \nu_{wr}(i, \tau), \tag{4.13}$$

and $1 - \nu_{wr}(i, \tau) \leqslant \frac{(\Delta - 1)^{\Delta - 1}}{\Delta^\Delta}$, then there exists $\rho > 0$ such that

$$\nu_{wr}(\cdot, \tau) \succcurlyeq \pi_\rho(\cdot),$$

where ρ satisfies the following

$$\begin{aligned}
\rho &= \left(1 - \frac{\mathbb{P}(A_3^c \cup A_4^c)^{\frac{1}{\Delta}}}{(\Delta - 1)^{\frac{\Delta - 1}{\Delta}}}\right)\left(1 - ((\Delta - 1)\,\mathbb{P}(A_3^c \cup A_4^c))^{\frac{1}{\Delta}}\right) \\
&\geqslant \left(1 - \frac{\mathbb{P}(A_3^c \cup A_4^c)^{\frac{1}{5^{d+1}-1}}}{(5^{d+1} - 2)^{\frac{5^{d+1}-2}{5^{d+1}-1}}}\right)\left(1 - \left(\left(5^{d+1} - 2\right)\mathbb{P}(A_3^c \cup A_4^c)\right)^{\frac{1}{5^{d+1}-1}}\right) \\
&\geqslant \left(1 - e^{-\frac{C_2}{5^{d+1}-1}\log p^{-1}\log^2(\ell)}\right)\left(1 - (5^{d+1} - 2)^{\frac{1}{5^{d+1}-1}} e^{-\frac{C_2}{5^{d+1}-1}\log p^{-1}\log^2(\ell)}\right) \\
&\geqslant \left(1 - e^{-\frac{C_2}{6^{d+1}}\log p^{-1}\log^2(\ell)}\right)^2 \\
&\geqslant 1 - e^{-\frac{C_2}{10^{d+1}}\log p^{-1}\log^2(\ell)} = p_\xi.
\end{aligned}$$

This proves the thesis of the Lemma, provided we can show that the inequalities (4.12) and (4.13) above are satisfied. But this is easily verified by noticing that, from Lemma 4.7,

$$1 - \nu_{wr}(i, \tau) = \mathbb{P}(A_3^c \cup A_4^c) \leqslant e^{-C_2 \log p^{-1} \log^2(\ell)}$$

can be made arbitrarily close to 0, choosing p small enough. □

We now introduce two new sequences of random variables $\{\Theta_i^\tau\}_{i,\tau}$ associated to the box $R_1(i, \tau)$; $\Theta_i^\tau = \mathbb{1}_{A_1 \cap A_2}$ the indicator random variable that the events A_1 and A_2 happen in $R_1(i, \tau)$. From Lemma 4.6,

$$p_\theta := \mathbb{P}(\Theta_i^\tau = 1) \geqslant e^{-C_1 \ell^{d + \frac{1}{2}}(1 - p_\star)}. \tag{4.14}$$

We will also consider $\{\Xi_i^\tau\}_{i,\tau}$ distributed as i.i.d. Bernoulli random variables of parameter p_ξ,

where

$$p_\xi = 1 - e^{-\frac{C_2}{10^{d+1}} \log p^{-1} \log^2 \ell}. \tag{4.15}$$

We can now introduce the concept of **good boxes**.

Definition 4.10. *If $\Theta_i^\tau = \Xi_i^\tau = 1$ the box $R_1(i,\tau)$ is said to be **good**, otherwise it is said to be **bad**.*

In particular, by Strassen's Theorem 2.45 together with Lemma 4.9, we can construct a coupling between good and almost good boxes in such a way that if a box is good then it is also almost good.
Recall that X_t denotes the position of the random walker at time t. In the lemma below, we will show that if the walker happens to be inside a good box, then it cannot move very quickly. This will allow us to have a better control on where the random walker can be when it crosses a good box. The idea of the proof essentially is using event A_4 which ensure the clusters in the box are small enough uniformly in time and space.

Lemma 4.11. *Let $t \geqslant 0$ be any given time and suppose $(X_t, t) \in R_1^{core}(i,\tau)$, where $R_1(i,\tau)$ is a good box. Assume that also $R_1(i,\tau-2)$ is good. Then*

$$\max_{s\in[t,(\tau+2)t_1]} \|X_t - X_s\|_1 \leqslant \frac{\ell}{3},$$

where $\|x-y\|_1$ denotes the L^1 distance in the torus between the positions $x, y \in \mathbb{T}_n^d$, in particular, it does not depend on whether edges are open or closed.

Proof. We first split the time interval of the box $R_1(i,\tau)$ which is $(\tau t_1, (\tau+2)\, t_1)$, into smaller sub-intervals of length $\frac{\gamma}{\mu}$ where γ is the constant in the event (A_4) of Definition 4.4. Note that in the box $R_1(i,\tau)$ there are $2t_1\frac{\mu}{\gamma}$ sub-intervals. Since $R_1(i,\tau-2)$ is good, by event A_4 the clusters at time τt_1 in $S_1^{\text{inn}}(i)$ are small, and because of event A_2 they will not increase during $T_1^1(\tau)$, namely $\mathcal{C}_x(\tau t_1) < \frac{\gamma}{6}\log^2\ell$ for all $x \in S_1^{\text{inn}}(i)$, therefore

$$\max_{s\in[t,(\tau+2)t_1]} \|X_t - X_s\|_1 \leqslant \sum_{j=0}^{2\frac{\sqrt{\ell}}{\gamma}-1} \|X_{s_j(\tau)} - X_{s_{j+1}(\tau)}\|_1 \leqslant 2\frac{\sqrt{\ell}}{\gamma}\cdot\frac{\gamma}{6}\log^2(\ell) < \frac{\ell}{3},$$

where $s_j(\tau)$ are as defined in (4.7). In words, under A_4 in each interval of length $\frac{\gamma}{\mu}$ the walker is able to move at most $\frac{\gamma}{6}\log^2(\ell)$; summing over all the intervals yields the lemma. □

4.3.3 Larger Scales

In this subsection we define the concept of good and bad boxes of scale larger than 1. We will need these to be able to give a qualitative description of the dynamics in larger portions of the space without always having to look at each single box of scale 1.

Definition 4.12. *A k-box $R_k(i,\tau)$ with $k \geq 2$ is said to be bad if it contains at least two non intersecting bad boxes of scale $k-1$.*

Remark 4.13. *The event $\{R_k(i,\tau)$ is bad$\}$ is restricted to the cube $S_k(i)$ and the time interval $T_k(\tau)$. Moreover, we will really only look at the event $\{R_k(i,\tau)$ is bad$\}$ for $\tau > 0$, therefore, by translation invariance, for any pair (i,τ), $(i'\tau')$ and any scale k we have $\mathbb{P}(R_k(i,\tau)$ is bad$) = \mathbb{P}(R_k(i',\tau')$ is bad$)$. Recall, moreover, that the event that a box of level 1 is almost good*

depends solely on the updates that occur within the box itself (see Definition 4.4), and by construction this property extends to every box of any level. Therefore if $R_k(i,\tau)$ and $R_k(i',\tau')$ are two non intersecting boxes then

$$\mathbb{P}(R_k(i,\,\tau) \text{ and } R_k(i',\,\tau') \text{ are bad}) = \mathbb{P}(R_k(i,\,\tau) \text{ is bad})^2.$$

Definition 4.14. *Let ρ_k be the probability $\rho_k := \mathbb{P}\left(R_k(i,\,\tau) \text{ is bad}\right)$ that a k-box $R_k(i,\,\tau)$ is bad. As noted in Remark 4.13, for $\tau > 0$ this probability does not depend on (i,τ).*

Recall that m is the variable that appears in the definition of ℓ_k from (4.4).

Lemma 4.15. *For any $m > 0$, there exists $p_0 > 0$ such that, for all $p \leqslant p_0$,*

$$\rho_k \leq \rho_1^{2^{k-2}}$$

Proof. We now prove the statement of the lemma in a slightly different version: we prove we can set $c_k \geqslant \frac{1}{2}$ for all k, so that

$$\rho_k \leq \rho_1^{c_k 2^{k-1}}.$$

We prove this by induction. For $k = 1$ the statement is trivially satisfied by setting $c_1 = 1$. Assume the statement is true up to k. Now, by the definition of bad box we have

$$\begin{aligned}
\rho_{k+1} &\leqslant (mk^2)^{2d+2}\rho_k^2 \\
&\leqslant (mk^2)^{2d+2}\rho_1^{2c_k 2^{k-1}} \\
&= (mk^2)^{2d+2}\rho_1^{(c_k - c_{k+1})2^k}\rho_1^{c_{k+1}2^k}.
\end{aligned}$$

Setting $c_{k+1} = c_k - \frac{1}{10k^2}$ gives that

$$(mk^2)^{2d+2}\rho_1^{(c_k-c_{k+1})2^k} = (mk^2)^{2d+2}\rho_1^{\frac{2^k}{10k^2}} \leqslant 1,$$

for all $k \geqslant 1$, provided ρ_1 is small enough with respect to m.
Notice that $c_k > c_\infty = c_1 - \sum_{i=1}^{\infty}\frac{1}{10i^2} \geqslant c_1 - \frac{1}{10} - \int_1^\infty \frac{1}{10x^2}dx \geqslant \frac{1}{2}$, which proves the lemma. □

4.3.4 Affected Area and Enlargement of boxes

Definition 4.16. *The* affected area *of a box $R_k(i,\,\tau)$ of scale k is the set of boxes*

$$R_k^{\text{aff}}(i,\,\tau) := \bigcup_{(j_1,\ldots,j_d,\beta)\in\{-1,0,1\}^{d+1}} R_k(i+j,\tau+\beta).$$

We also denote

$$S_k^{\text{aff}}(i) := \bigcup_{(j_1,\ldots,j_d)\in\{-1,0,1\}^d} S_k(i+j), \quad \text{and} \quad T_k^{\text{aff}}(\tau) := \bigcup_{\beta\in\{-1,0,1\}} T_k(\tau+\beta).$$

For $k = 1$ the definition of affected area takes into account boxes that we will not consider, ($R_1(i,\tau+\beta)$ with $\tau+\beta$ odd) but that are still well-defined and therefore describe a well defined region in the space-time slab. In the following Lemma we will prove that once a bad k-box $R_k(i',\tau')$ has been identified in a good box $R_{k+1}(i,\tau)$, the affected area $R_k^{\text{aff}}(i',\tau')$ contains all the space-time points that could be inside a bad box $R_k(i'',\tau'')$ inside $R_{k+1}(i,\tau)$, see Figure 4.2. Thus $R_k(i'',\tau'')$ is good for all $R_k(i'',\tau'') \subset R_{k+1}(i,\tau) \setminus R_k^{\text{aff}}(i',\tau')$.

Lemma 4.17. *Let $R_{k+1}(i,\tau)$, with $\tau > 0$, be a good box. If there is a bad box of scale k inside $R_{k+1}(i,\tau)$, then there exists (i',τ') such that $R_k(i'',\tau'')$ is good for all $R_k(i'',\tau'') \subset R_{k+1}(i,\tau) \setminus R_k^{\mathrm{aff}}(i',\tau')$.*

Proof. From the definition of good boxes, a $(k+1)$-box $R_{k+1}(i,\tau)$ that is good does not contain two non intersecting pairs $R_k(i',\tau')$ and $R_k(i'',\tau'')$ that are both bad. We will prove the lemma by contradiction. Suppose that for all bad boxes $R_k(i',\tau') \subset R_{k+1}(i,\tau)$ there exists a box $R_k(i'',\tau'') \subset R_{k+1}(i,\tau) \setminus R_k^{\mathrm{aff}}(i',\tau')$ that is bad. Then $R_k(i',\tau')$ and $R_k(i'',\tau'')$ are two bad boxes whose intersection is empty. □

Definition 4.18. *The 1-enlargement of a box $R_k(i,\tau)$ of level k, is the set of boxes*

$$R_k^{\mathrm{enl1}}(i,\tau) := \bigcup_{(j_1,\dots,j_d,\beta)\in\{-3,-2,\dots,3\}^{d+1}} R_k(i+j,\tau+\beta).$$

We also denote

$$S_k^{\mathrm{enl1}}(i) := \bigcup_{(j_1,\dots,j_d)\in\{-3,-2,\dots,3\}^{d}} S_k(i+j), \quad \text{and} \quad T_k^{\mathrm{enl1}}(\tau) := \bigcup_{\beta\in\{-3,-2,\dots,3\}} T_k(\tau+\beta).$$

The reason why we define the side length of the 1-enlargement that way is explained in the following Lemma.

Lemma 4.19. *The 1-enlargement is a $(d+1)$-dimensional parallelogram centered in $R_k(i,\tau)$ so that if every box $R_k(i',\tau')$ that intersects $R_k^{\mathrm{enl1}}(i,\tau)$ and does not intersect $R_k^{\mathrm{aff}}(i,\tau)$ is good, then any event outside of $R_k^{\mathrm{enl1}}(i,\tau)$ is independent of the updates occurring in the region $R_k^{\mathrm{aff}}(i,\tau)$.*

Proof. To see this, let $k=1$, denote by u_j the d dimensional vector where the jth entry equals to 1 and all other entries equal to zero, and suppose an edge $e \in E(S_1(i+5u_1))$ receives an update during the time interval $T_1(\tau)$. If the outcome of this update depended from the updates in the region $R_1^{\mathrm{aff}}(i,\tau)$ it would mean that there exists a connected component of length at least $5\ell - 3\ell = 2\ell$, but then it would follow that by definition of almost good box the event (A_4) is not satisfied in the box $R_1(i+5u_1,\tau)$ and the said box is bad. To see this at higher scales we can simple use the definition of good box at higher scale. Let $k>1$ and suppose an edge $e \in E(S_k(i+5u_1))$ receives an update during the time interval $T_k(\tau)$. If the outcome of this update depended from the updates in the region $R_k^{\mathrm{aff}}(i,\tau)$ it would mean that there exists a connected component of length at least $5\ell_k - 3\ell_k = 2\ell_k$, but then it would follow that there exists a sequence of intersecting boxes of scale 1 such that the first box of the sequence intersects $R_k^{\mathrm{aff}}(i,\tau)$ and the last one of the sequence intersects $R_k^{\mathrm{enl1}}(i,\tau)$ for which the event (A_4) is not satisfied. These boxes are all bad and the box $R_k(i+5u_1,\tau)$ is bad as well. □

Definition 4.20. *The 2-enlargement of a box $R_k(i,\tau)$ of scale k, is the set of boxes*

$$R_k^{\mathrm{enl2}}(i,\tau) := \bigcup_{\substack{(j_1,\dots,j_d)\in\{-20,-19,\dots,20\}^d,\\ \beta\in\{-18,-17,\dots,3\}}} R_k(i+j,\tau+\beta),$$

and we also denote

$$S_k^{\mathrm{enl2}}(i) := \bigcup_{(j_1,\dots,j_d)\in\{-20,-19,\dots,20\}^{d}} S_k(i+j), \qquad T_k^{\mathrm{enl2}}(\tau) := \bigcup_{\beta\in\{-18,-17,\dots,3\}} T_k(\tau+\beta).$$

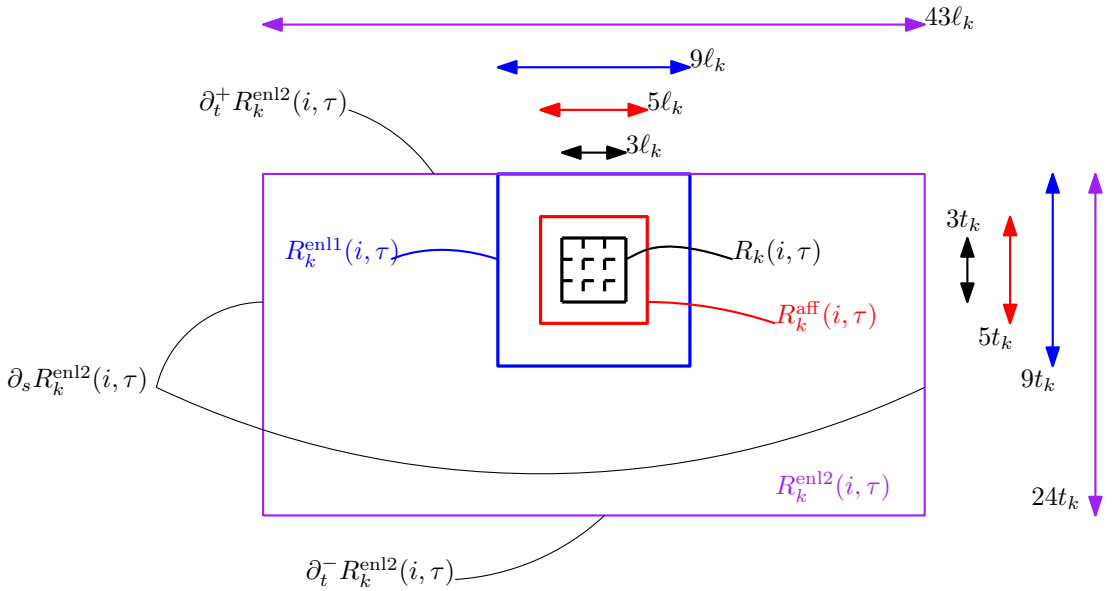


FIGURE 4.2: In black a box $R_k(i,\tau)$, with its affected area in red, the 1-enlargement in blue and finally its 2-enlargement in purple.

Note that the 2-enlargement is a larger $(d+1)$-dimensional parallelogram centered in $R_k(i,\tau)$ so that $\max T_k^{\text{enl1}}(\tau) = \max T_k^{\text{enl2}}(\tau)$.

Remark 4.21. *Later, we will use the enlargements of a box that is bad to define the coupling of the two processes and make sure they stay as much coupled as possible despite them traversing a region that does not behave well.*

4.3.5 Feasible Paths

In this subsection we introduce the concept of feasible paths. For any graph $G = (V, E)$, we denote the neighbors of a vertex $v \in V$ by $\mathcal{N}_G(v) = \{w \in V : (v, w) \in E\}$.

Definition 4.22. *A path $\mathcal{P} : \mathbb{R}^+ \to V$ on a graph $G = (V, E)$ is a cadlag function of time such that for any $s \in \mathbb{R}^+$, if we take s' to be the smallest value that is larger than s and such that $\mathcal{P}(s') \neq \mathcal{P}(s)$ then $\mathcal{P}(s') \in \mathcal{N}_G(\mathcal{P}(s))$.*

Now, we introduce a particular set of paths, the **feasible paths**, that we will use to control the trajectory of the walker in the process $\{M_t = (X_t, \eta_t)\}_{t\geqslant 0}$. In fact, as we will see in the definition, the path will be allowed to move instantaneously inside bad boxes of any scale or even good boxes of scale 1 which are preceded by bad boxes (this is because a bad box $R_1(i,\tau)$ will affect the connected components in $S_1(i)$ during the interval $T_1^1(\tau+2)$ before every edge has time to close); in good boxes of scale 2 or higher and good boxes of scale 1 which are preceded by good boxes of scale 1 the path will not be able to move as much, in particular its displacement will be controlled by the result stated in Lemma 4.11.

Definition 4.23. *A path $\mathcal{P}$ on $\mathbb{T}_n^d$ is said to be* feasible *if for any s, s' such that*

- $s' - s \leqslant \frac{\gamma}{\mu}$;
- $(\mathcal{P}(t), t) \in R_1^{\text{core}}(i,\tau)$ *for all* $t \in [s, s']$, *for some* $i \in \mathbb{Z}_{\frac{n}{\ell}}^d$ *and* $\tau > 0$;
- $R_1(i,\tau)$ *and* $R_1(i,\tau-2)$ *are good boxes;*

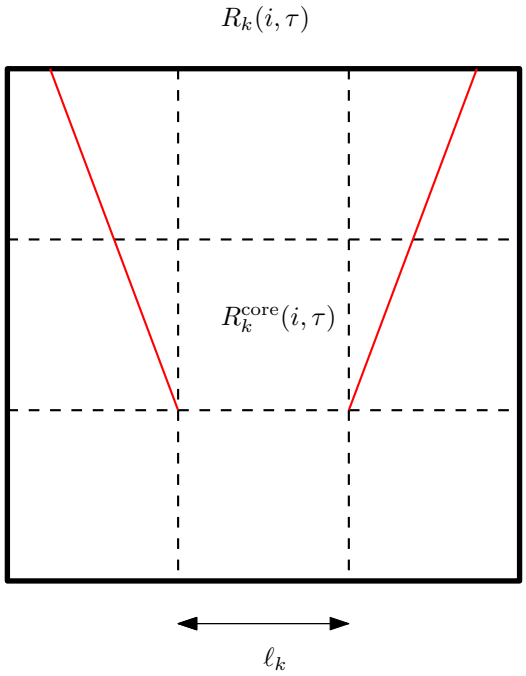


FIGURE 4.3: A feasible path starting anywhere in the core of a good box is contained in the area delimited by the red lines and it therefore can exit the box only from the time boundary.

then

$$\|(\mathcal{P}(s') - \mathcal{P}(s))\|_1 < \frac{\gamma}{3}\log^2 \ell,$$

where γ is the constant from the event (A_4) in Definition 4.4.

We will refer to *a path that can leave the box $R_k(i,\tau)$ only from the time boundary* $(\partial_t^+ R_k(i,\tau))$ as a path $\mathcal{P}$ such that $(\mathcal{P}(s), s) \in R_k^{\text{core}}(i,\tau)$ for some $s \geqslant 0$ and if s' is the smallest value such that $s' > s$ and $(\mathcal{P}(s'), s') \notin R_k(i,\tau)$ then $\mathcal{P}(s') \in S_k(i)$. In other words, it is a path that exits $R_k(i,\tau)$ through $\partial_t^+ R_k(i,\tau)$. In the following two lemmas we will prove that a feasible path always leaves good boxes only from the time boundary.

Lemma 4.24. *There exists m large enough such that for any feasible path such that there exists $s \geqslant 0$ for which $(\mathcal{P}(s), s) \in R_1^{core}(i,\tau)$ of a good box $R_1(i,\tau)$, and $R_1(i,\tau-2)$ is good as well, it must be the case that the path can leave $R_1(i,\tau)$ only from the time boundary, see Figure 4.3.*

Proof. For any $(v,s) \in R_1^{core}(i,\tau)$ and any $(v',s') \in \partial_s R_1(i,\tau)$, $\|v - v'\|_1 \geqslant \ell$ as well as $|s' - s| \leqslant 2t_1$. Let $\mathcal{P}$ be a feasible path such that $(\mathcal{P}(s), s) \in R_1^{core}(i,\tau)$ for some s, and $R_1(i,\tau)$ is a good box. We first split the time interval of the box $R_1(i,\tau)$, $(\tau t_1, (\tau+2)\,t_1)$, into smaller intervals of length $\frac{\gamma}{\mu}$ (in a box R_1 there are at most $2t_1\lceil\frac{\mu}{\gamma}\rceil$ of them). Since $R_1(i,\tau)$, $R_1(i,\tau-2)$ are both good and $\mathcal{P}$ is a feasible path, by definition we get

$$\|\mathcal{P}(s) - \mathcal{P}((\tau+2)t_1)\|_1 \leqslant \sum_{j=0}^{2\lceil\frac{\sqrt{\ell}}{\gamma}\rceil} \|\mathcal{P}(s_j(\tau)) - \mathcal{P}(s_{j+1}(\tau))\|_1 \leqslant \left(2\left\lceil\frac{\sqrt{\ell}}{\gamma}\right\rceil + 1\right)\cdot\frac{\gamma}{3}\log^2\ell \leqslant \frac{\ell}{3},$$

where $s_j(\tau)$ are as defined in (4.7). This tells us that this feasible path $\mathcal{P}$ is such that

$$\|\mathcal{P}(\tau t_1) - \mathcal{P}((\tau+2)t_1)\|_1 \leqslant \frac{\ell}{3}.$$

□

Lemma 4.25. *For all $k \geqslant 2$, there exists m large enough such that the following is true: for any good box $R_k(i,\tau)$, if a feasible path $\mathcal{P}$ is in that box at some time s, $(\mathcal{P}(s),s) \in R_k^{core}(i,\tau)$, then $\mathcal{P}$ can leave $R_k(i,\tau)$ only from the time boundary, see Figure 4.3.*

Proof. For any $(v,s) \in R_k^{core}(i,\tau)$ and any $(v',s') \in \partial_s R_k(i,\tau)$, $\|v-v'\|_1 \geqslant \ell_k$ as well as $|s'-s| \leqslant 2t_k$. We do a proof by induction on k. The case $k=1$ is proven by Lemma 4.24 assuming $R_1(i,\tau-2)$ is good too. The boxes of scale $k \geqslant 2$ do not require this assumption because it is already implicit in the definition of the boxes of scale higher than 1, being that $T_k^{\text{core}}(\tau-1) \subset T_k(\tau)$. Now, we do the proof for $k \geqslant 2$. Suppose the statement is true for good boxes up to level k, and more precisely that for any $j \leqslant k$ there exists $c_j < 1$ such that for any feasible path $\mathcal{P}$ for which $(\mathcal{P}(s),s) \in R_j^{\text{core}}(i,\tau)$, and $R_j(i,\tau)$ is a good box (when $j=1$ we also require that $R_j(i,\tau-2)$ is good), then

$$\|\mathcal{P}(\tau t_j) - \mathcal{P}((\tau+2)t_j)\|_1 \leqslant c_j \ell_j.$$

So we set $c_1 = \frac{1}{3}$. Let now $\mathcal{P}$ be a feasible path such that $(\mathcal{P}(s),s) \in R_{k+1}^{core}(i,\tau)$, and $R_{k+1}(i,\tau)$ is a good box. Thus there are no pairs of non intersecting bad boxes of level k inside $R_{k+1}(i,\tau)$. Equivalently, by Lemma 4.17, if $R_{k+1}(i,\tau)$ contains at least one bad box, then there exists a bad box $R_k(i',\tau')$ such that for all $R_k(i'',\tau'') \subset R_{k+1}(i,\tau) \setminus R_k^{aff}(i',\tau')$, $R_k(i'',\tau'')$ is good. Inside $R_k^{aff}(i',\tau')$ a feasible path has no restriction on how quickly it can move and it could potentially traverse $S_k^{aff}(i')$ instantaneously. The remaining boxes of scale k that are in $R_{k+1}(i,\tau)$ are good and by the inductive hypothesis we can use that in these ones the maximum displacement of the path is bounded above by $c_k\ell_k$, so it follows that

$$\begin{aligned}\|\mathcal{P}(\tau t_{k+1}) - \mathcal{P}((\tau+2)t_{k+1})\|_1 \leqslant & 9\ell_k + \frac{2t_{k+1}}{2t_k} c_k \ell_k \\ \leqslant & 9\ell_k + k^2 m c_k \ell_k = \left(c_k + \frac{9}{mk^2}\right)\ell_{k+1} =: c_{k+1}\ell_{k+1}.\end{aligned}$$

Where $c_{k+1} = \frac{1}{3} + \frac{9}{m}\sum_{i=1}^{k}\frac{1}{i^2} < 1$ for any k, for m large enough. □

Corollary 4.26. *If $\mathcal{P}$ is a feasible path, for any $s, s' > 0$ such that $s' > s$ and $s' - s \leqslant 2t_k$, and $(\mathcal{P}(t),t) \in R_k(i,\tau)$ for all $t \in [s,s']$ with $R_k(i,\tau)$ being good, then $\|\mathcal{P}(s') - \mathcal{P}(s)\|_1 \leqslant \ell_k$.*

Next we will prove that if a feasible path enters the 2-enlargement of a box $R_k(i,\tau)$ from $\partial_s R_k^{\text{enl2}}(i,\tau)$ and all the k-boxes inside are good, then the path remains far from the box $R_k(i,\tau)$.

Lemma 4.27. *Let $\mathcal{P}$ be a feasible path such that $(\mathcal{P}(t),t) \in \partial_s R_k^{enl2}(i,\tau)$ for some $t \in T_k^{enl2}(\tau)$ and $R_k(i',\tau')$ is good for all $R_k(i',\tau') \subset R_k^{\text{enl2}}(i,\tau) \setminus R_k^{\text{aff}}(i,\tau)$. Then for all $v \in S_k^{enl1}(i)$, for all $t' \in T_k^{enl2}(\tau)$ with $t' \geqslant t$ one has $\|\mathcal{P}(t') - v\|_1 \geqslant 5\ell_k$.*

Proof. By hypothesis every box $R_k(i',\tau') \subset R_k^{enl2}(i,\tau) \setminus R_k^{enl1}(i,\tau)$ is good. For any $t,t' \in T_k^{enl2}(\tau)$, $|t-t'| \leqslant 8 \times 3t_k = 24t_k$. Assume that the path remains inside $S_k^{\text{enl2}}(i)$ during $[t,t']$, if the lemma holds for this path, then the same reasoning applied to every segment of the path completely contained inside $R_k^{\text{enl2}}(i,\tau)$ yields the result. Therefore, from Corollary 4.26 $\|\mathcal{P}(t) - \mathcal{P}(t')\|_1 \leqslant 12\ell_k$. It follows for any $v \in S_k^{enl1}(i)$ one has

$$\|v - \mathcal{P}(t')\|_1 \geqslant \|v - \mathcal{P}(t)\|_1 - \|\mathcal{P}(t) - \mathcal{P}(t')\|_1 \geqslant 17\ell_k - 12\ell_k = 5\ell_k.$$

□

4.3.6 Great Boxes

In this subsection we introduce the concept of k-great boxes, these are *extremely good* boxes of scale 1, as they are surrounded by many good boxes of scale 1 (how many depends on k). Later, we will use the k-great boxes to define the coupling between the processes so that we can attempt to bring them closer to each other.

Definition 4.28. *A box $R_1(i,\tau)$ is said to be k-great if for all $k' \leqslant k$, for all $R_{k'}(i',\tau')$ such that $R^{enl2}_{k'}(i',\tau') \cap R_1(i,\tau) \neq \emptyset$ then $R_{k'}(i',\tau')$ is good. Moreover, we define*

$$G_k := \{(i,\tau) : R_1(i,\tau) \text{ is } k\text{-great}\}$$

will be the set of k-great boxes.

Now, we prove that a feasible path $\mathcal{P}$ that is on $\partial t^- R^{core}_k(i,\tau)$ will cross $C\frac{t_k}{t_1}$ distinct k-great boxes, for some constant C, provided $R_k(i,\tau)$ as well as *enough* neighbouring boxes are good. This result will be important later because it will allow us to effectively count how many times we can try to change the distance between the walkers in the coupling.

Lemma 4.29. *Let $\mathcal{P}$ be a feasible path and suppose $(\mathcal{P}(\tau t_k), \tau t_k) \in \partial t^- R^{core}_k(i,\tau)$ such that all k-boxes $R_k(i',\tau')$ with $R^{\text{enl2}}_k(i',\tau') \cap R_k(i,\tau) \neq \emptyset$ are good. Then there exists $C_3 > 0$ such that letting $\kappa = C_3\frac{t_k}{t_1}$ we can find times $s_1 < \cdots < s_\kappa$ and distinct space-time indices $(i_1,\tau_1),\ldots,(i_\kappa,\tau_\kappa)$ for which $(\mathcal{P}(s_j), s_j) \in \partial t^- R^{\text{core}}_1(i_j,\tau_j)$ for all j. Moreover, we have that $R_1(i_j,\tau_j) \subset R_k(i,\tau)$ are k-great for all j, and $R_1(i_j,\tau_j) \cap R_1(i_{j'},\tau_{j'}) = \emptyset$ for all $j \neq j'$.*

Proof. We prove the statement of the lemma for $C_3 = c_k$ being a function of k, then the lemma follows by showing that there is a universal value C_3 such that $0 < C_3 \leq c_k$ for all k. We will do a proof by induction on k. Case $k = 1$ is trivially verified by choosing $c_1 = 1$ because $R_1(i,\tau)$ is 1-great.

Assume it is true up to scale k and now consider a feasible path that crosses $\partial t^- R^{core}_{k+1}(i,\tau)$ such that every box of scale $k+1$ whose 2-enlargement intersects $R_{k+1}(i,\tau)$ is good. By Lemma 4.17 the bad boxes of scale k inside $R_{k+1}(i,\tau)$ are all contained in $R^{\text{aff}}_k(i',\tau')$ for some i',τ'; the boxes of level 1, that are in at least one of the 2-enlargement of the boxes contained in $R^{\text{aff}}_k(i',\tau')$, are by definition not k-great. By Lemma 4.25 we know that $\mathcal{P}$ crosses $\partial^+_t R_{k+1}(i,\tau)$ before $\partial_s R_{k+1}(i,\tau)$. In words the path stays for $2t_{k+1}$ time in the box $S_{k+1}(i)$. Therefore, by the fact that the bad boxes of scale k inside $R_{k+1}(i,\tau)$ are all contained in $R^{\text{aff}}_k(i',\tau')$ for some i',τ', we know that at most $24t_k + 2t_k$ of the $2t_{k+1}$ will be spent in the enlargement of a bad box $R^{\text{enl2}}_k(i',\tau')$.

The path will spend at least $2t_{k+1} - 26t_k$ time in good boxes of level k. Starting on $\partial^-_t R^{core}_{k+1}(i,\tau)$, we can say it starts on $\partial^-_t R^{core}_k(i'',\tau'')$ for some i'' and τ'' and here applying the inductive hypothesis we know after $2t_k$ the path has gone through at least $c_k\frac{t_k}{t_1}$ distinct k-great boxes. When the path reaches $\partial^+_t R_k(i'',\tau'')$ we have that the path is now on $\partial^-_t R_k(i''+j,\tau''+1)$ for some $j \in \{0,1\}^d$ and from here we can reapply the inductive hypothesis. We only need to count how many times we can do this before $2t_{k+1}$ time has passed:

the path will traverse at least $\frac{2t_{k+1}-26t_k}{2t_k}$ good boxes R_k of level k, which means that, by the inductive hypothesis, the path will traverse at least $c_k\frac{t_{k+1}-13t_k}{t_1} = (1-\frac{13}{mk^2})c_k\frac{t_{k+1}}{t_1}$ good boxes of level 1.

We set $c_{k+1} = c_k(1-\frac{13}{mk^2})$, $c_\infty = c_1\prod_{i=1}^\infty(1-\frac{13}{mi^2}) > 0$, and the lemma is proved for $C_3 = c_\infty$. □

4.4 Overview of the Proof

In this section we give a high-level description of the proof. Consider two processes $\{\tilde{M}_t^1\}_{t\geq 0} = \{(X_t^1, \tilde{\eta}_t^1)\}_{t\geq 0}$ and $\{\tilde{M}_t^2\}_{t\geq 0} = \{(X_t^2, \tilde{\eta}_t^2)\}_{t\geq 0}$ with starting states $\tilde{M}_0^1, \tilde{M}_0^2 \in \mathbb{T}_n^d \times \{0,1\}^{E(\mathbb{T}_n^d)}$. We will construct a coupling of the two processes so that for some random stopping time T with mean of order at most $\frac{n^2}{\mu}$ the two configurations agree with positive probability. Since $\{M_t\}_{t\geqslant 0}$ can be recovered from $\{\tilde{M}_t\}_{t\geqslant 0}$, by sampling the edges with status $\star$, we will obtain our result.

The coupling will consist of three different phases which we will describe in a high level way below. The detailed analysis of each phase will be given in sections 4.5, 4.6 and 4.7. Then in Section 4.8, we will put all phases together and complete the proof of Theorem 4.1.

4.4.1 First phase: the local coupling

Denote by $X_t + e$ the edge e translated in $E(\mathbb{T}_n^d)$ by the vector X_t, or equivalently, $X_t + e$ is the edge that corresponds to e in the graph $\mathbb{T}_n^d$ centered in X_t. During the first phase we let the two processes evolve *independently*, and wait for the first time the graphs of the two processes agree on a ball of radius 3ℓ around the walkers, that is, we wait for a time t such that

$$\tilde{\eta}_t^1(X_t^1 + e) = \tilde{\eta}_t^2(X_t^2 + e),$$

for all edges $e \in E(B_{3\ell}(0))$, where $B_r(x)$ is the vertices inside the ball of radius r around x. We will show in Lemma 4.30 that this time will happen within time $\frac{C_4}{\mu}$ with probability at least $\frac{1}{10}$, where C_4 is a fixed, large enough constant. If the first phase does not end within time $\frac{C_4}{\mu}$, we declare the whole three-phase procedure to have failed. More details will be given in Section 4.5.

4.4.2 Second phase: the non-Markovian coupling of the walkers

After the first phase has been completed successfully, the graphs of the two processes are the same on a ball of radius 3ℓ around the walkers. Then, in the second phase we wish to couple the motion of the walkers. We use the information from the entire tessellation (which tells us whether $R_1(i,\tau)$ are good or bad for each i and for each τ which identifies a box that belongs to the second phase) to decide when to couple the walkers identically (so that they jump in the same way) and when to perform a better coupling aiming to bring the walkers together. Intuitively, whenever the walkers are passing through a "bad" region of the environment (which in our case will be the 2-enlargement of a bad box) we will just do the identity coupling to make sure the distance between the walkers does not increase. In fact, this will only happen because we will use the space between the enlargement of the bad box and the bad box itself (which is a good region of the environment) to ensure that the graph around the walkers is equal in both configurations, allowing the identity coupling to be carried out. If, instead the two walkers are in a great box, then we try to do what we call a **simple random walk moment**.

More precisely, to do this, at the beginning of the second phase we create the multi-scale tessellation described in Section 4.3 up to time $2c_2\frac{n^2}{\mu}$. We will fix a largest scale $k_{\max}$ and will look at how many times the walkers enter $k_{\max}$-great boxes. When the walkers are in great boxes, Lemma 4.43 will give that the environment is favourable enough so that with probability $C_7(p) > 0$ the displacement of the walkers will have the same distribution as that

one of a simple random walk on the graph $\mathbb{T}_n^d$ (i.e., where all edges are open). Phase 2 ends at time $c_2\frac{n^2}{\mu}$ where we check whether the walkers are coupled and the graphs are coupled on a ball of radius 3ℓ around the walkers. Lemma 4.29 says that the walkers will cross at least c_3n^2 great boxes and therefore after time $c_2\frac{n^2}{\mu}$ the walkers are expected to have done about $C_7(p)c_2c_3n^2$ simple random walk steps. The constants can be tuned so that with probability at least $\frac{1}{8}$ phase 2 ends successfully. More details will be given in Section 4.6.

4.4.3 Third phase: the coupling of the graphs

At the beginning of the third phase the walkers are coupled and the graphs are coupled as well on a ball of radius 3ℓ around them. We note that even though the walkers are coupled, we are not yet done because since the graphs are not yet guaranteed to be equal everywhere, it could happen that at some point we cannot perform the identity coupling of the walkers. The idea of this phase is exactly to keep performing identity coupling while the graphs couple together. There is another caveat. In the second phase, we needed to construct the tessellation all the way to time $2c_2\frac{n^2}{\mu}$, while the second phase ends already at time $c_2\frac{n^2}{\mu}$. The reason for this is that, in order to know whether we can perform a simple random walk moment, we need to observe a little bit of future information about the enlargement. Therefore, as we performed the second phase, we observed some information from the updates after the end of phase two. The coupling time of the two process must be a stopping time. So we also use the third phase as a well to let time pass until we get to a point where no information regarding future times have been observed.
During the third phase we then aim to keep the walkers coupled at all times, while we finish to couple the graphs before time $2c_2\frac{n^2}{\mu}$. We do not use any further information from the tessellation. The delicate point is that in order to apply identity coupling of the walkers, as we explained in the second phase, we have to ensure that the graphs around the walkers are coupled. We were doing this by observing good and bad boxes, but now we cannot observe anything beyond what we have already observed; otherwise we will keep observing future information and will never get to a stopping time. So we just proceed with the identity coupling "blindly". That is, we perform identity coupling assuming that any information that we have not yet observed is good, and just hope for the best. It will turn out that this procedure succeeds with probability at least $\frac{1}{2}$ leaving the two processes completely coupled (both the graphs and the walkers).
If any of the three phases does not successfully end we let the two processes run independently until the end of the third phase (this is needed as we might have observed some information about the environment up to that time), and then repeat the procedure from phase 1. Since the three phases succeed with positive probability, we only need to repeat the whole procedure a geometric number of times. More details will be given in Section 4.7.

4.5 The First Phase

During the first phase we let the processes $\tilde{M}_t^1 = (X_t^1, \tilde{\eta}_t^1)$ and $\tilde{M}_t^2 = (X_t^2, \tilde{\eta}_t^2)$ evolve independently.
Let $\Psi_t : V \to V$ be the translation that maps X_t^1 into X_t^2, we will abuse notation and use the same Ψ_t to represent the translation map $E \to E$ of the edges. Let

$$E(B_r(v)) = \{e = (v_1, v_2) \in E : \|v - v_1\|_1 \leqslant r, \text{ and } \|v - v_2\|_1 \leqslant r\}$$

be the ball of edges with endpoints at graph-distance less than r from the vertex v. Define the event

$$\mathcal{B}_t := \left\{\forall e \in E(B_1(X_t^1)),\, \tilde{\eta}_t^1(e) = \tilde{\eta}_t^2(\Psi_t(e)) = 0\right\} \cap \left\{\forall e \in E(B_{3\ell}(X_t^1) \setminus B_1(X_t^1)),\, \tilde{\eta}_t^1(e) = \tilde{\eta}_t^2(\Psi_t(e)) = \star\right\}$$

that the edges in a ball of radius 3ℓ around the walkers are all $\star$ at time t, besides the ones adjacent to the walkers, which are closed. Let

$$\tau_B := \inf\{t \geqslant 0 : \mathcal{B}_t \text{ holds}\}. \tag{4.16}$$

Define $T_1 := \frac{C_4}{\mu}$, for some constant $C_4(p) > 0$, and define the event

$$F_1 := \{\tau_B < T_1\}, \tag{4.17}$$

which we shall take as the event that phase 1 succeeds. This event is a bit more restricted than the one announced in the previous section, but this will be convenient for us in the next phase. If it turns out that F_1 does not occur, we declare the whole procedure to have failed at time T_1 and do not proceed to the second phase. Now we show that phase 1 succeeds with good probability.

Lemma 4.30. *There exists $p_0 > 0$ such that for any $p < p_0$, there exists $C_4(p) > 0$ in the definition of T_1 so that for any $\tilde{\eta}_0^1, \tilde{\eta}_0^2 \in \Omega$ we obtain*

$$\mathbb{P}(F_1^c) \leqslant \frac{1}{10}.$$

Proof. For each process $\tilde{M}_t^1$ and $\tilde{M}_t^2$, create a tessellation from time 0 to time T_1 of boxes of scale 1 only, like the one defined in Section 4.3. There will be no need for boxes of scale larger than 1 at this phase, and for scale 1 boxes we will use the value for ℓ in (4.3). All boxes in the tessellation will be entirely contained in $\mathbb{T}_n^d \times [0, T_1]$. Then let the two processes $\tilde{M}_t^1$, $\tilde{M}_t^2$ evolve independently until the first walker enters a box $R_1^{\text{core}}(i, \tau)$ with $\tau \geqslant 2$ (to avoid dependence from time 0), and the second walker is at a box $R_1^{\text{core}}(i', \tau)$ such that every box contained in $R_1^{\text{enl1}}(i, \tau)$ and every box contained in $R_1^{\text{enl1}}(i', \tau)$ is good, call this event E_{enl}. Then once the two walkers find themselves in boxes satisfying the conditions above, we check whether the edges adjacent to X_s^1, X_s^2 at time $s = \tau t_1 + |T_1^1(\tau)|$ remain closed in their respective graphs until $(\tau + 2)t_1$. Call this latter event E_{stuck}. Essentially, if the first walker enters a box $R_1(i, \tau)$ and the second walker is in $R_1(i', \tau)$, for some i, i', τ, that are good and that every box in their enlargement is good then by Lemma 4.11 the walkers will remain in their box for the whole time interval of the box. Thus, by the event A_2 in Definition 4.4 at time $\tau t_1 + |T_1^1(\tau)|$ each walker will get stuck in a vertex, namely the edges adjacent to the walkers will be closed. If E_{stuck} happens, then the walkers will remain stuck in their position until $(\tau + 2)t_1$ and at the same time the edges in a ball of radius 3ℓ around them will refresh $\star$. This latter fact is guaranteed by the boxes where the walkers are located being good. If the above events all happen simultaneously in the two processes, at time $(\tau + 2)t_1$, the two configurations $\tilde{\eta}^1_{(\tau+2)t_1}$ and $\tilde{\eta}^2_{(\tau+2)t_1}$ agree on the balls $E(B_{3\ell}(X^1_{(\tau+2)t_1}))$ and $E(B_{3\ell}(X^2_{(\tau+2)t_1}))$. Now, up to time T_1 the walkers must cross at least $\frac{T_1\mu}{10\sqrt{\ell}} = \frac{C_4}{10\sqrt{\ell}}$ distinct $R_1^{\text{core}}(\cdot, \cdot)$ for which the 1-enlargements are disjoint. Consider a sequence of $R_1^{\text{core}}(i_j, \tau_j)$ with $j = 1, \ldots, \frac{C_4}{10\sqrt{\ell}} - 1$ whose 1-enlargement are disjoint and $0 \notin T_1(\tau_j)$. We can choose $C_4(p)$ large enough so that

$\frac{C_4}{10\sqrt{\ell}} - 1 > 1$. Denote $E_j := E_{\text{enl}} \cap E_{\text{stuck}}$ the event that every box in $R_1^{\text{enl1}}(i_j, \tau_j)$ is good and that E_{stuck} happens. Using the definition of p_{ce} in (2.15)

$$\mathbb{P}(E_{\text{stuck}}) \geqslant e^{-2dp_{ce}(2\sqrt{\ell})} \geqslant e^{-\frac{4d}{q}p\sqrt{\ell}};$$

We recall the choice we made for ℓ in (4.3) and notice that because the boxes $R_1(i_j, \tau_j)$ are well separated, the E_j are independent (recall Definition 4.4, that we defined almost good as an event that depends only on the update within the box itself and therefore independent of what happens outside of it). Thus

$$\begin{aligned}
\mathbb{P}(\tau_B > T_1) &\leqslant \mathbb{P}\left(\bigcap_{j=1}^{\frac{C_4}{10\sqrt{\ell}}-1} E_j^c\right) \\
&= \prod_{j=1}^{\frac{C_4}{10\sqrt{\ell}}-1} \mathbb{P}\left(E_j^c \middle| \bigcap_{r=1}^{j-1} E_r^c\right) \\
&= \mathbb{P}(E_1^c)^{\frac{C_4}{10\sqrt{\ell}}-1} \\
&= \mathbb{P}(E_{\text{stuck}}^c \cup E_{\text{enl}}^c)^{\frac{C_4}{10\sqrt{\ell}}-1} \\
&\leqslant 3^{d+2}\rho_1 + \left(1 - e^{-\frac{4d}{q}p\sqrt{\ell}}\right) \\
&< \frac{1}{10},
\end{aligned}$$

choosing p_0 small enough. □

4.6 The Second Phase / The Non Markovian Coupling

To describe the coupling during the second phase we will use the full multi-scale space-time tessellation described in Section 4.3.

4.6.1 Largest scale

We will begin by creating the multi-scale space-time tessellation of $\mathbb{T}_n^d \times [\tau_B, \tau_B + T_2]$ and with largest scale

$$k_{max} := \log_2 \log n,$$

$T_3 := T_2 + \frac{n^2}{\mu}$ and $T_2 := C_5 \frac{n^2}{\mu}$, where $C_5(p) > 0$ is a positive constant to be chosen later so that $t_{k_{\max}}$ divides $C_5 \frac{n^2}{\mu}$. In order to simplify the notation we will translate time so that the interval $[\tau_B, \tau_B + T_2]$ will be in this section referred to as the interval $[0, T_2]$. The following Lemma shows that with large probability there are no bad boxes of scale $k_{\max}$ or larger. This will allow us to restrict our analysis to boxes of scale at most $k_{\max}$. We will consider all the boxes contained within the tessellation $\mathbb{T}_n^d \times [0, T_2 + \frac{n^2}{\mu}]$, which in particular include all the boxes intersecting the tessellation of $\mathbb{T}_n^d \times [0, T_2]$.

Lemma 4.31. *Let $T = (C_5 + 1)\frac{n^2}{\mu}$, there exists p_0 such that for all $p < p_0$ and n large*

$$\mathbb{P}(R_k(i,\tau) \text{ is bad for some } R_k(i,\tau) \subset \mathbb{T}_n^d \times [0,T], \text{ with } k \geqslant k_{\max}) \leqslant \frac{1}{10}.$$

Proof. We start by deriving bounds on ℓ_k and t_k, the size of the boxes of scale k. When

$k \geqslant k_{\max}$, we can choose n large enough so that for any m, ℓ fixed

$$2\ell k^{2k} \leqslant \ell_k = m^k (k!)^2 \ell \leqslant \ell k^{3k},$$
$$2\sqrt{\ell} k^{2k} \leqslant \mu t_k = m^k (k!)^2 \sqrt{\ell} \leqslant \sqrt{\ell} k^{3k}.$$

Therefore the number ζ_k of boxes of scale k in $\mathbb{T}_n^d \times [0,T]$ is bounded above and below by

$$\zeta_k \geqslant \left(\frac{n}{3\ell_k}\right)^d \frac{T}{3t_k} \geqslant \frac{(C_5+1)n^{d+2}}{9^{d+1}\ell^{d+\frac{1}{2}}k^{k(3d+3)}},$$
$$\zeta_k \leqslant 1 + \left(\frac{n}{\ell_k}\right)^d \frac{T}{t_k} \leqslant 1 + \frac{(C_5+1)n^{d+2}}{\ell^{d+\frac{1}{2}}k^{k(2d+2)}}.$$

In the upper bound of ζ_k we add a 1 to the fraction to consider the case when k is so large that we cannot find a box all contained in the tessellation. Using Lemma 4.15 the probability that there exists a box of scale $k_{\max}$ or bigger that is bad is bounded above by

$$\sum_{k \geqslant k_{\max}} \zeta_k \rho_k \leqslant \sum_{k \geqslant k_{\max}} \zeta_k \rho_1^{2^{k-2}},$$

moreover using the inequalities above for ζ_k it is easy to see that, for any $k \geqslant k_{\max}$,

$$\zeta_k \rho_1^{2^{k-2}} \leqslant 2^{-(k-k_{\max})} \zeta_{k_{\max}} \rho_1^{2^{k_{\max}-2}},$$

so that

$$\begin{aligned}\sum_{k \geqslant k_{\max}} \zeta_k \rho_k &\leqslant 2\zeta_{k_{\max}} \rho_1^{2^{k_{max}}-2} \\ &\leqslant 2\frac{(C_5+1)}{\ell^{d+\frac{1}{2}}} n^{d+2} \rho_1^{2^{\log\log n}-2} \\ &\leqslant \frac{1}{10},\end{aligned}$$

where we choose p_0 small enough so that Lemma 4.15 can be applied and $\rho_1 < e^{-2d}$. □

4.6.2 The coupling

During this phase we will change the map Ψ_t only whenever the walker is not in the enlargement of a bad box. More precisely, denote with

$$\bar{s}_t = \sup\{s \leqslant t \,:\, (X_s^1, s) \text{ is in a } k_{\max} \text{ great box}\}$$

the last time before t the walker is in a $k_{\max}$ great box, we will consider Φ_t such that

$$\Phi_t := \Psi_{\bar{s}_t}.$$

Later we will prove that $\Phi_t \equiv \Psi_t$ for all t because when the walkers are in the enlargement of a bad box their distance does not change and therefore the translation map remains constant. As soon as the second phase begins we check whether the box $R_1(i,0)$, such that $(X_0, 0) \in R_1^{\text{core}}(i,0)$, is $k_{\max}$-great (the reason we do this will be clarified later, see Remark 4.34). If that is the case then we can begin the coupling procedure relative to the second

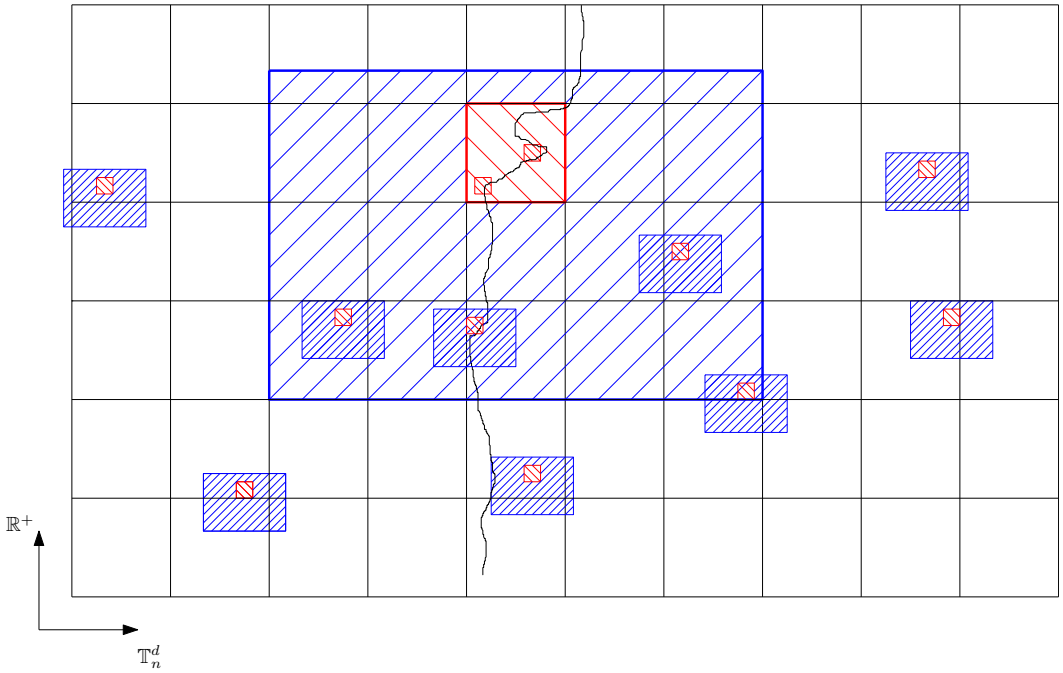


FIGURE 4.4: In red the bad boxes, in blue their enlargement, in black the tessellation and the walker's trajectory. In bad boxes there is no control over the displacement of the walker, whereas in good boxes, the walker always leaves the box only from its time boundary. Whenever the walker enters the enlargement of a bad box we start doing identity coupling, otherwise, in great boxes, if a SRWM (see Definition 4.39) occurs we do a simple random walk coupling, if not we keep doing identity coupling.

phase.

The coupling is composed of two parts: the coupling of the graphs (that is, the coupling of $\tilde{\eta}_t^1$ and $\tilde{\eta}_t^2$) and the coupling of the walkers. We first describe the coupling of the graphs.

We let the process $\{\tilde{\eta}_t^1\}_{t\geq 0}$ evolve. Denote with $\mathcal{C}_v^i(t)$ the cluster that contains vertex v at time t in the process $\tilde{\eta}_t^i$. When an update (s, U', U) occurs at an edge e we update the process $\{\tilde{\eta}_t^2\}_{t\geq 0}$ as follows.

- If the update is a $\star$-update we refrain from looking at U and instead simply set $\tilde{\eta}_s^1(e) = \star$ and $\tilde{\eta}_s^2(\Phi_s(e)) = \star$.
- If the update is not a $\star$-update we must check in both configurations $\tilde{\eta}_s^1$ and $\tilde{\eta}_s^2$ whether e is a cut-edge or not. We do this by looking at the connected components of the endpoints v_1, v_2 of the edge e. If an edge e' is such that $\tilde{\eta}_s^1(e') = \star$ and $e' \in \mathcal{C}_{v_1}^1(s) \cup \mathcal{C}_{v_2}^1(s)$, we sample its current status, open or closed, according to its last update. Note that this last update is itself a tuple $(\bar{s}, \bar{U}', \bar{U})$, so this step boils down to checking the value of $\bar{U}$. If $\tilde{\eta}_s^2(\Phi_s(e')) = \star$ we set $\tilde{\eta}_s^2(\Phi_s(e')) = \tilde{\eta}_s^1(e')$ as well. We continue this procedure until the components of v_1 and v_2 have been fully explored in $\tilde{\eta}_s^1$ and proceed analogously for the process $\tilde{\eta}_s^2$ until the components of $\Phi_s(v_1)$ and $\Phi_s(v_2)$ have been fully explored. A potential disagreement $\tilde{\eta}_s^1(e) \neq \tilde{\eta}_s^2(\Phi_s(e))$ can happen only if by revealing the components of v_1 and v_2 we find that e is a cut-edge in $\tilde{\eta}_s^1$ but not in $\tilde{\eta}_s^2$ or vice-versa.

In this way edges whose status is $\star$ can always be coupled equivalently whereas non $\star$-updates cause the reveal of the status of other edges, potentially creating disagreement between the two configurations. This concludes the coupling of the graphs.

We now describe the coupling of the walkers; during this discussion the reader should refer to Figure 4.4.

Our goal is to define a coupling that can bring the walkers together. For this we will use the multi-scale tessellation. When the walker X_s^1 enters the core of a great box $R_1^{\text{core}}(i,\tau)$, we will try to take advantage of this to perform a coupling that we refer to as a *simple random walk moment*. On the other hand, whenever X_s^1 is not in a great box, then we do not have a good enough control on the environment around the walker to do this coupling; instead we will just resort to an identity coupling that keeps the distance between the walkers unchanged. We proceed to the details. Consider the event

$$\mathcal{B}'_t := \left\{\forall e \in E(B_{3\ell}(X_t^1)),\, \tilde{\eta}_t^1(e) = \tilde{\eta}_t^2(\Phi_t(e))\right\}. \tag{4.18}$$

We now prove that, by doing identity coupling, as long as the particle X_t^1 is in a point $(v,t) \in V(\mathbb{T}_n^d) \times \mathbb{R}^+$ in space-time that is part of a box $R_1(\cdot,\cdot)$ that is good, it is always possible to keep the distance between X_t^1 and X_t^2 constant.

Lemma 4.32. *Assume $(X_{s_1}^1, s_1) \in R_1^{\text{core}}(i,\tau)$, for some $s_1 \in T_1(\tau)$, such that $R_1(i',\tau')$ is good for all $R_1(i',\tau') \subset R_1^{\text{enl1}}(i,\tau)$ and $\mathcal{B}'_{s_1}$ holds. Let $s_2 \in T_1(\tau)$, $s_2 \geqslant s_1$. If we attempt to do identity coupling for the entire time interval $[s_1,s_2]$, then $\Psi_{s_1} \equiv \Psi_{s_2}$. Moreover, if $s_2 - s_1 \geqslant t_1$, then $\tilde{\eta}_t^1(e) = \tilde{\eta}_t^2(\Phi_t(e))$ for all $t \in (s_1+t_1, s_2)$ and all $e \in E(S_1^{\text{enl1}}(i))$.*

Proof. Let B be the ball of radius 3ℓ around $X_{s_1}^1$ (B is a fixed region is space, not changing as time goes). The edges in $E(B)$ are coupled at time s_1, by definition of $\mathcal{B}'_{s_1}$. By Lemma 4.11 the displacement of the walker at time $(\tau+2)t_1$ is not more than $\frac{\ell}{3}$, so the walker never leaves B, during the time interval (s_1,s_2). Moreover, because $R_1(i',\tau')$ is good for all $R_1(i',\tau') \subset R_1^{\text{enl1}}(i,\tau)$, there is no non-$\star$ update in $E(B)$ during (s_1,s_2), therefore B remains coupled throughout. Hence, identity coupling is successful.
Finally, we notice that $R_1(i',\tau)$ is good for all $R_1(i',\tau) \subset R_1^{\text{enl1}}(i,\tau)$. It follows that during (s_1,s_2) every $e \in E(S_1^{\text{enl1}}(i))$ has never received a non-$\star$ update. Therefore, if $s_2 - s_1 < t_1$, for all $e \in E(S_1^{\text{enl1}}(i))$, for all $t \in (s_1,s_2)$,

$$\tilde{\eta}_t^1(e) = \tilde{\eta}_t^2(\Phi_t(e)).$$

Whereas, if $s_2 - s_1 \geqslant t_1$, every $e \in E(S_1^{\text{enl1}}(i))$ receives at least one $\star$-update during (s_1, s_1+t_1), by event A_2 in the Definition 4.4 of almost good box, so the claim of the lemma holds. □

Lemma 4.33. *Let $R_1(i,\tau)$ be a good box and suppose that for some $s_1 \in T_1(\tau)$*

$$\tilde{\eta}_{s_1}^1(e) = \tilde{\eta}_{s_1}^2(\Phi_{s_1}(e)),\ \text{for all } e \in E(S_1(i)). \tag{4.19}$$

Let $s_2 \in T_1(\tau)$, $s_2 \geqslant s_1$. If we do identity coupling for the entire time interval $[s_1,s_2]$ and $\Psi_t \equiv \Psi_{s_1}$ for all $t \in [s_1,s_2]$, then $\tilde{\eta}_t^1(e) = \tilde{\eta}_t^2(\Phi_t(e))$ for all $e \in E(S_1(i))$ for all $t \in (s_1,s_2)$.

Proof. For all $s_2 \in T_1(\tau)$, $s_2 \geqslant s_1$, if the map Ψ_{s_1} does not change, meaning the identity coupling is successful, revealed edges do not create disagreements. Moreover, the updates occurring on each $e \in E(S_1(i))$ at all times in $T_1(\tau)$ are $\star$, so that $\tilde{\eta}_t^1(e) = \tilde{\eta}_t^2(\Phi_t(e))$ for all $e \in E(S_1(i))$ for all $t \in (s_1,s_2)$. □

Remark 4.34. *The 2-enlargement of a bad box is chosen so that whenever the walker crosses it, by doing identity coupling the two processes have time to couple the environment before the walker crosses the bad box. For this exact reason we want the first box whose core the walker is at, at the beginning of the second phase, to be $k_{\max}$-great, so we know that the walker does*

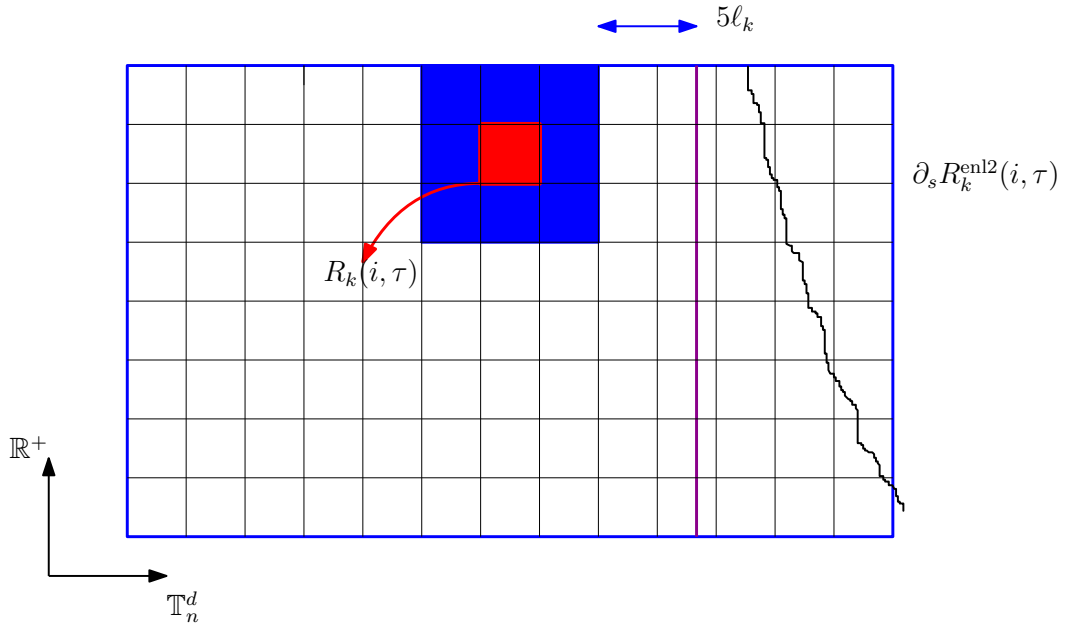


FIGURE 4.5: In red the bad box $R_k(i,\tau)$, in blue its enlargements, in black the tessellation and the walker's trajectory. If the walker traverses $\partial_s R_k^{\mathrm{enl2}}(i,\tau)$, then it will never reach $R_k(i,\tau)$ and it only traverses good boxes of scale k.

not start inside the enlargement of a bad box, meaning that if the walker enters into a bad box during the second phase, it must first traverse its enlargement.

Now we prove that if the walkers were to cross the 2-enlargement of a bad box of scale 1 from the space boundary $\partial_s R_1^{\mathrm{enl2}}(\cdot,\cdot)$, then the walkers would never get closer than 3ℓ to the bad box and $\mathcal{B}'_t$ would still be verified for all t and, as long as the walkers are in the 2-enlargement, the relative distance between them does not change, see Figure 4.5. Whereas if they were to cross it from $\partial_t^- R_1^{\mathrm{enl2}}(\cdot,\cdot)$, they would be able to reach the bad box but the environment in the 2-enlargement of the bad box is coupled at all times in $T_1^{\mathrm{enl1}}(\cdot,\cdot)$ thanks to the abundance of $\star$-updates in $T_1^{\mathrm{enl2}}(\cdot,\cdot) \setminus T_1^{\mathrm{enl1}}(\cdot,\cdot)$, see Figure 4.6. Moreover if the map Φ_t does not change the environments remain coupled also for all the bad boxes the walkers are not crossing: this is because the abundance of $\star$-updates in the 2-enlargement couples the graphs before any non-$\star$ updates occur in the bad box (the non-$\star$-updates do not create disagreement if the environments are coupled), furthermore, as long as the map does not change, revealed edges remain coupled in the two processes. These reasonings prove that the relative distance between the walkers does not change and the graphs remain coupled in $E(S_1^{\mathrm{enl2}}(i))$ when the walker cross a bad box of scale 1.

Denote with $\tau_1^+ = \max T_1^{\mathrm{enl2}}(\tau)$, and with $\tau_1^- = \min T_1^{\mathrm{enl2}}(\tau)$. Moreover, let $s_c := \inf\{t : t \in T_1^{\mathrm{enl2}}(\tau),\, X_t^1 \in S_1^{\mathrm{enl2}}(i)\}$ be the first time the walker crosses $R_1^{\mathrm{enl2}}(i,\tau)$, and $s_e := \inf\{t : s_c < t \leqslant \tau_1^+,\, X_t^1 \notin S_1^{\mathrm{enl2}}(i)\}$ the first time the walker exits $R_1^{\mathrm{enl2}}(i,\tau)$ after having entered it; we take the convention that $s_c = +\infty$ (as well as $s_e = +\infty$) if the walker does not enter $S_1^{\mathrm{enl2}}(i)$ during the time interval $T_1^{\mathrm{enl2}}(\tau)$, and we take the convention that $s_e = \tau_1^+$ if $X_t^1 \in S_1^{\mathrm{enl2}}(i)$ for all $t \in (s_c, \tau_1^+]$.

Lemma 4.35. *Let $R_1(i,\tau)$ be a bad box of scale 1 such that $R_2(i',\tau')$ is good for all $R_2(i',\tau') \supset R_1(i,\tau)$. Assume $\mathcal{B}'_{s_c}$ happens. Thus*

$$\text{if } s_c < \infty, \text{ then we may choose } \Psi_t \text{ to remain constant for all } s_c \leqslant t \leqslant s_e. \tag{4.20}$$

Moreover, let $s \in T_1^{\mathrm{enl2}}(\tau)$, and assume $\Phi_t \equiv \Phi_{\tau_1^-}$ for all $t \in [\tau_1^-, s]$; we do not assume in this

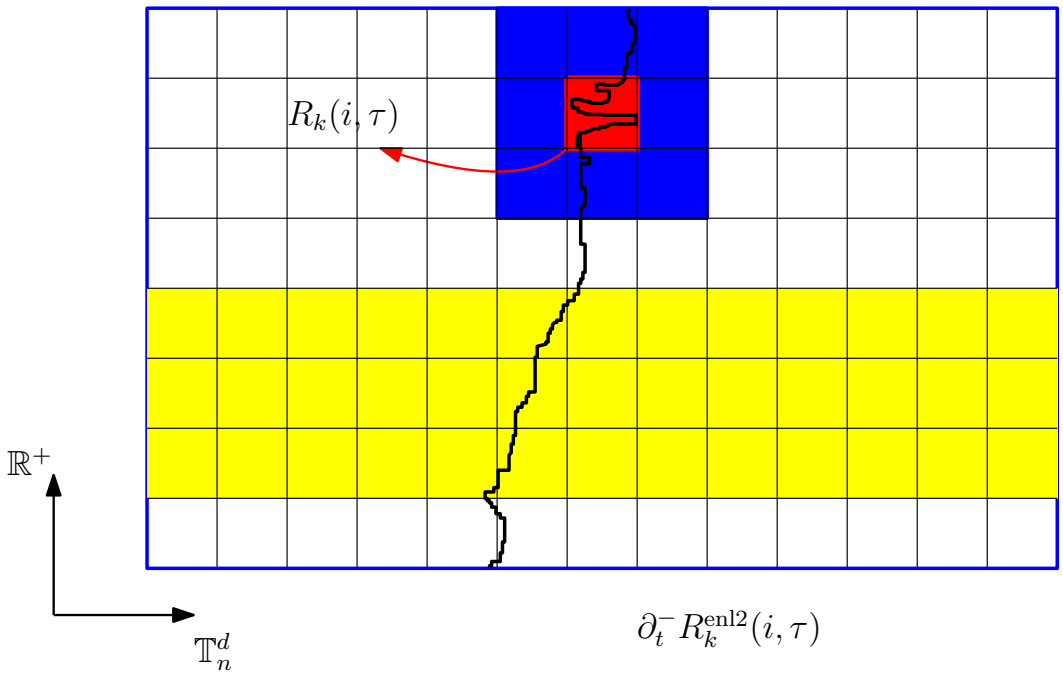


FIGURE 4.6: In red the bad box $R_k(i,\tau)$, in blue its enlargements, in black the tessellation and the walker's trajectory. If the walker traverses $\partial_t^- R_k^{\mathrm{enl2}}(i,\tau)$, then the environment is recoupled (during the yellow area) before the walker crosses $R_k(i,\tau)$.

part that the walker enters $R_1^{\mathrm{enl2}}(i,\tau)$, so s_c, s_e could be equal to ∞, we only assume that the map Φ_t does not change. Then

$$\tilde{\eta}_t^1(e) = \tilde{\eta}_t^2(\Phi_t(e)) \quad \textit{for all } t \in [\tau_1^- + t_1, s], \textit{ and all } e \in E(S_1^{\mathrm{enl2}}(i)) \tag{4.21}$$

and, in addition, if $\tilde{\eta}_{\tau_1^-}^1(e) = \tilde{\eta}_{\tau_1^-}^2(\Phi_{\tau_1^-}(e))$ for all $e \in E(S_1^{\mathrm{enl2}}(i))$, then (4.21) occurs for all $t \in [\tau_1^-, s]$.

Remark 4.36. *In this lemma we defined $[s_c, s_e]$ as the time interval of the* first *excursion of the walker through $R_1^{\mathrm{enl2}}(i,\tau)$, but if the walker enters and exits $R_1^{\mathrm{enl2}}(i,\tau)$ several times, then (4.20) clearly holds by defining s_c and s_e as the time interval of any such excursion.*

Proof. We prove that crossing the 2-enlargement of the bad box from $\partial_s R_1^{\mathrm{enl2}}(i,\tau)$ ($s_c > \tau_1^-$), the walker never gets closer than 5ℓ from $S_1^{\mathrm{enl1}}(i)$, and, because it traverses only good boxes of scale 1, we can apply the previous two lemmas, see Figure 4.5; on the other hand if $s_c = \tau_1^-$, the abundance of $\star$ updates let us couple the graph before $\min T_1^{\mathrm{enl1}}(\tau)$, see Figure 4.6.

Suppose $s_c > \tau_1^-$. From Lemma 4.27, for all $t \in T_1^{enl2}(\tau)$, for all $v \in S_1^{enl1}(i)$, $\|X_t^1 - v\| \geqslant 5\ell$. Moreover, because every box in $R_1^{\mathrm{enl2}}(i,\tau) \setminus R_1^{\mathrm{enl1}}(i,\tau)$ is good, Lemma 4.32 together with Lemma 4.33 yields the lemma.

Suppose now $s_c = \tau_1^-$. The statement is easily verified if $s_e < \min T_1^{\mathrm{enl1}}(i,\tau)$ as the walker traverses only good boxes of scale 1, therefore we can simply apply Lemma 4.32. Whereas if $s_e > \min T_1^{\mathrm{enl1}}(\tau)$, every $e \in E(S_1^{\mathrm{enl2}}(i))$ receives at least a $\star$ update during $(\tau_1^-, \tau_1^- + t_1)$, so that by $\tau_1^- + t_1$ they are all coupled, namely $\tilde{\eta}_{\tau_1^- + t_1}^1(e) = \tilde{\eta}_{\tau_1^- + t_1}^2(\Phi_{\tau_1^- + t_1}(e))$ for all $e \in E(S_1^{\mathrm{enl2}}(i))$. Moreover, before $\min T_1^{\mathrm{enl1}}(i,\tau)$ they only receive $\star$-updates, so that $\tilde{\eta}_t^1(e) = \tilde{\eta}_t^2(\Phi_t(e))$ for all $e \in E(S_1^{\mathrm{enl2}}(i))$, for all $t \in (\tau_1^- + t_1, \min T_1^{\mathrm{enl1}}(i,\tau))$. We use that Φ_t does not change, then non-$\star$ updates during $T_1^{\mathrm{aff}}(\tau)$ cannot create disagreements. In fact, suppose $e = (v_1, v_2) \in E(S_1^{\mathrm{aff}}(i))$ receives a non-$\star$ update at time $t \in T_1^{\mathrm{aff}}(\tau)$, suppose for simplicity this is the first non-$\star$ update occurring in the region $R_1^{\mathrm{aff}}(i,\tau)$: to decide its outcome

we check whether e is a cut-edge, and, if e is a cut-edge in one process but not in the other, $\tilde{\eta}^1_t(e) \neq \tilde{\eta}^2_t(\Phi_t(e))$. So we explore the connected components in the first process $\mathcal{C}^1_{v_1}(t)$ and $\mathcal{C}^1_{v_2}(t)$ and the connected components in the second process $\mathcal{C}^1_{\Phi_t(v_1)}(t)$ and $\mathcal{C}^2_{\Phi(v_2)}(t)$, but, as explained in Remark 4.19, these components are all contained in $S^{\text{enl1}}_1(i)$, and in these region the edges are all coupled, which concludes the proof.

For the final statement of the proof we use the same argument used in the case where the walker does cross $R^{\text{enl2}}_1(i,\tau)$ and $s_c = \tau_1^-$, as we only need the map Φ_t not to change. □

We now prove that the argument above can be iterated to prove an equivalent result for any box of scale k. Denote with $\tau_k^+ = \max T^{\text{enl2}}_k(\tau)$, and with $\tau_k^- = \min T^{\text{enl2}}_k(\tau)$. Moreover, let $s_c := \inf\{t : t \in T^{\text{enl2}}_k(\tau),\, X^1_t \in S^{\text{enl2}}_k(i)\}$ be the first time the walker crosses $R^{\text{enl2}}_k(i,\tau)$, and $s_e := \inf\{t :\ s_c < t \leqslant \tau_k^+,\, X^1_t \notin S^{\text{enl2}}_k(i)\}$ the first time the walker exits $R^{\text{enl2}}_k(i,\tau)$ after having entered it; we take the convention that $s_c = +\infty$ if the walker does not enter $S^{\text{enl2}}_k(i)$ during the time interval $T^{\text{enl2}}_k(\tau)$, and we take the convention that $s_e = \tau_k^+$ if $X^1_t \in S^{\text{enl2}}_k(i)$ for all $t \in (s_c, \tau_k^+]$.

Lemma 4.37. *Let $R_k(i,\tau)$ be a bad box of scale k, such that $R_{k+1}(i',\tau')$ is good for all $R_{k+1}(i',\tau') \supset R_k(i,\tau)$ ($R_{k+1}(i'\tau')$ are not unique because the boxes overlap). Assume $\mathcal{B}'_{s_c}$ happens. Thus*

$$\text{if } s_c < \infty, \text{ then } \Psi_t \text{ remains the same for all } s_c \leqslant t \leqslant s_e. \tag{4.22}$$

Moreover, let $s \in T^{\text{enl2}}_k(\tau)$, and assume $\Phi_t \equiv \Phi_{\tau_k^-}$ for all $t \in [\tau_k^-, s]$; we do not assume in this part that the walker enters $R^{\text{enl2}}_k(i,\tau)$, so s_c, s_e could be equal to ∞, we only assume that the map Φ_t does not change. Then

$$\tilde{\eta}^1_t(e) = \tilde{\eta}^2_t(\Phi_t(e)) \quad \text{for all } t \in [\tau_k^- + t_k, s] \text{ and all } e \in E(S^{\text{enl2}}_k(i)), \tag{4.23}$$

and, in addition, if $\tilde{\eta}^1_{\tau_k^-}(e) = \tilde{\eta}^2_{\tau_k^-}(\Phi_{\tau_k^-}(e))$ for all $e \in E(S^{\text{enl2}}_k(i))$, then (4.23) occurs for all $t \in [\tau_k^-, s]$.

Proof. We will prove the lemma by induction on the scale k. The case $k = 1$ is exactly Lemma 4.35. Assume all claims of the lemma are proved for $k-1$ and consider a bad box $R_k(i,\tau)$ such that $R_{k+1}(i',\tau')$ is good for all $R_{k+1}(i',\tau') \supset R_k(i,\tau)$. We first prove the case $s_c > \tau_k^-$. In this case we use the same argument used in Lemma 4.35. From Lemma 4.27, for all $t \in T^{enl2}_k(\tau)$, for all $v \in S^{enl1}_k(i)$, $\|X^1_t - v\| \geqslant 5\ell_k$, see Figure 4.5. Moreover, in the interval $[s_c, s_e]$ the map Φ_t does not change and because every box of scale k in $R^{\text{enl2}}_k(i,\tau) \setminus R^{\text{enl1}}_k(i,\tau)$ is good, the walker may only cross bad boxes of scale $k-1$ or lower for which the hypothesis of the lemma are satisfied, so by induction we have the lemma for scale k.

We now prove the case $s_c = \tau_k^-$, see Figure 4.6. Because the walker remains in $S^{\text{enl2}}_k(i)$ until time s_e, then the map Φ_t does not change until time s_e. If $s_e < \min T^{\text{enl1}}_k(i,\tau)$, then the walker traverses only bad boxes of scale $k-1$ for which the hypothesis of the lemma are satisfied; otherwise using induction for boxes of scale $k-1$, the $\tilde{\eta}^1_t$ and $\tilde{\eta}^2_t$ are coupled on $E(S^{\text{enl2}}_k(i))$ by time $\tau_k^- + t_k$ (if they were not coupled at time τ_k^-) or remain coupled at all times (if they were). Equivalently $\tilde{\eta}^1_{\tau_k^- + t_k}(e) = \tilde{\eta}^2_{\tau_k^- + t_k}(\Phi_{\tau_k^- + t_k}(e))$ for all $e \in E(S^{\text{enl2}}_k(i))$ and by using induction over and over again for boxes of scale $k-1$ or lower, the graphs remain coupled at all times the walker is in the enlargement.

For the final statement of the proof we use the same argument used in the case where the walker does cross $R^{\text{enl2}}_k(i,\tau)$ and $s_c = \tau_1^-$, as we only need the map Φ_t not to change. □

Remark 4.38. *If B'_s holds for all $s \in (s_1, s_2)$ then in this time interval the walkers can perform the same jumps and not change their relative distance. In fact if the environment around them is the same (as a matter of fact we only need the environments to agree on a ball of radius 1 around the walkers at all times), by doing identity coupling they are always able to perform the same jumps.*

4.6.3 Simple random walk moment

Now, we introduce a concept that will be key for the analysis of the coupling. It will give us a method to effectively identify when to try to bring the two chains closer. This is an event that does not depend on the walkers, but only on the updates of the graph. Recall the definition of $T_1^1(\tau)$ from Definition 4.4.

Definition 4.39. *Let $R_1^{\text{core}}(i,\tau)$ be a $k_{\max}$-great box. We consider three consecutive intervals I_1, I_2, I_3 whose lengths will be defined below and such that I_1 begins at time $\tau t_1 + |T_1^1(\tau)|$ and $\tau t_1 + |T_1^1(\tau)| + \sum_{j=1}^3 |I_j| < (\tau+2)t_1$. Let $v \in S_1(i)$ be the position of the walker $X^1_{\tau t_1 + |T_1^1(\tau)|}$.* A simple random walk moment *(SRWM) occurs in $R_1(i,\tau)$ when the following events happen consecutively:*

(E_1) In the interval I_1, $|I_1| := t_1$, one of the edges adjacent to v opens, say e, and the edges adjacent to e with status $\star$ are sampled closed, moreover after e opens, e does not close for at least $\frac{C_6}{\mu}$ and the adjacent edges do not open;

(E_2) In the interval I_2, with $|I_2| := \frac{1}{\mu}$, edge e closes and does not open, the edges adjacent to e, that were closed, do not open;

(E_3) Finally in the interval I_3, $|I_3| := \frac{1}{\mu}$, the edges adjacent to e but not to the walker refresh with a $\star$ update, whereas the ones adjacent to the walker do not open.

See Figure 4.7 for a possible realization of simple random walk moment. We let SRWM_i^τ be the indicator random variable for the event that the simple random walk moment occurs successfully in the box $R_1(i,\tau)$.

Remark 4.40. *Given $X_{\tau t_1 + |T_1^1(\tau)|} = v$, the position of the walker at time $\tau t_1 + |T_1^1(\tau)|$, the event SRWM depends only from the updates in $E(S_1(i))$ during the time interval $\bigcup_j I_j$.*

We only do the SRWM in great boxes. This will ensure that during I_1 every edge is updated $\star$ at least once.

In words, we essentially want the walkers to get stuck in a vertex, and then wait for a random adjacent edge e to open. Recall that the walker tries to jump at rate 1 and we are assuming $\mu \leqslant 1$. We refresh the environment around, and then we let the walker jump along e for a time long enough so that when e closes there is a positive probability that the walker is at either endpoint of e. If e is uniformly random over the set of the edges adjacent to $X^1_{\tau t_1 + |T_1^1(\tau)|}$, at time $\tau t_1 + T_1^1(\tau) + \sum_{j=1}^3 |I_j|$ the walker will have essentially performed a simple random walk jump from the place where it was at time $\tau t_1 + |T_1^1(\tau)|$. When the simple random walk moment occurs we couple them in the following way: from time τt_1 to time $\tau t_1 + |T_1^1(\tau)|$ we do identity coupling, then, as the distribution at time $\tau t_1 + |T_1^1(\tau)| + \sum_{j=1}^3 |I_j|$ is equivalent to a step of lazy simple random walk we let each walker choose one of the d coordinates uniformly at random, if they agree in this coordinate, then they both stay still, or both move "right" in this coordinate or both move "left" in this coordinate, where the

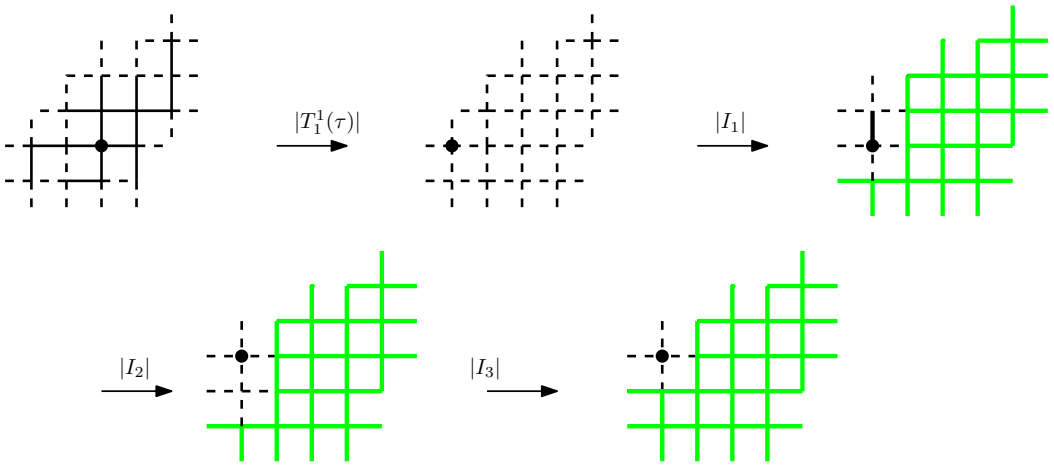


FIGURE 4.7: A possible realization of SRWM in a $k_{\max}$-great box. Dashed the edges that are closed, in green the edges that are updated $\star$, not drawn the edges that remain coupled and whose last update was $\star$, the position of the walker shown by a black circle.

probability of each such case is $\delta \geqslant \frac{1}{2}$ (which depends on how long the edge e remains open in the interval I_1, but is equivalent in both processes), $\frac{1-\delta}{2}$ and $\frac{1-\delta}{2}$, respectively. If the two walkers disagree in this coordinate, then with probability $(1-\delta)$ the first one jumps and the second one stays still, with probability $(1-\delta)$ the second one jumps and the first stays still, and both stay still with probability $2\delta - 1$. It is easy to check that this is a coupling of the two lazy simple random walks, see Theorem 5.5 of [LPwcbELW17].

Denote $s_b = \tau t_1$, the beginning time of the SRWM and $s_f = \tau t_1 + |T_1^1(\tau)| + \sum_{j=1}^3 |I_j|$ the end time of the SRWM. If the simple random walk moment occurs then $\|X^1_{s_f} - X^2_{s_f}\|_1$ may differ from $\|X^1_{s_b} - X^2_{s_b}\|_1$, the distance may change, and as a result of that the translation map Φ_s may change as well. This implies that, unless

$$\tilde{\eta}^1_{s_b}(e') = \tilde{\eta}^1_{s_f}(e') = \star \quad \text{and} \quad \tilde{\eta}^2_{s_b}(\Phi_{s_b}(e')) = \tilde{\eta}^2_{s_f}(\Phi_{s_f}(e')) = \star,$$

it could be the case that $\tilde{\eta}^1_{s_f}(e')$ and $\tilde{\eta}^2_{s_f}(\Phi_{s_f}(e'))$ are different, that is, an edge that was coupled before the simple random walk moment (in the sense that $\tilde{\eta}^1_{s_b}(e') = \tilde{\eta}^2_{s_b}(\Phi_{s_b}(e'))$ may get uncoupled because the map Φ_{s_b} changes. On the other hand, after I_1 all the edges in the box receive a $\star$ update. So the at the end of the SRWM, all edges in the box of the walker are $\star$ with the only exception being the edges adjacent to the walker which are closed, allowing us to recouple the graph configurations locally, i.e. around the walker in $S_1(i)$, so that when an edge adjacent to $X^1_{s_f}$ and $X^2_{s_f}$ opens the particles can perform the same steps again. Moreover, as $R_1(i,\tau)$ is great (so it is also good) the particles will stay in $S_1(i)$ for the whole time interval $T_1(\tau)$. In other words we obtain that the edges in $S_1(i)$, where the random walk moment is occurring, are coupled after the simple random walk moment ends; similarly we can assume the same for any edge that at that moment is inside a good box.
More formally, we will implement this by assigning a closed *envelope* to each box (which in fact is just a random variable) that tells us whether the processes will perform a SRWM in the said box, should the walker pass there. The main point is that we can obtain a lower bound on the probability of the event $SRWM_i^\tau = 1$ that is uniform on the location of the walker. So, if the random walker enters a great box, we open the envelope of that box (revealing the value of its random variable) getting to know whether the walker will perform a simple random walk moment in that box. Because of the uniform bound on the probability that $SRWM_i^\tau = 1$, we can couple the outcome of the envelope with the evolution of the process

$\tilde{M}_t^1$ so that the simple random walk moment takes place, regardless of the location of the walker within the box. The content of the envelope is distributed as a Bernoulli of parameter C_7, for some constant $C_7(p) > 0$ defined in Lemma 4.43 so that the event of successfully performing a SRWM stochastically dominates the random variable of the envelope. Thus, if the envelope of a box $R_1(i,\tau)$ has value 1, the walker X^1 is in $R_1^{\text{core}}(i,\tau)$ at time τt_1 and the box $R_1(i,\tau)$ is $k_{\max}$ great, then the two processes perform a SRWM, otherwise they do not and we just perform the identity coupling.

Lemma 4.41. *Let $R_1(i,\tau)$ be a $k_{\max}$ great box such that $X^1_{\tau t_1} \in S_1^{\text{core}}(i)$ and $\tilde{\eta}^1_{\tau t_1}(e) = \tilde{\eta}^2_{\tau t_1}(\Phi_{\tau t_1}(e))$ for all $e \in E(S_1^{\text{enl1}}(i))$. Suppose the walkers perform successfully a simple random walk moment. Then $\tilde{\eta}^1_t(e) = \tilde{\eta}^2_t(\Phi_t(e))$ for all $e \in E(S_1^{\text{enl1}}(i))$, for all $t \in T_1(\tau) \setminus (\bigcup_{j=1}^3 I_j)$.*

Proof. $R_1(i,\tau)$ is $k_{\max}$-great, and in particular 1-great, it follows that every box $R_1(\cdot,\cdot) \subset R_1^{\text{enl2}}(i,\tau)$ is good. We do identity coupling until an edge e adjacent to the walkers open during the interval I_1, by Lemma 4.32, $\tilde{\eta}^1_t(e) = \tilde{\eta}^2_t(\Phi_t(e))$ for all $e \in E(S_1^{\text{enl1}}(i))$ and for all $\tau t_1 \leqslant t \leqslant \tau t_1 + |T_1^1(\tau)|$. Moreover every $e' \in E(S_1^{\text{enl2}}(i))$ is updated $\star$ at least once during I_1, therefore $\tilde{\eta}^1_t(e') = \tilde{\eta}^2_t(\Phi_t(e'))$ for all edges not adjacent to e, for all $t \in T_1(\tau)$ (these edges are never revealed during the simple random walk moment). At the end of I_3 the edges adjacent to the walkers are closed in the two processes (Φ_t may have changed during $\bigcup_{j=1}^3 I_j$), and every other edge in $E(S_1^{\text{enl2}}(i))$ is $\star$. □

Remark 4.42. *$\mathcal{B}'_t$ may not be verified during $\bigcup_{j=1}^3 I_j$, however the walkers are only allowed to jump along e since the adjacent edges are closed, allowing us to control the coupling.*

Whenever the walkers are in a great box, they have a chance at doing a SRWM, the following Lemma says that they will successfully attempt a SRWM with a probability at least $C_7(p) > 0$. Recall the definition of Θ_i^τ and Ξ_i^τ from Definition 4.10, we also recall they are all independent. Let $\mathcal{I}_\mathcal{T}$ be the set of indices (i,τ) for which $R_1(i,\tau)$ is a box in the tessellation, more precisely

$$\mathcal{I}_\mathcal{T} := \{(i,\tau) : R_1(i,\tau) \subset \mathbb{T}_n^d \times [0,T_3]\}.$$

Let $\Omega_\mathcal{T} := \{0,1\}^{2|\mathcal{I}_\mathcal{T}|}$ be the set of all possible configurations of tessellation of $\mathbb{T}_n^d \times [0,T_3]$ (attributions of 1 or zero to the variables $\{\Theta_i^\tau\}_{i,\tau}$ and $\{\Xi_i^\tau\}_{i,\tau}$). Note that for all $\omega_\mathcal{T} = (\theta_i^\tau, \chi_i^\tau)_{(i,\tau)\in\mathcal{I}_\mathcal{T}} \in \Omega_\mathcal{T}$

$$\mathcal{T}(\omega_\mathcal{T}) := \bigcap_{(i,\tau)\in\mathcal{I}_\mathcal{T}} \{\Theta_i^\tau = \theta_i^\tau, \Xi_i^\tau = \chi_i^\tau\}$$

contains the information of whether each box of scale 1 is good or bad, and consequently the same information can be obtained for higher scales as they are deterministically defined given the information for the scale 1 boxes. Let $\mathcal{F}_t := \sigma\left(\{X_s^1\}_{s\in[0,t)}, \{\tilde{\eta}_s\}_{s\in[0,t)}\right)$ be the σ-algebra generated by the trajectory of the walker and the updates of the graph up to time t.

Lemma 4.43. *There exists $p_0 > 0$ such that for all $p < p_0$, for all $\omega_\mathcal{T} \in \Omega_\mathcal{T}$, realizations of the tessellation for which $R_1(i,\tau)$ is $k_{\max}$-great, and $F \in \mathcal{F}_{\tau t_1}$ for which $\mathbb{P}(\mathcal{T}(\omega_\mathcal{T}) \cap F) > 0$, the probability of performing a simple random walk moment in $R_1(i,\tau)$ is*

$$\mathbb{P}(SRWM_i^\tau = 1 \,|\, \mathcal{T}(\omega_\mathcal{T}) \cap F) \geqslant C_7(p), \tag{4.24}$$

for some $C_7(p) > 0$.

Proof. If we denote with J_1 the indices (i',τ') for which $\Xi_{i'}^{\tau'}=1$, with J_2 the set for which $\Xi_{i'}^{\tau'}=0$ and J_3 (respectively, J_4) the set of indices for which $\Theta_{i'}^{\tau'}=1$ (respectively, $\Theta_{i'}^{\tau'}=0$), then we can write

$$\mathcal{T}(\omega_{\mathcal{T}})=\bigcap_{(i',\tau')\in J_1}\left\{\Xi_{i'}^{\tau'}=1\right\}\bigcap_{(i',\tau')\in J_2}\left\{\Xi_{i'}^{\tau'}=0\right\}\bigcap_{(i',\tau')\in J_3}\left\{\Theta_{i'}^{\tau'}=1\right\}\bigcap_{(i',\tau')\in J_4}\left\{\Theta_{i'}^{\tau'}=0\right\}.$$

$J_1,\dots,J_4$ are so that $R_1(i,\tau)$ is $k_{\max}$-great. Let $J:=\{(i',\tau'):R_1(i',\tau')\cap R_1(i,\tau)\neq\emptyset\}$. Then we denote with S the event

$$S:=\bigcap_{(i',\tau')\in J_1\setminus J}\left\{\Xi_{i'}^{\tau'}=1\right\}\bigcap_{(i',\tau')\in J_2}\left\{\Xi_{i'}^{\tau'}=0\right\}\bigcap_{(i',\tau')\in J_3}\left\{\Theta_{i'}^{\tau'}=1\right\}\bigcap_{(i',\tau')\in J_4}\left\{\Theta_{i'}^{\tau'}=0\right\},$$

which fixes the values of all the $\Theta_{i'}^{\tau'}$ and the $\Xi_{i'}^{\tau'}$ for $(i',\tau')\notin J$. Note that the event $\{R_1(i,\tau)$ is $k_{\max}$ great$\}$ implies the event that $\Xi_{i'}^{\tau'}=1$ for all $(i',\tau')\in J$ (i.e. $J\cap J_2=\emptyset$). Thus,

$$\begin{aligned}\mathbb{P}(SRWM_i^{\tau}=1\,|\,\mathcal{T}(\omega_{\mathcal{T}})\cap F)&=\mathbb{P}\left(SRWM_i^{\tau}=1\,\middle|\,\bigcap_{(i',\tau')\in J}\{\Xi_{i'}^{\tau'}=1\}\cap S\cap F\right)\\&\geqslant\frac{\mathbb{P}(SRWM_i^{\tau}=1\,|\,S\cap F)-\mathbb{P}(\bigcup_{(i',\tau')\in J}\{\Xi_{i'}^{\tau'}=0\}\,|\,S\cap F)}{\mathbb{P}(\bigcap_{(i',\tau')\in J}\{\Xi_{i'}^{\tau'}=1\}\,|\,S\cap F)}.\end{aligned}$$

First of all we bound $\mathbb{P}(\bigcap_{(i',\tau')\in J}\{\Xi_{i'}^{\tau'}=1\}\,|\,S\cap F)\leqslant 1$. Next, we notice that $(i',\tau')\in J$ if and only if $\tau'=\tau$, so that $\bigcup_J\{\Xi_j^{\tau}=0\}$ is independent of F, moreover, by construction of the variables Θ_i^{τ} and Ξ_i^{τ}, it is independent of S as well. Recall the definition of p_ξ from (4.11), then

$$\mathbb{P}(\bigcup_{(i',\tau')\in J}\{\Xi_{i'}^{\tau'}=0\}\,|\,S\cap F)\leqslant 2d(1-p_\xi).$$

Finally, we bound $\mathbb{P}(SRWM_i^{\tau}=1\,|\,S\cap F)$. We write

$$\begin{aligned}&\mathbb{P}(SRWM_i^{\tau}=1\,|\,S\cap F)\\&\quad=\sum_{v\in S_1(i)}\mathbb{P}(SRWM_i^{\tau}=1\,|\,X_{\tau t_1+|T_1^1(\tau)|}=v,S\cap F)\mathbb{P}(X_{\tau t_1+|T_1^1(\tau)|}=v\,|\,S\cap F).\end{aligned}$$

Consider $\mathbb{P}(SRWM_i^{\tau}=1\,|\,X_{\tau t_1+|T_1^1(\tau)|}=v,S\cap F)$. As explained in Remark 4.40, $SRWM_i^{\tau}$ is independent of F and S if conditioned on the position of the walker at time $\tau t_1+|T_1^1(\tau)|$. Next, we break $\mathbb{P}(SRWM(i,\tau)=1\,|\,X^1_{\tau t_1+|T_1^1(\tau)|}=v)$ into

$$\mathbb{P}(E_3|E_2,E_1,X^1_{\tau t_1+|T_1^1(\tau)|}=v)\mathbb{P}(E_2|E_1,X^1_{\tau t_1+|T_1^1(\tau)|}=v)\mathbb{P}(E_1|X^1_{\tau t_1+|T_1^1(\tau)|}=v),$$

where the E_j are from Definition 4.39, and bound each term one by one. There exists $p_0>0$ such that for all $p<p_0$

1. During this step an edge e opens within $\tau t_1+T_1^1(\tau)+|I_1|-\frac{C_6}{\mu}$, from the definition of ℓ in (4.3)

$$\begin{aligned}\mathbb{P}(E_1|X^1_{\tau t_1+|T_1^1(\tau)|}=v)&\geqslant\left(1-e^{-2dp_{ce}\mu\left(|I_1|-\frac{C_6}{\mu}\right)}\right)\left(\frac{1-p}{p_\star}\right)^{2d-1}e^{-(1-p)C_6-(4d-2)p_{ce}\mu|I_1|}\\&\geqslant c_1\left(1-e^{-dp_{ce}\sqrt{\ell}}\right),\end{aligned}$$

for some $c_1 > 0$;

2. $\mathbb{P}(E_2|E_1, X^1_{\tau t_1+|T^1_1(\tau)|} = v) \geqslant (1 - e^{-(1-p)\mu|I_2|})e^{-(4d-1)p_{ce}\mu|I_2|} > c_2$, for some $c_2 > 0$;

3. $\mathbb{P}(E_3|E_1, E_2, X^1_{\tau t_1+|T^1_1(\tau)|} = v) \geqslant (1-e^{-p_\star\mu|I_3|})^{2d-1}e^{-(2d-1)p_{ce}\mu|I_3|} > c_3$, for some $c_3 > 0$.

Putting together all the bounds above we find that we can set $c_4 > 0$ such that

$$\begin{aligned}\mathbb{P}(SRWM^\tau_i = 1 \,|\, S \cap F) &\\ &\geqslant (1 - e^{-c_4 p\sqrt{\ell}}) \sum_{v\in S_1(i)} \mathbb{P}(X_{\tau t_1+|T^1_1(\tau)|} = v|S \cap F)\\ &= 1 - e^{-c_4 p\sqrt{\ell}}.\end{aligned}$$

Now we have all the missing pieces to conclude. Recall the definition of p_ξ from (4.11) and the definition of ℓ from (4.3) to find that

$$\begin{aligned}\mathbb{P}(SRWM^\tau_i = 1 \,|\, \mathcal{T}(\omega_\mathcal{T}) \cap F) &\geqslant (1 - e^{-c_4 p\sqrt{\ell}}) - 2d(1 - p_\xi)\\ &\geqslant (1 - e^{-c_4 p\sqrt{\ell}}) - e^{\frac{C_2}{11^d}\log p \log^2 \ell}\\ &\geqslant \frac{c_4}{2}p^{1-\frac{1}{6d}} - p^{\frac{C_2}{9d^2 11^d}\log^2 p}\\ &\geqslant p^{1-\frac{1}{10d}} - p^{\frac{C_2}{9d^2 11^d}\log^2 p}\\ &\geqslant C_7(p) > 0,\end{aligned}$$

there exists $p_0 > 0$ such that for all $p < p_0$ one has that $1 - \frac{1}{10d} < \frac{C_2}{9d^2 11^d}\log^2 p$. □

Recall from Definition 4.28 that G_k is the set of indexes (i, τ) of k great boxes. Let $(i, 0)$ be the pair for which $(X^1_0, 0) \in R^{\text{core}}_1(i, 0)$ and denote with $I_d : V \to V$ the identity map, then we define

$$F_2 := \{(i, 0) \in G_{k_{\max}}\} \cap \{\Phi_{T_2} = I_d\} \cap \mathcal{B}'_{T_2}. \tag{4.25}$$

If F_2 is verified, the third phase can start, otherwise we let the two processes evolve independently until the end of phase 3 and start the coupling from stage 1.

Lemma 4.44. *Assume F_1 is verified at time 0. There exists $C_5 > 0$, $p_0 > 0$, $n_0 < \infty$ such that for all $p < p_0$ and $n > n_0$*

$$\mathbb{P}(F_2) \geqslant \frac{1}{8}.$$

Proof. Let $\mathcal{P}$ be a feasible path and consider

$$\begin{aligned}\Upsilon^\mathcal{P}_1 &:= \inf\{\tau : \tau > 0, (\mathcal{P}(\tau t_1), \tau t_1) \in R^{\text{core}}_1(i, \tau) \text{ and } R^{\text{core}}_1(i, \tau) \text{ is } k_{\max} \text{ great}\},\\ \Upsilon^\mathcal{P}_j &:= \inf\{\tau : \tau t_1 > \Upsilon^\mathcal{P}_{j-1}, (\mathcal{P}(\tau t_1), \tau t_1) \in R^{\text{core}}_1(i, \tau) \text{ and } R^{\text{core}}_1(i, \tau) \text{ is } k_{\max} \text{ great}\}, \text{ if } j > 1.\end{aligned}$$

For any feasible path $\mathcal{P}$ we let $\kappa_\mathcal{P}$ be the largest value such that $\Upsilon^\mathcal{P}_{\kappa_\mathcal{P}} \leqslant \frac{T_2}{t_1} - 2$.
Define the following events

(E1) The number of simple random walk moments successfully performed is at least $c_0 d^2 n^2$, c_0 to be chosen later;

(E2) For all feasible paths $\mathcal{P}$,

$$\kappa_\mathcal{P} \geqslant \kappa := C_3\frac{C_5}{2\sqrt{\ell}}n^2, \tag{4.26}$$

with C_3 from Lemma 4.29 and C_5 from the definition of T_2.

In this stage we want to couple the position of the walkers. From Lemma 4.37, by doing identity coupling whenever the walkers are not in a $k_{\max}$ great box, their relative distance does not change. Their relative distance changes only when they are in a $k_{\max}$ great box and a simple random walk moment is successfully performed. Let $E_{\text{coup}} = \{\Phi_{T_2} = I_d\} \cap \mathcal{B}'_{T_2}$, and let i be such that $X_0^1 \in R_1(i,0)$. Hence

$$\begin{aligned}\mathbb{P}(F_2^c) &\leqslant \mathbb{P}((i,0) \notin G_{k_{\max}}) + \mathbb{P}(E_1 \cap E_{\text{coup}}^c) + \mathbb{P}(E_1^c) \\ &\leqslant \mathbb{P}((i,0) \notin G_{k_{\max}}) + \mathbb{P}(E_1 \cap E_{\text{coup}}^c) + \mathbb{P}(E_1^c \cap E_2) + \mathbb{P}(E_2^c).\end{aligned}$$

We start by bounding the first term, once the tessellation of $\mathbb{T}_n^d \times [0, T_3]$ is created we check whether the box $R_1(i,0)$, such that $(X_0^1, 0) \in R_1^{\text{core}}(i,0)$, is $k_{\max}$-great. We notice first of all that the event $\{(i,0) \notin G_{k_{\max}}\}$ does not depend on the configuration at time 0. Moreover at time 0 the walkers are stuck in a vertex, so, they can leave $R_1(i,0)$ only from the time boundary. Using the definition of bad box in Definition 4.12 to bound ρ_2 and Lemma 4.15 to bound ρ_j for $j > 2$, we can find $p_0 > 0$ small enough so that this happens with probability at most

$$\begin{aligned}\mathbb{P}((i,0) \notin G_{k_{\max}}) &\leqslant c_d \sum_{j=1}^{k_{\max}} \rho_j \\ &\leqslant c_d \left(\rho_1 + m^{2d+2}\rho_1^2 + \sum_{j=3}^{k_{\max}} \rho_1^{2^{j-2}} \right) \\ &\leqslant 5 c_d \rho_1.\end{aligned}$$

where c_d is a constant that counts the number of boxes whose 2-enlargement intersects $R_1(i,0)$. Next we bound $\mathbb{P}(E_1 \cap E_{\text{coup}}^c) \leqslant \frac{1}{4c_0}$, as this can be bounded by the probability that two random walkers performing SRW on $\mathbb{T}_n^d$ are not coupled after $c_0 d^2 n^2$ steps. Next, we proceed with the fourth term. From Lemma 4.31, with probability at least $\frac{9}{10}$ all the $k_{\max}$ boxes in the tessellation are good and therefore all feasible paths will traverse at least $\frac{T_2}{2t_{kmax}}$ unique good boxes $R_{k_{max}}^{core}(\cdot,\cdot)$. Moreover, under the condition that all boxes of scale $k_{\max}$ are good, Lemma 4.29 states that any feasible path will go through at least $C_3 \frac{t_{kmax}}{t_1} \frac{T_2}{2t_{kmax}} = \kappa$ great boxes, therefore

$$\mathbb{P}(E_2^c) \leqslant \frac{1}{10}.$$

Finally we bound $\mathbb{P}(E_1^c \cap E_2)$. Under E_2, the walkers, whose path is a feasible path, will cross at least κ disjoint great boxes $R_1(i_j, \tau_j)$, with $j = 1, \dots, \kappa$, and κ from (4.26). Let $\{E_i^\tau\}_{i,\tau}$ be i.i.d. Bernoulli random variables of parameter C_7, with C_7 from Lemma 4.43, we will refer to these as envelopes whose outcome we will couple with the decision of performing a simple random walk moment successfully in each box $R_1(i,\tau)$. Recall

$$\mathcal{F}_t = \sigma\left(\{X_s^1\}_{s\in[0,t)}, \{\tilde{\eta}_s\}_{s\in[0,t)}\right)$$

is the filtration of σ-algebras generated by the trajectory of the walker and the refreshes of the graph up to time t, and

$$\mathcal{F}_\mathcal{T} := \sigma\left(\{\Theta_i^\tau\}_{i,\tau}, \{\Xi_i^\tau\}_{i,\tau} : R_1(i,\tau) \subset \mathbb{T}_n^d \times [0,T_3]\right)$$

the σ-algebra generated by the variables Θ_i^τ and Ξ_i^τ, for all i, τ in the tessellation. Next,

Let $\Omega_{\mathcal{T}} := \{0,1\}^{2|\mathcal{I}_{\mathcal{T}}|}$ be the set of all possible configurations of tessellation of $\mathbb{T}_n^d \times [0, T_3]$ (attributions of 1 or zero to the variables $\{\Theta_i^\tau\}_{i,\tau}$ and $\{\Xi_i^\tau\}_{i,\tau}$). Note that for all $\omega_{\mathcal{T}} = (\theta_i^\tau, \chi_i^\tau)_{(i,\tau)\in\mathcal{I}_{\mathcal{T}}} \in \Omega_{\mathcal{T}}$

$$\mathcal{T}(\omega_{\mathcal{T}}) := \bigcap_{(i,\tau)\in\mathcal{I}_{\mathcal{T}}} \{\Theta_i^\tau = \theta_i^\tau, \Xi_i^\tau = \chi_i^\tau\}$$

is the event of whether each box of scale 1 is good or bad, and consequently the same information can be obtained for higher scales as they are deterministically defined given the information for the scale 1 boxes. To simplify the notation we will also omit X_t when we want to refer to the $\Upsilon_i^{X_t}$. Then, $\{\Upsilon_i\}_i$ are stopping times w.r.t. the filtration $\mathcal{F}_t \times \mathcal{F}_{\mathcal{T}}$. We will couple the decision of $SRWM_{i_j}^{\Upsilon_j}$ with the outcome of $E_{i_j}^{\Upsilon_j}$ in the following way. Let the walkers move until time $\Upsilon_1 t_1$, when $(X^1_{\Upsilon_1 t_1}, \Upsilon_1 t_1) \in R_1^{\text{core}}(i_1, \Upsilon_1)$. From Lemma 4.43, for all $F \in \mathcal{F}_{\Upsilon_1 t_1}$ such that $\mathbb{P}(F \cap \mathcal{T}(\omega_{\mathcal{T}})) > 0$

$$\mathbb{P}(SRWM_{i_1}^{\Upsilon_1} = 1 | F \cap \mathcal{T}(\omega_{\mathcal{T}})) \geqslant \mathbb{P}(E_{i_1}^{\Upsilon_1} = 1).$$

so we couple them in such a way that if $E_{i_1}^{\Upsilon_1} = 1$ then $SRWM_{i_1}^{\Upsilon_1} = 1$. Then, we let the walkers move until $\Upsilon_2 t_1$ and repeat the same argument. In particular, for all $j = 1, \dots, \kappa$, if $SRWM_{i_j}^{\Upsilon_j} = S_{i_j}^{\Upsilon_j}$, we have that $\bigcap_{\iota=1}^{j-1}\{SRWM_{i_\iota}^{\Upsilon_\iota} = S_{i_\iota}^{\Upsilon_\iota}\} \in \mathcal{F}_{\Upsilon_j t_1} \times \mathcal{F}_{\mathcal{T}}$. Then, from Lemma 4.43

$$\mathbb{P}\left(SRWM_{i_j}^{\Upsilon_j} = 1 \middle| \bigcap_{\iota=1}^{j-1}\{SRWM_{i_\iota}^{\Upsilon_\iota} = S_{i_\iota}^{\Upsilon_\iota}\} \cap \mathcal{T}(\omega_{\mathcal{T}})\right) \geqslant \mathbb{P}(E_{i_j}^{\Upsilon_j} = 1),$$

It follows that the sequence $\{S_{i_j}^{\Upsilon_j}\}_j$ stochastically dominates a sequence of i.i.d. Bernoulli random variables of parameter C_7. Define then the event

$$E_3 := \left\{\sum_{j=1}^{\kappa} E_{i_j}^{\Upsilon_j} \geqslant c_0 d^2 n^2\right\}.$$

First, we notice that, intersected with E_2, E_1^c implies E_3^c.

$$\begin{aligned}
\mathbb{P}(E_1^c \cap E_2) &\leqslant \mathbb{P}(E_3^c \cap E_2) \\
&= \sum_{\omega_{\mathcal{T}}\in\Omega_{\mathcal{T}}} \mathbb{P}(E_3^c \cap E_2 | \mathcal{T}(\omega_{\mathcal{T}}))\mathbb{P}(\mathcal{T}(\omega_{\mathcal{T}})) \\
&= \sum_{\substack{\omega_{\mathcal{T}}\in\Omega_{\mathcal{T}}, \\ E_2 \text{ happens}}} \mathbb{P}(E_3^c | \mathcal{T}(\omega_{\mathcal{T}}))\mathbb{P}(\mathcal{T}(\omega_{\mathcal{T}})).
\end{aligned}$$

Choose C_5 large enough so that $c_0 d^2 n^2 < \frac{C_7}{2} C_3 \frac{C_5}{2\sqrt{\ell}} n^2 = \frac{C_7}{2}\kappa$,

$$\mathbb{P}(E_3^c | \mathcal{T}(\omega_{\mathcal{T}})) \leqslant \mathbb{P}\left(\sum_{j=1}^{\kappa} E_{i_j}^{\Upsilon_j} < \frac{C_7(p)}{2}\kappa \,\middle|\, \mathcal{T}(\omega_{\mathcal{T}})\right).$$

Then by a standard application of a Chernoff bound for independent Bernoulli random variables we get that for all $\mathcal{T}(\omega_{\mathcal{T}})$ such that E_2 happens

$$\mathbb{P}(E_3^c | \mathcal{T}(\omega_{\mathcal{T}})) \leqslant e^{-\frac{C_7(p)}{4}\kappa}.$$

Which yields that

$$\begin{aligned}\mathbb{P}(E_1^c \cap E_2) &\leqslant e^{-\frac{C_7(p)}{4}\kappa} \sum_{\substack{\omega_{\mathcal{T}} \in \Omega_{\mathcal{T}}, \\ E_2 \text{ happens}}} \mathbb{P}(\mathcal{T}(\omega_{\mathcal{T}})) \\ &= \mathbb{P}(E_2) e^{-\frac{C_7(p)}{4}\kappa} \\ &\leqslant e^{-\frac{C_7(p)}{4}\kappa}.\end{aligned}$$

Putting everything together

$$\mathbb{P}(F_2) \geqslant 5c_d\rho_1 + \frac{1}{4c_0} + e^{-\frac{C_7(p)}{4}\kappa} + \frac{1}{10} < \frac{1}{8},$$

and we can choose p_0 small so that ρ_1 is small enough, next we choose c_0, C_5 large enough, n_0 large enough to prove the Lemma. □

4.7 Third Phase

At the beginning of the third phase the walkers are coupled and the graphs are coupled on $B_{3\ell}(X^1_{T_2})$. During the third phase we let $\tilde{M}^2_t$ mimic the steps of $\tilde{M}^1_t$, we do then an identity coupling and check whether the processes are fully coupled after time $T_3 = T_2 + \frac{n^2}{\mu}$.

Let F_3 be the event

$$F_3 := \{X^1_{T_3} = X^2_{T_3},\, \tilde{\eta}^1_{T_3}(e) = \tilde{\eta}^2_{T_3}(e)\, \forall e \in E\},$$

if F_3 is not verified, we restart the coupling from phase 1.

Lemma 4.45. *There exists p_0, m such that there exists $n_0 > 0$, for which for all $p < p_0$, $n > n_0$*

$$\mathbb{P}(F_3|F_2, F_1) \geqslant \frac{1}{2}. \tag{4.27}$$

Proof. During the second phase we sampled the variables Θ^τ_i and Ξ^τ_i for all (i,τ) such that $R_1(i,\tau) \subset \mathbb{T}^d_n \times [0, T_3]$. By Lemma 4.37 and as explained in Remark 4.34, if we do only identity coupling when the walkers enter the 2-enlargement of a bad box (of any scale), when they exit such 2-enlargement their relative distance does not change and the environments around the walkers remain coupled throughout. During phase 3, we only do identity coupling, therefore for the walkers to get uncoupled before time T_3 there must exist a bad box of scale k that has not been observed during the second phase (or equivalently, there exists at least one variable $\Xi^\tau_i = 0$ or $\Theta^\tau_i = 0$ for some τ such that $\tau t_1 \geqslant T_3$) and whose 2-enlargement intersects $\mathbb{T}^d_n \times [0, T_2]$. We want to count the number of such boxes. Following a similar argument used in Lemma 4.31, we start by deriving bounds on ℓ_k and t_k, the size of the boxes of scale k. When $k \geqslant k_{\max}$, we can choose n large enough so that for any m, ℓ fixed

$$\begin{aligned}2\ell k^{2k} \leqslant \ell_k = m^k (k!)^2 \ell \leqslant \ell k^{3k}, \\ 2\sqrt{\ell} k^{2k} \leqslant \mu t_k = m^k (k!)^2 \sqrt{\ell} \leqslant \sqrt{\ell} k^{3k}.\end{aligned}$$

We notice that because $T_3 - T_2 = \frac{n^2}{\mu}$, k must be greater than $k_{\max}$. Therefore

$$\begin{aligned}\mathbb{P}(F_3|F_2, F_1) &\geqslant 1 - \mathbb{P}(\exists\, k > 0,\, R_k(i,\tau) \text{ bad box} : T_3 \in T_k(\tau) \text{ and } T_2 \in T_k^{\text{enl2}}(\tau)) \\ &\geqslant 1 - \mathbb{P}(\exists\, k > k_{\max},\, R_k(i,\tau) \text{ bad box} : T_3 \in T_k(\tau)).\end{aligned}$$

Next, using the bounds we derived above for ℓ_k and t_k, the number ζ_k of boxes of scale k that intersect $\mathbb{T}_n^d \times T_3$ is bounded above and below by

$$\begin{aligned}\zeta_k &\geqslant \left(\frac{n}{3\ell_k}\right)^d \geqslant \frac{n^d}{3^{d+1}\ell^d k^{3dk}},\\ \zeta_k &\leqslant 1 + 2\left(\frac{n}{\ell_k}\right)^d \leqslant 1 + \frac{n^d}{\ell^d k^{2dk}}.\end{aligned}$$

In the upper bound of ζ_k we add a 1 to the fraction to consider the case when k is so large that we cannot find a box all contained in the tessellation. Using Lemma 4.15 the probability that there exists a box of scale $k_{\max}$ or bigger that is bad is bounded above by

$$\sum_{k \geqslant k_{\max}} \zeta_k \rho_k \leqslant \sum_{k \geqslant k_{\max}} \zeta_k \rho_1^{2^{k-2}},$$

moreover using the inequalities above for ζ_k it is easy to see that, for any $k \geqslant k_{\max}$,

$$\zeta_k \rho_1^{2^{k-2}} \leqslant 2^{-(k-k_{\max})} \zeta_{k_{\max}} \rho_1^{2^{k_{\max}-2}},$$

so that

$$\begin{aligned}\mathbb{P}(F_3|F_2,F_1) &\geqslant 1 - \sum_{k \geqslant k_{\max}} \zeta_k \rho_k\\ &\geqslant 1 - 2\zeta_{k_{\max}} \rho_1^{2^{kmax}-2}\\ &\geqslant 1 - 2\frac{n^d}{\ell^d} \rho_1^{2^{\log\log n}-2}\\ &\geqslant \frac{1}{2},\end{aligned}$$

where we choose p_0 small enough so that Lemma 4.15 can be applied and $\rho_1 < e^{-2d}$. □

4.8 The Proof of the Theorem

Let $\{\tilde{M}_t^1\}_{t\geqslant 0}$ and $\{\tilde{M}_t^2\}_{t\geqslant 0}$ denote two copies of our process $\{\tilde{M}_t\}_{t\geqslant 0}$, each starting from an arbitrary configuration in $\mathbb{T}_n^d \times \{0,1\}^{E(\mathbb{T}_n^d)}$. We construct a coupling $(\{\tilde{M}_t^1\}_{t\geqslant 0}, \{\tilde{M}_t^2\}_{t\geqslant 0})$ of our two processes and a nonnegative random variable T so that

1. $\tilde{M}_t^1 = \tilde{M}_t^2$ for all $t \geqslant T$;
2. $\mathbb{E}[T] \leqslant O_{d,p}(1)\frac{n^2}{\mu}$.

From here, it is standard that this gives a bound on the mixing time as follows. If $t \geqslant 4\mathbb{E}[T]$, then

$$\|\mathcal{L}(M_t^1) - \mathcal{L}(M_t^2)\|_{TV} \leqslant \|\mathcal{L}(\tilde{M}_t^1) - \mathcal{L}(\tilde{M}_t^2)\|_{TV} \leqslant \mathbb{P}(\tilde{M}_t^1 \neq \tilde{M}_t^2) \leqslant \mathbb{P}[T > t] \leqslant \frac{1}{4}$$

by Markov's inequality. The coupling ends successfully when $F_1 \cap F_2 \cap F_3$ therefore the number of times we need to repeat the coupling procedure until the two processes are fully coupled is distributed as a geometrical random variable. From Lemma 4.30, Lemma 4.44 and Lemma 4.45

$$\mathbb{P}(F_1 \cap F_2 \cap F_3) = \mathbb{P}(F_3|F_2,F_1)\mathbb{P}(F_2|F_1)\mathbb{P}(F_1) \geqslant \frac{1}{160}.$$

So that $\mathbb{E}[T] \leqslant 160 T_3$, which concludes the proof of the theorem.

Chapter 5

The Random Cluster model

In this chapter, we will refer to $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ for a polynomial growth lattice, $\Lambda := \Lambda_n = (V_n, E_n)$ a ball of radius n around vertex 0. The internal boundary $\partial\Lambda$ will be the set of vertices in V_n adjacent to an edge in $\mathbb{E} \setminus E_n$.

In this chapter, we extend the result in [BS17] on the mixing time of Glauber Dynamics for the Random-Cluster to a larger class of graphs. More precisely, in their paper, they prove an upper bound for the mixing time of the Glauber dynamics for the Random Cluster on $\mathbb{Z}_2$. Using their machinery we extend this result to all lattices (recall the definition of lattices from Definition 2.31) with **polynomial growth**, that is

Definition 5.1. *We say that an infinite lattice has polynomial growth if $\Lambda_n = (V_n, E_n)$ the ball of radius n around vertex 0 is such that $|V_n| \leqslant C_0 n^d$, for some positive constants C_0, and d and for all n.*
Moreover $|E_n| \leqslant C_1 n^d$ for some constant $C_1 > 0$ because a lattice is locally finite and vertex transitive.

Recall the definition of Random Cluster Glauber Dynamics from Section 2.6.

Theorem 5.2. *Let $q \geqslant 1$, $p < p_c(q)$, $\mathbb{G}$ polynomial growth lattice and $\Lambda_n = (V_n, E_n)$ a ball of radius n with $m = |E_n|$. Consider the discrete time Glauber Dynamics for random cluster on Λ_n with either free or wired boundary condition.*
There exists a fixed constant $C_5 > 0$ such that the mixing time is at most

$$T_{mix} \leqslant T(m) := C_5 m \log m.$$

We now recall the definition of **Weak Spatial Mixing**, which we previously presented in Chapter 2.

Definition 5.3. *Let $\mathbb{G}$ be a lattice, let $\Lambda_n = (V_n, E_n)$ be the ball of radius n around 0. Consider now $B_{\Lambda_n}(e, r) = (V(B_{\Lambda_n}(e, r)), E(B_{\Lambda_n}(e, r))) := B(e, r) \cap \Lambda_n \subseteq \Lambda_n$, the ball of radius r around $e \in E_n$ intersected with Λ_n itself, see Figure 2.5 (c); and let $\partial B_{\Lambda_n}(e, r)$ the set of vertices $v \in B(e, r)$ connected by an edge in E_n to $V_n \setminus V(B(e, r))$. Then, if there exist c_{sm}, C_{sm} positive constants such that for all $n > 0$, $e \in E_n$, $r \geqslant c_{sm}$ and every pair of random cluster configurations A_1, A_2 in $\{0,1\}^{E_n \setminus E(B_{\Lambda_n}(e,r))}$, we have*

$$|\nu^{\mathsf{s}}_{\Lambda_n,p,q}(\eta(e) = 1 |\, A_1) - \nu^{\mathsf{s}}_{\Lambda_n,p,q}(\eta(e) = 1 |\, A_2)| \leqslant e^{-C_{sm} r}, \tag{5.1}$$

we say that the random-cluster measure $\nu^{\mathsf{s}}_{\Lambda_n,p,q}$ has weak spatial mixing.

For a fixed $e = (u, v) \in E_n$ we will let $B := B_\Lambda(e, r) = (V(B), E(B))$ as defined in Definition 5.3.

Definition 5.4. *Let η be any random-cluster configuration in Λ. Define*

$$\Gamma(\eta, B) := V(B) \setminus \bigcup_{v \in \partial B} \mathcal{C}_v(\eta),$$

where $\mathcal{C}_v(\eta)$ is the connected component of v in η.

We first prove the two main ingredients of the proof, namely **Weak Spatial Mixing** (recall Definition 5.3), and **Disagreement Percolation**. We will use the notion of homogeneous boundary condition, which was introduced in Definition 2.36.

5.1 Weak spatial mixing and disagreement percolation

First, we present a counter-example produced in [BS17] that shows that for some non homogeneous boundary condition weak spatial mixing does not hold.

Remark 5.5. *[BS17] Call $E^1_{B^c}$ (resp., $E^0_{B^c}$) to denote the event that all the edges in $E_n \setminus E(B)$ are open (resp., closed). Suppose $e = (u, v) \in E_n$ is such that both u and v are in $\partial\Lambda$, and let ς be the boundary condition on Λ where u is wired to a vertex $u' \in \partial\Lambda \setminus V(B)$ and v is wired to a different vertex $v' \in \partial\Lambda \setminus V(B)$, see Figure 5.1. When $p = \frac{1}{2}$ and $q = 3$, we have $\nu^\varsigma_{\Lambda,p,q}(e = 1 | E^1_{B^c}) = \frac{1}{2}$. Also, by considering r, in the definition of B, small enough so that $u', v' \notin V(B)$, it is easy to check that $\nu^\varsigma_{\Lambda,p,q}(e = 1 | E^0_{B^c}) = \frac{1}{4}$. Both these bounds are independent of r and n; consequently, ς does not have the spatial mixing property.*

Now, we are ready to prove weak spatial mixing for random cluster measure with homogeneous boundary conditions.

Theorem 5.6 (Weak Spatial Mixing)**.** *Let ς be a homogeneous boundary condition for Λ, then for any $q \geqslant 1$ and $p \leqslant p_c(q)$, the random-cluster measure $\nu^\varsigma_{\Lambda,p,q}$ has weak spatial mixing.*

Proof. Following [BS17], let A be any random-cluster configuration in Λ. For a fixed $e = (u, v) \in E_n$ we will let $B := B_\Lambda(e, r) = (V(B), E(B))$ as defined in Definition 5.3. Consider $\Gamma(A, B)$ as defined in Definition 5.4. Call $E^1_{B^c}$ the event that all edges in $E(B)^c := E_n \setminus E(B)$ are open. Denote $\nu^w := \nu^\varsigma_{\Lambda,p,q}(\cdot \,|\, E^1_{B^c})$.
Consider any two configurations A^c_1, A^c_2 on $E(B)^c$, then there exists a coupling π of $\nu_1 := \nu^\varsigma_{\Lambda,p,q}(\cdot \,|\, A^c_1)$, $\nu_2 := \nu^\varsigma_{\Lambda,p,q}(\cdot \,|\, A^c_2)$, and ν^w such that if A_1, A_2, A_w are random configurations corresponding to the random cluster configuration with respect to the measures ν_1, ν_2 and ν^w, respectively, then $\pi(A_1, A_2, A_w) > 0$, only if $A_1 \subseteq A_w$, $A_2 \subseteq A_w$ and A_1, A_2 agree on all edges $e \in E(B)$ with both endpoints in $\Gamma(A_w, B)$.
In fact, consider the coupling π built by deciding the status of the edges adopting the following manner

1. from ∂B explore edge by edge of $E(B)$ in the configuration A_w (according to ν^w) and, each time the status of an edge is revealed, decide its status in A_1 and A_2 using the monotonic coupling, so that we have A_1, $A_2 \subseteq A_w$;

2. repeat this process until we have explored all connected components of A_w whose vertices intersect ∂B;

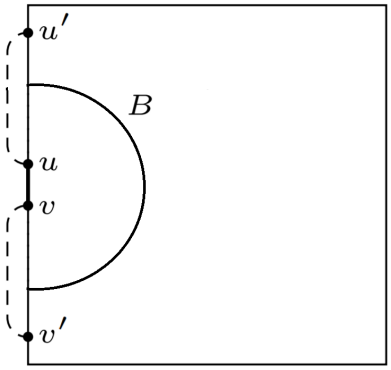


FIGURE 5.1: A boundary condition ς where the spatial mixing property does not hold, see Remark 5.5 for a full explanation. [BS17].

when the process ends, the only edges that have not been revealed yet are the ones with both endpoints in $\Gamma(A_w, B)$; moreover on the edges that have been revealed we have coupled A_1, A_2, A_w in such a way that A_1, $A_2 \subseteq A_w$. Note that all edges in the edge boundary of $\Gamma(A_w, B)$ (the set of edges with exactly one endpoint in $\Gamma(A_w, B)$) are closed in A_w, A_1, and A_2. Hence if we call ς_w, ς_1, and ς_2 the boundary conditions induced on the set of edges with both endpoints in $\Gamma(A_w, B)$ by A_w, A_1, A_2 respectively, we obtain that $\varsigma_w = \varsigma_1 = \varsigma_2$ because they are simply the free boundary conditions. Also, if $\Gamma(A_w, B) \cap \partial\Lambda \neq \emptyset$, then pairs of vertices may be wired if and only if they are wired in ς, that is because they are either wired or one of them is a singleton (recall that ς being a homogeneous boundary condition implies that $|P_i| > 1$ for at most one i, cf. Definition 2.36), therefore the boundary conditions remain the same regardless of the configurations of A_w, A_1, A_2. Because $\varsigma_w = \varsigma_1 = \varsigma_2$, the **domain Markov property** of random-cluster measures, see [Gri06], allows us to couple A_1, A_2, A_w exactly on the edges with both endpoints in $\Gamma(A_w, B)$ without affecting the correct marginal distributions.
Now that we have such a coupling π,

$$\begin{aligned}|\nu^{\varsigma}_{\Lambda_n,p,q}(A_1(e) = 1 \,|\, A_1^c) - \nu^{\varsigma}_{\Lambda_n,p,q}(A_2(e) = 1 \,|\, A_2^c)| &\leqslant \pi(A_1(e) \neq A_2(e)) \\ &\leqslant \pi(u \notin \Gamma(A_w, B),\, v \notin \Gamma(A_w, B)) \\ &\leqslant \nu^{w}(\{e\} \overset{B}{\leftrightarrow} \partial B) \\ &\leqslant |\partial B| e^{-C_{spt} r} \leqslant e^{-C_{sm} r},\end{aligned}$$

where in the last line we used (2.12): the exponential decay of finite volume connectivities, and a union bound over the vertices on the boundary. □

Now we prove a bound on the probability that two configurations that agree on a ball around an edge e disagree on e after km steps, this is also known as **Disagreement Percolation**.

Theorem 5.7 (Disagreement Percolation)**.** *Let $\mathbb{G}$ be a polynomial growth lattice, let $\Lambda_n = (V_n, E_n)$ be a ball of radius n around vertex 0, $m = |E_n|$, and $B := B_{\Lambda_n}(e, r)$, for some $e \in E_n$, and $r \geq 1$, like the ones defined in Definition 2.42. Let $p \leqslant p_c(q)$ and $q \geqslant 1$ and consider two copies $\{X_t\}$, $\{Y_t\}$ of the Glauber Dynamics in Λ_n with homogeneous boundary condition ς, where Y_0 has law $\nu^{\varsigma}_{\Lambda_n,p,q}$, and X_0 is an arbitrary configuration satisfying $X_0(B) = Y_0(B)$.*

Recall C_0 from Definition 5.1. If the evolutions of $\{X_t\}$ and $\{Y_t\}$ are coupled using the

identity coupling, then there exist, absolute constants $C_2, C_3 > 0$ *such that, if* $r \geqslant C_2$ *and* $1 \leqslant k \leqslant \frac{1}{C_0 e^2} r^{\frac{1}{d+2}}$,

$$\mathbb{P}(X_{km}(e) \neq Y_{km}(e)) \leqslant e^{-C_3 r^{\frac{1}{d+2}}}.$$

Proof. For some fixed $\ell > 0$ to be chosen later and $t \geqslant 0$, consider

$$\mathcal{E}_{\ell,t} := \{u \overset{Y_t}{\not\leftrightarrow} v, \quad \forall\, u, v \in B \,:\, \mathrm{dist}(u, v) > \ell\},$$

the event that any pair of vertices at distance larger than ℓ are not connected by a path in $Y_t(B)$. Let then $\mathcal{E}_\ell = \bigcap_{t=0}^{km} \mathcal{E}_{\ell,t}$ be the event that $\mathcal{E}_{\ell,t}$ occurs for all time from 0 to km. Then,

$$\mathbb{P}(X_{km}(e) \neq Y_{km}(e)) \leqslant \mathbb{P}(\{X_{km}(e) \neq Y_{km}(e)\} \cap \mathcal{E}_\ell) + \mathbb{P}(\mathcal{E}_\ell^c), \tag{5.2}$$

and now we will bound the two terms separately.
For A some random cluster configuration in Λ_n let $\Gamma(A, B)$ be as defined in Definition 5.4, and for $D \subseteq V_n$ let $\partial E(D) := \{(u, v) \in E_n \,:\, u \in D,\, v \notin D\}$.
Consider the decreasing sequence of graphs $R_0 \supseteq R_1 \supseteq \dots$ such that $V(R_0) = \Gamma(Y_0, B)$ and $E(R_t) = \{(u, v) \in E_n \,:\, u, v \in \Gamma(Y_t, B),\, Y_t((u, v)) = 1\}$. Moreover let (a_t, b_t) be the edge updated from step t to $t + 1$. Then, for $t \geqslant 0$, define

$$V(R_{t+1}) = \begin{cases} V(R_t) & \text{if } (a_t, b_t) \notin \partial E(R_t); \\ V(R_t) \setminus \Gamma(a_t, Y_t) & \text{if } (a_t, b_t) \in \partial E(R_t) \text{ and } a_t \in V(R_t),\, b_t \notin V(R_t). \end{cases}$$

Observe that $\partial E(R_t)$ contains only closed edges and R_t does not need to be a connected graph. We claim that for all $t \geqslant 0$, $X_t(E(R_t)) = Y_t(E(R_t))$.
To prove this observe that, obviously, R_t changes if an edge is updated, therefore R_t changes at discrete time-points, so that we can use induction to prove our statement.
Denote with $E(R_t^c)$ the set of edges in E_n with at most one endpoint in $V(R_t)$ Assume $X_t(E(R_t)) = Y_t(E(R_t))$, and call ξ_X, ξ_Y the boundary conditions induced in R_t by $X_t(E(R_t^c))$ and $Y_t(E(R_t^c))$, respectively. As usual we regard ξ_X, ξ_Y as partitions of ∂R_t. Let (a_t, b_t) be the edge updated at time t. We have now three possibilities:

1. The endpoints $a_t, b_t \notin V(R_t)$, then clearly $X_{t+1}(E(R_{t+1})) = Y_{t+1}(E(R_{t+1}))$, because, although the update might be different, $R_{t+1} = R_t$ as the edge is not in $\partial E(R_t)$;

2. Only one endpoint of (a_t, b_t) is in $V(R_t)$, assume it is a_t. In this case we update $V(R_{t+1}) = V(R_t) \setminus \Gamma(a_t, Y_t)$, and because $\mathcal{C}_{a_t}(X_t) \cap V(R_t) = \mathcal{C}_{a_t}(Y_t) \cap V(R_t)$ we have again $X_{t+1}(E(R_{t+1})) = Y_{t+1}(E(R_{t+1}))$;

3. Both $a_t, b_t \in V(R_t)$, in this case we will show below that $\xi_X = \xi_Y$, which implies that $X_{t+1}(E(R_{t+1})) = Y_{t+1}(E(R_{t+1}))$.

The argument to prove that $\xi_X = \xi_Y$ is similar to the one used in the proof of the weak spatial mixing. First of all, every edge in $\partial E(R_t)$ is closed in both X_t and Y_t, therefore if $|V(R_t) \cap \partial\Lambda| \leqslant 1$ then the boundary conditions are both free.

Assume now that $|V(R_t) \cap \partial\Lambda| \geqslant 2$, consider u, v two vertices in $V(R_t) \cap \partial\Lambda$, if they are wired in ς then they are also wired in ξ_X and ξ_Y, whereas if they are not wired in ς, then because $|P_i| > 1$ for at most one i, then at least one between u and v is free in ς, and it is a singleton also in ξ_X, ξ_Y because there are no paths from v to $V_n \setminus V(R_t)$ neither in X_t nor in Y_t. Consequently, ξ_X and ξ_Y induce the same boundary conditions.

Now that we have that for all $t \geqslant 0$, $X_t(E(R_t)) = Y_t(E(R_t))$, then if both extremes of e are in $V(R_k)$, $X_k(e) = Y_k(e)$, and because also $\mathcal{E}_\ell$ holds, if we choose $\ell \ll r$, then $e \in E(R_0)$. If $X_k(e) \neq Y_k(e)$ we can define a "sequence of witnesses" for the fact that $X_k(e) \neq Y_k(e)$. Consider v_0 to be the first endpoint of e to be removed from R_{t_0}, at some time $t_0 \leqslant km$, and call $e_1 \in \partial E(R_{t_0})$ the unique edge whose update is responsible for removing v_0 from $V(R_{t_0})$ (v_0 is in the same connected component of e_1). Then starting from e_1, we repeat this process and construct a sequence of edges $e_2, e_3, \dots$ such that, $e_i = (u_i, v_i)$ with $u_i \in V(R_{t_{i-1}})$ and $v_i \notin V(R_{t_{i-1}})$ is the edge responsible for removing $v_{t_{i-1}}$ from $V(R_{t_{i-1}})$ at time t_{i-1}.

We are building the sequence exploring the updates backwards in time and the sequence stops once we reach an edge incident to a vertex not in $V(R_0)$.

Because we $\mathcal{E}_\ell$ has to hold, $\operatorname{dist}(v_i, u_{i+1}) < \ell$, and therefore the number of possible sequence of L witnesses is at most $(C_0(\ell+1)^d)^L$. Moreover every sequence of L witnesses must satisfy $L \geqslant \frac{r}{\ell+1}$, because it must reach the vertices not in $V(R_0)$. Finally, the probability that a given sequence of L witnesses occur within km steps is $\binom{km}{L}(\frac{1}{m})^L$, therefore

$$\begin{aligned}
\mathbb{P}(\{X_{km}(e) \neq Y_{km}(e)\} \cap \mathcal{E}_\ell) &\leqslant \sum_{L \geqslant \frac{r}{\ell+1}} \binom{km}{L} \left(\frac{1}{m}\right)^L (C_0(\ell+1)^d)^L \\
&\leqslant \sum_{L \geqslant \frac{r}{\ell+1}} \left(\frac{C_0 ek(\ell+1)^d}{L}\right)^L \\
&\leqslant c_0^{\frac{r}{\ell+1}} \sum_{L \geqslant 0} c_0^L,
\end{aligned}$$

where $c_0 := \frac{C_0 ek(\ell+1)^{d+1}}{r} < \frac{1}{e}$ if we choose $\ell := r^{\frac{1}{d+2}} - 1$; so that the sum converges and we have

$$\mathbb{P}(\{X_{km}(e) \neq Y_{km}(e)\} \cap \mathcal{E}_\ell) \leqslant \frac{e}{e-1} e^{-r^{\frac{d+1}{d+2}}}. \tag{5.3}$$

This gives the bound for the first term of the r.h.s. of (5.2).

To give a bound for the second term in the r.h.s. of (5.2), our first thought is to take a union bound over $\mathcal{E}_\ell^c = \bigcap_{t=0}^{km} \mathcal{E}_{\ell,t}^c$, however m depends on n and could be much larger than r. Nevertheless notice that an update outside B does not change $\mathcal{E}_{\ell,t}^c$, because updating an edge outside B cannot create nor break a path in B.

Let $M := |E(B)|$ be the number of the edges in B, we should expect about kM updates in B after km steps, so that, if we call N the number of updates in B up to time km, we can write

$$\mathbb{P}(\mathcal{E}_\ell^c) \leqslant \mathbb{P}(\mathcal{E}_\ell^c |\, N \leq 2kM) + \mathbb{P}(N > 2kM).$$

N is a binomial of parameters km and $\frac{M}{m}$, using a Chernoff bound we obtain

$$\begin{aligned}
\mathbb{P}(N > 2kM) &= e^{-2kM} \sum_{i=0}^{km} \binom{km}{i} \left(\frac{Me}{m}\right)^i \left(1 - \frac{M}{m}\right)^{km-i} \\
&\leqslant e^{-2kM} \left(1 - \frac{M}{m} + \frac{eM}{m}\right)^{km} \leqslant e^{-(3-e)kM}.
\end{aligned}$$

Therefore we can now bound

$$\mathbb{P}(\mathcal{E}_l^c) \leqslant 2kM \max_{0 \leqslant t \leqslant km} \mathbb{P}(\mathcal{E}_{l,t}^c) + e^{-\frac{1}{5}kM}.$$

Thus, using (2.12), the fact that $l = r^{\frac{1}{d+2}} - 1$, that in B there are at most $(C_0 r^d)^2$ pair of vertices and $M \leqslant C_1 r^d$,

$$\mathbb{P}(\mathcal{E}_l^c) \leqslant \frac{2}{C_0 e^2} r^{\frac{1}{d+2}} C_1 r^d C_0^2 r^{2d} C_0 e^{-C_{spt} l} + e^{-\frac{1}{5} C_1 r^d} \leqslant e^{-c_1 r^{\frac{1}{d+2}}}, \tag{5.4}$$

there exists $c_1 > 0$ subject to $r \geqslant C_2$, for some $C_2 > 0$. Finally, we can plug (5.3) and (5.4) in (5.2) and get the correct bound for some positive constant C_3. □

5.2 Upper bound for the Mixing Time in subcritical regime

In this section we prove an upper bound for the Glauber Dynamics in subcritical regime of order $O(m \log m)$, where m is the number of the edges. This result was also proved independently by Harel and Spinka, see Theorem 16 in [HS18]. Although their proof is slightly different from ours using a different machinery which does not involve Disagreement Percolation, but a technique called *finitary coding* which is essentially using the *Coupling from the past*. We will make great use of the two results proved in the previous section, namely Spatial Mixing (Theorem 5.6) and Disagreement Percolation (Theorem 5.7).

We will first prove a weaker bound (with an extra poly-loglog factor w.r.t. the result presented at the beginning of the chapter), Theorem 5.8 and then using a recursion argument we will push the bound down to $O(m \log m)$ which is Theorem 5.2, the main result of this chapter.

Theorem 5.8. *Let $q \geqslant 1$, $p < p_c(q)$, $\mathbb{G}$ polynomial growth lattice and $\Lambda_n = (V_n, E_n)$ a ball of radius n with $m = |E_n|$. Consider the discrete time Glauber Dynamics for random cluster on Λ_n with either free or wired boundary condition.*

There exists a fixed constant $C_4 > 0$ such that the mixing time

$$T_{\text{mix}}(m) \leqslant C_4 m \log m \, (\log \log m)^2 .$$

Proof. Consider two copies of the Glauber Dynamics X_t, Y_t coupled with the identity coupling, starting from $X_0 = \{1\}^{E_n}$, $Y_0 = \{0\}^{E_n}$.

We prove that for $t \geqslant T(m)$

$$\mathbb{P}(X_t(e) \neq Y_t(e)) \leqslant 5m^{-2}, \tag{5.5}$$

so that by taking a union bound over the edges we have that $\mathbb{P}(X_t \neq Y_t) \leqslant 5\frac{1}{m} \leqslant \frac{1}{4}$; and because the mixing time is smaller than the coupling time, we have the theorem.

Consider $B = B_{\Lambda_n}(e, r)$ as defined previously in Definition 2.42, with $r = c_0 \log m$ and $M := |E(B)|$, for some c_0 to be chosen later. Define two additional Markov Chains Z_t^+ and Z_t^-, starting from X_0 and Y_0, respectively, but following a slightly different dynamic: if the edge to be refreshed is in $E(B)$, then update in Z_t^+ and Z_t^- according to the identity coupling; otherwise, if the edge is not in $E(B)$, suppress the update.

It is clear that by monotonicity of the identity coupling we have for all $t \geqslant 0$

$$Z_t^+ \geqslant X_t \geqslant Y_t \geqslant Z_t^- .$$

Therefore

$$\begin{aligned}\mathbb{P}(X_t(e) \neq Y_t(e)) &\leqslant \mathbb{P}(Z_t^+(e) \neq Z_t^-(e)) = \mathbb{P}(Z_t^+(e) = 1, Z_t^-(e) = 0) \\ &= \mathbb{P}(Z_t^+(e) = 1) - \mathbb{P}(Z_t^-(e) = 1).\end{aligned}$$

Call now $E^1_{B^c}$ (respectively, $E^0_{B^c}$) the event that all the edge in $E_n \setminus E(B)$ are open (respectively, closed) and, then by triangle inequality

$$\begin{aligned}\mathbb{P}(X_t(e) \neq Y_t(e)) \leqslant &\left|\mathbb{P}\left(Z_t^+(e) = 1\right) - \nu^{\varsigma}_{\Lambda_n}\left(\eta(e) = 1 \mid E^1_{B^c}\right)\right| \\ &+ \left|\nu^{\varsigma}_{\Lambda_n}\left(\eta(e) = 1 \mid E^1_{B^c}\right) - \nu^{\varsigma}_{\Lambda_n}\left(\eta(e) = 1 \mid E^0_{B^c}\right)\right| \\ &+ \left|\nu^{\varsigma}_{\Lambda_n}\left(\eta(e) = 1 \mid E^0_{B^c}\right) - \mathbb{P}\left(Z_t^-(e) = 1\right)\right|.\end{aligned}$$

Notice that the chains Z_t^+ and Z_t^- are regular Glauber Dynamics on B and the induced boundary conditions are still homogeneous (we can still apply spatial mixing and disagreement percolation bounds). In particular, spatial mixing implies that

$$\left|\nu^{\varsigma}_{\Lambda_n}\left(\eta(e) = 1 \mid E^1_{B^c}\right) - \nu^{\varsigma}_{\Lambda_n}\left(\eta(e) = 1 \mid E^0_{B^c}\right)\right| \leqslant e^{-C_{\mathrm{sm}} r} = m^{-c_0 C_{\mathrm{sm}}},$$

where the last step follows from our choice of r. We will only explain the first term, since the other one is bounded in an analogous way. We will proceed inductively.

The base of the induction is a fixed box of size m_0; consider the quantity

$$T(m_0) := C_4 m_0 \log m_0 \left(\log\log m_0\right)^2.$$

Then there exists a constant $C_4 = C_4(m_0)$ large enough for which the statement of the theorem is verified: $T_{\mathrm{mix}}(m_0) \leqslant T(m_0)$.

For the inductive step: suppose $T(M) = C_4 M \log M \left(\log\log M\right)^2$ is the time needed to mix in B, we want to prove $T_{\mathrm{mix}}(m) \leqslant T(m)$ for Λ with m edges.

Let $U_B(T(m))$ be the number of updates in B up to time $T(m)$. Using that $M \leqslant C_1(c_0 \log m)^d$ (and therefore that m is exponential in M), the expected number of updates in B up to time $T(m)$ is

$$\begin{aligned}\mathbb{E}[U_B(T(m))] &= T(m)\frac{M}{m} = C_4 M \log m \left(\log\log m\right)^2 \\ &\geqslant 4M \log_2 m \log M (\log\log M)^2 = 4T(M)\log_2 m,\end{aligned}$$

provided m is large enough. This means that using a Chernoff bound we have

$$\begin{aligned}\mathbb{P}(U_B(T(m)) \geqslant 2T(M)\log_2 m) &\geqslant \mathbb{P}\left(U_B\left(T\left(m\right)\right) \geqslant \frac{1}{2}\mathbb{E}\left[U_B\left(T\left(m\right)\right)\right]\right) \\ &= 1 - \mathbb{P}\left(U_B\left(T\left(m\right)\right) < \frac{1}{2}\mathbb{E}\left[U_B\left(T\left(m\right)\right)\right]\right) \\ &\geqslant 1 - e^{-\frac{\mathbb{E}[U_B(T(m))]}{8}} \geqslant 1 - e^{-\frac{1}{2}T(M)\log_2 m} \geqslant 1 - m^{-2}.\end{aligned}$$

Finally call $\mathcal{J} = \{U_B(T(m)) \geq 2T(M)\log_2 m\}$, using Proposition 2.13 (i.e. $T_{\mathrm{mix}}(\frac{1}{m}) \leqslant$

$\lceil \log_2 m \rceil T_{\text{mix}})$ we have

$$\begin{aligned}
&\|\mathbb{P}(Z_t^+ \in \cdot\,) - \nu^{\varsigma}_{\Lambda_n}(\,\cdot\,|E^1_{B^c})\|_{TV} \\
&\leqslant \left\|\mathbb{P}(Z_t^+ \in \cdot\,|\mathcal{J}) - \nu^{\varsigma}_{\Lambda_n}(\,\cdot\,|E^1_{B^c})\right\|_{TV} + \mathbb{P}(\mathcal{J}^c) \\
&\leqslant m^{-2} + m^{-2} = 2m^{-2}.
\end{aligned}$$

We are now ready to put all the pieces together

$$\begin{aligned}
\mathbb{P}(X_t(e) \neq Y_t(e)) &\leqslant m^{-c_0 C_{sm}} + 2|\mathbb{P}(Z_t^+(e) = 1) - \nu^{\varsigma}_{\Lambda_n}(e = 1|E^1_{B^c})| \\
&\leqslant m^{-c_0 C_{sm}} + 2\|\mathbb{P}(Z_t^+ \in \cdot\,) - \nu^{\varsigma}_{\Lambda_n}(\,\cdot\,|E^1_{B^c})\|_{TV} \\
&= m^{-c_0 C_{sm}} + 4m^{-2} \leqslant 5m^{-2},
\end{aligned}$$

by choosing $c_0 = \frac{2}{C_{sm}}$.

The only thing left to say is that $C_4 = C_4(m_0)$, and the theorem is true for all $m \geqslant m_0$. Therefore we can just choose the minimal possible m_0 to conclude. □

At this point we have not used yet the result for disagreement percolation. This last ingredient is going to give a final boost to improve the bound on the mixing time. We are now ready to prove Theorem 5.2.

Proof of Theorem 5.2. Consider two copies of the Glauber Dynamics X_t, Y_t coupled with the identity coupling starting from $X_0 = \{1\}^{E_n}$, $Y_0 \sim \nu^{\varsigma}_{\Lambda_n}$.
The coupling time for $X_0 = \{0\}^{E_n}$, $Y_0 \sim \nu^{\varsigma}_{\Lambda_n}$ can be bounded in the same way, and then by monotonicity, this bound holds also for any pair of initial configurations.
Just like in the previous theorem we prove that for $t = T(m)$, for any $e \in E_n$

$$\mathbb{P}(X_t(e) \neq Y_t(e)) \leqslant \frac{1}{4m}, \tag{5.6}$$

so that by taking a union bound over the edges we have that $\mathbb{P}(X_t \neq Y_t) \leqslant \frac{1}{4}$; and because the mixing time is smaller than the coupling time, we have the theorem.

Consider

$$\rho(k) := \max_{e \in E_n} \mathbb{P}(X_{km}(e) \neq Y_{km}(e)).$$

We want to prove that $\rho(C_5 \log m) \leqslant \frac{1}{4m}$.
Let $B_\Lambda(e, r_k)$ be a ball of radius r_k (yet to be chosen) around e intersected with Λ. Let

$$\mathcal{E}_k := \{X_{km}(E(B_\Lambda(e, r_k))) = Y_{km}(E(B_\Lambda(e, r_k)))\},$$

the event that X_{km} and Y_{km} agree on the edges in $B_\Lambda(e, r_k)$.

Using Theorem 5.7, provided $r \geqslant C_2$ for any $k \leqslant \frac{1}{C_0 e^2} r^{\frac{1}{d+2}}$

$$\rho(2k) \leqslant \mathbb{P}(X_{2km}(e) \neq Y_{2km}(e)|\,\mathcal{E}_k^c)\mathbb{P}(\mathcal{E}_k^c) + e^{-C_3 r_k^{\frac{1}{d+2}}} \tag{5.7}$$

$$\leqslant \rho(k)\left(C_1 r_k^d \rho(k)\right) + e^{-C_3 r_k^{\frac{1}{d+2}}} = C_1 r_k^d \rho(k)^2 + e^{-C_3 r_k^{\frac{1}{d+2}}}. \tag{5.8}$$

Choose r_k to maximize k and so that disagreement percolation can still be applied, namely

$$r_k = \frac{1}{C_3}(c_0 k)^{d+2}, \tag{5.9}$$

for some $c_0 > C_3^{\frac{1}{d+2}} C_0 e^2$ to be chosen later so that the requirement for disagreement percolation is satisfied.

With this choice for r_k we get

$$e^{-C_3 r_k^{\frac{1}{d+2}}} \leqslant e^{-c_0 k}.$$

Call $D := d(d+2)$, there exists $c_1 > 0$ such that

$$\rho(2k) \leqslant C_1 \left(\frac{1}{C_3}(c_0 k)^{d+2}\right)^d \rho(k)^2 + e^{-c_0 k} \leqslant c_1 k^D \rho(k)^2 + e^{-c_0 k}. \tag{5.10}$$

We now have a relation between $\rho(2k)$ and $\rho(k)^2$ and we want to keep on halving k to find a suitable bound for $\rho(k)$.

Each time we apply the recursion (5.10), we obtain the sum of two terms: if $c_1 k^D \rho(k)^2 > e^{-c_0 k}$ then

$$\rho(2k) \leqslant 2c_1 k^D \rho(k)^2,$$

and we can keep on applying the recursion, otherwise

$$\rho(2k) \leqslant 2e^{-c_0 k},$$

and we can tune c_0 to get the desired bound.

Let $C_5 \log m = 2^i \ell$ for some large $\ell > 0$ fixed, yet to be chosen, $2^i = \frac{C_5}{\ell} \log m$. We bound the quantity $\rho(2^i \ell)$ using the recursion the number of times the first term in the r.h.s of (5.10) dominates. Let $0 \leqslant J \leqslant i$ be this number. If $J < i$

$$\begin{aligned}
\rho(2^J 2^{i-J}\ell) \leqslant & 2^{2^J} c_1^{2^J} \left(\prod_{j=1}^{J} (2^{J-j} 2^{i-J}\ell)^{2^{j-1}}\right)^D \rho(2^{i-J}\ell)^{2^J} \\
\leqslant & 2^{2^J} c_1^{2^J} \left(2^{2^J} (2^{i-J}\ell)^{2^J}\right)^D 2(e^{-c_0 2^{i-J-1}\ell})^{2^J} \\
\leqslant & 2\left(2c_1 (2^{i-J+1}\ell)^D e^{-c_0 2^{i-J-1}\ell}\right)^{2^J} \\
\leqslant & e^{2^{i-2}\ell} = m^{-\frac{C_5}{4}},
\end{aligned}$$

provided ℓ is large enough. Set $C_5 = 8$ to prove the theorem. Otherwise, assume $J = i$, then

$$\begin{aligned}
\rho(2^i \ell) \leqslant & 2^{2^i} c_1^{2^i} \left(\prod_{j=1}^{i} (2^{i-j}\ell)^{2^{j-1}}\right)^D \rho(\ell)^{2^i} \\
\leqslant & 2^{2^i} c_1^{2^i} \left(2^{2^i} \ell^{2^i}\right)^D \rho(\ell)^{2^i} \\
\leqslant & \left(2c_1 (2\ell)^D \rho(\ell)\right)^{2^i}
\end{aligned}$$

We only need to prove that there exists $\ell > 0$ fixed for which

$$\rho(\ell) \leqslant \frac{1}{2ec_1(2\ell)^D}.$$

Following the same steps in Theorem 5.8, with $r = r_0$ fixed, such that $|E(B_\Lambda(e, r_0))| = m_0$, define $E^1_{B^c}$ (respectively, $E^0_{B^c}$) the event that every edge in $E_n \setminus E(B_\Lambda(e, r_0))$ is open

(respectively, closed) then

$$\begin{aligned}\rho(\ell) \leqslant &\left|\mathbb{P}\left(Z_t^+(e)=1\right)-\nu_{\Lambda_n}^{\varsigma}\left(\eta(e)=1 \mid E_{B^c}^1\right)\right|+\\ &+\left|\nu_{\Lambda_n}^{\varsigma}\left(\eta(e)=1 \mid E_{B^c}^1\right)-\nu_{\Lambda_n}^{\varsigma}\left(\eta(e)=1 \mid E_{B^c}^0\right)\right|+\\ &+\left|\nu_{\Lambda_n}^{\varsigma}\left(\eta(e)=1 \mid E_{B^c}^0\right)-\mathbb{P}\left(Z_t^-(e)=1\right)\right|.\end{aligned}$$

Consider $T_{\text{mix}}(m_0)$ the mixing time in a ball of size m_0, and call $U_B(t)$ the number of updates in $B(e,r_0)$ within time t, then

$$\mathcal{J}:=\left\{U_B(\ell m_0)\leqslant \frac{1}{2}\ell m_0\ \left(=T_{\text{mix}}(m_0)\frac{\ell m_0}{2T_{\text{mix}}(m_0)}\right)\right\}.$$

Provided $\frac{1}{2}\ell m_0 > T_{\text{mix}}(m_0)$ (true for $\ell=\ell(m_0)$ fixed large enough), using Theorem 5.6, and a Chernoff bound we have that for some $c_3>0$

$$\begin{aligned}\rho(l)\leqslant& e^{-C_{sm}r_0}+2\left(\mathbb{P}(\mathcal{J})+\left(\frac{1}{4}\right)^{\frac{\ell m_0}{2T_{\text{mix}}(m_0)}}\right)\leqslant\\ \leqslant& e^{-C_{sm}\left(\frac{m_0}{C_1}\right)^{\frac{1}{d}}}+2e^{-\frac{\ell m_0}{8}}+2\left(\frac{1}{4}\right)^{\frac{\ell m_0}{2T_{\text{mix}}(m_0)}}\leqslant\\ \leqslant& e^{-C_{sm}\left(\frac{m_0}{C_1}\right)^{\frac{1}{d}}}+e^{-c_3\frac{\ell m_0}{T_{\text{mix}}(m_0)}}\leqslant\frac{1}{2ec_1(2\ell)^D},\end{aligned}$$

where the last inequality is verified provided

$$\min\left\{C_{sm}\left(\frac{m_0}{C_1}\right)^{\frac{1}{d}},\ c_3\frac{\ell m_0}{T_{\text{mix}}(m_0)}\right\}\geqslant\log\ell,$$

because if this is verified then we can find ℓ large enough for which the last inequality is true.

From this last request, we can work out the conditions for ℓ and $T_{\text{mix}}(m_0)$, in particular from the first one, we find that we need

$$C_{sm}\left(\frac{m_0}{C_1}\right)^{\frac{1}{d}}\geqslant\log\ell\Rightarrow\ell\leqslant e^{c_4 m_0^{\frac{1}{d}}},$$

for some $c_4>0$.
Whereas from the second one, we find that we need:

$$c_3\frac{\ell m_0}{T_{\text{mix}}(m_0)}\geqslant\log\ell\Rightarrow T_{\text{mix}}(m_0)\leqslant c_3\frac{\ell m_0}{\log\ell};$$

and to conclude we see that by choosing $\ell=e^{c_4 m_0^{\frac{1}{d}}}$, it follows that we actually need

$$T_{\text{mix}}(m_0)\leqslant e^{c_4 m_0^{\frac{1}{d}}}m_0^{1-\frac{1}{d}},$$

which is true by Theorem 5.8.

As a final comment observe that also the choice we made for ℓ large enough so that $\frac{1}{2}\ell m_0 > T_{\text{mix}}(m_0)$ is still verified; also, because of the choice we made for $r_k=\Theta(k^{d+2})$, our

method works as long as $B(e, r_k) \subset \Lambda_n$, which means

$$C_1 r_k^d \leqslant m \Leftrightarrow c_0 k^D \leqslant m \Leftrightarrow k = o(m^{\frac{1}{D}}).$$

However we start our recursion from $k = c_2 \log m$ which is legitimate. □

Chapter 6

Future Work

6.1 Random Walk in Dynamical Percolation.

In this chapter we state a few open problems and conclusive remarks from our work.

Consider a sequence of graphs $G_n = (V_n, E_n)$, with $|V_n| = N_n$, that monotonically increases towards $G_\infty = (V_\infty, E_\infty)$ and $\max_{x \in V_n} \deg(x) \leqslant k$, for all $n \in \mathbb{N} \cup \{\infty\}$, and some $k < \infty$. We assume that G_n are **vertex transitive**.

Definition 6.1. *A graph $G = (V, E)$ is vertex transitive if for any two vertices $v, w \in V$, there exists a graph automorphism $\phi : V \to V$ that maps v to w.*

In words, the neighborhoods of v and w look the same, and in particular we get that all vertices have the same degree, which we denote by k.

Following the proof in Proposition 2.28 it is easy to show that the random walk in dynamical percolation process $\{M_t\}_{t=0}^{\infty} := \{(X_t, \eta_t)\}_{t=0}^{\infty}$ on G_n is reversible and has stationary distribution $\pi := u \times \pi_p$, with u being the uniform distribution and π_p the product of Bernoulli with parameter p.

Conjecture 1. *Let $G_n = (V_n, E_n)$, be a sequence of vertex-transitive graphs, with bounded degree $k > 0$. For any $p \in (0, p_c(G_\infty))$, there exists $C_1 < \infty$, such that for all n, and for all $\mu \leqslant 1$, we have that*

$$T_{\text{mix}}^{p,\mu}(G_n) \leqslant \frac{C_1 T_{\text{mix}}^{\text{RW}}(n)}{\mu} \tag{6.1}$$

Where $T_{\text{mix}}^{\text{RW}}(n)$ is the mixing time of a Lazy Simple Random Walk on G_n (with all open edges), and $T_{\text{mix}}^{p,\mu}(G_n)$ is the mixing time of the full system M_t on G_n with parameter p and refresh rate μ.

In [HS19] Hermon and Sousi prove this conjecture with the less general setup of vertex transitive graphs of *moderate growth*, while also eliminating the requirement that p is subcritical; but a bound as in (6.1) is more relevant in the subcritical regime, where it is expected to be tight, whereas in the supercritical regime one expects $T_{\text{mix}}^{p,\mu}(G_n)$ to behave more like $C(\frac{1}{\mu} + T_{\text{mix}}^{\text{RW}}(n))$. Following the same argument used for the proof of the case $G_n = \mathbb{T}_n^d$ in Chapter 3, it is easy to see that, up to some minor corrections, the **first stage** of the proof can be adapted for vertex-transitive graphs, because Theorem 3.3 stated in [AN84] holds also for vertex-transitive graphs. However, in the **second stage**, we encounter a first issue: in general it is not possible to preserve the distance between the particles with a coupling of the two dynamics after a non-simple regeneration time step. For example, regular graphs of large girth that are vertex transitive (they are locally tree-like because of the large girth), or

the honeycombe lattice (imagine two walkers on two adjacent vertices, if they both take one step their distance cannot be 1).

6.1.1 Abelian groups.

Because translations are commutative, whatever move the walker does on $\mathbb{T}_n^d$, the trivial coupling in which one walker moves independently and then the second one mimics its movements on the lattice, preserves the distance between the two. For this reason, it is easy to construct a coupling that preserves the distance during a non-simple regeneration step, for random walks on dynamical percolation on a Cayley graph of any abelian group.

Proposition 6.2. *If $G = (V, E)$ is a Cayley graph of an abelian group $(G', \cdot)$, and X_t, $Y_t \in V$ two particles performing random walk on dynamical percolation, then there exists a coupling for which after a non-simple regeneration time step* $\mathrm{dist}(X_t, Y_t) = \mathrm{dist}(X_{t+1}, Y_{t+1})$.

Proof. Let S be a set of generators of the group. Assume $X_t = v_x \in V$, and $Y_t = v_y \in V$ and after a non-simple regeneration time step, $X_{t+1} = X_t s_0$, for some $s_0 \in S$. Then update $Y_{t+1} = Y_t s_0$, it is easy to see that $\mathrm{dist}(X_t, Y_t) = \mathrm{dist}(X_{t+1}, Y_{t+1})$.
Let $s_1 s_2 \cdot \ldots s_k$ be a path from X_t to Y_t, for some $s_1, \ldots, s_k \in S$. More precisely $v_x s_1 \cdot \ldots s_k = v_y$. Then $s_1 s_2 \cdot \ldots s_k$ is also a path from X_{t+1} to Y_{t+1} since $(v_x s_0) s_1 s_2 \cdot \ldots s_k = (v_x s_1 s_2 \cdot \ldots s_k) s_0 = v_y s_0$. □

Let P be the transition matrix associated to the simple random walk on G_n and Q the transition matrix associated to a non-simple regeneration time step. We can commute the products of the matrices by Abelianess so that, after t regeneration time steps, we get a term of the form $P^k Q^{t-k}$. Then, for Q^{t-k} we do identity coupling, so that the relative distance between the particles does not change, and for P^k we do the optimal coupling of SRW. This is enough to prove that the mixing time of RW on Dynamical Percolation is at most $\frac{1}{\mu}$ times the bound on the mixing time we get from the coupling of SRW on the graph.
The same strategy cannot be employed for graphs without the Abelian property, and some new idea is required in such cases.

6.1.2 Lower bound for the mixing time.

It would be interesting to find an equivalent result as in [PSS15] for the lower bound of the mixing time when the vertices of the graph have uniformly bounded degree.

Conjecture 2. *For any $k < \infty$, $p \in (0, p_c(G_\infty))$, there exists $C < \infty$, such that for all n, and for all μ, we have that:*

$$T_{\mathrm{mix}}^{p,\mu}(G_n) \geqslant \frac{C T_{\mathrm{mix}}^{\mathrm{RW}}(n)}{\mu},$$

where $T_{\mathrm{mix}}^{\mathrm{RW}}(n)$ is the mixing time of a Lazy Simple Random Walk on G_n (with all edges open).

In particular the above conjecture would imply a tight lower bound for the mixing time implying that the upper bound claimed in Conjecture 3.1 is also a lower bound up to constants.

It would also be interesting to understand whether Theorem 3.1 and Conjecture 2 hold when G_n is not necessarily vertex transitive.

The next step then would be to study graphs where the degree is not uniformly bounded and could be for example a function of the number of vertices N, with $\deg(v) = f(N) \to \infty$ as $N \to \infty$.

Conjecture 3. *For any $p \in (0, p_c(G_\infty))$, for all n, and for all μ, considering the full system $\{M_t\}_{t\geqslant 0}$ as we defined in (2.9) on G_n with $\deg(v) = f(N) \to \infty$ as $N \to \infty$ we have that*

$$T^{p,\mu}_{\text{mix}}(G_n) = O_{n,p,f(\cdot)}(1)\frac{T^{RW}_{MIX}(n)}{\mu},$$

where $T^{RW}_{MIX}(n)$ is the mixing time of a Lazy Simple Random Walk on G_n (with all edges open).

Also, the refreshing parameter μ is always considered to be fixed, another possible direction for future research is to study what happens if μ is allowed to change through time. For example, what if $\mu(t) \to 0$, as $t \to \infty$?

6.2 Random cluster model

We have seen in Chapter 5 how to prove bounds for the mixing time of the Glauber Dynamics for the random cluster model in subcritical setting and how . When $p > p_c(q)$ instead, the supercritical phase, we do not know whether exponential decay of finite volume connectivities holds, namely

$$\nu^{\varsigma}_{\Lambda,p,q}(u \overset{\Lambda}{\leftrightarrow} v) \leqslant e^{-c_0 \operatorname{dist}(u,v)},$$

for any boundary condition ς and all $u, v \in \Lambda$. This was a key ingredient in the proof of Spatial Mixing and Disagreement Percolation, and if we are hoping to extend results proved for $p < p_c(q)$ to $p > p_c(q)$ using the same machinery, we need to replace exponential decay of finite volume connectivities with a similar property that shows the exponential decay of dependence between the status of two different edges. Remember that in random cluster model, the statuses of two edges e, e' influence each other if changing the status of e decides whether e' is cut-edge or not and vice versa. In particular, two edges are independent if either there are no cycles of open edges that contain both of them or there is more than one cycle. When $p < p_c(q)$, the probability that even just a single path between e, e' exists, decays exponentially by exponential decay of connectivities. When $p > p_c(q)$ we can hope instead that so many edges open that with high probability we can find at least two cycles of open edges containing both e and e'.With this in mind, when p is large enough, we would expect to see "very dense" configurations, so that if we call

$$\mathcal{E}_\ell := \{\exists\, e = (u, v) \in E \,|\, \mathcal{C}^e_u \cap \partial B(e, \ell) \neq \emptyset, \mathcal{C}^e_v \cap \partial B(e, \ell) \neq \emptyset, \mathcal{C}^e_u \cap \mathcal{C}^e_v = \emptyset\},$$

where with $\mathcal{C}^e_v$ we denote the connected component deprived of the edge e, this event should be exponentially small in $\ell > 0$. In words, $\mathcal{E}_\ell$ is the event that there exists an edge e for which the connected components of its endpoints (deprived of the edge e itself) both have diameter at least ℓ but there is no open path between them. In supercriticality the probability to have high diameter is large, therefore what really gives exponential decay to $\mathcal{E}_\ell$ is the requirement of no intersection between the two clusters.

If we can prove that there exists a constant $C > 0$ and $p_{\mathcal{E}}$ such that for all $p > p_{\mathcal{E}}$ and all $\ell > 0$

$$\mathbb{P}(\mathcal{E}_\ell) \leqslant e^{-C\ell}, \tag{6.2}$$

then we can prove Disagreement Percolation. It is easy to prove Spatial Mixing as well if we assume (6.2) and add extra hypothesis that require $p > p_0 > p_c(q)$, but we do not know yet how to do it assuming only (6.2). If we could do that, then we would have a proof for Mixing Time for all $p > p_{\mathcal{E}}$.

We can prove $\mathcal{E}_\ell$ has exponential decay for $p_{\mathcal{E}} > p_c(q)$, but proving $p_{\mathcal{E}} = p_c(q)$ is a major open problem.

6.3 Random Walks in Dynamical Random Cluster

In Chapter 4 we identify the mixing time of the RWRC only for p very small, a natural question is to identify the mixing time for any $p < p_c(q)$. This is a much harder problem as the *great boxes*, defined previously, no longer percolate. We are no longer able to easily describe regions of space and time that are *safe* from the bad influence of non-$\star$ updates and the techniques we use in Chapter 4, fail to work. In particular, we require for a box $R_1(i, \tau)$ to be good (see Definition 4.4) that the following properties are satisfied, roughly speaking:

- Every edge is updated *enough* times;
- The connected components are not larger than $\log^2 \ell$;
- Every update is $\star$;
- Every edge closes within an interval of length $\frac{\log^2 \ell}{\mu}$.

The first event does not depend on p and so it will not represent a problem. The second event depends on p, but we can use exponential decay of connectivities (2.12), which holds for all $p < p_c(q)$. The last two events depend on p and surely do not hold for all $p < p_c(q)$ and we will need to figure out a way to work around this issue.

However, our proof does not use *weak spatial mixing*, which essentially says that the influence between the status of two edges decays exponentially with their distance. So, even though many steps in our proof require p to be much smaller than $p_c(q)$, we are hopeful that weak spatial mixing could be a key ingredient to bound the *information percolation* caused by non-$\star$ updates and push our result to all $p < p_c(q)$.

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