Limit theory of combinatorial optimization for random geometric graphs

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Abstract

In the random geometric graph $G(n, r_n)$, n vertices are placed randomly in Euclidean d-space and edges are added between any pair of vertices distant at most r_n from each other. We establish strong laws of large numbers (LLNs) for a large class of graph parameters, evaluated for $G(n, r_n)$ in the thermodynamic limit with $nr_n^d =$ const., and also in the dense limit with $nr_n^d \to \infty$, $r_n \to 0$. Examples include domination number, independence number, clique-covering number, eternal domination number and triangle packing number. The general theory is based on certain subadditivity and superadditivity properties, and also yields LLNs for other functionals such as the minimum weight for the traveling salesman, spanning tree, matching, bipartite matching and bipartite traveling salesman problems, for a general class of weight functions with at most polynomial growth of order $d - \varepsilon$, under thermodynamic scaling of the distance parameter.

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1 Introduction

Random geometric graphs (RGGs) are a well-known baseline stochastic model for combinatorial structures with spatial or multivariate content. Starting with the seminal paper of Gilbert [20], random geometric graphs have in recent decades received a lot of attention as a model for large communication networks such as sensor networks, see [3]. Network agents are represented by the vertices of the graph, and direct connectivity is represented by edges. Applications arise in other fields including theoretical computer science, geography, biology, topological data analysis, network science and astronomy - for more on applications of random geometric graphs, we refer to Chapter 3 of [23], as well as to [35].

The classic random geometric graph [35], also called the Gilbert graph, has its vertex set given by taking the points of a sample of size n from some specified probability distribution in Euclidean d-space, and edges between any two points distant at most r from each other. In the terminology of [35], a *thermodynamic* limiting regime involves taking $r = r(n) = \Theta(n^{-1/d})$ as n grows large, so that average degrees remain bounded away from zero and infinity as $n \to \infty$, while a *dense* limiting regime is one with $n^{-1/d} \ll r(n) \ll 1$, i.e. $nr(n)^d \to \infty$ and $r(n) \to 0$ as $n \to \infty$.

The first order (and in some cases, second order) limit theory for RGGs in the thermodynamic limit, of quantities such as number of edges, number of components, and number of isolated vertices, is described in [35]. Loosely speaking, these enjoy linear growth in n because they are the sum of locally determined quantities. The order of the largest component (also considered in [35]) also falls into this category if nr^d is above a certain continuum percolation threshold, in which case we say it is *supercritical*.

In the present paper we derive laws of large numbers (LLNs), in thermodynamic or dense limiting regimes, for a variety of other graph invariants. These include independence number, domination number, and many others (to be listed shortly). Each of these quantities is obtained as the solution of some constrained optimization problem in the graph, and enjoys linear growth in n in the thermodynamic regime. If nr^d is subcritical, they can be given as a sum of locally determined quantities, but if nr^d is supercritical they cannot, and the methods of [35] are not applicable.

Our LLNs (Theorems 2.1 and 2.7) come with a rather simple set of conditions, applicable to a wide range of parameters of interest for RGGs, for which LLNs were previously available only in the special case of the domination number with $d = 2$ and $nr(n)^d \gg \log n$ [9], which we shall relax to $nr(n)^d \gg 1$. Moreover, the dense limit theory relates these graph parameters to certain classic quantities of interest in deterministic combinatorial optimization such as sphere packing and sphere covering densities. In the thermodynamic limit, our LLN provides a bound on the rate of convergence.

Furthermore, we shall use our methods to derive LLNs for weighted traveling salesman, minimum spanning tree, minimum matching and minimum bipartite matching problems with edge weights determined by inter-point distances, via some arbitrary weight function that is either bounded, or grows at most polynomially of order $d - \varepsilon$, for some $\varepsilon > 0$, under thermodynamic scaling (for the unbounded case we require μ to have compact support). These results (Theorems 4.3, 4.6, 4.9 and 4.11) generalize earlier work on these problems, see e.g. [8, 46], in which only power law weight functions were considered.

As in the earlier work such as [8, 46], we use methods based on *subadditivity*. Here, however, we develop a method for using subadditivity which does not require the spatial homogeneity assumptions used in [46] (in the sense that the parameter ζ of a point set $\mathcal{X}$ satisfies $\zeta(a\mathcal{X}) = a^p \zeta(\mathcal{X})$ for some $p > 0$); instead we use the thermodynamic scaling of the extraneous distance parameter r. This is what enables us to deal with the larger class of weight functions than those considered in [46], and moreover to apply subadditivity to RGG functionals of the type considered here, for which subadditivity does not appear to have been used before.

Another methodology for deriving LLNs in the thermodynamic limit was developed in [39], based on *stabilization*. A point process functional is said to be stabilizing if it is a sum of locally determined contributions from the points. Most of the graph parameters considered here do not appear to be stabilizing in the supercritical case.

We now define various graph parameters for an arbitrary finite graph $G = (V, E)$. We are concerned here with the limit theory of these parameters for RGGs.

- *Independence number:* A set $A \subset V$ is said to be *independent* (or *stable*), if for any $u, v \in A$, $uv \notin E$. The *independence number* (or *stability number*) of G, here denoted $\alpha(G)$, is the maximum possible cardinality of an independent set $A \subset V$ (A could be V itself).

- *Domination number:* The *domination number* of G, here denoted $\gamma(G)$, is the minimum number m such that there exists a set A of m vertices with every vertex of G at graph distance at most 1 from A; such a set A is known as a *dominating set* for G.

- *Clique-covering number:* The clique-covering number of G, $\theta(G)$, is defined to be the chromatic number of the complement of G, that is, the minimum number of colors needed for coloring the vertices of G in such a way that no two adjacent vertices in the complement of G (i.e. non-adjacent vertices in G) obtain the same color. In other words, $\theta(G)$ is the minimum possible size (i.e. cardinality) of a *clique-partition* of V, where by a clique-partition we mean a partition π of V such that for each set $W \in \pi$, the subgraph of G induced by W is the complete graph on W.

- *Eternal domination number:* If $A \subset V$ is a dominating set, we may think of A as representing the locations of a set of 'guards', such that for any 'attack' on an unguarded vertex, there is a guard that can defend by moving to that vertex from an adjacent vertex. We say A is *eternally dominating* if guards, placed initially on the vertices of A, can defend against any finite or infinite *sequence* of attacks (so after each attack, the defender can move a guard from an adjacent vertex to the attacked vertex leaving a new configuration which is also eternally dominating). Here we allow each attack in the sequence to be decided on the spot by the attacker[1].

  The eternal domination number of G, $\gamma^\infty(G)$, is the minimum number of vertices possible in an eternally dominating set of vertices A. See e.g. [29] for further discussion.

It is well known (see [11]) and easy to see that

$$\gamma(G) \le \alpha(G) \le \gamma^\infty(G) \le \theta(G). \tag{1.1}$$

We shall also consider some further graph parameters, namely vertex cover number, H-packing number, edge cover number, number of components, and number of isolated vertices. We shall define these later on, in Section 3.

A second class of applications is to optimization problems over the weighted complete graph on the sample of n random points. The classic example [8] is the *traveling salesman problem* (TSP): find a tour (i.e. a Hamiltonian cycle) through the points of minimum total edge-length. It is natural to consider the analogous problem when the cost of each edge e is some *function* of the edge-length $\|e\|$, denoted $f(\|e\|)$ say, and one chooses the tour which minimizes the total cost. The case where $f(\|e\|) = \|e\|^p$ for some fixed constant p (i.e. power-weighted edges) has been much studied, see [46] and references therein, but many other functions are available. One way to take the limit is by thermodynamic scaling, i.e. consider a weight of the form $f(r_n^{-1}\|e\|)$ for a sample of size n, in the thermodynamic limit. Here we obtain LLNs under thermodynamic scaling for a general class of weight functions f, not only for the TSP but also for the minimum-weight matching (MM), minimum spanning tree (MST) and minimum bipartite matching (BM) problems (for exact definitions of these problems, see Section 4). In the case of BM, and bounded f, our results go beyond what was previously available for power-weighted edges.

[1] There is also a variant, famous in computer science, where the whole attack sequence is given in advance, the so called k-server problem. We will not consider this here.

We now describe some of the motivation and relevant past work. On the one hand, the study of the parameters in a random setup is motivated by the fact that several of the decision problems studied here (that is, to decide whether a certain parameter is at least k), in particular independence number, domination number, clique-covering number, vertex cover number and H-packing number are well known to be NP-complete even for restricted graph classes, see [19, 31, 13, 16]. Therefore unless $P = NP$, one cannot expect a polynomial-time algorithm for such problems. This motivates looking for polynomial-time algorithms which are 'near-optimal most of the time'. To quantify such terminology requires the study of these problems in a random setup (such as the current one).

We briefly discuss previous work on the corresponding problems for other random graph models. Consider first the Erdös-Rényi random graph $G(n, p)$ with $p = c/n$ for some constant c, which corresponds to our thermodynamic limit. When $c < 1$, a weak LLN for any of the graph parameters considered here can readily be obtained by computing the first two moments, using standard branching process approximation arguments for the exploration process. In the case of the independence number $\alpha(\cdot)$, the value of the limit in the LLN is determined in [40], where a central limit theorem is also provided. For $c \geq 1$, a weak LLN for $\alpha(G(n, p))$ was established in [7], resolving a long-standing open problem. In fact, [7] gives this LLN for the 'other' Erdös-Rényi model $G(n, m)$ with a deterministic number of edges m and with $m \propto n$, but the result for $G(n, p)$ with $p = c/n$ can then be readily derived from this. Similar LLNs for the independence number have also been obtained for the random d-regular graph [7] and the configuration model [42]. For $p = c/n$ with arbitrary c, a LLN is known for the *maximum matching number*, which is a special case of the H-packing number that we shall define later; see [26].

On the other hand, the parameters have several practical applications. Finding dominating sets is important in finding 'central' or 'important' sets of nodes in a network, in contexts such as facility location [22], molecular biology [34] for detecting significant proteins in protein-protein interaction networks, network controllability [15] and in wireless networks as centrality measure for efficient routing [45]. Dominating sets have attracted considerable attention in the combinatorics literature; see the monograph [22]. Domination numbers of random geometric graphs have been considered in [9].

Finding large independent sets has different applications: for example vertices might represent intervals of tasks, and there is an edge between vertices if the corresponding intervals overlap; in job scheduling one likes to find a maximum number of jobs to be scheduled on one machine [30], corresponding to the independence number. Also, the problem of finding maximum independent sets in geometric graphs has been studied, for example, in the context of automatic label placement: given a set of locations in a map, find a maximum set of disjoint rectangular labels near these locations [1].

Covering a graph with cliques can be seen as coloring the complement of the graph, and applications of coloring apply to the clique-covering number. An iterative procedure of covering a graph with 'almost' cliques was proposed in an influential work of [43], as a way of defining the fractal dimension of networks.

Regarding the eternal domination number, its study was motivated by ancient problems in military defence (see for example [4]). It is known to be NP-hard [28], but the decision problem of having an eternal domination number of at most k is not known to be NP-complete since it is not known whether it belongs to NP: it is not clear how to confirm in polynomial time that a given initial configuration of guards can defend all possible sequences of attacks, see [28, 29].

All of the classic optimization problems we consider here such as the weighted TSP, MM and BM functionals, and the weighted MST functional have numerous applications in operations research (for example, in food delivery) and computer science. Among these, only the TSP is NP-complete [19]; see [17] and [33] for polynomial-time algorithms for the matching and MST problems respectively.

Jaillet [25] has emphasized the importance in applications of estimating the rate of convergence in LLNs. His remarks refer to the classic optimization problems just mentioned, but also apply to the graph parameters discussed earlier.

Organization of the paper. In Section 2 we state the general results of this paper. In Section 3 we then give applications of the general results to graph parameters; in Section 4 we show how to apply the results to classic optimization problems such as TSP. Section 5 is devoted to the proof of the general results. Finally, Section 6 contains the proofs of additional results about the domination number in the dense regime.

2 Statement of general results

2.1 Preliminaries

We now describe our setup in more detail. Let $d \in \mathbb{N}$. The d-dimensional random geometric graph $G(\mathcal{X}_n, r)$ is defined as follows. We take $\mathcal{X}_n$ to be a set of n independent identically distributed (i.i.d.) random points in $\mathbb{R}^d$ with a specified common probability distribution μ, and assume $(r_n)_{n\geq 1}$ is a sequence of positive real numbers. For any locally finite $\mathcal{X} \subset \mathbb{R}^d$ and distance parameter $r > 0$, the graph $G(\mathcal{X}, r)$ is defined to have vertex set $\mathcal{X}$, with two vertices connected by an edge if and only if their spatial locations are at (Euclidean) distance at most r from each other.

Throughout this paper we assume the measure μ is diffuse (i.e. $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}^d$), so that the points of $\mathcal{X}_n$ are almost surely distinct. (Actually, our general results carry through to the case where μ is not diffuse, provided we allow for $\mathcal{X}_n$

to be a *multiset*, that is we count each repeated point of $\mathcal{X}_n$ as many times as it arises, but we shall not pursue this further here.) Unlike in [35], we do *not* generally assume μ has a probability density function.

Let $\mathbb{R}_+ := (0, \infty)$. Given sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ taking values in $\mathbb{R}_+$, we write $a_n \ll b_n$, or $b_n \gg a_n$, or $a_n = o(b_n)$, or $b_n = \omega(a_n)$, to mean that $a_n/b_n \to 0$ as $n \to \infty$. Also we write $a_n \sim b_n$ if $a_n/b_n \to 1$ as $n \to \infty$, and $a_n = \Theta(b_n)$ if a_n/b_n is bounded away from 0 and ∞. For $x \in \mathbb{R}$ let $\lfloor x \rfloor$ denote the integer part of x, i.e. the largest integer not exceeding x. Let $\lceil x \rceil$ denote the smallest integer not less than x. For any $x \in \mathbb{R}^d$ and $r > 0$, let $B_r(x)$ denote the closed Euclidean ball (disk) centred on x of radius r. Let o denote the origin in $\mathbb{R}^d$. For $k \in \mathbb{N}$ set $[k] := \{1, 2, \ldots, k\}$.

For $s \in \mathbb{R}_+$, define Q_s, a half-open cube of side s in $\mathbb{R}^d$ centred at the origin, by

$$Q_s := [-s/2, s/2)^d.$$

Let Leb denote the Lebesgue measure on $\mathbb{R}^d$. We set $\pi_d := \mathrm{Leb}(B_1(o))$, the volume of the unit ball. Let μ_U denote the uniform distribution on Q_1, i.e. the restriction of the Lebesgue measure to Q_1. We write $\mu = \mu_U$ in the case where the common distribution μ of the points of $\mathcal{X}_n$ is this uniform distribution.

Given a sequence of random variables ξ_n and a constant c, we write $\xi_n \xrightarrow{\text{a.s.}} c$ (respectively $\xi_n \xrightarrow{L^2} c$) if ξ_n converges to c almost surely (respectively, in mean-square) as $n \to \infty$. We write $\xi_n \xrightarrow{c.c.} c$ if we have *complete convergence* of ξ_n to c, by which we mean that for all $\varepsilon > 0$ we have $\sum_{n=1}^{\infty} \mathbb{P}[|\xi_n - c| > \varepsilon] < \infty$. By the Borel-Cantelli lemma, if $\xi_n \xrightarrow{c.c.} c$ then $\xi'_n \xrightarrow{\text{a.s.}} c$ for *any* sequence $(\xi'_n)_{n\geq 1}$ of random variables on a common probability space with ξ'_n having the same distribution as ξ_n for each n. For further discussion of complete convergence, see [46]. If both $\xi_n \xrightarrow{c.c.} c$ and $\xi_n \xrightarrow{L^2} c$, we write $\xi_n \xrightarrow[L^2]{c.c.} c$.

Besides $\mathcal{X}_n$, we now define two further point processes, denoted $\mathcal{P}_t$ and $\mathcal{H}_\lambda$. For $t > 0$ we define the Poissonized point process $\mathcal{P}_t$, coupled to $\mathcal{X}_n$ as follows. Let $X_0, X_1, X_2, \ldots$ be a sequence of independent identically distributed random d-vectors with common distribution μ, and let N_t be a Poisson random variable with mean t, independent of $(X_1, X_2, \ldots)$. Set $\mathcal{X}_n := \{X_1, \ldots, X_n\}$ and $\mathcal{P}_t := \{X_1, X_2, \ldots, X_{N_t}\}$. Then $\mathcal{P}_t$ is a Poisson point process in $\mathbb{R}^d$ with intensity measure $t\mu$ (see [32, Proposition 3.5] or [35, Proposition 1.5]), coupled to $\mathcal{X}_n$. Also, for $\lambda > 0$, let $\mathcal{H}_\lambda$ be a homogeneous Poisson point process of intensity λ on $\mathbb{R}^d$, and for $s > 0$ set $\mathcal{H}_{\lambda,s} := \mathcal{H}_\lambda \cap Q_s$. For any $\mathcal{X} \subset \mathbb{R}^d$ and $a > 0$, we write $a\mathcal{X}$ for $\{ax : x \in \mathcal{X}\}$, and for $y \in \mathbb{R}^d$ set $y + \mathcal{X} := \{y + x : x \in \mathcal{X}\}$. We write $|\mathcal{X}|$ for the number of elements of $\mathcal{X}$.

2.2 A class of functionals on point sets

Let $\zeta(\cdot)$ be a non-negative real-valued function defined on the collection of all finite subsets of $\mathbb{R}^d$, with $\zeta(\varnothing) = 0$, and having the following properties:

- P1 *Measurability:* For all $k \in \mathbb{N}$, the function $(x_1, \dots, x_k) \mapsto \zeta(\{x_1, \dots, x_k\})$ is Lebesgue-measurable from $D^{(k)}$ to $\mathbb{R}$, where $D^{(k)}$ denotes the set of $(x_1, \dots, x_k) \in (\mathbb{R}^d)^k$ such that $x_1, \dots, x_k$ are all distinct.
- P2 *Translation invariance*: $\zeta(x + \mathcal{X}) = \zeta(\mathcal{X})$ for all finite $\mathcal{X} \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$.
- P3 *Almost subadditivity:* There exists a constant $c_1 \in [0, \infty)$, such that $\zeta(\mathcal{Y} \cup \mathcal{Z}) \leq \zeta(\mathcal{Y}) + \zeta(\mathcal{Z}) + c_1$ for any two disjoint finite subsets $\mathcal{Y}, \mathcal{Z}$ of $\mathbb{R}^d$.
- P4 *Superadditivity up to boundary:* There exists a constant $c_2 \in [0, \infty)$ such that for any ordered pair $(\mathcal{Y}, \mathcal{Z})$ of disjoint finite sets in $\mathbb{R}^d$, with $\partial_{\mathcal{Z}}(\mathcal{Y})$ denoting the set of points of $\mathcal{Y}$ that lie at Euclidean distance at most 1 from the set $\mathcal{Z}$,

$$\zeta(\mathcal{Y} \cup \mathcal{Z}) \geq \zeta(\mathcal{Y}) + \zeta(\mathcal{Z}) - c_2 |\partial_{\mathcal{Z}}(\mathcal{Y})|.$$

Given the functional ζ, and given a scaling parameter $r > 0$, we define the scaled functional $\zeta_r(\mathcal{X})$ for all finite $\mathcal{X} \subset \mathbb{R}^d$ by

$$\zeta_r(\mathcal{X}) := \zeta(r^{-1}\mathcal{X}).$$

Sometimes we shall assume ζ has one or more of the following further properties:

- P5 *Local sublinear growth:* $\lim_{\delta \downarrow 0}(\sup\{\frac{\zeta(\mathcal{X})}{|\mathcal{X}|} : \mathcal{X} \subset B_\delta(o), \delta^{-1} \leq |\mathcal{X}| < \infty\}) = 0$.
- P5′ *Local uniform boundedness:* It is the case that $\sup_{\mathcal{X} \subset B_{1/2}(o), |\mathcal{X}| < \infty} |\zeta(\mathcal{X})| < \infty$.
- P6 *Upward monotonicity in $\mathcal{X}$*: For all finite $\mathcal{X} \subset \mathbb{R}^d$ and $\mathcal{Y} \subset \mathbb{R}^d$ we have $\zeta(\mathcal{X}) \leq \zeta(\mathcal{X} \cup \mathcal{Y})$.
- P7 *Downward monotonicity in r*: For all finite $\mathcal{X} \subset \mathbb{R}^d$ we have that $\zeta_r(\mathcal{X})$ is nonincreasing in r.
- P8 *RGG function:* ζ is a graph invariant of $G(\mathcal{X}, 1)$, that is, $\zeta(\mathcal{X}) = \zeta(\mathcal{Y})$ whenever $G(\mathcal{X}, 1)$ is isomorphic to $G(\mathcal{Y}, 1)$.

Note that Property P5′ implies Property P5. Many of our examples satisfy P5′.

For many of our examples, Property P8 holds, that is, $\zeta(\mathcal{X})$ is some specified graph invariant, evaluated on the geometric graph $G(\mathcal{X}, 1)$. Then $\zeta_r(\mathcal{X})$ is the same graph invariant evaluated for $G(\mathcal{X}, r)$. Note that if P8 applies then both the measurability condition P1 and the translation invariance P2 follow automatically. In particular, if P8 holds and we define $f : D^{(k)} \to \mathbb{R}$ by $f((x_1, \dots, x_k)) = \zeta(\{x_1, \dots, x_k\})$, then f takes only finitely many values and for each $t \in \mathbb{R}$, $f^{-1}(\{t\}) = C \cap D^{(k)}$, for some closed set $C \subset (\mathbb{R}^d)^k$, so that $f^{-1}(\{t\})$ is a Borel set; thus P1 holds.

2.3 General results for the thermodynamic limit

Theorem 2.1 below provides a law of large numbers, in the thermodynamic limit, for a general functional ζ satisfying properties P1–P5, applied to $\mathcal{X}_n$. The limit is expressed in terms of a function $\rho : (0, \infty) \to [0, \infty)$, defined as follows for all $\lambda > 0$:

$$\rho(\lambda) := \lim_{s\to\infty} \mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s})/(\lambda s^d)]. \tag{2.2}$$

Given $(r_n)_{n\geq1}$, we use r_n as the scaling parameter when applying ζ to $\mathcal{X}_n$. Let f_μ denote the density function (with respect to Leb) of the absolutely continuous part of μ, and let $\mu^\perp$ denote the singular part of μ, in its Lebesgue decomposition.

Theorem 2.1 (General LLN in the thermodynamic limit)**.** *(a) Let $\zeta(\cdot)$ be a functional satisfying Properties P1–P4 and P5. Then, for all $\lambda > 0$, the limit in (2.2) exists in $\mathbb{R}$. Moreover, if $nr_n^d \to t$ as $n \to \infty$, for some $t \in (0, \infty)$, we have*

$$n^{-1}\zeta_{r_n}(\mathcal{X}_n) \xrightarrow[L^2]{c.c.} \int_{\mathbb{R}^d} \rho(tf_\mu(x))f_\mu(x)dx \quad \text{as} \quad n \to \infty, \tag{2.3}$$

where for $\lambda > 0$, we define $\rho(\lambda)$ by (2.2), and moreover

$$n^{-1}\zeta_{r_n}(\mathcal{P}_n) \xrightarrow[L^2]{c.c.} \int_{\mathbb{R}^d} \rho(tf_\mu(x))f_\mu(x)dx \quad \text{as} \quad n \to \infty. \tag{2.4}$$

Moreover, almost surely,

$$n^{-1}|\zeta_{r_n}(\mathcal{X}_n) - \mathbb{E}\,[\zeta_{r_n}(\mathcal{X}_n)]| = O(((\log n)/n)^{1/2}). \tag{2.5}$$

(b) If we assume additionally that $\mu = \mu_U$ and $|nr_n^d - t| = O(n^{-1/d})$ as $n \to \infty$, then as $n \to \infty$,

$$|\mathbb{E}\,[n^{-1}\zeta_{r_n}(\mathcal{P}_n)] - \rho(t)| = O(n^{-1/d}), \tag{2.6}$$

$$|\mathbb{E}\,[n^{-1}\zeta_{r_n}(\mathcal{X}_n)] - \rho(t)| = O(n^{-\min(1/d,1/2)}). \tag{2.7}$$

Remarks. If $\zeta(\cdot)$ satisfies only P1–P4, then provided μ is absolutely continuous, the conclusion of Theorem 2.1 still holds. See Proposition 2.3(i) below, and the remark just before Lemma 5.6.

We conjecture that the rates in (2.6) and (2.7) are sharp in most examples, but that the factor of $(\log n)^{1/2}$ in (2.5) is an artifact of the proof and is not needed.

In some examples, the functional ζ does not satisfy P3 and P4 because the inequalities are the wrong way around, but can be obtained as a linear combination of functionals which do satisfy P3 and P4, possibly including the counting functional $\mathcal{X} \mapsto |\mathcal{X}|$. The following corollary deals with these cases.

Corollary 2.2. *Suppose for some $c_3 \geq 0$ that we can write $\zeta(\cdot) = c_3|\cdot| - \zeta'(\cdot) + \zeta''(\cdot)$, where $\zeta'(\cdot), \zeta''(\cdot)$ are non-negative functionals both satisfying Properties P1–P5. Then, for all $\lambda > 0$, the limit $\rho(\lambda)$ defined by (2.2) exists in $\mathbb{R}$. Also, if $nr_n^d \to t$ as $n \to \infty$, for some $t \in (0, \infty)$, we have*

$$n^{-1}\zeta_{r_n}(\mathcal{X}_n) \xrightarrow[L^2]{c.c.} \int_{\mathbb{R}^d} \rho(tf_\mu(x))f_\mu(x)dx + c_3\mu^\perp(\mathbb{R}^d) \quad \text{as} \quad n \to \infty. \tag{2.8}$$

Proof. For $\lambda > 0$, let $\rho'(\lambda), \rho''(\lambda)$ be defined analogously to $\rho(\lambda)$ at (2.2), viz., as the large-s limit of $\mathbb{E}[\zeta'(\mathcal{H}_{\lambda,s})/(\lambda s^d)]$, respectively $\mathbb{E}[\zeta''(\mathcal{H}_{\lambda,s})/(\lambda s^d)]$; these limits exist by applying Theorem 2.1 to ζ' and ζ'', respectively. Hence the limit $\rho(\lambda)$ defined by (2.2) also exists, with $\rho(\lambda) = c_3 - \rho'(\lambda) + \rho''(\lambda)$. Applying Theorem 2.1 to ζ' and ζ'', and using the fact that for any two sequences of random variables $(\xi_n)_n$ and $(\xi'_n)_n$, we have that $\xi_n \xrightarrow[L^2]{c.c.} c$ and $\xi'_n \xrightarrow[L^2]{c.c.} d$ together imply $\xi_n + \xi'_n \xrightarrow[L^2]{c.c.} c + d$, we obtain the complete and L^2 convergence

$$\begin{aligned}
\lim_{n\to\infty} n^{-1}\zeta_{r_n}(\mathcal{X}_n) &= \lim_{n\to\infty} (c_3 - n^{-1}\zeta'_{r_n}(\mathcal{X}_n) + n^{-1}\zeta''_{r_n}(\mathcal{X}_n)) \\
&= c_3 + \int_{\mathbb{R}^d} [-\rho'(tf_\mu(x)) + \rho''(tf_\mu(x))]f_\mu(x)dx,
\end{aligned}$$

which is equal to the right hand side of (2.8), as required. □

Our next result gives some of the properties of the function $\rho(\cdot)$.

Proposition 2.3 (Properties of $\rho(\cdot)$)**.** *Suppose ζ satisfies Properties P1–P4. Then:*
(i) For all $\lambda > 0$ the limit in (2.2) exists in $\mathbb{R}$. In fact, given $\lambda_1 > 0$,

$$\sup_{\lambda\in[\lambda_1,\infty)} \left(s^{-d}\mathbb{E}[\zeta(\mathcal{H}_\lambda \cap Q_s)] - \lambda\rho(\lambda)\right) = O(s^{-1}) \quad \text{as } s \to \infty. \tag{2.9}$$

(ii) The function $\lambda \mapsto \lambda\rho(\lambda)$ is Lipschitz continuous on $(0, \infty)$ with Lipschitz constant at most $K := \max(c_1 + \zeta(\{o\}), c_2 - \zeta(\{o\}))$, where c_1, c_2 are the constants in P3 and P4 respectively. Hence the function $\lambda \mapsto \rho(\lambda)$ is also continuous.
(iii) For all $\lambda > 0$ we have $\rho(\lambda) \leq c_1 + \zeta(\{o\})$.
(iv) If ζ satisfies P6, then the function $\lambda \mapsto \lambda\rho(\lambda)$ is nondecreasing.
(v) If ζ satisfies P7, then the function $\lambda \mapsto \rho(\lambda)$ is nonincreasing.
(vi) If we can take $c_1 = 0$ in P3, then $\rho(\lambda) \to \zeta(\{o\})$ as $\lambda \downarrow 0$.
(vii) If $\zeta(\{o\}) > 0$, then $\lambda \mapsto \lambda\rho(\lambda)$ is strictly increasing for $0 < \lambda < \zeta(\{o\})/(c_2\pi_d)$.

In some cases we can improve on Part (vii): see Remark 5.4.

Let $\lambda_c := \lambda_c(d)$ be the critical value for continuum percolation, namely the supremum of all λ such that all components of $G(\mathcal{H}_\lambda, 1)$ are finite. It is known that $\lambda_c(d) \in (0, \infty)$ for $d \geq 2$ and $\lambda_c(1) = +\infty$. See [35] for further discussion.

Proposition 2.4 (Alternative characterization of $\rho(\lambda)$ for subcritical λ)**.** *Suppose $\lambda \in (0, \lambda_c)$. Suppose ζ satisfies Properties P1–P4, with $c_1 = 0$ in P3. Then the constant $\rho(\lambda)$ given at (2.2) satisfies*

$$\rho(\lambda) = \mathbb{E}\left[\zeta(\mathcal{C}_o(\lambda))/|\mathcal{C}_o(\lambda)|\right], \tag{2.10}$$

where $\mathcal{C}_o(\lambda)$ denotes the set of vertices of the component containing o of the graph $G(\mathcal{H}_\lambda \cup \{o\}, 1)$.

It follows from Proposition 2.4 that if the hypotheses of Theorem 2.1 hold with $c_1 = 0$, and also $tf_\mu(x) < \lambda_c$ for Lebesgue-almost all $x \in \mathbb{R}^d$ then the limit in (2.3) is equal to $\int_{\mathbb{R}^d} \mathbb{E}\left[\zeta(\mathcal{C}_o(tf_\mu(x)))/|\mathcal{C}_o(tf_\mu(x))|\right] f_\mu(x)dx$. In the case where μ is absolutely continuous, this can also be proved using the methods of [39].

If the hypotheses of Theorem 2.1 hold with $c_1 = 0$, and also $\sup_{x\in\mathbb{R}^d} tf_\mu(x) < \lambda_c$, then we expect that one could prove a Gaussian limit for $\zeta_{r_n}(\mathcal{X}_n)$ (suitably scaled and centred), using the methods of [37] for example. However, this is beyond the remit of the present paper.

2.4 General results for dense limiting regimes

In Theorem 2.6 below, we provide information about the limiting behaviour of $\rho(\lambda)$ for large λ, under the extra conditions P5$'$ (local uniform boundedness), P6 (monotonicity) and P8 (ζ is a RGG functional). This complements the information provided for small λ in parts (vi) and (vii) of Proposition 2.3. Before stating this result, we first need to describe the deterministic limiting behaviour of $\zeta^*(Q_s)$, where for $A \subset \mathbb{R}^d$, we set

$$\zeta^*(A) := \sup\{\zeta(\mathcal{X}) : \mathcal{X} \subset A, |\mathcal{X}| < \infty\}. \tag{2.11}$$

Lemma 2.5. *Suppose $\zeta(\cdot)$ satisfies P1–P4 and P5$'$. Then the limit*

$$\overline{\zeta} := \lim_{s\to\infty} \left(s^{-d}\zeta^*(Q_s)\right) \tag{2.12}$$

exists in $[0, \infty)$. Also for all $\lambda > 0$, we have

$$\lambda\rho(\lambda) \le \overline{\zeta}. \tag{2.13}$$

Theorem 2.6. *Suppose ζ satisfies Properties P3, P4, P5$'$, P6 and P8. Then*

$$\lim_{\lambda\to\infty} (\lambda\rho(\lambda)) = \overline{\zeta}. \tag{2.14}$$

Our main result in this subsection gives the limiting behaviour of $\zeta_{r_n}(\mathcal{X}_n)$ in the *dense limit* with $nr_n^d \to \infty$ and $r_n \to 0$. Here we consider only the case where μ has a density f_μ, and where moreover $f_\mu^{-1}((0,\infty))$ is Riemann measurable. Recall that we say a set $A \subset \mathbb{R}^d$ is *Riemann measurable* if $\mathbf{1}_A$ is Riemann integrable, i.e. if A is bounded and ∂A has Lebesgue measure zero.

Theorem 2.7 (General LLN in the dense limit)**.** *Suppose ζ satisfies Properties P3, P4, P5′, P6 and P8. Suppose μ is absolutely continuous, and f_μ can be chosen in such a way that $f_\mu^{-1}((0,\infty))$ is Riemann measurable. Suppose $r_n \to 0$ and $nr_n^d \to \infty$ as $n \to \infty$. Then, almost surely*

$$\lim_{n\to\infty} r_n^d \zeta_{r_n}(\mathcal{X}_n) = \overline{\zeta}\ \mathrm{Leb}(f_\mu^{-1}((0,\infty))).$$

We shall prove the general results stated so far in Section 5.

3 Applications to graph parameters

In this section, we seek to apply Theorems 2.1, 2.6 and 2.7 to the four graph parameters described in the Introduction, and also to the following graph parameters:

- *Vertex cover number:* A set $W \subset V$ is called a *vertex cover* of G if every edge of G is incident to at least one vertex in W. The *vertex cover number* of G is the smallest possible number of vertices required for a vertex cover of G.

- *H-packing number:* Let H be a fixed connected graph with k vertices, $2 \leq k < \infty$. Given $m \in \mathbb{N}$, we refer to any collection of m vertex-disjoint H-subgraphs of G as an *H-packing* of size m. Here an H-subgraph means a subgraph isomorphic to H (it does not need to be an induced subgraph). The *maximum H-packing number* $\psi_H(G)$ is defined to be the largest possible size of H-packing in G. In a remark just after Theorem 3.15, we shall describe a generalization that allows for more than one H.

- *Edge cover number:* If G has no isolated vertex, the edge cover number $\eta(G)$ is the smallest number of edges that can be selected such that every vertex is incident to at least one of the selected edges. For general G, we let $\eta(G)$ denote the edge cover number of the graph obtained from G by removing all isolated vertices.

- *Number of connected components*, and *number of isolated vertices.* The definitions of these graph parameters are well known.

We consider in turn each of these graph parameters, evaluated on the geometric graph $G(\mathcal{X}, 1)$. Clearly, in each case this gives a functional $\zeta(\cdot)$ satisfying Property P8, and hence both P1 and P2. Also, each of these graph parameters, except vertex cover number and edge cover number, satisfies Property P7, because it can only be reduced by adding an edge.

3.1 Independence number

For finite $\mathcal{X} \subset \mathbb{R}^d$, let $\alpha(\mathcal{X}) := \alpha(G(\mathcal{X}, 1))$, the independence number of the geometric graph $G(\mathcal{X}, 1)$. Thus if we take $\zeta(\mathcal{X}) = \alpha(\mathcal{X})$, then $\zeta_r(\mathcal{X}) = \alpha(G(\mathcal{X}, r))$ for all $r > 0$. Let $\mathcal{B}$ denote the class of bounded Borel subsets of $\mathbb{R}^d$. Given $A \in \mathcal{B}$, set

$$\begin{aligned} \alpha^*(A) &:= \sup\{\alpha(\mathcal{X}) : \mathcal{X} \subset A, |\mathcal{X}| < \infty\} \\ &= \sup\{|\mathcal{X}| : \mathcal{X} \subset A, |\mathcal{X}| < \infty, E(G(\mathcal{X}, 1)) = \varnothing\}, \end{aligned} \tag{3.1}$$

where $E(\cdot)$ denotes the set of edges of the graph in question. Thus $\alpha^*(A)$ is the maximum number of disjoint closed balls of radius 1/2 which can be packed into A (or at least, with their centres in A), which is clearly finite.

Theorem 3.1 (LLNs for independence number of RGG)**.** *Theorems 2.1, 2.6 and 2.7 all apply when choosing* $\zeta(\mathcal{X}) := \alpha(\mathcal{X})$.

Remarks. (a) A Poissonized version of the case $\mu = \mu_U$ of Theorem 2.1 for the present choice of $\zeta(\cdot)$ was given in [39, Theorem 2.7], but that result required the limit of nr_n^d to be below the percolation threshold λ_c.

(b) With the current choice of ζ, the quantity $\lambda\rho(\lambda)$ defined at (2.2) is the intensity of a maximum hard-core thinning of the restriction of $\mathcal{H}_\lambda$ to a box, in the limit of a large box. Here, by 'hard-core thinning' we mean a sub-point process with all inter-point distances greater than 1. A related quantity is the maximum possible intensity of a *hard-core stationary thinning* of the whole of $\mathcal{H}_\lambda$; we denote this quantity here by $\hat{\rho}(\lambda)$. See [24] for further details on hard-core stationary thinnings. It is not hard to see that $\hat{\rho}(\lambda) \leq \lambda\rho(\lambda)$, and we conjecture that in fact equality holds here.

(c) By Lemma 2.5, the limit

$$\overline{\alpha} := \lim_{s \to \infty} (s^{-d}\alpha^*(Q_s)), \tag{3.2}$$

exists in $\mathbb{R}$. It is the optimal (i.e. maximal) *packing density* of balls of radius 1/2 in $\mathbb{R}^d$, where we measure density here by 'number of packed balls per unit volume', not 'volume of packed balls per unit volume'. For the present choice of ζ, the limit $\overline{\zeta}$ appearing in the statement of Theorems 2.6 and 2.7 is equal to $\overline{\alpha}$.

It can be seen that $\overline{\alpha} = 1$ for $d = 1$, and $\overline{\alpha} = \sqrt{4/3}$ for $d = 2$ by Thue's theorem on disk packing. As a consequence of the Kepler conjecture [21], for $d = 3$ we have $\overline{\alpha} = \sqrt{2}$. For further discussion of packing densities, see [41, 14].

Proof of Theorem 3.1. Given disjoint finite $\mathcal{Y}, \mathcal{Z} \subset \mathbb{R}^d$, and given independent $\mathcal{A} \subset \mathcal{Y} \cup \mathcal{Z}$ with $|\mathcal{A}| = \alpha(\mathcal{Y} \cup \mathcal{Z})$, the set $\mathcal{A} \cap \mathcal{Y}$ is independent in $G(\mathcal{Y}, 1)$ and $\mathcal{A} \cap \mathcal{Z}$ is independent in $G(\mathcal{Z}, 1)$. Therefore $\alpha(\mathcal{Y} \cup \mathcal{Z}) \leq \alpha(\mathcal{Y}) + \alpha(\mathcal{Z})$, so Property P3 (almost subadditivity) holds with $c_1 = 0$.

To check Property P4, let $\mathcal{I} \subset \mathcal{Y}$ be an independent set of $G(\mathcal{Y}, 1)$ with $|\mathcal{I}| = \alpha(\mathcal{Y})$, and let $\mathcal{J} \subset \mathcal{Z}$ be an independent set of $G(\mathcal{Z}, 1)$ with $|\mathcal{J}| = \alpha(\mathcal{Z})$. Let $\mathcal{I}' := \mathcal{I} \setminus \partial_{\mathcal{Z}}\mathcal{Y}$. Then $\mathcal{I}' \cup \mathcal{J}$ is an independent set in $G(\mathcal{Y} \cup \mathcal{Z}, 1)$. Hence

$$\begin{aligned} \alpha(\mathcal{Y} \cup \mathcal{Z}) &\geq |\mathcal{I}' \cup \mathcal{J}| = |\mathcal{I}'| + |\mathcal{J}| \\ &\geq |\mathcal{I}| - |\partial_{\mathcal{Z}}\mathcal{Y}| + |\mathcal{J}| = \alpha(\mathcal{Y}) + \alpha(\mathcal{Z}) - |\partial_{\mathcal{Z}}\mathcal{Y}|, \end{aligned}$$

which gives us Property P4 with $c_2 = 1$. Also for any finite nonempty $\mathcal{X} \subset B_{1/2}(o)$, $G(\mathcal{X}, 1)$ is a complete graph so $\alpha(\mathcal{X}) \leq 1$. Hence P5′ (locally uniform boundedness), and hence also P5, holds here. Thus Theorem 2.1 applies.

Property P6 clearly holds here, since the independence number can only be increased by adding vertices. Therefore Theorems 2.6 and 2.7 also apply. □

3.2 Domination number

For finite $\mathcal{X} \subset \mathbb{R}^d$, set $\gamma(\mathcal{X}) := \gamma(G(\mathcal{X}, 1))$, the domination number of $G(\mathcal{X}, 1)$. Then taking $\zeta(\cdot) := \gamma(\cdot)$, we have $\zeta_r(\mathcal{X}) = \gamma(G(\mathcal{X}, r))$ for all $r > 0$.

For this choice of ζ, it is shown in [9, Theorem 2(c)] that when $d = 2$ and $\mu = \mu_U$, if nr_n^2 converges to a positive finite limit, then $n^{-1}\zeta_{r_n}(\mathcal{X}_n) = \Theta(1)$ in probability. Our next result improves on this by showing almost sure convergence to a limit, and allowing for general d and μ.

Theorem 3.2 (LLN for domination number of RGG in the thermodynamic limit)**.** *Theorem 2.1 holds when choosing $\zeta(\mathcal{X})$ as $\gamma(\mathcal{X})$.*

Proof. Setting $\zeta(\mathcal{X}) := \gamma(\mathcal{X})$, we need to check Properties P3–P5. For P3 (almost subadditivity), let $\mathcal{Y}, \mathcal{Z}$ be disjoint finite subsets of $\mathbb{R}^d$. Taking a minimum dominating set for the graph induced by $\mathcal{Y}$ together with a minimum dominating set for the graph induced by $\mathcal{Z}$ yields a dominating set for the graph induced by $\mathcal{Y} \cup \mathcal{Z}$, and hence $\gamma(\cdot)$ satisfies Property P3 with $c_1 = 0$. Also by Lemma 3.6, which we give at the end of this subsection, $\gamma(\cdot)$ satisfies Property P4 (superadditivity up to boundary).

Finally $\gamma(\mathcal{X}) \leq 1$ for all $\mathcal{X} \subset B_{1/2}(o)$, so P5′ (and hence P5) holds for $\gamma(\cdot)$. Thus, $\gamma(\cdot)$ satisfies all the conditions for Theorem 2.1. □

Property P6 (upward monotonicity in $\mathcal{X}$) does *not* hold here, since adding vertices might make the domination number smaller. Therefore Theorems 2.6 and 2.7, concerning high density limits, are not applicable here. Nevertheless, we are able to provide certain similar results for the domination number too. First we need to recall the notion of *covering density*, adapted from [41].

For $A \in \mathcal{B}$ (the class of bounded Borel subsets of $\mathbb{R}^d$), let $\kappa(A)$ denote the smallest possible number of unit radius balls required to cover A, i.e.

$$\kappa(A) = \inf\{|\mathcal{X}| : \mathcal{X} \subset \mathbb{R}^d, |\mathcal{X}| < \infty, A \subset \cup_{x \in \mathcal{X}} B_1(x)\}. \tag{3.3}$$

Then $\kappa(A \cup A') \leq \kappa(A) + \kappa(A')$ for any $A, A' \in \mathcal{B}$ with $A \cap A' = \varnothing$, since for any covering of A and any covering of A', they can be combined to cover $A \cup A'$. Also $\kappa(A) \leq \kappa(Q_1) < \infty$ for all Borel $A \subset Q_1$. Hence by a simple deterministic version of the subadditive limit theorem (Lemma 5.1 below),

$$\lim_{s \to \infty} \left(s^{-d} \kappa(Q_s)\right) = \inf_{s \geq 1} \left(s^{-d} \kappa(Q_s)\right) =: \overline{\kappa}. \tag{3.4}$$

The quantity $\overline{\kappa}$ is the optimal (i.e. minimal) covering density of unit balls in $\mathbb{R}^d$. We can now state our results for the domination number that are analogous to Theorems 2.6 and 2.7.

Theorem 3.3. *Choosing $\zeta(\mathcal{X})$ as $\gamma(\mathcal{X})$, and defining $\rho(\lambda)$ by (2.2), we have that $\lim_{\lambda \to \infty}(\lambda \rho(\lambda)) = \overline{\kappa}$.*

Theorem 3.4 (LLN for domination number of RGG in the dense limit)**.** *Suppose $\mu = \mu_U$ and $n^{-1/d} \ll r_n \ll 1$. Then*

$$r_n^d \gamma(G(\mathcal{X}_n, r_n)) \xrightarrow{\text{a.s.}} \overline{\kappa} \quad \text{as } n \to \infty. \tag{3.5}$$

When $d = 2$, it is known that $\overline{\kappa} = \sqrt{4/27}$. See [27], or [41, page 16]. Therefore, Theorem 3.4 extends [9, Theorem 2(b)], which is concerned only with convergence in probability in the case where $d = 2$ and $r_n \gg ((\log n)/n)^{1/2}$.

We defer the proofs of Theorems 3.3 and 3.4 to Section 6. It is likely that Theorem 3.4 can be generalized from the special case given here with $\mu = \mu_U$, to general absolutely continuous μ such that f_μ can be taken to be Lebesgue-almost everywhere continuous with $f_\mu^{-1}((0,\infty))$ Riemann measurable. In this generality the limit at (3.5) should be $\overline{\kappa}\, \mathrm{Leb}(f_\mu^{-1}((0,\infty)))$.

We now give some relations between $\overline{\kappa}$, and $\overline{\zeta}$ defined at (2.12), and the packing density $\overline{\alpha}$ from the previous subsection.

Proposition 3.5. *Choosing $\zeta(\mathcal{X})$ as $\gamma(\mathcal{X})$, we have $\overline{\kappa} \leq \overline{\alpha} = \overline{\zeta}$.*

Proof. Given $A \in \mathcal{B}$, we have $\kappa(A) \le \alpha^*(A)$. Indeed, for any $\mathcal{X} \subset A$ with $|\mathcal{X}| = \alpha^*(A)$ and $E(G(\mathcal{X}, 1)) = \varnothing$, we have $A \subset \cup_{x \in \mathcal{X}} B_1(x)$ (else we could find a point $y \in A \setminus \mathcal{X}$ with $E(G(\mathcal{X} \cup \{y\}, 1)) = \varnothing$). Hence by (3.4) and (3.2), $\overline{\kappa} \le \overline{\alpha}$.

Since the domination number of a graph with no edges equals the number of vertices, with our current choice of ζ we have $\alpha^*(A) \le \zeta^*(A)$. But also by (1.1) we have $\zeta(\mathcal{X}) \le \alpha(\mathcal{X})$ for any finite $\mathcal{X} \subset \mathbb{R}^d$, and hence $\zeta^*(A) \le \alpha^*(A)$. Thus $\alpha^*(A) = \zeta^*(A)$. Hence by (2.12) and (3.2), $\overline{\alpha} = \overline{\zeta}$. □

The inequality in Proposition 3.5 is strict, at least for low dimensions. Indeed, for $d = 1$ it can be seen that $\overline{\kappa} = 1/2$, while $\overline{\alpha} = 1$. For $d = 2$, $\overline{\kappa} = \sqrt{4/27}$ while $\overline{\alpha} = \sqrt{4/3}$ as mentioned already. For $d = 3$ we have $\overline{\alpha} = \sqrt{2}$ as mentioned already, while $\overline{\kappa} \le \sqrt{125/1024}$ by consideration of a body-centred cubic array; see for example [6, 18]. Thus, at least in low dimensions we have $\overline{\kappa} < \overline{\zeta}$, so that the limit in Theorems 3.3 and 3.4 (which equals $\overline{\kappa}$) is not equal to $\overline{\zeta}$. This distinguishes the domination number from a number of other graph parameters (see Subsections 3.1, 3.3 and 3.4) for which we *can* apply Theorems 2.6 and 2.7 to obtain limiting results analogous to Theorems 3.3 and 3.4, but there with the limit equal to the relevant $\overline{\zeta}$.

Lemma 3.6. *Let $\mathcal{Y} \subset \mathbb{R}^d$ and $\mathcal{Z} \subset \mathbb{R}^d$ be finite and disjoint. Then $\gamma(\mathcal{Y} \cup \mathcal{Z}) \ge \gamma(\mathcal{Y}) + \gamma(\mathcal{Z}) - (1 + \kappa(B_2(o)))|\partial_{\mathcal{Z}} \mathcal{Y}|$.*

Proof. Let $\mathcal{S} \subset \mathcal{Y} \cup \mathcal{Z}$ be a dominating set in $\mathcal{Y} \cup \mathcal{Z}$ with $|\mathcal{S}| = \gamma(\mathcal{Y} \cup \mathcal{Z})$. Enumerate the points of $\mathcal{Y}$ distant at most 1 from $\mathcal{Z}$ as $y_1, \ldots, y_m$, where $m = |\partial_{\mathcal{Z}} \mathcal{Y}|$. Then $(\mathcal{S} \cap \mathcal{Y}) \cup \{y_1, \ldots, y_m\}$ is a dominating set in $\mathcal{Y}$.

Now set $k := \kappa(B_2(o))$, the number of closed balls of radius 1 required to cover a closed ball of radius 2 centred at the origin. For $1 \le i \le m$, let $B_{i,1}, \ldots B_{i,k}$ be closed balls of radius 1/2 that cover $B_1(y_i)$. For $1 \le j \le k$, if $\mathcal{Z} \cap B_{i,j} \cap B_1(y_i) \ne \varnothing$ then pick one element of $\mathcal{Z} \cap B_{i,j} \cap B_1(y_i)$ and denote it $z_{i,j}$. Then for each $z \in B_1(y_i) \cap \mathcal{Z}$, there exists j such that $z \in B_{i,j}$, and then $z_{i,j}$ is defined and $\|z - z_{i,j}\| \le 1$. Hence the set

$$\mathcal{T} := \cup_{i=1}^m \{z_{i,j} : j \in \{1, \ldots, k\}, \mathcal{Z} \cap B_{i,j} \cap B_1(y_i) \ne \varnothing\}$$

is a dominating set for $\mathcal{Z} \cap (\cup_{i=1}^m B_1(y_i))$ (which equals $\partial_{\mathcal{Y}} \mathcal{Z}$), and hence $(\mathcal{S} \cap \mathcal{Z}) \cup \mathcal{T}$ is a dominating set for $\mathcal{Z}$. Also $|\mathcal{T}| \le mk$. Then

$$\begin{aligned}
\gamma(\mathcal{Y}) + \gamma(\mathcal{Z}) &\le |(\mathcal{S} \cap \mathcal{Y}) \cup \{y_1, \ldots, y_m\}| + |(\mathcal{S} \cap \mathcal{Z}) \cup \mathcal{T}| \\
&\le (|\mathcal{S} \cap \mathcal{Y}| + m) + (|\mathcal{S} \cap \mathcal{Z}| + km) \\
&= |\mathcal{S}| + (1 + k)m \\
&= \gamma(\mathcal{Y} \cup \mathcal{Z}) + (1 + k)|\partial_{\mathcal{Z}} \mathcal{Y}|,
\end{aligned}$$

which gives us the result. □

Although Proposition 2.3(iv) does not apply with the current choice of ζ (because P6 fails), by Proposition 2.3(vii) and Lemma 3.6, we know at least that $\lambda \mapsto \lambda\rho(\lambda)$ is increasing on $(0, 1/(\pi_d(1+\kappa(B_2(o)))))$, and in fact Remark 5.4 yields even a larger range of λ for which this holds.

3.3 Clique-covering number

For finite $\mathcal{X} \subset \mathbb{R}^d$, let $\theta(\mathcal{X}) := \theta(G(\mathcal{X}, 1))$, the clique-covering number of $G(\mathcal{X}, 1)$.

Theorem 3.7 (LLNs for clique-covering number of RGG)**.** *Theorems 2.1, 2.6 and 2.7 all apply when choosing $\zeta(\mathcal{X}) := \theta(\mathcal{X})$.*

Denote by $\overline{\theta}$ the quantity $\overline{\zeta}$ appearing in Theorems 2.6 and 2.7 when we take $\zeta(\cdot) = \theta(\cdot)$. Then $\overline{\theta}$ is the minimum density (number of sets per unit volume) of a partition of $\mathbb{R}^d$ into sets of Euclidean diameter at most 1. It can be seen that $\overline{\theta} = 1$ for $d = 1$. For $d = 2$, by partitioning the plane into regular hexagons of Euclidean diameter 1, it can be shown that $\overline{\theta} \leq \sqrt{64/27}$. We conjecture that in fact equality holds here, i.e. the most efficient packing of the plane by sets of Euclidean diameter 1 uses regular hexagons. For $d = 3$, an upper bound for $\overline{\theta}$ can be obtained for example by tiling the three-dimensional space into trapezo-rhombic dodecahedra of diameter 1.

Proof of Theorem 3.7. Given finite disjoint $\mathcal{Y}, \mathcal{Z} \subset \mathbb{R}^d$, we have $\theta(\mathcal{Y} \cup \mathcal{Z}) \leq \theta(\mathcal{Y}) + \theta(\mathcal{Z})$: indeed, color all vertices in $\mathcal{Y}$ with $\theta(\mathcal{Y})$ many colors, and all vertices in $\mathcal{Z}$ with $\theta(\mathcal{Z})$ fresh colors. The coloring obtained is clearly an upper bound on $\theta(\mathcal{Y} \cup \mathcal{Z})$, and Property P3 (with $c_1 = 0$) follows.

For Property P4, let π be a clique-partition of $\mathcal{Y} \cup \mathcal{Z}$ with $|\pi| = \theta(\mathcal{Y} \cup \mathcal{Z})$. Set

$$\pi_1 := \{S \cap \mathcal{Y} : S \in \pi, S \cap \mathcal{Y} \neq \varnothing\}; \qquad \pi_2 := \{S \cap \mathcal{Z} : S \in \pi, S \cap \mathcal{Z} \neq \varnothing\}.$$

Then π_1, π_2 are clique-partitions of $\mathcal{Y}, \mathcal{Z}$ respectively, with

$$|\pi_1| + |\pi_2| - |\pi| = |\{S \in \pi : S \cap \mathcal{Y} \neq \varnothing \text{ and } S \cap \mathcal{Z} \neq \varnothing\}| \leq |\partial_{\mathcal{Z}} \mathcal{Y}|,$$

and rearranging this shows that P4 holds with $c_2 = 1$. Also P5$'$ holds, since $\theta(\mathcal{X}) \leq 1$ if $\mathcal{X} \subset B_{1/2}(o)$ (so that $G(\mathcal{X}, 1)$ is complete). Therefore Theorem 2.1 is applicable.

Clearly P6 and P7 also hold, since the clique-covering number can only be increased by adding vertices and can only be decreased by adding edges. Since P6 holds, Theorems 2.6 and 2.7 also apply. $\square$

3.4 Eternal domination number

For finite $\mathcal{X} \subset \mathbb{R}^d$, set $\gamma^\infty(\mathcal{X}) := \gamma^\infty(G(\mathcal{X}, 1))$, the eternal domination number.

Theorem 3.8 (LLNs for eternal domination number of RGG)**.** *Theorems 2.1, 2.6 and 2.7 all apply when choosing $\zeta(\mathcal{X}) := \gamma^\infty(\mathcal{X})$.*

Denote by $\overline{\gamma^\infty}$ the quantity $\overline{\zeta}$ appearing in Theorems 2.6 and 2.7 when we take $\zeta(\mathcal{X}) = \gamma^\infty(\mathcal{X})$. Then $\overline{\gamma^\infty}$ is the minimum density (number of guards per unit volume in the large-s limit) of a finite set of guards placed in Q_s that can defend against any sequence of attacks on locations in Q_s with moves of Euclidean size at most 1 in each step. By (1.1) and (2.12) we have $\overline{\alpha} \leq \overline{\gamma^\infty} \leq \overline{\theta}$ in all dimensions. Hence $\overline{\gamma^\infty} = 1$ for $d = 1$, and for $d = 2$ we have $\sqrt{4/3} \leq \overline{\gamma^\infty} \leq \sqrt{64/27}$. It would be of interest to find sharper upper and lower bounds when $d = 2$.

Proof of Theorem 3.8. Given finite disjoint $\mathcal{Y}, \mathcal{Z} \subset \mathbb{R}^d$, if $\mathcal{A} \subset \mathcal{Y}$ is eternally dominating in $\mathcal{Y}$, and $\mathcal{A}' \subset \mathcal{Z}$ is eternally dominating in $\mathcal{Z}$, then $\mathcal{A} \cup \mathcal{A}'$ is eternally dominating in $\mathcal{Y} \cup \mathcal{Z}$, since one can defend all attacks on $\mathcal{Y}$ using guards from $\mathcal{A}$ and all attacks on $\mathcal{Z}$ using guards from $\mathcal{A}'$. Therefore $\gamma^\infty(\mathcal{Y} \cup \mathcal{Z}) \leq \gamma^\infty(\mathcal{Y}) + \gamma^\infty(\mathcal{Z})$, so Property P3 holds with $c_1 = 0$. We check P4 in Lemma 3.10 below.

Property P5$'$ (local uniform boundedness) follows from the fact that $\gamma^\infty(\mathcal{X}) \leq \theta(\mathcal{X})$ by (1.1), and the uniform boundedness of $\theta(\cdot)$ checked earlier. Property P6 holds by Lemma 3.11 below. Therefore Theorems 2.1, 2.6 and 2.7 all apply. $\square$

Before checking P4 we give a further lemma. Given a finite graph $G = (V, E)$, consider a function $\phi : V \to \mathbb{N} \cup \{0\}$ representing an assignment of finitely many guards to the vertices, now with *more than one guard allowed* on each vertex. Let us say that ϕ is eternally dominating if starting from ϕ, one can defend against any sequence of attacks on unoccupied vertices, moving one guard at a time. In the special case where ϕ takes values only in $\{0, 1\}$, this reduces to our earlier definition of eternal domination. Let $\gamma_*^\infty(G)$ be the minimum number of guards (i.e. the minimal value of $\sum_{v \in V} \phi(v)$) required for an eternally dominating ϕ.

Lemma 3.9. *Let $G = (V, E)$ be a finite graph. Then $\gamma_*^\infty(G) = \gamma^\infty(G)$.*

Proof. Essentially this is shown in [11], but for completeness we include a more detailed explanation than is given there. The inequality $\gamma_*^\infty(G) \leq \gamma^\infty(G)$ is obvious.

Let $\phi : V \to \mathbb{N} \cup \{0\}$ be eternally dominating with $\sum_{v \in V} \phi(v) = \gamma_*^\infty(G)$, and also with maximum total *coverage* out of all such functions, where the coverage of ϕ is defined to be the total number of $v \in V$ for which $\phi(v) > 0$. Suppose for some $v_0 \in V$ that $\phi(v_0) > 1$. We shall derive a contradiction.

For any finite sequence of attacks, the assignment of guards after defending against these attacks must still be eternally dominating. Also the coverage never goes down as one defends against the successive attacks, because one moves a guard to an empty vertex each time.

The coverage never goes up as one defends against successive attacks, because we assumed the original ϕ had maximum coverage out of all eternally dominating assignments of guards. Therefore in defending against any sequence of attacks, we never move any guard off from v_0, because if we did then the coverage would go up.

But this shows we could still defend ourselves if we reduced $\phi(v_0)$ to 1 (leaving $\phi(v)$ unchanged for all other $v \in V$) and never moving the (single) guard at v_0. This change reduces the total number of guards, contradicting the earlier assertion that originally $\sum_{v\in V} \phi(V) = \gamma_*^\infty(G)$. Thus by contradiction, $\phi(v) \leq 1$ for all $v \in V$, so $\gamma^\infty(G) \leq \sum_{v\in V} \phi(v) = \gamma_*^\infty(G)$. □

Lemma 3.10. *Property P4 (superadditivity up to boundary) holds for* $\gamma^\infty(\cdot)$.

Proof. Let $\mathcal{Y}, \mathcal{Z}$ be disjoint finite subsets of $\mathbb{R}^d$, and let $\mathcal{A}$ be an eternally dominating set in $\mathcal{Y} \cup \mathcal{Z}$ with $|\mathcal{A}| = \gamma^\infty(\mathcal{Y} \cup \mathcal{Z})$.

Define $\phi : \mathcal{Y} \to \mathbb{N} \cup \{0\}$ by $\phi := \mathbf{1}_{\mathcal{A}\cap\mathcal{Y}} + \mathbf{1}_{\partial_{\mathcal{Z}}\mathcal{Y}}$. That is, ϕ is an assignment of guards to $\mathcal{Y}$ obtained by adding to the guards of $\mathcal{A} \cap \mathcal{Y}$, a guard at each vertex in $\partial_{\mathcal{Z}}\mathcal{Y}$ (in addition to any guards that were already there).

Then ϕ is eternally dominating for $\mathcal{Y}$. Indeed, since $\mathcal{A}$ is eternally dominating for $\mathcal{Y} \cup \mathcal{Z}$, the guards of $\mathcal{A}$ can defend against any sequence of attacks on vertices of $\mathcal{Y} \setminus \partial_{\mathcal{Z}}\mathcal{Y}$, and for any such sequence of attacks the guards on $\mathcal{A} \cap \mathcal{Z}$ are unable to help with the defence, so therefore the guards of $\mathcal{A} \cap \mathcal{Y}$ are able to defend against any such sequence. On the other hand, for any sequence of attacks on vertices in $\partial_{\mathcal{Z}}\mathcal{Y}$, the added guards in $\partial_{\mathcal{Z}}\mathcal{Y}$ are able to defend without moving at all. So the combined guards of $\mathbf{1}_{\mathcal{A}\cap\mathcal{Y}} + \mathbf{1}_{\partial_{\mathcal{Z}}\mathcal{Y}}$ can defend against any sequence of attacks on $\mathcal{Y}$. Together with Lemma 3.9, this shows that

$$\gamma^\infty(\mathcal{Y}) \leq |\mathcal{A} \cap \mathcal{Y}| + |\partial_{\mathcal{Z}}\mathcal{Y}|. \tag{3.6}$$

For each $y \in \partial_{\mathcal{Z}}\mathcal{Y}$, let π_y be a clique-partition of $\mathcal{Z} \cap B_1(y)$, of size k_y with $k_y \leq k := \kappa(B_2(o))$. This can be found by a similar argument to the one in the proof of Lemma 3.6. Enumerate the sets of ϕ_y as $\mathcal{S}_{y,1}, \ldots, \mathcal{S}_{y,k_y}$. For each $j = 1, \ldots, k_y$, pick an element $z_{y,j}$ of $\mathcal{S}_{y,j}$. Then define $\phi : \mathcal{Z} \to \mathbb{N} \cup \{0\}$ by

$$\phi' = \mathbf{1}_{\mathcal{A}\cap\mathcal{Z}} + \sum_{y\in\partial_{\mathcal{Z}}\mathcal{Y}} \sum_{j=1}^{k_y} \mathbf{1}_{z_{y,j}}.$$

In other words we take the set of guards in $\mathcal{A} \cap \mathcal{Z}$, and for each $y \in \partial_{\mathcal{Z}}\mathcal{Y}$, and each set $\mathcal{S}_{y,j}$ in the clique-partition π_y, we add one guard at a vertex in $\mathcal{S}_{y,j}$.

Then we claim ϕ' is eternally dominating for $\mathcal{Z}$. Indeed, similarly to before, the guards of $\mathcal{A} \cap \mathcal{Z}$ can defend against any sequence of attacks on vertices in $\mathcal{Z} \setminus \partial_{\mathcal{Y}}\mathcal{Z}$.

Also for each $y \in \partial_{\mathcal{Z}}\mathcal{Y}$, and each set $\mathcal{S}_{y,j}$ of the partition π_y, the added guard placed at $z_{y,j}$ can defend against any sequence of attacks on vertices in $\mathcal{S}_{y,j}$, since the subgraph induced by $\mathcal{S}_{y,j}$ is complete.

Since $k_y \leq k$ for each $y \in \partial_{\mathcal{Z}}\mathcal{Y}$, we have $\sum_{z\in\mathcal{Z}} \phi'(z) \leq |\mathcal{A} \cap \mathcal{Z}| + k|\partial_{\mathcal{Z}}\mathcal{Y}|$. Hence, by Lemma 3.9, $\gamma^{\infty}(\mathcal{Z}) \leq |\mathcal{A} \cap \mathcal{Z}| + k|\partial_{\mathcal{Z}}\mathcal{Y}|$. Combining this with (3.6) yields

$$\gamma^{\infty}(\mathcal{Y}) + \gamma^{\infty}(\mathcal{Z}) \leq |\mathcal{A}| + (1+k)|\partial_{\mathcal{Z}}\mathcal{Y}| = \gamma^{\infty}(\mathcal{Y} \cup \mathcal{Z}) + (1+k)|\partial_{\mathcal{Z}}\mathcal{Y}|,$$

and rearranging this gives us Property P4 with $c_2 = 1 + \kappa(B_2(o))$. $\square$

Lemma 3.11. *Property P6 (upward monotonicity in $\mathcal{X}$) holds for $\gamma^{\infty}(\cdot)$.*

Proof. Given finite $\mathcal{X} \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d \setminus \mathcal{X}$, let $\mathcal{A} \subset \mathcal{X} \cup \{x\}$ be eternally dominating for $\mathcal{X} \cup \{x\}$ (i.e. for $G(\mathcal{X} \cup \{x\}, 1)$) with $|\mathcal{A}| = \gamma^{\infty}(\mathcal{X} \cup \{x\})$.

If $x \notin \mathcal{A}$, then starting from $\mathcal{A}$ we can defend against any sequence of attacks on vertices in $\mathcal{X}$, so $\mathcal{A}$ is eternally dominating for $\mathcal{X}$, and hence $\gamma^{\infty}(\mathcal{X}) \leq |\mathcal{A}|$.

Suppose $x \in \mathcal{A}$. If any sequence of attacks on vertices in $\mathcal{X}$ can be defended without moving the guard at x, then $\mathcal{A} \setminus \{x\}$ is eternally dominating for $\mathcal{X}$, so that $\gamma^{\infty}(\mathcal{X}) \leq |\mathcal{A}| - 1$. Otherwise, there exists some finite sequence of attacks on vertices in $\mathcal{X}$, such that the eternally dominating defence against these attacks ends with moving the guard at x. After defending against this sequence of attacks we are left with a configuration $\mathcal{A}' \subset \mathcal{X}$ that is eternally dominating (for $\mathcal{X} \cup \{x\}$) and satisfies $|\mathcal{A}'| = |\mathcal{A}|$. Then, as for the first case considered above, $\mathcal{A}'$ is eternally dominating for $\mathcal{X}$, so $\gamma^{\infty}(\mathcal{X}) \leq |\mathcal{A}'| = |\mathcal{A}|$.

Thus in all cases we have $\gamma^{\infty}(\mathcal{X}) \leq |\mathcal{A}| = \gamma^{\infty}(\mathcal{X} \cup \{x\})$, which gives us P6. $\square$

3.5 Vertex cover number

Theorem 3.12 (LLN for vertex cover number of RGG)**.** *Corollary 2.2 (with $c_3 = 1$) applies when choosing $\zeta(\mathcal{X})$ to be the vertex cover number of $G(\mathcal{X}, 1)$.*

Proof. Let $\zeta(\mathcal{X})$ here denote the vertex cover number of $G(\mathcal{X}, 1)$. Suppose $\mathcal{Y}$ and $\mathcal{Z}$ are disjoint finite subsets of $\mathbb{R}^d$. If $\mathcal{D}$ is a vertex cover of $G(\mathcal{Y} \cup \mathcal{Z}, 1)$ with $|\mathcal{D}| = \zeta(\mathcal{Y} \cup \mathcal{Z})$, then $\mathcal{D} \cap \mathcal{Y}$ and $\mathcal{D} \cap \mathcal{Z}$ are vertex covers of $G(\mathcal{Y}, 1)$ and of $G(\mathcal{Z}, 1)$ respectively. Therefore $\zeta(\mathcal{Y}) + \zeta(\mathcal{Z}) \leq \zeta(\mathcal{Y} \cup \mathcal{Z})$.

Let $\mathcal{A}$, $\mathcal{A}'$ be vertex covers of $G(\mathcal{Y}, 1)$ and $G(\mathcal{Z}, 1)$ respectively, with $|\mathcal{A}| = \zeta(\mathcal{Y})$ and $|\mathcal{A}'| = \zeta(\mathcal{Z})$. Then the set $\mathcal{A} \cup \mathcal{A}' \cup \partial_{\mathcal{Z}}(\mathcal{Y})$ is a vertex cover for $G(\mathcal{Y} \cup \mathcal{Z}, 1)$. Indeed, the set $\mathcal{A}$ covers all edges of $G(\mathcal{Y} \cup \mathcal{Z}, 1)$ having both endpoints in $\mathcal{Y}$, while $\mathcal{A}'$ covers all edges of $G(\mathcal{Y} \cup \mathcal{Z}, 1)$ having both endpoints in $\mathcal{Z}$, and $\partial_{\mathcal{Z}}(\mathcal{Y})$ covers all edges of $G(\mathcal{Y} \cup \mathcal{Z}, 1)$ having one endpoint in $\mathcal{Y}$ and one endpoint in $\mathcal{Z}$. Therefore $\zeta(\mathcal{Y} \cup \mathcal{Z}) \leq \zeta(\mathcal{Y}) + \zeta(\mathcal{Z}) + |\partial_{\mathcal{Z}}(\mathcal{Y})|$.

Thus we have Properties P3, P4 for ζ but with the inequalities the wrong way round. Therefore taking $\zeta'(\mathcal{X}) := |\mathcal{X}| - \zeta(\mathcal{X})$ gives us a functional satisfying Properties P3 and P4 with the inequalities the right way round and with $c_1 = 0$, $c_2 = 1$. Moreover, clearly $\zeta(\mathcal{X}) \leq |\mathcal{X}|$, so $\zeta'(\cdot)$ is nonnegative. Also, if $G(\mathcal{X}, 1)$ is a complete graph then $\zeta(\mathcal{X}) = |\mathcal{X}| - 1$ and $\zeta'(\mathcal{X}) = 1$, so $\zeta'(\cdot)$ satisfies P5′. Therefore, taking $\zeta''(\cdot) \equiv 0$, we can apply Corollary 2.2 to ζ, with $c_3 = 1$. □

3.6 Number of connected components

When we take $\zeta(\mathcal{X})$ to be the number of components of $G(\mathcal{X}, 1)$, and μ is absolutely continuous, a result along the lines of Theorem 2.1 was already known to be true (see [35, Theorem 13.25]). Here we extend that result to cases where μ has a singular part.

Theorem 3.13 (LLN for number of components of RGG)**.** *Theorem 2.1 applies when choosing $\zeta(\mathcal{X})$ to be the number of connected components of $G(\mathcal{X}, 1)$.*

As we shall discuss in Section 4.4, this result can be obtained as a special case of Theorem 4.11 below. Alternatively, one can check Properties P1–P4 and P5′ directly.

By comparing (2.3) with [35, Theorem 13.25], one can see that $\rho(\lambda) = \mathbb{E}\left[|\mathcal{C}_o(\lambda)|^{-1}\right]$ in this case. Therefore $\rho(\lambda)$ is decreasing in λ. Also $\lambda\rho(\lambda)$ is increasing in λ for small λ by Proposition 2.3(vii) (see also Remark 5.4 below). However, by [36, Theorem 3], $\rho(\lambda) \sim e^{-\pi_d \lambda}$ as $\lambda \to \infty$ here, so $\lambda\rho(\lambda) \to 0$ as $\lambda \to \infty$ and $\lambda\rho(\lambda)$ cannot be increasing for all $\lambda > 0$.

3.7 Number of isolated vertices

Let $\sigma(G)$ denote the number of isolated vertices of a graph G, and for finite $\mathcal{X} \subset \mathbb{R}^d$ set $\sigma(\mathcal{X}) := \sigma(G(\mathcal{X}, 1))$.

When μ is absolutely continuous, it was already known from [35, Theorem 3.15] that $n^{-1}\sigma(G(\mathcal{X}_n, r_n))$ converges in the thermodynamic limit. The following result adds to that result by allowing μ to have a singular part.

Theorem 3.14 (LLN for number of isolated vertices of RGG)**.** *When choosing $\zeta(\mathcal{X})$ to be $\sigma(\mathcal{X})$, Theorem 2.1 applies, and also $\rho(\lambda) = \exp(-\pi_d \lambda)$ for all $\lambda > 0$.*

Proof. Suppose $\mathcal{Y}, \mathcal{Z}$ are finite disjoint subsets of $\mathbb{R}^d$. Then every isolated point in $\mathcal{Y} \cup \mathcal{Z}$ is either an isolated point in $\mathcal{Y}$ or an isolated point in $\mathcal{Z}$, so $\sigma(\mathcal{Y} \cup \mathcal{Z}) \leq \sigma(\mathcal{Y}) + \sigma(\mathcal{Z})$; thus P3 holds with $c_1 = 0$. Moreover, if $\mathcal{I}$ denotes the set of isolated vertices in $\mathcal{Y} \cup \mathcal{Z}$, then every isolated vertex in $\mathcal{Y}$ is either in $\mathcal{I}$ or lies within unit distance of $\mathcal{Z}$, so that

$$\sigma(\mathcal{Y}) \leq |\mathcal{I} \cap \mathcal{Y}| + |\partial_{\mathcal{Z}} \mathcal{Y}|. \tag{3.7}$$

Similarly, each isolated vertex in $\mathcal{Z}$ is either in $\mathcal{I}$ or lies within unit distance of $\mathcal{Y}$. Moreover, for each $y \in \mathcal{Y}$ the number of isolated vertices of $\mathcal{Z}$ within unit distance of y is bounded by 3^d, since this is an upper bound for the number of disjoint balls of radius $1/2$ which can be fitted inside a ball of radius $3/2$. Therefore the number of isolated vertices of $\mathcal{Z}$ within unit distance of $\mathcal{Y}$ is bounded by $3^d |\partial_{\mathcal{Z}} \mathcal{Y}|$. Hence,

$$\sigma(\mathcal{Z}) \leq |\mathcal{I} \cap \mathcal{Z}| + 3^d |\partial_{\mathcal{Z}} \mathcal{Y}|.$$

Combined with (3.7) this shows that

$$\sigma(\mathcal{Y} \cup \mathcal{Z}) = |\mathcal{I}| \geq \sigma(\mathcal{Y}) + \sigma(\mathcal{Z}) - (1 + 3^d)|\partial_{\mathcal{Z}} \mathcal{Y}|,$$

so P4 holds with $c_2 = 1 + 3^d$. Also P5$'$ holds since $\sigma(\mathcal{X}) \leq 1$ for any $\mathcal{X} \subset B_{1/2}(o)$.

It can be seen from (2.2) and the Palm-Mecke formula from the theory of Poisson point processes (see [35] or [32]) that $\rho(\lambda) = \exp(-\pi_d \lambda)$ here. □

3.8 H-packing number

Let H be a fixed connected graph with h vertices, $2 \leq h < \infty$. Recall that the H-packing number $\psi_H(G)$ is defined to be the largest possible size of H-packing in G. For finite $\mathcal{X} \subset \mathbb{R}^d$ set $\psi_H(\mathcal{X}) := \psi_H(G(\mathcal{X}, 1))$.

Theorem 3.15 (LLN for H-packing number of RGG)**.** *Corollary 2.2 applies, with $c_3 = 1/h$, when choosing $\zeta(\mathcal{X})$ to be $\psi_H(\mathcal{X})$.*

Remark. A generalization of H-packing allows for more than one H. Suppose we are given a collection of pairs $\{(H_1, v_1), \ldots, (H_m, v_m)\}$, where for each $i \in \{1, \ldots, m\}$, H_i is a finite connected graph and $v_i > 0$, with $H_1, \ldots, H_m$ pairwise non-isomorphic. We call v_i the 'value' of H_i. Given a graph $G = (V, E)$, the aim now is to pack vertex-disjoint copies of the sets $H_1, \ldots, H_m$ into G (allowing repetitions) with maximum possible total value. That is, a packing of G is a collection $\pi = \{J_i, i \in \mathcal{I}\}$ of vertex-disjoint subgraphs, each of which is isomorphic to one of $H_1, \ldots, H_m$, and the total value $v(\pi)$ of the packing equals $\sum_{i \in \mathcal{I}} w(J_i)$, where we set $w(J_i)$ to be $v(H_j)$ if J_i is isomorphic to H_j. Then define $MP_{(H_1,v_1),\ldots,(H_m,v_m)}(G)$ to be the maximum of $v(\pi)$, over all packings π. Theorem 3.15 can be generalized to this setting by minor modifications of the proof. In this generality we need to take $c_3 = \max_{i \in \{1,\ldots,m\}}(v_i/h_i)$, where h_i denotes the number of vertices of H_i.

Proof of Theorem 3.15. Let $\mathcal{Y}, \mathcal{Z} \subset \mathbb{R}^d$ be disjoint and finite. Then any H-packing of $\mathcal{Y}$ and H-packing of $\mathcal{Z}$ can be put together to give an H-packing of $\mathcal{Y} \cup \mathcal{Z}$; hence $\psi_H(\mathcal{Y} \cup \mathcal{Z}) \geq \psi_H(\mathcal{Y}) + \psi_H(\mathcal{Z})$.

For any H-packing of $\mathcal{Y} \cup \mathcal{Z}$, if we remove all H-subgraphs in the packing that contain at least one vertex in $\mathcal{Y}$ and at least one vertex in $\mathcal{Z}$, we are left with an H-packing of $\mathcal{Y}$ and an H-packing of $\mathcal{Z}$. The number of removed H-subgraphs is at most $|\partial_{\mathcal{Z}} \mathcal{Y}|$, and hence

$$\psi_H(\mathcal{Y} \cup \mathcal{Z}) - |\partial_{\mathcal{Z}} \mathcal{Y}| \leq \psi_H(\mathcal{Y}) + \psi_H(\mathcal{Z}).$$

Thus, taking $\zeta(\cdot) = \psi_H(\cdot)$, Properties P3 and P4 hold with the inequalities the wrong way round. Therefore, the functional ζ', defined by $\zeta'(\mathcal{X}) := (|\mathcal{X}|/h) - \zeta(\mathcal{X})$, satisfies P3 and P4 with $c_1 = 0$, $c_2 = 1$. Moreover clearly $\zeta'(\mathcal{X}) \geq 0$ for all $\mathcal{X}$.

If $G(\mathcal{X}, 1)$ is a complete graph, then $h\zeta(\mathcal{X}) \geq |\mathcal{X}| - h$ so $\zeta'(\mathcal{X}) \leq 1$; hence P5′ holds for ζ'. Thus, taking $\zeta''(\cdot) \equiv 0$, we can apply Corollary 2.2 with $c_3 = 1/h$. □

3.9 Edge cover number

Recall that $\eta(G)$ denotes the edge cover number of the graph obtained from G by removing all isolated vertices. For $\mathcal{X} \subset \mathbb{R}^d$, write $\eta(\mathcal{X})$ for $\eta(G(\mathcal{X}, 1))$.

Theorem 3.16 (LLN for edge cover number of RGG)**.** *Corollary 2.2 applies, with $c_3 = 1/2$, when choosing $\zeta(\mathcal{X}) := \eta(\mathcal{X})$.*

Proof. For any finite graph $G = (V, E)$, let $\theta_2(G)$ denote the minimum size of partition π of V such that every set $W \in \pi$ is either a single vertex or two vertices connected by an edge. We shall refer to any such partition as an *edge-partition* of V. For $\mathcal{X} \subset \mathbb{R}^d$ set $\theta_2(\mathcal{X}) := \theta_2(G(\mathcal{X}, 1))$.

Recalling from Subsection 3.7 the notation $\sigma(G)$ for the number of isolated vertices of a graph G, we assert that $\eta(G) = \theta_2(G) - \sigma(G)$ for any finite graph G (and hence $\eta(\mathcal{X}) = \theta_2(\mathcal{X}) - \sigma(\mathcal{X})$ for any finite $\mathcal{X} \subset \mathbb{R}^d$). It clearly suffices to check this assertion in the case where $\sigma(G) = 0$. In this case $\eta(G) \leq \theta_2(G)$ since when $\sigma(G) = 0$, starting with any edge-partition of V we can replace each singleton in the partition by an edge incident to it, to get an edge cover of the same cardinality. But also given any edge cover, we can think of each edge as a set of two vertices and go through these sets one by one, at each step removing any vertex that was already counted (i.e. possibly sometimes changing the pair of vertices to a singleton, or even removing the pair altogether if the original edge cover was of non-minimum cardinality), to end up with an edge-partition of V consisting of at most the same number of sets as there were edges in the original edge cover. Hence $\theta_2(G) \leq \eta(G)$ so $\theta_2(G) = \eta(G)$ as asserted.

Now let H be a complete graph on 2 vertices, i.e. a graph with two vertices and one edge. For all finite $\mathcal{X} \subset \mathbb{R}^d$, we claim that

$$\eta(\mathcal{X}) = |\mathcal{X}| - \psi_H(\mathcal{X}) - \sigma(\mathcal{X}). \tag{3.8}$$

This is because any minimum edge-partition of $\mathcal{X}$ consists of a maximum H-packing in $\mathcal{X}$, together with $|\mathcal{X}|-2\psi_H(\mathcal{X})$ singletons, making a total of $\psi_H(\mathcal{X})+|\mathcal{X}|-2\psi_H(\mathcal{X})$ sets in the partition. Thus we have $\theta_2(\mathcal{X}) = |\mathcal{X}| - \psi_H(\mathcal{X})$, and (3.8) follows.

Define $\zeta(\cdot) := \eta(\cdot)$, $\zeta'(\cdot) = \sigma(\cdot)$ and $\zeta''(\cdot) = (|\cdot|/2) - \psi_H(\cdot)$. By Theorem 3.14, $\zeta'(\cdot)$ satisfies Properties P1–P5, and by the proof of Theorem 3.15, so does $\zeta''(\cdot)$. By (3.8), for all $\mathcal{X}$ we have $\zeta(\mathcal{X}) = (|\mathcal{X}|/2) - \zeta'(\mathcal{X}) + \zeta''(\mathcal{X})$. We may thus apply Corollary 2.2 with $c_3 = 1/2$, as asserted. □

4 Applications to weighted complete graphs

In this section we consider the limit theory of four classic combinatorial optimization problems (traveling salesman, minimum matching, minimum bipartite matching, minimum spanning tree), defined on a weighted complete graph on vertex set $\mathcal{X}_n$, where the edge weights are determined by the inter-point displacements via a specified measurable *weight function* $w : \mathbb{R}^d \to [0,\infty)$. Given $w(\cdot)$, define $w_{\max} \in [0,\infty]$ by $w_{\max} := \sup\{w(x) : x \in \mathbb{R}^d\}$. We shall require $w(\cdot)$ to satisfy the following conditions (but allow it to be otherwise arbitrary).

- W1 (*symmetry*): $w(x) = w(-x)$ for all $x \in \mathbb{R}^d$.
- W2 (w is small near the origin): $w(o) = 0$, and $w(x) \to 0$ as $x \to o$.
- W3 (w is large far from the origin): $w(x) \to w_{\max}$ as $\|x\| \to \infty$. (Here and throughout this paper, $\|\cdot\|$ denotes the L^2 (Euclidean) norm on $\mathbb{R}^d$.)
- W4 (Polynomial growth bound): There exist $c_4 \in (0,\infty)$ and $p \in (0,d)$ such that $w(x) \le c_4 \max(\|x\|^p, 1)$ for all $x \in \mathbb{R}^d$.

In some lemmas we shall require the following stronger versions of W3, W4:

- W5 (w achieves its maximum far from the origin). There is a constant $c_5 \in (0,\infty)$ such that $w(x) = w_{\max} < \infty$ for all $x \in \mathbb{R}^d$ with $\|x\| > c_5$.
- W6 There exists $p \in (0,d)$, $c_6 \in (0,\infty)$ such that $w(x) \le c_6\|x\|^p$ for all $x \in \mathbb{R}^d$.

Given $a \in [0, w_{\max})$, define the truncated weight function $w_a(x) = \min(w(x), a)$, $x \in \mathbb{R}^d$. Note that if $w(\cdot)$ satisfies W3, then $w_a(\cdot)$ satisfies W5. Moreover, if $w(\cdot)$

satisfies W4, then, for any $\delta > 0$, $w(\cdot)(1-\mathbf{1}_{B_\delta(o)}(\cdot))$ satisfies W6. Also, if $w_{\max} < \infty$, then W4 follows automatically.

As usual we are given a sequence of scaling parameters r_n. We shall take the complete graph on $\mathcal{X}_n$, with each edge $e = \{x, y\}$ having weight $w_n(e) := w(r_n^{-1}(y - x))$ (the unscaled weight function is $w(e) := w(y - x)$.) One example of interest is to take $w(x) = 1 - \mathbf{1}_{B_1(o)}(x)$, so that $w_n(e)$ is 0 if e is an edge of $G(\mathcal{X}_n, r_n)$, and otherwise it is 1.

Our results in this section extend results given in [46], where attention is restricted to weight functions of the form $w(x) = \|x\|^\alpha$. Here we allow for a much more general class of weight functions $w(\cdot)$; for example, if $d \geq 2$, with $|\!|\!|\cdot|\!|\!|$ denoting an arbitrary norm on $\mathbb{R}^d$, one could take $w(x) = \log(|\!|\!|x|\!|\!|+1)$, or $w(x) = |\!|\!|x|\!|\!|(2+\sin|\!|\!|x|\!|\!|)$, or $w(x) = |\!|\!|x|\!|\!|^{3/2} + |\!|\!|x|\!|\!|^{1/2}$. Also we consider the bipartite matching problem, which is not addressed in [46].

Recall that for $(\lambda, s) \in (0, \infty)^2$, we write $\mathcal{H}_{\lambda,s}$ for $\mathcal{H}_\lambda \cap Q_s$. We say that μ has *bounded support* if $\mu(Q_s) = 1$ for some $s \in (0, \infty)$. The next two lemmas are messy to state, but will be repeatedly useful for proving LLNs for certain functionals via approximating functionals.

Lemma 4.1. *Suppose that for each $k \in \mathbb{N}$, $\zeta^{(k)}(\mathcal{X})$ is a measurable nonnegative real-valued functional defined on all finite $\mathcal{X} \subset \mathbb{R}^d$. Assume for each $\mathcal{X}$ that $\zeta^{(k)}(\mathcal{X})$ is nondecreasing in k, and set $\zeta(\mathcal{X}) := \lim_{k\to\infty} \zeta^{(k)}(\mathcal{X})$. Suppose for each $k \in \mathbb{N}, \lambda > 0$ that the limit $\rho_k(\lambda) := \lim_{s\to\infty} \mathbb{E}[\zeta^{(k)}(\mathcal{H}_{\lambda,s})/(\lambda s^d)]$ exists and is finite, and that if $nr_n^d \to t \in (0, \infty)$ as $n \to \infty$, then*

$$\zeta^{(k)}(r_n^{-1}\mathcal{X}_n) \xrightarrow[L^2]{c.c.} \int \rho_k(tf_\mu(x))f_\mu(x)dx. \tag{4.1}$$

Let $(h(k), k \in \mathbb{N})$ be a real-valued sequence with $h(k) \to 0$ as $k \to \infty$. Suppose moreover that either (a) μ has bounded support and there exist constants $p \in (0, d), c_7 \in (0, \infty)$ such that for all $s \geq 1$, $k \in \mathbb{N}$ and all finite $\mathcal{X} \subset Q_s$, we have

$$\zeta^{(k)}(\mathcal{X}) \leq c_7 s^p |\mathcal{X}|^{1-(p/d)}; \tag{4.2}$$

$$\zeta(\mathcal{X}) - \zeta^{(k)}(\mathcal{X}) \leq s^p h(k)(\zeta^{(k)}(\mathcal{X}))^{1-(p/d)}, \tag{4.3}$$

or (b) $\zeta(\mathcal{X}) - \zeta^{(k)}(\mathcal{X}) \leq h(k)|\mathcal{X}|$ for all finite $\mathcal{X} \subset \mathbb{R}^d$, $k \in \mathbb{N}$. Then for each $\lambda > 0$ the limit $\rho(\lambda) := \lim_{s\to\infty} \mathbb{E}[\zeta(\mathcal{H}_{\lambda,s})/(\lambda s^d)]$ exists and is finite, and if $nr_n^d \to t \in (0, \infty)$ as $n \to \infty$, then $\zeta(r_n^{-1}\mathcal{X}_n) \xrightarrow[L^2]{c.c.} \int \rho(tf_\mu(x))f_\mu(x)dx$, i.e. (2.3) holds.

Proof. First we assume condition (a) holds. Using (4.2), taking expectations and using Jensen's inequality yields for all $k, \lambda, s > 0$ that

$$\mathbb{E}[\zeta^{(k)}(\mathcal{H}_{\lambda,s})] \leq c_7 s^p \mathbb{E}[|\mathcal{H}_{\lambda,s}|^{(d-p)/d}] \leq c_7 s^p (\lambda s^d)^{1-(p/d)} = c_7 \lambda^{1-(p/d)} s^d.$$

Hence using (4.3) and Jensen's inequality again yields that

$$\begin{aligned}\mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s}) - \zeta^{(k)}(\mathcal{H}_{\lambda,s})] &\le s^p h(k)(\mathbb{E}\,[\zeta^{(k)}(\mathcal{H}_{\lambda,s})])^{1-(p/d)} \\ &\le c_7^{1-(p/d)} s^p h(k)\lambda^{(1-(p/d))^2} s^{d-p} \\ &= c_7^{1-(p/d)} h(k)\lambda^{(1-(p/d))^2} s^d. \end{aligned} \tag{4.4}$$

Let $\lambda > 0$. By (4.4), given $\varepsilon > 0$ we can choose $k_0 \in \mathbb{N}$ such that

$$0 \le \mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s}) - \zeta^{(k)}(\mathcal{H}_{\lambda,s})]/(\lambda s^d) \le \varepsilon, \quad \forall\ s \ge 1, k \ge k_0.$$

Note that this is also true under condition (b). Taking the large-s limit yields for $k \ge k_0$ that

$$\rho_k(\lambda) \le \liminf_{s\to\infty} \mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s})/(\lambda s^d)] \le \limsup_{s\to\infty} \mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s})/(\lambda s^d)] \le \rho_k(\lambda) + \varepsilon.$$

Therefore since $\rho_k(\lambda)$ is nondecreasing in k, the limit $\rho(\lambda) := \lim_{k\to\infty} \rho_k(\lambda)$ exists, is finite, and is equal to $\lim_{s\to\infty} \mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s})/(\lambda s^d)]$ under either condition (a) or (b). By monotone convergence,

$$\lim_{k\to\infty} \int \rho_k(tf_\mu(x))f_\mu(x)dx = \int \rho(tf_\mu(x))f_\mu(x)dx. \tag{4.5}$$

Under condition (a), by the assumption of bounded support we can and do choose $C = C(\mu) \in (0,\infty)$ such that $r_n^{-1}\mathcal{X}_n \subset Q_{r_n^{-1}C}$ almost surely, for all $n \in \mathbb{N}$. Then by (4.2), $\zeta^{(k)}(r_n^{-1}\mathcal{X}_n) \le c_7 C^p r_n^{-p} n^{1-(p/d)}$, almost surely. Using (4.3), therefore we have, almost surely,

$$\zeta(r_n^{-1}\mathcal{X}_n) - \zeta^{(k)}(r_n^{-1}\mathcal{X}_n) \le C^p r_n^{-p} h(k)(c_7 C^p r_n^{-p} n^{1-(p/d)})^{1-(p/d)}.$$

Since we are taking the thermodynamic limit $nr_n^d \to t$, we have $r_n^p n^{p/d} \to t^{p/d}$, so there are constants C', C'' (depending only on μ and t) such that, almost surely,

$$\zeta(r_n^{-1}\mathcal{X}_n) - \zeta^{(k)}(r_n^{-1}\mathcal{X}_n) \le C' r_n^{-p} h(k) n^{1-(p/d)} \sim C'' h(k) n.$$

Hence given $\varepsilon > 0$, we can choose $k_0, n_0 \in \mathbb{N}$ such that for all $k \ge k_0, n \ge n_0$, almost surely $\zeta(r_n^{-1}\mathcal{X}_n) - \zeta^{(k)}(r_n^{-1}\mathcal{X}_n) \le \varepsilon n$; this is also true under condition (b). Hence using (4.1) and (4.5) we can obtain the desired result (2.3) here. $\square$

Lemma 4.2. *Suppose μ has bounded support. Suppose for each $\ell, m \in \mathbb{N}$ that $\tilde{\zeta}^{(\ell,m)}(\mathcal{X})$ is a measurable non-negative real-valued functional defined for all all finite $\mathcal{X} \subset \mathbb{R}^d$, such that for all $\lambda > 0$ the limit $\rho^{(\ell,m)}(\lambda) := \lim_{s\to\infty} \mathbb{E}\,[\tilde{\zeta}^{(\ell,m)}(\mathcal{H}_{\lambda,s})/(\lambda s^d)]$*

exists and is finite, and if $nr_n^d \to t \in (0,\infty)$ *as* $n \to \infty$, *then* $n^{-1}\tilde{\zeta}^{(\ell,m)}(r_n^{-1}\mathcal{X}_n) \xrightarrow[L^2]{c.c.} \int \rho^{(\ell,m)}(tf_\mu(x))f_\mu(x)dx$. *Assume for all* $\mathcal{X}$ *that* $\tilde{\zeta}^{(\ell,m)}(\mathcal{X})$ *is nondecreasing both in* ℓ *and in* m, *and set* $\tilde{\zeta}^{(\ell)}(\mathcal{X}) := \lim_{m\to\infty}\tilde{\zeta}^{(\ell,m)}(\mathcal{X})$ *and* $\tilde{\zeta}(\mathcal{X}) := \lim_{\ell\to\infty}\tilde{\zeta}^{(\ell)}(\mathcal{X})$.

Let $p \in (0,d)$. *Suppose for each* $\ell \in \mathbb{N}$ *that there is a constant* $C(\ell)$ *and a real-valued sequence* $(h_\ell(m), m \in \mathbb{N})$ *with* $h_\ell(m) \to 0$ *as* $m \to \infty$, *such that for all* $s \geq 1$ *and all finite* $\mathcal{X} \subset Q_s$, *we have* $\tilde{\zeta}^{(\ell,m)}(\mathcal{X}) \leq C(\ell)s^p|\mathcal{X}|^{1-(p/d)}$ *and*

$$\tilde{\zeta}^{(\ell)}(\mathcal{X}) - \tilde{\zeta}^{(\ell,m)}(\mathcal{X}) \leq h_\ell(m)s^p(\tilde{\zeta}^{(\ell,m)}(\mathcal{X}))^{1-(p/d)}.$$

Suppose moreover that there is a sequence $(\tilde{h}(\ell), \ell \in \mathbb{N})$ *such that* $\tilde{h}(\ell) \to 0$ *as* $\ell \to \infty$, *and such that* $\tilde{\zeta}(\mathcal{X}) - \tilde{\zeta}^{(\ell)}(\mathcal{X}) \leq \tilde{h}(\ell)|\mathcal{X}|$ *for all finite* $\mathcal{X} \subset \mathbb{R}^d$. *Then for all* $\lambda > 0$ *the limit* $\rho(\lambda) := \lim_{s\to\infty}\mathbb{E}[\tilde{\zeta}(\mathcal{H}_{\lambda,s})/(\lambda s^d)]$ *exists and is finite, and if* $nr_n^d \to t \in (0,\infty)$ *as* $n \to \infty$, *then* $n^{-1}\tilde{\zeta}(r_n^{-1}\mathcal{X}_n) \xrightarrow[L^2]{c.c.} \int \rho(tf_\mu(x))f_\mu(x)dx$.

Proof. Let $\ell \in \mathbb{N}$. Then taking $\zeta(\cdot) := \tilde{\zeta}^{(\ell)}(\cdot)$ and $\zeta^{(k)}(\cdot) := \tilde{\zeta}^{(\ell,k)}(\cdot)$ gives us functionals which together satisfy (4.2) and (4.3) if we take $C = C(\ell)$ and $h(k) := h_\ell(k)$. By Lemma 4.1 (a), for all $\lambda > 0$ the limit $\rho^{(\ell)}(\lambda) := \lim_{s\to\infty}\mathbb{E}[\tilde{\zeta}^{(\ell)}(\mathcal{H}_{\lambda,s})/(\lambda s^d)]$ exists and is finite, and if $nr_n^d \to t \in (0,\infty)$ then $\tilde{\zeta}^{(\ell)}(r_n^{-1}\mathcal{X}_n) \xrightarrow[L^2]{c.c.} \int \rho^{(\ell)}(tf_\mu(x))f_\mu(x)dx$.

Now taking instead $\zeta(\cdot) := \tilde{\zeta}(\cdot)$ and $\zeta^{(k)}(\cdot) := \tilde{\zeta}^{(k)}(\cdot)$ for $k \in \mathbb{N}$, we can apply Lemma 4.1 (b), now with $\tilde{h}(k)$ playing the role of $h(k)$, to obtain the desired conclusion. □

4.1 Weighted traveling salesman problem

Given finite $\mathcal{X} \subset \mathbb{R}^d$, the (weighted) traveling salesman problem (TSP) is to find a tour of minimum weight (using weight function $w(\cdot)$) that passes exactly once through all points and returns to the starting vertex. Formally, a *tour* of $\mathcal{X}$ is defined to be a Hamilton cycle in the complete graph on vertex set $\mathcal{X}$. For each tour τ of $\mathcal{X}$, define $w(\tau) = \sum_{e\in E(\tau)} w(e)$, where $E(\tau)$ denotes the set of edges of τ. Define $TSP_w(\mathcal{X}) := \min w(\tau)$, where the minimum is over all tours of $\mathcal{X}$.

Theorem 4.3 (LLN for weighted TSP)**.** *Suppose* $w(\cdot)$ *satisfies conditions W1–W4, and either (a)* $w_{\max} < \infty$ *or (b)* μ *has bounded support,* $w_{\max} = \infty$. *Then when choosing* $\zeta(\mathcal{X})$ *as* $TSP_w(\mathcal{X})$, *the limit* $\rho(\lambda)$ *given by (2.2) exists and is finite for all* $\lambda \in (0,\infty)$, *and if* $nr_n^d \to t \in (0,\infty)$ *as* $n \to \infty$, *then (2.3) holds.*

Remark. One could also consider a more general *directed* version of the TSP, dropping condition W1 so that the weight of an edge depends on the direction of travel. Theorem 4.3(a) and its proof apply without change to this directed setting (we do not know if the directed version of Theorem 4.3(b) holds). For example, in $d = 2$ one could take $w(\cdot) = 1 - \mathbf{1}_S$ with S being the right half of $B_1(o)$ (a semi-circle); in this case some minor modifications to the proof are needed because W2 fails too, but the directed version of Theorem 4.3 still holds. A different spatial directed TSP problem has been considered in [44].

We first prove Theorem 4.3 under the extra condition W5.

Lemma 4.4. *Suppose $w(\cdot)$ satisfies W1, W2 and W5. Then when choosing $\zeta(\mathcal{X})$ as $TSP_w(\mathcal{X})$, the limit $\rho(\lambda)$ given by (2.2) exists and is finite for all $\lambda \in (0, \infty)$, and if $nr_n^d \to t \in (0, \infty)$ as $n \to \infty$, then (2.3) holds.*

Proof. Assume first that W5 holds with $c_5 = 1$ (as well as W1 and W2). We will show that Theorem 2.1 holds for $\zeta(\cdot) := TSP_w(\cdot)$ in this case. Clearly, $TSP_w(\cdot)$ satisfies Properties P1 and P2. Next, for any disjoint finite $\mathcal{Y}, \mathcal{Z} \subset \mathbb{R}^d$, let τ' be a tour of $\mathcal{Y}$ and τ'' a tour of $\mathcal{Z}$ with $w(\tau') = TSP_w(\mathcal{Y})$ and $w(\tau'') = TSP_w(\mathcal{Z})$. Pick an edge of τ', and denote its endpoints y_1, y_2. Pick an edge of τ'', and denote its endpoints z_1, z_2. Then $(E(\tau') \cup E(\tau'') \cup \{y_1z_1, y_2z_2\}) \setminus \{y_1y_2, z_1z_2\}$ is the edge-set of a tour of $\mathcal{Y} \cup \mathcal{Z}$, and hence

$$TSP_w(\mathcal{Y} \cup \mathcal{Z}) \leq TSP_w(\mathcal{Y}) + TSP_w(\mathcal{Z}) + 2w_{\max}.$$

Thus, Property P3 holds with $c_1 = 2w_{\max}$.

For Property P4, let τ be a tour of $\mathcal{Y} \cup \mathcal{Z}$ with $w(\tau) = TSP_w(\mathcal{Y} \cup \mathcal{Z})$. Suppose without loss of generality that the tour τ starts at a vertex in $\mathcal{Y}$. Let τ_1 be the tour through $\mathcal{Y}$ obtained by taking the vertices of $\mathcal{Y}$ in the same order as they arise in τ, while leaving out from τ all vertices belonging to $\mathcal{Z}$, thereby introducing some new edges within $\mathcal{Y}$. The number of new edges equals half the number of edges of τ that go from $\mathcal{Y}$ to $\mathcal{Z}$. Similarly let τ_2 be the tour though $\mathcal{Z}$ obtained from τ by going through the vertices of $\mathcal{Z}$ in the order they arise in τ.

Now writing just τ for the set of edges of τ (i.e. identifying a tour with its set of edges), we have $|\tau_1 \cup \tau_2| = |\mathcal{Y}| + |\mathcal{Z}| = |\tau|$, and hence

$$|\tau \setminus (\tau_1 \cup \tau_2)| = |(\tau_1 \cup \tau_2) \setminus \tau|. \tag{4.6}$$

Also, $TSP_w(\mathcal{Y}) \leq w(\tau_1)$ and $TSP_w(\mathcal{Z}) \leq w(\tau_2)$. Hence

$$\begin{aligned} TSP_w(\mathcal{Y}) + TSP_w(\mathcal{Z}) - TSP_w(\mathcal{Y} \cup \mathcal{Z}) &\leq w(\tau_1) + w(\tau_2) - w(\tau) \\ &= \left(\sum_{e \in (\tau_1 \cup \tau_2) \setminus \tau} w(e) \right) - \sum_{e \in \tau \setminus (\tau_1 \cup \tau_2)} w(e) \\ &\leq \sum_{e \in \tau \setminus (\tau_1 \cup \tau_2)} (w_{\max} - w(e)), \end{aligned} \tag{4.7}$$

where for the second inequality we have used both (4.6) and the assumption that all edge weights are at most $w_{\max}$.

Let us say that an edge uv of τ is *short* if $\|u-v\| \leq 1$, *long* if $\|u-v\| > 1$. The last sum in (4.7) is at most $w_{\max}$ times the number of *short* edges in τ that go from $\mathcal{Y}$ to $\mathcal{Z}$ (because all long edges have weight $w_{\max}$ so do not contribute to the sum), and this is bounded above by $2w_{\max}|\partial_{\mathcal{Z}}\mathcal{Y}|$. Therefore (4.7) gives us

$$TSP_w(\mathcal{Y}) + TSP_w(\mathcal{Z}) - TSP_w(\mathcal{Y} \cup \mathcal{Z}) \leq 2w_{\max}|\partial_{\mathcal{Z}}\mathcal{Y}|,$$

and hence Property P4 with $c_2 = 2w_{\max}$.

To check Property P5 (local sublinear growth), note that given any $\delta > 0, n \in \mathbb{N}$ and finite $\mathcal{X} \subset B_\delta(o)$, for *any* tour τ of $\mathcal{X}$ we have $w(\tau) \leq |\mathcal{X}| \sup_{x \in B_{2\delta}(o)} w(x)$. Therefore $\sup\{TSP_w(\mathcal{X})/|\mathcal{X}| : \mathcal{X} \subset B_\delta(o)\} \leq \sup_{x \in B_{2\delta}(o)} w(x)$, which tends to 0 as $\delta \downarrow 0$ by W2, and P5 follows.

Thus, Theorem 2.1 is applicable for $\zeta(\cdot) := TSP_w(\cdot)$, in the special case where $c_5 = 1$ in W5. In the general case where $c_5 \in (0, \infty)$, consider the rescaled weight function $w'(x) := w(c_5 x)$, so that $w'(x) = w_{\max}$ for $\|x\| > 1$. By the special case already considered, Theorem 2.1 applies to $\zeta(\cdot) = TSP_{w'}(\cdot)$. Thus for all $\lambda > 0$ the limit $\rho'(\lambda) := \lim_{s\to\infty} \mathbb{E}\left[TSP_{w'}(\mathcal{H}_{\lambda,s})/(\lambda s^d)\right]$ exists. Since $TSP_w(\mathcal{X}) = TSP_{w'}(c_5^{-1}\mathcal{X})$ for all $\mathcal{X}$, by the Mapping theorem for Poisson point processes [32],

$$\begin{aligned}\mathbb{E}\left[TSP_w(\mathcal{H}_{\lambda,s})\right]/(\lambda s^d) &= \mathbb{E}\left[TSP_{w'}(c_5^{-1}\mathcal{H}_{\lambda,s})\right]/(\lambda s^d) \\ &= \mathbb{E}\left[TSP_{w'}(\mathcal{H}_{c_5^d\lambda, c_5^{-1}s})\right]/(\lambda s^d),\end{aligned}$$

which converges to $\rho'(c_5^d\lambda) =: \rho(\lambda)$ as $s \to \infty$.

Now suppose $nr_n^d \to t \in (0, \infty)$. Then as $n \to \infty$ we have $n(c_5 r_n)^d \to c_5^d t$, so by Theorem 2.1 applied to $\zeta(\cdot) = TSP_{w'}(\cdot)$,

$$\begin{aligned}n^{-1}TSP_w(r_n^{-1}\mathcal{X}_n) = n^{-1}TSP_{w'}((c_5 r_n)^{-1}\mathcal{X}_n) &\xrightarrow[L^2]{c.c.} \int \rho'(c_5^d t f_\mu(x)) f_\mu(x) dx \\ &= \int \rho(t f_\mu(x)) f_\mu(x) dx,\end{aligned}$$

completing the proof in the general case. $\square$

Proof of Theorem 4.3(a). Assume that $w(\cdot)$ satisfies W1–W3 with $0 < w_{\max} < \infty$ (and hence also W4). Then the function $w_a(\cdot) := \min(w(\cdot), a)$ satisfies condition W5. Therefore we can apply Lemma 4.4 to the weight function w_a. In particular, for all $\lambda \in (0, \infty)$ the limit $\rho_a(\lambda) := \lim_{s\to\infty} \mathbb{E}\left[TSP_{w_a}(\mathcal{H}_{\lambda,s})/(\lambda s^d)\right]$ exists, and if $nr_n^d \to t \in (0, \infty)$ then (2.3) holds for $\zeta(\cdot) = TSP_{w_a}(\cdot)$, i.e.

$$n^{-1}TSP_{w_a}(r_n^{-1}\mathcal{X}_n) \xrightarrow[L^2]{c.c.} \int \rho_a(t f_\mu(x)) f_\mu(x) dx. \tag{4.8}$$

For any finite $\mathcal{X} \subset \mathbb{R}^d$ and $a \in (0, w_{\max})$ we have $0 \le TSP_w(\mathcal{X}) - TSP_{w_a}(\mathcal{X}) \le (w_{\max} - a)|\mathcal{X}|$, since for each edge e of any tour of $\mathcal{X}$, we have $0 \le w(e) - w_a(e) \le w_{\max} - a$. Therefore setting $a(k) := \max(w_{\max} - 1/k, 0)$ for each $k \in \mathbb{N}$ and setting $\zeta^{(k)}(\cdot) := TSP_{w_{a(k)}}(\cdot)$, gives us a sequence of functionals satisfying the conditions of Lemma 4.1(b). By that result we obtain that $\mathbb{E}\,[TSP_w(\mathcal{H}_{\lambda,s})/(\lambda s^d)]$ converges to $\rho(\lambda)$ as $s \to \infty$ for all $\lambda \in (0,\infty)$, and if $nr_n^d \to t \in (0,\infty)$, then (4.8) holds with $w_a(\cdot)$ replaced by $w(\cdot)$, ρ_a replaced by ρ, as required. □

Now we aim to prove Theorem 4.3(b), so we assume W1–W4, with $w_{\max} = +\infty$. Initially we assume W6 too. Again set $w_a(\cdot) := \min(w(\cdot), a)$.

Lemma 4.5. *Suppose that $w(\cdot)$ satisfies W1–W4 and W6 with $w_{\max} = +\infty$. There is a constant C such that for all $(a, s) \in (0,\infty)^2$, for all finite $\mathcal{X} \subset Q_s$ we have*

$$TSP_{w_a}(\mathcal{X}) \le Cs^p|\mathcal{X}|^{(d-p)/d}, \tag{4.9}$$

and moreover

$$TSP_w(\mathcal{X}) - TSP_{w_a}(\mathcal{X}) \le Cs^p a^{(p/d)-1}(TSP_{w_a}(\mathcal{X}))^{(d-p)/d}. \tag{4.10}$$

Proof. By [46, eqn (3.6) or (3.7)], there is a constant $C \in (0,\infty)$ such that for any finite $\mathcal{Y} \subset Q_1$, we have $TSP_{c_6\|\cdot\|^p}(\mathcal{Y}) \le C|\mathcal{Y}|^{(d-p)/d}$, where c_6 is the constant in assumption W6. Hence for any $s > 0$, by scaling $TSP_w(s\mathcal{Y}) \le TSP_{c_6\|\cdot\|^p}(s\mathcal{Y}) \le Cs^p|\mathcal{Y}|^{(d-p)/d}$. Since $TSP_{w_a}(\mathcal{X}) \le TSP_w(\mathcal{X})$ for all $\mathcal{X}$, this implies (4.9).

Given $s, a > 0$ and given finite $\mathcal{X} \subset Q_s$, let $\tau_a(\mathcal{X})$ be (the set of edges in) a minimum-weight tour of $\mathcal{X}$, using weight function w_a (and using some deterministic rule to choose if there exist several tours of minimum weight). That is, let $\tau_a(\mathcal{X})$ be a tour of $\mathcal{X}$ with $w_a(\tau_a(\mathcal{X})) = TSP_{w_a}(\mathcal{X})$. Then let $\tau_{(=a)}(\mathcal{X}) = \{e \in \tau_a(\mathcal{X}) : w_a(e) = a\}$, and let $N_a(\mathcal{X}) := |\tau_{(=a)}(\mathcal{X})|$. Thus $N_a(\mathcal{X})$ is the total number of edges of weight a in the minimum w_a-weight tour of $\mathcal{X}$ (in the top figure of Figure 1 all edges of weight a are drawn as thick edges). Let $\mathcal{Y}_a(\mathcal{X})$ be the set of those vertices in $\mathcal{X}$ that are incident to at least one edge in $\tau_{(=a)}(\mathcal{X})$, and set $r = |\mathcal{Y}_a(\mathcal{X})|$. Note that $N_a(\mathcal{X}) \le r \le 2N_a(\mathcal{X})$.

We now remove the edges of $\tau_{(=a)}(\mathcal{X})$ from $\tau_a(\mathcal{X})$, and add further edges to make a new tour of $\mathcal{X}$, for which we can bound the total weight of the added edges. To do this, we use the *space-filling curve heuristic* (see [46]). Let $\phi : [0,1] \to [-1/2, 1/2]^d$ be a continuous surjection. Assume moreover that ϕ is Lipschitz of order $1/d$, that is, for any $t, t' \in [0,1]$, we have $\|\phi(t') - \phi(t)\| \le c_8|t' - t|^{1/d}$ for some absolute constant $c_8 > 0$ (c_8 depends on the norm). Such a ϕ is well known to exist; see for example [10]. Consider a tour of $s^{-1}\mathcal{Y}_a(\mathcal{X})$, obtained by visiting the points in order of this curve. Let $y_1, \ldots, y_r$ be the corresponding points of $\mathcal{Y}_a(\mathcal{X})$ taken in this order, and let $t_1 < \ldots < t_r \in [0,1]$ be such that $s\phi(t_i) = y_i$; to be definite, if $\phi^{-1}(\{s^{-1}y_i\})$ has more than one element, let t_i be the minimal element.

We now create a multigraph on vertex set $[r] := \{1, \ldots, r\}$ as follows. Starting with the empty graph on this vertex set, add an edge from i to j for each $\{i,j\} \subset [r]$ such that there exists a path from y_i to y_j in the graph with vertex set $\mathcal{X}$ and edge set $\tau_a \setminus \tau_{(=a)}$, i.e. a path from y_i to y_j in the original tour τ_a with all edge weights less than a along this path. We shall refer to the edges added so far as *red* edges. At this stage, each vertex has degree 0 or 1, and the number of red edges equals $r - N_a(\mathcal{X})$.

Next, if 1 has odd degree, add a *blue* edge from 1 to 2; otherwise add two blue edges from 1 to 2. Then if 2 has odd degree, add a blue edge from 2 to 3; otherwise, add two blue edges from 2 to 3. Then if 3 has an odd degree, add a blue edge from 3 to 4, and otherwise add two such edges, and so on.

Continuing in this way up to vertex $r-1$, we add one or two blue edges between i and $i+1$ for each $i \in \{1, \ldots, r-1\}$ in turn (see the middle figure of Figure 1 for an example). We end up with a multigraph which we denote $\mathcal{G}$, in which each $i \in \{1, \ldots, r-1\}$ has even degree, so vertex r must also have even degree since the sum of degrees is even. Thus, $\mathcal{G}$ is connected and all of its degrees are even; in fact, each vertex has degree 2 or 4. We define the *red-degree* of a vertex in $\mathcal{G}$ to be the number of red vertices incident to that vertex (which is either 0 or 1).

By Euler's "Königsberg bridge" theorem, we can and do choose an Eulerian circuit through $\mathcal{G}$, starting and ending at vertex 1. Choose a direction of travel through this circuit. We denote this directed Eulerian circuit by γ. All edges of $\mathcal{G}$ now become arcs (i.e. directed edges), with the direction of an edge determined by the direction of travel through the edge in γ.

List the vertices and arcs of $\mathcal{G}$ in the order they appear in the circuit γ. This list is an alternating sequence of vertices and arcs, starting and ending at vertex 1. Vertices of degree 4 on $\mathcal{G}$ appear twice in the list (as does vertex 1); other vertices appear once. Now reduce this list to a list of vertices only, in which each vertex other than vertex 1 appears once, as follows.

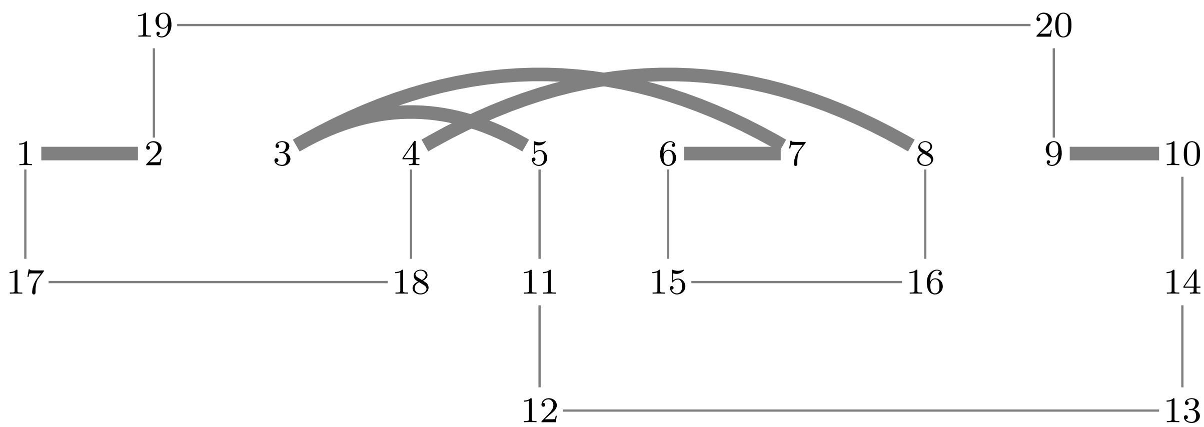
Suppose vertex $i > 1$ appears twice in the old list. If i has red-degree 0 then omit the second appearance of i in the old list. If i has red-degree 1, then retain the appearance of i in the old list which is adjacent in that list to the red arc incident to i, and omit the other appearance of vertex i.

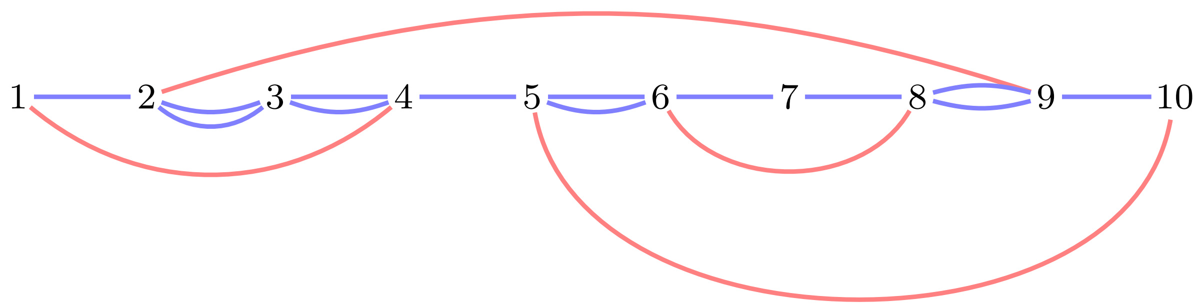
Write the vertices in the order of this new list as $i(1) = 1, i(2), \ldots, i(r), i(r+1) = 1$. Let $\mathcal{K}$ be the set of indices $k \in [r]$ such that $(i(k), i(k+1))$ is *not* a red arc in $\mathcal{G}$.

We now create our new tour of $\mathcal{X}$, by taking the set of edges

$$\{\{y_{i(k)}, y_{i(k+1)}\}, k \in \mathcal{K}\} \cup (\tau \setminus \tau_{(=a)})$$

(see the bottom figure of Figure 1 for the edges added corresponding to the Eulerian





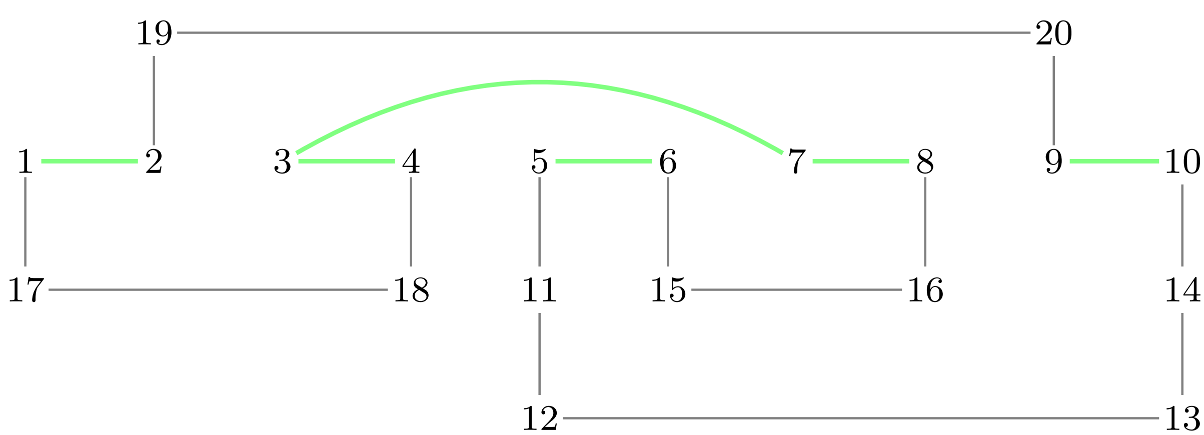


Figure 1: Top: Example of an optimal tour τ_a with respect to w_a (on $|\mathcal{X}| = 20$ vertices); thick edges have weight a ($r = 10$ here, the space-filling curve visits vertices $1, \ldots, 10$ in this order). Middle: construction of red and blue edges of $\{1, \ldots, 10\}$. Bottom: an Eulerian circuit visits vertices in this order: 1,2,9,10,5,6,8,9,8,7,6,5,4,3,2,3,4,1. New list of vertices: 1,2,9,10,5,6,8,7,3,4,1. In green: not-red edges corresponding to the new list added to the new tour.

circuit chosen in the text below the figure). In other words, we take the edges in the tour $(y_{i(1)}, y_{i(2)}, \ldots, y_{i(r)}, y_{i(1)})$ of $\mathcal{Y}_a(\mathcal{X})$, and then replace each red edge of this tour (i.e. a step $\{y_{i(k)}, y_{i(k+1)}\}$ for which $\{i(k), i(k+1)\}$ is a red edge of $\mathcal{G}$) with the portion of the original tour $\tau_{(a)}$ that goes from $y_{i(k)}$ to $y_{i(k+1)}$ and consists entirely of edges of weight less than a.

The total w-weight of this new tour of $\mathcal{X}$ is an upper bound for $TSP_w(\mathcal{X})$, so $TSP_w(\mathcal{X}) - TSP_{w_a}(\mathcal{X})$ is bounded above by the total weight of edges in this tour that are not in τ. Thus using W6, we have

$$TSP_w(\mathcal{X}) - TSP_{w_a}(\mathcal{X}) \leq \sum_{k \in \mathcal{K}} w(y_{i(k+1)} - y_{i(k)}) \leq c_6 \sum_{k \in \mathcal{K}} \|y_{i(k+1)} - y_{i(k)}\|^p.$$

Using the Lipschitz continuity of ϕ, and then Hölder's inequality, we obtain that

$$\begin{aligned} TSP_w(\mathcal{X}) - TSP_{w_a}(\mathcal{X}) &\leq c_6 c_8^p s^p \sum_{k \in \mathcal{K}} |t_{i(k+1)} - t_{i(k)}|^{p/d} \\ &\leq c_6 c_8^p s^p |\mathcal{K}|^{1-(p/d)} \sum_{k \in \mathcal{K}} |t_{i(k+1)} - t_{i(k)}|. \end{aligned} \tag{4.11}$$

In the multigraph $\mathcal{G}$, each blue edge goes between two neighbouring vertices in the list $\{1, \ldots, r\}$. For $1 \leq \ell \leq r-1$, let each blue edge from ℓ to $\ell+1$ be given weight $|t_{\ell+1} - t_\ell|$. Then for $k \in \mathcal{K}$, the value of $|t_{i(k+1)} - t_{i(k)}|$ is bounded above by the sum of the weights of the edges making up the portion of the Eulerian circuit γ that goes from $i(k)$ to $i(k+1)$ (all of which are blue). Since this circuit is Eulerian, no blue edge is traversed more than once, so the sum in the last line of (4.11) is at most the total weight of all the blue edges, which is at most 2 because there are at most two blue edges from i to $i+1$, $1 \leq i \leq r-1$. Also $|\mathcal{K}| = N_a(\mathcal{X})$, so

$$TSP_w(\mathcal{X}) - TSP_{w_a}(\mathcal{X}) \leq 2c_6 c_8^p s^p (N_a(\mathcal{X}))^{1-(p/d)}, \qquad \mathcal{X} \subset Q_s, \mathcal{X} \text{ finite}.$$

Since $aN_a(\mathcal{X}) \leq TSP_{w_a}(\mathcal{X})$, this yields (4.10). □

Proof of Theorem 4.3(b). Assume that $w(\cdot)$ satisfies W1–W4 with $w_{\max} = +\infty$, and that μ has bounded support. Given $\delta > 0$, define the new weight function $w'_\delta(x) := w(x)(1 - \mathbf{1}_{B_\delta(o)}(x))$. For $m \in \mathbb{N}$ set $w'_{\delta,m}(x) := \min(w_\delta(x), m)$.

Then $w'_{\delta,m}(\cdot)$ satisfies W2 and W5. Hence by Lemma 4.4, for all $\lambda > 0$ the limit $\rho'_{\delta,m}(\lambda) := \lim_{s\to\infty} \mathbb{E}[TSP_{w'_{\delta,m}}(\mathcal{H}_{\lambda,s})/(\lambda s^d)]$ exists, and if $nr_n^d \to t \in (0,\infty)$ as $n \to \infty$, then $n^{-1} TSP_{w'_{\delta,m}}(r_n^{-1}\mathcal{X}_n) \xrightarrow[L^2]{c.c.} \int \rho'_{\delta,m}(tf_\mu(x)) f_\mu(x) dx$.

Since we assume $w(\cdot)$ satisfies W4, for each $\delta > 0$ the function $w'_\delta(\cdot)$ satisfies W6. Therefore by Lemma 4.5, there is a constant $C(\delta)$ such that for all $s > 0$ and all finite $\mathcal{X} \in Q_s$, we have $TSP_{w'_{\delta,m}}(\mathcal{X}) \leq C(\delta) s^p |\mathcal{X}|^{(d-p)/d}$ and

$$TSP_{w'_\delta}(\mathcal{X}) - TSP_{w'_{\delta,m}}(\mathcal{X}) \leq C(\delta) s^p m^{(p/d)-1} (TSP_{w'_{\delta,m}}(\mathcal{X}))^{(d-p)/d}. \tag{4.12}$$

Now for $\ell, m \in \mathbb{N}$, set $\tilde{\zeta}^{(\ell,m)}(\mathcal{X}) := TSP_{w'_{1/\ell,m}}(\mathcal{X})$, which is nondecreasing both in ℓ and in m. Set $\tilde{\zeta}^{(\ell)}(\mathcal{X}) := \lim_{m\to\infty} \tilde{\zeta}^{(\ell,m)}(\mathcal{X}) = TSP_{w'_{1/\ell}}(\mathcal{X})$, and set $\tilde{\zeta}(\mathcal{X}) := \lim_{\ell\to\infty} \tilde{\zeta}^{(\ell)}(\mathcal{X}) = TSP_w(\mathcal{X})$. Set $\tilde{h}(\ell) := \sup_{x\in B_{1/\ell}(o)} w(x)$. Then $\tilde{h}(\ell) \to 0$ as $\ell \to \infty$, by W2, and

$$\tilde{\zeta}(\mathcal{X}) - \tilde{\zeta}^{(\ell)}(\mathcal{X}) = TSP_w(\mathcal{X}) - TSP_{w'_{1/\ell}}(\mathcal{X}) \leq \tilde{h}(\ell)|\mathcal{X}|. \tag{4.13}$$

By (4.12) and (4.13) we can apply Lemma 4.2 to the functionals $\tilde{\zeta}^{(\ell,m)}(\cdot)$, $\ell, m \in \mathbb{N}$, taking $h_\ell(m) = C(1/\ell)m^{(p/d)-1}$. Hence the limit $\lim_{s\to\infty} \mathbb{E}\left[TSP_w(\mathcal{H}_{\lambda,s})/(\lambda s^d)\right] =: \rho(\lambda)$ exists and is finite, for all $\lambda > 0$, and if $nr_n^d \to t \in (0,\infty)$ as $n \to \infty$, then $TSP_w(r_n^{-1}\mathcal{X}_n) \xrightarrow[L^2]{c.c.} \int \rho(tf_\mu(x))f_\mu(x)dx$, as required. □

4.2 Minimum-weight matching

Given finite $\mathcal{X} \subset \mathbb{R}^d$, the minimum-weight matching (MM) problem is to find a near-perfect matching (that is, a perfect matching if $|\mathcal{X}|$ is even and a matching excluding one vertex if $|\mathcal{X}|$ is odd) of minimum weight. More precisely, we define a (near-perfect) *matching* of $\mathcal{X}$ to be a collection τ of $\lfloor |\mathcal{X}|/2 \rfloor$ pairwise disjoint edges of the complete graph on vertex set $\mathcal{X}$, where two edges uv and xy are said to be disjoint if u, v, x, y are all distinct. Given such a matching τ, we define $w(\tau) := \sum_{e\in\tau} w(e)$. We define $MM_w(\mathcal{X}) := \min_\tau w(\tau)$, where the minimum is taken over all matchings τ of $\mathcal{X}$.

Theorem 4.6 (LLN for minimum-weight matching)**.** *Suppose $w(\cdot)$ satisfies conditions W1–W4, and either (a) $w_{\max} < \infty$ or (b) μ has bounded support, $w_{\max} = \infty$. Then when choosing $\zeta(\mathcal{X})$ as $MM_w(\mathcal{X})$, the limit $\rho(\lambda)$ given by (2.2) exists and is finite for all $\lambda \in (0,\infty)$, and if $nr_n^d \to t \in (0,\infty)$ as $n \to \infty$, then (2.3) holds.*

We first prove Theorem 4.6 under the extra condition W5.

Lemma 4.7. *Suppose $w(\cdot)$ satisfies W1, W2 and W5. Then when choosing $\zeta(\mathcal{X})$ as $MM_w(\mathcal{X})$, the limit $\rho(\lambda)$ given by (2.2) exists and is finite for all $\lambda \in (0,\infty)$, and if $nr_n^d \to t \in (0,\infty)$ as $n \to \infty$, then (2.3) holds.*

Proof. Assume first that W5 holds with $c_5 = 1$ (as well as W1 and W2). We check that Theorem 2.1 is applicable when taking $\zeta(\cdot) := MM_w(\cdot)$ in this case.

Properties P1 and P2 are clear. To check P3, let $\mathcal{Y}, \mathcal{Z}$ be disjoint finite subsets of $\mathbb{R}^d$. If τ is an optimal matching of $\mathcal{Y}$ (i.e. a matching of $\mathcal{Y}$ with $w(\tau) = MM_w(\mathcal{Y})$), and τ' is an optimal matching of $\mathcal{Z}$, then we can create a matching τ'' of $\mathcal{Y} \cup \mathcal{Z}$ by taking all edges of $\tau \cup \tau'$, together with possibly one more edge. Then

$$MM_w(\mathcal{Y} \cup \mathcal{Z}) \leq w(\tau'') \leq w(\tau) + w(\tau') + w_{\max} = MM_w(\mathcal{Y}) + MM_w(\mathcal{Z}) + w_{\max},$$

so that P3 holds with $c_1 = w_{\max}$ here.

Now we check P4. Suppose τ is an optimal matching of $\mathcal{Y} \cup \mathcal{Z}$, and enumerate the edges in τ that have one endpoint in $\mathcal{Y}$ and one endpoint in $\mathcal{Z}$, as $e_1, \ldots e_k$. For $1 \leq i \leq k$ denote the endpoints of e_i by y_i, z_i with $y_i \in \mathcal{Y}$ and $z_i \in \mathcal{Z}$.

Now create a matching τ' on $\mathcal{Y}$ as follows. Choose a matching τ_1 on $\{y_1, \ldots, y_k\}$, and let τ' consist of all edges of τ that have both endpoints in $\mathcal{Y}$, together with the edges of τ_1, together with one further edge if k is odd and $\mathcal{Y}$ has an unmatched vertex in τ.

Similarly, create a matching τ'' on $\mathcal{Z}$ as follows. Choose a matching τ_2 on $\{z_1, \ldots, z_k\}$, and let τ'' consist of all edges of τ that have both endpoints in $\mathcal{Z}$, together with the edges of τ_2, together with one further edge if k is odd and $\mathcal{Z}$ has an unmatched vertex in τ.

Since $\tau' \cup \tau''$ is either a matching of $\mathcal{Y} \cup \mathcal{Z}$, or is contained in such a matching, $|\tau' \cup \tau''| \leq |\tau|$, so that $|(\tau' \cup \tau'') \setminus \tau| \leq |\tau \setminus (\tau' \cup \tau'')| = k$. Then

$$\begin{aligned} MM_w(\mathcal{Y}) + MM_w(\mathcal{Z}) &\leq w(\tau') + w(\tau'') \\ &= w(\tau) + \left(\sum_{e \in (\tau' \cup \tau'') \setminus \tau} w(e) \right) - \sum_{i=1}^{k} w(e_i) \\ &\leq w(\tau) + \sum_{i=1}^{k} (w_{\max} - w(e_i)). \end{aligned}$$

As before, we say that an edge uv of τ is *short* if $\|u-v\| \leq 1$, *long* if $\|u-v\| > 1$. In the last sum, the ith term is zero whenever e_i is a long edge. The number of short edges e_i is at most $|\partial_{\mathcal{Z}} \mathcal{Y}|$, and therefore $MM_w(\mathcal{Y}) + MM_w(\mathcal{Z}) \leq MM_w(\mathcal{Y} \cup \mathcal{Z}) + w_{\max} |\partial_{\mathcal{Z}} \mathcal{Y}|$. This gives us P4 with $c_2 = w_{\max}$.

Since one can obtain a matching from a tour by removing alternate edges, $MM_w(\mathcal{X}) \leq TSP_w(\mathcal{X})$ for all $\mathcal{X}$. Hence Property P5 for MM_w holds as a consequence of the corresponding property for TSP_w, which we checked earlier.

Thus Theorem 2.1 is directly applicable in the case where $c_5 = 1$, giving us the desired conclusion in that case. We can then extend the result to the general case $c_5 \in (0, \infty)$ in the same manner as we did in the proof of Lemma 4.4. □

Proof of Theorem 4.6(a). Assume that $w(\cdot)$ satisfies W1–W3 with $0 < w_{\max} < \infty$ (and hence also W4). Then for $a \in (0, w_{\max})$ we have that $w_a(\cdot) := \min(w(\cdot), a)$ satisfies W5 (as well as W1 and W2). So by Lemma 4.7, the limit $\rho_a(\lambda) := \lim_{s \to \infty} \mathbb{E}\left[MM_{w_a}(\mathcal{H}_{\lambda,s})/(\lambda s^d) \right]$ exists and is finite for all $\lambda > 0$, and if $nr_n^d \to t \in (0, \infty)$, then $n^{-1} MM_{w_a}(r_n^{-1} \mathcal{X}_n) \xrightarrow[L^2]{c.c.} \int \rho_a(t f_\mu(x)) f_\mu(x) dx$.

For any finite $\mathcal{X} \subset \mathbb{R}^d$, $0 \leq MM_w(\mathcal{X}) - MM_{w_a}(\mathcal{X}) \leq |\mathcal{X}|(w_{\max} - a)$. Therefore, setting $a(k) := \max(w_{\max} - 1/k, 0)$ for each $k \in \mathbb{N}$, we can apply Lemma 4.1(b), taking $\zeta^{(k)}(\cdot) := MM_{w_{a(k)}}(\cdot)$ and $\zeta(\cdot) := MM_w(\cdot)$, to deduce the desired conclusion. $\square$

Now we aim to prove Theorem 4.6(b), so we assume W1–W4 with $w_{\max} = +\infty$. Initially we shall also assume W6. Again set $w_a(\cdot) := \min(w(\cdot), a)$.

Lemma 4.8. *Suppose $w(\cdot)$ satisfies W1–W4 and W6 with $w_{\max} = +\infty$. Then there exists $C \in (0, \infty)$ such that for all $a \in (0, \infty)$, $s \geq 1$ and all finite $\mathcal{X} \subset Q_s$ we have*

$$MM_{w_a}(\mathcal{X}) \leq MM_w(\mathcal{X}) \leq Cs^p|\mathcal{X}|^{(d-p)/d} \tag{4.14}$$

and

$$0 \leq MM_w(\mathcal{X}) - MM_{w_a}(\mathcal{X}) \leq Cs^p a^{(p/d)-1}(MM_{w_a}(\mathcal{X}))^{(d-p)/d}. \tag{4.15}$$

Proof. Let c_6 be as in W6. By [46, (3.6) or (3.7)], there is a constant $C > 0$ such that for any finite $\mathcal{Y} \subset Q_1$, $MM_{c_6\|\cdot\|^p}(\mathcal{Y}) \leq C|\mathcal{Y}|^{(d-p)/d}$. Hence by scaling, $MM_w(s\mathcal{Y}) \leq MM_{c_6\|\cdot\|^p}(s\mathcal{Y}) \leq Cs^p|\mathcal{Y}|^{(d-p)/d}$. Then (4.14) follows.

Now let $\tau_a(\mathcal{X})$ be (the set of edges in) a minimum-weight matching of $\mathcal{X}$, using weight function w_a (and using some deterministic rule to choose if there exist several matchings of minimum weight). That is, let $\tau_a(\mathcal{X})$ be a matching of $\mathcal{X}$ with $w_a(\tau_a(\mathcal{X})) = MM_{w_a}(\mathcal{X})$. Let $\tau_{(=a)}(\mathcal{X}) = \{e \in \tau_a(\mathcal{X}) : w_a(e) = a\}$, and let $N_a(\mathcal{X}) := |\tau_{(=a)}(\mathcal{X})|$. Thus $N_a(\mathcal{X})$ is the total number of edges of weight a in the minimum w_a-weight matching of $\mathcal{X}$. Let $\mathcal{Y}_a(\mathcal{X})$ be the set of vertices in $\mathcal{X}$ that are incident to edges in $\tau_{(=a)}(\mathcal{X})$.

Let τ' be a matching of $\mathcal{Y}_a(\mathcal{X})$ with $w(\tau') = MM_w(\mathcal{Y}_a(\mathcal{X}))$. Then $\tau' \cup (\tau_a(\mathcal{X}) \setminus \tau_{(=a)}(\mathcal{X}))$ is a matching on $\mathcal{X}$ with w-weight at most $w_a(\tau_a(\mathcal{X})) + w(\tau')$. Therefore $MM_w(\mathcal{X}) \leq MM_{w_a}(\mathcal{X}) + MM_w(\mathcal{Y}_a(\mathcal{X}))$, and hence by the second inequality of (4.14) applied to $\mathcal{Y}_a(\mathcal{X})$,

$$MM_w(\mathcal{X}) - MM_{w_a}(\mathcal{X}) \leq Cs^p(2N_a(\mathcal{X}))^{(d-p)/d}, \qquad \mathcal{X} \subset Q_s, \mathcal{X} \text{ finite.}$$

Since $aN_a(\mathcal{X}) \leq MM_{w_a}(\mathcal{X})$, this gives us (4.15). $\square$

Proof of Theorem 4.6(b). We can follow the same argument as in the proof of Theorem 4.3(b), now using Lemma 4.7 instead of Lemma 4.4 and Lemma 4.8 instead of Lemma 4.5. $\square$

4.3 Minimum-weight bipartite matching

Given two disjoint finite sets $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^d$, and given a near-perfect matching τ of $\mathcal{U} \cup \mathcal{V}$, define $w^*(\tau) := \sum_{e \in \tau} w^*(e)$, with $w^*(xy) := w(xy)$ if $x \in \mathcal{U}, y \in \mathcal{V}$ or $x \in \mathcal{V}, y \in \mathcal{U}$, and $w^*(xy) := w_{\max}$ if $x, y \in \mathcal{U}$ or $x, y \in \mathcal{V}$. In other words, if we say vertices in $\mathcal{U}$ are of type 1 and vertices in $\mathcal{V}$ are of type 2, we allow edges between vertices of the same type, but give them a weight which is at least as big as that of any edge between vertices of different types. We then define $BM_w(\mathcal{U}, \mathcal{V}) := \min_\tau w^*(\tau)$, where the minimum is taken over all near-perfect matchings of $\mathcal{U} \cup \mathcal{V}$. When $|\mathcal{U}| = |\mathcal{V}|$, clearly the minimum can be attained using a *bipartite matching*, i.e. a perfect matching in which each edge is between vertices of different types. (Hence, when $|\mathcal{U}| = |\mathcal{V}|$, $BM_w(\mathcal{U}, \mathcal{V}) = \min_\tau w(\tau)$ with the minimum taken over all bipartite matchings.) Given $r > 0$, we define $BM_{w,r}(\mathcal{U}, \mathcal{V}) := BM_w(r^{-1}\mathcal{U}, r^{-1}\mathcal{V})$.

For $\lambda, s > 0$, let $\mathcal{H}'_\lambda$ denote a homogeneous Poisson point process in $\mathbb{R}^d$ of intensity λ, independent of $\mathcal{H}_\lambda$, and set $\mathcal{H}'_{\lambda,s} := \mathcal{H}'_\lambda \cap Q_s$. Let $U_0, V_0, U_1, V_1, U_2, V_2, \ldots$ be independent random d-vectors with common distribution μ. For all $n \in \mathbb{N}$, let $\mathcal{U}_n := \{U_1, \ldots, U_n\}, \mathcal{V}_n := \{V_1, \ldots, V_n\}$. We have the following law of large numbers for $BM_w(\mathcal{U}_n, \mathcal{V}_n)$ in the thermodynamic limit, when $w_{\max} < \infty$.

Theorem 4.9 (LLN for BM). *Suppose $w(\cdot)$ satisfies W1–W3 with $w_{\max} < \infty$. Then for all $\lambda > 0$, the limit $\rho_{BM_w}(\lambda) := \lim_{s\to\infty} \mathbb{E}\,[BM_w(\mathcal{H}_{\lambda,s}, \mathcal{H}'_{\lambda,s})/(\lambda s^d)]$ exists and is finite. Also, if $nr_n^d \to t \in (0, \infty)$ as $n \to \infty$, then*

$$n^{-1} BM_{w,r_n}(\mathcal{U}_n, \mathcal{V}_n) \xrightarrow[L^2]{c.c.} \int_{\mathbb{R}^d} \rho_{BM_w}(t f_\mu(x)) f_\mu(x) dx. \tag{4.16}$$

Remarks. (a) The case with $w(x) = \|x\|^p$ for some fixed $p > 0$ is considered in [5] and references therein; it is *not* covered by our Theorem 4.9 because we require $w_{\max} < \infty$. For this case, results along the lines of (4.16) have been shown [5] for $p < d/2$, $d \leq 2$ or $p \leq 1, d \geq 3$ and f_μ being uniform over a bounded region, or for $p < d/2$ with μ being uniform over a bounded region with no singular part. In contrast, our result (4.16) for bounded w holds for *any* choice of μ.

(b) One may also consider the *Bipartite TSP* with weight function w (the case with $w(\cdot) = \|\cdot\|^p$ is considered in [5, 12]). To define this, let $BTSP_w(\mathcal{U}, \mathcal{V}) := \min_\tau w^*(\tau)$, now taking the minimum over all *tours* τ of $\mathcal{U} \cup \mathcal{V}$; let $BTSP_{w,r}(\mathcal{U}, \mathcal{V}) := BTSP_w(r^{-1}\mathcal{U}, r^{-1}\mathcal{V})$. If $|\mathcal{U}| = |\mathcal{V}|$, then $BTSP_w(\mathcal{U}, \mathcal{V})$ is the minimum weight of all tours through $\mathcal{U} \cup \mathcal{V}$ that alternate between $\mathcal{U}$ and $\mathcal{V}$.

The statement of Theorem 4.9 also holds with the functional $BM_w(\cdot)$ replaced by $BTSP_w(\cdot)$ and $BM_{w,r}(\cdot)$ replaced by $BTSP_{w,r}(\cdot)$. The proof is very similar.

Lemma 4.10. *Suppose $w(\cdot)$ satisfies W1–W3 and W5. Then the conclusion of Theorem 4.9 holds.*

Proof. First assume $c_5 = 1$ in W5. Given finite $\mathcal{X} \subset \mathbb{R}^d$, suppose each point is assigned a mark in $\{1, 2\}$. Write $\mathcal{X}^*$ for the resulting marked point set (a subset of $\mathbb{R}^d \times \{1, 2\}$). Define $\zeta^m(\mathcal{X}^*)$ to be $BM_w(\mathcal{U}, \mathcal{V})$ where $\mathcal{U}$ is the set of vertices of $\mathcal{X}$ with mark 1, and $\mathcal{V} := \mathcal{X} \setminus \mathcal{U}$. Then define $\zeta(\mathcal{X}) := \mathbb{E}\,[\zeta^m(\mathcal{X}^*)]$, where the expectation is over a random marking of $\mathcal{X}$ where each point is independently marked 1 or 2 with equal probability. Likewise, for $r > 0$ define $\zeta_r^m(\mathcal{X}^*) := BM_{w,r}(\mathcal{U},\mathcal{V})$ and $\zeta_r(\mathcal{X}) := \mathbb{E}\,[\zeta_r^m(\mathcal{X}^*)]$.

Given two marked sets $\mathcal{Y}^*$ and $\mathcal{Z}^*$, just as in the proof of Lemma 4.7 we have

$$\begin{aligned}\zeta^m(\mathcal{Y}^*) + \zeta^m(\mathcal{Z}^*) - w_{\max}|\partial_{\mathcal{Z}}\mathcal{Y}| &\leq \zeta^m(\mathcal{Y}^* \cup \mathcal{Z}^*) \\ &\leq \zeta^m(\mathcal{Y}^*) + \zeta^m(\mathcal{Z}^*) + w_{\max}, \end{aligned} \tag{4.17}$$

so for any disjoint finite $\mathcal{Y},\mathcal{Z} \subset \mathbb{R}^d$, taking expectations over the random markings of $\mathcal{Y},\mathcal{Z}$ gives us

$$\zeta(\mathcal{Y}) + \zeta(\mathcal{Z}) - w_{\max}|\partial_{\mathcal{Z}}\mathcal{Y}| \leq \zeta(\mathcal{Y} \cup \mathcal{Z}) \leq \zeta(\mathcal{Y}) + \zeta(\mathcal{Z}) + w_{\max}.$$

Therefore ζ satisfies P1–P4.

Next we verify P5 for ζ. Let $\varepsilon > 0$. By W2, we can and do choose $\delta \in (0, \varepsilon^2)$ such that $w(x) \leq \varepsilon$ for all $x \in B_{2\delta}(o)$. Given $\mathcal{X} \subset B_\delta(o)$ with $|\mathcal{X}| \geq \delta^{-1}$, set $n = |\mathcal{X}|$. Let M_n be a Binomial$(n, 1/2)$ variable, representing the number of elements of $\mathcal{X}$ assigned to type 1 rather than to type 2 in the randomly marked point set $\mathcal{X}^*$. Then

$$\zeta^m(\mathcal{X}^*) \leq \varepsilon \min(M_n, n - M_n) + |M_n - (n/2)|w_{\max},$$

so that taking expectations and using Jensen's inequality yields

$$\begin{aligned}\zeta(\mathcal{X}) \leq \varepsilon(n/2) + w_{\max}\mathbb{E}\,[|M_n - (n/2)|] &\leq \varepsilon(n/2) + w_{\max}\sqrt{\mathrm{Var} M_n} \\ &= \varepsilon(n/2) + w_{\max}\sqrt{n/4},\end{aligned}$$

and therefore $n^{-1}\zeta(\mathcal{X}) \leq (1/2)(\varepsilon + n^{-1/2}w_{\max}) \leq \varepsilon(1 + w_{\max})/2$, which yields P5.

Thus Theorem 2.1 is applicable here, so for all $\lambda \in (0,\infty)$ the limit $\rho_1(\lambda) := \lim_{s\to\infty} \mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s})/(\lambda s^d)]$ exists in $\mathbb{R}$, and by (2.4) we have as $n \to \infty$ that

$$(2n)^{-1}\mathbb{E}\,[\zeta_{r_n}(\mathcal{P}_{2n})] \to \int_{\mathbb{R}^d} \rho_1(2tf_\mu(x))f_\mu(x)dx. \tag{4.18}$$

By the Marking theorem [32], for $\lambda > 0$ the randomly marked point process $\mathcal{H}^*_{2\lambda}$ decomposes as the union of two independent copies of $\mathcal{H}_\lambda$. Therefore

$$\rho_1(2\lambda) = \lim_{s\to\infty} \mathbb{E}\,[BM_w(\mathcal{H}_{\lambda,s}, \mathcal{H}'_{\lambda,s})/(2\lambda s^d)] = (1/2)\rho_{BM_w}(\lambda),$$

where $\rho_{BM_w}(\lambda)$ was defined just after (4.16).

For $n > 0$, let N_n, N'_n be independent Poisson(n) random variables, independent of $(U_1, V_1, U_2, V_2, \ldots)$. Again by the Marking theorem, and by [32, Prop. 3.5], the left hand side of (4.18) is equal to $(2n)^{-1}\mathbb{E}\,[BM_{w,r_n}(\mathcal{U}_{N_n}, \mathcal{V}_{N'_n})]$. Thus (4.18) becomes

$$n^{-1}\mathbb{E}\,[BM_{w,r_n}(\mathcal{U}_{N_n}, \mathcal{V}_{N'_n})] \to \int_{\mathbb{R}^d} \rho_{BM_w}(tf_\mu(x))f_\mu(x)dx. \tag{4.19}$$

By taking $|\mathcal{Y}^*| = 1$ in (4.17), we see that adding one element to $\mathcal{U}$ (or $\mathcal{V}$) changes $BM_{w,r_n}(\mathcal{U}, \mathcal{V})$ by at most $w_{\max}$. Thus by iteration,

$$|BM_{w,r_n}(\mathcal{U}_{N_n}, \mathcal{V}_{N'_n}) - BM_{w,r_n}(\mathcal{U}_n, \mathcal{V}_n)| \leq w_{\max}(|N_n - n| + |N'_n - n|),$$

and hence by (4.19), since $\mathbb{E}\,[|N_n - n|] = o(n)$, as $n \to \infty$ we have

$$n^{-1}\mathbb{E}\,[BM_{w,r_n}(\mathcal{U}_n, \mathcal{V}_n)] \to \int_{\mathbb{R}^d} \rho_{BM_w}(tf_\mu(x))f_\mu(x)dx. \tag{4.20}$$

We now show the complete and L^2 convergence. Let $\varepsilon > 0$. For $1 \leq i \leq n$, let $\mathcal{U}_{n,i} := \{U_1, \ldots, U_0, \ldots, U_n\}$ where U_0 appears as the ith item in the list, and let $\mathcal{V}_{n,i} := \{V_1, \ldots, V_0, \ldots, V_n\}$ where V_0 appears as the ith item in the list. That is, $(\mathcal{U}_{n,i}, \mathcal{V}_{n,i})$ is the pair of point processes $(\mathcal{U}_n, \mathcal{V}_n)$ with (U_i, V_i) replaced by (U_0, V_0). Let $\mathcal{F}_i$ be the σ-algebra generated by $((U_1, V_1), \ldots, (U_i, V_i))$, and let $\mathcal{F}_0$ be the trivial σ-algebra. Then

$$BM_{w,r_n}(\mathcal{U}_n, \mathcal{V}_n) - \mathbb{E}\,[BM_{w,r_n}(\mathcal{U}_n, \mathcal{V}_n)] = \sum_{i=1}^{n} D_{n,i}, \tag{4.21}$$

where for each $i \in [n] := \{1, \ldots, n\}$ we set

$$\begin{aligned} D_{n,i} &:= \mathbb{E}\,[BM_{w,r_n}(\mathcal{U}_n, \mathcal{V}_n)|\mathcal{F}_i] - \mathbb{E}\,[BM_{w,r_n}(\mathcal{U}_n, \mathcal{V}_n)|\mathcal{F}_{i-1}] \\ &= \mathbb{E}\,[BM_{w,r_n}(\mathcal{U}_n, \mathcal{V}_n) - BM_{w,r_n}(\mathcal{U}_{n,i}, \mathcal{V}_{n,i})|\mathcal{F}_i]. \end{aligned}$$

By the conditional Jensen inequality, and since replacing (U_i, V_i) by (U_0, V_0) can change ζ_{r_n} by at most $4w_{\max}$ by (4.17),

$$|D_{n,i}| \leq \mathbb{E}\,[(|BM_{w,r_n}(\mathcal{U}_n, \mathcal{V}_n) - BM_{w,r_n}(\mathcal{U}_{n,i}, \mathcal{V}_{n,i})|)|\mathcal{F}_i] \leq 4w_{\max}, \tag{4.22}$$

almost surely. Also $(D_{n,1}, \ldots, D_{n,n})$ is a sequence of martingale differences with respect to the filtration $(\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_n)$. Therefore by Azuma's inequality (see e.g. [46] or [35]),

$$\mathbb{P}[|BM_{w,r_n}(\mathcal{U}_n, \mathcal{V}_n) - \mathbb{E}\,[BM_{w,r_n}(\mathcal{U}_n, \mathcal{V}_n)]| > \varepsilon n] \leq 2\exp\left(\frac{-(\varepsilon n)^2}{32w_{\max}^2 n}\right),$$

which is summable in n. Combined with (4.20), this demonstrates the complete convergence in (4.16).

For the L^2 convergence, note that by (4.21), and the orthogonality of martingale differences, and (4.22),

$$\mathrm{Var}[n^{-1}BM_{w,r_n}(\mathcal{U}_n, \mathcal{V}_n)] = n^{-2}\sum_{i=1}^{n} \mathbb{E}\,[D_{n,i}^2] \leq 16w_{\max}^2/n,$$

which tends to 0 as $n \to \infty$, so the L^2 convergence in (4.16) follows as well. $\square$

Proof of Theorem 4.9. Assume $w(\cdot)$ satisfies W1–W3 with $w_{\max} < \infty$. Given $a \in (0, w_{\max})$, $w_a(\cdot) := \min(w(\cdot), a)$ satisfies W5. By Lemma 4.10, for all $\lambda > 0$ the limit

$$\rho_{BM_{w_a}}(\lambda) := \lim_{s\to\infty} \mathbb{E}\,[BM_{w_a}(\mathcal{H}_{\lambda,s}, \mathcal{H}'_{\lambda,s})/(\lambda s^d)] \tag{4.23}$$

exists, and if $nr_n^d \to t \in (0, \infty)$ as $n \to \infty$, then

$$n^{-1} BM_{w_a}(\mathcal{U}_n, \mathcal{V}_n) \xrightarrow[L^2]{c.c.} \int \rho_{BM_{w_a}}(t f_\mu(x)) f_\mu(x) dx. \tag{4.24}$$

Also $0 \le BM_w(\mathcal{U}, \mathcal{V}) - BM_{w_a}(\mathcal{U}, \mathcal{V}) \le (w_{\max} - a)|\mathcal{U} \cup \mathcal{V}|$ for any finite $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^d$. Therefore setting $a(k) := \max(w_{\max} - 1/k, 0)$ for each $k \in \mathbb{N}$, using the fact that (4.23) and (4.24) hold for $a = a(k)$, taking $k \to \infty$ and arguing similarly to the proof of Lemma 4.1(b), we can replace w_a by w in (4.23) and (4.24), which is the desired conclusion. □

4.4 Weighted minimum spanning tree (MST)

Given finite $\mathcal{X} \subset \mathbb{R}^d$, the weighted minimum spanning tree (MST) problem is to find a spanning tree on vertex set $\mathcal{X}$ of minimum weight. More formally, for each spanning tree τ in the complete graph on vertex set $\mathcal{X}$, define $w(\tau) = \sum_{e \in E(\tau)} w(e)$, where $E(\tau)$ denotes the set of edges of τ. Define $MST_w(\mathcal{X}) := \min w(\tau)$, where the minimum is over all spanning trees in the complete graph on vertex set $\mathcal{X}$.

Theorem 4.11. *[LLN for weighted MST] Suppose $w(\cdot)$ satisfies conditions W1–W4, and either (a) $w_{\max} < \infty$ or (b) μ has bounded support, $w_{\max} = \infty$. Then when choosing $\zeta(\mathcal{X})$ as $MST_w(\mathcal{X})$, the limit $\rho(\lambda)$ given by (2.2) exists and is finite for all $\lambda \in (0, \infty)$, and if $nr_n^d \to t \in (0, \infty)$ as $n \to \infty$, then (2.3) holds.*

In the case where $w(\cdot) = 1 - \mathbf{1}_{B_1(o)}(\cdot)$, $MST_w(\mathcal{X})$ is the number of components of $G(\mathcal{X}, 1)$, minus 1. Thus Theorem 3.13 is a corollary of Theorem 4.11(a).

There is some overlap between Theorem 4.11 and [39, Theorem 2.3]. However, we here allow μ to have a singular part, and do not require $w(x)$ to be a monotonic increasing function of $\|x\|$, unlike in [39].

We work towards proving the preceding theorems. We shall first consider $w(\cdot)$ satisfying an extra condition, namely that *there exists $\delta > 0$ such that $w(x) = 0$ whenever $\|x\| \le \delta$*; we call this condition W7.

Given finite $\mathcal{X} \subset \mathbb{R}^d$ and given a spanning tree τ on vertex set $\mathcal{X}$, we shall say, as before, that an edge uv of τ is *short* if $\|u - v\| \le 1$, *long* if $\|u - v\| > 1$.

Lemma 4.12. *Suppose $w(\cdot)$ satisfies W1, W5 with $c_5 = 1$, and W7. There is a constant k with the following property. Given any finite $\mathcal{X} \subset \mathbb{R}^d$, there exists a spanning tree τ on $\mathcal{X}$ with $w(\tau) = MST_w(\mathcal{X})$, such that for each $x \in \mathcal{X}$ there are at most k short edges in τ incident to x.*

Proof. Let δ be as in W7. Partition $\mathbb{R}^d$ into half-open cubes of side δ/d. Label these cubes as $C_1, C_2, \ldots$.

Let $\mathcal{I} := \{i : |\mathcal{X} \cap C_i| \geq 2\}$. For each $i \in \mathcal{I}$, choose a path π_i through $\mathcal{X} \cap C_i$. The edges of this path all have weight zero. Therefore we can then create a minimum-weight spanning tree τ on $\mathcal{X}$ that includes the edges of all of the paths $\pi_i, i \in \mathcal{I}$. For any two distinct cubes C_i, C_j, τ has at most one edge from $\mathcal{X} \cap C_i$ to $\mathcal{X} \cap C_j$ (since it is a tree).

Let k' be the number of cubes in the partition lying within unit Euclidean distance of C_1. This is also the number of cubes in the partition lying within unit distance of C_i, for any $i \in \mathbb{N}$. For any $x \in \mathcal{X}$ the number of short edges in τ from x is bounded by $k := k' + 2$, as required. □

Lemma 4.13. *Suppose $w(\cdot)$ satisfies W1, W5 and W7. Then when choosing $\zeta(\mathcal{X})$ as $MST_w(\mathcal{X})$, the limit $\rho(\lambda)$ given by (2.2) exists and is finite for all $\lambda \in (0, \infty)$, and if $nr_n^d \to t \in (0, \infty)$ as $n \to \infty$, then (2.3) holds.*

Proof. We assume first that $c_5 = 1$ in condition W5. Clearly $\zeta(\cdot) := MST_w(\cdot)$ satisfies Properties P1 and P2. To check P3, let $\mathcal{Y}, \mathcal{Z}$ be disjoint finite subsets of $\mathbb{R}^d$. Let τ be a spanning tree on $\mathcal{Y}$ with weight $MST_w(\mathcal{Y})$, and let τ' be a spanning tree on $\mathcal{Z}$ with weight $MST_w(\mathcal{Z})$. If we combine the edges of τ and of τ', along with a single edge from a vertex of $\mathcal{Y}$ to a vertex of $\mathcal{Z}$, we obtain a spanning tree on $\mathcal{Y} \cup \mathcal{Z}$ with total weight at most $MST_w(\mathcal{Y}) + MST_w(\mathcal{Z}) + w_{\max}$. Hence, $MST_w(\mathcal{Y} \cup \mathcal{Z}) \leq MST_w(\mathcal{Y}) + MST_w(\mathcal{Z}) + w_{\max}$, so P3 holds with $c_1 = w_{\max}$.

To check Property P4, let τ be a spanning tree on $\mathcal{Y} \cup \mathcal{Z}$ with $w(\tau) = MST_w(\mathcal{Y} \cup \mathcal{Z})$, having the property described in Lemma 4.12. Remove from τ all edges going from $\mathcal{Y}$ to $\mathcal{Z}$. This leaves us with a forest, each component of which has either all vertices in $\mathcal{Y}$ or all vertices in $\mathcal{Z}$. Then add edges to this forest, connecting up the components in $\mathcal{Y}$ to make a spanning tree τ' in $\mathcal{Y}$ and connecting up the components in $\mathcal{Z}$ to make a spanning tree τ'' in $\mathcal{Z}$. The total number of added edges equals the total number of previously removed edges, minus 1.

Then $w(\tau') + w(\tau'') - w(\tau)$ equals the total weight of added edges, minus the total weight of removed edges. This is at most $w_{\max}$ times the number of added edges, minus the total weight of removed edges. Therefore it is at most $w_{\max}$ times the total number of short removed edges, since the long removed edges have weight $w_{\max}$, and there are more removed edges than added edges. Since τ has the property described in Lemma 4.12, for each $y \in \mathcal{Y}$ the number of short removed edges incident to y is at most k, and is zero if $y \notin \partial_{\mathcal{Z}} \mathcal{Y}$, so that

$$\begin{aligned} MST_w(\mathcal{Y}) + MST_w(\mathcal{Z}) - MST_w(\mathcal{Y} \cup \mathcal{Z}) &\leq w(\tau') + w(\tau'') - w(\tau) \\ &\leq k w_{\max} |\partial_{\mathcal{Z}} \mathcal{Y}|, \end{aligned}$$

which gives us Property P4 with $c_2 = kw_{\max}$. Property P5 follows because $MST_w(\cdot) \leq TSP_w(\cdot)$ (since removing one edge from a tour yields a spanning tree). $\square$

As usual, for $a \in (0, w_{\max})$ we set $w_a(\cdot) := \min(w(\cdot), a)$.

Proof of Theorem 4.11(a). Assume $w(\cdot)$ satisfies W1–W3 with $w_{\max} < \infty$ (and hence also W4). For $k \in \mathbb{N}$, set $a(k) := \max(w_{\max} - 1/k, 0)$ and set

$$w^{(k)}(x) := w_{a(k)}(x)(1 - \mathbf{1}_{B_{1/k}(o)}(x)), \qquad x \in \mathbb{R}^d.$$

Then $w^{(k)}(\cdot)$ satisfies W1, W5 and W7, so by Lemma 4.13, for all $\lambda > 0$ the limit $\rho_k(\lambda) := \lim_{s\to\infty} \mathbb{E}\,[MST_{w^{(k)}}(\mathcal{H}_{\lambda,s})/(\lambda s^d)]$ exists and if $nr_n^d \to t \in (0,\infty)$ as $n \to \infty$, then $n^{-1}MST_{w^{(k)}}(r_n^{-1}\mathcal{X}_n) \xrightarrow[L^2]{c.c.} \int \rho_k(tf_\mu(x))f_\mu(x)dx$.

Now, for all finite $\mathcal{X} \subset \mathbb{R}^d$, set $\zeta^{(k)}(\mathcal{X}) := MST_{w^{(k)}}(\mathcal{X})$ and $\zeta(\mathcal{X}) := MST_w(\mathcal{X})$. Then $\zeta^{(k)}(\mathcal{X})$ is nondecreasing in k with $\lim_{k\to\infty} \zeta^{(k)}(\mathcal{X}) = \zeta(\mathcal{X})$. Moreover, for all $x \in \mathbb{R}^d$, we have

$$0 \leq w(x) - w^{(k)}(x) \leq (w_{\max} - a(k)) + \sup_{x\in B_{1/k}(o)} w(x) =: h(k),$$

and $h(k) \to 0$ as $k \to \infty$ by assumption W2. Then $0 \leq \zeta(\mathcal{X}) - \zeta^{(k)}(\mathcal{X}) \leq h(k)|\mathcal{X}|$, and so we can apply Lemma 4.1(b) with this choice of $\zeta^{(k)}$, to deduce that for all $\lambda > 0$ the limit $\rho(\lambda) := \lim_{s\to\infty} \mathbb{E}\,[MST_w(\mathcal{H}_{\lambda,s})/(\lambda s^d)]$ exists and if $nr_n^d \to t \in (0,\infty)$ as $n \to \infty$, then $n^{-1}MST_w(r_n^{-1}\mathcal{X}_n) \xrightarrow[L^2]{c.c.} \int \rho(tf_\mu(x))f_\mu(x)dx$, as required. $\square$

Now we aim to prove Theorem 4.11(b), so we assume $w(\cdot)$ satisfies W1–W4 with $w_{\max} = +\infty$. Initially we shall also assume W6. Again set $w_a(\cdot) := \min(w(\cdot), a)$.

Lemma 4.14. *Suppose $w(\cdot)$ satisfies W1–W4 and W6. There is a constant $C > 0$ such that for all $a \in (0,\infty)$, all $s \geq 1$ and all finite $\mathcal{X} \subset Q_s$ we have*

$$MST_w(\mathcal{X}) \leq Cs^p|\mathcal{X}|^{(d-p)/d} \tag{4.25}$$

and

$$0 \leq MST_w(\mathcal{X}) - MST_{w_a}(\mathcal{X}) \leq Cs^p a^{(p/d)-1}(MST_{w_a}(\mathcal{X}))^{(d-p)/d}. \tag{4.26}$$

Proof. Let c_6 be as in W6. By [46, (3.6) or (3.7)], there exists a finite constant $C > 0$ such that for any finite $\mathcal{Y} \subset Q_1$, we have $MST_{c_6\|\cdot\|^p}(\mathcal{Y}) \leq C|\mathcal{Y}|^{(d-p)/d}$. Hence by scaling, $MST_w(s\mathcal{Y}) \leq MST_{c_6\|\cdot\|^p}(s\mathcal{Y}) \leq Cs^p|\mathcal{Y}|^{(d-p)/d}$, which implies (4.25).

Now let $s \geq 1$ and let $\mathcal{X} \subset Q_s$ be finite. Let $\tau_a(\mathcal{X})$ be (the set of edges in) a minimum-weight spanning tree on $\mathcal{X}$, using weight function w_a (and using some deterministic rule to choose if there exist several spanning trees of minimum weight).

That is, let $\tau_a(\mathcal{X})$ be a spanning tree of $\mathcal{X}$ with $w_a(\tau_a(\mathcal{X})) = MST_{w_a}(\mathcal{X})$. Let $\tau_{(=a)}(\mathcal{X}) = \{e \in \tau_a(\mathcal{X}) : w_a(e) = a\}$, and let $N_a(\mathcal{X}) := |\tau_{(=a)}(\mathcal{X})|$. Thus $N_a(\mathcal{X})$ is the total number of edges of weight a in the minimum w_a-weight spanning tree of $\mathcal{X}$. Let $\mathcal{Y}_a(\mathcal{X})$ be a set consisting of one vertex from each component of the graph $(\mathcal{X}, \tau_a(\mathcal{X}) \setminus \tau_{(=a)}(\mathcal{X}))$, chosen in an arbitrary measurable way. Note that $|\mathcal{Y}_a(\mathcal{X})| = |N_a(\mathcal{X})| + 1$.

Let τ' be (the edge-set of) a spanning tree on $\mathcal{Y}_a(\mathcal{X})$ with $w(\tau') = MST_w(\mathcal{Y}_a(\mathcal{X}))$. Then $\tau' \cup (\tau_a(\mathcal{X}) \setminus \tau_{(=a)}(\mathcal{X}))$ is (the edge-set of) a spanning tree on $\mathcal{X}$ with w-weight at most $w_a(\tau_a(\mathcal{X})) + w(\tau')$. Therefore $MST_w(\mathcal{X}) \leq MST_{w_a}(\mathcal{X}) + MST_w(\mathcal{Y}_a(\mathcal{X}))$, and hence by (4.25) applied to $\mathcal{Y}_a(\mathcal{X})$, if $N_a(\mathcal{X}) \geq 1$ then

$$MST_w(\mathcal{X}) - MST_{w_a}(\mathcal{X}) \leq Cs^p(N_a(\mathcal{X}) + 1)^{(d-p)/d} \leq Cs^p(2N_a(\mathcal{X}))^{(d-p)/d}.$$

Since $aN_a(\mathcal{X}) \leq MST_{w_a}(\mathcal{X})$, this gives us (4.26) (upon changing the constant C). If $N_a(\mathcal{X}) = 0$ then $MST_w(\mathcal{X}) = MST_{w_a}(\mathcal{X})$ so (4.26) holds then too. $\square$

Proof of Theorem 4.11(b). Assume that $w(\cdot)$ satisfies W1–W4 with $w_{\max} = +\infty$, and that μ has bounded support. Given $\delta > 0$, $m \in \mathbb{N}$, define the new weight functions $w'_\delta(x) := w(x)(1 - \mathbf{1}_{B_\delta(o)}(x))$ and $w'_{\delta,m}(x) := \min(w_\delta(x), m)$.

Then $w'_{\delta,m}$ satisfies W5 and W7. Hence by Lemma 4.13, for all $\lambda > 0$ the limit $\rho'_{\delta,m}(\lambda) := \lim_{s\to\infty} \mathbb{E}\,[MST_{w'_{\delta,m}}(\mathcal{H}_{\lambda,s})/(\lambda s^d)]$ exists, and if $nr_n^d \to t \in (0,\infty)$ as $n \to \infty$, then $n^{-1}MST_{w'_{\delta,m}}(r_n^{-1}\mathcal{X}_n) \xrightarrow[L^2]{c.c.} \int \rho'_{\delta,m}(tf_\mu(x))f_\mu(x)dx$.

Since $w(\cdot)$ satisfies W4, the function $w'_\delta(\cdot)$ satisfies W6. Therefore by Lemma 4.14, there is a constant $C(\delta)$ such that for all $s > 0$ and all finite $\mathcal{X} \in Q_s$, we have $MST_{w'_{\delta,m}}(\mathcal{X}) \leq MST_{w'_\delta}(\mathcal{X}) \leq C(\delta)s^p|\mathcal{X}|^{(d-p)/d}$ and

$$MST_{w'_\delta}(\mathcal{X}) - MST_{w'_{\delta,m}}(\mathcal{X}) \leq C(\delta)s^p m^{(p/d)-1}(MST_{w'_{\delta,m}}(\mathcal{X}))^{(d-p)/d}.$$

Also, setting $\tilde{h}(\ell) := \sup_{x\in B_{1/\ell}(o)} w(x)$, we have $MST_w(\mathcal{X}) - MST_{w'_{1/\ell}}(\mathcal{X}) \leq \tilde{h}(\ell)|\mathcal{X}|$. Thus the functionals $\tilde{\zeta}^{(\ell,m)}(\cdot) := MST_{w'_{1/\ell,m}}(\cdot)$ satisfy the conditions for Lemma 4.2 with $h_\ell(m) := C(1/\ell)m^{(p/d)-1}$. Hence by Lemma 4.2, for all $\lambda > 0$ the limit $\rho(\lambda) := \lim_{s\to\infty} \mathbb{E}\,[MST_w(\mathcal{H}_{\lambda,s})/(\lambda s^d)]$ exists and is finite, and also if $nr_n^d \to t \in (0,\infty)$ as $n \to \infty$, then $MST_w(r_n^{-1}\mathcal{X}_n) \xrightarrow[L^2]{c.c.} \int \rho(tf_\mu(x))f_\mu(x)dx$, as required. $\square$

5 Proof of the general results

In this section we prove the results stated in Section 2. Assume throughout this section that the function $\zeta(\cdot)$ has been specified, taking values in $[0,\infty)$ with $\zeta(\varnothing) = 0$, satisfying Properties P1–P4.

The following elementary lemma concerns deterministic subadditive functionals of Borel sets. As before, let $\mathcal{B}$ denote the class of bounded Borel sets in $\mathbb{R}^d$, and let $\mathcal{B}_1$ denote the class of all Borel sets that are contained in Q_1. Let us say that a set function $f : \mathcal{B} \to \mathbb{R}$ is *Borel subadditive* if $f(A \cup A') \leq f(A) + f(A')$ whenever $A, A' \in \mathcal{B}$ with $A \cap A' = \varnothing$.

Lemma 5.1. *Suppose $f : \mathcal{B} \to [0, \infty)$ is a Borel subadditive set function such that $f(x+A) = f(A)$ for all $A \in \mathcal{B}$, $x \in \mathbb{R}^d$, and also such that $\sup\{f(A) : A \in \mathcal{B}_1\} < \infty$. Then $\overline{f} := \inf_{s \geq 1}(s^{-d} f(Q_s)) \in [0, \infty)$ and $s^{-d} f(Q_s) \to \overline{f}$ as $s \to \infty$.*

More sophisticated stochastic subadditive limit theorems are available (see [2], or [46, Theorem 4.9]) but are not needed here. Since we use Lemma 5.1 repeatedly, for completeness we include a proof.

Proof of Lemma 5.1. Set $f^* := \sup\{f(A) : A \in \mathcal{B}_1\}$. Clearly $0 \leq \overline{f} \leq f^* < \infty$. Given $\varepsilon > 0$, choose $s_0 \geq 1$ such that $f(Q_{s_0}) < (\overline{f} + \varepsilon) s_0^d$.

Given $s > 0$, write $s = ns_0 + t$ with $t \in [0, s_0)$. Then there is a constant c such that for all such s we can write Q_s as a disjoint union of n^d translates of Q_{s_0}, together with at most cn^{d-1} further sets in $\mathcal{B}$ that are translates of sets in $\mathcal{B}_1$ (in fact, rectangles with all sides at most 1). By repeated use of Borel subadditivity,

$$f(Q_s) \leq n^d f(Q_{s_0}) + cn^{d-1} f^* \leq (ns_0)^d (\overline{f} + \varepsilon) + cn^{d-1} f^*.$$

Therefore since $s \geq ns_0$, for large enough s (and hence, large enough n) we have

$$s^{-d} f(Q_s) \leq \overline{f} + \varepsilon + cf^* s_0^{-d} n^{-1} < \overline{f} + 2\varepsilon,$$

which implies the result. □

Next we show that ζ_r enjoys a *smoothness* property; the effect on ζ_r of adding a single point to an existing point set is uniformly bounded.

Lemma 5.2. *Let $K := \max(c_1 + \zeta(\{o\}), c_2 - \zeta(\{o\}))$, where c_1, c_2 are as in P3, P4 respectively. Then $|\zeta_r(\mathcal{Y}) - \zeta_r(\mathcal{X})| \leq K |\mathcal{Y} \triangle \mathcal{X}|$ for all finite $\mathcal{X}, \mathcal{Y} \subset \mathbb{R}^d$, and all $r > 0$.*

Proof. It suffices to prove the result when $|\mathcal{Y} \triangle \mathcal{X}| = 1$ and $r = 1$. Let $\mathcal{X} \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d \setminus \mathcal{X}$. By Property P2 (translation invariance), $\zeta(\{x\}) = \zeta(\{o\})$, and by P3 (almost subadditivity)

$$\zeta(\mathcal{X} \cup \{x\}) \leq \zeta(\mathcal{X}) + \zeta(\{o\}) + c_1.$$

Similarly, since $|\partial_{\mathcal{X}}(\{x\})| \leq 1$, by P4 (superadditivity up to boundary)

$$\zeta(\mathcal{X} \cup \{x\}) \geq \zeta(\mathcal{X}) + \zeta(\{o\}) - c_2.$$

Combining these inequalities shows that $|\zeta(\mathcal{X} \cup \{x\}) - \zeta(\mathcal{X})| \leq K$, if we take $K = \max(c_1 + \zeta(\{o\}), c_2 - \zeta(\{o\}))$. This gives the result. □

Given $A \subset \mathbb{R}^d$ and $r > 0$, let $\partial_r A$ denote the set of all points in A at a Euclidean distance at most r from $\mathbb{R}^d \setminus A$. The following result is a consequence of almost subadditivity (Property P3) and superadditivity up to boundary (Property P4).

Lemma 5.3. *Let $k \in \mathbb{N}$ and let $A_1, \ldots, A_k$ be disjoint subsets of $\mathbb{R}^d$. For any $r > 0$ and any finite $\mathcal{X} \subset \cup_{i=1}^k A_i$ we have*

$$\zeta_r(\mathcal{X}) \leq \left(\sum_{i=1}^{k} \zeta_r(\mathcal{X} \cap A_i) \right) + c_1(k-1), \tag{5.1}$$

and

$$\zeta_r(\mathcal{X}) \geq \left(\sum_{i=1}^{k} \zeta_r(\mathcal{X} \cap A_i) \right) - c_2 \sum_{i=2}^{k} |\mathcal{X} \cap \partial_r A_i|, \tag{5.2}$$

where c_1, c_2 are the constants in Properties P3, P4 respectively, and the second sum in (5.2) is interpreted as zero if $k = 1$.

Proof. Since we assume Property P3 (almost subadditivity of ζ), an analogous property also holds for ζ_r. We then obtain (5.1) by a straightforward induction in k.

To prove (5.2), first suppose $k = 2$. For $i = 1, 2$ set $\mathcal{Y}_i := r^{-1}(\mathcal{X} \cap A_i)$. Then by P4,

$$\zeta_r(\mathcal{X}) = \zeta(\mathcal{Y}_1 \cup \mathcal{Y}_2) \geq \zeta(\mathcal{Y}_1) + \zeta(\mathcal{Y}_2) - c_2|\partial_{\mathcal{Y}_1}\mathcal{Y}_2|.$$

Also, since A_1 and A_2 are disjoint,

$$|\partial_{\mathcal{Y}_1}\mathcal{Y}_2| \leq |r^{-1}\mathcal{X} \cap \partial_1(r^{-1}A_2)| = |\mathcal{X} \cap \partial_r A_2|.$$

Together these two inequalities give us the result for $k = 2$, and the general result follows by a straightforward induction on k. $\square$

Proof of Proposition 2.3. (i) By Lemma 5.2, for all finite $\mathcal{X} \subset \mathbb{R}^d$ we have that $\zeta(\mathcal{X}) \leq K|\mathcal{X}|$. Hence for each $A \in \mathcal{B}$, we have

$$\mathbb{E}\left[\zeta(\mathcal{H}_\lambda \cap A)\right] \leq K\mathbb{E}\left[|\mathcal{H}_\lambda \cap A|\right] = K\lambda \mathrm{Leb}(A). \tag{5.3}$$

In particular, this expectation is finite. Given $\lambda \geq \lambda_1 > 0$ and $A \in \mathcal{B}$, let

$$f_\lambda(A) := \mathbb{E}\left[\zeta(\mathcal{H}_\lambda \cap A)\right] + c_1,$$

where $c_1 \in \mathbb{R}$ is the constant given in Property P3. By Property P3, for any disjoint $A, A' \in \mathcal{B}$ we have $f_\lambda(A \cup A') \leq f_\lambda(A) + f_\lambda(A')$. Also for all $A \in \mathcal{B}$ and all $x \in \mathbb{R}^d$ we have $f_\lambda(x+A) = f_\lambda(A)$. Moreover, by (5.3) f_λ has the other properties described in the hypothesis of Lemma 5.1. Hence by Lemma 5.1, the limit in (2.2) does exist in $\mathbb{R}$, and

$$\lambda\rho(\lambda) = \lim_{s\to\infty} s^{-d} f_\lambda(Q_s) = \inf_{s\geq 1} s^{-d} f_\lambda(Q_s).$$

Thus for all $s \geq 1$ we have that $s^{-d}(\mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s})] + c_1) \geq \lambda\rho(\lambda)$, or in other words,

$$\begin{aligned}\mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s})/(\lambda s^d)] &\geq \rho(\lambda) - c_1/(\lambda s^d). \\ &\geq \rho(\lambda) - c_1/(\lambda_1 s^d).\end{aligned} \tag{5.4}$$

Next, fix $s > 2$ and let $m \in \mathbb{N}$. Then we can partition Q_{ms} into m^d translates of Q_s, which we denote $Q_{m,1}, \ldots, Q_{m,m^d}$. By (5.2) from Lemma 5.3,

$$\zeta(\mathcal{H}_{\lambda,ms}) \geq \left(\sum_{i=1}^{m^d} \zeta(\mathcal{H}_\lambda \cap Q_{m,i})\right) - c_2 \sum_{i=2}^{m^d} |\mathcal{H}_\lambda \cap \partial_1 Q_{m,i}|.$$

Now, there is a constant c' depending only on d such that $\mathrm{Leb}(\partial_1 Q_s) \leq c' s^{d-1}$. Therefore taking expectations in the above yields

$$\mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,ms})] \geq m^d \mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s})] - c_2 m^d c' \lambda s^{d-1},$$

and hence

$$\mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s})/s^d] \leq \mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,ms})/(ms)^d] + c_2 c' \lambda s^{-1}.$$

Taking the large-m limit and using (2.2) yields that

$$\mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s})/(\lambda s^d)] \leq \rho(\lambda) + c_2 c' s^{-1},$$

and combined with (5.4) this yields (2.9).

(ii) Suppose $0 < \lambda < \lambda'$. By the Superposition theorem (see e.g. [32]) we can couple $\mathcal{H}_\lambda$ and $\mathcal{H}_{\lambda'}$ in such a way that $\mathcal{H}_\lambda \subset \mathcal{H}_{\lambda'}$ and $|(\mathcal{H}_{\lambda'} \setminus \mathcal{H}_\lambda) \cap Q_s|$ is Poisson with parameter $(\lambda' - \lambda)s^d$, for all $s \in (0,\infty)$. With this coupling, by smoothness (Lemma 5.2), with K as in that result we have for any $s > 0$ that

$$\mathbb{E}\,[|\zeta(\mathcal{H}_{\lambda',s}) - \zeta(\mathcal{H}_{\lambda,s})|] \leq K(\lambda' - \lambda)s^d, \tag{5.5}$$

and therefore by (2.2), we obtain that $|\lambda'\rho(\lambda') - \lambda\rho(\lambda)| \leq K(\lambda' - \lambda)$, so $\lambda\rho(\lambda)$ is Lipschitz continuous in λ with Lipschitz constant at most K. Hence $\rho(\lambda)$ is also continuous in λ.

(iii) By the proof of Lemma 5.2, for all finite $\mathcal{X}$ we have $\zeta(\mathcal{X}) \leq (c_1 + \zeta(\{o\}))|\mathcal{X}|$. Hence for all $s > 0$ we have

$$\mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s})] \leq (c_1 + \zeta(\{o\}))\mathbb{E}\,[|\mathcal{H}_{\lambda,s}|] = (c_1 + \zeta(\{o\}))\lambda s^d.$$

Hence by (2.2), $\rho(\lambda) \le c_1 + \zeta(\{o\})$.

(iv) Suppose $0 < \lambda < \lambda'$, and couple $\mathcal{H}_\lambda$ and $\mathcal{H}_{\lambda'}$ as in the proof of (ii). Under assumption P6, for all $s > 0$ we then have $\zeta(\mathcal{H}_{\lambda,s}) \le \zeta(\mathcal{H}_{\lambda',s})$, so using (2.2) we obtain that $\lambda\rho(\lambda) \le \lambda'\rho(\lambda')$, as required.

(v) Given $\lambda > 0$, by (2.2) we have

$$\rho(\lambda) = \lim_{s\to\infty} (\lambda^{1/d}s)^{-d}\mathbb{E}\,[\zeta_{\lambda^{1/d}}(\lambda^{1/d}\mathcal{H}_\lambda \cap Q_{\lambda^{1/d}s})],$$

and hence by the Mapping theorem for Poisson point processes (see e.g. [32]),

$$\rho(\lambda) = \lim_{s\to\infty} (\lambda^{1/d}s)^{-d}\mathbb{E}\,[\zeta_{\lambda^{1/d}}(\mathcal{H}_1 \cap Q_{\lambda^{1/d}s})] = \lim_{t\to\infty} t^{-d}\mathbb{E}\,[\zeta_{\lambda^{1/d}}(\mathcal{H}_1 \cap Q_t)].$$

Hence by Property P7 (downward monotonicity in r), $\rho(\lambda)$ is nonincreasing in λ.

(vi) Let $\mathcal{H}^0_\lambda$ be the set of points of $\mathcal{H}_\lambda$ that are isolated vertices in the graph $G(\mathcal{H}_\lambda, 1)$. Using smoothness (Lemma 5.2), and then P4 repeatedly, we have

$$\begin{aligned}\zeta(\mathcal{H}_\lambda \cap Q_s) &\ge \zeta(\mathcal{H}^0_\lambda \cap Q_s) - K|(\mathcal{H}_\lambda \setminus \mathcal{H}^0_\lambda) \cap Q_s| \\ &\ge \zeta(\{o\})|\mathcal{H}^0_\lambda \cap Q_s| - K|(\mathcal{H}_\lambda \setminus \mathcal{H}^0_\lambda) \cap Q_s|.\end{aligned}$$

Hence by the Palm-Mecke formula from the theory of Poisson point processes (see [35] or [32]),

$$\mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s})] \ge \lambda s^d \zeta(\{o\})e^{-\lambda\pi_d} - K\lambda s^d(1 - e^{-\lambda\pi_d}).$$

Hence by (2.2), $\rho(\lambda) \ge \zeta(\{o\})e^{-\lambda\pi_d} - K(1 - e^{-\lambda\pi_d})$, and hence $\liminf_{\lambda\downarrow 0}\rho(\lambda) \ge \zeta(\{o\})$. Combined with part (iii) this gives us the result.

(vii) Fix $0 < \lambda < \lambda' < \zeta(\{o\})/(c_2\pi_d)$. Our goal is to show that $\lambda'\rho(\lambda') > \lambda\rho(\lambda)$. Fix $s \in (0, \infty)$. By the Superposition theorem [32], the Poisson point process $\mathcal{H}_{\lambda'}$ may be obtained as the union of the Poisson point process $\mathcal{H}_\lambda$ and another fresh Poisson point process of intensity $\lambda' - \lambda$ added on top of it. Let N denote the number of points in Q_s of the second Poisson point process to be added, and let N_0 be the number of these points that are isolated in $G(\mathcal{H}_{\lambda'} \cap Q_s, 1)$, i.e. have no other point of $\mathcal{H}_{\lambda',s}$ within unit Euclidean distance.

For each added point, if that point is isolated then by P4 it increases the value of ζ by at least $\zeta(\{o\})$, and if it is not isolated then by P4 it decreases the value of ζ by at most $c_2 - \zeta(\{o\})$. Therefore adding the points one by one, we obtain that

$$\zeta(\mathcal{H}_{\lambda',s}) - \zeta(\mathcal{H}_{\lambda,s}) \ge \zeta(\{o\})N - c_2(N - N_0). \tag{5.6}$$

Now, $\mathbb{E}\,[N] = (\lambda' - \lambda)s^d$, and by the Palm-Mecke formula (see [35] or [32]),

$$\mathbb{E}\,[N - N_0] \le (\lambda' - \lambda)s^d(1 - \exp(-\lambda'\pi_d)) \le (\lambda' - \lambda)s^d\lambda'\pi_d.$$

Therefore taking expectations in (5.6) yields

$$s^{-d}(\mathbb{E}\,[\zeta(\mathcal{H}_{\lambda',s})] - \mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s})]) \geq (\lambda' - \lambda)(\zeta(\{o\}) - c_2\pi_d\lambda').$$

By the choice of λ and λ', the right hand side above is strictly positive. Taking the large-s limit and using (2.2) shows that $\lambda'\rho(\lambda') > \lambda\rho(\lambda)$, as required. □

Remark 5.4. We can improve Proposition 2.3(vii) as follows in cases where P6 (namely $\zeta(\mathcal{X} \cup \mathcal{Y}) \geq \zeta(\mathcal{X})$) fails in general (so Proposition 2.3(iv) does not apply), but does hold whenever $|\mathcal{Y}| = 1$ and $|\partial_{\mathcal{Y}}(\mathcal{X})| = 1$. These include domination number and number of connected components (see Section 3.2 and Section 3.6). We deal in this remark with these two: using the same notation of N and N_0 as before, define by N_1 the number of vertices of degree exactly one in $G(\mathcal{H}_{\lambda'} \cap Q_s, 1)$ stemming from the second Poisson point process of intensity $\lambda' - \lambda$ added. Now note, that for each added point, if that point is isolated then by P4 it increases the value of ζ by at least $\zeta(\{o\})$ (as before), if that point is of degree 1, it cannot decrease the value of ζ, and if it is of degree at least two, then by P4 it decreases the value of ζ by at most $c_2 - \zeta(\{o\})$. Hence, adding the points one by one, we obtain that for these examples

$$\begin{aligned} \zeta(\mathcal{H}_{\lambda',s}) - \zeta(\mathcal{H}_{\lambda,s}) &\geq \zeta(\{o\})N_0 - (c_2 - \zeta(\{o\}))(N - N_0 - N_1) \\ &= \zeta(\{o\})(N - N_1) - c_2(N - N_0 - N_1). \end{aligned} \tag{5.7}$$

By the Palm-Mecke formula,

$$\mathbb{E}\,[N - N_1] \geq (\lambda' - \lambda)(s-2)^d(1 - \lambda'\pi_d \exp(-\lambda'\pi_d)),$$

and

$$\mathbb{E}\,[N - N_0 - N_1] \leq (\lambda' - \lambda)s^d(1 - (1 + \lambda'\pi_d)\exp(-\lambda'\pi_d)).$$

By Taylor's theorem applied to e^x, if $x \geq 0$ then $1 - (1+x)e^{-x} \leq x^2/2$. Therefore $\mathbb{E}\,[N - N_0 - N_1] \leq (\lambda' - \lambda)s^d(\lambda'\pi_d)^2/2$.

Taking expectations in (5.7) and using that $xe^{-x} \leq x - x^2 + x^3/2$ for $x \geq 0$ (again by Taylor's theorem) yields, in cases with $\zeta(\{o\}) = 1$, that

$$\begin{aligned} \lim_{s\to\infty}\left(\frac{\mathbb{E}\,[\zeta(\mathcal{H}_{\lambda',s})] - \mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s})]}{s^d(\lambda' - \lambda)}\right) &\geq 1 - \lambda'\pi_d\exp(-\lambda'\pi_d) - (c_2/2)(\lambda'\pi_d)^2 \\ &\geq 1 - \lambda'\pi_d + (\lambda'\pi_d)^2 - (\lambda'\pi_d)^3/2 - (c_2/2)(\lambda'\pi_d)^2. \end{aligned} \tag{5.8}$$

Let λ^* be the smallest positive root of the equation

$$1 - \lambda^*\pi_d + (1 - c_2/2)(\lambda^*\pi_d)^2 - (\lambda^*\pi_d)^3/2.$$

If $0 < \lambda' < \lambda^*$, the right hand side of (5.8) is strictly positive. Hence by (2.2), for $0 < \lambda < \lambda' < \lambda^*$, we have $\lambda'\rho(\lambda') > \lambda\rho(\lambda)$, i.e. $\lambda\rho(\lambda)$ is increasing on $\lambda \in (0, \lambda^*)$.

This improves on Proposition 2.3(vii). For example, in the case of the domination number for $d = 2$, we have $\zeta(\{o\}) = 1$, and since $\kappa(B_2(o)) = 7$, by Lemma 3.6, $c_2 = 8$, so Proposition 2.3(vii) shows that $\lambda \mapsto \lambda\rho(\lambda)$ is increasing on $0 < \lambda < 1/(8\pi) \approx 0.0398$, whereas this remark improves this range to $0 < \lambda < \lambda^* \approx 0.1348$.

In the case of the number of connected components for $d = 2$, $\zeta(\{o\}) = 1$ as before, and we can take $c_2 = 5$ since adding one vertex reduces the number of connected components of a geometric graph by at most 5, by elementary geometric arguments. Hence by the present remark, $\lambda \mapsto \lambda\rho(\lambda)$ is increasing on $0 < \lambda < \lambda^* \approx 0.1660$, whereas Proposition 2.3(vii) gives a range only of $0 < \lambda < 1/(5\pi) \approx 0.0637$.

Proof of Proposition 2.4. Given a locally finite set $\mathcal{X} \subset \mathbb{R}^d$, define the *clusters* of $\mathcal{X}$ to be the vertex sets of the connected components of $G(\mathcal{X}, 1)$. For $x \in \mathcal{X}$ we define $\mathcal{C}(x, \mathcal{X})$ to be the cluster of $\mathcal{X}$ containing x.

By the assumptions P3 (with $c_1 = 0$) and P4, if $\mathcal{X}$ and $\mathcal{Y}$ are disjoint finite subsets of $\mathbb{R}^d$ such that $\partial_{\mathcal{Z}}\mathcal{Y} = \varnothing$, then $\zeta(\mathcal{Y} \cup \mathcal{Z}) = \zeta(\mathcal{Y}) + \zeta(\mathcal{Z})$. Thus for any finite $\mathcal{X} \subset \mathbb{R}^d$, denoting the clusters of $\mathcal{X}$ by $\mathcal{C}_1, \ldots, \mathcal{C}_m$ we have $\zeta(\mathcal{X}) = \sum_{i=1}^m \zeta(\mathcal{C}_i)$. Thus for $s > 0$,

$$\zeta(\mathcal{H}_{\lambda,s}) = \sum_{x \in \mathcal{H}_{\lambda,s}} \frac{\zeta(\mathcal{C}(x, \mathcal{H}_{\lambda,s}))}{|\mathcal{C}(x, \mathcal{H}_{\lambda,s})|}.$$

Hence by the Palm-Mecke formula

$$\mathbb{E}\left[\zeta(\mathcal{H}_{\lambda,s})\right] = \lambda \int_{Q_s} \mathbb{E}\left[\frac{\zeta(\mathcal{C}(x, \mathcal{H}_\lambda^x \cap Q_s))}{|\mathcal{C}(x, \mathcal{H}_\lambda^x \cap Q_s)|}\right] dx,$$

where we write $\mathcal{H}_\lambda^x$ for $\mathcal{H}_\lambda \cup \{x\}$. Taking $y = s^{-1}x$ we have

$$\lambda^{-1} s^{-d} \mathbb{E}\,\zeta(\mathcal{H}_{\lambda,s}) = \int_{Q_1} \mathbb{E}\left[\frac{\zeta(\mathcal{C}(sy, \mathcal{H}_\lambda^{sy} \cap Q_s))}{|\mathcal{C}(sy, \mathcal{H}_\lambda^{sy} \cap Q_s)|}\right] dy. \tag{5.9}$$

Let Q_1^o denote the interior of Q_1, and let $y \in Q_1^o$. By P2 (translation invariance) and the stationarity of $\mathcal{H}_\lambda$ we have

$$\mathbb{E}\left[\frac{\zeta(\mathcal{C}(sy, \mathcal{H}_\lambda^{sy} \cap Q_s))}{|\mathcal{C}(sy, \mathcal{H}_\lambda^{sy} \cap Q_s)|}\right] = \mathbb{E}\left[\frac{\zeta(\mathcal{C}(o, \mathcal{H}_\lambda^{o} \cap (-sy + Q_s)))}{|\mathcal{C}(o, \mathcal{H}_\lambda^{o} \cap (-sy + Q_s))|}\right]. \tag{5.10}$$

Since we assume $\lambda < \lambda_c$ the set $\mathcal{C}(o, \mathcal{H}_\lambda^o)$ is almost surely finite, and hence for large enough s this set is contained in the set $-sy + Q_s$ (which is equal to $s(-y + Q_1)$). Also, by smoothness (Lemma 5.2) there is a constant K such that $|\zeta(\mathcal{X})| \leq K|\mathcal{X}|$

for all finite $\mathcal{X} \subset \mathbb{R}^d$. Hence by the Dominated Convergence theorem, as $s \to \infty$ the expression in (5.10) converges to

$$\mathbb{E}\left[\frac{\zeta(\mathcal{C}(o,\mathcal{H}^o_\lambda))}{|\mathcal{C}(o,\mathcal{H}^o_\lambda)|}\right].$$

By Dominated Convergence again, the expression (5.9) converges to the same limit, and then the result follows from (2.2). $\square$

It remains, in this section, to prove Theorem 2.1. Hence, now and for the rest of this section we assume P5 as well as P1–P4. The existence in $\mathbb{R}$ of the limit $\rho(\lambda)$ given by (2.2), for all $\lambda > 0$, was already proved for Proposition 2.3(i).

Following [46], we say the distribution μ is *blocked* if for some $M \in \mathbb{N}$ and some $m \in \mathbb{N}$ a power of 2, the density function f_μ of the absolutely continuous part of μ is constant on each of the $(Mm)^d$ cubes in the subdivision of Q_M into half-open cubes of side $1/m$, which we call a *block-partition* of Q_M, and $f_\mu \equiv 0$ outside Q_M. Note that if μ is both blocked and absolutely continuous, then μ is supported by Q_M.

Lemma 5.5. *Suppose that μ is both blocked and absolutely continuous, and that $nr_n^d \to \lambda$ as $n \to \infty$, for some $\lambda \in (0,\infty)$. Let $t > 0$. Then*

$$n^{-1}\mathbb{E}\left[\zeta_{r_n}(\mathcal{P}_{nt})\right] \to \int_{\mathbb{R}^d} \rho(\lambda t f_\mu(x)) t f_\mu(x) dx \quad \text{as} \quad n \to \infty. \tag{5.11}$$

Proof. Let $\varepsilon > 0$. Let M and m be as in the description of the blocked distribution μ. Fix a cube C of side $1/m$ in the block-partition of Q_M, and let the constant value taken by f_μ on this cube be denoted b. Assume $b > 0$. Then

$$\mathbb{E}\left[\zeta_{r_n}(\mathcal{P}_{nt} \cap C)\right] = \mathbb{E}\left[\zeta(r_n^{-1}(\mathcal{P}_{nt} \cap C))\right] = \mathbb{E}\left[\zeta(\mathcal{H}_{nr_n^d bt} \cap r_n^{-1} C)\right],$$

because the restriction of $\mathcal{P}_{nt}$ to C is a homogeneous Poisson point process on C with intensity nbt, so by the Mapping theorem the restriction of $r_n^{-1}\mathcal{P}_{nt}$ to $r_n^{-1}C$ is a homogeneous Poisson point process on $r_n^{-1}C$ with intensity $nbtr_n^d$. Since $nr_n^d \to \lambda$, by Proposition 2.3(i) we have as $n \to \infty$ that

$$m^d r_n^d \mathbb{E}\left[\zeta_{r_n}(\mathcal{P}_{nt} \cap C)\right] \to \lambda bt\rho(\lambda bt),$$

so that

$$n^{-1}\mathbb{E}\left[\zeta_{r_n}(\mathcal{P}_{nt} \cap C)\right] \to bt\rho(\lambda bt)m^{-d} = \rho(\lambda bt)bt\mathrm{Leb}(C). \tag{5.12}$$

Let the cubes in the block-partition be denoted $C_1, \ldots, C_{(Mm)^d}$. Then for some $b_1, \ldots, b_{(Mm)^d} \in [0,\infty)$ we have $f_\mu = \sum_{i=1}^{(Mm)^d} b_i \mathbf{1}_{C_i}$. Assume the cubes are enumerated in such a way that for some $k \le (Mm)^d$ we have that $b_i > 0$ for $i = 1, \ldots, k$ and $b_i = 0$ for $i = k+1, \ldots, (Mm)^d$.

By (5.1) from Lemma 5.3, we have

$$\zeta_{r_n}(\mathcal{P}_{nt}) \leq \left(\sum_{i=1}^{k} \zeta_{r_n}(\mathcal{P}_{nt} \cap C_i) \right) + c_1 k,$$

where c_1 is the constant from Property P3. Therefore using (5.12), we obtain that

$$\limsup_{n\to\infty} \mathbb{E}\left[n^{-1}\zeta_{r_n}(\mathcal{P}_{nt})\right] \leq \sum_{i=1}^{k} \rho(\lambda b_i t) b_i t \mathrm{Leb}(C_i) \tag{5.13}$$

$$= \int_{\mathbb{R}^d} \rho(\lambda t f_\mu(x)) t f_\mu(x) dx. \tag{5.14}$$

For a lower bound, note that for each n, since $\mathcal{P}_{nt} \subset \cup_{i=1}^{k} C_i$, by (5.2) from Lemma 5.3 we have

$$\zeta_{r_n}(\mathcal{P}_{nt}) \geq \left(\sum_{i=1}^{k} \zeta_{r_n}(\mathcal{P}_{nt} \cap C_i) \right) - c_2 \sum_{i=2}^{k} |\mathcal{P}_{nt} \cap \partial_{r_n} C_i|.$$

Since μ is absolutely continuous and $r_n \to 0$, as $n \to \infty$ we have

$$n^{-1}\mathbb{E}\left[|\mathcal{P}_{nt} \cap \partial_{r_n} C_i|\right] = t\mu(\partial_{r_n} C_i) \to 0, \quad 1 \leq i \leq k.$$

Thus, using (5.12) we obtain that

$$\liminf_{n\to\infty} \mathbb{E}\left[n^{-1}\zeta_{r_n}(\mathcal{P}_{nt})\right] \geq \sum_{i=1}^{k} \rho(\lambda b_i t) b_i t \mathrm{Leb}(C_i).$$

Combined with (5.13) and (5.14) this gives us (5.11). □

Remark. The next lemma is the only place in the proof of Theorem 2.1 where we use Property P5. If we assume only P1–P4, but also assume μ is absolutely continuous, then we can use Lemma 5.5 instead of Lemma 5.6, and still obtain the conclusion of Theorem 2.1.

Lemma 5.6. *Suppose μ is blocked, and $nr_n^d \to \lambda$ as $n \to \infty$, $\lambda \in (0, \infty)$. Then*

$$\lim_{n\to\infty} \mathbb{E}\left[n^{-1}\zeta_{r_n}(\mathcal{P}_n)\right] = \int_{\mathbb{R}^d} \rho(\lambda f_\mu(x)) f_\mu(x) dx. \tag{5.15}$$

Proof. Let $\varepsilon > 0$. By P5, we can and do choose $\delta \in (0, 1)$ such that for any $\mathcal{X} \subset B_{\delta d}(o)$ with $\delta^{-1} \leq |\mathcal{X}| < \infty$ we have $\zeta(\mathcal{X}) \leq \varepsilon|\mathcal{X}|$.

Let μ' and (as before) $\mu^{\perp}$ denote the continuous and singular parts of μ, respectively (so μ' is a measure with density function f_μ). Assume that M (in the description of the blocked distribution μ) is chosen large enough so that $\mu^{\perp}(\mathbb{R}^d \setminus Q_M) < \varepsilon/2$.

As in the proof of [46, eqn. (7.3)], provided m (in the definition of a blocked distribution) is chosen large enough we can assume that the support of $\mu^{\perp}$ is contained in the union of two disjoint sets A and D, where $\mathrm{Leb}(A) = 0$ and $\mu(A) \leq \varepsilon$, while D is a union of cubes C_i in the block-partition of Q_M with total Lebesgue measure at most $\varepsilon\delta^{d+1}\lambda/2$. This is because the support of the restriction of $\mu^{\perp}$ to Q_M has zero Lebesgue outer measure, so we can find a countable collection of dyadic cubes contained in Q_M with total volume less than $\varepsilon\delta^{d+1}\lambda/2$ that contains this supporting set. Then we can take D to be the union of a sufficiently large finite subcollection of these dyadic cubes.

By the Superposition theorem for Poisson point processes (see for example [32]),

$$\mathbb{E}\left[\zeta_{r_n}(\mathcal{P}_n)\right] = \mathbb{E}\left[\zeta_{r_n}(\mathcal{P}_n^{(1)} \cup \mathcal{P}_n^{(2)} \cup \mathcal{P}_n^{(3)})\right], \tag{5.16}$$

where $\mathcal{P}_n^{(1)}$, $\mathcal{P}_n^{(2)}$ and $\mathcal{P}_n^{(3)}$ are independent Poisson point processes with intensities $n\mu'|_{D^c}$, $n\mu^{\perp}|_A$ and $n\mu|_D$ respectively. Here we set $D^c := \mathbb{R}^d \setminus D$ and for any Borel measure ν on $\mathbb{R}^d$ and Borel $E \subset \mathbb{R}^d$ we write $\nu|_E$ for the restriction of ν to E.

Set $t = \mu'(D^c)$. If $t > 0$, then $t^{-1}\mu'|_{D^c}$ is an absolutely continuous and blocked probability distribution with density $t^{-1}f_\mu \cdot \mathbf{1}_{D^c}$. Hence by Lemma 5.5 we have

$$\lim_{n\to\infty}(n^{-1}\mathbb{E}\left[\zeta_{r_n}(\mathcal{P}_n^{(1)})\right]) = \int_{D^c} \rho(\lambda f_\mu(x)) f_\mu(x)dx. \tag{5.17}$$

If $t = 0$, then $\mathcal{P}_n^{(1)} = \varnothing$, and (5.17) still holds because both sides are zero.

Let the cubes in the block-partition be denoted $C_1, \ldots, C_{(Mm)^d}$. Then for some $b_1, \ldots, b_{(Mm)^d} \in [0, \infty)$ we have $f_\mu = \sum_{i=1}^{(Mm)^d} b_i \mathbf{1}_{C_i}$. Assume the cubes are enumerated in such a way that for some $k \leq k' \leq \ell \leq (Mm)^d$ we have that $b_i > 0$ for $i = 1, \ldots, k'$ and $b_i = 0$ for $i = k'+1, \ldots, (Mm)^d$, while $D = \cup_{i=k+1}^{\ell} C_i$.

Now, for each $n \in \mathbb{N}$, divide D into smaller cubes (boxes) of equal side length s_n where $(1/(ms_n)) \in \mathbb{N}$, and $s_n \sim \delta r_n$ as $n \to \infty$, and $s_n \leq \delta r_n$ for all large enough n, say for $n \geq n_0$. Denote these boxes $B_{n,1}, \ldots, B_{n,m_n}$. Then the volume of each one of these boxes is asymptotic to $\delta^d r_n^d$, and hence to $\delta^d\lambda/n$, so that $m_n \sim \mathrm{Leb}(D)n/(\lambda\delta^d)$ as $n \to \infty$. Since $\mathrm{Leb}(D) \leq \varepsilon\delta^{d+1}\lambda/2$, therefore $m_n \leq \varepsilon\delta n$ for all large enough n.

By Lemma 5.3, writing $\mathcal{P}_n^{(1,3)}$ for $\mathcal{P}_n^{(1)} \cup \mathcal{P}_n^{(3)}$ and noting that $\mathcal{P}_n^{(3)} \subset D$ while $\mathcal{P}_n^{(1)} \subset \cup_{i=1}^{k} C_i$, we have

$$\zeta_{r_n}(\mathcal{P}_n^{(1,3)}) \leq \zeta_{r_n}(\mathcal{P}_n^{(1)}) + \left(\sum_{i=1}^{m_n} \zeta_{r_n}(\mathcal{P}_n^{(3)} \cap B_{n,i})\right) + c_1 m_n, \tag{5.18}$$

where c_1 is the constant from Property P3. We now show that the expected value of the sum on the right hand side of (5.18) is small. For each $i \in \{1, \ldots, m_n\}$, the box $B_{n,i}$ is contained in a ball of radius $\delta d r_n$, so by translation invariance (Property P2) and our choice of δ, for any finite $\mathcal{X} \subset B_{n,i}$ we have $\zeta_{r_n}(\mathcal{X}) \leq \varepsilon|\mathcal{X}|$ if $|\mathcal{X}| \geq \delta^{-1}$. Moreover, with K as given in Lemma 5.2, by that result $\zeta_{r_n}(\mathcal{X}) \leq K/\delta$ if $|\mathcal{X}| \leq \delta^{-1}$. Thus in all cases $\zeta_{r_n}(\mathcal{X}) \leq \varepsilon|\mathcal{X}| + K/\delta$. Hence for all large enough n,

$$\sum_{i=1}^{m_n} \mathbb{E}\left[\zeta_{r_n}(\mathcal{P}_n^{(3)} \cap B_{n,i})\right] \leq \sum_{i=1}^{m_n} (K/\delta + \varepsilon \mathbb{E}\left[|\mathcal{P}_n^{(3)} \cap B_{n,i}|\right]) \leq (K+1)\varepsilon n.$$

Also $\rho(r) \geq 0$ for all $r \geq 0$, by (2.2). Therefore using (5.17) and (5.18), and the assumption $\delta < 1$, we obtain that

$$\limsup_{n\to\infty} \mathbb{E}\left[n^{-1}\zeta_{r_n}(\mathcal{P}_n^{(1,3)})\right] \leq \int_{Q_M} \rho(\lambda f_\mu(x)) f_\mu(x) dx + (K+1+c_1)\varepsilon. \tag{5.19}$$

For a lower bound, we start with $\mathcal{P}_n^{(1)}$. By (2.2) and Lemma 5.2 we have $\rho(a) \leq K$ for all $a > 0$. Since μ is blocked, we have that $f_{\max} := \sup_{x\in\mathbb{R}^d} f_\mu(x) < \infty$. Therefore $\int_D \rho(\lambda f_\mu(x))\lambda f_\mu(x) dx \leq K\lambda f_{\max} \mathrm{Leb}(D) \leq K\lambda^2 f_{\max}\varepsilon$. Hence by (5.17), since f_μ is supported by Q_M,

$$\liminf_{n\to\infty} \mathbb{E}\left[n^{-1}\zeta_{r_n}(\mathcal{P}_n^{(1)})\right] \geq \int_{Q_M} \rho(\lambda f_\mu(x)) f_\mu(x) dx - K\lambda^2 f_{\max}\varepsilon. \tag{5.20}$$

Also since $\mathcal{P}_n^{(3)} \subset D$ and $\mathcal{P}_n^{(1)} \subset \cup_{i=1}^k C_i$ which is disjoint from D, by (5.2) from Lemma 5.3, and the non-negativity of ζ,

$$\begin{aligned}\zeta_{r_n}(\mathcal{P}_n^{(1,3)}) - \zeta_{r_n}(\mathcal{P}_n^{(1)}) &\geq \zeta_{r_n}(\mathcal{P}_n^{(3)}) - c_2|\mathcal{P}_n^{(1)} \cap \partial_{r_n}(\cup_{i=1}^k C_i)| \\ &\geq -c_2 \sum_{i=1}^k |\mathcal{P}_n^{(1)} \cap \partial_{r_n} C_i|.\end{aligned}$$

Therefore taking expectations we obtain that

$$\liminf_{n\to\infty} n^{-1}\mathbb{E}\left[\zeta_{r_n}(\mathcal{P}_n^{(1,3)}) - \zeta_{r_n}(\mathcal{P}_n^{(1)})\right] \geq -c_2 \lim_{n\to\infty} \sum_{i=1}^k \mu'(\partial_{r_n} C_i) = 0$$

Hence by (5.20),

$$\liminf_{n\to\infty} \mathbb{E}\left[n^{-1}\zeta_{r_n}(\mathcal{P}_n^{(1,3)})\right] \geq \int_{Q_M} \rho(\lambda f_\mu(x)) f_\mu(x) dx - K\lambda^2 f_{\max}\varepsilon. \tag{5.21}$$

Also by (5.16) and Lemma 5.2, for all n we have

$$|\mathbb{E}[\zeta_{r_n}(\mathcal{P}_n)] - \mathbb{E}[\zeta_{r_n}(\mathcal{P}_n^{(1,3)})]| \leq K\mathbb{E}[|\mathcal{P}_n^{(2)}|] \leq Kn\varepsilon.$$

Therefore using (5.19) and (5.21), since $\limsup |a_n| = \max(\limsup(a_n), -\liminf(a_n))$ for any real-valued sequence (a_n), we obtain that

$$\limsup_{n\to\infty} \left| n^{-1}\mathbb{E}[\zeta_{r_n}(\mathcal{P}_n)] - \int_{Q_M} \rho(\lambda f_\mu(x)) f_\mu(x) dx \right| \leq (2K + 1 + c_1 + K\lambda^2 f_{\max})\varepsilon,$$

and since $\varepsilon > 0$ is arbitrary this yields

$$\lim_{n\to\infty} \mathbb{E}[n^{-1}\zeta_{r_n}(\mathcal{P}_n)] = \int_{Q_M} \rho(\lambda f_\mu(x)) f_\mu(x) dx = \int_{\mathbb{R}^d} \rho(\lambda f_\mu(x)) f_\mu(x) dx,$$

which is (5.15), as required. □

We now drop the restriction, in the last lemma, to blocked density functions.

Lemma 5.7. *Suppose $nr_n^d \to \lambda$ as $n \to \infty$, for some $\lambda \in (0, \infty)$. Then (5.15) holds.*

Proof. Let $\varepsilon > 0$. As in the proof of Lemma 7.3 of [46], we can find a blocked distribution ν with the same singular part as μ, such that

$$\int |f_\nu(x) - f_\mu(x)| dx < \varepsilon,$$

where we now write $\int$ for $\int_{\mathbb{R}^d}$. Also as in that proof, we can find a pair of coupled random variables (X, Y) such that X has distribution μ, Y has distribution ν, and $\mathbb{P}[X \neq Y] \leq \varepsilon$. Then taking $(X_i, Y_i)_{i=1,2,3,\ldots}$ to be a sequence of i.i.d. coupled pairs with the distribution of (X, Y), we may consider coupled Poisson point processes $\mathcal{P}_n := \{X_1, \ldots, X_{N_n}\}$ and $\mathcal{Q}_n := \{Y_1, \ldots, Y_{N_n}\}$. By smoothness (Lemma 5.2),

$$\mathbb{E}[n^{-1}|\zeta_{r_n}(\mathcal{P}_n) - \zeta_{r_n}(\mathcal{Q}_n)|] \leq K\mathbb{E}[n^{-1}|\mathcal{P}_n \triangle \mathcal{Q}_n|] \leq 2K\varepsilon. \tag{5.22}$$

By Lemma 5.6, for large enough n we have

$$\left| \mathbb{E}[n^{-1}\zeta_{r_n}(\mathcal{Q}_n)] - \int \rho(\lambda f_\nu(x)) f_\nu(x) dx \right| < \varepsilon. \tag{5.23}$$

Moreover, by the Lipschitz continuity in λ of $\lambda\rho(\lambda)$ (see Proposition 2.3(ii)),

$$\begin{aligned}
&\left| \int \rho(\lambda f_\nu(x)) f_\nu(x) dx - \int \rho(\lambda f_\mu(x)) f_\mu(x) dx \right| \\
&= \lambda^{-1} \left| \int (\lambda f_\nu(x) \rho(\lambda f_\nu(x)) - \lambda f_\mu(x) \rho(\lambda f_\mu(x))) dx \right| \\
&\leq \lambda^{-1} K \int |\lambda f_\nu(x) - \lambda f_\mu(x)| dx \leq K\varepsilon.
\end{aligned} \tag{5.24}$$

Combining (5.22), (5.23), and (5.24), we obtain for all large enough n that

$$\left| \mathbb{E}\left[n^{-1}\zeta_{r_n}(\mathcal{P}_n) \right] - \int \rho(\lambda f_\mu(x)) f_\mu(x) dx \right| \leq \varepsilon(1+3K),$$

and since ε is arbitrarily small, (5.15) follows. $\square$

Proof of Theorem 2.1. Suppose $nr_n^d \to t \in (0,\infty)$. By Lemma 5.7, $n^{-1}\mathbb{E}\left[\zeta_{r_n}(\mathcal{P}_n)\right]$ converges to $\int_{\mathbb{R}^d} \rho(tf_\mu(x))f_\mu(x)dx$, as $n \to \infty$.

Recalling from Section 2.1 our coupling of $\mathcal{P}_n$ and $\mathcal{X}_n$, and that $N_n := |\mathcal{P}_n|$, by Lemma 5.2 we have

$$\mathbb{E}\left[|\zeta_{r_n}(\mathcal{P}_n) - \zeta_{r_n}(\mathcal{X}_n)|\right] \leq K\mathbb{E}\left[|\mathcal{P}_n \triangle \mathcal{X}_n|\right] = K\mathbb{E}\left[|N_n - n|\right] = o(n),$$

and therefore

$$n^{-1}\mathbb{E}\left[\zeta_{r_n}(\mathcal{X}_n)\right] \to \int_{\mathbb{R}^d} \rho(tf_\mu(x))f_\mu(x)dx \quad \text{as} \quad n \to \infty. \tag{5.25}$$

The proof of (2.3) starting from (5.25) is completed by martingale difference methods as in the proof of Theorem 4.9 starting from (4.20), with $(\mathcal{U}_n, \mathcal{V}_n)$ therein replaced by $\mathcal{X}_n$ and $(\mathcal{U}_{n,i}, \mathcal{V}_{n,i})$ replaced by $\mathcal{X}_{n,i}$, which is obtained from $\mathcal{X}_n$ by replacing X_i by an independent copy X_0. Due to the similarity to the corresponding part of the proof of Theorem 4.9, we do not give the details this time.

To prove (2.5), let $K' > 0$. Again by a martingale difference argument (in particular, Azuma's inequality), there is a constant c such that

$$\mathbb{P}[n^{-1}|\zeta_{r_n}(\mathcal{X}_n) - \mathbb{E}\left[\zeta_{r_n}(X_n)\right]| > K'((\log n)/n)^{1/2}] \leq \exp\left(\frac{-(K'(\log n)^{1/2}n^{1/2})^2}{cn} \right),$$

which is summable in n, provided K' is chosen large enough. Therefore by the Borel-Cantelli lemma, (2.5) holds.

By Lemma 5.2, for all $\varepsilon > 0, n \in \mathbb{N}$ we have

$$\mathbb{P}[|\zeta_{r_n}(\mathcal{P}_n) - \zeta_{r_n}(\mathcal{X}_n)| > \varepsilon n] \leq \mathbb{P}[|N_n - n| > \varepsilon n/K],$$

which is summable in n by a Chernoff bound (e.g. [35, Lemma 1.2]).

Using this, and the complete convergence in (2.3), gives us the complete convergence in (2.4). Also $\mathbb{E}\left[(n^{-1}(\zeta_{r_n}(\mathcal{P}_n) - \zeta_{r_n}(\mathcal{X}_n)))^2\right] \leq \mathbb{E}\left[n^{-2}K^2|N_n - n|^2\right]$, which tends to zero, so using the L^2 convergence in (2.3), we have the L^2 convergence in (2.4). Part (a) of Theorem 2.1 follows.

It remains to prove part (b), so now assume that $\mu = \mu_U$, and that $|nr_n^d - t| = O(n^{-1/d})$ as $n \to \infty$. Set $t_n = nr_n^d$. Then taking $\lambda_1 = t/2$, we have $t_n \geq \lambda_1$ for all large enough n. Then

$$\begin{aligned} |\mathbb{E}\,[n^{-1}\zeta_{r_n}(\mathcal{P}_n)] - \rho(t_n)| &= |\mathbb{E}\,[\zeta(r_n^{-1}\mathcal{P}_n)/n] - \rho(t_n)| \\ &= |\mathbb{E}\,[\zeta(\mathcal{H}_{nr_n^d} \cap Q_{r_n^{-1}})/n] - \rho(t_n)| \\ &= \left|\mathbb{E}\,[\zeta(\mathcal{H}_{t_n} \cap Q_{r_n^{-1}})/(t_n r_n^{-d})] - \rho(t_n)\right|, \end{aligned}$$

and therefore by (2.9) from Lemma 2.3,

$$|\mathbb{E}\,[n^{-1}\zeta_{r_n}(\mathcal{P}_n)] - \rho(t_n)| = O(r_n) = O(n^{-1/d}).$$

Also $|t_n - t| = O(n^{-1/d})$ by assumption, so also $|\rho(t_n) - \rho(t)| = O(n^{-1/d})$ by the local Lipschitz continuity of $\rho(\cdot)$ (this follows from Proposition 2.3(ii)). Thus we have (2.6). Also by smoothness (Lemma 5.2),

$$|\mathbb{E}\,[n^{-1}\zeta_{r_n}(\mathcal{X}_n)] - n^{-1}\mathbb{E}\,[\zeta_{r_n}(\mathcal{P}_n)]| \leq n^{-1}K\mathbb{E}\,[|N_n - n|] = O(n^{-1/2}).$$

Combined with (2.6), this yields (2.7). $\square$

Proof of Lemma 2.5. By (2.11) and P3, $\zeta^*(A \cup A') + c_1 \leq (\zeta^*(A) + c_1) + (\zeta^*(A') + c_1)$ for any $A, A' \subset \mathbb{R}^d$ (not necessarily disjoint). Also $\zeta^*(A) \leq \zeta^*(A')$ whenever $A \subset A'$.

Setting $k := \kappa(Q_2)$, we can cover Q_1 by k balls of radius $1/2$, and repeatedly using the above subadditivity and monotonicity properties we obtain for all $A \subset Q_1$ that $\zeta^*(A) + c_1 \leq k(\zeta^*(B_{1/2}(o)) + c_1)$, which is finite by Property P5$'$. Hence taking $f(A) = \zeta^*(A) + c_1$ gives a functional satisfying all the conditions of Lemma 5.1, and that result gives us (2.12).

By (2.11), $\mathbb{E}\,[\zeta(\mathcal{H}_{\lambda,s})] \leq \zeta^*(Q_s)$, $s > 0$. Then by (2.2) and (2.12), we have (2.13). $\square$

Proof of Theorem 2.6. Since we assume $\zeta(\cdot) \geq 0$, we have that $\overline{\zeta} \geq 0$ by (2.12), and $\rho(\lambda) \geq 0$ for all $\lambda \in \mathbb{R}_+$ by (2.2). If $\overline{\zeta} = 0$, then by (2.13) $\rho(\lambda) = 0$ for all λ so the result (2.14) holds. Therefore we may assume without loss of generality that $\overline{\zeta} > 0$.

Let $\varepsilon \in (0, 1)$. Choose $s_0 \in (0, \infty)$ with $(s_0/(s_0 + 2))^d > 1 - \varepsilon$ and $s_0^{-d}\zeta^*(Q_{s_0}) > (1 - \varepsilon)\overline{\zeta}$. Then choose a finite set $\mathcal{X} \subset Q_{s_0}$ with $\zeta(\mathcal{X}) > (1 - \varepsilon)\zeta^*(s_0)$.

Choose $\delta \in (0, 1)$ such that $\inf[\{\|x - y\| - 1 : x, y \in \mathcal{X}\} \cap (0, \infty)] > \delta$ (using the convention $\inf[\varnothing] := +\infty$ if needed); this can be done because the infimum is over a finite set of strictly positive numbers. Define the re-scaled point set $\mathcal{X}' := (1+\delta)^{-1}\mathcal{X}$; then $G(\mathcal{X}', 1)$ is isomorphic to $G(\mathcal{X}, 1)$ so $\zeta(\mathcal{X}') = \zeta(\mathcal{X})$ by assumption P8, and $\mathcal{X}'$ has no point on the boundary of Q_{s_0}, and $\|x - y\| \neq 1$ for all $x, y \in \mathcal{X}'$.

Enumerate the elements of $\mathcal{X}'$ as $x_1, \ldots, x_k$, say. By the preceding conclusion there exists $\delta' > 0$ such that the balls $B_i := B_{\delta'}(x_i)$ $1 \leq i \leq k$, are disjoint, are all contained in Q_i, and have the property that for any $\mathcal{Y} = \{y_1, \dots, y_k\}$ with $y_i \in B_i$ for each i we have that $G(\mathcal{Y}, 1)$ isomorphic to $G(\mathcal{X}', 1)$ and hence $\zeta(\mathcal{Y}) = \zeta(\mathcal{X}') = \zeta(\mathcal{X})$ by P8. By P6 (upwards monotonicity in $\mathcal{X}$),

$$\mathbb{P}[\zeta(\mathcal{H}_\lambda \cap Q_{s_0}) > (1-\varepsilon)\zeta^*(Q_{s_0})] \geq \mathbb{P}\left[\cap_{i=1}^k \{\mathcal{H}_\lambda \cap B_i \neq \varnothing\}\right] \\ \to 1 \quad \text{as } \lambda \to \infty.$$

Hence, using also the nonnegativity of ζ, we have that

$$\liminf_{\lambda \to \infty} \mathbb{E}\left[\zeta(\mathcal{H}_{\lambda, s_0})\right] > (1-\varepsilon)^2 \zeta^*(Q_{s_0}).$$

Pick $\lambda_0 \in (0, \infty)$ such that

$$\mathbb{E}\left[\zeta(\mathcal{H}_{\lambda, s_0})\right] > (1-\varepsilon)^2 \zeta^*(Q_{s_0}), \quad \forall\, \lambda \geq \lambda_0. \tag{5.26}$$

Given $s > 0$, write $s = n(s_0 + 2) + t$ with $0 \leq t < s_0 + 2$. Let $Q'_1, \dots, Q'_{n^d}$ be a collection of cubes of side s_0, that are disjoint and distant at least 2 from each other, and contained in Q_s. Then using P6 followed repeatedly by P4, we have

$$\zeta(\mathcal{H}_\lambda \cap Q_s) \geq \zeta\left(\mathcal{H}_\lambda \cap \left(\cup_{i=1}^{n^d} Q'_i\right)\right) \geq \sum_{i=1}^{n^d} \zeta(\mathcal{H}_\lambda \cap Q'_i).$$

Therefore by (5.26) for fixed $\lambda \geq \lambda_0$, provided s (and hence n) is large enough

$$s^{-d}\mathbb{E}\left[\zeta(\mathcal{H}_\lambda \cap Q_s)\right] \geq \frac{n^d \mathbb{E}\left[\zeta(\mathcal{H}_\lambda \cap Q_{s_0})\right]}{(n+1)^d (s_0+2)^d} \\ \geq \left(\frac{n}{n+1}\right)^d \left(\frac{s_0}{s_0+2}\right)^d \left(\frac{(1-\varepsilon)^2 \zeta^*(Q_{s_0})}{s_0^d}\right) > (1-\varepsilon)^5\, \overline{\zeta}.$$

Using (2.2), this shows that $\lambda\rho(\lambda) \geq (1-\varepsilon)^5\, \overline{\zeta}$ for $\lambda \geq \lambda_0$, so that $\liminf_{\lambda\to\infty}(\lambda\rho(\lambda)) \geq \overline{\zeta}$. Combined with (2.13), this gives us the result. □

Proof of Theorem 2.7. Assume that $r_n \to 0$ and $nr_n^d \to \infty$ as $n \to \infty$, and that μ is absolutely continuous with density f_μ chosen in such a way that the set $A := f_\mu^{-1}((0,\infty))$ is Riemann measurable. Then, given $\varepsilon > 0$, we can find $a > 0$ and $k \in \mathbb{N}$, such that we can find a covering of the set A by disjoint cubes $C_1, \dots, C_k$ of side a, with $ka^d < \mathrm{Leb}(A) + \varepsilon$. Then by (5.1) and (2.11), almost surely

$$\zeta_{r_n}(\mathcal{X}_n) \leq \left(\sum_{i=1}^k \zeta_{r_n}(\mathcal{X}_n \cap C_i)\right) + c_1 k \ \leq k\zeta^*(Q_{r_n^{-1}a}) + c_1 k,$$

so that by (2.12),

$$\limsup_{n\to\infty}(r_n^d \zeta_{r_n}(\mathcal{X}_n)) \le ka^d\,\overline{\zeta} \le (\mathrm{Leb}(A)+\varepsilon)\,\overline{\zeta}. \tag{5.27}$$

For an inequality the other way, let $\lambda \in (0,\infty)$ and let $(m(n), n \in \mathbb{N})$ be an $\mathbb{N}$-valued sequence with $m(n) \le n$ for all n and with $m(n)r_n^d \to \lambda$ as $n \to \infty$. Then by P6, almost surely

$$r_n^d \zeta_{r_n}(\mathcal{X}_n) \ge r_n^d \zeta_{r_n}(\mathcal{X}_{m(n)}) = (1+o(1))(\lambda/m(n))\zeta_{r_n}(\mathcal{X}_{m(n)}),$$

so by Theorem 2.1, almost surely

$$\liminf_{n\to\infty}(r_n^d \zeta_{r_n}(\mathcal{X}_n)) \ge \lambda \int_{\mathbb{R}^d} \rho(\lambda f_\mu(x)) f_\mu(x)dx. \tag{5.28}$$

By Theorem 2.6 and monotone convergence (which is applicable by Proposition 2.3(iv)), we have $\int_A \lambda f_\mu(x)\rho(\lambda f_\mu(x))dx \to \overline{\zeta}\,\mathrm{Leb}(A)$ as $\lambda \uparrow \infty$. Since (5.28) holds for arbitrarily large $\lambda \in (0,\infty)$, we obtain that $\liminf_{n\to\infty}(r_n^d\zeta_{r_n}(\mathcal{X}_n)) \ge \overline{\zeta}\,\mathrm{Leb}(A)$, and combined with (5.27) this gives us the result. □

6 Proofs for domination number at high density

In this section we shall prove Theorems 3.3 and 3.4. Throughout this section we take $\zeta(\mathcal{X}) = \gamma(G(\mathcal{X},1))$, the domination number of $G(\mathcal{X},1)$, for all finite $\mathcal{X} \subset \mathbb{R}^d$. Therefore $\zeta_r(\mathcal{X})$ is equal to $\gamma(G(\mathcal{X},r))$.

Throughout this section we assume that a sequence $(r_n)_{n\ge1}$ is given, taking values in $(0,\infty)$ and satisfying $r_n \to 0$ and $nr_n^d \to \infty$ as $n \to \infty$, but otherwise arbitrary (this is part of the hypothesis of Theorem 3.4, but since some of the preliminaries below will also be used in the proof of Theorem 3.3, and for the latter proof this assumption is not harmful, we assume it throughout the whole section).

6.1 Deterministic preliminaries

Lemma 6.1 (Smoothness of the domination number)**.** *There is a constant $K \in (0,\infty)$ such that if $\mathcal{X},\mathcal{Y}$ are finite subsets of $\mathbb{R}^d$, and $r > 0$, then*

$$|\zeta_r(\mathcal{Y}) - \zeta_r(\mathcal{X})| \le K|\mathcal{Y}\triangle\mathcal{X}|. \tag{6.1}$$

Proof. This follows from Lemma 5.2; we already checked in Section 3.2 that our current choice of ζ satisfies P1–P4. □

For $s > 0$ set $\overline{Q}_s := [-s/2, s/2]^d$, the closure of Q_s. The sequence $(r_n)_{n\in\mathbb{N}}$ was specified already. Also we fix a constant $\delta \in (0, 1/4)$. Recall the definitions (3.3) and (3.4) of $\kappa(\cdot)$ and $\overline{\kappa}$, respectively.

Lemma 6.2. *There exists a sequence $(\mathcal{L}_n)_{n\in\mathbb{N}}$ of finite sets with $\mathcal{L}_n \subset \overline{Q}_{r_n^{-1}}$ and $Q_{r_n^{-1}} \subset \cup_{x\in\mathcal{L}_n} B_{1-\delta}(x)$ for each n, such that $k_n := |\mathcal{L}_n|$ satisfies*

$$(1-\delta)^{-d}\overline{\kappa} \le \liminf_{n\to\infty}(r_n^d k_n) \le \limsup_{n\to\infty}(r_n^d k_n) \le (1-4\delta)^{-d}\overline{\kappa}, \tag{6.2}$$

and moreover with $\|x - y\| > 3\delta$ for all $x, y \in \mathcal{L}_n$ such that $x \ne y$.

Proof. Set $k_n^0 := \kappa(Q_{(1-4\delta)^{-1}r_n^{-1}})$, and let $\mathcal{L}_n^0 \subset \mathbb{R}^d$ with $Q_{r_n^{-1}} \subset \cup_{x\in\mathcal{L}_n^0} B_{1-4\delta}(x)$ and $|\mathcal{L}_n^0| = k_n^0$. We may assume without loss of generality that all elements of $\mathcal{L}_n^0$ lie in $\overline{Q}_{r_n^{-1}}$, since if $x \in \mathcal{L}_n^0 \setminus \overline{Q}_{r_n^{-1}}$, and y is the closest point of $\overline{Q}_{r_n^{-1}}$ to x, then $\|z - y\| \le \|z - x\|$ for all $z \in Q_{r_n^{-1}}$, so we could replace the point x in $\mathcal{L}_n^0$ by y.

Let $\mathcal{L}_n \subset \mathcal{L}_n^0$ be a maximum independent set of $G(\mathcal{L}_n^0, 3\delta)$, i.e. an independent set of $G(\mathcal{L}_n^0, 3\delta)$ with $|\mathcal{L}_n| = \alpha(G(\mathcal{L}_n^0, 3\delta))$, where $\alpha(\cdot)$ was defined in Section 3. Then $\mathcal{L}_n$ is a dominating set for $G(\mathcal{L}_n^0, 3\delta)$; otherwise, a further element of $\mathcal{L}_n^0 \setminus \mathcal{L}_n$ could be added to $\mathcal{L}_n$ without it ceasing to be an independent set, contradicting the maximality of $\mathcal{L}_n$.

Then $Q_{r_n^{-1}} \subset \cup_{x\in\mathcal{L}_n} B_{1-\delta}(x)$. Indeed, for each $x \in \mathcal{L}_n^0 \setminus \mathcal{L}_n$ there exists $y \in \mathcal{L}_n$ with $\|x - y\| \le 3\delta$, and therefore by the triangle inequality $B_{1-4\delta}(x) \subset B_{1-\delta}(y)$.

Set $k_n := |\mathcal{L}_n|$. By (3.4), $r_n^d k_n^0 \to (1-4\delta)^{-d}\overline{\kappa}$ as $n \to \infty$. Since $k_n \le k_n^0$, this gives the last inequality of (6.2). Since $Q_{r_n^{-1}} \subset \cup_{x\in\mathcal{L}_n} B_{1-\delta}(x)$, by (3.3) we have $k_n \ge \kappa(Q_{(1-\delta)^{-1}r_n^{-1}})$, and hence by (3.4), the first inequality of (6.2). $\square$

Given $n \in \mathbb{N}$, enumerate $\mathcal{L}_n$ as $\mathcal{L}_n = \{x_{n,1}, \ldots, x_{n,k_n}\}$. Choose $n_0 \in \mathbb{N}$ such that $r_n \le 1$ for all $n \ge n_0$. Then for all $n \ge n_0$ and each $i \in [k_n]$, since $x_{n,i} \in \overline{Q}_{r_n^{-1}}$, the ball $B_\delta(x_{n,i}) \cap Q_{r_n^{-1}}$ contains a cube of side δ/d with a corner at $x_{n,i}$, and hence

$$\mathrm{Leb}(B_\delta(x_{n,i}) \cap Q_{r_n^{-1}}) \ge (\delta/d)^d, \quad \forall\, n \ge n_0, i \in [k_n]. \tag{6.3}$$

Given finite $\mathcal{X} \subset Q_{r_n^{-1}}$, define the set of 'good' indices

$$I_n(\mathcal{X}) := \{i \in [k_n] : \mathcal{X} \cap B_\delta(x_{n,i}) \ne \varnothing\}. \tag{6.4}$$

Set $K_0 := \kappa(B_2(o))$.

Lemma 6.3. *For all $n \in \mathbb{N}$ and all finite $\mathcal{X} \subset Q_{r_n^{-1}}$,*

$$\zeta(\mathcal{X}) \le k_n + K_0|[k_n] \setminus I_n(\mathcal{X})|. \tag{6.5}$$

Proof. Let $n \in \mathbb{N}$. For each $i \in [k_n]$, cover the ball $B_1(x_{n,i})$ by balls $B_{n,i,1}, \ldots, B_{n,i,K_0}$ of radius $1/2$. We now define a set $\mathcal{A} \subset \mathcal{X}$ as follows.

For each 'good' index $i \in I_n(\mathcal{X})$, let $y_{n,i}$ be the first element of $\mathcal{X} \cap B_\delta(x_{n,i})$ in the lexicographic ordering. If $i \in [k_n] \setminus I_n(\mathcal{X})$ (so i is a 'bad' index), then for each $j \in [K_0]$ such that $\mathcal{X} \cap B_{n,i,j} \neq \varnothing$, let $z_{n,i,j}$ be the first element of $\mathcal{X} \cap B_{n,i,j}$ in the lexicographic ordering. Set

$$\mathcal{A} := \{y_{n,i} : i \in I_n(\mathcal{X})\} \cup \{z_{n,i,j} : i \in [k_n] \setminus I_n(\mathcal{X}), j \in [K_0], \mathcal{X} \cap B_{n,i,j} \neq \varnothing\}.$$

We assert that $\mathcal{A}$ is a dominating set for $G(\mathcal{X}, 1)$. Indeed, if i is a good index then the set $B_1(y_{n,i})$ contains the whole of the ball $B_{1-\delta}(x_{n,i})$. If i is a bad index, then for each $x \in \mathcal{X} \cap B_{1-\delta}(x_{n,i})$, choosing $j \in [K_0]$ such that $x \in B_{n,i,j}$ we have $x \in B_1(z_{n,i,j})$. Since $\mathcal{X} \subset Q_{r_n^{-1}}$, every point of $\mathcal{X}$ lies in at least one of the balls $B_{1-\delta}(x_{n,i})$, $i \in [k_n]$. Therefore $\mathcal{A}$ is a dominating set as asserted. Moreover, $|\mathcal{A}| \leq k_n + K_0 |[k_n] \setminus I_n(\mathcal{X})|$, and (6.5) follows. $\square$

Now we derive a bound the other way. Given $n \in \mathbb{N}$, partition $Q_{r_n^{-1}}$ into cubes of side $\delta_n := r_n^{-1}/\lceil r_n^{-1}/\delta \rceil$, denoted $S_{n,1}, S_{n,2}, \ldots, S_{n,\ell_n}$. Note that $\delta_n \leq \delta$ and $\delta_n \to \delta$ as $n \to \infty$, so that

$$\ell_n \sim (r_n^{-1}/\delta)^d \qquad \text{as} \quad n \to \infty. \tag{6.6}$$

For $i \in [\ell_n]$, let $v_{n,i}$ denote the centre of the cube $S_{n,i}$. Given finite $\mathcal{X} \subset Q_{r_n^{-1}}$, set

$$J_n(\mathcal{X}) := \{i \in [\ell_n] : \mathcal{X} \cap S_{n,i} \neq \varnothing\}.$$

Lemma 6.4. *Let $n \in \mathbb{N}$ and let $\mathcal{X} \subset Q_{r_n^{-1}}$ be finite. Then*

$$\zeta(\mathcal{X}) \geq r_n^{-d}(1 + d\delta)^{-d}\overline{\kappa} - |[\ell_n] \setminus J_n(\mathcal{X})|. \tag{6.7}$$

Proof. Suppose $\mathcal{A} \subset \mathcal{X}$ is a dominating set for $G(\mathcal{X}, 1)$. Then for each $i \in J_n(\mathcal{X})$ we have for some $x \in \mathcal{A}$ and $y \in \mathcal{X} \cap S_{n,i}$ that $y \in B_1(x)$, and hence $S_{n,i} \subset B_{1+d\delta}(x)$. Therefore

$$\cup_{i \in J_n(\mathcal{X})} S_{n,i} \subset \cup_{x \in \mathcal{A}} B_{1+d\delta}(x).$$

Also, since the cube $S_{n,i}$ has side at most δ, we have $S_{n,i} \subset B_{1+d\delta}(v_{n,i})$, for $1 \leq i \leq \ell_n$. Hence the union of balls of radius $1 + d\delta$ centred on the points of the set

$$\mathcal{A} \cup \{v_{n,i} : i \in [\ell_n] \setminus J_n(\mathcal{X})\}$$

contains the whole of $Q_{r_n^{-1}}$. Therefore by the definition (3.4) of $\overline{\kappa}$, we have

$$(1 + d\delta)^d r_n^d (|\mathcal{A}| + |[\ell_n] \setminus J_n(\mathcal{X})|) \geq \overline{\kappa},$$

for all choices of $\mathcal{A}$. Hence (6.7) holds. $\square$

6.2 Proof of Theorem 3.3

Proof of Theorem 3.3. Let $(r_n)_{n\geq 1}$ and $\delta \in (0, 1/4)$ be as specified already in this section. Let $\lambda > 0$. Let $\mathcal{L}_n := \{x_{n,1}, \ldots, x_{n,k_n}\}$ be as described in Lemma 6.2. For finite $\mathcal{X} \subset Q_{r_n^{-1}}$, define the set of 'good' indices $I_n(\mathcal{X})$ by (6.4). Then by Lemma 6.3,

$$\zeta(\mathcal{H}_\lambda \cap Q_{r_n^{-1}}) \leq k_n + K_0|[k_n] \setminus I_n(\mathcal{H}_\lambda \cap Q_{r_n^{-1}})|,$$

and $\limsup_{n\to\infty}(r_n^d k_n) \leq (1-4\delta)^{-d}\overline{\kappa}$ by (6.2). Then by (6.3),

$$\mathbb{E}\left[|[k_n] \setminus I_n(\mathcal{H}_\lambda \cap Q_{r_n^{-1}})|\right] \leq k_n \exp(-\lambda(\delta/d)^d).$$

Therefore, using also (2.2), we have

$$\lambda\rho(\lambda) = \lim_{n\to\infty}(r_n^d \mathbb{E}\left[\zeta(\mathcal{H}_\lambda \cap Q_{r_n^{-1}})\right]) \leq (1-4\delta)^{-d}\overline{\kappa}(1 + \exp(-\lambda(\delta/d)^d)),$$

so that

$$\limsup_{\lambda\to\infty}(\lambda\rho(\lambda)) \leq (1-4\delta)^{-d}\overline{\kappa}. \tag{6.8}$$

By Lemma 6.4 we have

$$\zeta(\mathcal{H}_\lambda \cap Q_{r_n^{-1}}) \geq (1+d\delta)^{-d} r_n^{-d}\overline{\kappa} - |[\ell_n] \setminus J_n(\mathcal{H}_\lambda \cap Q_{r_n^{-1}})|,$$

and $\ell_n \sim (r_n^{-1}/\delta)^d$ by (6.6). For fixed λ we have as $n \to \infty$ that

$$\mathbb{E}\left[|[\ell_n] \setminus J_n(\mathcal{H}_\lambda \cap Q_{r_n^{-1}})|\right] \sim (r_n^{-1}/\delta)^d \exp(-\lambda\delta^d).$$

Therefore,

$$\lambda\rho(\lambda) = \lim_{n\to\infty}(r_n^d \mathbb{E}\left[\zeta(\mathcal{H}_{\lambda,r_n^{-1}})\right]) \geq (1+d\delta)^{-d}\overline{\kappa} - \delta^{-d}\exp(-\lambda\delta^d),$$

so that

$$\liminf_{\lambda\to\infty}(\lambda\rho(\lambda)) \geq (1+d\delta)^{-d}\overline{\kappa}.$$

Combining this with (6.8), since $\delta \in (0, 1/4)$ is arbitrary, gives us Theorem 3.3. $\square$

6.3 Proof of Theorem 3.4

Assume in this subsection that $\mu = \mu_U$. We shall prove Theorem 3.4 using separate arguments according to whether nr_n^d grows faster or more slowly than $n^{1/8}$. Set

$$\mathcal{N} := \{n \in \mathbb{N} : r_n^d \geq n^{-7/8}\}. \tag{6.9}$$

Let $\delta \in (0, 1/4)$, and let $\mathcal{L}_n$ and k_n be as described in Lemma 6.2.

Lemma 6.5. *Let $\varepsilon > 0$. Then*

$$\sum_{n \in \mathcal{N}} \mathbb{P}[r_n^d \zeta_{r_n}(\mathcal{X}_n) > \overline{\kappa} + \varepsilon] < \infty. \tag{6.10}$$

Proof. Assume $\mathcal{N}$ is infinite (otherwise (6.10) is trivial). For all $n \in \mathcal{N}$ and $i \in [k_n]$, recalling the definition of $I_n(\mathcal{X})$ at (6.4), and using (6.3), if $n \geq n_0$ we have

$$\mathbb{P}[i \notin I_n(r_n^{-1}\mathcal{X}_n)] \leq (1 - (\delta/d)^d r_n^d)^n \leq \exp(-n(\delta/d)^d r_n^d) \leq \exp(-(\delta/d)^d n^{1/8}).$$

Since $r_n^{-d} \ll n$, $k_n \ll n$ by (6.2), and so by the union bound we have for all large enough $n \in \mathcal{N}$ that

$$\mathbb{P}[[k_n] \setminus I_n(r_n^{-1}\mathcal{X}_n) \neq \varnothing] \leq k_n \exp(-(\delta/d)^d n^{1/8}) \leq n \exp(-(\delta/d)^d n^{1/8}). \tag{6.11}$$

Also, using (6.5) and then (6.2), for all large enough $n \in \mathcal{N}$ we have

$$\begin{aligned} & \mathbb{P}[r_n^d \zeta(r_n^{-1}\mathcal{X}_n) > (1-4\delta)^{-d}\overline{\kappa} + \delta] \\ \leq \; & \mathbb{P}[r_n^d \left(k_n + K_0 |[k_n] \setminus I_n(r_n^{-1}\mathcal{X}_n)|\right) > (1-4\delta)^{-d}\overline{\kappa} + \delta] \\ \leq \; & \mathbb{P}[[k_n] \setminus I_n(r_n^{-1}\mathcal{X}_n) \neq \varnothing], \end{aligned}$$

which is summable in n by (6.11). Since $\delta > 0$ is arbitrarily small, (6.10) follows. □

Lemma 6.6. *Let $\varepsilon > 0$. Then*

$$\sum_{n \in \mathbb{N} \setminus \mathcal{N}} \mathbb{P}[r_n^d \zeta_{r_n}(\mathcal{X}_n) > \overline{\kappa} + \varepsilon] < \infty. \tag{6.12}$$

Proof. We first consider the point process $\mathcal{P}_n$ defined earlier; since $\mu = \mu_U$ here, $\mathcal{P}_n$ is a homogeneous Poisson point process in Q_1 of intensity n, so by the Mapping theorem $r_n^{-1}\mathcal{P}_n$ is a homogeneous Poisson point process in $Q_{r_n^{-1}}$ of intensity nr_n^d. By (6.3), we have for all $n \geq n_0$ and $i \in [k_n]$ that $\mathbb{P}[i \notin I_n(r_n^{-1}\mathcal{P}_n)] \leq p_n$, where we set

$$p_n := \exp(-n(\delta/d)^d r_n^d) = o(1),$$

because we assume $nr_n^d \to \infty$.

Also the balls $B_\delta(x_{n,i}), 1 \leq i \leq k_n$, are disjoint, and setting $M_n := |[k_n] \setminus I_n(r_n^{-1}\mathcal{P}_n)|$, we have that M_n is stochastically dominated by a Binomial with parameters k_n and p_n. Hence, given any fixed $\delta' > 0$, by (1.7) of [35],

$$\mathbb{P}[M_n \geq \delta' k_n] \leq \exp\left(-(\delta' k_n/2)\log(\delta'/p_n)\right) = \exp(-\omega(k_n)).$$

Therefore by (6.2), for large enough $n \in \mathbb{N} \setminus \mathcal{N}$ we have

$$\mathbb{P}[M_n \geq \delta' k_n] \leq \exp(-n^{7/8}). \tag{6.13}$$

By (6.5) and (6.2), for all large enough n we have

$$\begin{aligned}
&\mathbb{P}\left[r_n^d \zeta(r_n^{-1}\mathcal{P}_n) \geq (1-4\delta)^{-d}\overline{\kappa} + 2\delta\right] \\
&\quad \leq \mathbb{P}[r_n^d(k_n + K_0|[k_n] \setminus I_n(r_n^{-1}\mathcal{P}_n)|) > (1-4\delta)^{-d}\overline{\kappa} + 2\delta] \leq \mathbb{P}[r_n^d K_0 M_n \geq \delta].
\end{aligned}$$

Therefore since $\zeta(r_n^{-1}\mathcal{P}_n) = \zeta_{r_n}(\mathcal{P}_n)$, for large enough $n \in \mathbb{N} \setminus \mathcal{N}$ we have by (6.2) and (6.13) that

$$\mathbb{P}\left[r_n^d \zeta_{r_n}(\mathcal{P}_n) \geq (1-4\delta)^{-d}\overline{\kappa} + 2\delta\right] \leq \exp(-n^{7/8}). \tag{6.14}$$

With our coupling of $\mathcal{X}_n$ and $\mathcal{P}_n$, by [35, Lemma 1.2] and Taylor expansion about 1 of the function $h(x) := 1 - x + x \log x$ we have for large enough n that

$$\begin{aligned}
\mathbb{P}[|\mathcal{X}_n \triangle \mathcal{P}_n| > n^{3/4}] &\leq \exp\left(-nh\left(\frac{n+n^{3/4}}{n}\right)\right) + \exp\left(-nh\left(\frac{n-n^{3/4}}{n}\right)\right) \\
&\leq 2\exp(-n^{1/2}/3).
\end{aligned}$$

By Lemma 6.1, if $|\mathcal{X}_n \triangle \mathcal{P}_n| \leq n^{3/4}$ then $|\zeta_{r_n}(\mathcal{X}_n) - \zeta_{r_n}(\mathcal{P}_n)| \leq K n^{3/4}$. Hence

$$\mathbb{P}[r_n^d|\zeta_{r_n}(\mathcal{X}_n) - \zeta_{r_n}(\mathcal{P}_n)| > K r_n^d n^{3/4}] \leq 2\exp(-n^{1/2}/3). \tag{6.15}$$

Since $K r_n^d n^{3/4} \leq K n^{-1/8} < \delta$ for all large enough $n \in \mathbb{N} \setminus \mathcal{N}$, combining (6.15) with (6.14) shows that for all large enough $n \in \mathbb{N} \setminus \mathcal{N}$,

$$\mathbb{P}\left[r_n^d \zeta_{r_n}(\mathcal{X}_n) \geq (1-4\delta)^{-d}\overline{\kappa} + 3\delta\right] \leq \exp(-n^{7/8}) + 2\exp(-n^{1/2}/3),$$

which is summable in n. Since δ can be taken arbitrarily small, this gives us the result. □

Proof of Theorem 3.4. By Lemma 6.4, for all $n \in \mathbb{N}$ we have

$$\mathbb{P}[r_n^d \zeta_{r_n}(\mathcal{X}_n) < (1+d\delta)^{-d}\overline{\kappa}] \leq \mathbb{P}[[\ell_n] \setminus J_n(r_n^{-1}\mathcal{X}_n) \neq \varnothing]. \tag{6.16}$$

For all sufficiently large $n \in \mathcal{N}$ and all $i \in [\ell_n]$,

$$\mathbb{P}[i \notin J_n(r_n^{-1}\mathcal{X}_n)] \leq \left(1 - \left(\frac{\delta r_n}{2}\right)^d\right)^n \leq \exp\left[-\left(\frac{\delta}{2}\right)^d n r_n^d\right] \leq \exp\left[-\left(\frac{\delta}{2}\right)^d n^{1/8}\right].$$

Thus by the union bound, since $\ell_n = O(r_n^{-d}) = o(n)$ by (6.6), for all sufficiently large $n \in \mathcal{N}$ we have

$$\mathbb{P}[[\ell_n] \setminus J_n(r_n^{-1}\mathcal{X}_n) \neq \varnothing] \leq n \exp\left(-(\delta/2)^d n^{1/8}\right). \tag{6.17}$$

Suppose $n \in \mathbb{N} \setminus \mathcal{N}$, so that $nr_n^d \leq n^{1/8}$. Then $|[\ell_n] \setminus J_n(r_n^{-1}\mathcal{P}_n)|$ is binomial with parameters ℓ_n and q_n, where we set $q_n := \exp(-n\delta_n^d r_n^d)$ so $q_n \to 0$ as $n \to \infty$. Therefore, similarly to (6.13) and using (6.6), we have for all large enough $n \in \mathbb{N}\setminus\mathcal{N}$ that

$$\begin{aligned} \mathbb{P}[|[\ell_n] \setminus J_n(r_n^{-1}\mathcal{P}_n)| \geq \delta r_n^{-d}] &\leq \exp(-(\delta r_n^{-d}/2)\log[\delta r_n^{-d}/(\ell_n q_n)]) \\ &\leq \exp(-n^{7/8}). \end{aligned} \tag{6.18}$$

By Lemma 6.4, we have for all large enough $n \in \mathbb{N} \setminus \mathcal{N}$ that

$$\mathbb{P}[r_n^d \zeta(r_n^{-1}\mathcal{P}_n) < (1+d\delta)^{-d}\overline{\kappa} - \delta] \leq \mathbb{P}[|[\ell_n] \setminus J_n(r_n^{-1}\mathcal{P}_n)| \geq \delta r_n^{-d}]. \tag{6.19}$$

As before, we have (6.15) for all n, and $Kr_n^d n^{3/4} \leq Kn^{-1/8} \leq \delta$ for all large enough $n \in \mathbb{N} \setminus \mathcal{N}$. Hence by (6.18) and (6.19), for all large enough $n \in \mathbb{N} \setminus \mathcal{N}$ we have

$$\mathbb{P}[r_n^d \zeta_{r_n}(\mathcal{X}_n) < (1+d\delta)^{-d}\overline{\kappa} - 2\delta] \leq \exp(-n^{7/8}) + 2\exp(-n^{1/2}/3).$$

Together with (6.16) and (6.17), this shows that

$$\sum_{n=1}^{\infty} \mathbb{P}[r_n^d \zeta_{r_n}(\mathcal{X}_n) < (1+d\delta)^{-d}\overline{\kappa} - 2\delta] < \infty.$$

Combined with Lemmas 6.5 and 6.6 and using the Borel-Cantelli lemma, since $\delta > 0$ can be arbitrarily small this shows that $r_n^d \zeta_{r_n}(\mathcal{X}_n) \to \overline{\kappa}$ almost surely, which gives us (3.5) as required. □

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