A geometric Ginzburg-Landau problem

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Abstract

For surfaces embedded in a three-dimensional Euclidean space, consider a functional consisting of two terms: a version of the Willmore energy and an anisotropic area penalising the first component of the normal vector, the latter weighted with the factor $1/\epsilon^2$. The asymptotic behaviour of such functionals as ϵ tends to 0 is studied in this paper. The results include a lower and an upper bound on the minimal energy subject to suitable constraints. Moreover, for embedded spheres, a compactness result is obtained under appropriate energy bounds.

1 Introduction

1.1 Background

Given a closed surface M, smoothly embedded in $\mathbb{R}^3$, let ν denote its normal vector and A its second fundamental form. Let $\mathcal{H}^n$ denote the n-dimensional Hausdorff measure. For $\epsilon > 0$, consider the integral

$$\frac{1}{2} \int_M \left(|A|^2 + \frac{\nu_1^2}{\epsilon^2} \right) d\mathcal{H}^2.$$

This is the sum of a quantity closely related to the Willmore energy (an overview of which is given by Willmore [17]) and an anisotropic area functional. We study the behaviour of this functional, and of surfaces with reasonably small energy, as the parameter ϵ tends to 0.

Functionals of this type have been proposed as models for the free energy of crystal surfaces [10, 1, 6, 15, 8, 9]. The anisotropy reflects the underlying crystal structure in this context, and the inclusion of the curvature term is motivated sometimes by its regularisation properties and sometimes on physical grounds. A one-dimensional version (with curves rather than surfaces) has also been studied for its applications in image processing [3, 4, 5].

Some tools for the analysis of a similar problem have been developed in a previous paper [13]. They are motivated by the theory of Modica and Mortola [11, 12] (and others) on models of phase transitions, and they work only for an anisotropy given by a potential function $\Psi : S^2 \to \mathbb{R}$ with isolated minima. In this situation, it turns out that a sequence of surfaces satisfying a suitable

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energy bound always subconverges to a generalised polyhedron. A lower bound for the limiting energy can also be derived, and it depends linearly on the lengths of the edges.

In contrast, the functional studied here involves the function $\Psi(\nu) = \nu_1^2$, which attains its minimum on the whole equator $\{0\} \times S^1$. At a first glance, this problem looks closer to the theory of Ginzburg-Landau vortices, which was first developed by Bethuel, Brezis, and Hélein [2]. But despite the formal similarities, we will see that the analogy between these two problems is limited, too. Above all, the interesting phenomena occur at different scaling regimes for the two energies. Ginzburg-Landau vortices have a typical energy of order $|\log \epsilon|$. For the geometric problem, unless we allow the surfaces to shrink to a point in the limit, we need at least an energy of order $\epsilon^{-1/2}$. It is therefore convenient to renormalise the functional and define

$$E_\epsilon(M) = \frac{\sqrt{\epsilon}}{2} \int_M \left(|A|^2 + \frac{\nu_1^2}{\epsilon^2} \right) d\mathcal{H}^2.$$

Provided that this quantity remains bounded, and assuming that we have topological spheres, we will see that they approach a line segment parallel to the x_1-axis in the Hausdorff distance sense. If L is the length of the line segment, then we find a limiting energy of order $\sqrt{L}$. When we study connected surfaces of arbitrary genus, then we still find estimates for the energy indicating a similar behaviour.

1.2 Main results

For simplicity, we now assume that we have a surface $M \subset \mathbb{R}^3$ that is not only closed and smoothly embedded, but also connected. This is not a significant restriction, because in the case of a disconnected surface, we can apply the theory to each connected component individually; on the other hand, the assumption simplifies the presentation of the results.

We first note that the two terms in the definition of E_ϵ behave differently when we scale M. Therefore, the asymptotic behaviour of the energy will depend on the size of the surface. There are of course many ways to measure its size, but for our purpose the extension of M in the direction of the x_1-axis is the most relevant quantity. We define

$$\Lambda(M) = \sup_{x,y \in M} (y_1 - x_1).$$

Let Γ denote the traditional Γ-function. Furthermore, let

$$a = \frac{\Gamma(7/6)}{\Gamma(2/3)}, \quad b = \frac{\Gamma(13/18)}{\Gamma(2/9)} \quad \text{and} \quad c = \frac{2\pi^{5/4}}{\sqrt{3}} \left(\frac{6b}{\sqrt{a}} - \sqrt{a} \right). \tag{1}$$

Then we have the following estimates.

Theorem 1.1. *For any $L > 0$,*

$$2^{3/4}\pi\sqrt{L} \le \liminf_{\epsilon \searrow 0} \inf_{\Lambda(M) \ge L} E_\epsilon(M) \le \limsup_{\epsilon \searrow 0} \inf_{\Lambda(M) \ge L} E_\epsilon(M) \le c\sqrt{L}.$$

A numerical evaluation gives $2^{3/4}\pi \approx 5.28351$ and $c \approx 6.77995$. We will prove the last inequality by a construction involving surfaces of revolution. More precisely, they are obtained from functions $u : [-L/2, L/2] \to [0, \infty)$ with $u(-L/2) = u(L/2) = 0$ through a rotation of the graph about the x_1-axis. We find a relation between the energy of such a surface and the functional

$$F(u) = \pi \int_{-L/2}^{L/2} \left(\frac{1}{u} + u(u')^2 \right) dt,$$

and the constant c is the minimum of F on a suitable space of functions. In particular, it is the optimal value found with this method.

Since the limiting energy is of order $\sqrt{L}$, we see that it is energetically unfavourable for a connected surface to split into several parts (while conserving the total extension in the x_1-direction) if ϵ is small. This observation provides further justification for restricting our attention to connected surfaces.

We now consider a variant of the problem. Let H be the mean curvature vector of the surface M and suppose that K is its Gauss curvature and $\chi(M)$ its Euler characteristic. Using the identity

$$|H|^2 - |A|^2 = 2K$$

and the Gauss-Bonnet formula, we can express $E_\epsilon(M)$ in the form

$$E_\epsilon(M) = \frac{\sqrt{\epsilon}}{2} \int_M \left(|H|^2 + \frac{\nu_1^2}{\epsilon^2} \right) d\mathcal{H}^2 - 2\pi\sqrt{\epsilon}\chi(M).$$

If we have suitable control of the Euler characteristic, then the asymptotic behaviours of E_ϵ and the functional

$$E_\epsilon^*(M) = \frac{\sqrt{\epsilon}}{2} \int_M \left(|H|^2 + \frac{\nu_1^2}{\epsilon^2} \right) d\mathcal{H}^2$$

are the same. In particular, the estimates from Theorem 1.1 then apply to E_ϵ^* as well. But as one of the estimates used in the proof relies on the mean curvature rather than the second fundamental form, we obtain a better result under certain assumptions on the Euler characteristics.

Theorem 1.2. *Let $L > 0$. For every $\epsilon > 0$, let $M_\epsilon \subset \mathbb{R}^3$ be a closed, connected, smoothly embedded surface with $\Lambda(M) \geq L$. If $\lim_{\epsilon \searrow 0} \sqrt{\epsilon}\chi(M_\epsilon) = 0$, then*

$$2\pi\sqrt{L} \leq \liminf_{\epsilon \searrow 0} E_\epsilon(M_\epsilon). \tag{2}$$

The surfaces used for the proof of Theorem 1.1 are topological spheres. Thus there exists a sequence of surfaces satisfying the hypothesis, such that in addition to the inequality of the theorem,

$$\limsup_{\epsilon \searrow 0} E_\epsilon(M_\epsilon) \leq c\sqrt{L}, \tag{3}$$

where c is the constant from (1).

For the final main result we only consider topological spheres. This is for technical reasons. When we further assume that we have uniformly bounded energies, then we obtain convergence of a subsequence up to translation. The limit is always a line segment parallel to the x_1-axis.

Theorem 1.3. *For every $\epsilon > 0$, suppose that $M_\epsilon \subset \mathbb{R}^3$ is a smoothly embedded sphere such that*

$$\limsup_{\epsilon \searrow 0} E_\epsilon(M_\epsilon) < \infty.$$

Then there exist a sequence $\epsilon_k \searrow 0$, a sequence of points $x_k \in \mathbb{R}^3$, and a number $\ell \geq 0$ such that $M_{\epsilon_k} - x_k$ converges in Hausdorff distance to the set $\{(t, 0, 0) : -\ell \leq t \leq \ell\}$.

Note that Theorem 1.2 then implies

$$2\pi\sqrt{L} \leq \liminf_{k\to\infty} E_{\epsilon_k}(M_{\epsilon_k})$$

for $L = 2\ell$. If we consider the functionals

$$\mathcal{E}_\epsilon(M) = \begin{cases} E_\epsilon(M) & \text{if } M \text{ is a smoothly embedded sphere,} \\ \infty & \text{else,} \end{cases}$$

on the space of compact subsets of $\mathbb{R}^3$, then we may interpret the combination of Theorems 1.2 and 1.3 as a step towards Γ-convergence of $\mathcal{E}_\epsilon$ with respect to the Hausdorff distance. The expected limit functional is of the form

$$\mathcal{E}(M) = \begin{cases} c_0\sqrt{L} & \text{if } M \text{ is a line segment of length } L \text{ parallel to the } x_1\text{-axis,} \\ \infty & \text{else,} \end{cases}$$

with $2\pi \leq c_0 \leq c$. But in order to obtain a complete proof of Γ-convergence, we would need matching constants in (2) and (3).

2 A construction

The purpose of this section is to prove the last inequality in Theorem 1.1. We first note that the reason for the appearance of $\sqrt{L}$ in this formula is the scaling behaviour of E_ϵ. Suppose that M is a closed, connected, smoothly embedded surface in $\mathbb{R}^3$ and $\epsilon > 0$. For $\lambda > 0$, consider $M_\lambda = \lambda M$ and $\epsilon_\lambda = \lambda\epsilon$. Then we compute

$$E_{\epsilon_\lambda}(M_\lambda) = \sqrt{\lambda} E_\epsilon(M).$$

Thus in order to prove Theorem 1.1, it suffices to consider one specific value for L. For reasons that we will see later, it is convenient to choose

$$L = \frac{4\sqrt{\pi}\Gamma(7/6)}{3\Gamma(2/3)}.$$

We want to construct a sequence of surfaces M_ϵ with $\Lambda(M_\epsilon) = L$, such that

$$\limsup_{\epsilon \searrow 0} E_\epsilon(M_\epsilon) \leq c\sqrt{L}$$

for the constant c defined in (1).

2.1 Surfaces of revolution

Fix an $\ell > 0$ and consider a surface of revolution generated by a function $u \in C^\infty(-\ell,\ell) \cap C^0([-\ell,\ell])$ with $u > 0$ in $(-\ell,\ell)$ and $u(-\ell) = u(\ell) = 0$. We also assume that

$$\lim_{t\to\pm\ell} u'(t) = \mp\infty.$$

The corresponding surface is then

$$M = \left\{x \in [-\ell,\ell] \times \mathbb{R}^2 : x_2^2 + x_3^2 = (u(x_1))^2\right\}.$$

The above assumptions do not imply that M is smooth at the points $(\pm\ell,0,0)$, but it is readily seen that the singularities can be smoothed out while increasing the energy by an arbitrarily small amount. We extend E_ϵ in the obvious way so that surfaces of this type are included in its domain.

We then calculate the first component of the normal vector and the norm of the second fundamental form of M, namely

$$\nu_1 = -\frac{u'}{\sqrt{1+(u')^2}}, \quad |A|^2 = \frac{1}{u^2(1+(u')^2)} + \frac{(u'')^2}{(1+(u')^2)^3}.$$

Therefore,

$$E_\epsilon(M) = \pi\sqrt{\epsilon}\int_{-\ell}^{\ell}\left(\frac{u^{-1}+\epsilon^{-2}u(u')^2}{\sqrt{1+(u')^2}} + \frac{u(u'')^2}{(1+(u')^2)^{5/2}}\right)dt.$$

When we let $\epsilon \searrow 0$, the functions generating our surfaces must of course depend on ϵ. We use the ansatz $u_\epsilon = \sqrt{\epsilon}v$ for a fixed function v. If M_ϵ denotes the corresponding surface of revolution, then

$$E_\epsilon(M_\epsilon) = \pi\int_{-\ell}^{\ell}\left(\frac{v^{-1}+v(v')^2}{\sqrt{1+\epsilon(v')^2}} + \frac{\epsilon^2 v(v'')^2}{(1+\epsilon(v')^2)^{5/2}}\right)dt.$$

Fix $\sigma \in (0,2]$. Using the estimates $1+\epsilon(v')^2 \geq 1$ and $1+\epsilon(v')^2 \geq \epsilon(1+(v')^2)$ simultaneously, we find

$$E_\epsilon(M_\epsilon) \leq \pi\int_{-\ell}^{\ell}\left(\frac{1}{v}+v(v')^2\right)dt + \epsilon^\sigma\pi\int_{-\ell}^{\ell}\frac{v(v'')^2}{((v')^2+1)^{2-\sigma}}\,dt$$

when $\epsilon \leq 1$. Provided that

$$\int_{-\ell}^{\ell}\frac{v(v'')^2}{((v')^2+1)^{2-\sigma}}\,dt < \infty, \tag{4}$$

this implies

$$\limsup_{\epsilon\searrow 0} E_\epsilon(M_\epsilon) \leq \pi\int_{-\ell}^{\ell}\left(\frac{1}{v}+v(v')^2\right)dt. \tag{5}$$

2.2 Solving an auxiliary problem

These observations show that for any function v that satisfies the relevant assumptions, including (4), we obtain an upper bound similar to the last estimate in Theorem 1.1. In order to make the most of this information, we want to minimise the functional

$$F(v) = \pi \int_{-\ell}^{\ell} \left(\frac{1}{v} + v(v')^2 \right) dt$$

in an appropriate class of functions. Thus we have a new variational problem, and its Euler-Lagrange equation is

$$v'' + \frac{(v')^2}{2v} + \frac{1}{2v^3} = 0. \tag{6}$$

We now look for solutions of this equation.

It is convenient, however, to first apply the transformation $w = v^{3/2}$. Defining

$$G(w) = \pi \int_{-\ell}^{\ell} \left(\frac{4}{9}(w')^2 + w^{-2/3} \right) dt,$$

we then have $F(v) = G(w)$. Moreover, equation (6) is equivalent to

$$w'' + \frac{3}{4} w^{-5/3} = 0. \tag{7}$$

Suppose that $\ell > 0$ and that w is a solution of (7) in $(-\ell, \ell)$ with $w > 0$. Multiply both sides of the equation with w'. Then we find

$$\frac{d}{dt} \left(\frac{4}{9}(w')^2 - w^{-2/3} \right) = 0.$$

Thus there exists a constant γ such that

$$\frac{4}{9}(w')^2 - w^{-2/3} = \gamma. \tag{8}$$

Motivated by the underlying geometric problem, and expecting symmetry, we make the ansatz $w'(0) = 0$ and $w'(t) \leq 0$ for $t > 0$. In order to single out one specific solution, we also assume that $w(0) = 1$, which implies $\gamma = -1$. In the interval $[0, \ell)$, we then have

$$w' = -\frac{3}{2} \sqrt{w^{-2/3} - 1}.$$

We can solve this equation using separation of variables. Let

$$g(s) = \int_0^s \frac{dr}{\sqrt{r^{-3/2} - 1}}.$$

Then g is invertible on the interval $[0, 1]$ and we have

$$w(t) = g^{-1} \left(g(1) - \frac{3t}{2} \right).$$

If

$$\ell = \frac{2}{3} g(1) = \frac{2}{3} \int_0^1 \frac{dr}{\sqrt{r^{-3/2} - 1}},$$

then we have $w(\ell) = 0$. Because w solves (8), we also obtain

$$\lim_{t \nearrow \ell} w'(t) = -\infty.$$

We extend w as an even function to $[-\ell, \ell]$, and it remains a solution of (7).

2.3 Condition (4) is satisfied

Now we set $v = w^{2/3}$. This is a solution of (6), and potentially a good choice for the construction in section 2.1. We need to verify, however, that v satisfies (4) before we can draw any conclusions.

Note that

$$\lim_{s \searrow 0} s^{3/4} \sqrt{s^{-3/2} - 1} = 1.$$

By l'Hôpital's rule,

$$\lim_{s \searrow 0} \frac{4s^{7/4}}{7g(s)} = \lim_{s \searrow 0} \frac{s^{3/4}}{g'(s)} = 1.$$

Hence

$$\lim_{t \searrow 0} \frac{g^{-1}(t)}{(7t/4)^{4/7}} = \lim_{s \searrow 0} \frac{g^{-1}(g(s))}{(7g(s)/4)^{4/7}} = \lim_{s \searrow 0} \left(\frac{4s^{7/4}}{7g(s)} \right)^{4/7} = 1.$$

In particular, we have

$$w(t) \geq (\ell - t)^{4/7}$$

in a neighbourhood of ℓ.

Now consider the function $v = w^{2/3}$, which is a solution of (6). We have

$$v' = \frac{2}{3} w^{-1/3} w' = -w^{-1/3} \sqrt{w^{-2/3} - 1}$$

for $t > 0$, and

$$v'' = -\frac{1}{2} \left(\frac{(v')^2}{v} + \frac{1}{v^3} \right) = -\frac{1}{2} \left(w^{-4/3} (w^{-2/3} - 1) + w^{-2} \right) = -w^{-2} + \frac{1}{2} w^{-4/3}.$$

If $\ell - t$ is sufficiently small, then

$$|v''| \leq 2w^{-2} \quad \text{and} \quad (v')^2 + 1 \geq \frac{1}{2} w^{-4/3}.$$

Thus

$$\frac{v(v'')^2}{((v')^2 + 1)^{2-\sigma}} \leq 2^{4-\sigma} w^{-\frac{2+4\sigma}{3}} \leq 2^{4-\sigma} (\ell - t)^{-\frac{8+16\sigma}{21}}.$$

As long as $\sigma < \frac{13}{16}$, we find that (4) is indeed satisfied. This means that for this choice of the function v, inequality (5) is satisfied as well.

2.4 Calculating the energy

Finally, we want to compute $F(v)$. Or more precisely, we want to show that

$$c = \frac{F(v)}{\sqrt{2\ell}} \tag{9}$$

for the number c defined in (1).

First we note that by the symmetry and by (8),

$$G(w) = 2\pi \int_0^\ell \left(2w^{-2/3} - 1\right) dt = 4\pi \int_0^\ell \left(g^{-1}\left(\frac{3(\ell - t)}{2}\right)\right)^{-2/3} dt - 2\pi\ell.$$

Using the substitution $s = g^{-1}(3(\ell - t)/2)$, we find

$$\int_0^\ell \left(g^{-1}\left(\frac{3(\ell - t)}{2}\right)\right)^{-2/3} dt = \frac{2}{3}\int_0^1 s^{-2/3} g'(s)\, ds = \frac{2}{3}\int_0^1 \frac{ds}{s^{2/3}\sqrt{s^{-2/3} - 1}}.$$

Thus

$$F(v) = G(w) = \frac{8\pi}{3}\int_0^1 \frac{ds}{s^{2/3}\sqrt{s^{-2/3} - 1}} - 2\pi\ell,$$

and recall that

$$\ell = \frac{2}{3}\int_0^1 \frac{ds}{\sqrt{s^{-3/2} - 1}}.$$

Next we want to evaluate these integrals. To this end, we use the traditional B- (beta-) and Γ-functions

$$\mathrm{B}(p, q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\, dt, \quad p, q > 0,$$

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\, dt, \quad p > 0,$$

satisfying the well-known relations $\Gamma(p+1) = p\Gamma(p)$ and

$$\mathrm{B}(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Moreover, we know that $\Gamma(1/2) = \sqrt{\pi}$.

If $\phi : [0, 1) \to [0, \infty)$ is a continuous, increasing function with $\phi(0) = 0$ and $\lim_{s \nearrow 1} \phi(s) = \infty$, then Fubini's theorem implies

$$\int_0^1 \phi(s)\, ds = \int_0^1 \int_0^{\phi(s)} 1\, dr\, ds = \int_0^\infty \int_{\phi^{-1}(r)}^1 1\, ds\, dr = \int_0^\infty \left(1 - \phi^{-1}(r)\right) dr.$$

Thus

$$\int_0^1 \frac{ds}{\sqrt{s^{-3/2} - 1}} = \int_0^\infty \left(1 - \frac{r^{4/3}}{(r^2+1)^{2/3}}\right) dr.$$

We have

$$\frac{d}{dr}\left(1 - \frac{r^{4/3}}{(r^2+1)^{2/3}}\right) = -\frac{4}{3}\frac{r^{1/3}}{(r^2+1)^{5/3}}.$$

Hence an integration by parts gives

$$\int_0^1 \frac{ds}{\sqrt{s^{-3/2}-1}} = \frac{4}{3}\int_0^\infty \frac{r^{4/3}}{(r^2+1)^{5/3}}\,dr.$$

The substitution $\rho = r^2$ finally yields

$$\int_0^1 \frac{ds}{\sqrt{s^{-3/2}-1}} = \frac{2}{3}\int_0^\infty \frac{\rho^{1/6}}{(1+\rho)^{5/3}}\,d\rho = \frac{2}{3}\mathrm{B}(7/6,1/2) = \frac{\sqrt{\pi}\Gamma(7/6)}{\Gamma(2/3)}.$$

With the same method, we obtain

$$\int_0^1 \frac{ds}{s^{2/3}\sqrt{s^{-3/2}-1}} = \frac{3\sqrt{\pi}\Gamma(13/18)}{\Gamma(2/9)}.$$

Hence

$$\frac{F(v)}{\sqrt{2\ell}} = \frac{2\pi^{5/4}}{\sqrt{3}}\left(\frac{6b}{\sqrt{a}} - \sqrt{a}\right),$$

where

$$a = \frac{\Gamma(7/6)}{\Gamma(2/3)} \quad \text{and} \quad b = \frac{\Gamma(13/18)}{\Gamma(2/9)}.$$

That is, identity (9) is true.

The rest of the proof of Theorem 1.1 is rather different from the arguments used here, and so before we continue, we summarise the results so far. We have constructed a family of surfaces of revolution M_ϵ with $\Lambda(M_\epsilon) = L = 2\ell$, such that

$$\limsup_{\epsilon\searrow 0} E_\epsilon(M_\epsilon) \leq c\sqrt{L}$$

for the number c defined in (1). Although M_ϵ is not necessarily smooth, it is regular enough so that the same conclusion can be drawn for appropriate smooth surfaces approximating M_ϵ.

3 Slicing

In the rest of the paper, we study an arbitrary closed, connected, smoothly embedded surface $M \subset \mathbb{R}^3$. Most of the estimates that we use involve intersections of M with planes perpendicular to the x_1-axis. In this section we give a preliminary discussion of these slices.

3.1 Regular surfaces

Definition 3.1. *Suppose that $M \subset \mathbb{R}^3$ is a closed, connected, smoothly embedded surface with normal vector ν. Then for $t \in \mathbb{R}$, we define*

$$M_t = M \cap (\{t\} \times \mathbb{R}^2).$$

We say that $t \in \mathbb{R}$ is regular (for M) if the intersection is transversal everywhere on M_t. We say that M is regular if $\nu^{-1}(\{(\pm 1, 0, 0)\})$ is finite.

For a regular surface M, it follows that there exist only finitely many irregular values for t. For the problems studied in this paper, it suffices to consider regular surfaces.

Proposition 3.1. *Let $M \subset \mathbb{R}^3$ be a closed, connected, smoothly embedded surface and $\epsilon > 0$. Then there exists a sequence of regular surfaces M_k such that $E_\epsilon(M_k) \to E_\epsilon(M)$ and $M_k \to M$ in the Hausdorff distance sense as $k \to \infty$.*

Proof. Let ν be the normal vector of M. By the area formula, we have

$$\int_{S^2} \mathcal{H}^0(\nu^{-1}(\{\eta\}))\, d\mathcal{H}^2(\eta) < \infty.$$

Thus in any neighbourhood of $(1,0,0)$, there exists an η such that $\nu^{-1}(\{\pm\eta\})$ is finite. We can obtain the surfaces M_k by applying suitable rotations to M. $\square$

From now on, we focus on regular surfaces. It is also convenient to fix $\Lambda(M)$, say $\Lambda(M) = 2\ell$, and then we may assume that $M \subset [-\ell, \ell] \times \mathbb{R}^2$.

Definition 3.2. *Let $\ell > 0$. The set $\mathcal{M}_\ell$ consists of all closed, connected, smoothly embedded surfaces $M \subset \mathbb{R}^3$ such that M is regular, $\Lambda(M) = 2\ell$, and $M \subset [-\ell, \ell] \times \mathbb{R}^2$.*

3.2 Notation and preliminaries

Now let $\ell > 0$ and $M \in \mathcal{M}_\ell$. For a regular $t \in (-\ell, \ell)$, we introduce some notation. First, the map $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$ is the projection onto the (x_2, x_3)-plane:

$$\Pi(x_1, x_2, x_3) = (x_2, x_3).$$

As M_t is the result of a transversal intersection, it is a compact, smooth, one-dimensional submanifold of the plane $\{t\} \times \mathbb{R}^2$; in other words, M_t is a union of smooth Jordan curves. Moreover, it has a well-defined curvature κ_t. We use a sign convention such that $\kappa_t < 0$ if M_t is convex.

When we vary t, then M_t remains smooth in a neighbourhood of a regular point in $(-\ell, \ell)$. Interpreting t as time, we obtain a smoothly evolving submanifold $\Pi(M_t)$ in $\mathbb{R}^2$, and we can define the normal velocity v_t of the evolution. We choose the sign of v_t such that $v_t > 0$ for an expansion.

Fix a regular point t again. Suppose that f is a differentiable function on M and s denotes an arc length parameter locally on M_t. We introduce the quantities $\dot{f}$ and f', informally defined as

$$\dot{f} = \frac{\partial f}{\partial s} \quad \text{and} \quad f' = \frac{\partial f}{\partial t}.$$

More precisely, if γ is a parametrisation of a piece of M_t by arc length, then

$$\dot{f}(\gamma(s)) = \frac{d}{ds} f(\gamma(s)).$$

This leaves some ambiguity about the sign, which we will remove in a moment by fixing the orientation of γ. In order to give another representation of f', we can extend f to a neighbourhood of M in $\mathbb{R}^3$ such that $\nabla_\nu f = 0$ (i.e., the directional derivative of f in the normal direction vanishes), and then $f' = \frac{\partial f}{\partial x_1}$.

Locally on M_t, we can use the unit vectors $e^{(1)}$ and $e^{(2)}$, where $e^{(1)} = \dot{\gamma}$ and $e^{(2)} = \nu \times e^{(1)}$. Assuming that γ is oriented such that $e_1^{(2)} > 0$, we have

$$\nabla_{e^{(2)}} f = \sqrt{1 - \nu_1^2}\, f'.$$

Of course, we also have $\dot{f} = \nabla_{e^{(1)}} f$. In particular,

$$|A|^2 = |\nabla_{e^{(1)}} \nu|^2 + |\nabla_{e_{(1)}} \nu|^2 = |\dot{\nu}|^2 + (1 - \nu_1^2)|\nu'|^2.$$

Furthermore,

$$|\kappa_t| = |\nabla_{e^{(1)}} e^{(1)}| = |\dot{e}^{(1)}| = \frac{|\langle \nu, \dot{e}^{(1)} \rangle|}{\sqrt{1 - \nu_1^2}} \leq \frac{|A|}{\sqrt{1 - \nu_1^2}}$$

and

$$v_t = -\frac{\nu_1}{\sqrt{1 - \nu_1^2}}.$$

Note also that we have

$$\int_M f \, d\mathcal{H}^2 = \int_{-\ell}^{\ell} \int_{M_t} \frac{f}{\sqrt{1 - \nu_1^2}} \, d\mathcal{H}^1 \, dt.$$

Finally, we calculate

$$\frac{d}{dt} \int_{M_t} f \, d\mathcal{H}^1 = \int_{M_t} (f' - f v_t \kappa_t) \, d\mathcal{H}^1.$$

3.3 A simple estimate

We can immediately give an estimate for the length of the part of M_t where ν_1 is small.

Lemma 3.1. *Let $\ell > 0$ and $M \in \mathcal{M}_\ell$. For every $t \in [-\ell, \ell]$,*

$$\int_{M_t} (1 - \nu_1^2) \, d\mathcal{H}^1 \leq \frac{3}{2} \sqrt{\epsilon} E_\epsilon(M).$$

Proof. We compute

$$\frac{d}{dt} \int_{M_t} (1 - \nu_1^2) \, d\mathcal{H}^1 = -\int_{M_t} (2\nu_1 \nu_1' + (1 - \nu_1^2) \kappa_t v_t) \, d\mathcal{H}^1$$

for every regular $t \in (-\ell, \ell)$. Using the previous calculations, we obtain

$$(1 - \nu_1^2)|\kappa_t v_t| \leq |\nu_1||\dot{\nu}| \leq \frac{|A||\nu_1|}{\sqrt{1 - \nu_1^2}}$$

and

$$\left| \frac{d}{dt} \int_{M_t} (1 - \nu_1^2) \, d\mathcal{H}^1 \right| \leq \frac{3\epsilon}{2} \int_{M_t} \frac{|A|^2 + \epsilon^{-2} \nu_1^2}{\sqrt{1 - \nu_1^2}} \, d\mathcal{H}^1.$$

The function

$$t \mapsto \int_{M_t} (1 - \nu_1^2) \, d\mathcal{H}^1$$

is continuous even at an irregular t. As $\mathcal{H}^1(M_{-\ell}) = \mathcal{H}^1(M_\ell) = 0$, we have

$$\begin{aligned} \int_{M_t} (1 - \nu_1^2) \, d\mathcal{H}^1 &\leq \frac{3\epsilon}{2} \int_{-\ell}^{t} \int_{M_s} \frac{|A|^2 + \epsilon^{-2} \nu_1^2}{\sqrt{1 - \nu_1^2}} \, d\mathcal{H}^1 \, ds \\ &= \frac{3\epsilon}{2} \int_{\{x \in M : x_1 \leq t\}} \left(|A|^2 + \frac{\nu_1^2}{\epsilon^2} \right) \, d\mathcal{H}^2 \end{aligned}$$

and also

$$\int_{M_t} (1-\nu_1^2)\, d\mathcal{H}^1 \leq \frac{3\epsilon}{2} \int_{\{x\in M : x_1 \geq t\}} \left(|A|^2 + \frac{\nu_1^2}{\epsilon^2}\right) d\mathcal{H}^2$$

for every $t \in [-\ell, \ell]$. The smaller of these two numbers gives rise to the desired inequality. □

4 The lower bound

We now derive the first estimate in Theorem 1.1. The proof relies on the next two lemmas.

4.1 Preliminary estimates

Let $\ell > 0$. Throughout this section, we consider a fixed regular surface $M \in \mathcal{M}_\ell$ with normal vector ν and second fundamental form A. Its mean curvature vector is denoted by H.

Lemma 4.1. *The inequality*

$$\mathcal{H}^2(M) \leq \left(\ell\sqrt{2\epsilon} + 2\epsilon^{3/2}\right) E_\epsilon(M)$$

holds true.

Proof. Consider the vector field $\Phi(x) = (x_1, 0, 0)$ on $\mathbb{R}^3$. Writing id for the identity (3×3)-matrix, we calculate

$$\operatorname{div}_M \Phi = \operatorname{trace}\left((\mathrm{id} - \nu\otimes\nu)\nabla\Phi\right) = 1 - \nu_1^2$$

for the divergence of Φ with respect to M. We have the integration by parts formula

$$\int_M \operatorname{div}_M \Phi\, d\mathcal{H}^2 = -\int_M \langle \Phi, H\rangle\, d\mathcal{H}^2.$$

Note that $|H|^2 \leq 2|A|^2$. Hence

$$\int_M (1-\nu_1^2)\, d\mathcal{H}^2 \leq \sqrt{2}\int_M |x_1||\nu_1||A|\, d\mathcal{H}^2 \leq \frac{\epsilon\ell}{\sqrt{2}} \int_M \left(|A|^2 + \frac{\nu_1^2}{\epsilon^2}\right) d\mathcal{H}^2.$$

The desired inequality follows. □

Remark. With exactly the same method, we also obtain

$$\mathcal{H}^2(M) \leq \left(\ell\sqrt{\epsilon} + 2\epsilon^{3/2}\right) E_\epsilon^*(M). \tag{10}$$

The next statement is a variant of a well-known inequality [14]. (A more general discussion of similar inequalities is given by Topping [16].) We provide the details of the proof, although there is nothing essentially new here.

Lemma 4.2. *Let $\eta \in S^2$ and suppose that $J \subset \mathbb{R}$ is a measurable set. Let $\tilde{M} = \{x \in M : \langle x, \eta\rangle \in J\}$. Then*

$$4\pi^2(\mathcal{H}^1(J))^2 \leq \mathcal{H}^2(\tilde{M}) \int_{\tilde{M}} |A|^2\, d\mathcal{H}^2.$$

In particular,

$$16\pi^2\ell^2 \le \mathcal{H}^2(M)\int_M |A|^2\, d\mathcal{H}^2.$$

Proof. We assume that $\eta = (1,0,0)$, because we already have the appropriate notation for this case. Any other case can be reduced to this situation by a rotation.

Fix a regular $t \in (-\ell,\ell)$. Let $\mathcal{C}$ be a connected component of M_t. Then

$$\int_{\mathcal{C}} \frac{|A|}{\sqrt{1-\nu_1^2}}\, d\mathcal{H}^1 \ge \int_{\mathcal{C}} |\kappa_t|\, d\mathcal{H}^1 \ge 2\pi. \tag{11}$$

Now we obtain

$$2\pi\mathcal{H}^1(J) \le \int_J\int_{M_t} \frac{|A|}{\sqrt{1-\nu_1^2}}\, d\mathcal{H}^1\, dt = \int_{\tilde{M}} |A|\, d\mathcal{H}^2 \le \left(\mathcal{H}^2(\tilde{M})\int_{\tilde{M}} |A|^2\, d\mathcal{H}^2\right)^{\frac{1}{2}}.$$

Finally we take the square on both sides. The second inequality follows when we choose $J = [-\ell,\ell]$. ☐

4.2 Proofs of Theorems 1.1 and 1.2

We have already seen that the last inequality in Theorem 1.1 is a consequence of the construction in section 2.

For the first inequality, it suffices to show that

$$2^{3/4}\pi\sqrt{L} \le \liminf_{\epsilon\searrow 0}\ \inf_{\mathcal{M}_{L/2}} E_\epsilon$$

by Proposition 3.1. Let $\ell = L/2$ and fix $M \in \mathcal{M}_\ell$. Combining Lemma 4.1 and Lemma 4.2, we then infer

$$16\pi^2\ell^2 \le \mathcal{H}^2(M)\int_M |A|^2\, d\mathcal{H}^2 \le \left(2\sqrt{2}\ell + 4\epsilon\right)(E_\epsilon(M))^2.$$

Thus

$$E_\epsilon(M) \ge \frac{2^{5/4}\pi\ell}{\sqrt{\ell+\sqrt{2}\epsilon}},$$

and the desired inequality in Theorem 1.1 follows as $\epsilon \searrow 0$.

For the proof of Theorem 1.2, we use exactly the same method, but instead of Lemma 4.1, we use inequality (10). This then gives

$$16\pi^2\ell^2 \le (2\ell+4\epsilon)\, E_\epsilon(M_\epsilon)E_\epsilon^*(M_\epsilon).$$

Under the hypothesis on the Euler characteristics in Theorem 1.2, we have

$$\lim_{\epsilon\searrow 0} \frac{E_\epsilon(M_\epsilon)}{E_\epsilon^*(M_\epsilon)} = 1,$$

and so the required inequality follows.

5 A decomposition

For the proof of Theorem 1.3, it suffices to study regular surfaces as well. Again we fix ℓ and study a fixed surface $M \in \mathcal{M}_\ell$, and as usual we use the symbols ν and A to denote its normal vector and its second fundamental form, respectively.

Under a control of E_ϵ of the form $E_\epsilon(M) \leq C_0\sqrt{L}$, we expect that the typical slice M_t will be a curve with length of order $\sqrt{\epsilon}$ and with a shape not too dissimilar to a circle; or possibly the union of several curves of this type, but not too many. But in general there will be slices where this expectation is not fulfilled. The purpose of this section is to separate the 'good' from the 'bad' curves.

5.1 Definitions

Throughout this section, let $\delta \in (0, \frac{1}{4}]$ be a fixed number. Fix a regular $t \in (-\ell, \ell)$. Let $\mathcal{C}_1, \ldots, \mathcal{C}_m$ be the connected components of M_t. Moreover, let a_i be the area of the region enclosed by $\mathcal{C}_i$. Define

$$\begin{aligned}
I_1 &= \{i \in \{1, \ldots, m\} : \nu_1(\mathcal{C}_i) \subset [-1, -\delta/2] \cup [\delta/2, 1]\}\,, \\
I_2 &= \{i \in \{1, \ldots, m\} : i \notin I_1 \text{ and } \nu_1(\mathcal{C}_i) \not\subset [-\delta, \delta]\}\,, \\
I_3 &= \left\{i \in \{1, \ldots, m\} : i \notin I_1 \cup I_2 \text{ and } a_i < \delta(\mathcal{H}^1(\mathcal{C}_i))^2\right\}, \\
I_4 &= \left\{i \in \{1, \ldots, m\} : i \notin I_1 \cup I_2 \cup I_3 \text{ and } \mathcal{H}^1(\mathcal{C}_i) < \delta\sqrt{\epsilon}\right\}.
\end{aligned}$$

These are the indices of curves that we think of as exceptional in some sense: either the intersection of the surface M with the plane $\{t\} \times \mathbb{R}^2$ is not sufficiently transversal on all or part of $\mathcal{C}_i$, or $\mathcal{C}_i$ is not round enough (in terms of the isoperimetric quotient), or it is too short. Now let

$$M_t^j = \bigcup_{i \in I_j} \mathcal{C}_i, \quad j = 1, 2, 3, 4,$$

and

$$M_t^0 = M_t \backslash (M_t^1 \cup M_t^2 \cup M_t^3 \cup M_t^4).$$

Furthermore, let

$$\#_t^j = |I_j|, \quad j = 1, 2, 3, 4$$

(i.e., the number of indices in each category). The quantities M_t^j and $\#_t^j$ are then independent of the labelling of the connected components, and thus well-defined as functions of t. If t is irregular, then we set $M_t^j = \emptyset$ and $\#_t^j = 0$ for every j.

Finally, for $j = 0, \ldots, 4$, let

$$M^j = \bigcup_{-\ell < t < \ell} \{t\} \times M_t^j.$$

The union of $M^0, \ldots, M^4$ then comprises all of M except the irregular slices.

As $|\nu_1| \geq \frac{\delta}{2}$ on M^1, it is quite easy to estimate the area of M^1 in terms of $E_\epsilon(M)$. We now want to estimate the areas of M^2, M^3, and M^4 as well, and for this purpose we estimate the integrals of $\#_t^2$, $\#_t^3$ and $\#_t^4$ first.

5.2 The second exceptional set

Lemma 5.1. *The inequality*

$$\int_{-\ell}^{\ell} \#_t^2 \, dt \leq 4\sqrt{\epsilon}\delta^{-2} E_\epsilon(M)$$

holds true.

Proof. Fix a regular $t \in (-\ell, \ell)$ with $\#_t^2 \neq 0$. Suppose that $\mathcal{C}$ is a connected component of M_t^2. Consider a parametrisation $\gamma : [0, S] \to \mathcal{C}$ by arc length. Then there exist two points $s_1, s_2 \in [0, S]$ with $|\nu_1(\gamma(s_1))| = \frac{\delta}{2}$ and $|\nu_1(\gamma(s_2))| = \delta$, and we may assume that $s_1 < s_2$. Then

$$\frac{\delta^2}{2} \leq 2\int_{s_1}^{s_2} |\nu_1(\gamma(s))||\dot{\nu}_1(\gamma(s))| \, ds \leq 2\int_{\mathcal{C}} \frac{|\nu_1||\dot{\nu}|}{\sqrt{1-\nu_1^2}} \, d\mathcal{H}^1.$$

Hence

$$\int_{\mathcal{C}} \frac{|A|^2 + \epsilon^{-2}\nu_1^2}{\sqrt{1-\nu_1^2}} \, d\mathcal{H}^1 \geq \frac{\delta^2}{2\epsilon}.$$

Treating every connected component similarly, we obtain

$$\int_{M_t^2} \frac{|A|^2 + \epsilon^{-2}\nu_1^2}{\sqrt{1-\nu_1^2}} \, d\mathcal{H}^1 \geq \frac{\delta^2 \#_t^2}{2\epsilon}.$$

On the other hand,

$$\int_{-\ell}^{\ell} \int_{M_t^2} \frac{|A|^2 + \epsilon^{-2}\nu_1^2}{\sqrt{1-\nu_1^2}} \, d\mathcal{H}^1 \, dt = \frac{2E_\epsilon(M)}{\sqrt{\epsilon}},$$

and the claim follows. □

5.3 The third exceptional set

In order to prove a similar estimate for $\#_t^3$, we use an estimate of the isoperimetric quotient.

Lemma 5.2. *For any $\alpha > 0$ there exists a number $\delta_0 > 0$ with the following property. Suppose that $\mathcal{C}$ is a smooth Jordan curve in $\mathbb{R}^2$ with curvature κ and length λ. Let a denote the area of the region enclosed by $\mathcal{C}$. Then either $a \geq \delta_0 \lambda^2$ or*

$$\lambda \int_{\mathcal{C}} \kappa^2 \, d\mathcal{H}^1 \geq \alpha.$$

Remark. Incidentally, the inequality

$$\int_{\mathcal{C}} \kappa^2 \, d\mathcal{H}^1 \geq \frac{\pi\lambda}{a}$$

was proved by Gage [7] for convex curves.

Proof. We may assume that $\lambda = 2\pi$, as the inequalities are invariant under scaling. It is convenient to consider a parametrisation $\gamma : S^1 \to \mathbb{R}^2$ of $\mathcal{C}$ by arc length.

If the statement was false for some $\alpha > 0$, then we could find a sequence of curves with parametrisations $\gamma_k : S^1 \to \mathbb{R}^2$ by arc length, such that the enclosed areas a_k converge to 0, but

$$2\pi \int_{S^1} |\ddot{\gamma}_k|^2 \, ds < \alpha$$

for every $k \in \mathbb{N}$. Thus after a translation, we have a uniform bound for γ_k in the Sobolev space $W^{2,2}(S^1; \mathbb{R}^2)$. Hence we may choose a subsequence that converges weakly in $W^{2,2}(S^1; \mathbb{R}^2)$, and this implies uniform convergence as well by the Sobolev embedding theorem and the Arzelà-Ascoli theorem. Let $\gamma : S^1 \to \mathbb{R}^2$ be the limit. This curve may have self-intersections, but it still decomposes the plane into two (degenerate) regions. Moreover, the inner region has vanishing area. But we also have

$$2\pi \int_{S^1} |\ddot{\gamma}|^2 \, ds \leq \alpha.$$

In particular we have a $C^{1,1/2}$-immersion, and it is clear that no such curve exists. □

Lemma 5.3. *Let $\alpha > 0$. There exists a number $\delta_0 > 0$ such that if $\delta \leq \delta_0$, then*

$$\int_{-\ell}^{\ell} \#_t^3 dt \leq 4\sqrt{\frac{\ell + \epsilon}{\alpha}} E_\epsilon(M).$$

Proof. Choose δ_0 as in Lemma 5.2. Fix a regular t and suppose that $\mathcal{C}$ is a connected component of M_t^3. Then we have

$$\int_{\mathcal{C}} \kappa_t^2 \, d\mathcal{H}^1 \geq \frac{\alpha}{\mathcal{H}^1(\mathcal{C})},$$

provided that $\delta \leq \delta_0$. Thus

$$\frac{\alpha}{\mathcal{H}^1(\mathcal{C})} \leq \int_{\mathcal{C}} \frac{|A|^2}{1 - \nu_1^2} \, d\mathcal{H}^1 \leq \frac{1}{\sqrt{1 - \delta^2}} \int_{\mathcal{C}} \frac{|A|^2}{\sqrt{1 - \nu_1^2}} \, d\mathcal{H}^1.$$

Furthermore,

$$\mathcal{H}^1(\mathcal{C}) \leq \int_{\mathcal{C}} \frac{d\mathcal{H}^1}{\sqrt{1 - \nu_1^2}}.$$

Hence, recalling that we have assumed $\delta \leq \frac{1}{4}$, we find

$$\frac{\sqrt{\alpha}}{2} \leq \left(\int_{\mathcal{C}} \frac{|A|^2}{\sqrt{1 - \nu_1^2}} \, d\mathcal{H}^1 \right)^{1/2} \left(\int_{\mathcal{C}} \frac{d\mathcal{H}^1}{\sqrt{1 - \nu_1^2}} \right)^{1/2}.$$

Summing over all connected components and using the Cauchy-Schwarz inequality, we obtain

$$\frac{\#_t^3 \sqrt{\alpha}}{2} \leq \left(\int_{M_t^3} \frac{|A|^2}{\sqrt{1 - \nu_1^2}} \, d\mathcal{H}^1 \right)^{1/2} \left(\int_{M_t^3} \frac{d\mathcal{H}^1}{\sqrt{1 - \nu_1^2}} \right)^{1/2}.$$

Integrating over $(-\ell,\ell)$ and using Hölder's inequality and Lemma 4.1 now yields

$$\frac{\sqrt{\alpha}}{2}\int_{-\ell}^{\ell}\#_t^3\,dt \le \left(\int_M |A|^2\,d\mathcal{H}^2\right)^{1/2}\left(\mathcal{H}^2(M)\right)^{1/2} \le 2\sqrt{\ell+\epsilon}E_\epsilon(M).$$

The claim follows immediately. □

5.4 The fourth exceptional set

Lemma 5.4. *The inequality*

$$\int_{-\ell}^{\ell}\#_t^4\,dt \le \pi^{-2}\delta E_\epsilon(M)$$

holds true.

Proof. Fix a regular t with $\#_t^4 \neq 0$. Let $\mathcal{C}$ be a connected component of M_t^4. Then we have

$$4\pi^2 \le \left(\int_{\mathcal{C}}\kappa_t\,d\mathcal{H}^1\right)^2 \le \delta\sqrt{\epsilon}\int_{\mathcal{C}}\kappa_t^2\,d\mathcal{H}^1 \le \frac{\delta\sqrt{\epsilon}}{\sqrt{1-\delta^2}}\int_{\mathcal{C}}\frac{|A|^2}{\sqrt{1-\nu_1^2}}\,d\mathcal{H}^1.$$

Hence

$$2\pi^2\#_t^4 \le \delta\sqrt{\epsilon}\int_{M_t^4}\frac{|A|^2}{\sqrt{1-\nu_1^2}}\,d\mathcal{H}^1.$$

As usual, we integrate and find

$$2\pi^2\int_{-\ell}^{\ell}\#_t^4\,dt \le 2\delta E_\epsilon(M),$$

which implies the claim. □

5.5 An estimate of the area

The following proposition, giving an estimate for the area of the 'bad' part of M, is the main result of this section. We now set $L = 2\ell$ again.

Proposition 5.1. *Let $C_0 \ge 1$ and $\alpha > 0$. There exists a number $\delta_0 > 0$ such that if $\delta \le \delta_0$, $\epsilon \le L$, and $E_\epsilon(M) \le C_0\sqrt{L}$, then*

$$\mathcal{H}^2(M^1 \cup M^2 \cup M^3 \cup M^4) \le 16C_0^2\sqrt{\epsilon}L\left(\delta + \frac{\sqrt{\epsilon}}{\delta^2} + \sqrt{\frac{L}{\alpha}}\right).$$

Proof. As $\#_t^2$ is always an integer, it follows from Lemma 5.1 that

$$\mathcal{H}^1\left(\left\{t \in [-\ell,\ell] : \#_t^2 > 0\right\}\right) \le 4\sqrt{\epsilon}\delta^{-2}E_\epsilon(M).$$

Combining this estimate with Lemma 3.1, we obtain

$$\mathcal{H}^2\left(\left\{x \in M^2 : |\nu_1| \le \frac{1}{4}\right\}\right) \le 8\epsilon\delta^{-2}(E_\epsilon(M))^2. \tag{12}$$

Using Lemmas 5.3 and 5.4, we obtain similar estimates for M^3 and M^4, namely

$$\mathcal{H}^2(M^3) \le 8\sqrt{\frac{2L\epsilon}{\alpha}}(E_\epsilon(M))^2 \quad \text{and} \quad \mathcal{H}^2(M^4) \le \delta\sqrt{\epsilon}(E_\epsilon(M))^2.$$

(Recall that $|\nu_1| \le \frac{1}{4}$ on M^3 and M^4.) We have $|\nu_1| \ge \frac{\delta}{2}$ on M^1, and also on the part of M^2 not accounted for in (12). It is clear that

$$\mathcal{H}^2\left(\left\{x \in M : |\nu_1| \ge \frac{\delta}{2}\right\}\right) \le 8\delta^{-2}\epsilon^{3/2}E_\epsilon(M).$$

Now it suffices to add up all the terms and use the fact that $C_0 \ge 1$ and $\epsilon \le L$. □

6 Convergence in Hausdorff distance

Finally, we want to give a proof of Theorem 1.3. In the first step, we still study a fixed $L > 0$ and an fixed $M \in \mathcal{M}_{L/2}$. We consider the projection

$$D = \Pi(M)$$

onto the (x_2, x_3)-plane. In order to prove closeness to a line segment parallel to the x_1-axis, we need above all to estimate the diameter of D.

6.1 The area of D

We begin with something easier: to estimate the area of D.

Lemma 6.1. *For any $\epsilon > 0$,*

$$\mathcal{H}^2(D) \le \epsilon\sqrt{L+\epsilon}E_\epsilon(M).$$

Proof. Consider an open subset $\Omega \subset D$ and a function $f \in C^\infty(\Omega)$ such that the graph of f,

$$G = \{(f(x_2, x_3), x_2, x_3) : (x_2, x_3) \in \Omega\},$$

is contained in M. Then we compute

$$\int_G |\nu_1|\, d\mathcal{H}^2 = \mathcal{H}^2(\Omega).$$

We define the function $\# : D \to \mathbb{N} \cup \{\infty\}$ by

$$\#(x_2, x_3) = \mathcal{H}^0\left(M \cap (\mathbb{R} \times \{(x_2, x_3)\})\right),$$

and we infer

$$\int_D \# \, d\mathcal{H}^2 = \int_M |\nu_1|\, d\mathcal{H}^2 \le (\mathcal{H}^2(M))^{1/2}\left(\int_M \nu_1^2\, d\mathcal{H}^2\right)^{1/2}.$$

We use Lemma 4.1 to estimate the area of M. We find

$$\int_D \# \, d\mathcal{H}^2 \le 2\epsilon\sqrt{L+\epsilon}E_\epsilon(M).$$

But clearly, we have $\# \ge 2$ almost everywhere in D. □

6.2 The diameter of D

Next we decompose D according to the 'good' and the 'bad' sets from section 5. For a fixed $\delta \in (0, \frac{1}{4}]$, let $M^0, \ldots, M^4$ be defined as before. Set

$$D^0 = \Pi(M^0) \quad \text{and} \quad D^* = \Pi(M \backslash M^0).$$

Then $D = D^0 \cup D^*$.

Lemma 6.2. *Let $C_0 \geq 1$ and $R = C_0\sqrt{\epsilon L}$. Furthermore, suppose that $n \in \mathbb{N}$ with $2C_0 L\delta^{-3} < n+1$. If M is a topological sphere, and if $\epsilon \leq L$ and $E_\epsilon(M) \leq C_0\sqrt{L}$, then there exist certain points $y^{(1)}, \ldots, y^{(n)} \in \mathbb{R}^2$ such that*

$$D^0 \subset \bigcup_{i=1}^{n} \left\{ y \in \mathbb{R}^2 : |y - y^{(i)}| \leq R \right\}.$$

Proof. We argue by contradiction. Suppose that D^0 cannot be covered by n disks of radius R. Then there exist $y^{(1)}, \ldots, y^{(n+1)} \in D^0$ with

$$|y^{(i)} - y^{(j)}| > R \quad \text{for } i \neq j.$$

For every $i = 1, \ldots, n+1$ there exists a number t_i such that $(t_i, y^{(i)}) \in M^0_{t_i}$. For each i, let $\mathcal{C}_i$ be the connected component of $\Pi(M^0_{t_i})$ containing $y^{(i)}$. Let a_i be the area of the region enclosed by $\mathcal{C}_i$.

Note that $\{t_i\} \times \mathcal{C}_i$ is a loop in M. Since M is a sphere, the loop can be contracted within M, which gives rise to a contraction of $\mathcal{C}_i$ in D. In particular, the region bounded by $\mathcal{C}_i$ is contained in D. (This is the only place in the proof of this lemma and of Theorem 1.3 where the assumption on the topology of M is used.)

By the definition of $M^0_{t_i}$ and Lemma 3.1, we have

$$\mathcal{H}^1(M_{t_i}) \leq 2C_0\sqrt{\epsilon L} = 2R.$$

Hence the curves $\mathcal{C}_i$ are pairwise disjoint, as are the regions enclosed by them. Again using the definition of $M^0_{t_i}$, we also see that

$$a_i \geq \delta(\mathcal{H}^1(\mathcal{C}_i))^2 \geq \delta^3\epsilon.$$

Therefore,

$$\mathcal{H}^2(D) \geq \sum_{i=1}^{n+1} a_i \geq (n+1)\delta^3\epsilon > 2C_0 L\epsilon.$$

This contradicts Lemma 6.1. □

Proposition 6.1. *For any $C_0 \geq 1$ there exists a number $C > 0$ with the following property. Let $\alpha > 0$ and $L > 0$. Then there is a $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0]$ and $\epsilon \in (0, L]$, and for all embedded spheres $M \in \mathcal{M}_{L/2}$ with $E_\epsilon(M) \leq C_0\sqrt{L}$, the inequality*

$$\operatorname{diam} \Pi(M) \leq CL^{3/4} \left(\sqrt{\delta} + \frac{\epsilon^{1/4}}{\delta} + \frac{L^{1/4}}{\alpha^{1/4}} + \frac{L^{3/4}\sqrt{\epsilon}}{\delta^3} \right)$$

holds true.

Proof. We use the same notation as before, in particular $D = \Pi(M)$. Choose $y, z \in D$ with $\operatorname{diam} D = |y - z|$. Define

$$\eta = \frac{(0, y - z)}{|y - z|}$$

and

$$J = \{\langle x, \eta \rangle : x \in M\} \setminus \left\{\langle x, \eta \rangle : x \in M^0\right\}.$$

By Lemma 6.2, we have

$$\mathcal{H}^1(J) \geq \operatorname{diam} D - 4C_0^2 L^{3/2} \delta^{-3} \sqrt{\epsilon}.$$

According to Lemma 4.2 and Proposition 5.1, there exists a constant C_1, depending only on C_0, such that

$$\mathcal{H}^1(J) \leq C_1 L^{3/4} \left(\delta + \frac{\sqrt{\epsilon}}{\delta^2} + \sqrt{\frac{L}{\alpha}} \right)^{1/2}.$$

A combination of the two estimates gives the desired inequality. □

6.3 Proof of Theorem 1.3

By Proposition 3.1, we need only consider regular surfaces. Consider embedded spheres $M_\epsilon \subset \mathbb{R}^3$. Suppose that each M_ϵ is regular with $\Lambda(M_\epsilon) = L_\epsilon$ and

$$\limsup_{\epsilon \searrow 0} E_\epsilon(M_\epsilon) < \infty.$$

Let $D_\epsilon = \Pi(M_\epsilon)$. As we are free to apply a translation to each M_ϵ, we may assume that $M_\epsilon \in \mathcal{M}_{L_\epsilon/2}$ and $0 \in D_\epsilon$ for every ϵ.

By Theorem 1.1, we have

$$\limsup_{\epsilon \searrow 0} L_\epsilon < \infty.$$

Hence there exist a sequence $\epsilon_k \searrow 0$ and a number $L \geq 0$ such that $L_{\epsilon_k} \to L$.

Now consider the inequality of Proposition 6.1. Choosing α sufficiently large and δ and ϵ sufficiently small, we can make the right hand side arbitrarily small. Hence we have

$$\lim_{k \to \infty} \operatorname{diam} D_{\epsilon_k} = 0.$$

That is, the surface M_{ϵ_k} is contained in a cylinder about the x_1-axis of length L_{ϵ_k} and radius r_k with $\lim_{k\to\infty} r_k = 0$. As M_{ϵ_k} is connected, it follows that M_{ϵ_k} converges in Hausdorff distance to the line segment connecting the points $(\pm L/2, 0, 0)$.

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