Minimizers of a weighted maximum of the Gauss curvature

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Abstract

On a Riemann surface $\overline{\Sigma}$ with smooth boundary we consider Riemannian metrics conformal to a given background metric. Let κ be a smooth, positive function on $\overline{\Sigma}$. If K denotes the Gauss curvature, then the L^∞-norm of K/κ gives rise to a functional on the space of all admissible metrics. We study minimizers subject to an area constraint. Under suitable conditions, we construct a minimizer with the property that $|K|/\kappa$ is constant. The sign of K can change, but this happens only on the nodal set of the solution of a linear partial differential equation.

1 Introduction

There has been considerable interest in recent years in finding Riemannian metrics on a surface or manifold in a given conformal class such that the Gauss or scalar curvature has special properties. For example, the Yamabe problem is concerned with making the scalar curvature constant; for an overview see the discussion by Aubin [1, Chapter 5]. Also well-studied is Nirenberg's problem, which asks whether a prescribed function on a surface is the Gauss curvature of a metric in a given conformal class; a survey on existence and compactness results is given by Ma [9]. Less attention has been given to metrics such that the curvature has an extremal property. In this paper we study a variational problem of this sort and we see that it is loosely connected to Nirenberg's problem.

Consider the functional given by a weighted L^∞-norm of the Gauss curvature. We wish to minimize this among all metrics in a given conformal class with prescribed area. Under the conditions that we study, it is not difficult to see that a solution of the problem exists. More surprising is the fact that this solution can be chosen in a way such that the corresponding curvature coincides up to a constant and up to its sign with the reciprocal of the weight function. Thus to solve this variational problem comes close to prescribing the modulus of the Gauss curvature.

More precisely, the situation that we study is the following. Consider a Riemann surface (S, g_0) and an open subset $\Sigma \subset S$ with smooth boundary such that its closure $\overline{\Sigma}$ is compact. Then we may think of $(\overline{\Sigma}, g_0)$ as a Riemann surface with (possibly empty) boundary. For technical reasons, we assume that

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none of the connected components of Σ is a torus. We study conformal changes of the metric of the form $g = e^{2u} g_0$ on $\overline{\Sigma}$. If K_0 is the Gauss curvature and Δ_0 is the (negative semidefinite) Laplace-Beltrami operator for g_0, then this gives rise to the new Gauss curvature

$$K = e^{-2u}(-\Delta_0 u + K_0)$$

and the new Laplace-Beltrami operator

$$\Delta = e^{-2u}\Delta_0.$$

We write μ_0 for the measure on Σ induced by g_0. Then $\mu = e^{2u}\mu_0$ is the measure that belongs to g.

Fix a function $\kappa \in C^\infty(\overline{\Sigma})$ with $\inf_\Sigma \kappa > 0$. We are interested in minimizers of the functional

$$E_\infty(u) = \operatorname*{ess\,sup}_\Sigma \frac{|K|}{\kappa}.$$

It is obvious, however, that

$$\inf_{u \in C^\infty(\overline{\Sigma})} E_\infty(u) = 0,$$

as $E_\infty(u + \gamma) = e^{-2\gamma} E_\infty(u)$ for every constant $\gamma \in \mathbb{R}$. In order to obtain a reasonable variational problem, we therefore impose additional constraints. For a fixed number $c_1 > 0$, we require that

$$\mu(\Sigma) = c_1. \tag{1}$$

This rules out uniform scalings and avoids the previous problem. We have furthermore the freedom to prescribe boundary data. We choose a $\phi \in C^\infty(\partial\Sigma)$ and require that $u = \phi$ on $\partial\Sigma$. In addition, we prescribe the average geodesic curvature of the boundary with respect to g. By the Gauss-Bonnet formula, this amounts to fixing a constant $c_2 \in \mathbb{R}$ and requiring that

$$\int_\Sigma K \, d\mu = c_2. \tag{2}$$

For $1 < p < \infty$, let $\mathcal{U}^p(c_1, c_2, \phi)$ be the set of all $u \in W^{2,q}(\Sigma, g_0)$ such that $u = \phi$ on $\partial\Sigma$ and the metric $g = e^{2u}$ satisfies (1) and (2). Furthermore, the set $\mathcal{U}^\infty(c_1, c_2, \phi)$ consists of all $u \in \bigcap_{p<\infty} \mathcal{U}^p(c_1, c_2, \phi)$ with $K \in L^\infty(\mu_0)$.

Theorem 1.1. *Suppose that $\phi \in C^\infty(\partial\Sigma)$ and $c_1 > 0$, $c_2 \in \mathbb{R}$ satisfy*

$$\inf_{u \in \mathcal{U}^\infty(c_1,c_2,\phi)} E_\infty(u) < \frac{4\pi}{c_1 \sup_\Sigma \kappa}.$$

Then there exist a minimizer u of E_∞ in $\mathcal{U}^\infty(c_1, c_2, \phi)$ and a countably 1-rectifiable set $\Gamma \subset \Sigma$ that is closed relative to Σ, such that K/κ is locally constant with $|K|/\kappa = E_\infty(u)$ in $\Sigma \backslash \Gamma$.

The proof of the theorem gives in fact further information about Γ. It is the nodal set of a nontrivial solution to the linear boundary value problem

$$\begin{aligned} -\Delta w + 2Kw &= \alpha \quad \text{in } \Sigma, \\ w &= \beta \quad \text{on } \partial\Sigma, \end{aligned} \tag{3}$$

for two constants $\alpha, \beta \in [-1, 1]$. Moreover, we have

$$K = E_\infty(u) \frac{w\kappa}{|w|}$$

outside of Γ.

The set Γ may be empty, in which case we recover a solution of Nirenberg's problem for the function

$$\frac{c_2 \kappa}{\int_\Sigma \kappa \, d\mu}.$$

On the other hand, if Nirenberg's problem does not have a solution for the given boundary conditions, then Γ is necessarily non-empty and the curvature of the minimizer changes its sign. This is the typical case at least if $\partial\Sigma \neq \emptyset$. For a closed surface, there can be geometrical obstructions to solving Nirenberg's problem [8]. In the case of a non-empty boundary, the boundary conditions may give additional obstructions. The problem then amounts to a boundary value problem for a semilinear elliptic partial differential equation with an additional integral constraint (coming from the prescribed area). In the situation that we consider, this problem is typically overdetermined.

It would be interesting to study even more restrictive boundary conditions. From the variational point of view, it is natural to prescribe not just u but also its normal derivative on $\partial\Sigma$. Geometrically, this amounts to prescribing the metric on $\partial\Sigma$ and the geodesic curvature of the boundary curve as well. Furthermore, it is natural to ask whether it is possible to remove the bound for the infimum of the energy. Our methods are not sufficient to answer these questions, but this may be merely for technical reasons.

2 Preventing bubbling

The strategy for the proof of Theorem 1.1 is to approximate the L^∞-norm of K/κ by L^p-norms. When passing to the limit $p \to \infty$, the possibility of a lack of compactness arises, due to a concentration of energy, a phenomenon called bubbling in this context. Under the small energy assumption of the theorem, however, we can rule this out. The following tools are needed for this purpose.

Lemma 2.1. *Let $\phi \in C^\infty(\partial\Sigma)$ and $A > 0$. For any $\delta < 4\pi$, there exist two constants $p_0 \geq 2$ and C_0 with the following property. Suppose that $p \geq p_0$ and $K \in L^p(\mu_0)$. Let $u \in W^{2,p}(\Sigma, g_0)$ be a solution of*

$$\begin{aligned} \Delta_0 u + K e^{2u} &= K_0 \quad \text{in } \Sigma, \\ u &= \phi \quad \text{on } \partial\Sigma. \end{aligned} \tag{4}$$

If

$$\int_\Sigma e^{2u} \, d\mu_0 \leq A \quad \text{and} \quad A^{p-1} \int_\Sigma |K|^p e^{2u} \, d\mu_0 \leq \delta^p, \tag{5}$$

then $|u| \leq C_0$.

This result can be obtained as a consequence of an inequality mentioned by Chen [6]. His theory on surfaces with L^2-bounds for the curvature [4, 3, 5] provides the necessary tools for the proof, even though the arguments have not explicitly been formulated. For our situation, we do in fact not need this sophisticated machinery, and we therefore give a different proof. It relies on two ingredients: a concentration-compactness principle of Struwe [10] (which in turn is based on an inequality of Brezis and Merle [2]) and a version of the isoperimetric inequality due to Topping [11, 12]. The former is formulated in Struwe's paper for closed surfaces only, so we first give the statement that we need here. In the following we write $B_r(x_0)$ for the open ball in $\overline{\Sigma}$ of radius $r > 0$ about the point $x_0 \in \overline{\Sigma}$ with respect to the metric g_0.

Lemma 2.2. *Let $\phi \in C^\infty(\partial\Sigma)$. For $n \in \mathbb{N}$, let $u_n \in W^{2,2}(\Sigma_0, g_0)$ be solutions of*

$$\begin{aligned} \Delta_0 u_n + K_n e^{2u_n} &= K_0 \quad \text{in } \Sigma, \\ u_n &= \phi \quad \text{on } \partial\Sigma, \end{aligned} \tag{6}$$

where $K_n \in L^2(\mu_0)$ with

$$\sup_{n\in\mathbb{N}} \int_\Sigma \left(1 + K_n^2\right) e^{2u_n}\, d\mu_0 < \infty.$$

Then there exist a finite set of points $x_1, \dots, x_J \in \overline{\Sigma}$ (possibly empty) and a subsequence $(u_{n_i})_{i\in\mathbb{N}}$ such that for every $j = 1, \dots, J$,

$$\lim_{R\searrow 0} \liminf_{i\to\infty} \int_{B_R(x_j)} |K_{n_i}| e^{2u_{n_i}}\, d\mu_0 \geq 2\pi,$$

and for every connected compact set $\Omega \subset \overline{\Sigma}\backslash\{x_1, \dots, x_J\}$, either

$$\sup_{i\in\mathbb{N}} \sup_\Omega |u_{n_i}| < \infty \quad \text{or} \quad \lim_{i\to\infty} \sup_\Omega u_{n_i} = -\infty.$$

For closed surfaces, Struwe's arguments show that it suffices to consider functions on the unit disk in $\mathbb{R}^2$ with vanishing boundary data, and then an inequality of Brezis and Merle can be applied. If we have a boundary, then we have to replace the disk by a subset thereof (but with a piecewise smooth boundary). The results of Brezis and Merle do not require any assumptions on the shape of the domain, only on its size, and thus they can still be applied. We leave it to the reader to make the obvious modifications to the arguments.

Note also that in the case $J = 0$, we necessarily have

$$\sup_{i\in\mathbb{N}} \sup_\Sigma |u_{n_i}| < \infty.$$

Convergence to $-\infty$ is excluded by the boundary conditions (on connected components with boundary) and the Gauss-Bonnet formula (on the other components). Here we use the assumption that Σ has no tori as connected components.

Proof of Lemma 2.1. We argue by contradiction. Suppose that for a fixed $\delta < 4\pi$ and a fixed $p \geq 2$, we have a sequence of functions $K_n \in L^p(\mu_0)$ and corresponding solutions $u_n \in W^{2,p}(\Sigma, g_0)$ of (6), such that

$$\int_\Sigma e^{2u_n}\, d\mu_0 \leq A \quad \text{and} \quad A^{p-1} \int_\Sigma |K_n|^p e^{2u_n}\, d\mu_0 \leq \delta^p, \tag{7}$$

but

$$\sup_{\Sigma} |u_n| \to \infty \quad \text{as } n \to \infty.$$

Consider the measures $\mu_n = e^{2u_n}\mu_0$. We may assume that there exists a Radon measure μ_∞ on $\overline{\Sigma}$ such that $\mu_n \overset{*}{\rightharpoonup} \mu_\infty$ weakly* in the dual space of $C^0(\overline{\Sigma}, g_0)$. We write δ_x for the Dirac measure centred at $x \in \overline{\Sigma}$. If $x_1, \ldots, x_J$ are the blow-up points from Lemma 2.2, then we see that μ_∞ is of the form

$$\mu_\infty = \psi\mu_0 + \sum_{j=1}^{J} m_j \delta_{x_j}$$

for a function $\psi \in L^1(\mu_0)$ and certain weights $m_j \geq 0$. Furthermore, by our assumptions, we have at least one blow-up point, i.e., $J \geq 1$. We now examine the behaviour of the sequence near x_1.

Note that for every $R > 0$,

$$\begin{aligned} 2\pi &\leq \limsup_{n\to\infty} \int_{B_R(x_1)} |K_n|\, d\mu_n \\ &\leq \limsup_{n\to\infty} \left(\mu_n(B_R(x_1))\right)^{1-1/p} \left(\int_{B_R(x_1)} |K_n|^p\, d\mu_n\right)^{1/p} \\ &\leq \left(\frac{\mu_\infty(\overline{B_R(x_1)})}{A}\right)^{1-1/p} \delta. \end{aligned}$$

Hence

$$m_1 \geq \left(\frac{2\pi}{\delta}\right)^{\frac{p}{p-1}} A \geq \frac{A}{4}. \tag{8}$$

Fix $\epsilon > 0$. Let $\rho > 0$ be so small that $B_r(x)$ has the topology of a disk for every $x \in \overline{\Sigma}$ and every $r \in (0, \rho]$. Choose a radius $R \in (0, \rho]$ such that

$$\mu_\infty(\overline{B_{2R}(x_1)\backslash B_R(x_1)}) \leq \epsilon.$$

Then for every sufficiently large n, we have

$$\mu_n(B_{2R}(x_1)\backslash B_R(x_1)) \leq 2\epsilon.$$

In the following we write $\mathcal{H}^1$ for the 1-dimensional Hausdorff measure with respect to g_0. We first note that there exists a constant C_1, depending only on (Σ, g_0), such that whenever ρ is sufficiently small, we have

$$\begin{aligned} \mu_n(B_{2R}(x_1)\backslash B_R(x_1)) &= \int_R^{2R} \int_{\partial B_r(x_1)\backslash\partial\Sigma} e^{2u_n}\, d\mathcal{H}^1\, dr \\ &\geq \frac{1}{C_1} \int_R^{2R} \frac{1}{r} \left(\int_{\partial B_r(x_1)\backslash\partial\Sigma} e^{u_n}\, d\mathcal{H}^1\right)^2 dr. \end{aligned}$$

Hence we have another constant C_2 such that there always exists a radius $r_n \in [R, 2R]$ satisfying

$$\left(\int_{\partial B_{r_n}(x_1)} e^{u_n}\, d\mathcal{H}^1\right)^2 \leq C_2(\epsilon + \rho^2).$$

Set

$$a_n = \int_{B_{r_n}(x_1)} e^{2u_n} \, d\mu_0$$

(which is the area in $B_{r_n}(x_1)$ with respect to the metric $e^{2u_n} g_0$) and

$$\ell_n = \int_{\partial B_{r_n}(x_1)} e^{u_n} \, d\mathcal{H}^1$$

(the length of the corresponding boundary).

We now use an isoperimetric inequality of Topping [11, 12]. This inequality is formulated in terms of the nonincreasing rearrangement of the curvature. Combined with Hölder's inequality and applied to our situation, it immediately implies

$$4\pi a_n \leq \ell_n^2 + 2\left(\frac{p-1}{2p-1}\right)^{(p-1)/p} a_n^{(2p-1)/p} \left(\int_{B_{r_n}(x_1)} |K_n|^p \, d\mu_n\right)^{1/p}.$$

For the number $a = \limsup_{n\to\infty} a_n$, we obtain

$$4\pi a \leq C_2(\epsilon + \rho^2) + 2\left(\frac{p-1}{2p-1}\right)^{(p-1)/p} a^{(2p-1)/p} A^{1/p-1} \delta.$$

Of course a may depend on ϵ and ρ, but we always have $A/4 \leq a \leq A$ by (7) and (8). Letting $\epsilon \to 0$ and $\rho \to 0$, we conclude that

$$4\pi \leq 2\left(\frac{p-1}{2p-1}\right)^{(p-1)/p} \delta.$$

This gives a contradiction to the assumption $\delta < 4\pi$ if p is sufficiently large. □

3 Proof of Theorem 1.1

We may assume that there exists no $u \in \mathcal{U}^\infty(c_1, c_2, \phi)$ with $E_\infty(u) = 0$. We construct the minimizer u of E_∞ as a limit of minimizers of the functionals

$$E_p(u) = \left(\frac{1}{c_1} \int_\Sigma \kappa^{-p} |K|^p \, d\mu\right)^{1/p}$$

in $\mathcal{U}^p(c_1, c_2, \phi)$. The conditions of Theorem 1.1 imply $\mathcal{U}^p(c_1, c_2, \phi) \neq \emptyset$ for every $p \in [2, \infty]$. Furthermore, we have

$$\inf_{u \in \mathcal{U}^p(c_1, c_2, \phi)} E_p(u) < \frac{4\pi}{c_1 \sup_\Sigma \kappa}.$$

If $u \in \mathcal{U}^p(c_1, c_2, \phi)$ satisfies

$$E_p(u) \leq \frac{\delta}{c_1 \sup_\Sigma \kappa}$$

for a number $\delta < 4\pi$, then it follows that

$$c_1^{p-1} \int_\Sigma |K|^p e^{2u} \, d\mu_0 \leq \left(c_1 E_p(u) \sup_\Sigma \kappa\right)^p \leq \delta^p$$

for the corresponding curvature K. Thus we obtain the bound $|u| \leq C_0$ from Lemma 2.1 for $p \geq p_0$. Using equation (4) directly, and also taking into account that

$$\|Ke^{2u}\|_{L^p(\mu_0)} \leq \exp\left(\frac{2C_0(p-1)}{p}\right) \|K\|_{L^p(\mu)} \leq 4\pi \left(\frac{e^{2C_0}}{c_1}\right)^{\frac{p-1}{p}},$$

we derive the inequality

$$\|u\|_{C^{1,1/2}(\overline{\Sigma},g_0)} \leq C_1$$

for a constant C_1 that depends only on (Σ, g_0) and the data of the variational problem.

We can now use the direct method to construct a minimizer u_p of E_p in $\mathcal{U}^p(c_1, c_2, \phi)$ for every $p \in [p_0, \infty)$. We write $\Delta_p = e^{-2u_p}\Delta_0$ for the Laplace-Beltrami operator and $K_p = e^{-2u_p}(-\Delta_0 u_p + K_0)$ for the curvature of the metric $g_p = e^{2u_p} g_0$. Furthermore, we write $\mu_p = e^{2u_p}\mu_0$ for the corresponding measure.

Next we calculate the Euler-Lagrange equation for the above minimizing problem and we find that there exist $a_p, b_p \in \mathbb{R}$ (the Lagrange multipliers for the constraints (1) and (2), respectively), such that

$$-\Delta_p(\kappa^{-p}|K_p|^{p-2}K_p) + \left(2 - \frac{2}{p}\right)\kappa^{-p}|K_p|^p = a_p \quad \text{in } \Sigma,$$
$$\kappa^{-p}|K_p|^{p-2}K_p = b_p \quad \text{on } \partial\Sigma.$$

We use the notation $p' = \frac{p}{p-1}$ for the conjugate exponent to p. We define

$$\gamma_p = \max\left\{|a_p|, |b_p|, \|\kappa^{-p'}K_p\|_{L^p(\mu_p)}^{p-1}\right\}$$

and $\alpha_p = a_p/\gamma_p$, $\beta_p = b_p/\gamma_p$. Then $\gamma_p > 0$ by the assumption at the beginning of the proof. Furthermore, we define

$$w_p = \frac{|K_p|^{p-2}K_p}{\gamma_p \kappa^p}.$$

Now we can rewrite the Euler-Lagrange equation in the form

$$\begin{aligned} -\Delta_p w_p + \left(2 - \frac{2}{p}\right) K_p w_p &= \alpha_p \quad \text{in } \Sigma, \\ w_p &= \beta_p \quad \text{on } \partial\Sigma. \end{aligned} \tag{9}$$

We know that $\alpha_p, \beta_p \in [-1, 1]$ and $\|w_p\|_{L^{p'}(\mu_p)} \leq 1$. The coefficients of the operators Δ_p are uniformly bounded in $C^{1,1/2}(\overline{\Sigma}, g_0)$ and

$$\limsup_{p\to\infty} \|K_p\|_{L^p(\mu_p)} \leq \frac{4\pi}{c_1}.$$

Thus we obtain a uniform bound for w_p in $W^{1,q}(\Sigma, g_0)$ for every $q < 2$ from standard elliptic estimates. Using equation (9) again, we conclude that w_p is also uniformly bounded in $C^{0,\rho}(\overline{\Sigma}, g_0)$ for every $\rho \in (0, 1)$. Hence we may choose a sequence $p_k \to \infty$ such that

- $u_{p_k} \to u$ in $C^1(\overline{\Sigma}, g_0)$ and weakly in $W^{2,q}(\Sigma, g_0)$ for every $q < \infty$,

- $w_{p_k} \to w$ uniformly,
- $K_{p_k} \rightharpoonup K$ weakly in $L^q(\mu_0)$ for every $q < \infty$, and
- $\alpha_{p_k} \to \alpha$ and $\beta_{p_k} \to \beta$.

It is clear that K is the curvature belonging to the metric $g = e^{2u} g_0$. Moreover, we obtain (3) as a limiting equation for w, where $\Delta = e^{-2u}\Delta_0$. By construction we have either $(\alpha, \beta) \neq (0, 0)$ or $\|w\|_{L^1(\mu)} = 1$, so $w \neq 0$ in each case. Recall the definition of w_p, which implies

$$|K_p| = (\gamma_p |w_p|)^{1/(p-1)} \, \kappa^{p'}.$$

Define $\Gamma = w^{-1}(\{0\})$. In $\Sigma \backslash \Gamma$, we have

$$|w_{p_k}|^{1/(p_k - 1)} \to 1 \quad \text{as } k \to \infty$$

locally uniformly by the uniform convergence of w_{p_k}. We may assume that

$$\gamma_{p_k}^{1/(p_k-1)} \to \gamma_\infty \in [0, \infty].$$

Hence $|K_{p_k}| \to \kappa\gamma_\infty$ locally uniformly in $\Sigma \backslash \Gamma$. In particular, the limit K has locally constant sign and $|K| = \kappa\gamma_\infty$ in $\Sigma\backslash\Gamma$. It is clear that Γ is closed relative to Σ and $\Gamma \neq \Sigma$, so it follows that $\gamma_\infty < \infty$.

Next we claim that u is a minimizer of E_∞ in $\mathcal{U}^\infty(c_1, c_2, \phi)$. Note that for $p \leq q$, we have

$$E_p(u_p) \leq E_p(u_q) \leq E_q(u_q)$$

by the choice of u_p and Hölder's inequality. Hence the limit

$$e_\infty = \lim_{p\to\infty} E_p(u_p)$$

exists. Furthermore,

$$E_q(u) \leq \liminf_{k\to\infty} E_q(u_{p_k}) \leq \liminf_{k\to\infty} E_{p_k}(u_{p_k}) = e_\infty$$

for every $q < \infty$. We conclude that $E_\infty(u) \leq e_\infty$.

The constraints (1) and (2) are preserved under the type of convergence that we have for u_{p_k}. Thus $u \in \mathcal{U}^\infty(c_1, c_2, \phi)$. For any other $\tilde{u} \in \mathcal{U}^\infty(c_1, c_2, \phi)$, we have

$$E_\infty(\tilde{u}) = \lim_{p\to\infty} E_p(\tilde{u}) \geq \lim_{p\to\infty} E_p(u_p) = e_\infty.$$

That is, we have in fact $E_\infty(u) = e_\infty$ and u is a minimizer of E_∞ in $\mathcal{U}^\infty(c_1, c_2, \phi)$.

Finally, we examine Γ. We have a uniform bound for u_{p_k} in $C^{1,1/2}(\overline{\Sigma}, g_0)$, and this means that the leading order coefficients of equation (3) belong to $C^{1,1/2}(\overline{\Sigma}, g_0)$. Moreover, the coefficient $2K$ in the second term belongs to $L^\infty(\mu_0)$. Standard elliptic estimates therefore imply $w \in C^{1,\rho}(\overline{\Sigma}, g_0)$ for every $\rho \in (0, 1)$. But we can show more, observing that $Kw = \kappa\gamma_\infty|w|$. It follows that $Kw \in C^{0,1}(\overline{\Sigma}, g_0)$, and therefore $w \in C^{2,\rho}(\overline{\Sigma}, g_0)$ for every $\rho \in (0, 1)$.

If $\alpha = 0$, then we can use results of Hardt and Simon [7] to conclude that Γ has the required structure. If $\alpha \neq 0$, then we proceed as follows. We decompose Γ into

$$\Gamma^* = \{x \in \Sigma : \; w(x) = 0 \text{ and } dw(x) = 0\}$$

and

$$\Gamma^{\#} = \Gamma \backslash \Gamma^*.$$

Near every point of $\Gamma^{\#}$, we can use the implicit function theorem and we conclude that $\Gamma^{\#}$ is the union of countably many curves of class C^2. Near a given point $x_0 \in \Gamma^*$, we use local coordinates (x^1, x^2). As $\Delta w(x_0) \neq 0$, we have either $d\frac{\partial w}{\partial x^1}(x_0) \neq 0$ or $d\frac{\partial w}{\partial x^2}(x_0) \neq 0$. Thus we can apply the implicit function theorem to one of the partial derivatives and we conclude that Γ^* is contained in the union of countably many curves of class C^1. In particular Γ is countably 1-rectifiable. As this implies $\mu_0(\Gamma) = 0$, we now also see that $\gamma_\infty = E_\infty(u)$.

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