



Citation for published version:

de las Heras, I & Traustason, G 2021, 'Powerfully solvable and powerfully simple groups', *Journal of Pure and Applied Algebra*, vol. 225, no. 8, 106714. <https://doi.org/10.1016/j.jpaa.2021.106714>

DOI:

[10.1016/j.jpaa.2021.106714](https://doi.org/10.1016/j.jpaa.2021.106714)

Publication date:

2021

Document Version

Peer reviewed version

[Link to publication](#)

Publisher Rights

CC BY-NC-ND

University of Bath

Alternative formats

If you require this document in an alternative format, please contact:
openaccess@bath.ac.uk

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

POWERFULLY SOLVABLE AND POWERFULLY SIMPLE GROUPS

IKER DE LAS HERAS
UNIVERSITY OF THE BASQUE COUNTRY, SPAIN

GUNNAR TRAUSTASON
UNIVERSITY OF BATH, UK

ABSTRACT. We introduce the notion of a powerfully solvable group. These are powerful groups possessing an abelian series of a special kind. These groups include in particular the class of powerfully nilpotent groups. We will also see that for a certain rich class of powerful groups we can naturally introduce the term powerfully simple group and prove a Jordan-Hölder type theorem that justifies the term.

1. INTRODUCTION

In this paper p is always an odd prime.

Recall [2] that a finite p -group is *powerful* if $[G, G] \leq G^p$. More generally, a subgroup H of G is powerfully embedded in G if $[H, G] \leq H^p$.

The following useful property, known as Shalev's Interchange Lemma [3], will be used a number of times in this paper: If H and K are powerfully embedded in G then $[H^{p^i}, K^{p^j}] = [H, K]^{p^{i+j}}$ for $i, j \geq 0$.

Another term that we need is a *powerful basis*. For any powerful p -group G , there exist generators a_1, \dots, a_r such that $G = \langle a_1 \rangle \cdots \langle a_r \rangle$ and where $|G| = o(a_1) \cdots o(a_r)$. The number of generators of any given order is an invariant. We say that G is of *type* $(1, r_1, 1, 2, r_2, 2, \dots)$ if there are r_i generators of order p^i .

In [5] the notion of powerful nilpotence and powerfully central chain was introduced. If $K \leq H \leq G$, then a chain of subgroups

$$H = H_0 \geq H_1 \geq \cdots \geq H_n = K$$

is *powerfully central* in G if $[H_i, G] \leq H_{i+1}^p$ for $i = 0, \dots, n-1$. A finite p -group is said to be *powerfully nilpotent* if it has a powerfully central series $G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$. The smallest possible length of such a series is then called the *powerful nilpotence class* of G .

We will often use the shortening 'powerful class' for 'powerful nilpotence class'. Powerful nilpotence leads then naturally to a classification in terms of an ancestry tree and powerful co-class. In [5] it was shown that for every prime p there are finitely many powerfully nilpotent p -groups of each powerful co-class, and some general theory was developed for powerfully nilpotent groups. In [6] the powerfully nilpotent groups of maximal powerful class are studied.

In this paper we consider a natural larger class of powerful p -groups.

The first author is supported by the Spanish Government grant MTM2017-86802-P, partly with FEDER funds, and by the Basque Government grant IT974-16. He is also supported by a predoctoral grant of the University of the Basque Country.

Definition. Let G be a finite p -group and $K \leq H \leq G$. We say that a chain

$$H = H_0 \geq H_1 \geq \cdots \geq H_n = K$$

is *powerfully abelian* if $[H_i, H_i] = H_{i+1}^p$ for $i = 0, \dots, n-1$.

Definition. A finite p -group G is *powerfully solvable*, if there exists a powerfully abelian chain

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1.$$

The smallest possible length n is called the *powerful derived length* of G .

The structure of the paper is as follows. In Section 2 we show that all powerful p -groups of rank 2 are powerfully solvable and based on the work in [7] we provide a classification of all these groups as well as a closed formula for the number of such groups of order p^x . In Section 3 we introduce the notion of a powerfully solvable presentation that will be useful later on when going through some classification and calculating growth. In Section 4 we classify all powerful groups of order up to p^5 and see that these are all powerfully solvable. In Section 5 we discuss the growth of powerfully solvable groups, and various other classes of powerful groups, that are of exponent p^2 . In Section 6 we consider the rich class \mathcal{P} of all powerful p -groups of type $(2, 2, \dots, 2)$ and see that powerful nilpotence and powerful solvability play a similar role here as nilpotence and solvability for the class of all groups. The notion of a powerfully simple group arises naturally and we are able to prove a Jordan-Hölder like result that justifies the term. Finally in Section 7 we classify all the powerfully simple groups of order p^6 . The number turns out to depend on the prime p .

2. POWERFUL GROUPS OF RANK 2

It turns out that all powerful p -groups of rank 2 are powerfully solvable. In fact something stronger is true.

Proposition 2.1. *Let G be a powerful p -group. If $[G, G]$ is cyclic then G is powerfully solvable of powerful derived length at most 2.*

Proof As G is powerful we have that $[G, G] = \langle g^p \rangle$ for some $g \in G$. Therefore

$$G \geq \langle g \rangle \geq 1$$

is a powerfully abelian chain. \square

In [7] the powerfully nilpotent groups of rank 2 are classified and a closed formula is given for the number of powerfully nilpotent groups of order p^x . In fact there is implicitly the following classification of all powerful p -groups of rank 2. By Proposition 2.1 we know that these are all powerfully solvable.

Classification of the non-abelian powerful groups of rank 2.

(I) Semidirect products:

$$G = \langle a, b : a^{p^n} = b^{p^m} = 1, [a, b] = a^{p^r} \rangle$$

with $n - r \leq m$ and $1 \leq r \leq n - 1$.

(II) Non-semidirect products:

$$G = \langle a, b : a^{p^n} = 1, b^{p^m} = a^{p^l}, [a, b] = a^{p^r} \rangle$$

with $1 \leq r < l \leq n - 1$ and $n - r \leq l < m$.

From [7] we also know that a group above is powerfully nilpotent if and only if $r \geq 2$. Thus, it is easy to determine that there are $\lfloor \frac{x-1}{2} \rfloor$ semidirect products and $\lfloor \frac{x-4}{2} \rfloor$ non-semidirect products of order p^x that are not powerfully nilpotent (here $\lfloor \cdot \rfloor$ stands for the floor function). From this, the discussion above and [7, Proposition 2.2] we also get the following.

Enumeration. For $x \geq 3$, the number of powerful p -groups of rank 2 and order p^x is

$$\begin{aligned} & \frac{x^3 + 12x^2 + 12x}{72} \quad \text{if } x \equiv 0 \pmod{6} \\ & \frac{x^3 + 12x^2 + 3x - 16}{72} \quad \text{if } x \equiv 1 \pmod{6} \\ & \frac{x^3 + 12x^2 + 12x - 8}{72} \quad \text{if } x \equiv 2 \pmod{6} \\ & \frac{x^3 + 12x^2 + 3x}{72} \quad \text{if } x \equiv 3 \pmod{6} \\ & \frac{x^3 + 12x^2 + 12x - 16}{72} \quad \text{if } x \equiv 4 \pmod{6} \\ & \frac{x^3 + 12x^2 + 3x - 8}{72} \quad \text{if } x \equiv 5 \pmod{6}. \end{aligned}$$

3. PRESENTATIONS

Lemma 3.1. *Let G be a finite p -group and let $K < H \leq G$ where $[H, H] \leq K^p$. If for some positive integer n we have $K^{p^n} = H^{p^n}$, then there exists $x \in K \setminus H$ such that $x^{p^n} = 1$.*

Proof We prove this by induction on n . Suppose first that $n = 1$. We will show that for every $j \geq 1$, there exists $x \in K \setminus H$ such that $x^p \in H^{p^j}$. For $j = 1$ this is immediate from the hypothesis, so assume by induction on $j \geq 2$ that we know that $x^p \in H^{p^{j-1}}$ for some $x \in K \setminus H$. Then there exists $y \in H^{p^{j-2}}$ such that $x^p = y^p$. Now, by the Hall-Petresco Identity, as $p > 2$, we have

$$(xy^{-1})^p = x^p y^{-p} c_2^{\binom{p}{2}} c_3^{\binom{p}{3}} \cdots c_p^{\binom{p}{p}},$$

where $c_k \in [H^{p^{j-2}}, K, \dots, K]$ for $k = 2, \dots, p$. Notice that K is powerful and that H is powerfully embedded in K . We can thus use Shalev's Interchange Lemma. Therefore for $2 \leq k \leq p-1$ we have

$$c_k^{\binom{p}{k}} \in [H^{p^{j-2}}, K]^p = [H, K]^{p^{j-1}} \leq H^{p^j}$$

and

$$c_p \in [H^{p^{j-2}}, K, K] = [H, K, K]^{p^{j-2}} \leq [H^p, K]^{p^{j-2}} = [H, K]^{p^{j-1}} \leq H^{p^j}.$$

Since $x^p y^{-p} = 1$, we then have $(xy^{-1})^p \in H^{p^j}$. This finishes the inductive step. Taking j such that $H^{p^j} = 1$ we see that we can pick $x \in K \setminus H$ such that $x^p = 1$.

Now suppose $n > 1$ and that the result holds for smaller values of n . If $K^{p^{n-1}} = H^{p^{n-1}}$, then by induction hypothesis we know there exists $x \in K \setminus H$ where $x^{p^{n-1}} = 1$ and thus $x^{p^n} = 1$. We can thus assume that $K^{p^{n-1}} \neq H^{p^{n-1}}$. Now

$$[K^{p^{n-1}}, K^{p^{n-1}}] \leq [K, K]^{p^{2n-2}} \leq H^{p^{2n-1}} \leq (H^{p^{n-1}})^p.$$

Therefore by the induction hypothesis there exists an element $y \in K^{p^{n-1}} \setminus H^{p^{n-1}}$ such that $y^p = 1$. Since K is powerful we have $y = x^{p^{n-1}}$ for some $x \in K \setminus H$ and then $x^{p^n} = y^p = 1$. \square

Theorem 3.2. *Let G be a finite p -group of rank r and exponent p^e where G/G^{p^2} is powerfully solvable. Then G is powerfully solvable. Furthermore, we can choose our generators a_1, a_2, \dots, a_r such that $|G| = o(a_1) \cdots o(a_r)$ and such that the chain*

$$\begin{aligned} G &= \langle a_1, a_2, \dots, a_r \rangle && \geq \langle a_1^p, a_2, \dots, a_r \rangle && \geq \cdots \geq G^p \\ G^p &= \langle a_1^p, a_2^p, \dots, a_r^p \rangle && \geq \langle a_1^{p^2}, a_2^p, \dots, a_r^p \rangle && \geq \cdots \geq G^{p^2} \\ & && \vdots && \\ G^{p^{e-1}} &= \langle a_1^{p^{e-1}}, a_2^{p^{e-1}}, \dots, a_r^{p^{e-1}} \rangle && \geq \langle a_1^{p^e}, a_2^{p^{e-1}}, \dots, a_r^{p^{e-1}} \rangle && \geq \cdots \geq G^{p^e} = 1 \end{aligned}$$

is powerfully abelian.

Proof Suppose, using the fact that G/G^{p^2} is powerfully solvable, that $G = K_0 > K_1 > \cdots > K_m = G^{p^2}$ is a chain that is powerfully abelian modulo G^{p^2} . Notice that $[G, G] \leq K_1^p G^{p^2} \leq G^p$ and the group is thus powerful. In particular, we have $[G^p, G] \leq G^{p^2}$ and $(G^p)^p = G^{p^2}$. Therefore $G = K_0 G^p \geq K_1 G^p \geq \cdots \geq K_m G^p = G^p$ is also powerfully abelian. Removing redundant terms and refining if necessary, we get a powerfully abelian chain

$$G = H_0 > H_1 > \cdots > H_r = G^p$$

where the factors are of size p . Now notice that for $0 \leq i \leq r-1$ and $0 \leq j \leq e$ we have $[H_i^{p^j}, H_i^{p^j}] = [H_i, H_i]^{p^{2j}} \leq H_{i+1}^{p^{j+1}}$. This gives us the powerfully abelian chain we wanted. It remains to see that we can furthermore pick our generators such that a_1, \dots, a_r is a powerful basis for G . Let us pick our generators of G such that for every $1 \leq i \leq r-1$ we have $H_i = \langle a_{i+1}, \dots, a_r \rangle G^p$. If $H_i^p = H_{i+1}^p$ for some $1 \leq i \leq r-1$ then we can know from Lemma 3.1 that we can pick a_{i+1} such that $a_{i+1}^p = 1$. We can also in that case move the generator in front. We thus have

$$a_1^p = \cdots = a_{r-r_1}^p = 1 \text{ and } H_{r-r_1}^p > \cdots > H_r^p = G^{p^2},$$

where $r_1 = \text{rank}(G^p)$. Now consider the chain

$$G^{p^2} = H_{r-r_1}^{p^2} \geq \cdots \geq H_r^{p^2} = G^{p^3}.$$

Again if $H_i^{p^2} = H_{i+1}^{p^2}$, then we know by Lemma 3.1 that we can pick a_{i+1} such that $a_{i+1}^{p^2} = 1$. Continuing in this manner we see that we can choose our generators such that for $1 \leq i \leq r$ we have $o(a_i) = p^j$ where j is the smallest positive integer such that $H_{i-1}^{p^j} = H_i^{p^j}$. Also we have that $\text{rank}(G^{p^j})$ is then the number of $1 \leq i \leq r$ such that $a_i^{p^j} \neq 1$. Let r_j be the rank of G^{p^j} . Then ($r_e = 0$)

$$|G| = p^{r_0+r_1+\cdots+r_{e-1}} = p^{r_0-r_1} \cdot (p^2)^{r_1-r_2} \cdots (p^{e-1})^{r_{e-1}-r_e} = o(a_1) \cdots o(a_r).$$

This finishes the proof. \square

Powerfully solvable presentations. It follows in particular from Theorem 3.2 that a powerfully solvable group of order p^n and rank r has a presentation with generators a_1, \dots, a_r and relations

$$(1) \quad a_1^{p^{n_1}} = 1, \dots, a_r^{p^{n_r}} = 1$$

and

$$(2) \quad [a_j, a_i] = a_1^{m_1(i,j)} \cdots a_r^{m_r(i,j)}, \quad 1 \leq i < j \leq r,$$

where all the power indices $m_k(i, j)$ are divisible by p and where furthermore $p^2 | m_k(i, j)$ whenever $k \leq i$. Notice that G is the largest finite p -group satisfying these relations. To see this let H be the largest finite p -group satisfying these relations. The group H/H^{p^2} is powerfully solvable and thus H is powerfully solvable by Theorem 3.2. In particular H is powerful and therefore $|H| \leq o(a_1) \cdots o(a_r)$. However G is a homomorphic image of H and thus $|H| = o(a_1) \cdots o(a_r)$. Hence H is isomorphic to G .

A presentation with generators a_1, \dots, a_r and relations of the form (1) and (2) is called a *powerfully solvable presentation*. We say that such a presentation is *consistent* if the presentation determines a group of order $p^{n_1} \dots p^{n_r}$.

4. CLASSIFICATION OF POWERFUL GROUPS OF ORDER UP TO p^5

In this section we will find all powerful p -groups of order up to and including p^5 . It turns out that these are all powerfully solvable. We will see later that there are many powerful groups of order p^6 that are not powerfully solvable. Let us now turn to our task in this section. There are 2 non-abelian groups of order p^3 . The Heisenberg group of exponent p cannot be powerful as it is of exponent p . The other group is a semidirect product of a cyclic group of order p^2 by a cyclic group of order p :

$$G_1 = \langle a, b : a^{p^2} = b^p = 1, [a, b] = a^p \rangle.$$

Notice that this group is powerfully solvable with a powerfully abelian chain $G > \langle a \rangle > 1$. It is however not powerfully nilpotent as $Z(G)^p = 1$. Adding to this the 3 abelian groups of order p^3 , we see that there are in total 4 powerfully solvable groups of order p^3 .

Before moving on we consider a general setting like in [7] that includes a number of groups that will occur, namely the non-abelian groups of type $(1, \dots, 1, n)$ where n is an integer greater than 1. Suppose

$$G = \langle a_1, \dots, a_t, b \rangle$$

is a powerful group of this type where a_i is of order p and b of order p^n . Notice that $G^p = \langle b^p \rangle$ is cyclic and it follows from [5, Corollary 3.3] that $G^p \leq Z(G)$. In particular G is nilpotent of class at most 2 and $[G, G]^p = [G^p, G] = 1$. Next observe that $\Omega_1(G) = \langle a_1, \dots, a_t, b^{p^{n-1}} \rangle$ where $\Omega_1(G)$ is the subgroup consisting of all elements of order dividing p . Thus $[G, G] = \langle b^{p^{n-1}} \rangle$. Now, consider the vector space $V = \Omega_1(G)G^p/G^p$ over the field \mathbb{F}_p of p elements. The commutator operation naturally induces an alternating form on V through

$$(xG^p, yG^p) = \lambda \text{ if } [x, y] = b^{\lambda p^{n-1}}.$$

Without loss of generality we can suppose that our generators have been chosen such that we get the following orthogonal decomposition (see for example [1], Chapter 6)

$$V = \langle a_1 G^p, a_2 G^p \rangle \oplus \dots \oplus \langle a_{2s-1} G^p, a_{2s} G^p \rangle \oplus V^\perp$$

where $V^\perp = \langle a_{2s+1} G^p, \dots, a_t G^p \rangle$ and $(a_{2i-1} G^p, a_{2i} G^p) = 1$ for $i = 1, \dots, s$. There are now two cases to consider, depending on whether or not $Z(G) \leq \Omega_{n-1}(G)$.

Suppose first that $Z(G) \not\leq \Omega_{n-1}(G)$. This means that $Z(G)$ contains some element $b^l u$ with $u \in \langle a_1, \dots, a_t \rangle$ and $0 < l < p$. Thus without loss of generality we can assume that $b \in Z(G)$. We thus get a powerful group $G = A(n, t, s)$ with relations

$$\begin{aligned} a_1^p &= \dots = a_t^p = b^{p^n} = 1; \\ [a_{2i-1}, a_{2i}] &= b^{p^{n-1}} \text{ for } i = 1, \dots, s; \\ [a_i, a_j] &= 1 \text{ otherwise for } 1 \leq i < j \leq t; \\ [a_i, b] &= 1 \text{ for } 1 \leq i \leq t. \end{aligned}$$

Notice that we have $\langle b \rangle \leq Z(G)$ and $[G, G] \leq \langle b^p \rangle$ and thus these groups are all powerfully nilpotent, as was observed in [7]. Notice that for a fixed $n \geq 2$ and $t \geq 2$ we get $\lfloor t/2 \rfloor$ such groups.

We then consider the case when $Z(G) \leq \Omega_{n-1}(G)$. Notice first that replacing b by a suitable $ba_1^{\alpha_1} \dots a_{2s}^{\alpha_{2s}}$, we can assume that b commutes with a_1, \dots, a_{2s} . As $b \notin Z(G)$ we then must

have $t > 2s$ and similarly, replacing a_i by a suitable $a_{2s+1}^{\alpha_{2s+1}} \cdots a_t^{\alpha_t}$, we can pick our generators a_{2s+1}, \dots, a_t such that $[a_{2s+1}, b] = b^{p^{n-1}}$ and $[a_{2s+2}, b] = \cdots = [a_t, b] = 1$. We thus arrive at a group $G = B(n, t, s)$ satisfying the relations

$$\begin{aligned} a_1^p &= \cdots = a_t^p = b^{p^n} = 1; \\ [a_{2i-1}, a_{2i}] &= b^{p^{n-1}} \text{ for } i = 1, \dots, s; \\ [a_i, a_j] &= 1 \text{ otherwise for } 1 \leq i < j \leq t; \\ [a_1, b] &= \cdots = [a_{2s}, b] = [a_{2s+2}, b] = \cdots = [a_t, b] = 1 \text{ for } 1 \leq i \leq t; \\ [a_{2s+1}, b] &= b^{p^{n-1}}. \end{aligned}$$

Notice that for a fixed $n \geq 2$ and $t \geq 1$ there are $\lfloor (t+1)/2 \rfloor$ such groups. Notice also that when $n \geq 3$ then the group is powerfully nilpotent as $\langle b^p \rangle \leq Z(G)$ and $[G, G] \leq \langle b^{p^2} \rangle$. For $n = 2$ this is not the case but the group is still powerfully solvable as we have a powerfully abelian chain $G > \langle b \rangle > 1$ with $[G, G] \leq \langle b^p \rangle$. We are now ready for groups of order p^4 . In the following we will omit writing relations of the form $[x, y] = 1$.

Groups of order p^4 . From our analysis of non-abelian groups of rank 2 we get two such groups:

$$G_2 = \langle a, b : a^{p^2} = b^{p^2} = 1, [a, b] = a^p \rangle \text{ and } G_3 = \langle a, b : a^{p^3} = b^p = 1, [a, b] = a^{p^2} \rangle.$$

Here G_3 is furthermore powerfully nilpotent. The only non-abelian groups apart from these are of type $(1, 1, 2)$ and from the analysis of such groups above we know there are two groups:

$$G_4 = A(2, 2, 1) = \langle a, b, c : a^p = b^p = c^{p^2} = 1, [a, b] = c^p \rangle,$$

and

$$G_5 = B(2, 2, 0) = \langle a, b, c : a^p = b^p = c^{p^2} = 1, [a, c] = c^p \rangle.$$

Apart from these there are 5 abelian groups and we thus get in total **9** groups.

Groups of order p^5 . Again our analysis of groups of rank 2 and those of type $(1, 1, 3)$ and $(1, 1, 1, 2)$ gives us the following non-abelian powerfully solvable groups:

$$\begin{aligned} G_6 &= \langle a, b : a^{p^2} = b^{p^3} = 1, [a, b] = a^p \rangle, \quad G_7 = \langle a, b : a^{p^3} = b^{p^2} = 1, [a, b] = a^p \rangle, \\ G_8 &= \langle a, b : a^{p^3} = b^{p^2} = 1, [a, b] = a^{p^2} \rangle, \quad G_9 = \langle a, b : a^{p^4} = b^p = 1, [a, b] = a^{p^3} \rangle, \end{aligned}$$

and

$$\begin{aligned} G_{10} &= A(3, 2, 1) = \langle a, b, c : a^p = b^p = c^{p^3} = 1, [a, b] = c^{p^2} \rangle; \\ G_{11} &= B(3, 2, 0) = \langle a, b, c : a^p = b^p = c^{p^3} = 1, [a, c] = c^{p^2} \rangle; \\ G_{12} &= A(2, 3, 1) = \langle a, b, c, d : a^p = b^p = c^p = d^{p^2} = 1, [a, b] = d^p \rangle; \\ G_{13} &= B(2, 3, 0) = \langle a, b, c, d : a^p = b^p = c^p = d^{p^2} = 1, [a, b] = d^p, [c, d] = d^p \rangle; \\ G_{14} &= B(2, 3, 1) = \langle a, b, c, d : a^p = b^p = c^p = d^{p^2} = 1, [a, b] = d^p, [c, d] = d^p \rangle. \end{aligned}$$

Here $G_8, G_9, G_{10}, G_{11}, G_{12}$ are furthermore powerfully nilpotent. Apart from these 9 groups, there are 7 abelian groups. We are now only left with the non-abelian groups of type $(1, 2, 2)$ that will contain a number of different groups and we need to deal with a number of subcases.

Suppose that we have generators a, b, c of orders p, p^2, p^2 .

Case 1. ($Z(G)^p \neq 1$). Notice that we then must have $|Z(G)^p| = p$ as otherwise $G/Z(G)$ is cyclic and thus G abelian. We can assume that $c \in Z(G)$ and that $Z(G)^p = \langle c^p \rangle$. Notice also that $[G, G] = \langle [a, b] \rangle$ is cyclic. There are two possibilities. On the one hand, if $[G, G] \leq Z(G)^p$, then we can choose our generators so that we get a group with the following presentation:

$$G_{15} = \langle a, b, c : a^p = b^{p^2} = c^{p^2} = 1, [a, b] = c^p \rangle.$$

On the other hand if $[G, G] \not\leq Z(G)^p$, it is not difficult to see that we can pick our generators so that we get a group with the presentation

$$G_{16} = \langle a, b, c : a^p = b^{p^2} = c^{p^2} = 1, [a, b] = b^p \rangle.$$

Notice that both these groups are powerfully solvable and that G_{15} is furthermore powerfully nilpotent.

Case 2. ($Z(G)^p = 1$ and $G/Z(G)$ has rank 2). Then we must have $a \in Z(G)$. It is not difficult to see that in this case we can choose b, c such that $[b, c] = c^p$ and we get the powerfully solvable group

$$G_{17} = \langle a, b, c : a^p = b^{p^2} = c^{p^2} = 1, [b, c] = c^p \rangle.$$

Before considering further cases, we first show that if $Z(G)^p = 1$ and $G/Z(G)$ has rank 3, then we must have $[G, G] = G^p$. Note that $|G^p| = p^2$, so suppose by contradiction, that $|G'| = p$. Observe that $G^p \leq Z(G)$, so $G/Z(G)$ is a vector space over \mathbb{F}_p . Then, the commutator map in G induces a non-degenerate alternating form on $G/Z(G)$, and so $\dim_{\mathbb{F}_p}(G/Z(G))$ is even. This is a contradiction since $G/Z(G)$ has rank 3. We have thus shown that $[G, G] = G^p$. In order to distinguish further between different cases, we next turn our attention to $[\Omega_1(G), G]$. Notice that $\Omega_1(G) = \langle a \rangle G^p$. As $a \notin Z(G)$ either $|[\Omega_1(G), G]|$ is of size p or p^2 .

Case 3. ($Z(G)^p = 1$, $G/Z(G)$ of rank 3 and $|[\Omega_1(G), G]| = p$). Without loss of generality we can assume that $[\Omega_1(G), G] = \langle c^p \rangle$. There are two possibilities. Either $c \in C_G(\Omega_1(G)) = C_G(a)$ or not. Suppose first that $c \in C_G(\Omega_1(G))$. Then we have $[a, c] = 1$, and we can pick b such that $[a, b] = c^p$. Replacing b by bc^l does not change these relations and thus we can assume that $[b, c] = b^{\alpha}$ for some $0 < \alpha < p$. If we let β be the inverse of α modulo p and we replace a, c by a^β, c^β , then we arrive at a group with presentation

$$G_{18} = \langle a, b, c : a^p = b^{p^2} = c^{p^2} = 1, [a, b] = c^p, [b, c] = b^p \rangle.$$

Notice that this is a powerfully solvable group with a powerfully abelian chain $G > \langle b, c \rangle > \langle b \rangle > 1$. Suppose now $c \notin C_G(\Omega_1(G))$. Since $|[\Omega_1(G), G]| = |[a, G]| = p$, it follows that the conjugacy class of a has order p , and so $|G : C_G(a)| = p$. Thus, we can pick b such that $b \in C_G(a)$ and $[a, b] = 1$. Replacing a by a suitable power of a we can suppose that $[a, c] = c^p$. As before, replacing b by bc^l does not change these relations, so we can also assume $[b, c] = b^{\alpha p}$ for some $0 < \alpha < p$. Finally, if we let β be the inverse of α modulo p and we replace c by c^β , we arrive at a group with presentation

$$G_{19} = \langle a, b, c : a^p = b^{p^2} = c^{p^2} = 1, [a, c] = c^p, [b, c] = b^p \rangle.$$

This group is powerfully solvable with powerfully abelian chain $G > \langle b, c \rangle > \langle b \rangle > 1$.

Case 4. ($Z(G)^p = 1$, $G/Z(G)$ of rank 3 and $|[\Omega_1(G), G]| = p^2$). In this case, commutation with a induces a bijective linear map

$$\begin{aligned} F_a : G/\Omega_1(G) &\longrightarrow G^p \\ x\Omega_1(G) &\longmapsto [a, x]. \end{aligned}$$

Identifying $x\Omega_1(G)$ with x^p , we can think of F_a as a linear operator on a two dimensional vector space over \mathbb{F}_p . Also replacing b, c by a suitable ba^r, ca^s we can assume throughout that $[b, c] = 1$. All the groups are going to be powerfully solvable with powerfully abelian chain $G > \langle b, c \rangle > 1$.

Case 4.1. (F_a is a scalar multiplication). Notice that this property still holds if we replace a by any power of a and thus it is independent of what a we pick in $\Omega_1(G) \setminus G^p$. This is thus a

characteristic property of G . Replacing a with a power of itself we can assume that F_a is the identity map. This gives us the group

$$G_{20} = \langle a, b, c : a^p = b^{p^2} = c^{p^2} = 1, [a, b] = b^p, [a, c] = c^p \rangle.$$

Case 4.2. (F_a is not a scalar multiplication). Again we see that this is a characteristic property of G . We can now pick b and c such that

$$[a, b] = c^p, [a, c] = b^{p\alpha} c^{p\beta}.$$

Notice that the matrix for F_a is

$$\begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix}$$

with determinant $-\alpha$. This is an invariant for the given a that does not depend on our choice of b and c . If we replace a by a^r and c by c^r then we get

$$[a, b] = c^p, [a, c] = b^{p\alpha r^2} c^{p\beta r},$$

and the new determinant becomes $-\alpha r^2$. Pick some fixed τ such that $-\tau$ is a non-square in \mathbb{F}_p . With appropriate choice of r we can then assume that the determinant of F_a is $-\alpha$ where either $\alpha = -1$ or $\alpha = \tau$. We thus have a group with one of the two presentations

$$G_{21}(\beta) = \langle a, b, c : a^p = b^{p^2} = c^{p^2} = 1, [a, b] = c^p, [a, c] = b^{-1} c^{p\beta}, [b, c] = 1 \rangle,$$

and

$$G_{22}(\beta) = \langle a, b, c : a^p = b^{p^2} = c^{p^2} = 1, [a, b] = c^p, [a, c] = b^\tau c^{p\beta}, [b, c] = 1 \rangle.$$

Suppose we pick a different $\bar{b} = b^r c^s$. Then for $\alpha \in \{-1, \tau\}$ we have

$$[a, \bar{b}] = [a, b]^r [a, c]^s = c^{pr} (b^{p\alpha} c^{p\beta})^s = b^{ps\alpha} c^{p(r+s\beta)} = \bar{c}^p$$

where $\bar{c} = b^{s\alpha} c^{r+s\beta}$. Then

$$\begin{aligned} [a, \bar{c}] &= [a, b]^{s\alpha} [a, c]^{r+s\beta} \\ &= c^{ps\alpha} (b^{p\alpha} c^{p\beta})^{r+s\beta} \\ &= (b^r c^s)^{p\alpha} \cdot (b^{s\alpha} c^{r+s\beta})^{p\beta} \\ &= \bar{b}^{p\alpha} \bar{c}^{p\beta}. \end{aligned}$$

This shows that for the given $\alpha \in \{-1, \tau\}$, the constant $\beta \in \mathbb{F}_p$ is an invariant and we get p distinct groups $G_{21}(\beta)$ and p distinct groups $G_{22}(\beta)$.

Adding up we have 7 abelian groups and the groups $G_6, \dots, G_{20}, G_{21}(\beta), G_{22}(\beta)$, giving us in total $22 + 2p$ groups of order p^5 .

Notice that we have seen that all powerful groups of order up to and including p^5 are powerfully solvable. Now take a powerful group of order p^6 . Suppose it has a generator a of order p , say $G = \langle a, H \rangle$ where $H < G$. Notice that H is then powerful of order p^5 and thus powerfully solvable. As $[G, G] \leq H^p$ we then see that G is powerfully solvable. Thus all powerful groups of order p^6 are powerfully solvable with the possible exceptions of some groups of type $(2, 2, 2)$. We will see later that there are a number of groups of type $(2, 2, 2)$ that are not powerfully solvable.

5. GROWTH

Let G be a powerfully solvable group of order p^n . From Theorem 3.2 and the discussion in Section 3, we know that we may assume that $G = \langle a_1, \dots, a_y, a_{y+1}, \dots, a_{y+x} \rangle$ where $o(a_1) = \dots = o(a_y) = p$ and $o(a_{y+1}) = \dots = o(a_{y+x}) = p^2$. Furthermore the generators can be chosen such that $|G| = p^{y+2x}$ and

$$[a_j, a_i] = a_{i+1}^{p\alpha_{i+1}(i,j)} \dots a_{y+x}^{p\alpha_{y+x}(i,j)},$$

for $1 \leq i < j \leq y+x$, where $0 \leq \alpha_k(i, j) \leq p-1$ for $k = i+1, \dots, y+x$. For each such pair (i, j) where $1 \leq i \leq y$ there are p^x possible relations for $[a_j, a_i]$. There are $yx + \binom{y}{2}$ such pairs. On the other hand, for a pair (i, j) where $y+1 \leq i \leq y+x$, for each given i there are $y+x-i$ such pairs and p^{y+x-i} possible relations $[a_j, a_i]$. Adding up we see that the number of solvable presentations is $p^{h(x)}$ where

$$\begin{aligned} h(x) &= \left(yx + \binom{y}{2} \right) x + 1^2 + 2^2 + \dots + (x-1)^2 \\ &= \left((n-2x)x + \binom{n-2x}{2} \right) x + \frac{x(2x-1)(x-1)}{6} \\ &= \frac{1}{3}x^3 - \frac{(2n-1)}{2}x^2 + \frac{3n(n-1)+1}{6}x. \end{aligned}$$

Thus

$$h'(x) = x^2 - (2n-1)x + \frac{3n(n-1)+1}{6},$$

whose roots are $\frac{2n-1}{2} - \sqrt{\frac{1}{2}n^2 - \frac{n}{2} + \frac{1}{12}}$ and $\frac{2n-1}{2} + \sqrt{\frac{1}{2}n^2 - \frac{n}{2} + \frac{1}{12}}$. For large values of n we have that the first root is between 0 and $n/2$ whereas the latter is greater than n . Thus, for large n , the largest value of h in the interval between 0 and $n/2$ is $h(x(n))$ where $x(n) = \frac{2n-1}{2} - \sqrt{\frac{1}{2}n^2 - \frac{n}{2} + \frac{1}{12}}$. Now $\lim_{n \rightarrow \infty} x(n)/n = 1 - \frac{1}{\sqrt{2}}$. Therefore

$$\lim_{n \rightarrow \infty} \frac{h(x(n))}{n^3} = \lim_{n \rightarrow \infty} \frac{1}{3}(x(n)/n)^3 - (x(n)/n)^2 + \frac{1}{2}(x(n)/n) = \frac{-1 + \sqrt{2}}{6}.$$

We now argue in a similar way as in [5]. Let n be fixed. For any integer x where $0 \leq x \leq n/2$, let $\mathcal{P}(n, x)$ be the collection of all powerfully solvable presentations as above. It is not difficult to see that those presentations are consistent and thus the resulting group is of order p^n and rank $n-x$. Furthermore $a_1^p = \dots = a_{n-2x}^p = 1$ and $a_{n-2x+1}^{p^2} = \dots = a_{n-x}^{p^2} = 1$. We have just seen that, for large values of n , if we pick $x(n)$ such that the number of presentations is maximal then

$$|\mathcal{P}(n, x(n))| = p^{\alpha n^3 + o(n^3)}$$

where $\alpha = \frac{-1 + \sqrt{2}}{6}$. Let \mathcal{P}_n be the total number of the powerfully solvable presentations where $0 \leq x \leq n/2$. Then $\mathcal{P}_n = \mathcal{P}(n, 0) \cup \mathcal{P}(n, 1) \cup \dots \cup \mathcal{P}(n, \lfloor n/2 \rfloor)$ and thus

$$p^{\alpha n^3 + o(n^3)} = |\mathcal{P}(n, x(n))| \leq |\mathcal{P}_n| = |\mathcal{P}(n, 0)| + \dots + |\mathcal{P}(n, \lfloor n/2 \rfloor)| \leq n|\mathcal{P}(n, x(n))| = p^{\alpha n^3 + o(n^3)}.$$

This shows that $|\mathcal{P}_n| = p^{\alpha n^3 + o(n^3)}$. Let us show that this is also the growth of powerfully solvable groups of exponent p^2 with respect to the order p^n . Clearly $p^{\alpha n^3 + o(n^3)}$ gives us an upper bound. We want to show that this is also a lower bound. Let $x = x(n)$ be as above and let a_1, \dots, a_{n-x} be a set of generators for a powerfully solvable group G where $a_1^p = \dots = a_{n-2x}^p = 1$ and $a_{n-2x+1}^{p^2} = \dots = a_{n-x}^{p^2} = 1$. Notice that $\langle a_1, \dots, a_{n-2x} \rangle G^p = \Omega_1(G)$, which is a characteristic subgroup of G . It will be useful to consider a larger class of presentations for powerfully solvable groups of order p^n where we still require $a_1^p = \dots = a_{n-2x}^p = 1$ and $a_{n-2x+1}^{p^2} = \dots = a_{n-x}^{p^2} = 1$.

We let $\mathcal{Q}(n, x) = \mathcal{Q}(n, x(n))$ be the collection of all presentations with additional commutator relations

$$[a_i, a_j] = a_1^{p\alpha_1(i,j)} \cdots a_{n-x}^{p\alpha_{n-x}(i,j)}.$$

The presentation is included in $\mathcal{Q}(n, x)$ provided the resulting group is powerfully solvable of order p^n . Notice that $G^p \leq Z(G)$ and as a result the commutator relations above only depend on the cosets $\overline{a_1} = a_1 G^p, \dots, \overline{a_{n-x}} = a_{n-x} G^p$ and not on the exact values of a_1, \dots, a_{n-x} . Consider the vector space $V = G/G^p$ over \mathbb{F}_p and let $W = \mathbb{F}_p \overline{a_1} + \cdots + \mathbb{F}_p \overline{a_{n-2x}}$. Then let

$$H = \{ \phi \in \text{GL}(n-x, p) : \phi(W) = W \}.$$

There is now a natural action from H on $\mathcal{Q}(n, x)$. Suppose we have some presentation with generators a_1, \dots, a_{n-x} as above. Let $\phi \in H$ and suppose

$$\overline{a_i}^\phi = \beta_1(i) \overline{a_1} + \cdots + \beta_{n-x}(i) \overline{a_{n-x}}.$$

We then get a new presentation in $\mathcal{Q}(n, x)$ for G with respect to the generators b_1, \dots, b_{n-x} where $b_i = a_1^{\beta_1(i)} \cdots a_{n-x}^{\beta_{n-x}(i)}$.

Suppose there are l powerfully solvable groups of exponent p^2 and order p^n where furthermore $|G^p| = p^x$. Pick powerfully solvable presentations $p_1, \dots, p_l \in \mathcal{P}(n, x)$ for these. Let q be powerfully solvable presentation in $\mathcal{P}(n, x)$ of a group K with generators b_1, \dots, b_{n-x} . Then q is also a presentation for an isomorphic group G with presentation p_i and generators a_1, \dots, a_{n-x} . Let $\phi : K \rightarrow G$ be an isomorphism and let $\psi : K/K^p \rightarrow G/G^p$ be the corresponding linear isomorphism. This gives us a linear automorphism $\tau \in H$ induced by $\tau(\overline{a_i}) = \psi(\overline{b_i})$. Thus $q = p_i^\tau$. Therefore

$$\mathcal{P}(n, x) \subseteq p_1^H \cup p_2^H \cup \cdots \cup p_l^H.$$

From this we get

$$p^{an^3+o(n^3)} = |\mathcal{P}(n, x)| \leq |p_1^H| + \cdots + |p_l^H| \leq lp^{n^2},$$

and it follows that $l \geq p^{an^3+o(n^3)}$. We thus get the following result.

Theorem 5.1. *The number of powerfully solvable groups of exponent p^2 and order p^n is $p^{an^3+o(n^3)}$, where $\alpha = \frac{-1+\sqrt{2}}{6}$.*

As mentioned in [5] the growth of all powerful p -groups of exponent p^2 and order p^n is $p^{\frac{2}{27}n^3+o(n^3)}$. This claim was though not proved and we will fill in the details here.

As before we consider a group G of order $p^n = p^{y+2x}$ with generators a_1, \dots, a_{y+x} where $o(a_1) = \cdots = o(a_y) = p$ and $o(a_{y+1}) = \cdots = o(a_{y+x}) = p^2$. This time we can though include all powerful relations

$$[a_j, a_i] = a_1^{p\alpha_{y+1}(i,j)} \cdots a_{y+x}^{p\alpha_{y+x}(i,j)}$$

for $1 \leq i < j \leq y+x$, where $0 \leq \alpha_k(i, j) \leq p-1$ for $k = y+1, \dots, y+x$. For each such pair (i, j) there are p^x possible relations for $[a_j, a_i]$. We thus see that the number of presentations is $p^{h(x)}$ where

$$h(x) = \binom{y+x}{2} x = \binom{n-x}{2} x = \frac{x^3}{2} - \frac{(2n-1)}{2} x^2 + \frac{n(n-1)}{2} x.$$

Thus

$$h'(x) = \frac{3}{2} \left(x^2 - \frac{2(2n-1)}{3} x + \frac{n(n-1)}{3} \right)$$

and using the same kind of analysis as before we see that for a large n , h takes its maximal value for $x(n) = \frac{2n-1}{3} - \sqrt{\frac{n^2}{9} - \frac{n}{9} + \frac{1}{9}}$. Notice that $\lim_{n \rightarrow \infty} \frac{x(n)}{n} = 1/3$. Therefore

$$\lim_{n \rightarrow \infty} \frac{h(x(n))}{n^3} = \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \left(\frac{n - x(n)}{n} \right) \cdot \left(\frac{n - 1 - x(n)}{n} \right) \cdot \frac{x(n)}{n} = 2/27.$$

The same argument as above shows then that the growth of all powerful groups of exponent p^2 with respect to order p^n is $p^{\frac{2}{27}n^3 + o(n^3)}$.

Later on we will be working with a special subclass \mathcal{P} of powerful p -groups, namely those that are of type $(2, \dots, 2)$ with $r \geq 1$. In this case the number of presentations for groups of order p^n , n even, is $p^{h(n)}$ where $h(n) = \binom{n/2}{2}n/2$ and

$$\lim_{n \rightarrow \infty} \frac{h(n)}{n^3} = \lim_{n \rightarrow \infty} \frac{n/2(n/2 - 1)n/2}{2n^3} = 1/16.$$

Thus the growth here is $p^{\frac{1}{16}n^3 + o(n^3)}$.

6. GROUPS OF TYPE $(2, \dots, 2)$

We have seen that powerful nilpotence and powerful solvability is preserved under taking quotients. These properties however work badly under taking subgroups. Our next two results underscore this.

Proposition 6.1. *Let G be any powerful p -group of exponent p^2 . There exists a powerfully nilpotent group H of exponent p^2 and powerful class 2 such that G is powerfully embedded in H .*

Proof Suppose $G = \langle a_1, \dots, a_r \rangle$ where $a_1^p = \dots = a_s^p = 1$, $a_{s+1}^{p^2} = \dots = a_r^{p^2} = 1$ and where $|G| = p^{s+2(r-s)}$. Let $N = \langle x_{s+1} \rangle \times \dots \times \langle x_r \rangle$ be a direct product of cyclic groups of order p^2 . Let $H = (G \times N)/M$, where $M = \langle a_{s+1}^p x_{s+1}^{-p}, \dots, a_r^p x_r^{-p} \rangle$. Notice that $[G, H] = [G, G] \leq G^p$ and thus G is powerfully embedded in H . Also, as $[H, H] = G^p = N^p$, we see that

$$1 \leq \langle x_1, \dots, x_r \rangle \leq H$$

is a powerfully central chain and thus H is powerfully nilpotent of powerful class at most 2. \square

Remark. (1) There exist powerful p -groups of exponent p^2 that are not powerfully solvable and thus a powerfully embedded subgroup of a powerfully nilpotent group of powerful class 2 does not even need to be powerfully solvable.

(2) There exist powerfully nilpotent groups of exponent p^2 that are of arbitrary large powerful class and so the proposition above shows that a powerfully nilpotent group of powerful class 2 could have a powerfully embedded powerfully nilpotent subgroup of arbitrary large powerful class.

Next result shows that the subgroup structure of a powerfully nilpotent group of powerful class 2 is even more arbitrary. Notice that such a group is in particular nilpotent of class 2 and it turns out that any finite p -group of class 2 can occur as a subgroup.

Proposition 6.2. *Let G be any finite p -group of nilpotency class 2. There exists a powerfully nilpotent group H of powerful class 2 that contains G as a subgroup.*

Proof Suppose $[G, G]$ has a basis a_1, \dots, a_m as an abelian group where $o(a_i) = p^{j_i}$. Let $N = \langle x_1 \rangle \times \dots \times \langle x_m \rangle$ be a direct product of cyclic groups where $o(x_i) = p^{j_i+1}$. Now let $H = (G \times N)/M$ where $M = \langle a_1 x_1^{-p}, \dots, a_m x_m^{-p} \rangle$. Then G embeds as a subgroup of H . Notice also that

$$1 \leq \langle x_1, \dots, x_m \rangle \leq H$$

is powerfully central and thus H is powerfully nilpotent of powerful class 2. \square

Thus powerful nilpotence and powerful solvability are in general not as satisfactory as notions for powerful groups as nilpotence and solvability for the class of all groups. For a rich subclass of powerful groups things however turn out much better. This is the class \mathcal{P} of all powerful groups of type $(2, \dots, 2)$ that we considered in Section 5.

For a group $G \in \mathcal{P}$ we have that $G^p \leq Z(G)$. It follows that the map $G/G^p \rightarrow G^p$, $aG^p \mapsto a^p$ is a bijection and therefore, for any $H \geq G^p$, we have $|H/G^p| = |H^p|$.

Lemma 6.3. *Let $G \in \mathcal{P}$ and $H, K \leq G$ where $G^p \leq K$. Then $H^p \cap K^p = (H \cap K)^p$.*

Proof We have

$$\begin{aligned} |H^p \cap K^p| &= |(HG^p)^p \cap K^p| = \frac{|(HG^p)^p| \cdot |K^p|}{|(HK)^p|} \\ &= \frac{|HG^p/G^p| \cdot |K/G^p|}{|HK/G^p|} \\ &= |(HG^p \cap K)/G^p| \\ &= |(H \cap K)G^p/G^p| \\ &= |(H \cap K)^p|. \end{aligned}$$

As $(H \cap K)^p \leq H^p \cap K^p$ it follows that $H^p \cap K^p = (H \cap K)^p$. \square

Theorem 6.4. *Let G be a powerfully nilpotent group in \mathcal{P} and let H be a powerful subgroup of G . Then H is powerfully nilpotent of powerful class less than or equal to the powerful class of G .*

Proof Suppose G has powerful nilpotence class c and that we have a powerfully central chain $G = G_0 > G_1 > \dots > G_c = 1$. As $G^p \leq Z(G)$ and $(G^p)^p = 1$, multiplying a term by G^p makes no difference. Also as the powerful class is c we get a strictly decreasing powerfully central chain $G = G_0 > G_1 G^p > \dots > G_{c-1} G^p > 1$. Without loss of generality we can thus assume that G_1, \dots, G_{c-1} contain G^p as a subgroup. We claim that

$$H = H \cap G_0 \geq H \cap G_1 \geq \dots \geq H \cap G_{c-1} \geq 1$$

is powerfully central. Using Lemma 6.3 we have

$$[H \cap G_i, H] \leq [H, H] \cap [G_i, G] \leq H^p \cap G_{i+1}^p = (H \cap G_{i+1})^p,$$

for $0 \leq i \leq c-1$. Hence H is powerfully nilpotent of powerful class at most c .

Theorem 6.5. *Let G be a powerfully solvable group in \mathcal{P} and let H be a powerful subgroup of G . Then H is powerfully solvable of powerful derived length less than or equal to the powerful derived length of G .*

Proof Suppose the powerful derived length of G is d and that we have a powerfully abelian chain $G = G_0 > G_1 > \dots > G_d = 1$. Arguing like in the proof of the previous theorem, we can assume that G_{d-1} contains G^p . We show that

$$H = H \cap G_0 \geq H \cap G_1 \geq \dots \geq H \cap G_{d-1} \geq H \cap G_d = 1$$

is a powerfully abelian chain. Using Lemma 6.3, we have $[H \cap G_i, H \cap G_i] \leq [H, H] \cap [G_i, G_i] \leq H^p \cap G_{i+1}^p = (H \cap G_{i+1})^p$. This shows that H is powerfully solvable of powerful derived length at most d . \square

We introduce some useful notation. We use $H \leq_p G$ to stand for $H, G \in \mathcal{P}$ and $H \leq G$. We use $H \trianglelefteq_p G$ for $H, G \in \mathcal{P}$ and H powerfully embedded in G . The notations $H <_p G$ and

$H \triangleleft_p G$ are defined naturally in a similar way.

Let G be a powerful p -group in \mathcal{P} and let $V = G/G^p$ be the associated vector space over \mathbb{F}_p . The structure of G is determined by the commutator relations

$$(3) \quad [a, b] = c^p,$$

where there exists such $c \in G$ for each pair a, b in G . Notice that $[a, b]$ and c^p only depend on the cosets aG^p, bG^p and cG^p . Identifying the two vector spaces G/G^p and G^p under the map $G/G^p \rightarrow G^p, xG^p \mapsto x^p$, we get a natural alternating product on V with the relations (1) translating to

$$[aG^p, bG^p] = cG^p.$$

Let \mathcal{G} be the collection of all powerful subgroups of G that are of type $(2, \dots, 2)$ and let \mathcal{V} be the collection of all the alternating subalgebras of V . So for U to be a subalgebra of V it needs to be a subspace where $[U, U] \leq U$. Notice that $[H, H] \leq H^p$ translates to $[HG^p/G^p, HG^p/G^p] \leq HG^p/G^p$. Recall that for $H, K \in \mathcal{G}$ we write $H \trianglelefteq_p K$ for H powerfully embedded in K . For $U, W \in \mathcal{V}$ we likewise write $U \trianglelefteq W$ for U an ideal of W . Notice that $[H, K] \leq H^p$ translates to $[HG^p/G^p, KG^p/G^p] \leq HG^p/G^p$.

If $H, G \in \mathcal{P}$ such that G is powerfully nilpotent and $H \trianglelefteq_p G$, then the quotient G/H has naturally the structure of powerful group of type $(2, \dots, 2)$ with $[aH, bH] = [a, b]H$.

Definition. We say that a group $G \in \mathcal{P}$ is *powerfully simple* if $G \neq 1$ and if $H \triangleleft_p G$ implies that $H = 1$.

Definition. Let $H, G \in \mathcal{P}$ with $H \triangleleft_p G$. We say that H is a maximal powerfully embedded \mathcal{P} -subgroup of G if there is no $H < K < G$ such that $K \trianglelefteq_p G$.

Lemma 6.6. *Let $G \in \mathcal{P}$. We have that H is a maximal powerfully embedded \mathcal{P} -subgroup of G if and only if G/H is powerfully simple.*

Proof Let $H < K < G$. Now as H is powerful of type $(2, \dots, 2)$ we have $H \cap G^p = H^p$. Therefore $[K, G] \leq K^p H$ if and only if

$$[K, G] \leq (K^p H) \cap G^p = K^p (H \cap G^p) = K^p H^p = K^p.$$

The result follows from this. \square

Remark. Let $H, K \in \mathcal{G}$ and let U and W be the associated alternating algebras in \mathcal{V} . Suppose that H is powerfully embedded in K . Then K/H is powerfully simple if and only if W/U is a simple alternating algebra and the latter happens if and only if U is a maximal ideal of W .

We will next prove a Jordan-Hölder type theorem for alternating algebras. Suppose $A, B, a, b \in \mathcal{V}$ where $A \triangleleft B$ and $a \triangleleft b$. Let $\mathcal{I}_A^B = \{Z : A \leq Z \leq B\}$ and $\mathcal{I}_a^b = \{x : a \leq x \leq b\}$. We get natural projections $P : \mathcal{I}_a^b \rightarrow \mathcal{I}_A^B$ and $Q : \mathcal{I}_A^B \rightarrow \mathcal{I}_a^b$ given by

$$P(x) = A + B \cap x \text{ and } Q(Z) = a + b \cap Z.$$

Lemma 6.7. *We have $P(a) \trianglelefteq P(b)$ and $Q(A) \trianglelefteq Q(B)$. Furthermore $P(b)/P(a)$ is isomorphic to $Q(B)/Q(A)$.*

Proof Notice that $P(a) = A + B \cap a, P(b) = A + B \cap b, Q(A) = a + b \cap A$ and $Q(B) = a + b \cap B$. As $A \trianglelefteq B$, we have $[P(a), P(b)] = [A + B \cap a, A + B \cap b] \leq A + [B \cap a, B \cap b]$. Now as B is a subalgebra and $a \trianglelefteq b$ we have that this is contained in $A + B \cap a$. The second claim follows from this by symmetry.

Now for $P(b)/P(a)$, notice first that we have

$$B \cap b \cap (A + B \cap a) = B \cap b \cap A + B \cap a = A \cap b + B \cap a,$$

and for $u, v, w \in B \cap b$ it follows that

$$[u, v] + A + B \cap a = w + A + B \cap a \Leftrightarrow [u, v] + A \cap b + B \cap a = w + A \cap b + B \cap a.$$

By symmetry

$$[u, v] + a + b \cap A = w + a + b \cap A \Leftrightarrow [u, v] + a \cap B + b \cap A = w + a \cap B + b \cap A.$$

The isomorphism of $P(b)/P(a)$ and $Q(B)/Q(A)$ follows from this. \square

The Jordan-Hölder theorem for alternating algebras is proved from this in the standard way.

Definition. Let V be an alternating algebra. A chain $0 = U_0 \triangleleft U_1 \triangleleft \cdots \triangleleft U_n = V$ is a *composition series* for V if all the factors $U_1/U_0, \dots, U_n/U_{n-1}$ are simple alternating algebras.

Theorem 6.8. *Let V be an alternating algebra. Then all composition series have the same length and same composition factors up to order.*

Definition. Let $G \in \mathcal{P}$. A chain $1 = H_0 \triangleleft_p H_1 \triangleleft_p \cdots \triangleleft_p H_n = G$ is a *powerful composition series* for G if all the factors $H_1/H_0, \dots, H_n/H_{n-1}$ are powerfully simple.

Theorem 6.9. *Let G be a powerful p -group of type $(2, \dots, 2)$ with two powerful composition series, say*

$$1 = H_0 \triangleleft_p H_1 \triangleleft_p \cdots \triangleleft_p H_n = G$$

and

$$1 = K_0 \triangleleft_p K_1 \triangleleft_p \cdots \triangleleft_p K_m = G.$$

Then $m = n$ and the powerfully simple factors $H_1/H_0, H_2/H_1, \dots, H_n/H_{n-1}$ are isomorphic to $K_1/K_0, K_2/K_1, \dots, K_n/K_{n-1}$ (in some order).

Proof Replace the terms H_i, K_j by their associated alternating algebras U_i, V_j . The result now follows from the Jordan-Hölder theorem for alternating algebras. \square

Definition. We refer to the unique factors of a powerful composition series of a group $G \in \mathcal{P}$ as the *powerful composition factors* of G .

Corollary 6.10. *A group $G \in \mathcal{P}$ is powerfully solvable if and only if the powerful composition factors are cyclic of order p^2 .*

Proof Any powerful abelian chain of G can be refined to a chain with factors that are cyclic of order p^2 . \square

7. THE CLASSIFICATION OF POWERFULLY SIMPLE GROUPS OF TYPE $(2, 2, 2)$.

From previous section we know that this task is equivalent to classifying all simple alternating algebras of dimension 3.

Following [4], any given alternating algebra of dimension 3 over \mathbb{F}_p can be represented by a 3×3 matrix over \mathbb{F}_p . Here the matrix

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

corresponds to the 3-dimensional alternating algebra $\mathbb{F}_p v_1 + \mathbb{F}_p v_2 + \mathbb{F}_p v_3$ where

$$\begin{aligned} v_2 v_3 &= \alpha_{11} v_1 + \alpha_{21} v_2 + \alpha_{31} v_3 \\ v_3 v_1 &= \alpha_{12} v_1 + \alpha_{22} v_2 + \alpha_{32} v_3 \\ v_1 v_2 &= \alpha_{13} v_1 + \alpha_{23} v_2 + \alpha_{33} v_3. \end{aligned}$$

From last section we know that this corresponds to a powerful p -group of order p^6 with generators a_1, a_2, a_3 of order p^2 satisfying the relations:

$$\begin{aligned} [a_2, a_3] &= a_1^{p\alpha_{11}} a_2^{p\alpha_{21}} a_3^{p\alpha_{31}} \\ [a_3, a_1] &= a_1^{p\alpha_{12}} a_2^{p\alpha_{22}} a_3^{p\alpha_{32}} \\ [a_1, a_2] &= a_1^{p\alpha_{13}} a_2^{p\alpha_{23}} a_3^{p\alpha_{33}}. \end{aligned}$$

In [4] it is shown that two such matrices A, B represent the same alternating algebra with respect to different basis if and only there exists an invertible 3×3 matrix P such that

$$B = \frac{1}{\det(P)} P^t A P.$$

We write $B \simeq A$ if they are related in this way. This turns out to be slightly more general than being congruent (that is $B = P^t A P$).

Lemma 7.1. *Let $\lambda \in \mathbb{F}_p^*$. Then $\lambda A \simeq A$.*

Proof Let $P = \frac{1}{\lambda} I$. Then $\frac{1}{\det(P)} P^t A P = \lambda^3 \cdot \frac{1}{\lambda^2} A = \lambda A$. \square

From this we easily get the following corollary.

Proposition 7.2. *We have $B \simeq A$ if and only if there exists C such that A is congruent to C and $B = \lambda C$.*

In particular two matrices that are congruent are equivalent. We can write each such matrix A in a unique way as a sum of a symmetric and an anti-symmetric matrix, namely $A_s = \frac{A+A^t}{2}$ and $A_a = \frac{A-A^t}{2}$. As shown in [4] we have that $A \simeq B$ if and only if $A_s \simeq B_s$ and $A_a \simeq B_a$. We will determine all the equivalence classes and therefore all powerful p -groups of type $(2, 2, 2)$. From this we will then single out those that are powerfully simple.

Classification of the symmetric matrices. It is known that every symmetric matrix is congruent to a diagonal matrix and furthermore to exactly one of the following $D(1, 1, 1)$, $D(\tau, 1, 1)$, $D(1, 1, 0)$, $D(\tau, 1, 0)$, $D(1, 0, 0)$, $D(\tau, 0, 0)$ and $D(0, 0, 0)$, where τ is a fixed non-square in \mathbb{F}_p^* and $D(\alpha, \beta, \gamma)$ is the 3×3 matrix with α, β and γ on the diagonal (compare [1, Chapter 6, Theorem 2.7]). Now $D(1, 1, 1)$ is equivalent to $\tau D(1, 1, 1)$ where the latter has determinant τ modulo $(\mathbb{F}_p^*)^2$. Hence $D(1, 1, 1)$ and $D(\tau, 1, 1)$ are equivalent. Also $D(\tau, 0, 0) = \tau D(1, 0, 0)$ is equivalent $D(1, 0, 0)$. When the rank is 2 then multiplying the matrix by a constant $\lambda \in \mathbb{F}_p^*$ doesn't change the value of the determinant modulo $(\mathbb{F}_p^*)^2$. Hence $D(1, 1, 0)$ and $D(\tau, 1, 0)$ are not equivalent. Up to equivalence we thus get only 5 matrices:

$$D(1, 1, 1), D(1, 1, 0), D(\tau, 1, 0), D(1, 0, 0) \text{ and } D(0, 0, 0).$$

Classification of the anti-symmetric matrices. The situation regarding the anti-symmetric matrices is simpler as there are only two equivalence classes. One containing the zero matrix and one for the non-zero matrices. This comes from the fact that there are only two alternating forms (up to isomorphism) for a 3-dimensional algebra V . Either V^\perp has dimension 3 or 1.

Classification of the alternating algebras. Let A be some 3×3 matrix over \mathbb{F}_p and let V be

the corresponding alternating algebra. The symmetric part of A equips V with a corresponding symmetric bilinear form $\langle \cdot, \cdot \rangle_s$ and the anti-symmetric part of A equips V with a corresponding alternating form $\langle \cdot, \cdot \rangle_a$. Now there are two possibilities for $\langle \cdot, \cdot \rangle_a$. If it is zero then A is symmetric and we get **five** alternating algebras corresponding to the 5 diagonal matrices listed above. From now on we can thus assume that $\langle \cdot, \cdot \rangle_a$ is non-zero. Thus V^{\perp_a} is of dimension 1. Say

$$V^{\perp_a} = \mathbb{F}_p v_3$$

so that

$$V = (\mathbb{F}_p v_1 + \mathbb{F}_p v_2) \oplus_a \mathbb{F}_p v_3$$

for some $v_1, v_2 \in V$. For our classification we will divide first into 3 cases. For Case 1, we have $\langle v_3, v_3 \rangle_s \neq 0$. For Case 2, we have $\langle v_3, v_3 \rangle_s = 0$ and $(V^{\perp_a})^{\perp_s} = (\mathbb{F}_p v_3)^{\perp_s} = V$. Finally for Case 3, we have $\langle v_3, v_3 \rangle_s = 0$ and $(V^{\perp_a})^{\perp_s} = (\mathbb{F}_p v_3)^{\perp_s} < V$.

Case 1. We can here find a basis v_1, v_2, v_3 for V where

$$V = \mathbb{F}_p v_1 \oplus_s \mathbb{F}_p v_2 \oplus_s \mathbb{F}_p v_3.$$

Case 1.1. Suppose first that the rank of $\langle \cdot, \cdot \rangle_s$ is 1. In this case it is easy to see that we can pick our basis further so that

$$\begin{aligned} \langle v_1, v_2 \rangle_a &= 1, & \langle v_1, v_3 \rangle_a &= 0, & \langle v_2, v_3 \rangle_a &= 0, \\ \langle v_1, v_1 \rangle_s &= 0, & \langle v_2, v_2 \rangle_s &= 0, & \langle v_3, v_3 \rangle_s &= 1. \end{aligned}$$

Indeed, notice that we can always multiply the relevant matrix by a constant to get $\langle v_3, v_3 \rangle_s = 1$ and then it is easy to pick our v_1, v_2 such that $\langle v_1, v_2 \rangle_a = 1$. In this case we thus have only **1** algebra.

Case 1.2. Suppose next that the rank of $\langle \cdot, \cdot \rangle_s$ is 2. Here again by multiplying by a constant we can assume that $\langle v_3, v_3 \rangle_s = 1$ and we can assume that $\langle v_2, v_2 \rangle_s = 0$. Now $\langle v_1, v_1 \rangle_s = \lambda^2$ or $\langle v_1, v_1 \rangle_s = \tau \lambda^2$ for some $\lambda \in \mathbb{F}_p^*$. By replacing v_1 by $\frac{1}{\lambda} v_1$ we can assume that $\langle v_1, v_1 \rangle_s$ is either 1 or τ . Notice that we have also seen above that these cases are genuinely distinct. Now that v_1 has been chosen we can replace v_2 by a suitable multiple to ensure that $\langle v_1, v_2 \rangle_a = 1$. We thus get **2** algebras

$$\begin{aligned} \langle v_1, v_2 \rangle_a &= 1, & \langle v_1, v_3 \rangle_a &= 0, & \langle v_2, v_3 \rangle_a &= 0, \\ \langle v_1, v_1 \rangle_s &= 1, & \langle v_2, v_2 \rangle_s &= 0, & \langle v_3, v_3 \rangle_s &= 1. \end{aligned}$$

and

$$\begin{aligned} \langle v_1, v_2 \rangle_a &= 1, & \langle v_1, v_3 \rangle_a &= 0, & \langle v_2, v_3 \rangle_a &= 0, \\ \langle v_1, v_1 \rangle_s &= \tau, & \langle v_2, v_2 \rangle_s &= 0, & \langle v_3, v_3 \rangle_s &= 1. \end{aligned}$$

Case 1.3. We are then only left with the case where the rank of $\langle \cdot, \cdot \rangle_s$ is 3. It is not difficult to see that in this case we can pick v_1, v_2, v_3 such that

$$\begin{aligned} \langle v_1, v_2 \rangle_a &= 1, & \langle v_1, v_3 \rangle_a &= 0, & \langle v_2, v_3 \rangle_a &= 0, \\ \langle v_1, v_1 \rangle_s &= \alpha, & \langle v_2, v_2 \rangle_s &= 1, & \langle v_3, v_3 \rangle_s &= 1, \end{aligned}$$

where $\alpha \in \mathbb{F}_p^*$. We want to see when we get an equivalent algebra by changing α to β . If we multiply the presentation by a constant it must be by a square if we still want $\langle v_3, v_3 \rangle_s = 1$. Say, we multiply by λ^2 and then replace v_3 by $\frac{1}{\lambda} v_3$. Notice that we now have

$$\langle v_1, v_2 \rangle_a = \lambda^2, \quad \langle v_1, v_1 \rangle_s = \alpha \lambda^2, \quad \langle v_2, v_2 \rangle_s = \lambda^2.$$

We are now looking for all possible $\bar{v}_1 = av_1 + bv_2$ and $\bar{v}_2 = cv_1 + dv_2$ where $\langle \bar{v}_1, \bar{v}_2 \rangle_a = 1$, $\langle \bar{v}_1, \bar{v}_2 \rangle_s = 0$ and $\langle \bar{v}_2, \bar{v}_2 \rangle_s = 1$. This gives us the following system of equations:

$$\begin{aligned}\lambda^2(ad - bc) &= 1 \\ \lambda^2(\alpha ac + bd) &= 0 \\ \lambda^2(\alpha c^2 + d^2) &= 1.\end{aligned}$$

We look first for all the solutions where $c = 0$. Notice that in this case we must have $\lambda^2 ad = 1$, $\lambda^2 d^2 = 1$ and $b = 0$. Thus $\langle \bar{v}_1, \bar{v}_1 \rangle_s = \lambda^2(\alpha a^2 + b^2) = \lambda^2 \frac{\alpha}{d^2 \lambda^4} = \frac{\alpha}{\lambda^2 d^2} = \alpha$.

Next we look for solutions where $c \neq 0$ but $d = 0$. Then we must have $\lambda^2 bc = -1$, $\lambda^2 \alpha c^2 = 1$ and $a = 0$. Here $\langle \bar{v}_1, \bar{v}_1 \rangle_s = \lambda^2(\alpha a^2 + b^2) = \frac{\lambda^2}{c^2 \lambda^4} = \frac{\alpha}{\alpha c^2 \lambda^2} = \alpha$.

Finally we are left with finding all solutions where $cd \neq 0$. Then $a = -\frac{bd}{\alpha c}$, and the first equation above gives us

$$1 = -\lambda^2 \left(\frac{bd^2}{\alpha c} + bc \right) = -\frac{b}{\alpha c} \cdot \lambda^2(d^2 + \alpha c^2) = -\frac{b}{\alpha c}.$$

Thus $b = -\alpha c$ and $a = -\frac{bd}{\alpha c} = d$. Therefore

$$\begin{aligned}\langle \bar{v}_1, \bar{v}_1 \rangle &= \lambda^2(\alpha a^2 + b^2) \\ &= \lambda^2(\alpha d^2 + \alpha^2 c^2) \\ &= \alpha \lambda^2(\alpha c^2 + d^2) \\ &= \alpha.\end{aligned}$$

We have thus seen that the value of α doesn't change and we have $p - 1$ different algebras here.

Case 2. Here we are assuming that $\langle v_3, v_3 \rangle_s = 0$ and that v_3 is orthogonal to v_1, v_2 as well. Again we consider few subcases.

Case 2.1. Suppose that the rank of $\langle \cdot, \cdot \rangle_s$ is zero. Then clearly we have **1** algebra.

$$\begin{aligned}\langle v_1, v_2 \rangle_a = 1, \quad \langle v_1, v_3 \rangle_a = 0, \quad \langle v_2, v_3 \rangle_a = 0, \\ \langle v_1, v_1 \rangle_s = 0, \quad \langle v_2, v_2 \rangle_s = 0, \quad \langle v_3, v_3 \rangle_s = 0.\end{aligned}$$

Case 2.2. Suppose next that the rank of $\langle \cdot, \cdot \rangle_s$ is 1. By multiplying by a suitable constant we can assume that $\langle v_1, v_1 \rangle_s = 1$ and $\langle v_2, v_2 \rangle_s = 0$. Finally replacing v_2 by an appropriate multiple we can also assume that $\langle v_1, v_2 \rangle_a = 1$. We thus also get here only **1** algebra

$$\begin{aligned}\langle v_1, v_2 \rangle_a = 1, \quad \langle v_1, v_3 \rangle_a = 0, \quad \langle v_2, v_3 \rangle_a = 0, \\ \langle v_1, v_1 \rangle_s = 1, \quad \langle v_2, v_2 \rangle_s = 0, \quad \langle v_3, v_3 \rangle_s = 0.\end{aligned}$$

Case 2.3. Finally we are left with the case when the rank of $\langle \cdot, \cdot \rangle_s$ is 2. Here it is easy to see that we can pick our basis further so that

$$\begin{aligned}\langle v_1, v_2 \rangle_a = 1, \quad \langle v_1, v_3 \rangle_a = 0, \quad \langle v_2, v_3 \rangle_a = 0, \\ \langle v_1, v_1 \rangle_s = \alpha, \quad \langle v_2, v_2 \rangle_s = 1, \quad \langle v_3, v_3 \rangle_s = 0.\end{aligned}$$

Similar calculations as for Case 1.3 show that we get distinct algebras for different values of α . Thus here we have $p - 1$ algebras.

Case 3. Here we are assuming that $\langle v_3, v_3 \rangle_s = 0$ but that v_3 is not orthogonal to everything in V with respect to \langle , \rangle_s . Thus $(\mathbb{F}_p v_3)^{\perp_s}$ has dimension 2. Suppose

$$(\mathbb{F}_p v_3)^{\perp_s} = \mathbb{F}_p v_2 + \mathbb{F}_p v_3.$$

It is not difficult to see that we can pick our basis such that

$$\begin{aligned} \langle v_1, v_2 \rangle_a &= 1, & \langle v_1, v_3 \rangle_a &= 0, & \langle v_2, v_3 \rangle_a &= 0, \\ \langle v_1, v_1 \rangle_s &= 0, & \langle v_1, v_2 \rangle_s &= 0, & \langle v_1, v_3 \rangle_s &= 1, & \langle v_2, v_3 \rangle_s &= 0. \end{aligned}$$

Now there are two subcases.

Case 3.1. If the rank of \langle , \rangle_s is 2 then we must have $\langle v_2, v_2 \rangle_s = 0$ and this gives us **1** algebra.

Case 3.2. If the rank of \langle , \rangle_s is 3 then $\langle v_2, v_2 \rangle_s \neq 0$ and after multiplying by a suitable constant we can assume that this value is 1 (and then afterwards adjust things so that the other assumptions hold again). Thus we get again **1** algebra.

Adding up we see that in total we get $12 + 2(p - 1)$ algebras and thus the same number of powerful p -groups of type $(2, 2, 2)$. Before listing these we state and prove a proposition that shows how we can see which of these are powerfully simple.

Proposition 7.3. *An alternating algebra V over \mathbb{F}_p of dimension 3 is simple if and only if $V \cdot V = V$.*

Proof This condition is clearly necessary as $V \cdot V$ is an ideal of V . To see that it is sufficient, suppose $V \cdot V = V$ and let I be a proper ideal. We want to show that $I = 0$. We argue by contradiction and suppose I is an ideal of dimension either 1 or 2. If I is of dimension 2, then V/I is 1 dimensional and thus we get the contradiction that $V \cdot V \leq I < V$. Now suppose I is of dimension 1, say $V = I + \mathbb{F}_p v_1 + \mathbb{F}_p v_2$. Then $V \cdot V \leq I + \mathbb{F}_p v_1 v_2$. But the dimension of $I + \mathbb{F}_p v_1 v_2$ is at most 2 and we get the contradiction that $V \cdot V < V$. \square

We have thus determined the presentation matrices up to equivalence and got in total $12 + 2(p - 1)$. As we described at the beginning of the section this gives us a classification of all the alternating algebras of dimension 3 over \mathbb{F}_p that in turn gives us a classification of all the powerful p -groups of type $(2, 2, 2)$. Furthermore, last proposition tells us how we read from the presentation whether a given alternating algebra is simple and thus whether the corresponding powerful group is powerfully simple. The work above gives us the following list of powerful p -groups of type $(2, 2, 2)$. As the power relations for all of these are $a_1^{p^2} = a_2^{p^2} = a_3^{p^2} = 1$ we omit these below. Here τ is a fixed non-square in \mathbb{F}_p .

$$\begin{aligned}
A_1 &= \langle a_1, a_2, a_3 : [a_2, a_3] = a_1^p, [a_3, a_1] = a_2^p, [a_1, a_2] = a_3^p \rangle; \\
A_2 &= \langle a_1, a_2, a_3 : [a_2, a_3] = a_2^{-p}, [a_3, a_1] = a_1^p, [a_1, a_2] = a_3^p \rangle; \\
A_3 &= \langle a_1, a_2, a_3 : [a_2, a_3] = a_1^p a_2^{-p}, [a_3, a_1] = a_1^p, [a_1, a_2] = a_3^p \rangle; \\
A_4 &= \langle a_1, a_2, a_3 : [a_2, a_3] = a_1^{p\tau} a_2^{-p}, [a_3, a_1] = a_1^p, [a_1, a_2] = a_3^p \rangle; \\
A_5 &= \langle a_1, a_2, a_3 : [a_2, a_3] = a_2^{-p} a_3^p, [a_3, a_1] = a_1^p a_2^p, [a_1, a_2] = a_1^p \rangle; \\
A_6(\alpha) &= \langle a_1, a_2, a_3 : [a_2, a_3] = a_1^{p\alpha} a_2^{-p}, [a_3, a_1] = a_1^p a_2^p, [a_1, a_2] = a_3^p \rangle, \quad 1 \leq \alpha \leq p-2; \\
B_1 &= \langle a_1, a_2, a_3 : [a_2, a_3] = a_1^{-p} a_2^{-p}, [a_3, a_1] = a_1^p a_2^p, [a_1, a_2] = a_3^p \rangle; \\
B_2 &= \langle a_1, a_2, a_3 : [a_2, a_3] = a_1^p, [a_3, a_1] = a_2^p, [a_1, a_2] = 1 \rangle; \\
B_3 &= \langle a_1, a_2, a_3 : [a_2, a_3] = a_1^{p\tau}, [a_3, a_1] = a_2^p, [a_1, a_2] = 1 \rangle; \\
B_4 &= \langle a_1, a_2, a_3 : [a_2, a_3] = a_2^{-p}, [a_3, a_1] = a_1^p, [a_1, a_2] = 1 \rangle; \\
B_5 &= \langle a_1, a_2, a_3 : [a_2, a_3] = a_1^p a_2^{-p}, [a_3, a_1] = a_1^p, [a_1, a_2] = 1 \rangle; \\
B_6 &= \langle a_1, a_2, a_3 : [a_2, a_3] = a_2^{-p} a_3^p, [a_3, a_1] = a_1^p, [a_1, a_2] = a_1^p \rangle; \\
B_7(\alpha) &= \langle a_1, a_2, a_3 : [a_2, a_3] = a_1^{p\alpha} a_2^{-p}, [a_3, a_1] = a_1^p a_2^p, [a_1, a_2] = 1 \rangle, \quad 1 \leq \alpha \leq p-2; \\
C_1 &= \langle a_1, a_2, a_3 : [a_2, a_3] = a_1^{-p} a_2^{-p}, [a_3, a_1] = a_1^p a_2^p, [a_1, a_2] = 1 \rangle; \\
C_2 &= \langle a_1, a_2, a_3 : [a_2, a_3] = a_1^p, [a_3, a_1] = 1, [a_1, a_2] = 1 \rangle; \\
D &= \langle a_1, a_2, a_3 : [a_2, a_3] = 1, [a_3, a_1] = 1, [a_1, a_2] = 1 \rangle.
\end{aligned}$$

Of these $12 + 2(p-1)$ groups, the groups $A_1, \dots, A_6(\alpha)$ are the powerfully simple groups. There are $5 + (p-2)$ of these.

REFERENCES

- [1] P. M. Cohn, *Algebra*, Vol. 2, 2nd edition, University College London, 1989.
- [2] A. Lubotzky and A. Mann, Powerful p -groups, *J. Algebra*, **105** (1987), 485-505.
- [3] A. Shalev, On almost fixed point free automorphisms, *J. Algebra*, **157** (1993), 271-282.
- [4] G. Traustason, Symplectic alternating algebras, *Int. J. Algebra and Comp.*, **18** (2008), 719-757.
- [5] G. Traustason and J. Williams, Powerfully nilpotent groups, *J. Algebra*, **522** (2019), 80-100.
- [6] G. Traustason and J. Williams, Powerfully nilpotent groups of maximal powerful class, *Monatshefte für Mathematik*, **191**(4) (2020), 779-799.
- [7] G. Traustason and J. Williams, Powerfully nilpotent groups of rank 2 or small order, *J. Group Theory*, **23** (2020), 641-658.