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MULTIPOLE VORTEX PATCH EQUILIBRIA FOR ACTIVE SCALAR EQUATIONS

ZINEB HASSAINIA AND MILES H. WHEELER

ABSTRACT. We study how a general configuration of finitely-many point vortices, in a state of uniform rotation or translation with Newtonian interaction or generalized surface quasi-geostrophic interactions, can be desingularized into a steady configuration of vortex patches. Using a technique first introduced by Hmidi and Mateu for vortex pairs, we reformulate the problem for the patch boundaries so that it no longer appears singular, at which point classical techniques such as Lyapunov–Schmidt reduction can be used. Provided the point vortex equilibrium is non-degenerate in a natural sense, solutions can be constructed directly using the implicit function theorem, yielding asymptotics for the shape of the patch boundaries. As an application, we prove the existence of various families of solutions with patches arranged in asymmetric pairs, regular polygons, body-centered polygons, and nested regular polygons. These configurations are degenerate due to their additional symmetries, but this difficulty can be overcome by integrating the appropriate symmetries into the formulation of the problem.

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1. INTRODUCTION

1.1. Historical discussion. In this note we consider the generalized surface quasi-geostrophic (gSQG) equations, which describe the evolution of the potential temperature ω through the transport equation

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ v = -\nabla^\perp (-\Delta)^{-1+\frac{\alpha}{2}} \omega, \\ \omega|_{t=0} = \omega_0. \end{cases} \quad (1.1)$$

Here $\nabla^\perp = (-\partial_2, \partial_1)$ while α is a real parameter taken in the interval $[0, 1)$. The vector field v is the flow velocity and the fractional Laplacian operator $(-\Delta)^{-1+\frac{\alpha}{2}}$ is of convolution type, defined by

$$(-\Delta)^{-1+\frac{\alpha}{2}} \omega(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} K_\alpha(x-y) \omega(y) dy$$

with

$$K_\alpha(x) = \begin{cases} -\frac{1}{2\pi} \ln|x| & \text{if } \alpha = 0, \\ \frac{C_\alpha}{2\pi} \frac{1}{|x|^\alpha} & \text{if } \alpha \in (0, 1), \end{cases} \quad \text{with} \quad C_\alpha = \frac{\Gamma(\alpha/2)}{2^{1-\alpha} \Gamma(\frac{2-\alpha}{2})}, \quad (1.2)$$

where Γ is the gamma function. This model was proposed by Córdoba et al. in [12] as an interpolation between Euler equations and the surface quasi-geostrophic model, which correspond to $\alpha = 0$ and $\alpha = 1$, respectively.

The main purpose of this note is to show the existence of new families of periodic global solutions of (1.1) in the vortex patch setting, namely when $\omega(\cdot, t)$ is the characteristic function of finite collection of bounded domains. Such patterns are a special class of Yudovich solutions where $\omega(\cdot, t)$ is merely bounded and integrable. Yudovich solutions are known to be unique and to exist globally in time in the case of Euler equations [51], but for $\alpha > 0$ the situation is more delicate because the velocity field v is in general not Lipschitz. Nonetheless, when the initial datum has a patch structure, one can locally construct a unique solution which remains a patch. The motion of the boundary of the patch is governed by so-called contour dynamics equations; see [20, 44]. It is worth mentioning that, while the boundary's regularity is globally preserved for $\alpha = 0$ [11, 3], for $\alpha > 0$ numerical evidence [12] suggests singularity formation in finite time.

There are very few explicit solutions to the gSQG and Euler equations. The only known explicit simply-connected vortex patch solutions are the Rankine vortex, which is stationary, and the Kirchhoff ellipses [27] for the Euler equation, which are rotating. Nevertheless, a family of uniformly rotating patches with m -fold symmetry, called V-states, was numerically computed by Deem and Zabusky in [17]. Later, Burbua [4] gave an analytical proof of their existence, based on a conformal mapping parametrization and local bifurcation theory. Recently, Burbua's branches of solutions were extended to global ones in [30]. The regularity and the convexity of the V-states have been investigated [35, 30, 8]. Similar research has been carried out for the gSQG equations: The construction of simply connected V-states was established in [28, 7], and their boundary regularity was discussed in [8].

We point out that there is a large literature dealing with rotating vortex patches and related problems. For instance, we mention the existence results of rotating patches close to Kirchhoff's ellipses [32, 8], multiply-connected patches [36, 16, 33, 14, 43, 25], patches in bounded domains [15], non-trivial rotating smooth solutions [9] and rotating vortices with non-uniform densities [23]. The radial symmetry properties of stationary and uniformly-rotating solutions was studied in a series of works [19, 31, 26].

All of the above analytical results treat connected patches. The first numerical works revealing the existence of translating symmetric pairs of simply connected patches for Euler equation are due

to Deem and Zabusky [17] and Pierrehumbert [42]. Similar studies were performed by Saffman and Szeto in [47] for the symmetric co-rotating vortex pairs and by Dritschel [18] for asymmetric pairs. Later, Turkington gave in [48] an analytical proof using variational arguments, where he considered an unbounded fluid domain with N symmetrically arranged vortex patches rotating about the origin. Implementing the same approach, Keady [37] proved the existence of translating pairs of symmetric patches and Wan [49] studied the existence and stability of desingularizations of a general system of rotating point vortices. Very recently, Godard-Cadillac, Gravejat and Smets [24] extended Turkington's result for the gSQG equations, while Ao, Dávila, Del Pino, Musso and Wei [1] have obtained related families of smooth solutions via gluing techniques. See [39, 40, 41, 46, 50] for additional references on multiply connected patches.

The variational arguments [48, 37, 24] do not give much information about the shape of the vortex patches, or about the uniqueness of solutions. In [34], however, Hmidi and Mateu gave a direct proof showing the existence of co-rotating and counter-rotating vortex pairs, using an elegant desingularization of the contour dynamics equations and an application of the implicit function theorem. The same technique was implemented for the desingularization of the asymmetric pairs [29], Karman street [21] and, very recently, the vortex polygon [22]. See [6] for related results where point vortices are instead desingularized into doubly-connected patches.

In this note we show how the technique in [34] can be extended to arbitrary configurations of finitely-many point vortices. For a general configuration, the problem reduces via Lyapunov–Schmidt to a finite-dimensional nonlinear equation. Under a natural non-degeneracy assumption on the point vortex configuration alone, one can simply apply implicit function theorem. Many classical configurations are degenerate, however, due to additional reflection and rotational symmetries. For such configurations one can still hope to use the implicit theorem by reformulating the problem in spaces which take these symmetries into account.

1.2. Statement of the general result. Recall that the gSQG point vortex model for N interacting vortices in the complex plane \mathbb{C} is given by the Hamiltonian system

$$\frac{d}{dt}z_j(t) = i \frac{\alpha C_\alpha}{2} \sum_{k=1, k \neq j}^N \gamma_k \frac{z_j(t) - z_k(t)}{|z_j(t) - z_k(t)|^{\alpha+2}}, \quad j = 1, \dots, N, \quad (1.3)$$

where $\gamma_1, \dots, \gamma_N \in \mathbb{R} \setminus \{0\}$ are the circulations and C_α is defined in (1.2). The case $\alpha = 0$ corresponds to the classical point vortex Eulerian interaction. A general review about the N -vortex problem and vortex statics can be found in [2] for the Newtonian interaction and [45] for gSQG interactions. We are concerned with periodic solutions for which the configuration of vortices is instantaneously moving as a rigid body, so that

$$\frac{d}{dt}z_j(t) = iU + i\Omega z_j(t),$$

where here $U \in \mathbb{R}$ is the constant linear velocity and $\Omega \in \mathbb{R}$ is the constant angular velocity. Such solutions are known as *relative equilibria* or *vortex crystals*. Writing $z_j(0) = e^{i\theta_j} d_j$ in polar coordinates, (1.3) reduces to the algebraic system

$$\mathcal{P}_j^\alpha(\lambda) = \Omega d_j + U e^{-i\theta_j} - \frac{\alpha C_\alpha}{2} \sum_{k=1, k \neq j}^N \gamma_k \frac{d_j - d_k e^{i(\theta_k - \theta_j)}}{|d_j - d_k e^{i(\theta_k - \theta_j)}|^{\alpha+2}} = 0, \quad j = 1, \dots, N, \quad (1.4)$$

where here $\lambda = (\Omega, U, \gamma_1, \dots, \gamma_N, d_1, \dots, d_N, \theta_1, \dots, \theta_N)$. This defines a mapping $\mathcal{P}^\alpha(\lambda)$ with values in \mathbb{C}^N . We say that a solution λ^* of (1.4) is *non-degenerate* if the Jacobian matrix $D_\lambda \mathcal{P}^\alpha(\lambda^*)$ has full rank.

Informally stated, our first result is the following; see Proposition 2.1 for a precise version.

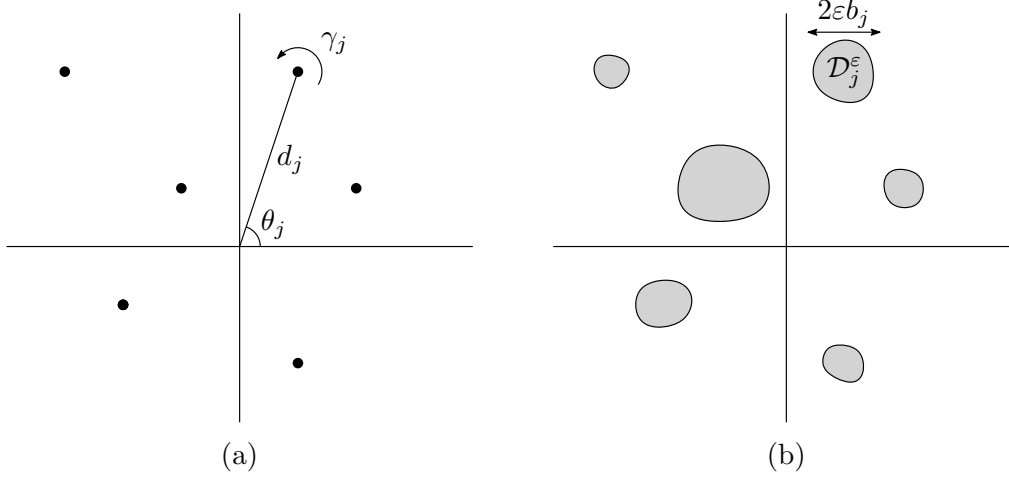


FIGURE 1. (a) A solution of the point vortex system (1.4). The vortices are located at the points $z_j(0) = e^{i\theta_j} d_j$ and have circulations γ_j . (b) A desingularization into vortex patches. The vortex at $z_j(0)$ has become a small nearly-circular patch \mathcal{D}_j^ϵ . To leading order in the small parameter ϵ , the radius of the patch is ϵb_j , and the net circulation is γ_j .

Theorem 1.1. *Let $\alpha \in [0, 1)$. Then any non-degenerate solution λ of (1.4) can be desingularized into a family of vortex patch equilibria depending smoothly on a small parameter ϵ measuring the size of the patches.*

See Figure 1 for an illustration.

Remark 1.2. As we shall see later in Proposition 2.1, the vortex patches in Theorem 1.1 are small $C^{1+\beta}$ perturbations of the unit disk, whose boundaries are given by conformal parametrizations which can be explicitly expanded to any order in the small parameter ϵ .

Remark 1.3. The leading-order ratios between the sizes of the patches can be specified a priori. Moreover, the range of ϵ is uniform as some of these ratios are sent to zero, allowing us to recover solutions involving a combination of point vortices and vortex patches. See Remark 2.4 for more details.

Remark 1.4. While the proof is valid for $\alpha \in [0, 1)$, we expect that the result remains true for $\alpha \in [1, 2)$ by using the spaces introduced in [7]; see [5].

1.3. Applications. Many important point vortex equilibria have additional symmetries, in which case the non-degeneracy assumption in Theorem 1.1 fails. However, one can still proceed by integrating these additional symmetries into the statement of the problem. We now give several applications of this type.

The most elementary solutions to (1.4) are the steady vortex pairs, namely the co-rotating pairs:

$$\begin{aligned} \theta_1 = 0, \quad \theta_2 = \pi, \quad U = 0, \\ d_2 = d_2^* := \frac{\gamma_1}{\gamma_2} d_1, \quad \Omega = \Omega^* := \frac{2^{\alpha-1} \Gamma(1 + \frac{\alpha}{2})}{d_1^{\alpha+2} \Gamma(1 - \frac{\alpha}{2})} \frac{\gamma_1 + \gamma_2}{|\gamma_1 + \gamma_2|^{\alpha+2}} |\gamma_2|^{\alpha+2}, \end{aligned} \quad (1.5)$$

with $d_1 > 0$ and $\gamma_1 + \gamma_2 \neq 0$, and the counter-rotating pairs:

$$\begin{aligned} \theta_1 = 0, \quad \theta_2 = \pi, \quad \Omega = 0, \quad \gamma_2 = -\gamma_1, \\ U = U^* := \frac{2^{\alpha-1} \Gamma(1 + \frac{\alpha}{2})}{\Gamma(1 - \frac{\alpha}{2})} \frac{\gamma_1}{|d_1 + d_2|^{\alpha+1}}, \end{aligned} \quad (1.6)$$

with $\gamma_1 \in \mathbb{R} \setminus \{0\}$ and $d_1 + d_2 \neq 0$. Thus in the time-dependent problem, the co-rotating pairs steadily rotate about the origin, while the counter-rotating pairs steadily translate along the y -axis. Note that, in both of these configurations, the configuration is invariant under reflections about the x -axis. Furthermore, (1.6) is invariant under horizontal translations, and (1.5) is invariant under rotations.

We shall construct small asymmetric co-rotating and counter-rotating pairs of vortex patches near the vortex pairs in (1.5) and (1.6). This extends the desingularization result of [29], obtained in the Eulerian case $\alpha = 0$, to gSQG equations (1.1) with $\alpha \in (0, 1)$. For the sake of clarity we shall give an elementary statement; for a complete statement see Propositions 3.1 and 3.2.

Theorem 1.5. *Let $\alpha \in (0, 1)$, $b_1, b_2 \in (0, \infty)$. Then, the following results hold true.*

- (i) *For any $d_1 \in (0, \infty)$, $\gamma_1, \gamma_2 \in \mathbb{R} \setminus \{0\}$ such that $\gamma_1 + \gamma_2 \neq 0$ and any $\varepsilon > 0$ sufficiently small, there are two strictly convex domains $\mathcal{O}_1^\varepsilon, \mathcal{O}_2^\varepsilon$, $C^{2-\alpha}$ perturbations of the unit disc, and a real number $d_2(\varepsilon)$ such that*

$$\omega_0^\varepsilon = \frac{\gamma_1}{\varepsilon^2 b_1^2} \chi_{\mathcal{D}_1^\varepsilon} + \frac{\gamma_2}{\varepsilon^2 b_2^2} \chi_{\mathcal{D}_2^\varepsilon} \quad \text{with} \quad \mathcal{D}_1^\varepsilon := \varepsilon b_1 \mathcal{O}_1^\varepsilon + d_1, \quad \mathcal{D}_2^\varepsilon := -(\varepsilon b_2 \mathcal{O}_2^\varepsilon + d_2(\varepsilon)),$$

generates a co-rotating vortex pair for (1.1) with some constant angular velocity $\Omega(\varepsilon)$.

- (ii) *For any $\gamma_1, d_1, d_2 \in \mathbb{R} \setminus \{0\}$ such that $d_1 + d_2 \neq 0$ and any $\varepsilon > 0$ sufficiently small, there are two strictly convex domains $\mathcal{O}_1^\varepsilon, \mathcal{O}_2^\varepsilon$, $C^{2-\alpha}$ perturbations of the unit disc, and a real number $\gamma_2(\varepsilon)$ such that*

$$\omega_0^\varepsilon = \frac{\gamma_1}{\varepsilon^2 b_1^2} \chi_{\mathcal{D}_1^\varepsilon} + \frac{\gamma_2(\varepsilon)}{\varepsilon^2 b_2^2} \chi_{\mathcal{D}_2^\varepsilon} \quad \text{with} \quad \mathcal{D}_1^\varepsilon := \varepsilon b_1 \mathcal{O}_1^\varepsilon + d_1, \quad \mathcal{D}_2^\varepsilon := -(\varepsilon b_2 \mathcal{O}_2^\varepsilon + d_2),$$

generates a counter-rotating vortex pair for (1.1) with some constant speed $U(\varepsilon)$.

Remark 1.6. By sending $b_1 \rightarrow 0$ or $b_2 \rightarrow 0$, we can recover configurations with one vortex patch and one point vortex; see Remark 2.4.

Remark 1.7. The domains $\mathcal{O}_1^\varepsilon, \mathcal{O}_2^\varepsilon$ are 1-fold symmetric, and the conformal parameterizations of their boundaries $\phi_j: \mathbb{T} \rightarrow \partial \mathcal{O}_j^\varepsilon$ have the Fourier asymptotic expansions

$$\phi_j(\varepsilon, w) = w + \Xi_\alpha \frac{\gamma_{3-j}}{\gamma_j} \left(\frac{\varepsilon b_j}{d_1} \frac{|\gamma_2|}{|\gamma_1 + \gamma_2|} \right)^{\alpha+2} \bar{w} + o(\varepsilon^2), \quad \Xi_\alpha := \frac{(\alpha+2)\Gamma(1-\frac{\alpha}{2})\Gamma(3-\frac{\alpha}{2})}{4\Gamma(2-\alpha)}$$

in the co-rotating case, and

$$\phi_j(\varepsilon, w) = w - \Xi_\alpha \left(\frac{\varepsilon b_j}{|d_1 + d_2|} \right)^{\alpha+2} \bar{w} + o(\varepsilon^2)$$

in the translating case.

Next, we shall be concerned with a configuration with m -fold symmetry. In particular, consider two concentric regular m -gons with a vortex at each vertex, and assume that the vortices of a same polygon have the same vorticity $\gamma_1 \in \mathbb{R} \setminus \{0\}$ or $\gamma_2 \in \mathbb{R}$. We place, in addition, a point vortex at the center of the regular m -gons with intensity $\gamma_0 \in \mathbb{R}$. More specifically, we are concerned with the system of point vortices

$$\omega_0^0(z) = \pi \gamma_0 \delta_0(z) + \sum_{j=1}^2 \pi \gamma_j \sum_{k=0}^{m-1} \delta_{z_{jk}(0)}(z), \quad z_{jk}(0) := \begin{cases} d_1 e^{\frac{2k\pi i}{m}} & \text{if } j = 1, \\ d_2 e^{\frac{(2k+\vartheta)\pi i}{m}} & \text{if } j = 2, \vartheta \in \{0, 1\}. \end{cases} \quad (1.7)$$

The value $\vartheta = 0$ corresponds to the configuration where the vertices of the polygons are radially aligned with each other and $\vartheta = 1$ refers to the case where the vertices are out-of-phase by an angle

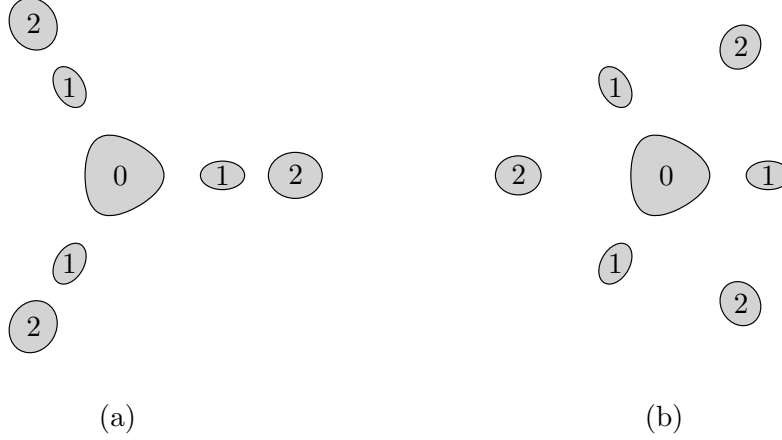


FIGURE 2. Sketch of the vortex patch solutions in Theorem 1.8 with $m = 3$. The point vortex of circulation γ_0 at the origin has been desingularized into a patch with m -fold symmetry, while the vortices with circulations γ_1, γ_2 at the vertices of two regular m -gons have been desingularized into vortex patches with 1-fold symmetry. (a) shows the case aligned case $\vartheta = 0$, while (b) shows the staggered case $\vartheta = 1$.

π/m . If we assume that (1.7) performs a perpetual uniform rotation, then we may easily check that the system of $2m + 1$ equations in (1.4) can be reduced to a system of two real equations:

$$\Omega d_j - \frac{\alpha C_\alpha}{2d_j^{1+\alpha}} \left(\gamma_0 + \gamma_j \sum_{k=1}^{m-1} \frac{1 - e^{\frac{2k\pi i}{m}}}{|1 - e^{\frac{2k\pi i}{m}}|^{2+\alpha}} + \gamma_{3-j} \sum_{k=0}^{m-1} \frac{1 - e^{\frac{(2k+\vartheta)\pi i}{m} \frac{d_{3-j}}{d_j}}}{|1 - e^{\frac{(2k+\vartheta)\pi i}{m} \frac{d_{3-j}}{d_j}}|^{2+\alpha}} \right) = 0, \quad j = 1, 2. \quad (1.8)$$

Observe that the last system is linear in Ω and γ_2 , and, thus, under a non-degeneracy condition we can explicitly solve the system (1.8) for Ω and $\gamma_2 \neq 0$.

Our desingularization result concerning these configurations is the following; for a complete statement see Proposition 4.2.

Theorem 1.8. *Let $\alpha \in [0, 1)$, $\vartheta \in \{0, 1\}$, $b_1, b_2 \in (0, 1)$, $\gamma_0, \gamma_1 \in \mathbb{R} \setminus \{0\}$, $d_1, d_2 \in (0, \infty)$ and let $(\Omega^*, \gamma_2^*) \in (\mathbb{R} \setminus \{0\})^2$ be a non-degenerate solution of (1.8). Then, for any $\varepsilon > 0$ sufficiently small, there are three strictly convex domains $\mathcal{O}_0^\varepsilon, \mathcal{O}_1^\varepsilon, \mathcal{O}_2^\varepsilon$, $C^{1+\beta}$ perturbations of the unit disc, and a real number $\gamma_2 = \gamma_2(\varepsilon)$ such that*

$$\omega_{0,\varepsilon}^\alpha = \frac{\gamma_0}{\varepsilon^2 b_0^2} \chi_{\varepsilon b_0 \mathcal{O}_0^\varepsilon} + \sum_{j=1}^2 \frac{\gamma_j}{\varepsilon^2 b_j^2} \sum_{k=0}^{m-1} \chi_{\mathcal{D}_{jk}^\varepsilon} \quad \text{with} \quad \mathcal{D}_{jk}^\varepsilon := \begin{cases} e^{\frac{2k\pi i}{m}} (\varepsilon b_1 \mathcal{O}_2^\varepsilon + d_1) & \text{if } j = 1, \\ e^{\frac{(2k+\vartheta)\pi i}{m}} (\varepsilon b_2 \mathcal{O}_2^\varepsilon + d_2) & \text{if } j = 2, \end{cases} \quad (1.9)$$

generates a rotating solution for (1.1) with some constant angular velocity $\Omega(\varepsilon)$. Moreover $\mathcal{O}_0^\varepsilon$ is m -fold symmetric and $\mathcal{O}_1^\varepsilon, \mathcal{O}_2^\varepsilon$ are 1-fold symmetric.

See Figure 2 for an illustration.

Remark 1.9. The complete statement of Theorem 1.8 given in Proposition 4.2 explicitly computes the asymptotic behavior of the conformal parametrizations $\phi_j^\varepsilon: \mathbb{T} \rightarrow \partial \mathcal{O}_j^\varepsilon$.

Remark 1.10. The parameter $\varepsilon > 0$ can be chosen uniformly as any of the parameters b_0, b_1 and b_2 tend to 0, and therefore we may recover the point vortex-vortex patch configurations discussed in [13, 50]; see Remark 2.4.

Remark 1.11. The proof can be easily adapted to the rotating vortex polygon (with $\gamma_0 = \gamma_2 = 0$ in (1.9)) and which has been studied in [1, 24, 48, 22]. This remains equally true for the body-centered

polygonal configurations ($\gamma_2 = 0$), treated in [49], as well as for the nested polygons without a central patch ($\gamma_0 = 0$). The latter solutions were first observed numerically in [50].

1.4. Idea of the proof. We shall briefly explain the basic ideas behind Theorem 1.1 for the Euler equations in the rotating case; a similar strategy is followed for the gSQG equations and for traveling patches. We seek simply connected bounded domains $\mathcal{O}_j^\varepsilon$ such that the initial datum

$$\omega_0^\varepsilon(z) = \sum_{j=1}^N \frac{\gamma_j}{\varepsilon^2 b_j^2} \chi_{\mathcal{D}_j^\varepsilon}(z), \quad \text{with } \mathcal{D}_j^\varepsilon := e^{i\theta_j}(\varepsilon b_j \mathcal{O}_j^\varepsilon + d_j),$$

performs a uniform rotation around the center of mass of the system, which is taken to be the origin, with an angular velocity Ω . Here the parameters b_j will allow us to specify the relative sizes of the patches; see Remark 1.3. After moving to the rotating frame, the boundaries of the system are subject to the following stationary system, see for instance [16, Page 1896],

$$\operatorname{Re} \{ \gamma_j (\overline{v^\varepsilon(z)} + i\Omega \bar{z}) \vec{n} \} = 0 \quad \text{for all } z \in \partial \mathcal{D}_j^\varepsilon, \quad (1.10)$$

where \vec{n} is the exterior unit normal vector to the boundary at the point z . In virtue of the Biot–Savart law and Green’s theorem, we may write

$$\overline{v^\varepsilon(z)} = \frac{1}{4\pi} \sum_{k=1}^N \frac{\gamma_k}{\varepsilon^2 b_k^2} \int_{\partial \mathcal{D}_k^\varepsilon} \frac{\bar{\xi} - \bar{z}}{\xi - z} d\xi$$

for all $z \in \mathbb{C}$.

Following Hmidi and Mateu [34], we reformulate (1.10) in terms of the conformal parametrizations of the boundaries $\phi_j: \mathbb{T} \rightarrow \partial \mathcal{O}_j^\varepsilon$, which we assume have the form

$$\phi_j(w) = w + \varepsilon b_j f_j(w).$$

In other words, we shall look for domains $\mathcal{O}_j^\varepsilon$ which are small perturbations of the unit disc with an amplitude of order εb_j . While the resulting problem initially appears to have terms which are singular in ε , as in [34] there is a cancellation — essentially due to the symmetry of the disk — which eliminates these terms. This leads to a nonlinear system

$$\mathcal{G}_j(\varepsilon, f; \lambda) = 0, \quad j = 1, \dots, N$$

for the perturbations $f := (f_1, \dots, f_N)$ of the patch boundaries, where the nonlinear operator $\mathcal{G}^\alpha = (\mathcal{G}_1^\alpha, \dots, \mathcal{G}_N^\alpha)$ is well-defined and of class C^1 in a small neighborhood of $(0, 0; \lambda^*)$ in some suitable Banach spaces. Here λ^* is any solution to the point vortex system (1.4) with $\alpha = 0$. Moreover, for $(\varepsilon, f; \lambda) = (0, 0; \lambda)$ we find

$$\mathcal{G}_j(0, 0; \lambda)(w) = \operatorname{Im} \{ \mathcal{P}_j^0(\lambda) \bar{w} \}, \quad (1.11)$$

so that $\mathcal{G}(0, 0; \lambda) = 0$ is equivalent to (1.4).

Intending to apply the implicit function theorem, we next linearize \mathcal{G} about the point vortex solution $(0, 0; \lambda^*)$. The linearized operator with respect to the patch boundaries f is

$$D_f \mathcal{G}_j(0, 0; \lambda^*) h(w) = \gamma_j \operatorname{Im} \{ h'_j(w) \},$$

which has a trivial kernel and a range with finite codimension $2N$. On the other hand, thanks to (1.11), the linearization with respect to the point vortex parameter λ can be expressed solely in terms of the Jacobian matrix $D_\lambda \mathcal{P}^0(\lambda^*)$. For non-degenerate configurations this matrix has full rank, and so we deduce that the linearized operator $D_{(f, \lambda_1)} \mathcal{G}(0, 0; \lambda^*)$ is invertible for some splitting $\lambda = (\lambda_1, \lambda_2)$ of the point vortex parameters. Thus the implicit function theorem allows us to locally express (f, λ_1) as a function of the small parameter ε , and indeed of the remaining parameters λ_2 .

1.5. Notation. Let us end this part by summarizing some notation to be used in the paper. We will denote the unit disc by \mathbb{D} and its boundary by \mathbb{T} . For continuous functions $f: \mathbb{T} \rightarrow \mathbb{C}$ we introduce the notation

$$\oint_{\mathbb{T}} f(\tau) d\tau := \frac{1}{2\pi i} \int_{\mathbb{T}} f(\tau) d\tau.$$

where $d\tau$ stands for complex integration. For any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we use the notation $(x)_n$ to denote the Pochhammer symbol defined by

$$(x)_n := \begin{cases} 1 & \text{if } n = 0, \\ x(x+1)\cdots(x+n-1) & \text{if } n \geq 1. \end{cases}$$

Finally, we use the notation δ_{ij} to denote the Kronecker delta defined by

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

1.6. Outline of the paper. In Section 2, we will deal with a completely general vortex equilibria, without any symmetry assumptions, show how the problem is desingularized and prove Theorem 1.1. In Section 3, we focus on the asymmetric pairs of patches and prove Theorem 1.5 by imposing reflection symmetry and using the non-degeneracy condition in this symmetry class. Finally, in Section 4, we prove Theorem 1.8 by imposing reflection symmetry and also rotational symmetry on the central vortex.

2. DESINGULARIZATION OF GENERAL VORTEX EQUILIBRIA

In this section we shall consider a general configuration of finitely many point vortices in a uniform rotation or translation, and show how these vortices can be desingularized into small vortex patches. Using the approach developed in [34], we first write down the contour dynamics equations governing the N steady vortex patches, and then find the suitable function spaces where the problem is well-posed. Finally, we prove Theorem 1.1 using the implicit function theorem.

2.1. Boundary equations. Consider N bounded simply connected domains $\mathcal{O}_j^\varepsilon$, $j = 1, \dots, N$, containing the origin and contained in the ball $B(0, 2)$. Given $b_j \in (0, \infty)$, $d_j, \theta_j \in \mathbb{R}$ and $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 \ll 1$, we define the domains

$$\mathcal{D}_j^\varepsilon := e^{i\theta_j} (\varepsilon b_j \mathcal{O}_j^\varepsilon + d_j) \tag{2.1}$$

such that the sets $\mathcal{D}_j^\varepsilon$ are pairwise disjoint,

$$\overline{\mathcal{D}_j^\varepsilon} \cap \overline{\mathcal{D}_k^\varepsilon} = \emptyset, \quad j \neq k. \tag{2.2}$$

Consider the initial vorticity

$$\omega_0^\varepsilon(z) = \frac{1}{\varepsilon^2} \sum_{j=1}^N \frac{\gamma_j}{b_j^2} \chi_{\mathcal{D}_j^\varepsilon}(z). \tag{2.3}$$

Note that, if $\varepsilon \rightarrow 0$, $|\mathcal{O}_j^\varepsilon| \rightarrow |\mathbb{D}|$ in (2.3) we find the point vortex distribution

$$\omega_0^0(z) = \sum_{j=1}^N \gamma_j \delta_{z_j^0}(z), \quad z_j^0 := e^{i\theta_j} d_j,$$

whose the evolution is described by (1.3).

Suppose that ω_0^ε gives rise to N rotating patches for the model (1.1) about the centroid of the system, assumed to be the origin, and with an angular velocity Ω . More precisely, we are looking for a solution $\omega^\varepsilon(t)$ of (1.1) of the form

$$\omega^\varepsilon(t, z) = \omega_0^\varepsilon(e^{it\Omega} z).$$

Inserting this expression into (1.1) we obtain

$$(v^\varepsilon(z) - i\Omega z) \cdot \nabla \omega_0^\varepsilon(z) = 0,$$

with v^ε is the velocity field associated to ω_0^ε . Using the patch structure, we conclude that [16, Page 1896]

$$\operatorname{Re} \{ \gamma_j (\overline{v^\varepsilon(z)} + i\Omega \bar{z}) \bar{n} \} = 0 \quad \text{for all } z \in \partial \mathcal{D}_j^\varepsilon, \quad j = 1, \dots, N \quad (2.4)$$

where \bar{n} is the exterior unit normal vector to the boundary at the point z . In the case where ω_0^ε travels steadily in the (Oy) direction with uniform velocity U , that is

$$\omega^\varepsilon(t, z) = \omega_0^\varepsilon(z + itU),$$

the analogue of (2.4) is

$$\operatorname{Re} \{ \gamma_j (\overline{v^\varepsilon(z)} + iU) \bar{n} \} = 0 \quad \text{for all } z \in \partial \mathcal{D}_j^\varepsilon, \quad j = 1, \dots, N. \quad (2.5)$$

For the sake of abbreviation and simplicity we shall unify (2.4) and (2.5) as follows

$$\operatorname{Re} \{ \gamma_j (\overline{v^\varepsilon(z)} + iU + i\Omega \bar{z}) \bar{n} \} = 0 \quad \text{for all } z \in \partial \mathcal{D}_j^\varepsilon, \quad j = 1, \dots, N, \quad (2.6)$$

and assume that either Ω or U vanishes.

2.1.1. *Euler equation.* In view of the Biot–Savart law one has

$$\overline{v^\varepsilon(z)} = -\frac{i}{2\pi} \sum_{k=1}^N \frac{\gamma_k}{\varepsilon^2 b_k^2} \int_{\mathcal{D}_k^\varepsilon} \frac{dA(\zeta)}{z - \zeta} \quad (2.7)$$

for all $z \in \mathbb{C}$. By Green–Stokes Theorem,

$$\int_D \partial_{\bar{\xi}} F(\xi, \bar{\xi}) dA(\xi) = \frac{1}{2i} \int_{\partial D} F(\xi, \bar{\xi}) d\xi,$$

we may replace the integral over \mathcal{D}_k in (2.7) with an integral along $\partial \mathcal{D}_k$:

$$\overline{v^\varepsilon(z)} = \frac{i}{2} \sum_{k=1}^N \frac{\gamma_k}{\varepsilon^2 b_k^2} \int_{\partial \mathcal{D}_k^\varepsilon} \frac{\bar{\xi} - \bar{z}}{\xi - z} d\xi.$$

Inserting the last identity into (2.6) leads to

$$\gamma_j \operatorname{Re} \{ (\Omega \bar{z} + U + V^\varepsilon(z)) z' \} = 0, \quad \forall z \in \partial \mathcal{D}_j^\varepsilon, \quad j = 1, \dots, N, \quad (2.8)$$

where z' denotes a tangent vector to the boundary at the point z and

$$V^\varepsilon(z) := \sum_{k=1}^N \frac{\gamma_k}{2\varepsilon^2 b_k^2} \int_{\partial \mathcal{D}_k^\varepsilon} \frac{\bar{\xi} - \bar{z}}{\xi - z} d\xi.$$

In view of (2.1), a suitable change of variables gives

$$V^\varepsilon(z) = \sum_{k=1}^N \frac{\gamma_k}{2\varepsilon b_k} \int_{\partial \mathcal{O}_k^\varepsilon} \frac{\varepsilon b_k \bar{\xi} + d_k - e^{i\theta_k} \bar{z}}{e^{i\theta_k} (\varepsilon b_k \xi + d_k) - z} d\xi.$$

Observe, from (2.2), that for any $j \neq k$ and $z \in \partial \mathcal{O}_j^\varepsilon$ one has

$$e^{i(\theta_j - \theta_k)} (\varepsilon b_j z + d_j) + d_k \notin \varepsilon b_k \overline{\mathcal{O}_k^\varepsilon}.$$

Thus, by the residue theorem, for every $z \in \partial\mathcal{O}_j^\varepsilon$, we may write

$$\begin{aligned} V^\varepsilon(e^{i\theta_j}(\varepsilon b_j z + d_j)) &= \sum_{k=1}^N \frac{\gamma_k e^{-i\theta_k}}{2\varepsilon b_k} \int_{\partial\mathcal{O}_k^\varepsilon} \frac{\varepsilon b_k \bar{\xi} - d_k - e^{i(\theta_k - \theta_j)}(\varepsilon b_j \bar{z} + d_j)}{\varepsilon b_k \xi + d_k - e^{i(\theta_j - \theta_k)}(\varepsilon b_j z + d_j)} d\xi \\ &= \frac{\gamma_j}{2\varepsilon b_j} \int_{\partial\mathcal{O}_j^\varepsilon} \frac{\bar{\xi} - \bar{z}}{\xi - z} d\xi + \sum_{\substack{k=1 \\ k \neq j}}^N \frac{\gamma_k}{2} \int_{\partial\mathcal{O}_k^\varepsilon} \frac{\bar{\xi}}{e^{i\theta_k}(\varepsilon b_k \xi + d_k) - e^{i\theta_j}(\varepsilon b_j z + d_j)} d\xi. \end{aligned}$$

Replacing z by $e^{i\theta_j}(\varepsilon b_j z + d_j)$ in (2.8) and using the last identity we get

$$\begin{aligned} \gamma_j \operatorname{Re} \left\{ \left(\Omega(\varepsilon b_j \bar{z} + d_j) + U e^{-i\theta_j} + \frac{\gamma_j}{2\varepsilon b_j} \int_{\partial\mathcal{O}_j^\varepsilon} \frac{\bar{\xi} - \bar{z}}{\xi - z} d\xi \right. \right. \\ \left. \left. + \sum_{\substack{k=1 \\ k \neq j}}^N \frac{\gamma_k}{2} \int_{\partial\mathcal{O}_k^\varepsilon} \frac{\bar{\xi}}{e^{i(\theta_k - \theta_j)}(\varepsilon b_k \xi + d_k) - (\varepsilon b_j z + d_j)} d\xi \right) z' \right\} = 0, \quad \forall z \in \partial\mathcal{O}_j^\varepsilon. \end{aligned} \quad (2.9)$$

We shall look for domains $\mathcal{O}_j^\varepsilon$, which are perturbations of the unit disc with an amplitude of order εb_j . More precisely, we shall consider $\phi_j: \mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C} \setminus \overline{\mathcal{O}_j^\varepsilon}$ the unique conformal map with the expansion

$$\phi_j(w) = w + \varepsilon b_j f_j(w) \quad \text{with} \quad f_j(w) = \sum_{m=1}^{\infty} \frac{a_m^j}{w^m}, \quad a_m^j \in \mathbb{C}. \quad (2.10)$$

By the Kellogg–Warschawski theorem [42, Theorem 3.6], since the boundary $\partial\mathcal{O}_j^\varepsilon$ is assumed to be a smooth Jordan Curve, ϕ_j extends to a smooth mapping $\mathbb{C} \setminus \overline{\mathbb{D}} \rightarrow \mathbb{C} \setminus \overline{\mathcal{O}_j^\varepsilon}$, and its trace, that we shall also denote by ϕ_j , is a smooth parametrization of $\partial\mathcal{O}_j^\varepsilon$. Thus, making the change of variable $z = \phi_j(w)$, $z' = iw\phi_j'(w)$ in (2.9), we obtain

$$\begin{aligned} \operatorname{Im} \left\{ \gamma_j \left(\Omega(\varepsilon b_j \overline{\phi_j(w)} + d_j) + U e^{-i\theta_j} + \frac{\gamma_j}{2\varepsilon b_j} \int_{\mathbb{T}} \frac{\overline{\phi_j(\tau)} - \overline{\phi_j(w)}}{\phi_j(\tau) - \phi_j(w)} \phi_j'(\tau) d\tau \right. \right. \\ \left. \left. + \sum_{k=1}^N \frac{\gamma_k}{2} \int_{\mathbb{T}} \frac{\overline{\phi_k(\tau)} \phi_k'(\tau)}{e^{i(\theta_k - \theta_j)}(\varepsilon b_k \phi_k(\tau) + d_k) - (\varepsilon b_j \phi_j(w) + d_j)} d\tau \right) w \phi_j'(w) \right\} = 0 \end{aligned} \quad (2.11)$$

for any $w \in \mathbb{T}$ and $j = 1, \dots, N$. To desingularize the last system in ε we follow the ideas of [34] and write, by virtue of (2.10),

$$\begin{aligned} \frac{1}{2\varepsilon b_j} \int_{\mathbb{T}} \frac{\overline{\phi_j(\tau)} - \overline{\phi_j(w)}}{\phi_j(\tau) - \phi_j(w)} \phi_j'(\tau) d\tau &= \frac{1}{2\varepsilon b_j} \int_{\mathbb{T}} \frac{\bar{w} - \bar{\tau} + \varepsilon b_j (\overline{f_j(\tau)} - \overline{f_j(w)})}{w - \tau + \varepsilon b_j (f_j(\tau) - f_j(w))} [1 + \varepsilon b_j f_j'(\tau)] d\tau \\ &= \frac{1}{2} \int_{\mathbb{T}} \frac{\bar{w} - \bar{\tau} + \varepsilon b_j (\overline{f_j(\tau)} - \overline{f_j(w)})}{w - \tau + \varepsilon b_j (f_j(\tau) - f_j(w))} f_j'(\tau) d\tau - \frac{1}{2\varepsilon b_j} \int_{\mathbb{T}} \frac{\bar{w} - \bar{\tau}}{w - \tau} d\tau \\ &\quad + \frac{1}{2} \int_{\mathbb{T}} \frac{(w - \tau)(\overline{f_j(\tau)} - \overline{f_j(w)}) - (\bar{w} - \bar{\tau})(f_j(\tau) - f_j(w))}{(w - \tau)(w - \tau + \varepsilon b_j (f_j(\tau) - f_j(w)))} d\tau \\ &:= \overline{\mathcal{I}^0}[\varepsilon, f_j](w) - \frac{1}{2\varepsilon b_j} \int_{\mathbb{T}} \frac{\bar{w} - \bar{\tau}}{w - \tau} d\tau. \end{aligned}$$

From the obvious identity

$$\int_{\mathbb{T}} \frac{\bar{w} - \bar{\tau}}{w - \tau} d\tau = \bar{w},$$

the expression of ϕ_j in (2.10) and the symmetry of the disc we can get rid of the singular term from the full nonlinearity,

$$\operatorname{Im} \left\{ \frac{1}{2\varepsilon b_j} \left(\int_{\mathbb{T}} \frac{\overline{\phi_j(\tau)} - \overline{\phi_j(w)}}{\phi_j(\tau) - \phi_j(w)} \phi_j'(\tau) d\tau \right) w \phi_j'(w) \right\} = \operatorname{Im} \left\{ \overline{\mathcal{I}^0}[\varepsilon, f_j](w) w \left(1 + \varepsilon f_j'(w) \right) - \frac{1}{2} f_j'(w) \right\}.$$

Inserting the last equation into (2.11), we conclude that

$$\begin{aligned} \gamma_j \mathcal{G}_j^0(\varepsilon, f; \lambda)(w) &:= -\gamma_j \operatorname{Im} \left\{ \left(\Omega(\varepsilon b_j \bar{w} + \varepsilon^2 b_j^2 \overline{f_j(w)} - d_j) + U + \gamma_j \overline{\mathcal{I}^0}[\varepsilon, f_j](w) \right. \right. \\ &\quad \left. \left. + \sum_{k=1, k \neq j}^N \gamma_k \overline{\mathcal{J}_{kj}^0}[\varepsilon, f_k, f_j](w) \right) w \left(1 + \varepsilon b_j f_j'(w) \right) - \frac{\gamma_j}{2} f_j'(w) \right\} = 0 \end{aligned} \quad (2.12)$$

for all $w \in \mathbb{T}$ and $j = 1, \dots, N$, where $f = (f_1, \dots, f_N)$,

$$\lambda = (\Omega, U, \gamma_1, \dots, \gamma_N, d_1, \dots, d_N, \theta_1, \dots, \theta_N)$$

and

$$\begin{aligned} \overline{\mathcal{I}^0}[\varepsilon, f_j](w) &:= \frac{1}{2} \int_{\mathbb{T}} \frac{\bar{w} - \bar{\tau} + \varepsilon b_j (\overline{f_j(\tau)} - \overline{f_j(w)})}{w - \tau + \varepsilon b_j (f_j(\tau) - f_j(w))} f_j'(\tau) d\tau \\ &\quad + \frac{1}{2} \int_{\mathbb{T}} \frac{2i \operatorname{Im} \left\{ (w - \tau) (\overline{f_j(\tau)} - \overline{f_j(w)}) \right\}}{(w - \tau) (w - \tau + \varepsilon b_j f_j(\tau) - \varepsilon b_j f_j(w))} d\tau, \end{aligned} \quad (2.13)$$

$$\overline{\mathcal{J}_{kj}^0}[\varepsilon, f_k, f_j](w) := \frac{1}{2} \int_{\mathbb{T}} \frac{(\bar{\tau} + \varepsilon b_k \overline{f_k(\tau)}) (1 + \varepsilon b_k f_k'(\tau))}{e^{i(\theta_k - \theta_j)} (\varepsilon b_k \tau + \varepsilon^2 b_k^2 \overline{f_k(\tau)} + d_k) - (\varepsilon b_j (w + \varepsilon b_j f_j(w) + d_j))} d\tau. \quad (2.14)$$

Here we have used the notation $\overline{\mathcal{I}^0}$ and $\overline{\mathcal{J}_{kj}^0}$ with a complex conjugate in order to unify the notation with the functions \mathcal{I}^α , \mathcal{J}_{kj}^α and \mathcal{G}_j^α that we shall introduce in next subsection for the gSQG equations. Furthermore, both sides of (2.12) are multiplied by γ_j to ensure that the system is valid even if we set some of the γ_j equal to zero.

2.1.2. gSQG equations. The velocity can be recovered from the boundary as follows

$$v^\varepsilon(z) = \frac{C_\alpha}{2\pi} \sum_{k=1}^N \frac{\gamma_k}{\varepsilon^2 b_k^2} \int_{\partial \mathcal{D}_k^\varepsilon} \frac{d\xi}{|z - \xi|^\alpha} \quad (2.15)$$

for all $z \in \mathbb{C}$, see for instance [28]. In view of (2.1), suitable change of variables gives

$$v^\varepsilon(z) = i \sum_{k=1}^N e^{i\theta_k} \frac{\gamma_k C_\alpha}{\varepsilon b_k} \int_{\partial \mathcal{O}_k^\varepsilon} \frac{1}{|e^{i\theta_k} (\varepsilon b_k \xi + d_k) - z|^\alpha} d\xi.$$

Inserting the last identity into (2.6) and taking a complex conjugate inside the real part leads to

$$\operatorname{Re} \left\{ \gamma_j \left(\Omega z + U - \sum_{k=1}^N e^{i\theta_k} \frac{\gamma_k C_\alpha}{\varepsilon b_k} \int_{\partial \mathcal{O}_k^\varepsilon} \frac{1}{|e^{i\theta_k} (\varepsilon b_k \xi + d_k) - z|^\alpha} d\xi \right) \overline{z'} \right\} = 0 \quad \forall z \in \partial \mathcal{D}_j^\varepsilon, \quad j = 1, \dots, N,$$

where z' denotes a tangent vector to the boundary at the point z . Replacing z by $e^{i\theta_j} (\varepsilon b_j z + d_j)$ in the last system gives

$$\begin{aligned} \operatorname{Re} \left\{ \gamma_j \left(\Omega(\varepsilon b_j z + d_j) + U e^{-i\theta_j} - \frac{\gamma_j C_\alpha}{\varepsilon |\varepsilon|^\alpha b_j^{1+\alpha}} \int_{\partial \mathcal{O}_j^\varepsilon} \frac{1}{|\xi - z|^\alpha} d\xi \right. \right. \\ \left. \left. - \sum_{k=1, k \neq j}^N \frac{\gamma_k C_\alpha}{\varepsilon b_k} \int_{\partial \mathcal{O}_k^\varepsilon} \frac{e^{i(\theta_k - \theta_j)}}{|e^{i\theta_k} (\varepsilon b_k \xi + d_k) - e^{i\theta_j} (\varepsilon b_j \xi + d_j)|^\alpha} d\xi \right) \overline{z'} \right\} = 0 \quad \forall z \in \partial \mathcal{O}_j^\varepsilon. \end{aligned} \quad (2.16)$$

We shall look for conformal parametrizations $\phi_j: \mathbb{T} \rightarrow \partial\mathcal{O}_j^\varepsilon$ having the expansions

$$\phi_j(w) = w + \varepsilon|\varepsilon|^\alpha b_j^{1+\alpha} f_j(w) \quad \text{with} \quad f_j(w) = \sum_{m=1}^{\infty} \frac{a_m^j}{w^m}, \quad a_m^j \in \mathbb{C}, \quad j = 1, \dots, N. \quad (2.17)$$

Here, the coefficient $|\varepsilon|^\alpha$ in the definition of the conformal mapping ϕ_j comes from the singularity of the gSQG kernel. For every $w \in \mathbb{T}$, the tangent vector is given by $z' = iw\phi_j'(w)$ and therefore (2.16) becomes

$$\begin{aligned} & \text{Im} \left\{ \gamma_j \left(\Omega(\varepsilon b_j \phi_j(w) + d_j) + U e^{-i\theta_j} - \frac{\gamma_j C_\alpha}{\varepsilon |\varepsilon|^\alpha b_j^{1+\alpha}} \int_{\mathbb{T}} \frac{\phi_j'(\tau)}{|\phi_j(\tau) - \phi_j(w)|^\alpha} d\tau \right. \right. \\ & \left. \left. - \sum_{k=1, k \neq j}^N \frac{\gamma_k C_\alpha}{\varepsilon b_k} \int_{\mathbb{T}} \frac{e^{i(\theta_k - \theta_j)} \phi_k'(\tau)}{|e^{i(\theta_k - \theta_j)}(\varepsilon b_k \phi_k(\tau) + d_k) - (\varepsilon b_j \phi_j(w) + d_j)|^\alpha} d\tau \right) \overline{w \phi_j'(w)} \right\} = 0. \end{aligned} \quad (2.18)$$

In order to desingularize the system (2.18) we shall use the following Taylor formula,

$$\frac{1}{|A+B|^\alpha} = \frac{1}{|A|^\alpha} - \alpha \int_0^1 \frac{\text{Re}(A\overline{B}) + t|B|^2}{|A+tB|^{2+\alpha}} dt \quad (2.19)$$

which is true for any complex numbers A, B such that $|B| < |A|$. Taking $A = e^{i(\theta_k - \theta_j)} d_k - d_j$ and $B = e^{i(\theta_k - \theta_j)} b_k \phi_k(\tau) - b_j \phi_j(w)$, one may write

$$\begin{aligned} & \frac{1}{|e^{i(\theta_k - \theta_j)}(\varepsilon b_k \phi_k(\tau) + d_k) - (\varepsilon b_j \phi_j(w) + d_j)|^\alpha} = \frac{1}{|e^{i(\theta_k - \theta_j)} d_k - d_j|^\alpha} \\ & - \alpha \varepsilon \int_0^1 \frac{\text{Re} \left[(e^{i(\theta_k - \theta_j)} d_k - d_j) (e^{-i(\theta_k - \theta_j)} b_k \overline{\phi_k(\tau)} - b_j \overline{\phi_j(w)}) \right] + t \varepsilon |e^{i(\theta_k - \theta_j)} b_k \phi_k(\tau) - b_j \phi_j(w)|^2}{|e^{i(\theta_k - \theta_j)}(\varepsilon b_k t \phi_k(\tau) + d_k) - (\varepsilon b_j t \phi_j(w) + d_j)|^{\alpha+2}} dt. \end{aligned}$$

It follows that

$$\begin{aligned} & \frac{1}{\varepsilon} \int_{\mathbb{T}} \frac{\phi_k'(\tau)}{|e^{i(\theta_k - \theta_j)}(\varepsilon b_k \phi_k(\tau) + d_k) - (\varepsilon b_j \phi_j(w) + d_j)|^\alpha} d\tau = \\ & - \alpha \int_{\mathbb{T}} \int_0^1 \frac{\text{Re} \left[(e^{i(\theta_k - \theta_j)} d_k - d_j) (e^{-i(\theta_k - \theta_j)} b_k \overline{\phi_k(\tau)} - b_j \overline{\phi_j(w)}) \right]}{|e^{i(\theta_k - \theta_j)}(\varepsilon b_k t \phi_k(\tau) + d_k) - (\varepsilon b_j t \phi_j(w) + d_j)|^{\alpha+2}} \phi_k'(\tau) dt d\tau \\ & - \alpha \int_0^1 \int_{\mathbb{T}} \frac{t \varepsilon |e^{i(\theta_k - \theta_j)} b_k \phi_k(\tau) - b_j \phi_j(w)|^2}{|e^{i(\theta_k - \theta_j)}(\varepsilon b_k t \phi_k(\tau) + d_k) - (\varepsilon b_j t \phi_j(w) + d_j)|^{\alpha+2}} \phi_k'(\tau) dt d\tau. \end{aligned} \quad (2.20)$$

On the other hand, from (2.17), one has

$$\begin{aligned} \frac{C_\alpha}{\varepsilon |\varepsilon|^\alpha} \int_{\mathbb{T}} \frac{\phi_j'(\tau)}{|\phi_j(\tau) - \phi_j(w)|^\alpha} d\tau &= \frac{C_\alpha}{\varepsilon |\varepsilon|^\alpha} \int_{\mathbb{T}} \frac{d\tau}{|\tau - w|^\alpha} + C_\alpha \int_{\mathbb{T}} \frac{b_j^{1+\alpha} f_j'(\tau) d\tau}{|w - \tau + t \varepsilon |\varepsilon|^\alpha b_j^{1+\alpha} (f_j(\tau) - f_j(w))|^\alpha} \\ &+ \frac{C_\alpha}{\varepsilon |\varepsilon|^\alpha} \int_{\mathbb{T}} \left(\frac{1}{|w - \tau + t \varepsilon |\varepsilon|^\alpha b_j^{1+\alpha} (f_j(\tau) - f_j(w))|^\alpha} - \frac{1}{|\tau - w|^\alpha} \right) d\tau. \end{aligned}$$

Using the identity, [28, Page 337],

$$C_\alpha \int_{\mathbb{T}} \frac{d\tau}{|\tau - w|^\alpha} = \frac{\alpha C_\alpha \Gamma(1 - \alpha)}{(2 - \alpha) \Gamma^2(1 - \frac{\alpha}{2})} w =: \mu_\alpha w, \quad (2.21)$$

and applying the formula (2.19) with $A = \tau - w$ and $B = \varepsilon|\varepsilon|^\alpha b_j^{1+\alpha}(f_j(\tau) - f_j(w))$, we find

$$\begin{aligned} \frac{C_\alpha}{\varepsilon|\varepsilon|^\alpha} \int_{\mathbb{T}} \frac{\phi_j'(\tau)}{|\phi_j(\tau) - \phi_j(w)|^\alpha} d\tau &= \frac{\mu_\alpha}{\varepsilon|\varepsilon|^\alpha} w + C_\alpha b_j^{1+\alpha} \int_{\mathbb{T}} \frac{f_j'(\tau)}{|w - \tau + t\varepsilon|\varepsilon|^\alpha b_j^{1+\alpha}(f_j(\tau) - f_j(w))|^\alpha} d\tau \\ &\quad - \alpha C_\alpha b_j^{1+\alpha} \int_{\mathbb{T}} \int_0^1 \frac{\operatorname{Re} \left((f_j(\tau) - f_j(w))(\bar{\tau} - \bar{w}) \right)}{|w - \tau + t\varepsilon|\varepsilon|^\alpha b_j^{1+\alpha}(f_j(\tau) - f_j(w))|^{2+\alpha}} d\tau dt \\ &\quad - \alpha C_\alpha \varepsilon|\varepsilon|^\alpha b_j^{1+\alpha} \int_{\mathbb{T}} \int_0^1 \frac{t|f_j(\tau) - f_j(w)|^2}{|w - \tau + t\varepsilon|\varepsilon|^\alpha b_j^{1+\alpha}(f_j(\tau) - f_j(w))|^{2+\alpha}} d\tau dt \\ &:= \frac{\mu_\alpha}{\varepsilon|\varepsilon|^\alpha} w - b_j^{1+\alpha} \mathcal{I}^\alpha[\varepsilon, f_j](w). \end{aligned}$$

As in the Euler case, by (2.17) and the symmetry of the disc the singular term disappears from the nonlinearity,

$$\begin{aligned} \operatorname{Im} \left\{ \frac{C_\alpha}{\varepsilon|\varepsilon|^\alpha b_j^{1+\alpha}} \int_{\mathbb{T}} \frac{\phi_j'(\tau)}{|\phi_j(\tau) - \phi_j(w)|^\alpha} d\tau \bar{w} \overline{\phi_j'(w)} \right\} &= \\ \operatorname{Im} \left\{ \mu_\alpha \overline{f_j'(w)} - \mathcal{I}^\alpha[\varepsilon, f_j](w) \bar{w} (1 + \varepsilon|\varepsilon|^\alpha b_j^{1+\alpha} \overline{f_j'(w)}) \right\}. \end{aligned} \quad (2.22)$$

Inserting (2.20) and (2.22) into (2.18) we get

$$\begin{aligned} \gamma_j \mathcal{G}_j^\alpha(\varepsilon, f; \lambda)(w) &:= \gamma_j \operatorname{Im} \left\{ \left(\Omega(\varepsilon b_j w + \varepsilon^2 |\varepsilon|^\alpha b_j^{2+\alpha} f_j(w) + d_j) + U e^{-i\theta_j} + \gamma_j \mathcal{I}^\alpha[\varepsilon, f_j](w) \right. \right. \\ &\quad \left. \left. + \sum_{k=1, k \neq j}^N \gamma_k \mathcal{J}_k^\alpha[\varepsilon, f_k, f_j](w) \right) \bar{w} (1 + \varepsilon|\varepsilon|^\alpha b_j^{1+\alpha} \overline{f_j'(w)}) - \mu_\alpha \gamma_j \overline{f_j'(w)} \right\} = 0 \end{aligned} \quad (2.23)$$

for all $w \in \mathbb{T}$ and $j = 1, \dots, N$, where $f = (f_1, \dots, f_N)$,

$$\lambda = (\Omega, U, \gamma_1, \dots, \gamma_N, d_1, \dots, d_N, \theta_1, \dots, \theta_N)$$

and

$$\begin{aligned} \mathcal{I}^\alpha[\varepsilon, f_j](w) &:= -C_\alpha \int_{\mathbb{T}} \frac{f_j'(\tau)}{|\tau - w + \varepsilon|\varepsilon|^\alpha b_j^{1+\alpha}(f_j(\tau) - f_j(w))|^\alpha} d\tau \\ &\quad + \alpha C_\alpha \int_{\mathbb{T}} \int_0^1 \frac{\operatorname{Re} \left[(f_j(\tau) - f_j(w))(\bar{\tau} - \bar{w}) \right] + \varepsilon|\varepsilon|^\alpha t |f_j(\tau) - f_j(w)|^2}{|w - \tau + t\varepsilon|\varepsilon|^\alpha b_j^{1+\alpha}(f_j(\tau) - f_j(w))|^{2+\alpha}} dt d\tau, \end{aligned} \quad (2.24)$$

$$\begin{aligned} \mathcal{J}_k^\alpha[\varepsilon, f_k, f_j; \lambda](w) &:= \frac{\alpha C_\alpha}{b_k} \left[\int_{\mathbb{T}} \int_0^1 \frac{\operatorname{Re} \left[(e^{i(\theta_k - \theta_j)} d_k - d_j) (e^{i(\theta_j - \theta_k)} b_k \overline{\phi_k(\tau)} - b_j \overline{\phi_j(w)}) \right]}{|e^{i(\theta_k - \theta_j)} (\varepsilon b_k t \phi_k(\tau) + d_k) - (\varepsilon b_j t \phi_j(w) + d_j)|^{\alpha+2}} \phi_k'(\tau) dt d\tau \right. \\ &\quad \left. + \int_{\mathbb{T}} \int_0^1 \frac{t \varepsilon |e^{i(\theta_k - \theta_j)} b_k \phi_k(\tau) - b_j \phi_j(w)|^2 \phi_k'(\tau)}{|e^{i(\theta_k - \theta_j)} (\varepsilon b_k t \phi_k(\tau) + d_k) - (\varepsilon b_j t \phi_j(w) + d_j)|^{\alpha+2}} dt d\tau \right] e^{i(\theta_k - \theta_j)}. \end{aligned} \quad (2.25)$$

2.2. Existence of the vortex patch equilibria. For any $\beta \in (0, 1)$, we denote by $C^\beta(\mathbb{T})$ the space of continuous functions $f: \mathbb{T} \rightarrow \mathbb{C}$ such that

$$\|f\|_{C^\beta(\mathbb{T})} := \|f\|_{L^\infty(\mathbb{T})} + \sup_{x \neq y \in \mathbb{T}} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$

For any integer n , the space $C^{n+\beta}(\mathbb{T})$ stands for the set of functions f of class C^n whose n -th order derivatives are Hölder continuous with exponent β . It is equipped with the usual norm,

$$\|f\|_{C^{n+\beta}(\mathbb{T})} := \|f\|_{L^\infty(\mathbb{T})} + \left\| \frac{d^n f}{dw^n} \right\|_{C^\beta(\mathbb{T})}.$$

We henceforth fix some $\beta \in (0, 1)$, and for $\alpha \in [0, 1)$ define

$$\eta_\alpha := \begin{cases} 1 - \alpha & \text{if } 0 < \alpha < 1, \\ \beta & \text{if } \alpha = 0. \end{cases}$$

Consider the Banach spaces

$$\begin{aligned} \mathcal{V}_0^\alpha &:= \underbrace{V_0^\alpha \times \cdots \times V_0^\alpha}_{N \text{ times}}, & \mathcal{W}_0^\alpha &:= \underbrace{W_0^\alpha \times \cdots \times W_0^\alpha}_{N \text{ times}}, & \text{and } \widetilde{\mathcal{W}}_0^\alpha &:= \underbrace{\widetilde{W}_0^\alpha \times \cdots \times \widetilde{W}_0^\alpha}_{N \text{ times}} \\ \text{with } V_0^\alpha &:= \left\{ f \in C^{1+\eta_\alpha}(\mathbb{T}) : f(w) = \sum_{n \geq 1} a_n \bar{w}^n, a_n \in \mathbb{C} \right\}, \\ W_0^\alpha &:= \left\{ g \in C^{1+\eta_\alpha}(\mathbb{T}) : g(w) = \frac{1}{2i} \sum_{n \neq 0} c_n w^n, c_{-n} = -\bar{c}_n \in \mathbb{C} \right\}, \\ \widetilde{W}_0^\alpha &:= \left\{ g \in W^\alpha : c_1 = 0 \right\}. \end{aligned} \tag{2.26}$$

Here the subscript 0 refers to the fact that no a priori symmetry assumption is made on the patch boundaries. We denote by B_0^α the unit ball in \mathcal{V}_0^α ,

$$B_0^\alpha := \{f \in \mathcal{V}^\alpha : \|f\|_{C^{\alpha+n}(\mathbb{T})} < 1\}.$$

We shall unify the expression of the function \mathcal{G}_j^α in (2.23) and the function \mathcal{G}_j^0 in (2.12) as follows: for every $f \in \mathcal{V}_0^\alpha$ and $w \in \mathbb{T}$,

$$\begin{aligned} \mathcal{G}_j^\alpha(\varepsilon, f; \lambda)(w) &:= \text{Im} \left\{ \left(\Omega(\varepsilon b_j w + \varepsilon^2 |\varepsilon|^\alpha b_j^{2+\alpha} f_j(w) + d_j) + U e^{-i\theta_j} \right) \bar{w} (1 + \varepsilon |\varepsilon|^\alpha b_j^{1+\alpha} \overline{f_j'(w)}) \right. \\ &\quad \left. - \mu_\alpha \gamma_j \overline{f_j'(w)} + \left(\gamma_j \mathcal{I}^\alpha[\varepsilon, f_j](w) + \sum_{k=1, k \neq j}^N \gamma_k \mathcal{J}_k^\alpha[\varepsilon, f_k, f_j](w) \right) \bar{w} (1 + \varepsilon |\varepsilon|^\alpha b_j^{1+\alpha} \overline{f_j'(w)}) \right\}, \end{aligned}$$

where μ_α is defined in (2.21), and $\mathcal{I}^\alpha[\varepsilon, f_j](w)$ and $\mathcal{J}_k^\alpha[\varepsilon, f_k, f_j](w)$ are given by (2.24)–(2.25) if $\alpha \in (0, 1)$ and by (2.13)–(2.14) if $\alpha = 0$. We then define the nonlinear operator

$$\mathcal{G}^\alpha(\varepsilon, f; \lambda) := (\mathcal{G}_1^\alpha(\varepsilon, f; \lambda), \dots, \mathcal{G}_N^\alpha(\varepsilon, f; \lambda)).$$

Proposition 2.1. *Let $\alpha \in [0, 1)$ and let λ^* solve (1.4).*

- (i) *There exists $\varepsilon_0 > 0$ and a small neighborhood Λ of λ^* such that \mathcal{G}^α can be extended to a C^1 mapping $(-\varepsilon_0, \varepsilon_0) \times B_0^\alpha \times \Lambda \rightarrow \mathcal{W}_0^\alpha$.*
- (ii) *For all $\lambda \in \Lambda$ one has*

$$\mathcal{G}_j^\alpha(0, 0; \lambda)(w) = \text{Im} \{ \mathcal{P}_j^\alpha(\lambda) \bar{w} \}$$

for $w \in \mathbb{T}$, where $\mathcal{P}_j^\alpha(\lambda)$ is given by (1.4).

- (iii) *The Fréchet derivative of \mathcal{G}^α with respect to f at $(0, 0; \lambda)$ is given by*

$$D_f \mathcal{G}^\alpha(0, 0; \lambda) h(w) = \sum_{n \geq 1} M_n^\alpha \begin{pmatrix} \gamma_1 \text{Im} \{ a_n^1 w^{n+1} \} \\ \vdots \\ \gamma_N \text{Im} \{ a_n^N w^{n+1} \} \end{pmatrix},$$

where here $h = (h_1, \dots, h_N) \in \mathcal{V}_0^\alpha$ with $h_j(w) := \sum_{n \geq 1} a_n^j \bar{w}^n$ and

$$M_n^\alpha := \frac{\Gamma(1 - \frac{\alpha}{2})\Gamma(1 - \alpha)}{2^{2-\alpha}\Gamma^3(1 - \frac{\alpha}{2})} \left(\frac{2(n+1)}{1 - \frac{\alpha}{2}} - \frac{(1 + \frac{\alpha}{2})_n}{(1 - \frac{\alpha}{2})_n} - \frac{(1 + \frac{\alpha}{2})_{n+1}}{(1 - \frac{\alpha}{2})_{n+1}} \right). \quad (2.27)$$

(iv) For any $\lambda \in \Lambda$, the linear operator $D_f \mathcal{G}^\alpha(0, 0; \lambda): \mathcal{V}_0^\alpha \rightarrow \widetilde{\mathcal{W}}_0^\alpha$ is an isomorphism.

Proof. Note that the term

$$\text{Im} \left\{ \left(\Omega(\varepsilon b_j w + \varepsilon^2 |\varepsilon|^\alpha b_j^{2+\alpha} f_j(w) + d_j) + U e^{-i\theta_j} + \gamma_j \mathcal{I}^\alpha[\varepsilon, f_j](w) \right) \bar{w} (1 + \varepsilon |\varepsilon|^\alpha b_j^{1+\alpha} \overline{f_j'(w)}) - \mu_\alpha \gamma_j \overline{f_j'(w)} \right\},$$

appears identically in the study of the vortex pairs in [34], where it was shown that it is C^1 in ε and f . The spaces used in [34] are V_0^α with the reflection symmetry $\overline{f_j(w)} = f_j(\bar{w})$ and W_0^α with the symmetry $g_j(\bar{w}) = -g_j(w)$, but the study of the regularity can be easily generalized to our case without further difficulties. The terms

$$\sum_{k=1, k \neq j}^N \text{Im} \left\{ \gamma_k \mathcal{J}_k^\alpha[\varepsilon, f_k, f_j; \lambda](w) \bar{w} (1 + \varepsilon |\varepsilon|^\alpha b_j^{1+\alpha} \overline{f_j'(w)}) \right\}$$

describe the interaction between the boundaries of the patches, which are disjoint provided ε is sufficiently small. Therefore the kernel involved is sufficiently smooth and can be treated in a very classical way.

Next, we shall prove (ii) and (iii) in the case $\alpha = 0$. Substituting $\varepsilon = 0$ in (2.13) and (2.14) gives

$$\begin{aligned} \overline{\mathcal{J}_{kj}^0}[0, f_k, f_j; \lambda](w) &= \frac{1}{2} \frac{1}{d_j - e^{i(\theta_k - \theta_j)} d_k}, \\ \overline{\mathcal{I}^0}[0, f_j](w) &= \frac{1}{2} \int_{\mathbb{T}} \frac{\bar{w} - \bar{\tau}}{w - \tau} f_j'(\tau) d\tau + \int_{\mathbb{T}} \frac{i \text{Im} \left\{ (w - \tau) (\overline{f_j(\tau)} - \overline{f_j(w)}) \right\}}{(w - \tau)^2} d\tau = 0, \end{aligned}$$

where we have used the residue theorem in the last identity. Then, from (2.12), we get

$$\mathcal{G}_j^0(0, f; \lambda)(w) = -\text{Im} \left\{ \left(\Omega d_j + U e^{-i\theta_j} + \frac{1}{2} \sum_{k=0, k \neq j}^n \frac{\gamma_k}{d_j - e^{i(\theta_k - \theta_j)} d_k} \right) w - \frac{\gamma_j}{2} f_j'(w) \right\} \quad (2.28)$$

and hence

$$\begin{aligned} \mathcal{G}_j^0(0, 0; \lambda)(w) &= -\text{Im} \left\{ \left(\Omega d_j + U e^{-i\theta_j} + \frac{1}{2} \sum_{k=0, k \neq j}^n \frac{\gamma_k}{d_j - e^{i(\theta_k - \theta_j)} d_k} \right) w \right\} \\ &\stackrel{(1.4)}{=} -\text{Im} \left\{ \overline{\mathcal{P}_j^0(\lambda)} w \right\}, \end{aligned}$$

which shows (ii). Differentiating (2.28) with respect to f gives

$$\partial_{f_k} \mathcal{G}_j^0(0, 0; \lambda) h_j(w) = \delta_{jk} \frac{\gamma_j}{2} \text{Im} \left\{ h_j'(w) \right\},$$

getting (iii) in the case $\alpha = 0$.

To prove (ii) and (iii) in the case $\alpha \in (0, 1)$ we substitute $\varepsilon = 0$ in (2.24) and (2.25) and obtain

$$\begin{aligned} \mathcal{I}^\alpha[0, f_j](w) &= -C_\alpha \int_{\mathbb{T}} \frac{f_j'(\tau)}{|\tau - w|^\alpha} d\tau + \alpha C_\alpha \int_{\mathbb{T}} \int_0^1 \frac{\text{Re} \left[(f_j(\tau) - f_j(w)) (\bar{\tau} - \bar{w}) \right]}{|w + \tau|^{2+\alpha}} d\tau dt, \\ \mathcal{J}_k^\alpha[0, f_k, f_j; \lambda](w) &= \frac{\alpha C_\alpha}{2} \frac{e^{i(\theta_k - \theta_j)} d_k - d_j}{|e^{i(\theta_k - \theta_j)} d_k - d_j|^{\alpha+2}}. \end{aligned}$$

Thus, by (2.23), one has

$$\begin{aligned} \mathcal{G}_j^\alpha(0, f; \lambda)(w) &= \operatorname{Im} \left\{ \left(\Omega d_j + U e^{-i\theta_j} + \frac{\alpha C_\alpha}{2} \sum_{k=1, k \neq j}^N \frac{e^{i(\theta_k - \theta_j)} d_k - d_j}{|e^{i(\theta_k - \theta_j)} d_k - d_j|^{\alpha+2}} \right) \bar{w} \right. \\ &\quad \left. + \gamma_j \bar{w} \mathcal{I}^\alpha[0, f_j](w) - \mu_\alpha \gamma_j \overline{f_j'(w)} \right\}. \end{aligned} \quad (2.29)$$

It follows that

$$\mathcal{G}_j^\alpha(0, 0; \lambda)(w) = \operatorname{Im} \left\{ \left(\Omega d_j + U e^{-i\theta_j} + \frac{\alpha C_\alpha}{2} \sum_{k=1, k \neq j}^N \frac{e^{i(\theta_k - \theta_j)} d_k - d_j}{|e^{i(\theta_k - \theta_j)} d_k - d_j|^{\alpha+2}} \right) \bar{w} \right\}.$$

Comparing the last expression with (1.4) concludes (ii). Next, differentiating (2.29) with respect to f gives

$$\partial_{f_k} \mathcal{G}_j^\alpha(0, 0; \lambda) h_j(w) = \delta_{jk} \gamma_j \operatorname{Im} \left\{ \mathcal{I}^\alpha[0, h_j](w) \bar{w} - \mu_\alpha \overline{h_j'(w)} \right\}. \quad (2.30)$$

The last expression was explicitly computed in [34, Pages 726–728] and takes the form

$$\partial_{f_j} \mathcal{G}_j^\alpha(0, 0; \lambda) h_j(w) = \gamma_j \sum_{n \geq 1} \frac{\alpha C_\alpha \Gamma(1 - \alpha)}{4\Gamma^2(1 - \frac{\alpha}{2})} \left(\frac{2(n+1)}{1 - \frac{\alpha}{2}} - \frac{(1 + \frac{\alpha}{2})_n}{(1 - \frac{\alpha}{2})_n} - \frac{(1 + \frac{\alpha}{2})_{n+1}}{(1 - \frac{\alpha}{2})_{n+1}} \right) \operatorname{Im} \{ a_n^j w^{n+1} \},$$

getting the announced result.

The proof of (iv) is elementary in the case $\alpha = 0$, since

$$D_f \mathcal{G}^\alpha(0, 0; \lambda) h(w) = \sum_{n \geq 1} \frac{1}{2} \begin{pmatrix} \gamma_1 \operatorname{Im} \{ h_1'(w) \} \\ \vdots \\ \gamma_N \operatorname{Im} \{ h_N'(w) \} \end{pmatrix}.$$

For the case $\alpha \in (0, 1)$, we reproduce similar arguments to [34]. In particular, we observe from (2.30) that $D_f \mathcal{G}_j^\alpha(0, 0; \lambda): \mathcal{V}_0^\alpha \rightarrow \widetilde{\mathcal{W}}_0^\alpha$ is a compact perturbation of a Fredholm operator of index zero, since $\mathcal{I}^\alpha: \mathcal{V}_0^\alpha \rightarrow \widetilde{\mathcal{W}}_0^\alpha$ is smoothing. Therefore $\partial_f \mathcal{G}_j^\alpha(0, 0; \lambda): \mathcal{V}_0^\alpha \rightarrow \widetilde{\mathcal{W}}_0^\alpha$ is Fredholm with index zero. To check that it has a trivial kernel we use the bound, [34, page 728],

$$\gamma_n^\alpha > C_0 n, \quad n \geq 1,$$

for some constant $C_0 > 0$. This concludes the proof of the proposition. \square

Remark 2.2. Since $\widetilde{\mathcal{W}}_0^\alpha \subset \mathcal{W}_0^\alpha$ has codimension $2N$, Proposition 2.1(iv) implies that $D_f \mathcal{G}^\alpha(0, 0; \lambda)$ is Fredholm with finite index $-2N$. Thus, in a small neighborhood of $(0, 0; \lambda^*)$, the nonlinear problem $\mathcal{G}^\alpha(\varepsilon, f, \lambda) = 0$ admits a Lyapunov–Schmidt reduction to an equation in finite dimensions; see for instance [38]. Of course, studying this reduced equation may still be quite challenging, and may in particular involve evaluating further Fréchet derivatives of \mathcal{G}^α . When the point vortex equilibrium represented by λ is non-degenerate, however, one can simply apply the implicit function theorem, as in the proposition below.

Proposition 2.3. *Let $\alpha \in [0, 1)$ and let λ^* be a non-degenerate solution to the N -vortex problem (1.4), so that after a reordering of the entries of λ we can write*

$$\lambda = (\lambda_1, \lambda_2) \quad \text{with} \quad \lambda_1 \in \mathbb{R}^{2N} \quad \text{and} \quad D_{\lambda_1} \mathcal{P}^\alpha(\lambda^*) \text{ invertible.}$$

Then the following hold true.

- (i) *There exists $\varepsilon_1 > 0$ and a unique C^1 function $(f, \lambda_1): (-\varepsilon_1, \varepsilon_1) \rightarrow B_1^\alpha \times \mathbb{R}^{2N}$ satisfying*

$$\mathcal{G}^\alpha(\varepsilon, f(\varepsilon); \lambda_1(\varepsilon), \lambda_2^*) = 0,$$

$$f(0) = 0, \text{ and } \lambda_1(0) = \lambda_1^*.$$

(ii) *These solutions enjoy the symmetries*

$$f(-\varepsilon)(w) = f(\varepsilon)(-w), \quad \lambda_1(-\varepsilon) = \lambda_1(\varepsilon).$$

(iii) *For all $\varepsilon \in (-\varepsilon_1, \varepsilon_1) \setminus \{0\}$ the domains $\mathcal{O}_j^\varepsilon$, whose boundaries are given by the conformal parameterizations $\phi_j^\varepsilon = \text{Id} + \varepsilon|\varepsilon|^\alpha b_j^{1+\alpha} f_j: \mathbb{T} \rightarrow \partial\mathcal{O}_j^\varepsilon$, are strictly convex.*

Proof. In view of Proposition 2.1, for any $h \in \mathcal{V}_0^\alpha$ and $\dot{\lambda}_1 \in \mathbb{R}^{2N}$, we have

$$D_{(f;\lambda_1)}\mathcal{G}^\alpha(0, 0; \lambda^*) \begin{pmatrix} h \\ \dot{\lambda}_1 \end{pmatrix} (w) = D_f\mathcal{G}^\alpha(0, 0; \lambda^*)h(w) + \text{Im} \{ D_{\lambda_1}\mathcal{P}^\alpha(\lambda^*)\dot{\lambda}_1\bar{w} \},$$

where $D_f\mathcal{G}^\alpha(0, 0; \lambda^*)$ is an isomorphism from \mathcal{V}_0^α to $\widetilde{\mathcal{W}}_0^\alpha$. Because $D_{\lambda_1}\mathcal{P}^\alpha(\lambda^*)$ is invertible, the second linear operator $D_{\lambda_1}\mathcal{G}^\alpha(0, 0; \lambda^*)$ on the right hand side is also an isomorphism from \mathbb{R}^{2N} to \mathbb{W} where $\mathbb{W} := \{w \mapsto \text{Im}[c_1 w] \in C^{\eta\alpha}(\mathbb{T}) : c_1 \in \mathbb{C}^N\}$. Moreover, it is easy to see that

$$\mathcal{W}_0^\alpha = \widetilde{\mathcal{W}}_0^\alpha \oplus \mathbb{W}.$$

Thus, the linear operator $D_{(f;\lambda_1)}\mathcal{G}^\alpha(0, 0; \lambda^*): \mathcal{V}_0^\alpha \times \mathbb{R}^{2N} \rightarrow \mathcal{W}_0^\alpha$ is an isomorphism, and (i) follows from the implicit function theorem.

By the uniqueness in (i), in order to prove (ii) it suffices to show that

$$\mathcal{G}_j^\alpha(\varepsilon, f; \lambda)(-w) = \mathcal{G}_j^\alpha(-\varepsilon, \tilde{f}; \lambda)(w), \quad (2.31)$$

where $\tilde{f}(w) = f(-w)$. From (3.6) one has

$$\begin{aligned} \mathcal{G}_j^\alpha(-\varepsilon, \tilde{f}; \lambda)(w) &= \text{Im} \left\{ \left(\Omega(-\varepsilon b_j w + \varepsilon^2 |\varepsilon|^\alpha b_j^{2+\alpha} \tilde{f}_j(w) + d_j) + U e^{-i\theta_j} \right. \right. \\ &\quad \left. \left. + \gamma_j \mathcal{I}^\alpha[-\varepsilon, \tilde{f}_j](w) + \gamma_{3-j} \mathcal{J}_{3-j}^\alpha[-\varepsilon, \tilde{f}_{3-j}, \tilde{f}_j; \lambda](w) \right) \bar{w} (1 - \varepsilon |\varepsilon|^\alpha b_1^{1+\alpha} \overline{\tilde{f}'_j(w)}) - \mu_\alpha \gamma_j \overline{\tilde{f}'_j(w)} \right\}. \end{aligned} \quad (2.32)$$

Since $\tilde{f}'_j(w) = -f'_j(-w)$, by (2.24) we have

$$\begin{aligned} \mathcal{I}^\alpha[-\varepsilon, \tilde{f}_j](w) &:= C_\alpha \int_{\mathbb{T}} \frac{f'_j(-\tau)}{|\tau - w - \varepsilon |\varepsilon|^\alpha b_j^{1+\alpha} (f_j(-\tau) - f_j(-w))|^\alpha} d\tau \\ &\quad + \alpha C_\alpha \int_{\mathbb{T}} \int_0^1 \frac{\text{Re} [(f_j(-\tau) - f_j(-w))(\bar{\tau} - \bar{w})] - \varepsilon |\varepsilon|^\alpha t |f_j(-\tau) - f_j(-w)|^2}{|\tau - w - t\varepsilon |\varepsilon|^\alpha b_j^{1+\alpha} (f_j(-\tau) + f_j(-w))|^{2+\alpha}} dt d\tau. \end{aligned}$$

Making the change of variable $\tau \mapsto -\tau$ we get

$$\begin{aligned} \mathcal{I}^\alpha[-\varepsilon, \tilde{f}_j](w) &= -C_\alpha \int_{\mathbb{T}} \frac{f'_j(\tau)}{|\tau + w + \varepsilon |\varepsilon|^\alpha b_j^{1+\alpha} (f_j(\tau) - f_j(-w))|^\alpha} d\tau \\ &\quad + \alpha C_\alpha \int_{\mathbb{T}} \int_0^1 \frac{\text{Re} [(f_j(\tau) - f_j(-w))(\bar{\tau} + \bar{w})] + \varepsilon |\varepsilon|^\alpha t |f_j(\tau) - f_j(-w)|^2}{|w + \tau + t\varepsilon |\varepsilon|^\alpha b_j^{1+\alpha} (f_j(\tau) + f_j(-w))|^{2+\alpha}} dt d\tau \\ &= \mathcal{I}^\alpha[\varepsilon, f_j](-w). \end{aligned}$$

In a similar way we can check that

$$\mathcal{J}_k^\alpha[-\varepsilon, \tilde{f}_k, \tilde{f}_j; \lambda](w) = \mathcal{J}_k^\alpha[\varepsilon, f_k, f_j](-w).$$

Inserting the two last identities into (2.32) and using the fact that $\tilde{f}_j(w) = f_j(-w)$ yields (2.31) as desired.

To prove the convexity of the domains $\mathcal{O}_j^\varepsilon$, we shall reproduce the same arguments of [34]. Recall that the curvature can be expressed, in terms of the conformal mapping, by the formula

$$\kappa(w) = \frac{1}{|\phi_j^\varepsilon(w)|} \text{Re} \left(1 + w \frac{\phi_j''(w)}{\phi_j'(w)} \right).$$

As $\phi_j(w) = w + \varepsilon|\varepsilon|^\alpha b_j^{1+\alpha} f_j(w)$, we easily verify that

$$1 + w \frac{\phi_j''(w)}{\phi_j'(w)} = 1 + O(\varepsilon|\varepsilon|^\alpha),$$

uniformly in w . Thus the curvature is strictly positive and therefore the domain $\mathcal{O}_j^\varepsilon$ is strictly convex. \square

Remark 2.4. In this section we have suppressed the dependence of \mathcal{G}^α on the parameters $b_j \in (0, \infty)$. Just as with ε , one can check that \mathcal{G}^α is in fact C^1 in b_j . This is true even for $b_j = 0$, corresponding to the case where j -th point vortex is not desingularized into a vortex patch but instead remains a point vortex. Applying the implicit function theorem as in the proof of Proposition 2.3, one obtains families of solutions made up of a combination of point vortices and small vortex patches. The same can be done in Propositions 3.1, 3.2, and 4.2 below.

3. ASYMMETRIC PAIRS

In this section we will give a precise statement of Theorem 1.5, which describes the structure of the set of vortex patch solutions in a neighborhood of the point vortex pairs. We shall only treat the case $\alpha \in (0, 1)$, since the limiting case $\alpha = 0$ was addressed in [29].

In view of (1.4), the interaction of two point vortices located initially at $z_1(0) = d_1 > 0$ and $z_2(0) = d_2 e^{i\pi}$ with circulations $\gamma_1 \in \mathbb{R} \setminus \{0\}$ and $\gamma_2 \in \mathbb{R} \setminus \{0\}$ is described by the system

$$\begin{aligned} \mathcal{P}_1^\alpha(\lambda) &= d_1 \Omega + U - \frac{\alpha C_\alpha}{2} \frac{\gamma_2 (d_1 + d_2)}{|d_1 + d_2|^{\alpha+2}} = 0, \\ \mathcal{P}_2^\alpha(\lambda) &= d_2 \Omega - U - \frac{\alpha C_\alpha}{2} \frac{\gamma_1 (d_1 + d_2)}{|d_1 + d_2|^{\alpha+2}} = 0. \end{aligned} \quad (3.1)$$

Co-rotating pairs. Setting $U = 0$ and solving the system (3.1) for Ω and d_2 , we obtain

$$d_2 = d_2^* := \frac{\gamma_1}{\gamma_2} d_1 \quad \text{and} \quad \Omega = \Omega^* := \frac{\alpha C_\alpha}{2} \frac{\gamma_1 + \gamma_2}{|d_1 + d_2^*|^{\alpha+2}}. \quad (3.2)$$

The differential of the mapping $\mathcal{P}^\alpha := (\mathcal{P}_1^\alpha, \mathcal{P}_2^\alpha)$ with respect to $\lambda = (\Omega, d_2)$ at the point $\lambda^* = (\Omega^*, d_2^*)$ is

$$D_\lambda \mathcal{P}^\alpha(\lambda^*) \begin{pmatrix} \dot{\Omega} \\ \dot{d}_2 \end{pmatrix} = \begin{pmatrix} d_1 & \frac{\alpha C_\alpha (\alpha + 1) \gamma_2 |\gamma_2|^{\alpha+2}}{2 |\gamma_1 + \gamma_2|^{\alpha+2} d_1^{\alpha+2}} \\ \frac{\gamma_1}{\gamma_2} d_1 & \frac{\alpha C_\alpha (\gamma_2 + (\alpha + 2) \gamma_1) |\gamma_2|^{\alpha+2}}{2 |\gamma_1 + \gamma_2|^{\alpha+2} d_1^{\alpha+2}} \end{pmatrix} \begin{pmatrix} \dot{\Omega} \\ \dot{d}_2 \end{pmatrix}, \quad (3.3)$$

and the Jacobian determinant is non-zero if $\gamma_1 + \gamma_2 \neq 0$.

Counter-rotating pairs. Setting $\Omega = 0$ and solving the system (3.1), for U and γ_2 , we get

$$\gamma_2 = -\gamma_1 \quad \text{and} \quad U = U^* := \frac{\alpha C_\alpha}{2} \frac{\gamma_1}{|d_1 + d_2|^{\alpha+1}}. \quad (3.4)$$

The differential of the mapping $\mathcal{P}^\alpha := (\mathcal{P}_1^\alpha, \mathcal{P}_2^\alpha)$ with respect to $\lambda = (U, \gamma_2)$ at the point $\lambda^* = (U^*, -\gamma_1)$ is

$$D_\lambda \mathcal{P}^\alpha(\lambda^*) \begin{pmatrix} \dot{U} \\ \dot{\gamma}_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{\alpha C_\alpha}{2} \frac{d_1 + d_2}{|d_1 + d_2|^{\alpha+2}} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dot{U} \\ \dot{\gamma}_2 \end{pmatrix}, \quad (3.5)$$

and the Jacobian determinant is non-zero if $d_1 + d_2 \neq 0$.

Let $\mathcal{O}_j^\varepsilon$, $j = 1, 2$ be two bounded domains containing the origin and contained in the ball $B(0, 2)$. Given $b_1, b_2, d_1 \in (0, \infty)$, $d_2 \in \mathbb{R}$ and $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 \ll 1$, we define the domains

$$\begin{aligned}\mathcal{D}_1^\varepsilon &:= \varepsilon b_1 \mathcal{O}_1^\varepsilon + d_1, \\ \mathcal{D}_2^\varepsilon &:= -(\varepsilon b_2 \mathcal{O}_2^\varepsilon + d_2).\end{aligned}$$

Consider the initial vorticity

$$\omega_0^\varepsilon = \frac{\gamma_1}{\varepsilon^2 b_1^2} \chi_{\mathcal{D}_1^\varepsilon} + \frac{\gamma_2}{\varepsilon^2 b_2^2} \chi_{\mathcal{D}_2^\varepsilon}.$$

Following the study done in Section 2.1.2, more precisely (2.23), the conformal parametrizations $\phi_j = \text{Id} + \varepsilon |\varepsilon|^\alpha b_j^{1+\alpha} f_j: \mathbb{T} \rightarrow \partial \mathcal{O}_j^\varepsilon$ are subject to the system

$$\begin{aligned}\mathcal{G}_j^\alpha(\varepsilon, f; \lambda) &:= \text{Im} \left\{ \left(\Omega(\varepsilon b_j w + \varepsilon^2 |\varepsilon|^\alpha b_j^{2+\alpha} f_j(w) + d_j) + (-1)^{3-j} U + \gamma_j \mathcal{I}^\alpha[\varepsilon, f_j](w) \right. \right. \\ &\quad \left. \left. + \gamma_{3-j} \mathcal{J}_{3-j}^\alpha[\varepsilon, f_{3-j}, f_j; \lambda](w) \right) \overline{w} (1 + \varepsilon |\varepsilon|^\alpha b_1^{1+\alpha} \overline{f_j'(w)}) - \mu_\alpha \gamma_j \overline{f_j'(w)} \right\} = 0\end{aligned}\quad (3.6)$$

for all $w \in \mathbb{T}$, $j = 1, 2$, where μ_α is defined in (2.21) and

$$\begin{aligned}\mathcal{I}^\alpha[\varepsilon, f_j](w) &:= -C_\alpha \int_{\mathbb{T}} \frac{f_j'(\tau)}{|\phi_j(\tau) - \phi_j(w)|^\alpha} d\tau \\ &\quad + \alpha C_\alpha \int_{\mathbb{T}} \int_0^1 \frac{\text{Re} [(f_j(\tau) - f_j(w))(\overline{\tau} - \overline{w})] + \varepsilon |\varepsilon|^\alpha t |f_j(\tau) - f_j(w)|^2}{|w + \tau + t\varepsilon |\varepsilon|^\alpha (f_j(\tau) + f_j(w))|^{2+\alpha}} dt d\tau,\end{aligned}\quad (3.7)$$

$$\begin{aligned}\mathcal{J}_k^\alpha[\varepsilon, f_k, f_j](w) &:= -\frac{\alpha C_\alpha}{b_k} \int_0^1 \int_{\mathbb{T}} \frac{(d_1 + d_2) \text{Re} [b_k \overline{\phi_k(\tau)} + b_j \overline{\phi_j(w)}]}{|\varepsilon b_k t \phi_k(\tau) + \varepsilon b_j t \phi_j(w) + d_1 + d_2|^{\alpha+2}} \phi_k'(\tau) dt d\tau \\ &\quad - \frac{\alpha C_\alpha}{b_k} \int_0^1 \int_{\mathbb{T}} \frac{t\varepsilon |b_k \phi_k(\tau) + b_j \phi_j(w)|^2}{|\varepsilon b_k t \phi_k(\tau) + \varepsilon b_j t \phi_j(w) + d_1 + d_2|^{\alpha+2}} \phi_k'(\tau) dt d\tau.\end{aligned}\quad (3.8)$$

Consider the Banach spaces

$$\begin{aligned}\mathcal{V}_1^\alpha &:= V_1^\alpha \times V_1^\alpha & \text{and} & & \mathcal{W}_1^\alpha &:= W_1^\alpha \times W_1^\alpha, \\ \text{with } V_1^\alpha &:= \{f \in V_0^\alpha : \overline{f(w)} = f(\overline{w})\} & \text{and} & & W_1^\alpha &:= \{g \in W_0^\alpha : \overline{g(w)} = -g(\overline{w})\},\end{aligned}\quad (3.9)$$

where V_0^α and W_0^α were defined in (2.26). Note that the reflection symmetry $\overline{f(w)} = f(\overline{w})$ implies that the Fourier coefficients of $f \in V_1^\alpha$ are real and the domain associated to the conformal mapping $\phi = \text{Id} + \varepsilon |\varepsilon|^\alpha f$ is symmetric with respect to the real axis. Denote by B_1^α the open unit ball in \mathcal{V}_1^α .

3.1. Co-rotating pairs. In this subsection we assume that $U = 0$ and set $\lambda = (\Omega, d_2)$ and

$$\mathcal{G}^\alpha(\varepsilon, f; \lambda) := (\mathcal{G}_1^\alpha(\varepsilon, f; \lambda), \mathcal{G}_2^\alpha(\varepsilon, f; \lambda)),\quad (3.10)$$

where $\mathcal{G}_1^\alpha, \mathcal{G}_2^\alpha$ are defined in (3.6). The following proposition describes the set of solutions of the equation $\mathcal{G}^\alpha(\varepsilon, f; \lambda) = 0$ around the point $(0, 0, \lambda^*)$, where $\lambda^* = (\Omega^*, d_2^*)$ is given by (3.2), as a one-parameter smooth curve.

Proposition 3.1. *Let $\alpha \in (0, 1)$, $\gamma_1, \gamma_2 \in \mathbb{R} \setminus \{0\}$, such that $\gamma_1 + \gamma_2 \neq 0$ and $d_1 \in (0, \infty)$. The following assertions hold true.*

- (i) *There exists $\varepsilon_0 > 0$ and a neighborhood Λ of λ^* in \mathbb{R}^2 such that \mathcal{G}^α can be extended to a C^1 mapping $(-\varepsilon_0, \varepsilon_0) \times B_1^\alpha \times \Lambda \rightarrow \mathcal{W}_1^\alpha$.*
- (ii) *$\mathcal{G}^\alpha(0, 0; \lambda^*) = 0$.*
- (iii) *The linear operator $D_{(f, \lambda)} \mathcal{G}^\alpha(0, 0; \lambda^*): \mathcal{V}_1^\alpha \times \mathbb{R}^2 \rightarrow \mathcal{W}_1^\alpha$ is an isomorphism.*
- (iv) *There exists $\varepsilon_1 > 0$ and a unique C^1 function $(f, \lambda): (-\varepsilon_1, \varepsilon_1) \rightarrow B_1^\alpha \times \mathbb{R}^2$ such that*

$$\mathcal{G}^\alpha(\varepsilon, f(\varepsilon); \lambda(\varepsilon)) = 0,$$

with $\lambda(\varepsilon) = \lambda^* + o(\varepsilon^2)$ and

$$f_j(\varepsilon, w) = \frac{\varepsilon b_j}{d_1^{\alpha+2}} \frac{\gamma_{3-j}}{\gamma_j} \frac{\Xi_\alpha |\gamma_2|^{\alpha+2}}{|\gamma_1 + \gamma_2|^{\alpha+2}} \bar{w} + o(\varepsilon^2), \quad \Xi_\alpha := \frac{(\alpha+2)\Gamma(1-\frac{\alpha}{2})\Gamma(3-\frac{\alpha}{2})}{4\Gamma(2-\alpha)}.$$

Proof. Note that the regularity of the nonlinear mapping \mathcal{G}^α is ensured by Proposition 2.1. Therefore, it remains only to check the reflection symmetry property: If the Fourier coefficients of f_1, f_2 are real, that is

$$\overline{f_j(w)} = f_j(\bar{w}), \quad (3.11)$$

then

$$\mathcal{G}_j^\alpha(\varepsilon, f; \lambda)(\bar{w}) = -\mathcal{G}_j^\alpha(\varepsilon, f)(w). \quad (3.12)$$

It is obvious that if f_1, f_2 satisfy (3.11), then

$$w \mapsto \text{Im} \left\{ \Omega(\varepsilon b_j w + \varepsilon^2 |\varepsilon|^{\alpha} b_j^{2+\alpha} f_j(w) + d_j) \bar{w} (1 + \varepsilon |\varepsilon|^{\alpha} b_1^{1+\alpha} \overline{f_j'(w)}) - \mu_\alpha \gamma_j \overline{f_j'(w)} \right\}$$

satisfies (3.12). Thus, it is sufficient to check that the Fourier coefficients of $\mathcal{I}^\alpha[\varepsilon, f_j](w)$ and $\mathcal{J}_k^\alpha[\varepsilon, f_k, f_j](w)$ are also real for every f satisfying (3.11), namely,

$$\overline{\mathcal{I}^\alpha[\varepsilon, f_j](w)} = \mathcal{I}^\alpha[\varepsilon, f_j](\bar{w}), \quad \text{and} \quad \overline{\mathcal{J}_k^\alpha[\varepsilon, f_k, f_j](w)} = \mathcal{J}_k^\alpha[\varepsilon, f_k, f_j; \lambda](\bar{w}).$$

According to (3.7) one has

$$\begin{aligned} \overline{\mathcal{I}^\alpha[\varepsilon, f_j](w)} &= -C_\alpha \int_{\mathbb{T}} \frac{\overline{f_j'(\bar{\tau})}}{|\bar{\tau} - w + \varepsilon |\varepsilon|^{\alpha} b_j^{1+\alpha} (f_j(\bar{\tau}) - f_j(w))|^\alpha} d\tau \\ &\quad + \alpha C_\alpha \int_{\mathbb{T}} \int_0^1 \frac{\text{Re} \left[(f_j(\bar{\tau}) - f_j(w))(\tau - \bar{w}) \right] + \varepsilon |\varepsilon|^{\alpha} t |f_j(\bar{\tau}) - f_j(w)|^2}{|w - \bar{\tau} + t \varepsilon |\varepsilon|^{\alpha} b_j^{1+\alpha} (f_j(\bar{\tau}) - f_j(w))|^{2+\alpha}} dt d\tau. \end{aligned}$$

Then, thanks to (3.11) we conclude that

$$\begin{aligned} \overline{\mathcal{I}^\alpha[\varepsilon, f_j](w)} &= -C_\alpha \int_{\mathbb{T}} \frac{f_j'(\tau)}{|\tau - \bar{w} + \varepsilon |\varepsilon|^{\alpha} b_j^{1+\alpha} (f_j(\tau) - f_j(\bar{w}))|^\alpha} d\tau \\ &\quad + \alpha C_\alpha \int_{\mathbb{T}} \int_0^1 \frac{\text{Re} \left[(f_j(\tau) - f_j(\bar{w}))(\bar{\tau} - w) \right] + \varepsilon |\varepsilon|^{\alpha} t |f_j(\tau) - f_j(\bar{w})|^2}{|\bar{w} - \tau + t \varepsilon |\varepsilon|^{\alpha} b_j^{1+\alpha} (f_j(\tau) - f_j(\bar{w}))|^{2+\alpha}} dt d\tau \\ &= \mathcal{I}^\alpha[\varepsilon, f_j](\bar{w}). \end{aligned}$$

As for $\mathcal{J}_k^\alpha[\varepsilon, f_k, f_j; \lambda](w)$ we write, by (3.8),

$$\begin{aligned} \overline{\mathcal{J}_k^\alpha[\varepsilon, f_k, f_j; \lambda](w)} &= \\ &\quad - \frac{\alpha C_\alpha}{b_k} \int_0^1 \int_{\mathbb{T}} \frac{(d_1 + d_2) \text{Re} \left[b_k \overline{\phi_k(\bar{\tau})} + b_j \overline{\phi_j(w)} \right] + t \varepsilon |b_k \phi_k(\bar{\tau}) + b_j \phi_j(w)|^2}{|\varepsilon b_k t \phi_k(\bar{\tau}) + \varepsilon b_j t \phi_j(w) + d_1 + d_2|^{\alpha+2}} \overline{\phi_k'(\bar{\tau})} dt d\tau. \end{aligned}$$

By virtue of (3.11) we find

$$\begin{aligned} \overline{\mathcal{J}_k^\alpha[\varepsilon, f_k, f_j; \lambda](w)} &= \\ &\quad - \frac{\alpha C_\alpha}{b_k} \int_0^1 \int_{\mathbb{T}} \frac{(d_1 + d_2) \text{Re} \left[b_k \overline{\phi_k(\tau)} + b_j \overline{\phi_j(\bar{w})} \right] + t \varepsilon |b_k \phi_k(\tau) - b_j \phi_j(\bar{w})|^2}{|\varepsilon b_k t \phi_k(\tau) + \varepsilon b_j t \phi_j(\bar{w}) + d_1 + d_2|^{\alpha+2}} \phi_k'(\tau) dt d\tau. \end{aligned}$$

This achieves the proof of (i).

Next, from Proposition 2.1(ii) one has

$$\mathcal{G}_j^\alpha(0, 0; \lambda)(w) = \text{Im} \left\{ \mathcal{P}_j^\alpha(\lambda) \bar{w} \right\}, \quad (3.13)$$

where $\mathcal{P}_j^\alpha(\lambda)$ is given by (3.1). Thus, by (3.2) we get (ii).

Now we shall compute the differential of \mathcal{G}^α . In view of Proposition 2.1(iii), (3.13) and using the fact that $\mathcal{P}_j^\alpha(\lambda)$ is real, we get

$$D_{(f;\lambda)}\mathcal{G}^\alpha(0,0,\lambda^*) \begin{pmatrix} h \\ \dot{\lambda} \end{pmatrix} (w) = -D_\lambda \mathcal{P}^\alpha(\lambda^*) \begin{pmatrix} \dot{\Omega} \\ \dot{d}_2 \end{pmatrix} \text{Im}\{w\} + \sum_{n \geq 1} M_n^\alpha \begin{pmatrix} \gamma_1 a_n^0 \\ \gamma_2 a_n^1 \end{pmatrix} \text{Im}\{w^{n+1}\} \quad (3.14)$$

for all $h \in \mathcal{V}_1^\alpha$ and $\dot{\lambda} = (\dot{\Omega}, \dot{d}_2) \in \mathbb{R}^2$, where M_n^α given by (2.27). From Proposition 2.1(iv) and (3.3) we infer that $D_{(f;\lambda)}\mathcal{G}^\alpha(0,0;\lambda^*): \mathcal{V}_1^\alpha \times \mathbb{R}^2 \rightarrow \mathcal{W}_1^\alpha$ is an isomorphism if and only if $\gamma_1 + \gamma_2 \neq 0$.

Using (i), (ii), (iii) and the implicit function theorem we conclude the existence of $\varepsilon_1 > 0$ and a unique C^1 parametrization $(f;\lambda): (-\varepsilon_1, \varepsilon_1) \rightarrow B_1^\alpha \times \mathbb{R}^2$ satisfying

$$\mathcal{G}^\alpha(\varepsilon, f(\varepsilon); \lambda(\varepsilon)) = 0 \quad \text{and} \quad (f(0); \lambda(0)) = (0, \lambda^*). \quad (3.15)$$

Differentiating with (3.15) with respect to ε using the chain rule, we get

$$\partial_\varepsilon(f(\varepsilon); \lambda(\varepsilon))|_{\varepsilon=0} = -D_{(f;\lambda)}\mathcal{G}^\alpha(0,0;\lambda^*)^{-1} \partial_\varepsilon \mathcal{G}^\alpha(0,0;\lambda^*). \quad (3.16)$$

We shall write down the explicit expression of $D_{(f;\lambda)}\mathcal{G}^\alpha(0,0;\lambda^*)^{-1}$. In view of (3.14), for any $H \in \mathcal{W}_1^\alpha$ with the expansion

$$H(w) = \sum_{n \geq 0} \begin{pmatrix} A_n^1 \\ A_n^2 \end{pmatrix} \text{Im}\{w^{n+1}\},$$

one has

$$D_{(f;\lambda)}\mathcal{G}^\alpha(0,0;\lambda^*)^{-1} H = \left(\sum_{n \geq 1} \frac{A_n^1}{M_n^\alpha \gamma_1} \bar{w}^n, \sum_{n \geq 1} \frac{A_n^2}{M_n^\alpha \gamma_2} \bar{w}^n, \frac{A_0^1}{d_1} - (\alpha + 1)\gamma_2 \frac{A_0^2 \gamma_2 - A_0^1 \gamma_1}{d_1(\gamma_1 + \gamma_2)}, \frac{A_0^2 \gamma_2 - A_0^1 \gamma_1}{\Omega^*} \right). \quad (3.17)$$

On the other hand, by (3.6) and (3.7), we have

$$\mathcal{G}_j^\alpha(\varepsilon, 0; \lambda)(w) = \text{Im} \left\{ \left(\Omega d_j + \gamma_{3-j} \mathcal{J}_{3-j}^\alpha[\varepsilon, 0, 0; \lambda](w) \right) \bar{w} \right\}.$$

From (3.8) and using formula (2.19) we get

$$\begin{aligned} \mathcal{J}_{j-3}^\alpha[\varepsilon, 0, 0; \lambda](w) &= -\frac{\alpha C_\alpha}{b_{j-3}} \int_0^1 \int_{\mathbb{T}} \frac{(d_1 + d_2) \text{Re}[b_{j-3} \bar{\tau} + b_j \bar{w}] + t\varepsilon |b_{j-3} \tau + b_j w|^2}{|\varepsilon b_{j-3} t \tau + \varepsilon b_j t w + d_1 + d_2|^{\alpha+2}} dt d\tau \\ &= -\frac{\alpha C_\alpha}{2|d_1 + d_2|^{\alpha+2}} \left(d_1 + d_2 - \varepsilon b_j w + \frac{(\alpha + 2)}{2} \varepsilon b_j (w + \bar{w}) \right) + O(\varepsilon^2). \end{aligned} \quad (3.18)$$

It follows that

$$\mathcal{G}_j^\alpha(\varepsilon, 0; \lambda)(w) = \left(\Omega d_j - \gamma_{3-j} \frac{\alpha C_\alpha (d_1 + d_2)}{2|d_1 + d_2|^{\alpha+2}} \right) \text{Im}\{\bar{w}\} + \varepsilon b_j \gamma_{3-j} \frac{\alpha(\alpha + 2) C_\alpha}{4|d_1 + d_2|^{\alpha+2}} \text{Im}\{\bar{w}^2\} + O(\varepsilon^2),$$

and hence in particular that

$$\partial_\varepsilon \mathcal{G}_j^\alpha(0, 0; \lambda^*)(w) = -\frac{\gamma_{3-j}(\alpha + 2)\alpha C_\alpha}{4|d_1 + d_2^*|^{\alpha+2}} b_j \text{Im}\{w^2\}. \quad (3.19)$$

Combining the identities (3.16), (3.17) and (3.19) we obtain

$$\partial_\varepsilon(f(\varepsilon); \lambda(\varepsilon))|_{\varepsilon=0} = \left(0, \frac{\gamma_2}{\gamma_1} \frac{\alpha C_\alpha}{M_1^\alpha} \frac{(\alpha + 2)}{4|d_1 + d_2^*|^{\alpha+2}} b_1 \bar{w}, \frac{\gamma_1}{\gamma_2} \frac{\alpha C_\alpha}{M_1^\alpha} \frac{(\alpha + 2)}{4|d_1 + d_2^*|^{\alpha+2}} b_2 \bar{w} \right).$$

Finally, straightforward computations yield

$$\frac{\alpha C_\alpha}{M_1^\alpha} = \frac{\Gamma(1 - \frac{\alpha}{2})\Gamma(3 - \frac{\alpha}{2})}{\Gamma(2 - \alpha)}, \quad (3.20)$$

completing the proof of (iv). \square

3.2. Counter-rotating pairs. Similar to the co-rotating case, we define the nonlinear operator

$$\mathcal{G}^\alpha(\varepsilon, f; \lambda) := (\mathcal{G}_1^\alpha(\varepsilon, f; \lambda), \mathcal{G}_2^\alpha(\varepsilon, f; \lambda)), \quad (3.21)$$

where $\mathcal{G}_1^\alpha, \mathcal{G}_2^\alpha$ are defined by (3.6), $\lambda = (U, \gamma_2)$, $f = (f_1, f_2)$ and $\Omega = 0$. We seek the solutions of the equation $\mathcal{G}^\alpha(\varepsilon, f; \lambda) = 0$ near $(0, 0; \lambda^*)$, where $\lambda^* = (U^*, -\gamma_1)$ is given by (3.4).

Proposition 3.2. *Let $\alpha \in (0, 1)$, $d_1 \in (0, \infty)$ and $\gamma_1, d_2 \in \mathbb{R} \setminus \{0\}$, such that $d_1 + d_2 \neq 0$. The following assertions hold true.*

- (i) *There exists $\varepsilon_0 > 0$ and a neighborhood Λ of λ^* in \mathbb{R}^2 such that \mathcal{G}^α can be extended to a C^1 mapping $(-\varepsilon_0, \varepsilon_0) \times B_1^\alpha \times \Lambda \rightarrow \mathcal{W}_1^\alpha$.*
- (ii) *$\mathcal{G}^\alpha(0, 0, \lambda^*) = 0$.*
- (iii) *The linear operator $D_{(f; \lambda)} \mathcal{G}^\alpha(0, 0, \lambda^*): \mathcal{V}_1^\alpha \times \mathbb{R}^2 \rightarrow \mathcal{W}_1^\alpha$ is an isomorphism.*
- (iv) *There exists $\varepsilon_1 > 0$ and a unique C^1 function $(f, \lambda): (-\varepsilon_1, \varepsilon_1) \rightarrow B_1^\alpha \times \mathbb{R}^2$ such that*

$$\mathcal{G}^\alpha(\varepsilon, f(\varepsilon); \lambda(\varepsilon)) = 0, \quad (3.22)$$

with $\lambda(\varepsilon) = \lambda^* + o(\varepsilon^2)$ and

$$f_j(\varepsilon, w) = -\frac{\varepsilon b_j \Xi_\alpha}{|d_1 + d_2|^{\alpha+2}} \bar{w} + o(\varepsilon^2), \quad \Xi_\alpha := \frac{(\alpha + 2)\Gamma(1 - \frac{\alpha}{2})\Gamma(3 - \frac{\alpha}{2})}{4\Gamma(2 - \alpha)}.$$

Proof. The proof of the proposition follows the same lines of the of Proposition 3.1. Therefore, we shall only compute the expansion of the solution. Combining Proposition 2.1 and (3.5) we get, for all $h \in \mathcal{V}_1^\alpha$ and $\dot{\lambda} = (\dot{U}, \dot{\gamma}_2) \in \mathbb{R}^2$,

$$D_{(f; \lambda)} \mathcal{G}^\alpha(0, 0, \lambda^*) \begin{pmatrix} h \\ \dot{\lambda} \end{pmatrix} (w) = \begin{pmatrix} 1 & \frac{\alpha C_\alpha}{2} \frac{d_1 + d_2}{|d_1 + d_2|^{\alpha+2}} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \dot{U} \\ \dot{\gamma}_2 \end{pmatrix} \text{Im}\{\bar{w}\} + \sum_{n \geq 1} M_n^\alpha \begin{pmatrix} \gamma_1 a_n^0 \\ \gamma_2 a_n^1 \end{pmatrix} \text{Im}\{w^{n+1}\},$$

where M_n^α is given by (2.27). Then, for any $H \in \mathcal{W}_1^\alpha$ with the expansion

$$H(w) = \sum_{n \geq 0} \begin{pmatrix} A_n^1 \\ A_n^2 \end{pmatrix} \text{Im}\{w^{n+1}\},$$

one has

$$D_{(f; \lambda)} \mathcal{G}^\alpha(0, 0, \lambda^*)^{-1} H(w) = \left(\sum_{n \geq 1} \frac{A_n^1}{M_n^\alpha \gamma_1} \bar{w}^n, \sum_{n \geq 1} \frac{A_n^2}{M_n^\alpha \gamma_2} \bar{w}^n; -A_0^2, \frac{2|d_1 + d_2|^{\alpha+2}}{\alpha C_\alpha (d_1 + d_2)} (A_0^1 + A_0^2) \right). \quad (3.23)$$

On the other hand, by (3.6), (3.7) and (3.18), we have

$$\mathcal{G}_j^\alpha(\varepsilon, 0; \lambda)(w) = \left((-1)^{3-j} U - \gamma_{3-j} \frac{\alpha C_\alpha (d_1 + d_2)}{2|d_1 + d_2|^{\alpha+2}} \right) \text{Im}\{\bar{w}\} + \varepsilon b_j \frac{\gamma_{3-j} \alpha (\alpha + 2) C_\alpha}{4|d_1 + d_2|^{\alpha+2}} \text{Im}\{\bar{w}^2\} + O(\varepsilon^2),$$

and hence in particular that

$$\partial_\varepsilon \mathcal{G}_j^\alpha(0, 0; \lambda)(w) = -\frac{\gamma_{3-j} (\alpha + 2) \alpha C_\alpha}{4|d_1 + d_2|^{\alpha+2}} b_j \text{Im}\{w^2\}. \quad (3.24)$$

Differentiating (3.22) using the chain rule, we deduce

$$\partial_\varepsilon (f(\varepsilon); \lambda(\varepsilon))|_{\varepsilon=0} = -D_{(f; \lambda)} \mathcal{G}^\alpha(0, 0; \lambda^*)^{-1} \partial_\varepsilon \mathcal{G}^\alpha(0, 0; \lambda^*).$$

This combined with (3.23) and (3.24) yields

$$\partial_\varepsilon (f(\varepsilon); \lambda(\varepsilon))|_{\varepsilon=0} = -\left(\frac{\alpha C_\alpha}{M_1^\alpha} \frac{(\alpha + 2)}{4|d_1 + d_2|^{\alpha+2}} b_1 \bar{w}, \frac{\alpha C_\alpha}{M_1^\alpha} \frac{(\alpha + 2)}{4|d_1 + d_2|^{\alpha+2}} b_2 \bar{w}; 0, 0 \right).$$

Finally, using the identity (3.20) we get the desired result. \square

4. NESTED POLYGONS

In this section we shall construct $2m + 1$ multipolar vortex equilibria in which a central patch is surrounded by $2m$ satellite patches centered at the vertices of two nested regular m -gons. The vertices of the polygons are either radially aligned with each other or out of phase by an angle π/m , and the patches on each polygon are identical with the same strength; see Figure 2. More precisely, we shall desingularize the following system of point vortices

$$\omega_0^0(z) = \pi \left(\gamma_0 \delta_{z_0(0)}(z) + \gamma_1 \sum_{k=0}^{m-1} \delta_{z_{1k}(0)}(z) + \gamma_2 \sum_{k=0}^{m-1} \delta_{z_{2k}(0)}(z) \right) \quad (4.1)$$

with

$$z_0(0) = 0 \quad \text{and} \quad z_{jk}(0) := \begin{cases} d_1 e^{\frac{2k\pi i}{m}} & \text{if } j = 1 \text{ and } 0 \leq k \leq m-1, \\ d_2 e^{\frac{(2k+\vartheta)\pi i}{m}} & \text{if } j = 2 \text{ and } 0 \leq k \leq m-1, \end{cases} \quad (4.2)$$

where $\gamma_0, \gamma_1, \gamma_2 \in \mathbb{R} \setminus \{0\}$, $d_2 > d_1 > 0$ and $\vartheta = 0$ corresponds to the aligned configuration while $\vartheta = 1$ refers to the staggered configuration. Assuming that $z_0(t) = z_0(0)$ and $z_{jk}(t) = e^{i\Omega t} z_{jk}(0)$, one may easily check that the system of $2m + 1$ equations in (1.4) can be reduced to

$$\gamma_j \mathcal{P}_j^\alpha(\lambda) = 0, \quad j = 0, 1, 2, \quad (4.3)$$

where $\lambda = (\Omega, \gamma_2)$ and

$$\begin{aligned} \mathcal{P}_0^\alpha(\lambda) &:= \frac{\alpha C_\alpha}{2} \left(\frac{\gamma_1}{d_1^{1+\alpha}} + \frac{\gamma_2}{d_2^{1+\alpha}} e^{\frac{\vartheta\pi i}{m}} \right) \sum_{k=0}^{m-1} e^{\frac{2k\pi i}{m}}, \\ \mathcal{P}_1^\alpha(\lambda) &:= \Omega d_1 - \frac{\alpha C_\alpha}{2d_1^{1+\alpha}} \left(\gamma_0 + \gamma_1 \sum_{k=1}^{m-1} \frac{1 - e^{\frac{2k\pi i}{m}}}{|1 - e^{\frac{2k\pi i}{m}}|^{2+\alpha}} + \gamma_2 \sum_{k=0}^{m-1} \frac{1 - e^{\frac{(2k+\vartheta)\pi i}{m}} d}{|1 - e^{\frac{(2k+\vartheta)\pi i}{m}} d|^{2+\alpha}} \right), \\ \mathcal{P}_2^\alpha(\lambda) &:= \Omega d_2 - \frac{\alpha C_\alpha}{2d_2^{1+\alpha}} \left(\gamma_0 + \gamma_1 \sum_{k=0}^{m-1} \frac{1 - e^{\frac{(2k-\vartheta)\pi i}{m}} d^{-1}}{|1 - e^{\frac{(2k-\vartheta)\pi i}{m}} d^{-1}|^{2+\alpha}} + \gamma_2 \sum_{k=1}^{m-1} \frac{1 - e^{\frac{2k\pi i}{m}}}{|1 - e^{\frac{2k\pi i}{m}}|^{2+\alpha}} \right), \end{aligned} \quad (4.4)$$

with $d := \frac{d_2}{d_1}$. By symmetry arguments, one may easily check that

$$\begin{aligned} \sum_{k=0}^{m-1} e^{\frac{2k\pi i}{m}} &= 0, & \sum_{k=1}^{m-1} \frac{1 - e^{\frac{2k\pi i}{m}}}{|1 - e^{\frac{2k\pi i}{m}}|^{2+\alpha}} &= \frac{1}{2} \sum_{k=1}^{m-1} \left(2 \sin\left(\frac{k\pi}{m}\right) \right)^{-\alpha} =: \frac{S_\alpha}{2}, \\ \sum_{k=0}^{m-1} \frac{1 - d^{\pm 1} e^{\frac{(2k\pm\vartheta)\pi i}{m}}}{|1 - d^{\pm 1} e^{\frac{(2k\pm\vartheta)\pi i}{m}}|^{\alpha+2}} &= \sum_{k=0}^{m-1} \frac{1 - d^{\pm 1} \cos\left(\frac{(2k\pm\vartheta)\pi}{m}\right)}{\left(1 + (d^{\pm 1})^2 - 2d^{\pm 1} \cos\left(\frac{(2k\pm\vartheta)\pi}{m}\right)\right)^{\frac{\alpha}{2}+1}} =: T_\alpha^\pm(d, \vartheta). \end{aligned}$$

Thus, the identities in (4.4) become

$$\begin{aligned} \mathcal{P}_0^\alpha(\lambda) &= 0, \\ \mathcal{P}_1^\alpha(\lambda) &= \Omega d_1 - \frac{\alpha C_\alpha}{2d_1^{\alpha+1}} \left[\gamma_0 + \frac{\gamma_1}{2} S_\alpha + \gamma_2 T_\alpha^+(d, \vartheta) \right], \\ \mathcal{P}_2^\alpha(\lambda) &= \Omega d_2 - \frac{\alpha C_\alpha}{2d_2^{\alpha+1}} \left[\gamma_0 + \gamma_1 T_\alpha^-(d, \vartheta) + \frac{\gamma_2}{2} S_\alpha \right]. \end{aligned} \quad (4.5)$$

Moreover, the differential of the mapping $\mathcal{P}^\alpha := (\mathcal{P}_1^\alpha, \mathcal{P}_2^\alpha)$ with respect to $\lambda = (\Omega, \gamma_2)$ is given by

$$D_\lambda \mathcal{P}^\alpha(\lambda) \begin{pmatrix} \dot{\Omega} \\ \dot{\gamma}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ d_1 & -\frac{\alpha C_\alpha}{2} \frac{T_\alpha^+(d, \vartheta)}{d_1^{1+\alpha}} \\ d_2 & -\frac{\alpha C_\alpha}{2} \frac{S_\alpha}{2d_2^{1+\alpha}} \end{pmatrix} \begin{pmatrix} \dot{\Omega} \\ \dot{\gamma}_2 \end{pmatrix}. \quad (4.6)$$

If the Jacobian determinant is non-trivial,

$$\det(D_\lambda \mathcal{P}^\alpha(\lambda)) = -\frac{\alpha C_\alpha d_1}{2d_2^{\alpha+1}} \left(\frac{S_\alpha}{2} - d^{\alpha+2} T_\alpha^+(d, \vartheta) \right) \neq 0, \quad (4.7)$$

then the system (4.5) has a unique solution $\lambda^* = (\Omega^*, \gamma_2^*)$ given by

$$\begin{aligned} \gamma_2^* &:= \frac{(d^{\alpha+2} - 1)\gamma_0 + \left(\frac{1}{2}S_\alpha d^{\alpha+2} - T_\alpha^-(d, \vartheta)\right)\gamma_1}{\frac{1}{2}S_\alpha - T_\alpha^+(d, \vartheta)d^{\alpha+2}}, \\ \Omega^* &:= \frac{\alpha C_\alpha}{2(d_1^{\alpha+2} + d_2^{\alpha+2})} \left(\gamma_0 + \gamma_1 \left(\frac{1}{2}S_\alpha + T_\alpha^-(d, \vartheta)\right) + \gamma_2^* \left(T_\alpha^+(d, \vartheta) + \frac{1}{2}S_\alpha\right) \right). \end{aligned} \quad (4.8)$$

In order to ensure that γ_2 is non-vanishing, one has to assume that γ_0 and γ_1 verify the condition

$$(d^{\alpha+2} - 1)\gamma_0 + \left(\frac{1}{2}S_\alpha d^{\alpha+2} - T_\alpha^-(d, \vartheta)\right)\gamma_1 \neq 0. \quad (4.9)$$

Remark 4.1. For the Eulerean interaction $\alpha = 0$, one may easily check that

$$S_\alpha = m - 1 \quad \text{and} \quad T_\alpha^\pm(d, \vartheta) = \frac{m}{1 - (-1)^\vartheta d^{\pm m}}.$$

It follows that (4.7) and (4.9) can be written as

$$\frac{m-1}{2} + \frac{md^2}{1 - (-1)^\vartheta d^m} \neq 0, \quad (d^2 - 1)\gamma_0 + \left(\frac{m-1}{2}d^2 - \frac{md^m}{d^m - (-1)^\vartheta}\right)\gamma_1 \neq 0.$$

This amounts to the study of the roots of two polynomials of order m . A detailed analysis is given in [10, Pages 10–13] and [2, Pages 18–22].

4.1. Boundary equations. Let m be a positive integer and $\mathcal{O}_0^\varepsilon, \mathcal{O}_1^\varepsilon, \mathcal{O}_2^\varepsilon$ be three bounded simply connected domains containing the origin and contained in the ball $B(0, 2)$. Assume in addition that $\mathcal{O}_0^\varepsilon$ is m -fold symmetric, that is

$$e^{\frac{2\pi i}{m}} \mathcal{O}_0^\varepsilon = \mathcal{O}_0^\varepsilon, \quad (4.10)$$

and $\mathcal{O}_1^\varepsilon, \mathcal{O}_1^\varepsilon$ and $\mathcal{O}_2^\varepsilon$ are symmetric about the real axis. Given $b_0, b_1, b_2 \in \mathbb{R}_+$ and $d_1, d_2 \in \mathbb{R}_+$ and $\varepsilon \in (0, \varepsilon_0)$, with $\varepsilon_0 \ll 1$, we define the domains

$$\mathcal{D}_{00}^\varepsilon := \varepsilon b_0 \mathcal{O}_0^\varepsilon, \quad (4.11)$$

$$\mathcal{D}_{1j}^\varepsilon := e^{\frac{2j\pi i}{m}} (\varepsilon b_1 \mathcal{O}_1^\varepsilon + d_1), \quad j = 0, \dots, m-1, \quad (4.12)$$

$$\mathcal{D}_{2j}^\varepsilon := e^{\frac{(2j+\vartheta)\pi i}{m}} (\varepsilon b_2 \mathcal{O}_2^\varepsilon + d_2), \quad j = 0, \dots, m-1, \quad (4.13)$$

where we recall that $\vartheta = 0$ corresponds to the aligned configuration and $\vartheta = 1$ refers to the staggered configuration. Let $\gamma_0, \gamma_1, \gamma_2 \in \mathbb{R} \setminus \{0\}$ and consider the initial vorticity

$$\omega_0^\varepsilon = \frac{\gamma_0}{\varepsilon^2 b_0^2} \chi_{\mathcal{D}_{00}^\varepsilon} + \frac{\gamma_1}{\varepsilon^2 b_1^2} \sum_{j=0}^{m-1} \chi_{\mathcal{D}_{1j}^\varepsilon} + \frac{\gamma_2}{\varepsilon^2 b_2^2} \sum_{j=0}^{m-1} \chi_{\mathcal{D}_{2j}^\varepsilon}. \quad (4.14)$$

Now assume that the evolution of ω_0^ε is prescribed by the (2.6) with $U = 0$. While this initially gives $N = 2m + 1$ equations, one for each patch, using the fact that

$$\mathcal{D}_{1j}^\varepsilon = e^{\frac{2j\pi i}{m}} \mathcal{D}_{10}^\varepsilon, \quad \text{and} \quad \mathcal{D}_{2j}^\varepsilon = e^{\frac{(2j+\vartheta)\pi i}{m}} \mathcal{D}_{20}^\varepsilon,$$

we shall show that this system can be reduced to a system of three equations, on the boundaries of $\mathcal{O}_0^\varepsilon, \mathcal{O}_1^\varepsilon$, and $\mathcal{O}_2^\varepsilon$.

4.1.1. *Euler equation.* From (2.8) one has

$$\begin{aligned} \operatorname{Re} \{ \gamma_0 (\Omega \bar{z} + V^\varepsilon(z)) z' \} &= 0, \quad \forall z \in \partial \mathcal{D}_{00}^\varepsilon, \\ \operatorname{Re} \{ \gamma_1 (\Omega \bar{z} + V^\varepsilon(z)) z' \} &= 0, \quad \forall z \in \partial \mathcal{D}_{1n}^\varepsilon, \quad n = 0, \dots, m-1, \\ \operatorname{Re} \{ \gamma_2 (\Omega \bar{z} + V^\varepsilon(z)) z' \} &= 0, \quad \forall z \in \partial \mathcal{D}_{2n}^\varepsilon, \quad n = 0, \dots, m-1, \end{aligned} \quad (4.15)$$

where z' denotes a tangent vector to the boundary at the point z and

$$V^\varepsilon(z) = \frac{\gamma_0}{2\varepsilon^2 b_0^2} \int_{\partial \mathcal{D}_{00}^\varepsilon} \frac{\bar{\xi} - \bar{z}}{\xi - z} d\xi + \sum_{\ell=1}^2 \frac{\gamma_\ell}{2\varepsilon^2 b_\ell^2} \sum_{k=0}^{m-1} \int_{\partial \mathcal{D}_{\ell k}^\varepsilon} \frac{\bar{\xi} - \bar{z}}{\xi - z} d\xi.$$

In view of (4.12) and (4.13), the change of variables $\xi \mapsto e^{\frac{2k\pi i}{m}} \xi$ leads to

$$V^\varepsilon(z) = \frac{\gamma_0}{2\varepsilon^2 b_0^2} \int_{\partial \mathcal{D}_{00}^\varepsilon} \frac{\bar{\xi} - \bar{z}}{\xi - z} d\xi + \sum_{\ell=1}^2 \frac{\gamma_\ell}{2\varepsilon^2 b_\ell^2} \sum_{k=0}^{m-1} \int_{\partial \mathcal{D}_{\ell 0}^\varepsilon} \frac{\bar{\xi} - e^{\frac{2k\pi i}{m}} \bar{z}}{e^{\frac{2k\pi i}{m}} \xi - z} d\xi. \quad (4.16)$$

For any $n \in \{1, \dots, m-1\}$ one has

$$V^\varepsilon(e^{\frac{2n\pi i}{m}} z) = \frac{\gamma_0}{2\varepsilon^2 b_0^2} \int_{\partial \mathcal{D}_{00}^\varepsilon} \frac{\bar{\xi} - e^{-\frac{2n\pi i}{m}} \bar{z}}{\xi - e^{\frac{2n\pi i}{m}} z} d\xi + e^{-\frac{2n\pi i}{m}} \sum_{\ell=1}^2 \frac{\gamma_\ell}{2\varepsilon^2 b_\ell^2} \sum_{k=0}^{m-1} \int_{\partial \mathcal{D}_{\ell 0}^\varepsilon} \frac{\bar{\xi} - e^{\frac{2(k-n)\pi i}{m}} \bar{z}}{e^{\frac{2(k-n)\pi i}{m}} \xi - z} d\xi.$$

In view of (4.10), making the change of variables $\xi \mapsto e^{\frac{2n\pi i}{m}} \xi$ in the first integral gives

$$V^\varepsilon(e^{\frac{2n\pi i}{m}} z) = e^{-\frac{2n\pi i}{m}} V^\varepsilon(z). \quad (4.17)$$

From (4.12), (4.13) and (4.17) we conclude that if (4.15) is satisfied for $n = 0$, then it is also satisfied for all $n \in \{1, \dots, m-1\}$. Thus, the system (4.15) is reduced to

$$\operatorname{Re} \{ \gamma_j (\Omega \bar{z} + V^\varepsilon(z)) z' \} = 0, \quad \forall z \in \partial \mathcal{D}_{j0}^\varepsilon, \quad j = 0, 1, 2. \quad (4.18)$$

We assume that the boundaries of the domains $\mathcal{O}_j^\varepsilon$, $j = 0, 1, 2$ in (4.12), (4.13) are parametrized by conformal mappings $\phi_j: \mathbb{T} \rightarrow \partial \mathcal{O}_j^\varepsilon$ satisfying

$$\phi_j(w) = w + \varepsilon b_j f_j(w) \quad \text{with} \quad f_j(w) = \sum_{m=1}^{\infty} \frac{a_m^j}{w^m}, \quad a_m^j \in \mathbb{R}.$$

Following the steps established in Section 2.1.1, more precisely (2.12), we may conclude that the dynamics on the three boundaries is governed by the system

$$\begin{aligned} \gamma_0 \mathcal{G}_0^0(\varepsilon, f; \lambda)(w) &= -\gamma_0 \operatorname{Im} \left\{ \left(\Omega (\varepsilon b_0 \bar{w} + \varepsilon^2 b_0^2 f_0(\bar{w})) + \gamma_0 \bar{\mathcal{T}}^0[\varepsilon, f_0](w) \right. \right. \\ &\quad \left. \left. + \sum_{\ell=1}^2 \gamma_\ell \sum_{k=0}^{m-1} \mathcal{K}_k^0[\varepsilon, f_\ell, f_0](w) \right) w (1 + \varepsilon b_0 f_0'(w)) - \frac{\gamma_0}{2} f_0'(w) \right\} = 0, \end{aligned} \quad (4.19)$$

$$\begin{aligned} \gamma_j \mathcal{G}_j^0(\varepsilon, f; \lambda)(w) &:= -\gamma_j \operatorname{Im} \left\{ \left(\Omega (\varepsilon b_j \bar{w} + \varepsilon^2 b_j^2 f_j(\bar{w}) + d_j) + \gamma_j \bar{\mathcal{T}}^0[\varepsilon, f_j](w) + \gamma_0 \mathcal{K}_0^0[\varepsilon, f_0, f_j](w) \right. \right. \\ &\quad \left. \left. + \sum_{\ell=1}^2 \gamma_\ell \sum_{k=\delta_{\ell j}}^{m-1} \mathcal{K}_k^0[\varepsilon, f_\ell, f_j](w) \right) w (1 + \varepsilon b_j f_j'(w)) - \frac{\gamma_j}{2} f_j'(w) \right\} = 0, \end{aligned} \quad (4.20)$$

for all $w \in \mathbb{T}$ and $j = 1, 2$, where

$$\begin{aligned} \overline{\mathcal{I}}^0[\varepsilon, f_j](w) &= \frac{1}{2} \int_{\mathbb{T}} \frac{\overline{w} - \overline{\tau} + \varepsilon b_j (f_j(\overline{\tau}) - f_j(\overline{w}))}{w - \tau + \varepsilon b_j (f_j(\tau) - f_j(w))} f_j'(\tau) d\tau \\ &\quad + \int_{\mathbb{T}} \frac{i \operatorname{Im} \{ (w - \tau) (f_j(\overline{\tau}) - f_j(\overline{w})) \}}{(w - \tau) (w - \tau + \varepsilon b_j f_j(\tau) - \varepsilon b_j f_j(w))} d\tau, \end{aligned} \quad (4.21)$$

$$\overline{\mathcal{K}}_k^0[\varepsilon, f_\ell, f_n](w) := \frac{1}{2} \int_{\mathbb{T}} \frac{(\overline{\tau} + \varepsilon b_\ell f_\ell(\overline{\tau})) (1 + \varepsilon b_\ell f_\ell'(\tau))}{\nu_{k\ell j} (\varepsilon b_\ell \tau + \varepsilon^2 b_\ell^2 f_\ell(\tau) + d_\ell) - \varepsilon b_n (w + \varepsilon b_n f_n(w)) - d_j} d\tau, \quad (4.22)$$

with the convention $d_0 = 0$ and $\nu_{k\ell j} := \exp(2k\pi i/m + (\delta_{2\ell} - \delta_{2j})\vartheta\pi i/m)$.

4.1.2. *gSQG equations.* From (2.6) and (2.15) one has

$$\begin{aligned} \operatorname{Re} \{ \gamma_0 (\Omega z + iv^\varepsilon(z)) \overline{z'} \} &= 0 \quad \forall z \in \partial \mathcal{D}_{00}^\varepsilon, \\ \operatorname{Re} \{ \gamma_1 (\Omega z + iv^\varepsilon(z)) \overline{z'} \} &= 0 \quad \forall z \in \partial \mathcal{D}_{1n}^\varepsilon, \quad n = 0, \dots, m-1, \\ \operatorname{Re} \{ \gamma_2 (\Omega z + iv^\varepsilon(z)) \overline{z'} \} &= 0 \quad \forall z \in \partial \mathcal{D}_{2n}^\varepsilon, \quad n = 0, \dots, m-1, \end{aligned} \quad (4.23)$$

where z' denotes a tangent vector to the boundary at the point z and

$$v^\varepsilon(z) = \frac{C_\alpha}{2\pi} \frac{\gamma_0}{\varepsilon^2 b_0^2} \int_{\partial \mathcal{D}_{00}^\varepsilon} \frac{d\xi}{|z - \xi|^\alpha} + \sum_{\ell=1}^2 \frac{\gamma_\ell}{\varepsilon^2 b_\ell^2} \sum_{k=0}^{m-1} \frac{C_\alpha}{2\pi} \int_{\partial \mathcal{D}_{\ell k}^\varepsilon} \frac{d\xi}{|z - \xi|^\alpha}$$

for all $z \in \mathbb{C}$. In view of (4.12) and (4.13), a suitable change of variables gives

$$v^\varepsilon(z) = \frac{C_\alpha}{2\pi} \frac{\gamma_0}{\varepsilon^2 b_0^2} \int_{\partial \mathcal{D}_{00}^\varepsilon} \frac{d\xi}{|z - \xi|^\alpha} + \sum_{\ell=1}^2 \frac{\gamma_\ell}{\varepsilon^2 b_\ell^2} \sum_{k=0}^{m-1} \frac{C_\alpha}{2\pi} \int_{\partial \mathcal{D}_{\ell 0}^\varepsilon} \frac{e^{\frac{2\pi k i}{m}} d\xi}{|z - e^{\frac{2\pi k i}{m}} \xi|^\alpha}.$$

Observe that for any $n \in \{1, \dots, m-1\}$ one has

$$v^\varepsilon(e^{\frac{2\pi n i}{m}} z) = \frac{C_\alpha}{2\pi} \frac{\gamma_0}{\varepsilon^2 b_0^2} \int_{\partial \mathcal{D}_{00}^\varepsilon} \frac{d\xi}{|e^{\frac{2\pi n i}{m}} z - \xi|^\alpha} + e^{\frac{2\pi n i}{m}} \sum_{\ell=1}^2 \frac{\gamma_\ell}{\varepsilon^2 b_\ell^2} \sum_{k=0}^{m-1} \frac{C_\alpha}{2\pi} \int_{\partial \mathcal{D}_{\ell 0}^\varepsilon} \frac{e^{\frac{2\pi(k-n)i}{m}}}{|e^{-\frac{2\pi(k-n)i}{m}} z - \xi|^\alpha} d\xi.$$

From (4.10), the change of variable $\xi \mapsto e^{\frac{2\pi n i}{m}} \xi$ in the first integral leads to

$$v^\varepsilon(e^{\frac{2\pi n i}{m}} z) = e^{\frac{2\pi n i}{m}} v^\varepsilon(z).$$

From the last identity and by (4.12) and (4.13), we conclude that the system (4.23) of $2m+1$ equations can be reduced to a system of three equations,

$$\gamma_j \operatorname{Re} \{ (\Omega z + iv^\varepsilon(z)) \overline{z'} \} = 0 \quad \text{for all } z \in \partial \mathcal{D}_{j0}^\varepsilon, \quad j = 0, 1, 2.$$

Assume that the boundaries of the domains $\mathcal{O}_j^\varepsilon$, $j = 0, 1, 2$ in (4.12), (4.13) are parametrized by the conformal mappings $\phi_j: \mathbb{T} \rightarrow \partial \mathcal{O}_j^\varepsilon$ satisfying

$$\phi_j(w) = w + \varepsilon |\varepsilon|^\alpha b_j^{1+\alpha} f_j(w) \quad \text{with} \quad f_j(w) = \sum_{m=1}^{\infty} \frac{a_m^j}{w^m}, \quad a_m^j \in \mathbb{R}.$$

Then, from (2.23) one may conclude that the dynamics of three boundaries is described by

$$\begin{aligned} \gamma_0 \mathcal{G}_0^\alpha(\varepsilon, f; \lambda)(w) &:= \gamma_0 \operatorname{Im} \left\{ \left(\Omega(\varepsilon b_0 w + b_0^{\alpha+2} \varepsilon^2 |\varepsilon|^\alpha f_0(w)) + \gamma_0 \mathcal{I}^\alpha[\varepsilon, f_0](w) \right. \right. \\ &\quad \left. \left. + \sum_{\ell=1}^2 \gamma_\ell \sum_{k=0}^{m-1} \mathcal{K}_k^\alpha[\varepsilon, f_\ell, f_0](w) \right) \bar{w} (1 + b_0^{\alpha+1} \varepsilon |\varepsilon|^\alpha f_0'(\bar{w})) - \gamma_0 \mu_\alpha f_0'(\bar{w}) \right\} = 0, \end{aligned} \quad (4.24)$$

$$\begin{aligned} \gamma_j \mathcal{G}_j^\alpha(\varepsilon, f; \lambda)(w) &:= \gamma_j \operatorname{Im} \left\{ \left(\Omega(\varepsilon b_j w + b_j^{\alpha+2} \varepsilon^2 |\varepsilon|^\alpha f_j(w) + d_j) + \gamma_j \mathcal{I}^\alpha[\varepsilon, f_j](w) + \gamma_0 \mathcal{K}_0^\alpha[\varepsilon, f_0, f_j](w) \right. \right. \\ &\quad \left. \left. + \sum_{\ell=1}^2 \gamma_\ell \sum_{k=\delta_{\ell j}}^{m-1} \mathcal{K}_k^\alpha[\varepsilon, f_\ell, f_j](w) \right) \bar{w} (1 + b_j^{\alpha+1} \varepsilon |\varepsilon|^\alpha f_j(\bar{w})) - \gamma_j \mu_\alpha f_j'(\bar{w}) \right\} = 0, \end{aligned} \quad (4.25)$$

for all $w \in \mathbb{T}$ and $j = 1, 2$, where μ_α is defined in (2.21) and

$$\begin{aligned} \mathcal{I}^\alpha[\varepsilon, f_j](w) &= -C_\alpha \int_{\mathbb{T}} \frac{f_j'(\tau)}{|\phi_j(\tau) - \phi_j(w)|^\alpha} d\tau \\ &\quad + \alpha C_\alpha \int_{\mathbb{T}} \int_0^1 \frac{\operatorname{Re}[(f_j(\tau) - f_j(w))(\bar{\tau} - \bar{w})] + \varepsilon |\varepsilon|^\alpha t |f_j(\tau) - f_j(w)|^2}{|w - \tau + t\varepsilon |\varepsilon|^\alpha (f_j(\tau) - f_j(w))|^{2+\alpha}} dt d\tau, \end{aligned} \quad (4.26)$$

$$\begin{aligned} \mathcal{K}_k^\alpha[\varepsilon, f_\ell, f_j](w) &:= \frac{\alpha C_\alpha}{b_\ell} \left[\int_{\mathbb{T}} \int_0^1 \frac{\operatorname{Re}[(\nu_{k\ell j} d_\ell - d_j)(b_\ell \bar{\nu}_{k\ell j} \phi_\ell(\tau) - b_j \phi_j(w))]}{|\nu_{k\ell j} (t\varepsilon b_\ell \phi_\ell(\tau) + d_\ell) - (t\varepsilon b_j \phi_j(w) + d_j)|^{\alpha+2}} \phi_\ell'(\tau) dt d\tau \right. \\ &\quad \left. + \int_{\mathbb{T}} \int_0^1 \frac{t\varepsilon |\nu_{k\ell j} b_\ell \phi_\ell(\tau) - b_j \phi_j(w)|^2 \phi_\ell'(\tau)}{|\nu_{k\ell j} (t\varepsilon b_\ell \phi_\ell(\tau) + d_\ell) - (t\varepsilon b_j \phi_j(w) + d_j)|^{\alpha+2}} dt d\tau \right] \nu_{k\ell j}, \end{aligned} \quad (4.27)$$

with the convention $d_0 = 0$ and $\nu_{k\ell j} = \exp(2k\pi i/m + (\delta_{2\ell} - \delta_{2j})\vartheta\pi i/m)$.

4.2. Existence of the nested polygonal vortex patch equilibria. For any $m \geq 2$ we define the Banach spaces

$$\mathcal{V}^\alpha := V_m^\alpha \times V_1^\alpha \times V_1^\alpha \quad \text{and} \quad \mathcal{W}^\alpha := W_m^\alpha \times W_1^\alpha \times W_1^\alpha, \quad (4.28)$$

$$\text{with } V_m^\alpha := \{f \in V_1^\alpha : f(e^{i2\pi/m} w) = f(w)\} \quad \text{and} \quad W_m^\alpha := \{g \in W_1^\alpha : g(e^{i2\pi/m} z) = g(z)\},$$

where V_1^α and W_1^α were defined in (3.9). Note that if $f_0 \in V_m^\alpha$, the expansion of the associated conformal mapping is given by

$$\phi_0(w) = w + \varepsilon |\varepsilon|^\alpha f_0(w) = w \left(1 + \varepsilon |\varepsilon|^\alpha \sum_{n=1}^{\infty} \frac{a_{nm-1}}{w^{nm}} \right).$$

which provides the m -fold symmetry of the associated patch. We denote by B^α the open unit ball in \mathcal{V}^α . Define the mapping

$$\mathcal{G}^\alpha(\varepsilon, f; \lambda) := (\mathcal{G}_0^\alpha(\varepsilon, f; \lambda), \mathcal{G}_1^\alpha(\varepsilon, f; \lambda), \mathcal{G}_2^\alpha(\varepsilon, f; \lambda)),$$

where $f = (f_0, f_1, f_2)$, $\lambda = (\Omega, \gamma_2)$, \mathcal{G}_j^α is given by (4.24)–(4.25) for $\alpha \in (0, 1)$ and by (4.19)–(4.20) for $\alpha = 0$.

The proof of the existence of the co-rotating nested polygons follows from the next proposition, which gives the full statement of Theorem 1.8.

Proposition 4.2. *Let $\alpha \in [0, 1)$, $b_1, b_2, d_1, d_2 \in (0, \infty)$ such that $d = d_2/d_1 > 0$ satisfies (4.7), and let $\gamma_0, \gamma_1 \in \mathbb{R} \setminus \{0\}$ such that (4.9) holds. Then*

- (i) *There exists $\varepsilon_0 > 0$ and a neighborhood Λ of λ^* in \mathbb{R}^2 such that \mathcal{G}^α can be extended to a C^1 mapping $(-\varepsilon_0, \varepsilon_0) \times B^\alpha \times \Lambda \rightarrow \mathcal{W}^\alpha$.*
- (ii) *$\mathcal{G}^\alpha(0, 0; \lambda^*) = 0$, where $\lambda^* = (\Omega^*, \gamma_2^*)$ is given by (4.8).*
- (iii) *The linear operator $D_{(f; \lambda)} \mathcal{G}^\alpha(0, 0; \lambda^*) : \mathcal{V}^\alpha \times \mathbb{R}^2 \rightarrow \mathcal{W}^\alpha$ is an isomorphism.*

(iv) There exists $\varepsilon_1 > 0$ and a unique C^1 function $(f, \lambda): (-\varepsilon_1, \varepsilon_1) \rightarrow B^\alpha \times \mathbb{R}^2$ such that

$$\mathcal{G}^\alpha(\varepsilon, f(\varepsilon); \lambda(\varepsilon)) = 0,$$

with $\lambda(\varepsilon) = \lambda^* + o(\varepsilon^2)$ and

$$f(\varepsilon) = \Xi_\alpha \left(0, \frac{\varepsilon b_1 \mathcal{Q}_1^\alpha}{\gamma_1 d_1^{\alpha+2}} \bar{w}, \frac{\varepsilon b_2 \mathcal{Q}_2^\alpha}{\gamma_2^* d_2^{\alpha+2}} \bar{w} \right) + o(\varepsilon^2),$$

$$\mathcal{Q}_j^\alpha := \gamma_0 + \sum_{\ell=1}^2 \gamma_\ell \sum_{k=\delta_{\ell j}}^{m-1} \frac{(e^{\frac{2k\pi i}{m} + (\delta_{2\ell} - \delta_{2j}) \frac{\vartheta\pi i}{m}} d_\ell / d_j - 1)^2}{|e^{\frac{2k\pi i}{m} + (\delta_{2\ell} - \delta_{2j}) \frac{\vartheta\pi i}{m}} d_\ell / d_j - 1|^{\alpha+4}}, \quad \Xi_\alpha := \frac{(\alpha + 2)\Gamma(1 - \frac{\alpha}{2})\Gamma(3 - \frac{\alpha}{2})}{4\Gamma(2 - \alpha)}. \quad (4.29)$$

Proof. The regularity of the nonlinear operator \mathcal{G}^α follows from Proposition 2.1. The reflection symmetry property can be easily checked as in the proof of Proposition 3.1. Thus, it remains to check the m -fold symmetry property of \mathcal{G}_0^α , namely, that if

$$f_0(e^{\frac{2\pi i}{m}} w) = f_0(w) \quad \forall w \in \mathbb{T}, \quad (4.30)$$

then

$$\mathcal{G}_0^\alpha(\varepsilon, f; \lambda)(e^{\frac{2\pi i}{m}} w) = \mathcal{G}_0^\alpha(\varepsilon, f; \lambda)(w) \quad \forall w \in \mathbb{T}. \quad (4.31)$$

We shall give the details of the proof in the case $\alpha \in (0, 1)$; the case $\alpha = 0$ can be checked in a similar way. From (4.21) one has

$$\begin{aligned} \overline{\mathcal{I}^0}[\varepsilon, f_0](e^{\frac{2\pi i}{m}} w) &= \frac{1}{2} \int_{\mathbb{T}} \frac{e^{-\frac{2\pi i}{m}} \bar{w} - \bar{\tau} + \varepsilon b_0 (f_0(\bar{\tau}) - f_0(e^{-\frac{2\pi i}{m}} \bar{w}))}{e^{\frac{2\pi i}{m}} w - \tau + \varepsilon b_0 (f_0(\tau) - f_0(e^{\frac{2\pi i}{m}} w))} f_0'(\tau) d\tau \\ &\quad + \int_{\mathbb{T}} \frac{i \operatorname{Im} \{ (e^{\frac{2\pi i}{m}} w - \tau) (f_0(\bar{\tau}) - f_0(e^{-\frac{2\pi i}{m}} \bar{w})) \}}{(e^{\frac{2\pi i}{m}} w - \tau) (e^{\frac{2\pi i}{m}} w - \tau + \varepsilon b_0 f_0(\tau) - \varepsilon b_0 f_0(e^{\frac{2\pi i}{m}} w))} d\tau. \end{aligned}$$

Using the change of variables $\tau \mapsto e^{\frac{2\pi i}{m}} \tau$, we find

$$\begin{aligned} \overline{\mathcal{I}^0}[\varepsilon, f_0](e^{\frac{2\pi i}{m}} w) &= \frac{1}{2} \int_{\mathbb{T}} \frac{e^{-\frac{2\pi i}{m}} \bar{w} - e^{-\frac{2\pi i}{m}} \bar{\tau} + \varepsilon b_0 (f_0(e^{-\frac{2\pi i}{m}} \bar{\tau}) - f_0(e^{-\frac{2\pi i}{m}} \bar{w}))}{e^{\frac{2\pi i}{m}} w - e^{\frac{2\pi i}{m}} \tau + \varepsilon b_0 (f_0(e^{\frac{2\pi i}{m}} \tau) - f_0(e^{\frac{2\pi i}{m}} w))} f_0'(e^{\frac{2\pi i}{m}} \tau) e^{\frac{2\pi i}{m}} d\tau \\ &\quad + \int_{\mathbb{T}} \frac{i \operatorname{Im} \{ (e^{\frac{2\pi i}{m}} w - e^{\frac{2\pi i}{m}} \tau) (f_0(e^{-\frac{2\pi i}{m}} \bar{\tau}) - f_0(e^{-\frac{2\pi i}{m}} \bar{w})) \}}{e^{\frac{2\pi i}{m}} (w - \tau) (e^{\frac{2\pi i}{m}} w - e^{\frac{2\pi i}{m}} \tau + \varepsilon b_0 f_0(e^{\frac{2\pi i}{m}} \tau) - \varepsilon b_0 f_0(e^{\frac{2\pi i}{m}} w))} d\tau. \end{aligned}$$

Then, by (4.30), we deduce that

$$\overline{\mathcal{I}^0}[\varepsilon, f_0](e^{\frac{2\pi i}{m}} w) = e^{-\frac{2\pi i}{m}} \overline{\mathcal{I}^0}[\varepsilon, f_0](w).$$

In view of (4.22) and (4.30) we have

$$\overline{\mathcal{K}_k^0}[\varepsilon, f_\ell, f_0](e^{\frac{2\pi i}{m}} w) = \frac{e^{-\frac{2\pi i}{m}}}{2} \int_{\mathbb{T}} \frac{(\bar{\tau} + \varepsilon b_\ell f_\ell(\bar{\tau})) (1 + \varepsilon b_\ell f_\ell'(\tau))}{e^{(k-1)\frac{2\pi i}{m} + \delta_{2\ell} \frac{\vartheta\pi i}{m}} (\varepsilon b_\ell \tau + \varepsilon^2 b_\ell^2 f_\ell(\tau) + d_\ell) - \varepsilon b_0 (w + \varepsilon b_0 f_0(w))} d\tau.$$

Summing over k then gives

$$\sum_{k=0}^{m-1} \overline{\mathcal{K}_k^0}[\varepsilon, f_\ell, f_0](e^{\frac{2\pi i}{m}} w) = e^{-\frac{2\pi i}{m}} \sum_{k=0}^{m-1} \overline{\mathcal{K}_k^0}[\varepsilon, f_\ell, f_0](w),$$

concluding the proof of (i). The proof of (ii) follows immediately from Proposition 2.1(ii), (4.5) and (4.8). In order to show (iii) we use Proposition 2.1(ii) and (4.6) to get, for all $h = (h_0, h_1, h_2) \in \mathcal{V}^\alpha$

and $(\dot{\Omega}, \dot{\gamma}_2) \in \mathbb{R}^2$,

$$D_{(f;\lambda)}\mathcal{G}^\alpha(0, 0; \lambda^*) \begin{pmatrix} \dot{\Omega} \\ \dot{\gamma}_2 \\ h \end{pmatrix} (w) = - \begin{pmatrix} 0 & 0 \\ d_1 & \frac{\alpha C_\alpha}{2} \frac{T_\alpha^+(d, \vartheta)}{d_1^{1+\alpha}} \\ d_2 & \frac{\alpha C_\alpha}{2} \frac{S_\alpha}{2d_2^{1+\alpha}} \end{pmatrix} \begin{pmatrix} \dot{\Omega} \\ \dot{\gamma}_2 \end{pmatrix} \text{Im}\{w\} + \sum_{n \geq 1} M_n^\alpha \begin{pmatrix} \gamma_0 a_n^0 \\ \gamma_1 a_n^1 \\ \gamma_2^* a_n^2 \end{pmatrix} \text{Im}\{w^{n+1}\},$$

where M_n^α is given by (2.27). Proposition 2.1(iv) and the assumption (4.7) then imply (iii).

The existence and uniqueness in (iv) follow from the implicit function theorem. In order to compute the asymptotic of the solution, we shall use the formula

$$\partial_\varepsilon(f(\varepsilon), \lambda(\varepsilon))|_{\varepsilon=0} = -D_g\mathcal{G}^\alpha(0, 0; \lambda^*)^{-1} \partial_\varepsilon\mathcal{G}^\alpha(0, 0; \lambda^*). \quad (4.32)$$

For any $H \in \mathcal{W}^\alpha$ with the expansion

$$H(w) = \sum_{n \geq 0} \begin{pmatrix} A_n^0 \\ A_n^1 \\ A_n^2 \end{pmatrix} \text{Im}\{w^{n+1}\},$$

with $A_n^0 = 0$ if n is not a multiple of m , we have

$$D_{(f;\lambda)}\mathcal{G}^\alpha(0, 0; \lambda^*)^{-1} H(w) = \left(\frac{1}{\gamma_0} \sum_{n \geq 1} \frac{A_n^0}{M_n^\alpha} \bar{w}^n, \frac{1}{\gamma_1} \sum_{n \geq 1} \frac{A_n^1}{M_n^\alpha} \bar{w}^n, \frac{1}{\gamma_2^*} \sum_{n \geq 1} \frac{A_n^2}{M_n^\alpha} \bar{w}^n; \frac{d^{\alpha+2} T_\alpha^+(d, \vartheta) A_0^2 - \frac{1}{2} S_\alpha A_0^1}{\det(D_\lambda \mathcal{P}^\alpha(\lambda^*))}, \frac{A_0^2 d_1 - A_0^1 d_2}{\det(D_\lambda \mathcal{P}^\alpha(\lambda^*))} \right), \quad (4.33)$$

where $\det(D_\lambda \mathcal{P}^\alpha(\lambda^*))$ was calculated in (4.7). On the other hand, from (4.24)–(4.25) and (4.19)–(4.20) we have

$$\begin{aligned} \mathcal{G}_0^\alpha(\varepsilon, 0; \lambda)(w) &= \text{Im} \left\{ \left(\sum_{\ell=1}^2 \gamma_\ell \sum_{k=0}^{m-1} \mathcal{K}_k^\alpha[\varepsilon, f_\ell, f_0](w) \right) \bar{w} \right\}, \\ \mathcal{G}_j^\alpha(\varepsilon, 0; \lambda)(w) &= \text{Im} \left\{ \Omega d_j \bar{w} + \left(\gamma_0 \mathcal{K}_0^\alpha[\varepsilon, f_0, f_j](w) + \sum_{\ell=1}^2 \gamma_\ell \sum_{k=\delta_{\ell j}}^{m-1} \mathcal{K}_k^\alpha[\varepsilon, f_\ell, f_j](w) \right) \bar{w} \right\}, \end{aligned} \quad (4.34)$$

with $j = 1, 2$.

Case $\alpha = 0$. Differentiating (4.22) with respect to ε gives

$$\begin{aligned} \partial_\varepsilon \overline{\mathcal{K}_k^0}[\varepsilon, 0, 0](w)|_{\varepsilon=0} &= \frac{1}{2(\nu_{k\ell j} d_\ell - d_j)^2} \int_{\mathbb{T}} \bar{\tau} (\nu_{k\ell j} b_\ell \tau - b_n w) d\tau \\ &= \frac{b_n w}{2(\nu_{k\ell j} d_\ell - d_j)^2}, \end{aligned}$$

with $\nu_{k\ell j} = \exp(2k\pi i/m + (\delta_{2\ell} - \delta_{2j})\vartheta\pi i/m)$. It follows from (4.34) that

$$\begin{aligned} \partial_\varepsilon \mathcal{G}_0^0(\varepsilon, 0; \lambda)|_{\varepsilon=0}(w) &= \frac{b_0}{2} \text{Im} \left\{ \sum_{\ell=1}^2 \frac{\gamma_\ell}{d_\ell^2} \sum_{k=0}^{m-1} e^{\frac{4k\pi i}{m} + (\delta_{2\ell} - \delta_{2j})\frac{2\vartheta\pi i}{m}} \bar{w}^2 \right\} = 0, \\ \partial_\varepsilon \mathcal{G}_j^0(\varepsilon, 0; \lambda)|_{\varepsilon=0}(w) &= \frac{b_j}{2d_j^2} \left(\gamma_0 + \sum_{\ell=1}^2 \sum_{k=\delta_{\ell j}}^{m-1} \frac{\gamma_\ell}{(\nu_{k\ell j} \frac{d_\ell}{d_j} - 1)^2} \right) \text{Im}\{\bar{w}^2\} = \frac{b_j}{2d_j^2} \mathcal{Q}_j^0 \text{Im}\{\bar{w}^2\}. \end{aligned}$$

Combining the two last identities with (4.32) and (4.33) yields

$$\partial_\varepsilon(f(\varepsilon), \lambda(\varepsilon))|_{\varepsilon=0} = \left(0, \frac{b_1 \mathcal{Q}_1^0}{\gamma_1 d_1^{\alpha+2}} \bar{w}, \frac{b_2 \mathcal{Q}_2^0}{\gamma_2^* d_2^{\alpha+2}} \bar{w}; 0, 0 \right).$$

Case $\alpha \in (0, 1)$. From (4.27) we have

$$\begin{aligned} \mathcal{K}_k^\alpha[\varepsilon, 0, 0](w) = \alpha C_\alpha \frac{\nu_{k\ell j}}{b_\ell} & \left[\int_{\mathbb{T}} \int_0^1 \frac{\operatorname{Re}[(\nu_{k\ell j} d_\ell - d_j)(\overline{\nu_{k\ell j}} b_\ell \bar{\tau} - b_j \bar{w})]}{|\nu_{k\ell j}(t\varepsilon b_\ell \tau + d_\ell) - (t\varepsilon b_j w + d_j)|^{\alpha+2}} dt d\tau \right. \\ & \left. + \int_{\mathbb{T}} \int_0^1 \frac{t\varepsilon |\nu_{k\ell j} b_\ell \tau - b_j w|^2}{|\nu_{k\ell j}(t\varepsilon b_\ell \tau + d_\ell) - (t\varepsilon b_j w + d_j)|^{\alpha+2}} dt d\tau \right]. \end{aligned}$$

with $\nu_{k\ell j}$ defined as for $\alpha = 0$. Applying formula (2.19) gives

$$\mathcal{K}_k^\alpha[\varepsilon, f_\ell, f_j](w) = \frac{\alpha C_\alpha}{4} \left[\frac{2(\nu_{k\ell j} d_\ell - d_j) + \varepsilon b_j w}{|\nu_{k\ell j} d_\ell - d_j|^{\alpha+2}} + (\alpha + 2) \frac{(\nu_{k\ell j} d_\ell - d_j)^2 \varepsilon b_j \bar{w}}{|\nu_{k\ell j} d_\ell - d_j|^{\alpha+4}} \right] + O(\varepsilon^2).$$

Inserting the last identity into (4.34) and then differentiating with respect ε , we obtain

$$\begin{aligned} \partial_\varepsilon \mathcal{G}_0^\alpha(\varepsilon, 0; \lambda)|_{\varepsilon=0}(w) &= \frac{(\alpha + 2)\alpha C_\alpha}{4} \operatorname{Im} \left\{ b_0 \bar{w}^2 \sum_{\ell=1}^2 \frac{\gamma_\ell}{d_\ell^{\alpha+2}} \sum_{k=0}^{m-1} e^{(2k+\sigma(\ell,j)\vartheta)\frac{2\pi i}{m}} \right\} = 0, \\ \partial_\varepsilon \mathcal{G}_j^\alpha(\varepsilon, 0; \lambda)|_{\varepsilon=0}(w) &= \frac{(\alpha + 2)\alpha C_\alpha}{4d_j^{\alpha+2}} b_j \left(\gamma_0 + \sum_{\ell=1}^2 \gamma_\ell \sum_{k=\delta_{\ell j}}^{m-1} \frac{(\nu_{k\ell j} \frac{d_\ell}{d_j} - 1)^2}{|\nu_{k\ell j} \frac{d_\ell}{d_j} - 1|^{\alpha+4}} \right) \operatorname{Im}\{\bar{w}^2\}. \end{aligned}$$

Combining the two last identities with (4.32), (4.33) and (3.20), we get

$$\partial_\varepsilon (f(\varepsilon), \lambda(\varepsilon))|_{\varepsilon=0} = \Xi_\alpha \left(0, \frac{b_1 \mathcal{Q}_1^\alpha}{\gamma_1 d_1^{\alpha+2}} \bar{w}, \frac{b_2 \mathcal{Q}_2^\alpha}{\gamma_2^* d_2^{\alpha+2}} \bar{w}; 0, 0 \right),$$

where \mathcal{Q}_j^α and Ξ_α are given by (4.29). This ends the proof of the proposition. \square

Remark 4.3. Note that, by setting $\gamma_2 = 0$ and $\lambda = \Omega$ in (2.3), (4.24)–(4.25) and (4.19)–(4.20), we can recover the existence and uniqueness result of the body-centered polygonal configuration through Proposition 4.2. This remains equally true for the rotating vortex polygon by setting $\gamma_0 = \gamma_2 = 0$ and $\lambda = \Omega$.

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DEPARTMENT OF MATHEMATICS, NEW YORK UNIVERSITY IN ABU DHABI, SAADIYAT ISLAND,
P.O. BOX 129188, ABU DHABI, UNITED ARAB EMIRATES ,
E-mail address: zh14@nyu.edu

DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF BATH, BATH BA2 7AY, UK,
E-mail address: mw2319@bath.ac.uk