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Isothermic surfaces and their generalisations

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Abstract

This thesis concerns the theory of isothermic submanifolds in symmetric R-spaces. We generalise several results for isothermic surfaces in S^3 to this more general class of submanifolds.

We develop a special class of submanifold of symmetric R-space, the cyclide, and characterise and study envelopes of congruences of cyclides. In particular, we show that Darboux pairs of isothermic submanifolds are common envelopes of a single congruence of cyclides.

We also describe a generalisation of fanning curves and show that they are natural isothermic submanifolds and this class is preserved under the transformations of isothermic submanifolds. We use this to define a semi-discretisation of isothermic submanifolds as iterated Darboux transforms of fanning curves and demonstrate how the transformation theory extends to these semi-discrete submanifolds.

Lastly we lay the groundwork for a theory of polynomial conserved quantities for isothermic submanifolds and explore this theory with the example of the self-dual Grassmannian.

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Contents

Chapter 1

Introduction

1.1 Background

Isothermic surfaces

Isothermic surfaces were of great interest to classical differential geometers. They were originally conceived by Bour [10] and were intensely studied by Bianchi [5], [6], Christoffel [27] and Darboux [29] among others. Classically, they are defined as surfaces in $\mathbb{R}^3$ admitting coordinates that are simultaneously conformal and curvature line. Natural examples include cones, quadrics and surfaces of revolution. Alternatively, we can characterise these as surfaces admitting a holomorphic quadratic differential $q = dz^2$ (where $z = x + iy$ and x, y are the conformal coordinates), which commutes with the trace-free second fundamental form. This is a conformally invariant property and so we may think of isothermic surfaces as surfaces in the conformal 3-sphere.

Both these conditions can be extended to higher codimension, although we must assume that f has a flat normal bundle in order to define the conformal coordinates.

Integrable geometry

An important theme of this thesis is the realisation of isothermic surfaces as an integrable system, originating in the work of Cieśliński–Goldstein–Sym [28]. Definitions of what constitutes an integrable system vary significantly[1], but there are a few common themes. These include explicit solutions, non-trivial transformations of solutions and spectral deformations. For our purposes, these are tied together by the existence of a zero-curvature representation. That is, there exists some family of connections dependent on a spectral

[1]See [39] for a PDE viewpoint and [12] for a differential-geometric perspective

parameter whose flatness encodes the integrability of the system.

For context, we examine this perspective on isothermic surfaces in the conformal 3-sphere S^3. Consider $\mathbb{R}^{4,1}$ a 5-dimensional real vector space equipped with a symmetric bilinear form of signature $(4,1)$. Let $\mathcal{L} \subset \mathbb{R}^{4,1}$ denote the set of null vectors and $\mathbb{P}(\mathcal{L})$ its projectivisation. Choosing a "time-like" vector $t \in \mathbb{R}^{4,1}$, $(t,t) = -1$ we obtain an identification of $\mathbb{P}(\mathcal{L})$ with $S_t = \{x \in \mathcal{L} | (x,t) = -1\}$ and then by orthoprojection onto $t^\perp \cong \mathbb{R}^4$ an identification with the unit sphere $S_t \cong S^3 \subset \mathbb{R}^4$. Naturally $G := O(4,1)$, the group of linear transformations preserving the bilinear form, must also preserve $\mathbb{P}(\mathcal{L})$ and indeed it acts as conformal diffeomorphisms of S^3. This realises the conformal geometry of S^n as a homogeneous space $O(4,1)/\operatorname{Stab}(\Lambda)$ where $\Lambda \in \mathbb{P}(\mathcal{L})$ and $\operatorname{Stab}(\Lambda)$ is its stabiliser, a parabolic subgroup. It thus has tangent spaces given by $T_\Lambda S^n \cong \mathfrak{g}/\operatorname{stab}(\Lambda)$ where $\mathfrak{g} := \mathfrak{so}(4,1)$ and $\operatorname{stab}(\Lambda)$ is the subalgebra corresponding to $\operatorname{Stab}(\Lambda)$. The Killing form on $\mathfrak{g}$ gives a dual identification $T^*_\Lambda S^3 \cong \operatorname{stab}(\Lambda)^\perp$. Under the natural identification of $\mathfrak{g}$ with $\bigwedge^2 \mathbb{R}^{4,1}$, $\operatorname{stab}(\Lambda)^\perp$ is given by $\Lambda \wedge \Lambda^\perp$.

Let $f : \Sigma \to S^3 = \mathbb{P}(\mathcal{L})$ be a surface and let q be a quadratic differential. That is $q \in \Gamma\, \mathrm{S}^2 T^*\Sigma \subset \Omega^1_\Sigma(T^*\Sigma)$. Then pulling back along f, we obtain an identification $T^*\Sigma \cong f^{-1}T^*S^3 \cong \operatorname{stab}(f)^\perp = f \wedge f^\perp$. Thus q defines a 1-form $\eta \in \Omega^1_\Sigma(\underline{\mathfrak{g}})$, on the trivial bundle $\underline{\mathfrak{g}} = \Sigma \times \mathfrak{g}$, taking values in $f \wedge f^\perp$. The condition that f is isothermic as noted above is that q is holomorphic and commutes with the second fundamental form of f. This is equivalent to η being closed. This 1-form is closed precisely when q is holomorphic and commutes with the second fundamental form of f.

Now we define a loop of connections $\nabla^t := d + t\eta$ on $\underline{\mathfrak{g}}$. The curvature of ∇^t is given by $td\eta + \frac{t^2}{2}[\eta \wedge \eta]$. The bundle $f \wedge f^\perp$ is a bundle of abelian subalgebras so $[\eta \wedge \eta] = 0$ and so ∇^t is flat for all $t \in \mathbb{R}$ if, and only if, f is isothermic.

Symmetric R-spaces

The important feature of S^3 that we used above is that $T^*_\Lambda S^3 \cong \operatorname{stab}(\Lambda)^\perp$ is a bundle of abelian subalgebras of $\mathfrak{g}$. In fact, this identification of the contangent space as an abelian subalgebra of is a defining property of symmetric R-spaces. From this observation, isothermic surfaces were generalised to isothermic submanifolds of symmetric R-spaces in Burstall et al. [17]. We define these by the existence of such a closed 1-form η taking values in the corresponding (pulled-back) cotangent bundle. Again, our notion of closed is induced by the realisation of these contangent spaces as subalgebras of a fixed semisimple, noncompact real Lie algebra. Equivalently, they have a family of flat connections $d + t\eta$. Thus they *a priori* form an integrable system, and indeed the transformation theory of isothermic surfaces extends to these almost completely.

Symmetric R-spaces lie at the intersection of homogeneous, Riemannian and algebraic geometry and as such can be described in many different ways [58]. They are compact Riemannian symmetric spaces which admit a group of diffeomorphisms G strictly larger than their isometry group. We refer to G as the big transformation group[2]. Indeed, by Nagano [48], they are the essentially the only such spaces. They comprise the Hermitian symmetric spaces and their real forms. From another point of view R-spaces are the homogeneous spaces which are also projective varieties [51, Definition 6.3]. A choice of maximal compact subgroup induces a Riemannian structure. Then symmetric R-spaces are those where this structure is symmetric. Most pertinently for us, they are the conjugacy classes of parabolic subalgebras of real semisimple Lie algebras whose Killing polar (or nilradical) is abelian.

Examples include the conformal projective quadrics as well as the Grassmannian and isotropic Grassmannians[3].

Darboux transforms

As noted earlier, an important characteristic of integrable systems is the existence of transformations between solutions. For isothermic surfaces, the relevant transformation is that of Darboux [29]. There are two classical approaches to this. Firstly, these arise as solutions to a linear system of equations. Let $f : \Sigma \to S^3$ be an isothermic surface with closed 1-form η. Then we can view $d + t\eta$ as connections on the trivial bundle $\mathbb{R}^{4,1}$. Darboux transforms of f are then given by parallel sections of $d + m\eta$ for $m \in \mathbb{R} \setminus \{0\}$. Specifically, if $(d + m\eta)\sigma = 0$ for $\sigma \in \Gamma f^*\mathcal{L}$ non-vanishing then $\hat{f} := \langle\sigma\rangle$ is a Darboux transformation of f. More geometrically, a Darboux transform $\hat{f}$ is characterised by the property that $f, \hat{f}$ both envelop a common sphere congruence which is conformal and Ribaucour. That is, $f, \hat{f}$ induce the same conformal structure on Σ and have the same curvature lines. The former description naturally extends to isothermic submanifolds of symmetric R-spaces and this is the definition used in [17]. The latter however is less clear as symmetric R-spaces, in general, do not have a canonical conformal structure or an appropriate concept of curvature lines.

Finally, we note another key part of the theory is that a pair of mutual Darboux transforms form a curved flat [30] in the symmetric space of distinct point pairs $S^3 \times S^3 \setminus \Delta \cong O(4, 1)/(O(3) \times O(1, 1))$.

Discrete and semi-discrete theories

[2]Note throughout we will work with the adjoint group of G which is in general a quotient of this group by some discrete subgroup

[3]for symplectic, symmetric bilinear and Hermitian forms

Discrete differential geometry has emerged recently from the union of discrete and smooth geometry. Isothermic surfaces [9] and isothermic submanifolds [17] admit a natural discretisation using their Darboux transformations. Integrability is replaced by multidimensional consistency, which follows from Bianchi's celebrated quadrilateral and cube permutability theorem. For isothermic surfaces there is also a semi-discrete theory intertwining the discrete and smooth theories together [19], [46]. Geometrically, this is the theory of a lattice of smooth curves where each pair of adjacent curves is a Darboux pair.

Polynomial conserved quantities

A useful tool for studying integrable geometrical theories is that of polynomial conserved quantities. For a family of flat connections ∇_t these are a corresponding family of sections polynomially dependent on t. These descend from the concept of polynomial Killing fields developed in [18] and were used in [15] to identify surfaces with constant mean curvature (or more broadly generalised H-surfaces) among isothermic surfaces. In [24] these are developed to include the special isothermic surfaces of Bianchi [6] as well as those contained in some hypersphere. Other integrable surface classes have also been investigated this way such as omega surfaces [21] and constrained Willmore surfaces [52]. This approach is also readily adapted to the discrete theories [16], [22].

1.2 Manifesto

The aims of this thesis can be broadly interpreted as generalising several results for isothermic surfaces to isothermic submanifolds. Firstly, we will develop a geometric characterisation of the Darboux transform in self-dual[4] symmetric R-spaces. This requires a few ingredients, the first of which is the analogous counterpart of the conformal structure. In [32], Gindikin–Kaneyuki describe a canonical choice of generalised conformal structure on a symmetric R-space. In fact, this has been studied from a few different viewpoints [47], [53], [59]. This structure arises by considering the tangent space as a prehomogeneous vector space (c.f [56]) for the action of the stabiliser at that point. More precisely, if R is a symmetric R-space, then points of R are parabolic subalgebras $\mathfrak{p} \leq \mathfrak{g}$ of some Lie algebra. Then $\mathrm{Stab}(\mathfrak{p})$ acts on $T_{\mathfrak{p}}R \cong \mathfrak{g}/\mathfrak{p}$ with a dense open orbit consisting of the elements $X \in \mathfrak{g}/\mathfrak{p}$ such that ad_X^2 defines an isomorphism $\mathfrak{p}^\perp \to \mathfrak{g}/\mathfrak{p}$. We call these elements

[4]so that Darboux transforms live in the same symmetric R-space as the original submanifold

regular and the complement of this orbit the generalised conformal structure. This structure is crucial to many aspects of this thesis. It is inherently G-invariant.

Another major component is a special class of submanifold: the cyclide. These are homogeneous embeddings of certain self-dual symmetric R-spaces into a larger self-dual symmetric R-space. For self-dual Hermitian symmetric spaces, these are precisely the polyspheres of Harish-Chandra (c.f. [45]). More generally, we define a class of submanifolds of self-dual symmetric spaces called splitting submanifolds. These are defined by the property that the generalised conformal structure intersects their complexified tangent spaces in a union of hyperplanes. We demonstrate that these are precisely the envelopes of a congruence of cyclides. Using the methods of [14], one can reformulate this as follows; splitting submanifolds are a parabolic subgeometry of R modelled on a cyclide. This yields canonical coordinates as well as a canonical enveloped cyclide congruence, generalising the central sphere congruence of Blaschke [7]. Then, we prove maximal non-degenerate isothermic submanifolds are splitting. A pair of these is a Darboux pair if, and only if, they envelop a common cyclide congruence and induce the same generalised conformal structure.

Next, we explore a convincing theory of semi-discrete isothermic submanifolds in self-dual symmetric R-spaces. This begins with a generalisation of fanning curves[5]. Explicitly, these are curves whose derivative everywhere avoids the generalised conformal structure. They are isothermic submanifolds in a natural way and the class of fanning curves are preserved under the transformations of isothermic submanifolds. A semi-discrete isothermic submanifold is then constructed as a lattice of iterated Darboux transforms of a fanning curve. We demonstrate that this theory has all the transformation theory of the smooth isothermic submanifolds. In [19], these curves can be seen to be curvature lines for a family of isothermic surfaces and we use this perspective to define a notion of curvature line on an isothermic submanifold.

Finally, we develop the theory of polynomial conserved quantities for isothermic submanifolds. The theory for isothermic surfaces depends on an implicit choice of representation, so we describe the basics of the theory in any representation and then give a natural example for the Grassmannian. Then we identify important properties of a representation for producing a valid theory of polynomial conserved quantities.

[5] see [3] for the definition in the Grassmannian

1.3 Road-map

In Chapter 2, we cover the algebraic preliminaries needed to use symmetric R-spaces. We cover parabolic subalgebras and filtrations as well as complementary parabolic subalgebras and gradings. We use these to define symmetric R-spaces, and R-spaces in general. We also define their dual space and space of complementary pairs. All examples of symmetric R-spaces for the classical simple groups are given, as well as the exceptional isomorphisms between them. Lastly, we define and investigate the generalised conformal structure.

In Chapter 3, we introduce isothermic submanifolds and recall the key aspects of their theory. We define them in terms of a closed 1-form, as well as in terms of flat connections. The Darboux, Christoffel and T-transforms are defined and we shall recall the permutability theorems between them. We give a new, direct proof of the permutability of Darboux and Christoffel transforms, using gauge transforms and an analogue of the Bianchi formula [6, p105]. We also demonstrate the link between Darboux pairs and another integrable system: the curved flat of Ferus–Pedit [30].

In Chapter 4, we define a special class of submanifolds of self-dual symmetric R-spaces: the cyclides. We define these in terms of the representation theory of their stabilisers and give examples of these, as well as proving their existence in general. We link these to Cartan subspaces for the space of complementary pairs and thus describe the space of cyclides as a finite union of reductive homogeneous spaces. Another class of submanifolds is defined by their interaction with the generalised conformal structure, and we show that these are precisely the envelopes of cyclide congruences. Then we define canonical coordinates on such a submanifold and define a canonical enveloped cyclide congruence generalising the central sphere congruence. We then show that Darboux pairs are given by common envelopes of cyclide congruences inducing the same generalised conformal structure. We give a full example of this in the Grassmannian and finally, we note that cyclides are locally isothermic submanifolds.

In Chapter 5, we develop a semi-discretisation of isothermic submanifolds in analogy to [19]. Firstly, we generalise the theory of fanning curves in the self-dual Grassmannian to self-dual symmetric R-spaces. In particular, we give the fundamental transformation and horizontal curve in this more general setting. We recognise fanning curves as isothermic submanifolds for a canonical choice of η and show that the transformations of isothermic submanifolds preserve this subclass. Iterated Darboux transformations of these give a theory of semi-discrete isothermic submanifolds. The permutability of the transformations of isothermic submanifolds then yield transformations of these semi-discrete analogues.

In Chapter 6, we develop the theory of polynomial conserved quantities for isothermic submanifolds. In general, the choice of representation we wish to use for this theory is less clear and so we begin by describing the aspects that do not depend on this choice. Then we consider in detail the example of the self-dual Grassmannian in the Plücker representation. We conclude by noting important features of a representation to guide the choice in general and discuss two natural choices.

Chapter 2

Symmetric R-spaces

In this chapter, we shall describe the fundamental theory of symmetric R-spaces. In particular, we shall construct them as certain conjugacy classes of parabolic subalgebras, demonstrate their useful properties and give the main families of examples. We will also define the space of complementary pairs, an associated symmetric space, and describe the properties of symmetric spaces. Finally, we will define a structure on a symmetric R-space which generalises the conformal structure of the conformal sphere.

We assume the reader is familiar with the basics of Lie theory, including the structure theory of semisimple Lie algebras, homogeneous geometry and some representation theory. For more information on these topics the reader is encouraged to refer to Humphreys [40] for all of the required Lie algebra knowledge and to Procesi [51] for the representation theory.

Throughout, we take $\mathfrak{g}$ to be a non-compact semisimple real Lie algebra with adjoint group G. Thus $G = \mathrm{Inn}(\mathfrak{g}) \leq \mathrm{Aut}(\mathfrak{g})$ is the Lie group of inner automorphisms and has Lie algebra $\mathrm{Der}(\mathfrak{g}) = \mathfrak{g}$ since $\mathfrak{g}$ is semisimple. Notably $\mathfrak{g}$ then has non-degenerate Killing form.

2.1 Parabolic subalgebras

2.1.1 Definition and filtrations

Definition 2.1.1. [25, Definition 2.1] *Let $\mathfrak{p} \leq \mathfrak{g}$. Then $\mathfrak{p}$ is a* ***parabolic subalgebra*** *if $\mathfrak{p}^\perp$, the polar with respect to the Killing form, is a nilpotent subalgebra of $\mathfrak{p}$. Furthermore, if $\mathfrak{p}^\perp$ is k-step nilpotent then we say $\mathfrak{p}$ is of height k.*

A parabolic subalgebra $\mathfrak{p} \leq \mathfrak{g}$ gives $\mathfrak{g}$ the structure of a filtered Lie algebra

[26, Proposition 2.4]:

$$\mathfrak{p}^{(0)} = \mathfrak{p}, \mathfrak{p}^{(-1)} = \mathfrak{p}^{\perp}, \mathfrak{p}^{(i)} := \begin{cases} [\mathfrak{p}^{\perp}, \mathfrak{p}^{(i+1)}], & i \leq -2, \\ \mathfrak{p}^{(-i-1)^{\perp}}, & i \geq 1. \end{cases} \tag{2.1.1}$$

Thus, $\{0\} = \mathfrak{p}^{(-k)} \leq \mathfrak{p}^{(-k)} \leq \cdots \leq \mathfrak{p}^{(k)} = \mathfrak{g}$ and $[\mathfrak{p}^{(i)}, \mathfrak{p}^{(j)}] \leq \mathfrak{p}^{(i+j)}$.

Indeed, filtrations or *flags* of representations of $\mathfrak{g}$ are a natural language with which to define parabolic subalgebras:

Example. Consider a (partial) flag $\{0\} \leq V_1 \leq \cdots \leq V_k \leq \mathbb{C}^n$. Then $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ acts on $\mathbb{C}^n$ as tracefree endomorphisms. Let $\mathfrak{p} \leq \mathfrak{g}$ be the stabiliser of this flag. Then $\mathfrak{p}^{(i)}$ comprises the maps sending each V_j to V_{j+i}. In particular, $\mathfrak{p}^{\perp} = \mathfrak{p}^{(-1)}$ is k-step nilpotent so that the height of $\mathfrak{p}$ is simply the length of the flag.

Example. If we replace $\mathfrak{sl}(n, \mathbb{C})$ by $\mathfrak{so}(n, \mathbb{C})$ or $\mathfrak{sp}(2n, \mathbb{C})$, then parabolic subalgebras are instead the stabilisers of flags of isotropic subspaces of $\mathbb{C}^n, \mathbb{C}^{2n}$ respectively.

The filtered Lie algebra structure on $\mathfrak{g}$, induced by $\mathfrak{p}$, defines a graded Lie algebra:

$$gr_{\mathfrak{p}}(\mathfrak{g}) = \bigoplus_{i=-n}^{n} \mathfrak{p}^{(i)}/\mathfrak{p}^{(i-1)}, \tag{2.1.2}$$

with bracket defined by:

$$[X + \mathfrak{p}^{(i-1)}, Y + \mathfrak{p}^{(j-1)}] = [X, Y] + \mathfrak{p}^{(i+j-1)}. \tag{2.1.3}$$

Then $gr_{\mathfrak{p}}(\mathfrak{g})$ is non-canonically isomorphic to $\mathfrak{g}$. In particular it is semisimple and so $\mathrm{ad} : gr_{\mathfrak{p}}(\mathfrak{g}) \cong \mathrm{Der}(gr_{\mathfrak{p}}(\mathfrak{g}))$. We note that the map $X \mapsto jX$ for $X \in \mathfrak{p}^{(j)}/\mathfrak{p}^{(j-1)}$ is a derivation and thus it is given by $\mathrm{ad}_{\xi_{\mathfrak{p}}}$ for some element $\xi_{\mathfrak{p}} \in gr_{\mathfrak{p}}(\mathfrak{g})$. Naturally, ξ_p lies in the centre of $\mathfrak{p}/\mathfrak{p}^{\perp}$.

Definition 2.1.2. [26, Definition 3.7] *We call $\xi_{\mathfrak{p}}$ the **grading element** of $gr_{\mathfrak{p}}(\mathfrak{g})$.*

2.1.2 Complementary parabolic subalgebras

Definition 2.1.3. [17, Definition 1.2] *Let $\mathfrak{p}, \mathfrak{q}$ be parabolic subalgebras of height k. Then we say they are **complementary** if:*

$$\mathfrak{g} = \mathfrak{p}^{(i)} \oplus \mathfrak{q}^{(-1-i)}, \tag{2.1.4}$$

for all $i \in \{-n, \ldots, n\}$. We denote the set of all parabolic subalgebras complementary to $\mathfrak{p}$ by $\Omega_{\mathfrak{p}}$.

A choice of complementary parabolic subalgebra $\mathfrak{q}$ to $\mathfrak{p}$ defines a unique isomorphism $\mathfrak{g} \cong gr_{\mathfrak{p}}(\mathfrak{g})$ such that:

$$\mathfrak{g}_i := \mathfrak{p}^{(i)} \cap \mathfrak{q}^{(-i)} \cong \mathfrak{p}^{(i)}/\mathfrak{p}^{(i-1)}. \tag{2.1.5}$$

Then $[\mathfrak{g}_i, \mathfrak{g}_j] \leq \mathfrak{g}_{i+j}$ so that $\mathfrak{g} = \bigoplus_{i=-n}^{n} \mathfrak{g}_i$ is a graded Lie algebra. Similarly, the grading element $\xi^{\mathfrak{p}} \in \mathfrak{p}/\mathfrak{p}^{\perp}$ lifts to a grading element $\xi^{\mathfrak{p}}_{\mathfrak{q}} \in \mathfrak{g}_0$ [26, Proposition 3.8].

Proposition 2.1.4 (c.f [26, Remark 3.15])**.** *The subgroup* $\exp(\mathfrak{p}^{\perp}) \leq G$ *acts simply transitively on parabolic subalgebras complementary to* $\mathfrak{p}$*. Equivalently, it acts simply transitively on lifts of the grading element* $\xi_{\mathfrak{p}}$*.*

Thus given a pair of complementary parabolic subalgebras $\mathfrak{p}, \mathfrak{q} \leq \mathfrak{g}$ we obtain charts:

$$\mathfrak{p}^{\perp} \to \Omega_{\mathfrak{p}}; X \mapsto \exp(X)\mathfrak{q}, \qquad \mathfrak{q}^{\perp} \to \Omega_{\mathfrak{q}}; Y \mapsto \exp(Y)\mathfrak{p}.$$

We call these maps inverse-stereoprojection with respect to $(\mathfrak{p}, \mathfrak{q})$.

Proposition 2.1.5. [17, Lemma 1.6] *Parabolic subalgebras are self-normalising. In other words, for* $X \in \mathfrak{g}$*,* $[X, \mathfrak{p}] \subset \mathfrak{p}$ *if, and only if,* $X \in \mathfrak{p}$*.*

2.2 R-spaces and symmetric R-spaces

Definition 2.2.1. [17, Definitions 2.1 and 2.2] *Let R denote a conjugacy class of parabolic subalgebras of height n. We call R an* ***R-space*** *of height n. We say that R is* ***irreducible*** *if $\mathfrak{g}$ is simple, and R is* ***symmetric*** *if $n = 1$.*

As might be expected, an R-space for $\mathfrak{g}$ can be decomposed into products of irreducible R-spaces for each of the simple ideals of $\mathfrak{g}$. Thus we may, without loss of generality, reduce to the case that $\mathfrak{g}$ is simple. The conjugacy classes of parabolic subalgebras can now be enumerated by subsets of nodes of the Dynkin[1] diagram of $\mathfrak{g}$. We describe this construction but see also: [4, Section 2.2]. Let $\mathfrak{g}$ be complex and let $\mathfrak{h}$ be a Cartan subalgebra. The nodes of the Dynkin diagram then correspond to the simple roots Δ' of $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$, after some choice of partial order on the roots. All choices of $\mathfrak{h}$ and orderings on the roots are conjugate under the action of $\mathfrak{g}$, so we may make these choices without loss of generality. Let $\Phi \subset \Delta'$, which we will denote by crossing the corresponding nodes on the Dynkin diagram. Any

[1] or Satake

root can be written as: $\sum_{\alpha\in\Delta'} m_\alpha \alpha$ where each $m_\alpha \in \mathbb{Z}$ and are either all positive or all negative. We denote by $\Phi^\pm$ the sets of roots where $m_\alpha \in \mathbb{Z}_\pm$ for each $\alpha \in \Phi$. Then, let

$$\mathfrak{p}_\Phi := \mathfrak{h} \oplus \bigoplus_{\beta \notin \Phi^-} \mathfrak{g}^\beta. \tag{2.2.1}$$

By [40, Proposition 8.1], $\mathfrak{g}^{\beta_1} \perp \mathfrak{g}^{\beta_2}$ unless $\beta_1 = -\beta_2$ and $\mathfrak{h} \perp \mathfrak{g}^\beta$ for any roots $\beta, \beta_1, \beta_2 \in \Delta$. Thus:

$$\mathfrak{p}_\Phi^\perp = \bigoplus_{\beta \in \Phi^+} \mathfrak{g}^\beta. \tag{2.2.2}$$

This is a nilpotent subalgebra of $\mathfrak{p}$ and thus $\mathfrak{p}$ is parabolic. Note that $\mathfrak{p}^\perp$ contains the root spaces of each of the simple roots in Φ as well as the highest root. Thus, the height of $\mathfrak{p}$ can be seen to be the sum of the coefficients in the highest root of each of the simple roots in Φ. In particular, $\mathfrak{p}$ has height 1 if Φ consists of a single root with coefficient 1 in the highest root. For $\mathfrak{g}$ semisimple, any parabolic subalgebra $\mathfrak{p} \leq \mathfrak{g}$ is the sum of parabolic subalgebras $\mathfrak{p}_i \leq \mathfrak{g}_i$ in each simple ideal, $\mathfrak{g}_i \leq \mathfrak{g}$. Then the height of $\mathfrak{p}$ is the maximum of the heights of the $\mathfrak{p}_i$.

For reference we include a table of the Dynkin diagrams with the coefficients of each simple root in the highest root marked (See Table 2.1)

For $\mathfrak{g}$ non-complex, we may perform a similar construction using the Satake diagram. We proceed as before with the caveat that the crossed nodes must be white and if a crossed node is joined to another by an arrow then that node must be crossed as well. Indeed, if $\mathfrak{p} \leq \mathfrak{g}$ is parabolic then $\mathfrak{p}^\mathbb{C} \leq \mathfrak{g}^\mathbb{C}$ is parabolic as well. If $R, R^\mathbb{C}$ denote the conjugacy classes of $\mathfrak{p}, \mathfrak{p}^\mathbb{C}$, respectively, then $R \hookrightarrow R^\mathbb{C} : \mathfrak{p} \mapsto \mathfrak{p}^\mathbb{C}$ embeds R as the fixed space of the involution induced by conjugation in $\mathfrak{g}^\mathbb{C}$ [44].

2.3 Duality and the space of complementary pairs

Definition 2.3.1. [17, Definition 2.4] *Let R be an R-space and define:*

$$R^* := \{\mathfrak{q} \leq \mathfrak{g} | \mathfrak{p}, \mathfrak{q} \textit{ complementary for some } \mathfrak{p} \in R\}. \tag{2.3.1}$$

*We call R^* the **dual space** to R. If $R = R^*$, we say R is **self-dual**.*

If $\mathfrak{p}, g \cdot \mathfrak{p} \in R$ and $\mathfrak{q}, \mathfrak{q}'$ are complementary to $\mathfrak{p}, g \cdot \mathfrak{p}$ respectively then $g^{-1} \cdot \mathfrak{q}'$ is complementary to $\mathfrak{p}$. Then by Proposition 2.1.4 there is a unique $h \in \exp(\mathfrak{p}^\perp)$ such that $\mathfrak{q} = hg^{-1} \cdot \mathfrak{q}'$. Therefore we have:

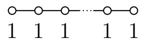
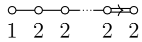
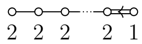
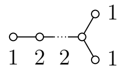
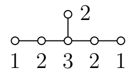
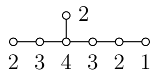
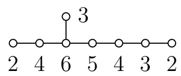
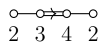
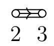
Root system	Dynkin diagram
A_n	
B_n	
C_n	
D_n	
E_6	
E_7	
E_8	
F_4	
G_2	

Table 2.1: Dynkin diagrams with root coefficients in highest root

Proposition 2.3.2. [17, Proposition 2.3] *R^* is an R-space of the same height as R.*

Lemma 2.3.3. [17, Section 2.1] *The set $\Omega_{\mathfrak{p}}$ of parabolic subalgebras complementary to $\mathfrak{p} \in R$ is dense and open in R^*. Moreover, the sets $\{\Omega_{\mathfrak{p}} | \mathfrak{p} \in R\}$ cover R^*.*

There is a natural reductive homogeneous space associated to R and R^* that, in the case R is of height 1 is a pseudo-Riemannian symmetric space.

Definition 2.3.4. [17, Section 2.4] *Let R be an R-space with dual space R^*. Define:*

$$Z_R := \{(\mathfrak{p}, \mathfrak{q}) \in R \times R^* | \mathfrak{p}, \mathfrak{q} \text{ are complementary}\}. \tag{2.3.2}$$

*We call Z_R the **space of complementary pairs**.*

2.3.1 Homogeneous geometry and symmetric spaces

Definition 2.3.5. [23, Chapter 1] *Let G/H be a homogeneous space. Then it is a **symmetric space** if there exists an involution σ of G such that H is open in the fixed set of σ. In Lie algebraic terms, if $\mathfrak{g}, \mathfrak{h}$ are the Lie algebras of G, H respectively, then G/H is symmetric if $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that:*

1. $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$.

2. $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$.

If only the first condition is satisfied we call G/H a ***reductive*** *homogeneous space. We refer to $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ as the symmetric (resp. reductive) decomposition.*

As a few examples we note the following:

Proposition 2.3.6. [17, Section 2.2] *Let R be a symmetric R-space. Then R is a symmetric space for a maximal compact subgroup $K \leq G$. That is, $R \cong K/(K \cap P)$ and $K/(K \cap P)$ is a Riemannian symmetric space.*

Proposition 2.3.7. [17, Proposition 2.5] *Let R be an R-space. Then Z_R is a pseudo-Riemannian reductive homogeneous space. If R is a symmetric R-space then Z_R is symmetric.*

To understand differentiation on homogeneous spaces we use the solder form:

Definition 2.3.8. [23, Chapter 1] *Let N be a homogeneous space with H_x the stabiliser of x and $\mathfrak{h}_x$ its Lie algebra. The* ***solder form*** *is the isomorphism $\beta^N_x : T_xN \cong \mathfrak{g}/\mathfrak{h}_x$ defined as the inverse of the map:*

$$X + \mathfrak{h}_x \mapsto \left.\frac{\partial}{\partial t}\right|_{t=0} \exp(tX)\mathfrak{h}_x. \tag{2.3.3}$$

By considering $\mathfrak{h} \leq \underline{\mathfrak{g}} = \mathfrak{g} \times N$ as the bundle over N with fibres $\mathfrak{h}_x$, the solder form is the bundle isomorphism $TN \cong \underline{\mathfrak{g}}/\mathfrak{h}$.

Using the Killing form we obtain a dual isomorphism $T^*N \cong (\underline{\mathfrak{g}}/\mathfrak{h})^* \cong \mathfrak{h}^\perp$. We may apply this reasoning to both R and Z_R.

The (infinitesimal) stabiliser of $\mathfrak{p} \in R$ is precisely $\mathfrak{p}$ and so the solder form identifies $TR \cong \underline{\mathfrak{g}}/\mathfrak{p}$ and $T^*R \cong \mathfrak{p}^\perp$.

We can also characterise the solder form as follows:

Lemma 2.3.9. [17, Lemma 2.6] *Let $s \in \Gamma\mathfrak{h}$. Then:*

$$ds \equiv [\beta^N, s] \mod \mathfrak{h}. \tag{2.3.4}$$

Moreover if $\mathfrak{h}$ is self-normalising, this uniquely determines β^N.

Furthermore, when N is reductive we have the decomposition $\underline{\mathfrak{g}} = \mathfrak{h} \oplus \mathfrak{m}$ for some $\mathrm{ad}_{\mathfrak{h}}$-invariant subbundle $\mathfrak{m}$. Therefore, we can further identify $TN \cong \underline{\mathfrak{g}}/\mathfrak{h} \cong \mathfrak{m}$. Thus we may view β^N as an $\mathfrak{m}$-valued 1-form and flat differentiation on $\underline{\mathfrak{g}}$ splits as:

$$d = \mathscr{D} + \beta^N, \tag{2.3.5}$$

where $\mathscr{D}$ restricts to a connection on $\mathfrak{h}$.

An important construction for symmetric spaces is the Cartan subspace:

Definition 2.3.10. [42, Corollary 5.1] *Let N be a symmetric space with symmetric decomposition $\mathfrak{g} = \mathfrak{h}_x \oplus \mathfrak{m}_x$ at $x \in N$. A **Cartan subspace** of $\mathfrak{m}_x$ is a maximal abelian subspace consisting of semisimple elements.*

Lemma 2.3.11. [42, Theorem 4.1 and Corollary 4.2] *All Cartan spaces of $\mathfrak{m}_x$ have the same dimension. If $\mathfrak{g}$ is complex then all Cartan subspaces of $\mathfrak{m}_x$ are conjugate.*

Definition 2.3.12. [42, Definition 4.1] *The **rank** of a symmetric space is the dimension of a Cartan subspace.*

Let R be a symmetric R-space and let $(\mathfrak{p}, \mathfrak{q}) \in Z_R$. We can construct Cartan subspaces of Z_R explicitly from root data: Let $\mathfrak{h}$ be a Cartan subspace of $\mathfrak{g}$ containing $\xi_{\mathfrak{p}}^{\mathfrak{q}}$. Then $\mathfrak{h} \leq \mathfrak{p} \cap \mathfrak{q}$ and we denote by Δ the corresponding set of roots. Note if we set $\Delta_i = \{\alpha \in \Delta | \alpha(\xi_{\mathfrak{p}}^{\mathfrak{q}}) = i\}$ then we have a decomposition:

$$\Delta = \Delta_{-1} \sqcup \Delta_0 \sqcup \Delta_1. \tag{2.3.6}$$

Definition 2.3.13. [1, Definition 1.5] *We say $\alpha, \beta \in \Delta$ are **strongly orthogonal** if $\alpha \pm \beta \notin \Delta \cup \{0\}$.*

Proposition 2.3.14. [35, Proposition 7.4] *Let $\mathfrak{g} =$ be complex. Let $\{\beta_1, \ldots, \beta_r\} \subset \Delta_1$ be a maximal subset of strongly orthogonal long roots. Let $E_i \in \mathfrak{g}^{\beta_i}, E_{-i} \in \mathfrak{g}^{-\beta_i}$. Then:*

$$\mathfrak{c} := \{E_i + E_{-i} | i = 1, \ldots, r\}, \tag{2.3.7}$$

is a Cartan subspace of $\mathfrak{p}^\perp \oplus \mathfrak{q}^\perp$.

In particular we note that $\operatorname{rank} Z_R = r$ the cardinality of $\{\beta_1, \ldots, \beta_r\}$.

2.4 Examples of symmetric R-spaces

In this section, we explore the practical results of the classification[2] in Section 2.2 by describing each of the infinite families of irreducible symmetric

[2]That is, they correspond to a single simple root with coefficient 1 in the highest root

R-spaces. These are presented by considering each of the 5 families[3] of Hermitian symmetric spaces and then describing their real forms. There remain 6 exceptional symmetric R-spaces not included in this list including the Cayley projective plane and its complexification.

2.4.1 Grassmannians

A prototypical example of a symmetric R-space is given by the Grassmannian of k-planes in $\mathbb{C}^n$. The Lie group $G = \mathrm{PSL}(n, \mathbb{C})$ acts transitively on $G_k(\mathbb{C}^n)$ for all $k \in \{1, \ldots, n-1\}$. The infinitesimal stabiliser of a k-plane in $\mathbb{C}^n$ is a parabolic subalgebra of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, whose nilradical is abelian. Indeed for $V \in G_k(\mathbb{C}^n)$:

$$\mathrm{stab}(V)^\perp = \mathrm{Hom}(\mathbb{C}^n/V, V) = \{\varphi \in \mathrm{End}(\mathbb{C}^n) | \operatorname{Im}\varphi \le V \le \operatorname{Ker}\varphi\}. \quad (2.4.1)$$

This map $V \mapsto \mathrm{stab}(V)$ is a G-isomorphism and identifies $G_k(\mathbb{C}^n)$ with a symmetric R-space.

If we choose $W \in G_{n-k}(\mathbb{C}^n)$ such that $V \oplus W = \mathbb{C}^n$ we can see that $\mathrm{stab}(V), \mathrm{stab}(W)$ are complementary. In fact, $(G_k(\mathbb{C}^n))^* = G_{n-k}(\mathbb{C}^n)$ and in particular, $G_k(\mathbb{C}^n)$ is self-dual precisely when $n = 2k$. The grading associated to V, W has grading element:

$$\xi_W^V = \begin{cases} \frac{k}{n} & \text{on } V, \\ \frac{k-n}{n} & \text{on } W. \end{cases} \quad (2.4.2)$$

The real forms of $\mathfrak{g}$ can all be described in terms of real, quaternionic or Hermitian structures on $\mathbb{C}^n$ and so we can use this to describe the real forms of $G_k(\mathbb{C}^n)$. Firstly, a real or quaternionic structure is a complex antilinear endomorphism j on $\mathbb{C}^n$ such that $j^2 = 1$ or $j^2 = -1$ respectively. If $j^2 = 1$, restricting $G_k(\mathbb{C}^n)$ to those k-planes which are j-stable yields the real Grassmannian $G_k(\mathbb{R}^n)$. If $j^2 = -1$, this identifies $\mathbb{C}^{2n}$ with $\mathbb{H}^n$. Again the j-stable $2k$-planes form a symmetric R-space $G_k(\mathbb{H}^n)$. Finally, if we fix a Hermitian inner product on $\mathbb{C}^{2n}$ we obtain the corresponding Hodge star operation $* \in \mathrm{End}(\bigwedge^n \mathbb{C}^{2n})$ which preserves decomposable forms and so $G_n(\mathbb{C}^{2n})$ via the Plücker embedding. Moreover, if the inner product has signature (n, n) then the $*$-stable lines correspond to the maximal isotropic subspaces. We denote this space $G^\perp(\mathbb{C}^{n,n})$.

[3]We consider the quadrics in even and odd dimension together

2.4.2 Quadrics

Equip $\mathbb{C}^{n+2}$ with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$. Consider the lightcone:

$$\mathcal{L} := \{x \in \mathbb{C}^{n+2} | (x,x) = 0\}. \tag{2.4.3}$$

Then $G = \mathrm{Ad}\ SO(n+2,\mathbb{C})$ acts transitively on $Q^n = \mathbb{P}(\mathcal{L})$, the complex projective quadric. We may identify $\mathfrak{g} = \mathfrak{so}(n,\mathbb{C})$ with $\bigwedge^2 \mathbb{C}^n$:

$$v \wedge w \mapsto (v,\cdot)w - (w,\cdot)v. \tag{2.4.4}$$

Then for any null line $\Lambda \leq \mathbb{C}^n$ its infinitesimal stabiliser is:

$$\mathrm{stab}(\Lambda) = \Lambda \wedge \mathbb{C}^{n+2} + \textstyle\bigwedge^2 \Lambda^\perp, \tag{2.4.5}$$

with abelian nilradical:

$$\mathrm{stab}(\Lambda)^\perp = \Lambda \wedge \Lambda^\perp. \tag{2.4.6}$$

Again, $\Lambda \mapsto \mathrm{stab}(\Lambda)$ is a G-equivariant map into a symmetric R-space.

Naturally $\mathrm{stab}(\Lambda)$ also preserves $\Lambda^\perp$ and indeed we could also think of this symmetric R-space as the flags of the form $\Lambda \leq \Lambda^\perp$. It is then clear that the complementary parabolic subalgebras are precisely the stabilisers of null lines $\hat{\Lambda} \not\leq \Lambda^\perp$. The grading element corresponding to a choice of complementary $\Lambda, \hat{\Lambda}$ is:

$$\xi^{\Lambda}_{\hat{\Lambda}} = \begin{cases} 1 & \text{on } \Lambda, \\ 0 & \text{on } (\Lambda \oplus \hat{\Lambda})^\perp, \\ -1 & \text{on } \hat{\Lambda}. \end{cases} \tag{2.4.7}$$

It is clear then to see $\xi^{\Lambda}_{\hat{\Lambda}} = l \wedge \hat{l} \in \Lambda \wedge \hat{\Lambda}$ such that $(l,\hat{l}) = -1$.

Again the real forms of $\mathfrak{g}$ are identified by additional structure on $\mathbb{C}^n$. This takes the form of real or quaternionic structures j compatible with the inner product:

$$(jv, jw) = \overline{(v,w)}. \tag{2.4.8}$$

We obtain several possible real structures which can be distinguished by the signature of $\mathrm{Fix}(j)$. Thus we obtain $\mathbb{R}^{p+1,q+1} \leq \mathbb{C}^{n+2}$ and $\mathfrak{so}(p+1,q+1) \leq \mathfrak{so}(n+2,\mathbb{C})$ for each $p+q=n$. Correspondingly, we obtain the real projective quadrics in each signature $S^{p,q} = \mathbb{P}(\mathcal{L} \cap \mathbb{R}^{p+1,q+1})$. Note that a real subspace of the form $\mathbb{R}^{n+2} = \mathbb{R}^{n+2,0}$ intersects $\mathcal{L}$ trivially and the corresponding Lie algebra $\mathfrak{so}(n+2)$ is compact and does not have parabolic subalgebras. A quaternionic structure fixes the real Lie algebra $\mathfrak{so}^*(n+2)$ but does not preserve complex lines in $\mathbb{C}^{n+2}$ and so yields no real form of Q^n.

2.4.3 Isotropic Grassmannians

Consider $\mathbb{C}^{2n}$ equipped with a non-degenerate symmetric bilinear form $(\cdot,\cdot)$. The Hodge star operator on $\Lambda^n \mathbb{C}^{2n}$ commutes with the action of $G = \mathrm{Ad}\, SO(2n,\mathbb{C})$ and $*^2 = -1$. As in the above case of a Hermitian inner product, the Hodge star fixes each $\Lambda^n W$ for W maximal isotropic. Thus the set of maximal isotropic subspaces comprises two orbits of G in the ± 1-eigenspaces of $*$ and we denote these orbits $J^+(\mathbb{C}^{2n}), J^-(\mathbb{C}^{2n})$. For $V \in J^\pm(\mathbb{C}^{2n})$, it is a simple matter to verify:

$$\mathrm{stab}(V) = V \wedge \mathbb{C}^{2n},\ \mathrm{stab}(V)^\perp = \Lambda^2 V. \tag{2.4.9}$$

Since $\mathrm{stab}(V)^\perp$ is abelian the conjugacy class of $\mathrm{stab}(V)$ is a symmetric R-space and so by extension are $J^\pm(\mathbb{C}^{2n})$.

Similarly to the traditional Grassmannian, any parabolic subalgebra complementary to $\mathrm{stab}(V)$ is of the form $\mathrm{stab}(W)$ for W a maximal isotropic subspace complementary to V. If $V \in J^+(\mathbb{C}^{2n})$ and $W \in J^-(\mathbb{C}^{2n})$, then the dimension of $V \cap W$ has parity opposite to n. Thus we conclude that $J^+(\mathbb{C}^{2n}), J^-(\mathbb{C}^{2n})$ are each self-dual if n is even and are dual to each other if n is odd.

The associated grading element ξ^V_W is the same as for the subspaces considered in the traditional Grassmannian. However we may now view it as an element of $V \wedge W$. For any bases $v_1,\dots,v_n$ of V, $w_1,\dots,w_n$ of W chosen such that $(v_i, w_j) = -\delta_{i,j}$:

$$\xi^V_W = \sum_{i=1}^{n} v_i \wedge w_i. \tag{2.4.10}$$

Comparing to the example of the quadric above we can see that a real structure j on $\mathbb{C}^{2n}$ with $\mathrm{Fix}(j) \cong \mathbb{R}^{n,n}$ induces real forms $J^\pm(\mathbb{R}^{n,n})$. A quaternionic structure on $\mathbb{C}^{4n}$ induces a real form but only on one of the two orbits. We denote this $J(\mathbb{H}^{2n})$.

2.4.4 Lagrangian Grassmannians

Considering a symplectic form ω on $\mathbb{C}^{2n}$ we obtain a very similar example. Here $G = PSp(2n,\mathbb{C})$, the projective symplectic group, acts transitively on maximal isotropic subspaces. We call such subspaces Lagrangian and the set of these the Lagrangian Grassmannian $\mathrm{Lag}(\mathbb{C}^{2n})$. We have an identification for $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$ similar to (2.4.4):

$$S^2\mathbb{C}^{2n} \cong \mathfrak{sp}(2n,\mathbb{C}); v \odot w \mapsto \omega(v,\cdot)w + \omega(w,\cdot)v. \tag{2.4.11}$$

Let $V \in \mathrm{Lag}(\mathbb{C}^{2n})$, under this identification it has infinitesimal stabiliser:

$$\mathrm{stab}(V) = V \odot \mathbb{C}^{2n}, \tag{2.4.12}$$

with abelian nilradical:

$$\mathrm{stab}(V)^{\perp} = S^2 V. \tag{2.4.13}$$

Again parabolic stabilisers complementary to $\mathrm{stab}(V)$ are of the form $\mathrm{stab}(W)$ for $W \in \mathrm{Lag}(\mathbb{C}^{2n})$ and $V \oplus W = \mathbb{C}^{2n}$. Similarly to (2.4.10), for any bases $v_1, \ldots, v_n$ of V, $w_1, \ldots, w_n$ of W chosen such that $\omega(v_i, w_j) = \delta_{i,j}$:

$$\xi_W^V = \sum_{i=1}^{n} v_i \odot w_i. \tag{2.4.14}$$

Then for j a real structure on $\mathbb{C}^n$ such that:

$$\omega(jv, jw) = \overline{\omega}(v, w), \tag{2.4.15}$$

the j-stable subspaces form the real Lagrangian Grassmannian $\mathrm{Lag}(\mathbb{R}^{2n})$. A compatible quaternionic structure on $\mathbb{C}^{4n}$ identifies it with $\mathbb{H}^{2n}$ and the symplectic form is then a Hermitian inner product on $\mathbb{H}^{2n}$. If this inner product has signature (n, n) the maximal isotropic subspaces are another real form we shall denote $\mathrm{Lag}(\mathbb{H}^{2n})$.

2.5 Exceptional isomorphisms

To help develop familiarity with the less commonly used symmetric R-spaces, we shall note some isomorphisms between low dimensional examples. This will also provide some geometrical context in Section 4.3 where we will consider symmetric R-spaces that are products of irreducible low-dimensional symmetric R-spaces. These all arise from the exceptional isomorphisms of the Lie groups and algebras concerned and so can be seen immediately via isomorphisms of the Dynkin or Satake diagrams. We shall discuss them as conversions between different representations of the associated Lie algebras but there are many ways in which to see these relations. Most of the representations we use can be described as alternating or symmetric tensor powers of some basic representation. In these cases, the representation is given can be understood in the following way. If G is some Lie group acting on V then G acts on $\bigotimes^k V$ diagonally:

$$g(v_1 \otimes \cdots \otimes v_k) := gv_1 \otimes \cdots \otimes gv_k, \tag{2.5.1}$$

for $g \in G$ and $v_i \in V$. This differentiates to the Lie algebra action:

$$X(v_1 \otimes \cdots \otimes v_k) := Xv_1 \otimes \cdots \otimes v_k + \cdots + v_1 \otimes \cdots \otimes Xv_k, \qquad (2.5.2)$$

where $X \in \mathfrak{g}$. These actions commute with permutations of the v_i and so, for example, $\Lambda^k V$, $S^k V$ are submodules.

2.5.1 The Riemann sphere: $A_1 \cong B_1 \cong C_1$

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ act on $\mathbb{C}^2$ as tracefree endomorphisms. We note that $\Lambda^2(\mathbb{C}^2)^*$ is a 1-dimensional module and therefore trivial. Thus there is a single symplectic form $\omega \in \Lambda^2(\mathbb{C}^2)^*$ up to scale and the action of $\mathfrak{g}$ is skew-symmetric with respect to ω. On the Lie group level we note that $SL(n, \mathbb{C})$ always acts trivially on $\Lambda^n \mathbb{C}^n$ (and its dual) since:

$$\begin{aligned} g(v_1 \wedge \cdots \wedge v_n) &= gv_1 \wedge \cdots \wedge gv_n, \\ &= \det(g) v_1 \wedge \cdots \wedge v_n, \\ &= v_1 \wedge \cdots \wedge v_n. \end{aligned}$$

Thus $\mathfrak{g} \cong \mathfrak{sp}(2, \mathbb{C})$.

Now consider the adjoint action of $\mathfrak{g}$ on itself. The Killing form on $\mathfrak{g}$ is a natural symmetric bilinear form and the action of $\mathfrak{g}$ is skew-symmetric with respect to this form. Thus $\mathrm{ad} : \mathfrak{g} \to \mathfrak{so}(\mathfrak{g}) \cong \mathfrak{so}(\mathbb{C}^3)$ is a Lie algebra homomorphism and is injective as its kernel is an ideal of $\mathfrak{g}$ which must be $\{0\}$. Thus $\mathfrak{g} \cong \mathfrak{so}(3, \mathbb{C})$. In summary we have shown:

$$\mathfrak{sl}(2, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sp}(2, \mathbb{C}). \qquad (2.5.3)$$

In terms of the appropriate symmetric R-spaces this gives us G-isomorphisms:

$$\mathbb{P}(\mathbb{C}^2) \cong S^2 \cong \mathrm{Lag}(\mathbb{C}^2). \qquad (2.5.4)$$

The former is simply the description of the Riemann sphere. Since $\mathfrak{g}$ has two real forms, we obtain two corresponding isomorphisms:

$$\begin{aligned} \mathfrak{sl}(2, \mathbb{R}) &\cong \mathfrak{so}(2,1) \cong \mathfrak{sp}(2, \mathbb{R}). \\ \mathfrak{su}(2) &\cong \mathfrak{so}(3) \cong \mathfrak{sp}(1). \end{aligned}$$

The second of these is compact, but from the first we obtain a corresponding isomorphism of symmetric R-spaces:

$$\mathbb{P}(\mathbb{R}^2) \cong S^1 \cong \mathrm{Lag}(\mathbb{R}^2). \qquad (2.5.5)$$

2.5.2 The Klein Correspondence: $A_3 \cong D_3$

Let $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{C})$ and consider its action on $\Lambda^2 \mathbb{C}^4$. There is a canonical symmetric bilinear form, up to scale, on $\Lambda^2 \mathbb{C}^4$ given by:

$$(v \wedge w, x \wedge y) = v \wedge w \wedge x \wedge y \in \Lambda^4 \mathbb{C}^4 \cong \mathbb{C}. \tag{2.5.6}$$

As in the previous example we can see that the action of $\mathfrak{g}$ is skew-symmetric since $\mathfrak{g}$ acts trivially on $\Lambda^4 \mathbb{C}^4$. Thus $\mathfrak{g} \leq \mathfrak{so}(6)$ and since they have the same dimension we conclude that they are equal:

$$\mathfrak{sl}(4, \mathbb{C}) \cong \mathfrak{so}(6, \mathbb{C}). \tag{2.5.7}$$

This gives us several symmetric R-space isomorphisms which we can also see directly:

$$G_2(\mathbb{C}^4) \cong Q^4, \tag{2.5.8}$$
$$\mathbb{P}(\mathbb{C}^4), G_3(\mathbb{C}^4) \cong J^{\pm}(\mathbb{C}^6). \tag{2.5.9}$$

The first is simply the complex Klein correspondence and can be seen as the map $V \mapsto \Lambda^2 V$. The latter two are then the realisations of the α and β planes in the Klein quadric. We send each line $L \leq \mathbb{C}^4$ to $L \wedge \mathbb{C}^4$ whose 1-dimensional subspaces represent all the planes containing L and similarly we send each hyperplane $H \leq \mathbb{C}^4$ to $\Lambda^2 H$ whose 1-dimensional subspaces represent all the planes contained in H. These form two disjoint families of isotropic 3-dimensional planes.

The split real form of $\mathfrak{g}$ is then:

$$\mathfrak{sl}(4, \mathbb{R}) \cong \mathfrak{so}(3, 3), \tag{2.5.10}$$

with similar symmetric R-space isomorphisms:

$$G_2(\mathbb{R}^4) \cong S^{2,2}, \tag{2.5.11}$$
$$\mathbb{P}(\mathbb{R}^4), G_3(\mathbb{R}^4) \cong J^{\pm}(\mathbb{R}^{3,3}). \tag{2.5.12}$$

This is precisely the real Klein correspondence and is identical *mutatis mutandis* to the complex case.

The other non-compact real forms are then:

$$\mathfrak{sl}(2, \mathbb{H}) \cong \mathfrak{so}(5, 1), \tag{2.5.13}$$
$$\mathfrak{su}(2, 2) \cong \mathfrak{so}(4, 2), \tag{2.5.14}$$
$$\mathfrak{su}(3, 1) \cong \mathfrak{so}^*(6). \tag{2.5.15}$$

The first case follows from the observation that a quaternionic structure j on $\mathbb{C}^4$ gives a real structure on $\Lambda^2 \mathbb{C}^4$ via:

$$j(v \wedge w) := jv \wedge jw. \tag{2.5.16}$$

The maximal isotropic subspaces stable under this real structure are 1-dimensional and so we get the isomorphism:

$$\mathbb{P}(\mathbb{H}^2) \cong S^4. \tag{2.5.17}$$

This identification forms the basis for the Quaternionic Holomorphic Geometry approach to S^4 (c.f. [36, Section 4.3]). For the other cases we note that a Hermitian form on $\mathbb{C}^4$ induces a Hodge star operator on $\Lambda^2 \mathbb{C}^4$. When the signature of this Hermitian form is $(2,2)$ the Hodge star is a real structure, $*^2 = 1$. The fixed set of $*$ then has signature $(4,2)$ and so we obtain an isomorphism:

$$G^{\perp}(\mathbb{C}^{2,2}) \cong S^{3,1}. \tag{2.5.18}$$

We note that the second term here is the Lie quadric and while this quadric does not have 3-dimensional isotropic subspaces, the geometry of its 2-dimensional isotropic subspaces is important in Lie sphere geometry and these form a (non-symmetric) R-space isomorphic to flags in $\mathbb{C}^{2,2}$ of the form $L \leq L^{\perp}$ for L a null line.

If instead we assume our Hermitian structure has signature $(3,1)$ the Hodge star is a quaternionic structure. However, we can see that there are no symmetric R-spaces in this example. Indeed, a quaternionic structure cannot preserve a complex line.

2.5.3 The "symplectic" Klein correspondence: $B_2 \cong C_2$

Here we proceed as in the previous example but we assume there is a distinguished symplectic form on $\mathbb{C}^4$, $\omega \in \Lambda^2(\mathbb{C}^4)^*$ with $\mathfrak{g} = \mathfrak{sp}(4,\mathbb{C})$ the corresponding Lie algebra. Let $W = \Lambda^2_o \mathbb{C}^4$ denote the subset of $\Lambda^2 \mathbb{C}^4$ which contracts to 0 with ω. Then W is 5-dimensional and the symmetric bilinear form on $\Lambda^2 \mathbb{C}^4$ descends to W. Now, $\mathfrak{sp}(4,\mathbb{C}) \leq \mathfrak{sl}(4,\mathbb{C})$ and it preserves W so we see that $\mathfrak{sp}(4,\mathbb{C}) \leq \mathfrak{so}(5,\mathbb{C})$. Again, we can see this is an equality by comparing dimensions or by considering the spin representation of $\mathfrak{so}(5,\mathbb{C})$. Broadly speaking, the spin representation is 4-dimensional and there is a natural symplectic form for which this action is skew. In conclusion, we see that:

$$\mathfrak{so}(5,\mathbb{C}) \cong \mathfrak{sp}(4,\mathbb{C}), \tag{2.5.19}$$

and the resulting isomorphism of symmetric R-spaces is:

$$Q^3 \cong \mathrm{Lag}(\mathbb{C}^4). \tag{2.5.20}$$

Said another way this isomorphism means that the restriction from $G_2(\mathbb{C}^4)$ to isotropic planes for some symplectic form is equivalent to the choice of a quadric hypersurface in Q^4. There are two real forms of note:

$$\mathfrak{so}(3,2) \cong \mathfrak{sp}(4,\mathbb{R}),$$
$$\mathfrak{so}(4,1) \cong \mathfrak{sp}(2,2).$$

Then the symmetric R-spaces isomorphisms are:

$$S^{2,1} \cong \mathrm{Lag}(\mathbb{R}^4), \tag{2.5.21}$$
$$S^3 \cong \mathrm{Lag}(\mathbb{H}^2). \tag{2.5.22}$$

The former we simply understand as the equivalent real construction restricting to a quadric hypersurface in the real Klein quadric. The latter however can be viewed through the lens of Quaternionic Holomorphic Geometry. A choice of Hermitian form of signature $(2,2)$ on $\mathbb{H}^2$ fixes a hypersphere in $S^4 \cong \mathbb{P}(\mathbb{H}^2)$ (c.f. [36, Lemma 4.3.12]).

2.5.4 The torus and the sphere $A_1 \times A_1 \cong D_2$

Firstly, we note that if $\mathfrak{g}_1, \mathfrak{g}_2$ are two Lie algebras with modules V_1, V_2 then $V_1 \otimes V_2$ is a module for $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ with action $(X_1 + X_2)(v_1 \otimes v_2) = X_1(v_1) \otimes v_2 + v_1 \otimes X_2(v_2)$ for $X_i \in \mathfrak{g}_i, v_i \in V_i$. We consider $\mathfrak{g}_1 = \mathfrak{sl}(2,\mathbb{C}) = \mathfrak{g}_2$ with modules $V_1 \cong \mathbb{C}^2 \cong V_2$. Now $V_1 \otimes V_2$ can be considered as a subspace of $\Lambda^2(V_1 \oplus V_2) \cong \Lambda^2 \mathbb{C}^4$ and inherits a non-degenerate $\mathfrak{g}_1 \oplus \mathfrak{g}_2$-invariant symmetric bilinear form, up to scale, from the wedge product as before. Thus we see that $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \leq \mathfrak{so}(\mathbb{C}^4)$ and this is an equality by comparing dimensions:

$$\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(\mathbb{C}^4). \tag{2.5.23}$$

The non-compact real forms are then:

$$\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2,2), \tag{2.5.24}$$
$$\mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(3,1). \tag{2.5.25}$$

The first of these is the split form given by real structures on V_1, V_2. The complex and split form give us natural isomorphisms of the complex and real torus:

$$\mathbb{P}(\mathbb{C}^2) \times \mathbb{P}(\mathbb{C}^2) \cong Q^2, \tag{2.5.26}$$
$$\mathbb{P}(\mathbb{R}^2) \times \mathbb{P}(\mathbb{R}^2) \cong S^{1,1}. \tag{2.5.27}$$

They also give realisations of the maximal isotropic Grassmannians:

$$\mathbb{P}(\mathbb{C}^2) \cong J^{\pm}(\mathbb{C}^4), \mathbb{P}(\mathbb{R}^2) \cong J^{\pm}(\mathbb{R}^{2,2}). \tag{2.5.28}$$

Finally, the second real form comes from the consideration of V_2 as the complex conjugate of V_1. Similarly then $\mathfrak{g}_2 = \overline{\mathfrak{g}_1}$ so that we are considering $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ as a complexification of $\mathfrak{sl}(2,\mathbb{C})$. The symmetric R-space isomorphism here is then just the realisation of the 1-dimensional complex quadric as the sphere:

$$Q^1 \cong S^2. \tag{2.5.29}$$

2.6 The generalised conformal structure

Throughout this section, we assume R is a self-dual symmetric R-space unless otherwise stated. We will define the *regular* elements of the tangent (and cotangent) space. These will then be used to construct special one dimensional submanifolds known as circles. Then we will use them to define the generalised conformal structure of R, a canonical flat G-invariant assignment of a prehomogeneous cone to the tangent space (or cotangent space) of R.

2.6.1 Regular elements and the inversion map

Recall that $\mathfrak{p} \in R$ defines a graded Lie algebra $gr_{\mathfrak{p}}(\mathfrak{g}) = \mathfrak{p}^{\perp} \oplus \mathfrak{p}/\mathfrak{p}^{\perp} \oplus \mathfrak{g}/\mathfrak{p}$ (c.f. (2.1.2)). This has Lie bracket as described in (2.1.3). That is $\mathfrak{p}^{\perp}$, $\mathfrak{g}/\mathfrak{p}$ are abelian, while $[\mathfrak{p}^{\perp}, \mathfrak{g}/\mathfrak{p}] \subset \mathfrak{p}/\mathfrak{p}^{\perp}$, $[\mathfrak{p}/\mathfrak{p}^{\perp}, \mathfrak{p}^{\perp}] = \mathfrak{p}^{\perp}$ and $[\mathfrak{p}/\mathfrak{p}^{\perp}, \mathfrak{g}/\mathfrak{p}] = \mathfrak{g}/\mathfrak{p}$.

$$P : \mathfrak{p}^{\perp} \to \mathrm{Hom}(\mathfrak{g}/\mathfrak{p}, \mathfrak{p}^{\perp}); X \mapsto \frac{1}{2}\mathrm{ad}_X^2. \tag{2.6.1}$$

In particular we see that for $X \in \mathfrak{p}^{\perp}, Y \in \mathfrak{g}/\mathfrak{p}$:

$$\exp(X)(Y) = Y + [X, Y] + P(X)Y. \tag{2.6.2}$$

Symmetrically, we define another function (which we will also call P):

$$P : \mathfrak{g}/\mathfrak{p} \to \mathrm{Hom}(\mathfrak{p}^{\perp}, \mathfrak{g}/\mathfrak{p}); Y \mapsto \frac{1}{2}\mathrm{ad}_Y^2. \tag{2.6.3}$$

Definition 2.6.1. [31, Section 7.1.2] *We call $X \in \mathfrak{p}^{\perp}$ (resp. $Y \in \mathfrak{g}/\mathfrak{p}$)* ***regular*** *if $P(X) : \mathfrak{g}/\mathfrak{p} \to \mathfrak{p}^{\perp}$ (resp. $P(Y) : \mathfrak{p}^{\perp} \to \mathfrak{g}/\mathfrak{p}$) is an isomorphism. We denote the set of these $\mathfrak{p}^{\perp}_{reg}$ (resp. $(\mathfrak{g}/\mathfrak{p})_{reg}$).*

We note that $X \in \mathfrak{p}^\perp$ is regular if, and only if, $\operatorname{Ker}\operatorname{ad}_X^2 = \mathfrak{p} \leq \mathfrak{g}$. Let $\mathfrak{q}, \mathfrak{r} \leq \mathfrak{g}$ be parabolic subalgebras both complementary to $\mathfrak{p}$, $\mathfrak{r} = \exp(X)\mathfrak{q}$ for a unique $X \in \mathfrak{p}^\perp$ Proposition 2.1.4. Then:

Lemma 2.6.2. [17, Lemma 4.1] $\mathfrak{q}, \mathfrak{r} = \exp(X)\mathfrak{q} \in \Omega_\mathfrak{p}$ *are complementary to each other if and only if X is regular.*

Now $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ are pairwise complementary and so we may equally write $\mathfrak{r} = \exp(Y)\mathfrak{p}$ for a unique $Y \in \mathfrak{q}^\perp_{reg}$. Thus there exists a natural bijection between $\mathfrak{p}^\perp_{reg}$ and $\mathfrak{q}^\perp_{reg}$ and with this as our motivation we define:

Definition 2.6.3. *Let $\mathfrak{p} \in R$. Let $X \in \mathfrak{g}/\mathfrak{p}$. We define $I_\mathfrak{p}(X) \in \mathfrak{p}^\perp$ as the unique element such that for any $\mathfrak{q} \in \Omega_\mathfrak{p}$ and $X_\mathfrak{q} \in \mathfrak{q}^\perp$ such that $X = X_\mathfrak{q} + \mathfrak{p}$:*

$$\exp(I_\mathfrak{p}(X))\mathfrak{q} = \exp(X_\mathfrak{q})\mathfrak{p}. \tag{2.6.4}$$

When $\mathfrak{q} \in \Omega_\mathfrak{p}$ is specified we will denote the induced map $\mathfrak{q}^\perp \to \mathfrak{p}^\perp$ by $I_\mathfrak{p}^\mathfrak{q}$. We call $I_\mathfrak{p}$ (or $I_\mathfrak{p}^\mathfrak{q}$) the ***inversion map****.*

Lemma 2.6.4. $I_\mathfrak{p} : (\mathfrak{g}/\mathfrak{p})_{reg} \to \mathfrak{p}^\perp_{reg}$ *is well defined.*

Proof. Recall from Proposition 2.1.4 that $\mathfrak{p}^\perp \to \Omega_\mathfrak{p}; X \mapsto \exp(X)\mathfrak{q}$ is an isomorphism. Thus for a given $\mathfrak{q}$, $I_\mathfrak{p}$ (and thus $I_\mathfrak{p}^\mathfrak{q}$) is uniquely defined by (2.6.4). We demonstrate that $I_\mathfrak{p}$ does not depend on our choice of $\mathfrak{q}$. If $\mathfrak{r}$ is another parabolic subalgebra complementary to $\mathfrak{p}$, $\mathfrak{r} = \exp(Y)\mathfrak{q}$ for a unique $Y \in \mathfrak{p}^\perp$ by Proposition 2.1.4. Thus since $Y, I_\mathfrak{p}(X) \in \mathfrak{p}^\perp$:

$$\begin{aligned}\exp(I_\mathfrak{p}(X))\mathfrak{r} &= \exp(I_\mathfrak{p}(X))\exp(Y)\mathfrak{q},\\ &= \exp(Y)\exp(I_\mathfrak{p}(X))\mathfrak{q}.\end{aligned}$$

Let $X_\mathfrak{r}$ be the representative of X in $\mathfrak{r}^\perp$. Then, $X_\mathfrak{r} = \exp(Y)X_\mathfrak{q}$. Note however that $\exp(gX_\mathfrak{q}) = g \circ \exp(X) \circ g^{-1}$ so that:

$$\begin{aligned}\exp(X_\mathfrak{r})\mathfrak{p} &= \exp(\exp(Y)X_\mathfrak{q}),\\ &= (\exp(Y) \circ \exp(X_\mathfrak{q}) \circ \exp(-Y))\,\mathfrak{p},\\ &= \exp(Y)\exp(X_\mathfrak{q})\mathfrak{p}.\end{aligned}$$

Thus:

$$\exp(I_\mathfrak{p}(X))\mathfrak{r} = \exp(X_\mathfrak{r})\mathfrak{p}. \tag{2.6.5}$$

Therefore $I_\mathfrak{p}(X)$ did not depend on the choice of $\mathfrak{q} \in \Omega_\mathfrak{p}$. □

This map will be important in several sections of this thesis and we collect a few of its properties here:

Proposition 2.6.5. *Let $X \in (\mathfrak{g}/\mathfrak{p})_{reg}$, $g \in G$.*

1. $[X, I_{\mathfrak{p}}(X)] = 2\xi_{\mathfrak{p}}$.
2. $I_{\mathfrak{p}}(X) = -P(X)^{-1}X = -\left(\frac{1}{2}\mathrm{ad}_X^2\right)^{-1} X$.
3. $I_{\mathfrak{p}}(tX) = \frac{1}{t} I_{\mathfrak{p}}(X)$.
4. $I_{g\cdot\mathfrak{p}}(g \cdot X) = g \cdot I_{\mathfrak{p}}(X)$.
5. $(X, I_{\mathfrak{p}}(X)) = 4 \dim R$.

Proof. For the first statement, take $\mathfrak{q} \in \Omega_{\mathfrak{p}}$ and $X_{\mathfrak{q}}$ the representative of X in $\mathfrak{q}^{\perp}$. Then [17, Proposition 4.3] says that $[X_{\mathfrak{q}}, I_{\mathfrak{p}}^{\mathfrak{q}}(X)] = 2\xi_{\mathfrak{p}}^{\mathfrak{q}}$. Thus, $[X, I_{\mathfrak{p}}(X)] = 2\xi_{\mathfrak{p}} = 2\xi_{\mathfrak{p}}^{\mathfrak{q}} + \mathfrak{p}^{\perp}$. The second statement then follows immediately by considering:

$$P(X)(I_{\mathfrak{p}}(X)) = \frac{1}{2}\mathrm{ad}_X[X, I_{\mathfrak{p}}(X)] = [X, \xi_{\mathfrak{p}}] = -X. \tag{2.6.6}$$

Then, since X is regular, $P(X)$ is invertible. Now $P(tX) = t^2 P(X)$ so that $I_{\mathfrak{p}}(tX) = \frac{1}{t} I_{\mathfrak{p}}(X)$. Similarly, $P(g \cdot X) = g \circ P(X) \circ g^{-1}$ for any $g \in G$ since $G \subset \mathrm{Aut}(\mathfrak{g})$. Thus,

$$I_{g\cdot\mathfrak{p}}(g \cdot X) = P(g \cdot X)^{-1} g \cdot X = g \cdot P(X)^{-1}X = g \cdot I_{\mathfrak{p}}(X). \tag{2.6.7}$$

Then, because ad_X is skew-adjoint for the Killing form:

$$\begin{aligned}
(X, I_{\mathfrak{p}}(X)) &= (-P(X)I_{\mathfrak{p}}(X), I_{\mathfrak{p}}(X)), \\
&= -\frac{1}{2}(\mathrm{ad}_X^2 I_{\mathfrak{p}}(X), I_{\mathfrak{p}}(X)), \\
&= \frac{1}{2}([X, I_{\mathfrak{p}}(X)], [X, I_{\mathfrak{p}}(X)]), \\
&= \frac{1}{2}(2\xi_{\mathfrak{p}}, 2\xi_{\mathfrak{p}}), \\
&= 4 \dim R.
\end{aligned}$$

The last step follows as $\mathrm{ad}_{\xi_{\mathfrak{p}}}^2 = \mathrm{id}_{\mathfrak{p}^{\perp} \oplus \mathfrak{g}/\mathfrak{p}}$ and so has trace $\dim \mathfrak{p}^{\perp} \oplus \mathfrak{g}/\mathfrak{p} = 2 \dim R$. □

Example. Let $V, W \in G_n(\mathbb{C}^{2n})$ be complementary and take $\mathfrak{p} := \mathrm{stab}(V)$, $\mathfrak{q} := \mathrm{stab}(W)$. Then $\mathfrak{p}^{\perp} \cong \mathrm{Hom}(W, V)$ and $\mathfrak{q}^{\perp} \cong \mathrm{Hom}(V, W)$. Let $X \in \mathfrak{p}^{\perp}$ and $Y \in \mathfrak{q}^{\perp}$. Now:

$$\mathrm{ad}_X^2(Y) = X^2Y - XYX - XYX + YX^2 = -2XYX, \tag{2.6.8}$$

so that:

$$P(X)(Y) = -XYX. \tag{2.6.9}$$

Then X is regular if it is invertible and for $X \in \mathfrak{p}^{\perp}_{reg}$:

$$I^{\mathfrak{p}}_{\mathfrak{q}}(X) = X^{-1}. \tag{2.6.10}$$

Example. Let $\Lambda, \hat{\Lambda} \in S^n$ be non-orthogonal and take $\mathfrak{p} := \mathrm{stab}(\Lambda), \mathfrak{q} := \mathrm{stab}(\hat{\Lambda})$. Then, for $W := (\Lambda \oplus \hat{\Lambda})^{\perp}$, $\mathfrak{p}^{\perp} \cong \Lambda \wedge W$, $\mathfrak{q}^{\perp} \cong \hat{\Lambda} \wedge W$. Choose $l \in \Lambda$, $\hat{l} \in \hat{\Lambda}$ such that $(l, \hat{l}) = -1$. Then for any $x, y \in W$:

$$\mathrm{ad}_{l\wedge x}(\hat{l} \wedge y) = (x, y) l \wedge \hat{l} - x \wedge y, \tag{2.6.11}$$

and consequently:

$$P(l \wedge x)(\hat{l} \wedge y) = \frac{1}{2}(x, x) l \wedge y - (x, y) l \wedge x. \tag{2.6.12}$$

Thus $l \wedge x$ is regular if x is not null. Then:

$$I^{\mathfrak{p}}_{\mathfrak{q}}(l \wedge x) = \frac{2}{(x, x)} \hat{l} \wedge x. \tag{2.6.13}$$

To interpret this we identify $\mathfrak{p}^{\perp}$ and $\mathfrak{q}^{\perp}$ with W using our choice of $l, \hat{l}$:

$$l \wedge x \mapsto x.$$
$$\hat{l} \wedge x \mapsto x.$$

Then $I^{\mathfrak{p}}_{\mathfrak{q}}(x) = \frac{2x}{(x,x)}$ and we recognise this as the Clifford algebra inverse of x multiplied by 2.

2.6.2 Circles in self-dual symmetric R-spaces

Burstall et al [17, Section 4.1] constructed a special class of curves, called circles, and these were explored further by Salvai [54]. These are invaluable in the permutation theory of isothermic submanifolds and will also prove vital in the semi-discretisation so we will describe their construction here.

Let $\mathfrak{p}, \mathfrak{q} \in R$ be complementary and let $X \in \mathfrak{q}^{\perp}_{reg}$. Then $t \mapsto \exp(tX)\mathfrak{p}$ describes a smooth curve $\mathbb{R} \to R$ which can be extended to $S^1 \cong \mathbb{R} \cup \{\infty\}$ by setting $\infty \mapsto \mathfrak{q}$. We call this curve a circle in R and denote it C.

Note that we could reparametrise this by instead considering $s \mapsto \exp(sI^{\mathfrak{q}}_{\mathfrak{p}}(X))\mathfrak{q}$. This defines the same curve as $\exp(sI^{\mathfrak{q}}_{\mathfrak{p}}(X)) = \exp(\frac{1}{s}X)$.

We can see immediately that the tangent space to this curve at $\exp(tX)\mathfrak{p}$ is given by $\langle X \rangle + \exp(tX)\mathfrak{p}$ for $t \neq \infty$.

Let:

$$\mathfrak{s} = \langle X, I^q_{\mathfrak{p}}(X), \xi^p_{\mathfrak{q}} = [X, I^q_{\mathfrak{p}}(X)]\rangle. \tag{2.6.14}$$

From [17, Proposition 4.3] or Proposition 2.6.5 above we can see that $\mathfrak{s} \cong \mathfrak{sl}(2,\mathbb{R})$. Then the set of Borel subalgebras $R_{\mathfrak{s}}$ of $\mathfrak{s}$ forms a symmetric R-space diffeomorphic to S^1. We can embed this in R by the map $\mathfrak{p}_{\mathfrak{s}} \mapsto \operatorname{Ker}\operatorname{ad}_{\mathfrak{p}_{\mathfrak{s}}^{\perp}}$ and the image of this is precisely the circle C. The analytic subgroup $S \leq G$ with Lie algebra $\mathfrak{s}$ acts transitively on C and on its image in R. This map is then equivariant under that action.

Each pair of distinct points on this circle is complementary by [17, Lemma 4.1]. Three distinct points determine the circle uniquely and indeed any three complementary points in R determine such a circle. Four points on the circle admit a cross ratio [17, Equation (4.4)]: Let $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_\infty, \in R$ and let $X \in \mathfrak{p}_\infty^\perp$ such that $\mathfrak{p}_1 = \exp(X)\mathfrak{p}_0$. Let $\mathfrak{p}_t := \exp(tX)\mathfrak{p}_0$. Then $C = \{\mathfrak{p}_t | t \in \mathbb{R} \cup \{\infty\}\}$ is the unique circle through $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_\infty$. The cross ratio is given by:

$$\operatorname{cr}(\mathfrak{p}_1, \mathfrak{p}_\infty, \mathfrak{p}_t, \mathfrak{p}_0) := t. \tag{2.6.15}$$

This cross ratio is invariant for the action of S.

2.6.3 Generalised conformal structure

Consider a manifold M equipped with a conformal structure. The conformal structure allows us to define, at each point, a complex quadratic cone of null tangent vectors. Conversely, a complex quadratic cone on the tangent space determines a conformal structure up to sign. Furthermore, the cone is the complement of the dense open orbit of the action of $CO\left((T_pM)^{\mathbb{C}}\right)$, the group of conformal transformations.

This is naturally described in the language of prehomogeneous vector spaces:

Definition 2.6.6 (c.f [56, Definition 2.1])**.** *Let G be an algebraic group acting on a vector space V. If V has a dense open orbit we call V a* ***prehomogeneous vector space****. Let $\mathscr{C}$ be the complement of this orbit. If $\mathscr{C}$ is conic (i.e. $\mathbb{F}^\times$-stable) then we call it a* ***prehomogeneous cone***

This leads to a natural generalisation of the conformal structure:

Definition 2.6.7. *A* ***generalised conformal structure*** *on M is a smooth assignment of a prehomogeneous cone to each $(T_pM)^{\mathbb{C}}$.*

On symmetric R-spaces there is a canonical choice due to Gindikin–Kaneyuki [32].

Definition 2.6.8. *Let R be a self-dual symmetric R-space. Let $\mathfrak{p} \in R$ and define:*

$$\mathscr{C}_{\mathfrak{p}} := \{X \in (T_{\mathfrak{p}}R)^{\mathbb{C}} = (\mathfrak{g}/\mathfrak{p})^{\mathbb{C}} | \operatorname{rank} P(X) \text{ not maximal}\}. \tag{2.6.16}$$

*Then, $\mathscr{C}$ is **the generalised conformal structure of** R. When R is self-dual this reads:*

$$\mathscr{C}_{\mathfrak{p}} := \{X \in (T_{\mathfrak{p}}R)^{\mathbb{C}} | \det P(X) = 0\}. \tag{2.6.17}$$

Firstly, we note, for R a self-dual symmetric R-space, $\mathscr{C}_{\mathfrak{p}}$ is simply the complement of $(T_{\mathfrak{p}}R)^{\mathbb{C}}_{reg}$ which is an open dense orbit (c.f [32, Theorem 2.9]). As justification of the assertion that this is a canonical choice we consider the following:

Proposition 2.6.9. [32, Theorem 2.9 and Lemma 3.2] *The generalised conformal structure $\mathscr{C}$ is G-invariant and locally diffeomorphic to the 'flat' generalised conformal structure $\mathscr{C}_{\mathfrak{p}} \subset T(\mathfrak{g}/\mathfrak{p})$.*

This structure will prove vital in later sections and so we shall investigate several of its properties. Henceforward, we consider only self-dual symmetric R-spaces.

Theorem 2.6.10. [53, Theorem 1.2] *Let $\mathfrak{g}$ be simple and complex, and $\mathfrak{p} \in R$ a point in a self-dual symmetric R-space. Let $P = \operatorname{Stab}(\mathfrak{p}) \leq G$ and let $L = P/\exp(\mathfrak{p}^{\perp})$. Then $\mathfrak{g}/\mathfrak{p}$ splits into a finite number of L-orbits:*

$$\mathfrak{g}/\mathfrak{p} = \mathscr{C}_0 \sqcup \cdots \sqcup \mathscr{C}_r, \tag{2.6.18}$$

where $r = \operatorname{rank} Z_R$ and $\mathscr{C}_{\mathfrak{p}} = \mathscr{C}_0 \sqcup \cdots \sqcup \mathscr{C}_{r-1}$. Furthermore, let $\mathfrak{h}$ be a Cartan subalgebra of $gr_{\mathfrak{p}}(\mathfrak{g})$ containing $\xi_{\mathfrak{p}}$ and denote by Δ_i the set of roots $\alpha \in \mathfrak{h}^*$ such that $\alpha(\xi_{\mathfrak{p}}) = i$ so that:

$$\Delta = \Delta_{-1} \sqcup \Delta_0 \sqcup \Delta_1. \tag{2.6.19}$$

Let $\{\beta_1, \ldots, \beta_r\} \subset \Delta_1$ be a maximal subset of pairwise strongly orthogonal long roots. Then $X_l := \sum_{i=1}^{l} E_i$ is a representative of $\mathscr{C}_l$ for $E_i \in \mathfrak{g}^{\beta_i}$.

In general, if two roots are strongly orthogonal then they are orthogonal under the Killing form but the converse does not always hold. However, it does hold if both roots are in Δ_1. This follows from [40, Section 9.4]: Simply put, α, β are orthogonal if, and only if, $\alpha + k\beta, \alpha - k\beta$ are either both in $\Delta \cup \{0\}$ or both not in $\Delta \cup \{0\}$. Then $\alpha + \beta \notin \Delta \cup \{0\}$ since they are in Δ_1, which in turn implies that α, β are strongly orthogonal if, and only if, they are orthogonal.

Proposition 2.6.11. [53, Proposition 3.12] *With Δ_1 and $\{\beta_1, \dots, \beta_r\}$ as defined in the previous theorem, every $\alpha \in \Delta_1$ is strongly orthogonal to all but two of the β_i.*

Proposition 2.6.12. [32, Theorem 2.9] *and* [53, Proposition 2.16] *We have:*

$$\mathscr{C}_l = \{X \in \mathfrak{g}/\mathfrak{p} |\, \mathrm{rank}\, P(X) = \kappa_l\}, \tag{2.6.20}$$

where:

$$\kappa_l = l + \frac{1}{2} l(l-1)a, \tag{2.6.21}$$

for a the number of roots not orthogonal to some β_i, β_j. Consequently:

$$\overline{\mathscr{C}}_l = \{X \in \mathfrak{g}/\mathfrak{p} |\, \mathrm{rank}\, P(X) \leq \kappa_l\} = \mathscr{C}_0 \sqcup \cdots \sqcup \mathscr{C}_l. \tag{2.6.22}$$

Moreover:

$$\dim \mathscr{C}_l = \dim R - \kappa_{l-1}. \tag{2.6.23}$$

Now we connect this with the theory of prehomogeneous vector spaces. Irreducible prehomogeneous vector spaces have been completely classified (up to, so called, castling transformations) by Sato–Kimura [56][4]. This will allow us to describe the generalised conformal structure as (pointwise) an algebraic variety and calculate its degree.

Definition 2.6.13. *Let G be a Lie group and let V be a G-module. Let p be a polynomial function on V. We call p a relative invariant if $p(g \cdot x) = \chi(g)p(x)$ for $\chi : G \to \mathbb{F}^\times$ a rational character.*

In a prehomogeneous vector space the relative invariants are determined, up to scale, by their characters [56, Proposition 4.3].

In our situation, the Levi factor $L = \mathrm{Stab}(\mathfrak{p})/\exp(\mathfrak{p}^\perp)$ acts on $\mathfrak{g}/\mathfrak{p}$ prehomogeneously, with the regular elements forming the open orbit. We have already seen one of the relative invariants $\det(P(X))$ and we can compute its character:

$$\begin{aligned}\det(P(\rho(g)X)) &= \det\left(\rho(g) \circ P(X) \circ \rho^*(g^{-1})\right), \\ &= \det(\rho(g)^2) \det(P(X)).\end{aligned}$$

Here ρ^* denotes the dual representation on $(\mathfrak{g}/\mathfrak{p})^* = \mathfrak{p}^\perp$, $\rho^*(g) = \rho(g^{-1})^*$. This is precisely the relative invariant described in [56, Proposition 4.8].

[4] The examples we are interested in comprise (1), (2), (3), (15) and (27) in [56, Section 7.I)]

Proposition 2.6.14. [56, Proposition 4.12] *If R is irreducible then all relative invariants are of the form $cp(x)^k$ for some relative invariant $p(x)$ and c a constant.*

Definition 2.6.15. *The relative invariant of smallest degree p is called the* ***fundamental relative invariant****.*

Proposition 2.6.16. [47, Proposition 2.16] *The degree of the fundamental relative invariant is $r = \operatorname{rank} Z_R$.*

Each relative invariant defines a map $\varphi_p : (\mathfrak{g}/\mathfrak{p})_{reg} \to \mathfrak{p}^\perp$, also known as grad log p. It is characterised by the following properties [56, Proposition 4.9]:

1. $\varphi_p(g \cdot X) = g \cdot \varphi_p(X)$.
2. $([A, X], \varphi_p(X)) = \delta_\chi(A)$.

Here, $A \in \mathfrak{p}/\mathfrak{p}^\perp$, χ is the associated character and $\delta_\chi : \mathfrak{p}/\mathfrak{p}^\perp \to \mathbb{F}$ is its derivative: $\chi(\exp(tA)) = \exp(t\delta_\chi(A))$. From Proposition 2.6.14, all relative invariants are of the form $p(x)^k$ and $\varphi_{p^k} = k\varphi_p$. Thus they are all scales of one another and, in fact, they are scales of the inversion map:

Proposition 2.6.17. $\varphi_{\det P} = \frac{1}{2} I_\mathfrak{p}$.

Proof. Let $A \in \mathfrak{p}/\mathfrak{p}^\perp$:

$$\begin{aligned}
([A, X], I_\mathfrak{p}(X)) &= (A, [X, I_\mathfrak{p}(X)]), \\
&= (A, 2\xi_\mathfrak{p}), \\
&= \operatorname{tr}(\operatorname{ad}_A \circ \operatorname{ad}_{2\xi_\mathfrak{p}}), \\
&= 2\operatorname{tr}(\operatorname{ad}_A|_{\mathfrak{g}/\mathfrak{p}}) - 2\operatorname{tr}(\operatorname{ad}_A|_{\mathfrak{p}^\perp}), \\
&= 4\operatorname{tr}(\operatorname{ad}_A|_{\mathfrak{g}/\mathfrak{p}}).
\end{aligned}$$

Differentiating $\chi_1 = \det(\rho(g)^2)$, we obtain $\delta_{\chi_1}(A) = 2\operatorname{tr}(\operatorname{ad}_A|_{\mathfrak{g}/\mathfrak{p}})$. Thus, $([A, X], \frac{1}{2} I_\mathfrak{p}(X)) = \delta_{\chi_1}(A)$. □

Chapter 3

Isothermic submanifolds

We now turn to the main object of consideration: isothermic submanifolds. We will define it in terms of a certain closed 1-form, and by the flatness of a certain family of connections. The Darboux and T-transform will be described, as well as the Christoffel dual and the permutability of these transformations.

3.1 Isothermic maps

As before, let $\mathfrak{g}$ be a noncompact semisimple real Lie algebra with adjoint group G and R a symmetric R-space of height 1 parabolic subalgebras of $\mathfrak{g}$. A key viewpoint that will be taken throughout this thesis is to view maps into symmetric R-spaces as vector bundles. Let $f : \Sigma \to R$. We may define a bundle, also called f, by:

$$f_x := f(x), \tag{3.1.1}$$

for all $x \in \Sigma$. Thus, a map into R is equated with a bundle of parabolic subalgebras of the trivial bundle $\underline{\mathfrak{g}} := \Sigma \times \mathfrak{g}$. Similarly, we may define a bundle of nilradicals $f^\perp \leq f \leq \underline{\mathfrak{g}}$ by $f_x^\perp := f(x)^\perp$.

A particular advantage of interpreting these bundles as subbundles of a trivial bundle is the existence of a canonical flat connection: the exterior derivative d.

Definition 3.1.1. [17, Definition 3.1] *Let $f : \Sigma \to R$. We say f is* ***isothermic*** *if there exists a non-zero $\eta \in \Omega^1_\Sigma(f^\perp)$ such that $d\eta = 0$. Furthermore, we say that (f, η) is an* ***isothermic submanifold*** *if it immerses.*

Note that this definition is clearly G-invariant as $(gf, g\eta)$ is also isothermic for all $g \in G$.

3.2 Zero curvature representation

Isothermic surfaces in $\mathbb{R}^3$ were shown to form an integrable system by Cieśliński–Goldstein–Sym [28]. Geometrically, we understand this property as the existence of a certain family of flat connections related to the surface (c.f. [39]). The definition of isothermic maps naturally extends this property.

Let $f : \Sigma \to R$ be any map and $\eta \in \Omega^1_\Sigma(f^\perp)$. We define connections:

$$\nabla^t := d + t\eta, \tag{3.2.1}$$

for all $t \in \mathbb{R}$. These connections have curvature:

$$R^{\nabla^t} = R^d + td\eta + \frac{t^2}{2}[\eta \wedge \eta]. \tag{3.2.2}$$

The first and last terms on the right hand side vanish as d is flat and η takes values in an abelian bundle. Consequently, we have:

Proposition 3.2.1. [17, Proposition 3.6] *For $f : \Sigma \to R$ and $\eta \in \Omega^1_\Sigma(f^\perp)$, ∇^t is flat for all $t \in \mathbb{R}$ if, and only if, (f, η) is isothermic.*

This approach lends itself effectively to the study of the transformations of isothermic submanifolds via the application of gauge theory. In the following sections we will describe each of the major transformations of isothermic submanifolds with a focus on the gauge theoretic characterisation. We will also note, for reference, the various permutability relations between these transformations.

3.3 Darboux transformation

Darboux transforms of isothermic surfaces in S^n traditionally arise as the other envelope of a conformal Ribaucour congruence of spheres. Another way to characterise these is as the solution to a Riccati-type equation (c.f. [38]). For our purposes, the most straightforward way is as parallel sections of one of the isothermic connections. Let $(f, \eta) : \Sigma \to S^n$. We can view $d + t\eta$ as a family of flat connections on $\underline{\mathbb{R}}^{n+1,1}$. Then any (null) parallel section $\sigma \in \Gamma\underline{\mathbb{R}}^{n+1,1}$ which is not orthogonal to f of the connection $d + m\eta$ is a lift of a Darboux transform of f. To define this in a general symmetric R-space, we first note what it means for a subbundle to be parallel for a given connection:

Definition 3.3.1. *Let V be a subbundle of $\underline{\mathfrak{g}}$ and let ∇ be a connection on $\underline{\mathfrak{g}}$. Then we say V is **parallel** for ∇ if $\nabla(\Gamma(V)) \subset \Omega^1(V)$.*

A null line bundle $\Lambda \leq \underline{\mathbb{R}}^{n+1,1}$ is defined by its stabiliser $\mathrm{stab}(\Lambda) \leq \underline{\mathfrak{g}}$. Then Λ is parallel for $d + m\eta$ precisely when $\mathrm{stab}(\Lambda)$ is parallel for $d + m\eta$ thought of as a connection on $\underline{\mathfrak{g}}$. This leads us to the following general definition:

Definition 3.3.2. [17, Definition 3.7] *Let $(f, \eta) : \Sigma \to R$ be an isothermic submanifold. Then we say that $\hat{f} : \Sigma \to R^*$ is a Darboux transform of f with parameter $m \in \mathbb{R} \setminus \{0\}$ if:*

1. *$f, \hat{f}$ are pointwise complementary parabolic subalgebras,*
2. *$\hat{f}$ is parallel for ∇^m.*

Note that the first condition here implies that $(f, \hat{f})$ is a map into Z_R.

To define the corresponding gauge transformation we need an important class of automorphisms of $\mathfrak{g}$:

$$\Gamma_{\mathfrak{q}}^{\mathfrak{p}}(s) := \begin{cases} s & \text{on } \mathfrak{p}^{\perp}, \\ 1 & \text{on } \mathfrak{p} \cap \mathfrak{q}, \\ s^{-1} & \text{on } \mathfrak{q}^{\perp}. \end{cases} \tag{3.3.1}$$

Proposition 3.3.3. [17, Theorem 3.10] *Let $(f, \eta) : \Sigma \to R$ be isothermic, and $\hat{f} : \Sigma \to R^*$ pointwise complementary to f. Then $\hat{f}$ is a Darboux transform of f if, and only if:*

$$\Gamma_f^{\hat{f}}\left(1 - \frac{t}{m}\right)(d + t\eta) = d + t\hat{\eta}, \tag{3.3.2}$$

where $\hat{\eta} \in \Omega^1_{\Sigma}(\hat{f}^{\perp})$ is given by $m\hat{\eta} \equiv df \mod f$.

Corollary 3.3.4. [17, Theorem 3.10] *If $\hat{f}$ is a Darboux transform of f it is isothermic. Symmetrically f is a Darboux transform of $(\hat{f}, \hat{\eta})$ with parameter m. Here $\hat{\eta}$ is defined by the property that $m\hat{\eta} \equiv df \mod f$.*

3.4 Christoffel dual

Classically, an isothermic surface and its Christoffel dual form the solution to Christoffel's problem (c.f. [11] or [27]). That is to say, any pair of surfaces in $\mathbb{R}^3$ that have parallel tangent planes and induce the same conformal structure, but opposite orientation on their domain, must be a Christoffel dual pair of isothermic surfaces. Moreover, this property characterises isothermic surfaces in $\mathbb{R}^3$.

The Christoffel dual of an isothermic surface in S^3 depends upon its realisation as a surface in $\mathbb{R}^3$. This property shall extend to the isothermic submanifolds.

Recall that a choice $\mathfrak{p}_0 \in R, \mathfrak{p}_\infty \in R^*$ defines dense open charts $\mathfrak{p}_0^\perp \cong \Omega_{\mathfrak{p}_0} \subset R^*, \mathfrak{p}_\infty^\perp \cong \Omega_{\mathfrak{p}_\infty} \subset R$ via $X \mapsto \exp(X)\mathfrak{p}_\infty$, $Y \mapsto \exp(Y)\mathfrak{p}_0$, respectively. Let $f : \Sigma \to R$ be some submanifold with image in $\Omega_{\mathfrak{p}_\infty}$. Then, the stereoprojection of f is $F : \Sigma \to \mathfrak{p}_\infty^\perp$ such that:

$$f = \exp(F)\mathfrak{p}_0. \tag{3.4.1}$$

Then, if $\eta \in \Omega^1_\Sigma(f^\perp)$, there exists a unique $\omega \in \Omega^1_\Sigma(\mathfrak{p}_0^\perp)$ such that:

$$\eta = \exp(F)\omega. \tag{3.4.2}$$

We calculate the exterior derivative of η:

$$\begin{aligned} d(\exp(F)\omega) &= d(\omega + [F,\omega] + \frac{1}{2}[F,[F,\omega]]), \\ &= \exp(F)(d\omega + [dF \wedge \omega]). \end{aligned}$$

Since $d\omega$ takes values in $\mathfrak{p}_0^\perp$ and $[dF \wedge \omega]$ takes values in $\mathfrak{p}_0 \cap \mathfrak{p}_\infty$, they must vanish independently if, and only if, η is closed. In summary:

Proposition 3.4.1 ([17, Proposition 3.3])**.** *Let $\mathfrak{p}_0 \in R, \mathfrak{p}_\infty \in R^*$ and $f : \Sigma \to \Omega_{\mathfrak{p}_\infty} \subset R$ with steroprojection $F : \Sigma \to \mathfrak{p}_0^\perp$. Then, f is isothermic if, and only if, there exists $\omega \in \Omega^1_\Sigma(\mathfrak{p}_0^\perp)$ such that:*

1. *$d\omega = 0$.*
2. *$[dF \wedge \omega] = 0$.*

Assuming f is isothermic, we can then integrate ω locally to give $F^c : \Sigma \to \mathfrak{p}_0^\perp$ such that $dF^c = \omega$. Then let:

$$f^c := \exp(F^c)\mathfrak{p}_\infty^\perp, \ \eta^c := \exp(F^c)dF. \tag{3.4.3}$$

Since $d\eta^c = \exp(F^c)(d^2F + [dF^c \wedge dF]) = 0$, it is clear that f^c is also isothermic.

Definition 3.4.2. [17, Section 3.2] *Let $f^c : \Sigma \to R^*$ be as constructed above. We call f^c the* ***Christoffel dual*** *of f with respect to $\mathfrak{p}_0, \mathfrak{p}_\infty$.*

As with the previous transformations, it is possible to demonstrate an explicit gauge transformation characterising the Christoffel dual.

Definition 3.4.3. *Let $f : \Sigma \to \Omega_{\mathfrak{p}_\infty}, f^* : \Sigma \to \Omega_{\mathfrak{p}_0}$ with stereoprojections $F : \Sigma \to \mathfrak{p}_\infty^\perp, F^* : \Sigma \to \mathfrak{p}_0^\perp$. We define the* ***Christoffel gauge****:*

$$\Gamma^c(t) := \exp(F^*)\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(t)\exp(-F). \tag{3.4.4}$$

Proposition 3.4.4. [17, Lemma 3.13] *Let $(f, \eta) : \Sigma \to \Omega_{\mathfrak{p}_\infty}, (f^*, \eta^*) : \Sigma \to \Omega_{\mathfrak{p}_0}$ be isothermic with stereoprojections $F : \Sigma \to \mathfrak{p}_\infty^\perp, F^* : \Sigma \to \mathfrak{p}_0^\perp$. Then, f, f^* are Christoffel dual if, and only if, $\Gamma^c(t) \cdot (d + t\eta) = d + t\eta^*$.*

3.5 T-transform

Isothermic submanifolds also admit a spectral deformation called the T-transform or Calapso transform. In the language of gauge theory, this is incredibly quick to see. Let $(f, \eta) : \Sigma \to R$ be an isothermic submanifold. For each $t \in \mathbb{R} \setminus \{0\}$, there exists a gauge transformation Φ_t, unique up to left multiplication by constant sections of G, such that $\Phi_t \cdot \nabla^t = d$.

Definition 3.5.1. [17, Section 3.3] *Let $(f_t, \eta_t) = (\Phi_t f, \Phi_t \eta) : \Sigma \to R$. Then we call f_t a T-transform of f with parameter t.*

The map f_t is also isothermic as can be seen from the following observation:

$$d + s\eta_t = \Phi_t \cdot (d + (t + s)\eta). \tag{3.5.1}$$

Changing viewpoint, we see that $d^{\nabla^t}\eta = d\eta + t[\eta \wedge \eta] = 0$. Thus the spectral deformation can also be described by the property that f remains isothermic when we replace d by $d + t\eta$.

3.6 The Bianchi formula

If $f : \Sigma \to S^3$ is an isothermic surface with Darboux transform $\hat{f}$ (with parameter m) and Christoffel dual f^c (with respect to some $(\Lambda_0, \Lambda_\infty)$) then there exists a unique surface $\hat{f}^c$ which is a Darboux transform of f^c (with parameter m)and Christoffel dual to $\hat{f}$ (with respect to $(\Lambda_0, \Lambda_\infty)$). This result is originally due to Bianchi [6] and he also proves that there is a natural relationship between four corresponding points on these surfaces. Let $p \in \Sigma$, and let $F, \hat{F}, F^c, \hat{F}^c$ be the respective stereoprojections. The line between $F(p)$ and $\hat{F}(p)$ is parallel to the line between $F^c(p)$ and $\hat{F}^c(p)$. Furthermore, we have the Bianchi formula:

$$\left\|\hat{F}(p) - F(p)\right\| \left\|\hat{F}^c(p) - F^c(p)\right\| = \frac{2}{m}. \tag{3.6.1}$$

See [11, Section 2.2] for a modern account of this in S^n, proved using Ricatti equations.

Burstall et al. [17, Theorem 4.11] generalise the permutability result to isothermic submanifolds of self-dual symmetric R-spaces. This is done by viewing Christoffel transforms as limits of Darboux transforms.

In this section we provide an alternative proof of [17, Theorem 4.11] by developing an analogue of the Bianchi formula (3.6.1).

Let R be a self-dual symmetric R-space. Throughout we assume $(f, \eta) : \Sigma \to \Omega_{\mathfrak{p}_\infty} \subset R$ is an isothermic submanifold, $(\hat{f}, \hat{\eta}) : \Sigma \to \Omega_{\mathfrak{p}_\infty} \subset R$ is a Darboux transform of f with parameter m and $(f^c, \eta^c) : \Sigma \to \Omega_{\mathfrak{p}_0} \subset R$ is Christoffel dual to f with respect to $(\mathfrak{p}_0, \mathfrak{p}_\infty) \in Z_R$. We denote the stereoprojections of $f, \hat{f}, f^c$ by $F, \hat{F} : \Sigma \to \mathfrak{p}_0^\perp$ and $F^c : \Sigma \to \mathfrak{p}_\infty^\perp$ respectively.

Theorem 3.6.1. *Let $\hat{f}^c : \Sigma \to \Omega_{\mathfrak{p}_0} \subset R$. Let $\hat{F}^c : \Sigma \to \mathfrak{p}_\infty^\perp$. Then $\hat{f}^c := \exp(\hat{F}^c)\mathfrak{p}_0$ is Christoffel dual to $\hat{f}$ with respect to $(\mathfrak{p}_0, \mathfrak{p}_\infty)$ if, and only if:*

$$\hat{F}^c - F^c = \frac{1}{m} I_{\mathfrak{p}_0}^{\mathfrak{p}_\infty} \left(\hat{F} - F \right). \tag{3.6.2}$$

Proof. From Proposition 3.3.3 we have:

$$d + t\hat{\eta} = \Gamma_f^{\hat{f}} \left(1 - \frac{t}{m} \right) \cdot (d + t\eta). \tag{3.6.3}$$

Differentiating with respect to t at $t = 0$ and using the fact that $\Gamma_f^{\hat{f}}(s) = \exp(\ln(s)\xi_f^{\hat{f}})$ then yields:

$$\hat{\eta} - \eta = \frac{1}{m} d\xi_f^{\hat{f}}. \tag{3.6.4}$$

Since $\eta, \hat{\eta}$ are closed we can find (locally) $\Phi, \hat{\Phi} \in \Gamma\underline{\mathfrak{g}}$ such that $d\Phi = \eta, d\hat{\Phi} = \hat{\eta}$. Note we can choose $\Phi, \hat{\Phi}$ up to addition of closed sections. In particular, for some choice of $\Phi, \hat{\Phi}$ we can ensure:

$$\hat{\Phi} - \Phi = \frac{1}{m} \xi_f^{\hat{f}}. \tag{3.6.5}$$

Then by [17, Proposition 3.4], if π_0 is the projection onto $\mathfrak{p}_0^\perp$ along $\mathfrak{p}_\infty$, $\pi_0\Phi = F^c$. Similarly, $\pi_0\hat{\Phi}$ is the stereoprojection of the Christoffel dual of $\hat{f}$. In other words, $\hat{F}^c = \pi_0\hat{\Phi}$ if, and only if, $\hat{f}^c$ is the Christoffel dual of $\hat{f}$.

We note that $f, \hat{f}$ are complementary so that $\mathfrak{p}_0 = \exp(-F)f, \exp(\hat{F} -$

$F)\mathfrak{p}_0 = \exp(-F)\hat{f}$. In particular, $\hat{F} - F$ is regular. Then:

$$\begin{aligned}
\xi_f^{\hat{f}} &= \xi_{\exp(F)\mathfrak{p}_0}^{\exp(\hat{F})\mathfrak{p}_0}, \\
&= \exp(F)\xi_{\mathfrak{p}_0}^{\exp(\hat{F}-F)\mathfrak{p}_0}, \\
&= \exp(F)\xi_{\mathfrak{p}_0}^{\exp\left(I_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(\hat{F}-F)\right)\mathfrak{p}_\infty}, \\
&= \exp(F)\exp\left(I_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(\hat{F}-F)\right)\xi_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}, \\
&= \exp(F)(\xi_{\mathfrak{p}_0}^{\mathfrak{p}_\infty} + I_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(\hat{F}-F)). \qquad (3.6.6)
\end{aligned}$$

Then, since $\operatorname{Im}\operatorname{ad}_F \subset \mathfrak{p}_\infty$ and $\xi_{\mathfrak{p}_0}^{\mathfrak{p}_\infty} \in \mathfrak{p}_\infty$ we have $\pi_0\xi_f^{\hat{f}} = I_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(\hat{F}-F))$. Thus:

$$\pi_0\hat{\Phi} - F^c = \frac{1}{m}I_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(\hat{F}-F). \qquad (3.6.7)$$

Consequently, $\hat{f}^c$ is Christoffel dual to $\hat{f}$ if, and only if, (3.6.2) holds. □

Remark. Let $R = S^n$ and let $l_0 \in \Lambda_0$, $l_\infty \in \Lambda_\infty$ such that $(l_0, l_\infty) = -1$. Let $\mathfrak{p}_0 = \operatorname{stab}(\Lambda_0)$, $\mathfrak{p}_\infty = \operatorname{stab}(\Lambda_\infty)$. We can identify both $\mathfrak{p}_0^\perp = \Lambda_0 \wedge \Lambda_0^\perp$, $\mathfrak{p}_\infty^\perp = \Lambda_\infty \wedge \Lambda_\infty^\perp$ with $\mathbb{R}^n \cong (\Lambda_0 \oplus \Lambda_\infty)^\perp$ via $l_0 \wedge x \mapsto x$, $l_\infty \wedge x \mapsto x$. Then $I_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(x) = \frac{2x}{(x,x)}$. Thus, (3.6.2) gives us the classical version (3.6.1).

Corollary 3.6.2. *Let $\hat{f}^c := \Gamma^c(m)\hat{f}$. Then $\hat{f}^c$ is Darboux transform of f^c with parameter m and a Christoffel dual of $\hat{f}$ with respect to $(\mathfrak{p}_0, \mathfrak{p}_\infty)$.*

Proof. Note that, since $\hat{f}$ is parallel for ∇^m, $\hat{f}^c$ is parallel for $\Gamma^c(m)\nabla^m = d + m\eta^c$. Equally, by our assumption that $\hat{f}$ is complementary to $\mathfrak{p}_\infty$, $\hat{f}^c$ is complementary to $\Gamma^c(m)\mathfrak{p}_\infty = \exp(F^c)\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(m)\exp(-F)\mathfrak{p}_\infty = f^c$. Thus $\hat{f}^c$ is a Darboux transform of f^c with parameter m.

Expanding out $\hat{f}^c = \Gamma^c(m)\hat{f}$, we obtain:

$$\exp(\hat{F}^c)\mathfrak{p}_\infty = \exp(F^c)\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(m)\exp(\hat{F}-F)\mathfrak{p}_0. \qquad (3.6.8)$$

Rearranging this gives:

$$\exp(\hat{F}^c - F^c)\mathfrak{p}_\infty = \exp(m(\hat{F}-F))\mathfrak{p}_0. \qquad (3.6.9)$$

But by the definition of $I_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}$ this is equivalent to (3.6.2). Thus by Theorem 3.6.1, $\hat{f}^c$ is Christoffel dual to $\hat{f}$. □

Alternatively, we can characterise this via gauge transformations:

Theorem 3.6.3. *Let $f, \hat{f}$ be a Darboux pair of isothermic submanifolds with parameter m, and let f^c be the Christoffel dual of f with respect to $(\mathfrak{p}_0, \mathfrak{p}_\infty)$. Let $\hat{f}^c := \Gamma^c(m)\hat{f}$ and let*

$$\hat{\Gamma}^c(t) := \exp(\hat{F}^c)\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(t)\exp(-\hat{F}) \tag{3.6.10}$$

denote the Christoffel gauge of $\hat{f}, \hat{f}^c$. Then,

$$\hat{\Gamma}^c(t)\Gamma_f^{\hat{f}}\left(1 - \frac{t}{m}\right) = \Gamma_{f^c}^{\hat{f}^c}\left(1 - \frac{t}{m}\right)\Gamma^c(t). \tag{3.6.11}$$

Proof. Let $L(t)$ and $R(t)$ denote the left and right sides of (3.6.11), respectively. We shall see that they are both in fact equal to the transformation:

$$\exp(F^c)\Gamma_{\mathfrak{p}_\infty}^{\mathfrak{p}_0}\left(\frac{1}{t} - \frac{1}{m}\right)\exp(-\hat{F}). \tag{3.6.12}$$

Firstly expanding out we see that:

$$L(t) = \exp(\hat{F}^c)\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(t)\exp(-\hat{F})\Gamma_f^{\hat{f}}\left(1 - \frac{t}{m}\right), \tag{3.6.13}$$

and thus

$$\begin{aligned} L'(t) &:= \exp(-F^c)L(t)\exp(\hat{F}), \\ &= \exp(\hat{F}^c - F^c)\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(t)\exp(-\hat{F})\Gamma_f^{\hat{f}}\left(1 - \frac{t}{m}\right)\exp(\hat{F}). \end{aligned}$$

Note that $g \circ \Gamma_{\mathfrak{q}}^{\mathfrak{p}}(s) \circ g^{-1} = \Gamma_{g\mathfrak{q}}^{g\mathfrak{p}}(s)$ so we can instead write this as:

$$L'(t) = \exp(\hat{F}^c - F^c)\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(t)\Gamma_{\exp(F-\hat{F})\mathfrak{p}_0}^{\mathfrak{p}_0}\left(1 - \frac{t}{m}\right). \tag{3.6.14}$$

At this point we can already see that $L'(t)$ acts as $\frac{1}{t}(1 - \frac{t}{m}) = \left(\frac{1}{t} - \frac{1}{m}\right)$ on $\mathfrak{p}_0^\perp$, since $\hat{F}^c - F^c$ takes values in $\mathfrak{p}_0^\perp$. Let $Y := \hat{F} - F$ and $Y^c := \hat{F}^c - F^c$, so that Theorem 3.6.1 gives,

$$I_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(Y) = m(Y^c). \tag{3.6.15}$$

Thus $\exp(F - \hat{F})\mathfrak{p}_0 = \exp(-Y)\mathfrak{p}_0 = \exp(-mY^c)\mathfrak{p}_\infty$ and we obtain:

$$L'(t) = \exp(Y^c)\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(t)\exp(-mY^c)\Gamma_{\mathfrak{p}_\infty}^{\mathfrak{p}_0}\left(1 - \frac{t}{m}\right)\exp(mY^c). \tag{3.6.16}$$

This is now in a form where we can straightforwardly calculate its values on $\mathfrak{p}_0^\perp, \mathfrak{p}_0 \cap \mathfrak{p}_\infty, \mathfrak{p}_\infty^\perp$. As we noted before $\exp(Y^c)$ acts as the identity on $\mathfrak{p}_0^\perp$ and so $L'(t)$ acts as $\left(\frac{1}{t} - \frac{1}{m}\right)$. Let $x \in \mathfrak{p}_0 \cap \mathfrak{p}_\infty$. Then we calculate in stages:

$$\exp(mY^c)(x) = x + m[Y^c, x].$$

$$\Gamma^{\mathfrak{p}_0}_{\mathfrak{p}_\infty}\left(1 - \frac{t}{m}\right)\exp(mY^c)(x) = x + (m-t)[Y^c, x].$$

$$\exp(-mY^c)\Gamma^{\mathfrak{p}_0}_{\mathfrak{p}_\infty}\left(1 - \frac{t}{m}\right)\exp(mY^c)(x) = x - t[Y^c, x].$$

$$\Gamma^{\mathfrak{p}_\infty}_{\mathfrak{p}_0}(t)\exp(-mY^c)\Gamma^{\mathfrak{p}_0}_{\mathfrak{p}_\infty}\left(1 - \frac{t}{m}\right)\exp(mY^c)(x) = x - [Y^c, x].$$

$$L'(t)(x) = x.$$

Then consider $x \in \mathfrak{p}_\infty^\perp$:

$$\exp(mY^c)(x) = x + m[Y^c, x] + \frac{m^2}{2}\mathrm{ad}^2_{Y^c}(x).$$

$$\Gamma^{\mathfrak{p}_0}_{\mathfrak{p}_\infty}\left(1 - \frac{t}{m}\right)\exp(mY^c)(x) = \frac{m}{m-t}x + m[Y^c, x] + \frac{m(m-t)}{2}\mathrm{ad}^2_{Y^c}(x).$$

$$\exp(-mY^c)\Gamma^{\mathfrak{p}_0}_{\mathfrak{p}_\infty}\left(1 - \frac{t}{m}\right)\exp(mY^c)(x) = \frac{m}{m-t}\left(x - t[Y^c, x] + \frac{t^2}{2}\mathrm{ad}^2_{Y^c}(x)\right).$$

$$\Gamma^{\mathfrak{p}_\infty}_{\mathfrak{p}_0}(t)\exp(-mY^c)\Gamma^{\mathfrak{p}_0}_{\mathfrak{p}_\infty}\left(1 - \frac{t}{m}\right)\exp(mY^c)(x) = \frac{mt}{m-t}\left(x - [Y^c, x] + \frac{1}{2}\mathrm{ad}^2_{Y^c}(x)\right).$$

$$L'(t)(x) = \frac{mt}{m-t}x = \left(\frac{1}{t} - \frac{1}{m}\right)^{-1}x.$$

Therefore:

$$L'(t) = \Gamma^{\mathfrak{p}_0}_{\mathfrak{p}_\infty}\left(\frac{1}{t} - \frac{1}{m}\right). \tag{3.6.17}$$

Now we consider the right hand side. Again we obtain:

$$\begin{aligned}
R'(t) &:= \exp(-F^c)R(t)\exp(\hat{F}), \\
&= \exp(-F^c)\Gamma^{\hat{f}^c}_{f^c}\left(1 - \frac{t}{m}\right)\exp(F^c)\Gamma^{\mathfrak{p}_\infty}_{\mathfrak{p}_0}(t)\exp(Y), \\
&= \Gamma^{\exp(Y^c)\mathfrak{p}_\infty}_{\mathfrak{p}_\infty}\left(1 - \frac{t}{m}\right)\Gamma^{\mathfrak{p}_\infty}_{\mathfrak{p}_0}(t)\exp(Y), \\
&= \Gamma^{\exp(mY)\mathfrak{p}_0}_{\mathfrak{p}_\infty}\left(1 - \frac{t}{m}\right)\Gamma^{\mathfrak{p}_\infty}_{\mathfrak{p}_0}(t)\exp(Y), \\
&= \exp(mY)\Gamma^{\mathfrak{p}_0}_{\mathfrak{p}_\infty}\left(1 - \frac{t}{m}\right)\exp(-mY)\Gamma^{\mathfrak{p}_\infty}_{\mathfrak{p}_0}(t)\exp(Y).
\end{aligned}$$

Immediately, we can see that $R'(t)$ acts as $t\left(1-\frac{t}{m}\right)^{-1}=\left(\frac{1}{t}-\frac{1}{m}\right)^{-1}$ on $\mathfrak{p}_\infty^\perp$. Let $x \in \mathfrak{p}_0 \cap \mathfrak{p}_\infty$:

$$\begin{aligned}
\exp(Y)(x) &= x + [Y, x].\\
\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(t)\exp(Y)(x) &= x + t[Y, x].\\
\exp(-mY)\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(t)\exp(Y)(x) &= x + (t-m)[Y, x].\\
\Gamma_{\mathfrak{p}_\infty}^{\mathfrak{p}_0}\left(1-\frac{t}{m}\right)\exp(-mY)\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(t)\exp(Y)(x) &= x - m[Y, x].\\
R'(t)(x) &= x.
\end{aligned}$$

Then for $x \in \mathfrak{p}_0^\perp$:

$$\begin{aligned}
\exp(Y)(x) &= x + [Y, x] + \frac{1}{2}\mathrm{ad}_Y^2(x).\\
\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(t)\exp(Y)(x) &= \frac{1}{t}x + [Y, x] + \frac{t}{2}\mathrm{ad}_Y^2(x).\\
\exp(-mY)\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(t)\exp(Y)(x) &= \frac{1}{t}\left(x - (m-t)[Y, x] + \frac{(m-t)^2}{2}\mathrm{ad}_Y^2(x)\right).\\
\Gamma_{\mathfrak{p}_\infty}^{\mathfrak{p}_0}\left(1-\frac{t}{m}\right)\exp(-mY)\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(t)\exp(Y)(x) &= \frac{m-t}{mt}\left(x - mY + \frac{m^2}{2}\mathrm{ad}_Y^2(x)\right).\\
R'(t)(x) &= \frac{m-t}{mt}x = \left(\frac{1}{t}-\frac{1}{m}\right)x.
\end{aligned}$$

Consequently:

$$R'(t) = \Gamma_{\mathfrak{p}_\infty}^{\mathfrak{p}_0}\left(\frac{1}{t}-\frac{1}{m}\right) = L'(t). \tag{3.6.18}$$

Therefore $L(t) = R(t)$ and we are done. $\square$

Combining this with Proposition 3.3.3 and Proposition 3.4.4 (the gauge theory characterisations of the Darboux and Christoffel transformations) gives an another proof of Corollary 3.6.2.

3.7 Other permutability results

In this section we will collect, for later reference, several other permutability results for transformations of isothermic submanifolds. These will be important for defining the semi-discrete analogues of the isothermic transformations. In general, these results are known and proofs are available in Burstall et al. [17, Sections 3 and 4].

Firstly, successive T-transformation are also T-transformations:

Proposition 3.7.1 ([37] for S^n, [17] for symmetric R-spaces). *Let $f : \Sigma \to R$ be an isothermic submanifold. Let f_t be a T-transform of f and let $(f_t)_s$ be a T-transform of f_t. Then, $(f_t)_s$ is a T-transform of f with parameter $t + s$.*

The Darboux and T-transformations commute:

Proposition 3.7.2 ([17, Theorem 3.12]). *Let $f : \Sigma \to R$ be an isothermic submanifold. Let f_t be a T-transform of f with parameter t and let $\hat{f}$ be a Darboux transform with parameter m. Let $\hat{f}_t$ be a T-transform of $\hat{f}$ with parameter t. Then, $\hat{f}_t$ is a Darboux transform of f_t with parameter $m - t$.*

The T-transform intertwines Darboux transforms with the Christoffel dual:

Proposition 3.7.3 ([17, Theorem 3.14]). *Let $f : \Sigma \to \Omega_{\mathfrak{p}_\infty} \subset R$ be an isothermic submanifold where $(\mathfrak{p}_0, \mathfrak{p}_\infty) \in Z_R$. Let f^c be the Christoffel dual of f with respect to $(\mathfrak{p}_0, \mathfrak{p}_\infty)$ and let f_t, f^c_t be T-transforms of f, f^c, respectively, with parameter t. Then, f^c_t is a Darboux transform of f_t with parameter $-t$.*

From this point we assume that R is a self-dual symmetric R-space. In particular this implies that isothermic submanifolds and their Darboux transforms live in the same symmetric R-space and two Darboux transforms of a fixed submanifold may be pointwise complementary to each other. Recall from Section 2.6.2 that there exist circles in R, which are particular 1-dimensional submanifolds diffeomorphic to S^1 with a well defined cross-ratio. Then we may see that the Darboux transform commutes with itself to form a Bianchi quadrilateral (c.f. [6]):

Proposition 3.7.4 ([17, Theorem 4.8]). *Let $f : \Sigma \to R$ be an isothermic submanifold into a self-dual symmetric R-space R. Let $f_1, f_2 : \Sigma \to R$ be pointwise complementary Darboux transforms of f with parameters $m_1 \neq m_2$, respectively. Then, there exists a common Darboux transform $f_{12} := \Gamma^{f_1}_{f_2}(m_2/m_1)f$ of f_1, f_2 with parameter m_2, m_1, respectively. Moreover, these 4 maps are pointwise concircular with constant cross ratio $cr(f, f_1, f_{12}, f_2) = m_2/m_1$.*

This result extends immediately to give Bianchi's cube theorem:

Proposition 3.7.5 ([17, Theorem 4.9]). *Let $f : \Sigma \to R$ be an isothermic map into a self-dual symmetric R-space R. Let $f_1, f_2, f_3 : \Sigma \to R$ be pairwise complementary Darboux transforms of f with distinct parameters m_1, m_2, m_3. Let f_{ij} be a simultaneous Darboux transform of f_i, f_j, as given by Proposition 3.7.4, for each $\{i, j\} \subset \{1, 2, 3\}$. Assume f_{12}, f_{13}, f_{23} are pairwise complementary. Then, there exists $(f_{123}, \eta_{123}) : \Sigma \to R$, an isothermic map,* *which is a simultaneous Darboux transform of each f_{ij} with parameter m_k where $i \neq j \neq k$. Moreover, f_1, f_2, f_3, f_{123} and $f, f_{12}, f_{23}, f_{13}$ are concircular respectively with the following cross ratio relations:*

$$cr(f_3, f_1, f_{123}, f_2) = \frac{1 - \frac{m_3}{m_1}}{1 - \frac{m_3}{m_2}} = cr(f_{12}, f_{23}, f, f_{13}). \tag{3.7.1}$$

Chapter 4

Cyclides

In this chapter, we will define a special class of submanifolds of self-dual symmetric R-spaces. Their interaction with the generalised conformal structure (as defined in Section 2.6.3) will be discussed and their relation to the Darboux transform of isothermic submanifolds. More broadly we will use these as model submanifolds for parabolic subgeometries of R.

4.1 Motivation

Surfaces in the conformal n-sphere S^n naturally form conformal geometries in their own right. In other words they envelop[1] a smooth family of 2-spheres $S^2 \subset S^n$ called a congruence. This observation allows us to study their geometry by studying this congruence and this approach can be seen in the work of Darboux [29] as well as Blaschke [7]. For example, a Darboux pair of isothermic surfaces envelop a common sphere congruence.

Extending this to more general submanifolds of symmetric R-spaces poses a problem. Subspaces of the tangent space of given dimension no longer lie in a single conjugacy class[2]. In particular, they may have various possible types of intersection with the generalised conformal structure. As a result, we do not have a single type of "model" homogeneous space that we can assume submanifolds are tangent to.

Since we are focused on isothermic submanifolds and in particular maximal non-degenerate examples in self-dual symmetric R-spaces, we infer a candidate from the particularly simple examples of these described in Burstall et al. [17, Section 6.2.2]. We recall the construction of those here. Let R be a symmetric R-space. Let $(\mathfrak{p}, \mathfrak{q}) \in Z_R$, a pair of complementary parabolic sub-

[1] at each point we have first order contact

[2] of the action of the stabiliser or its Levi component

algebras of $\mathfrak{g}$. Then Z_R is a symmetric space with symmetric decomposition $\mathfrak{g} = \mathfrak{p} \cap \mathfrak{q} \oplus (\mathfrak{p}^\perp \oplus \mathfrak{q}^\perp)$ at $(\mathfrak{p}, \mathfrak{q})$. Let $\mathfrak{a}$ be a Cartan subspace of $\mathfrak{p}^\perp \oplus \mathfrak{q}^\perp$ and define $F : \mathfrak{a} \to \mathfrak{q}^\perp$, $F^c : \mathfrak{a} \to \mathfrak{p}^\perp$ as the projections onto $\mathfrak{q}^\perp, \mathfrak{p}^\perp$ respectively. For any $H \in \mathfrak{a}$, $X \in T_X\mathfrak{a} = \mathfrak{a}$ we have $d_H F(X) = F(X)$, $d_H F^c(X) = F^c(X)$ and $F(X) + F^c(X) = X$. Then for any $X, Y, H \in \mathfrak{a}$:

$$\begin{aligned}
[d_H F \wedge d_H F^c](X,Y) &= [F(X), F^c(Y)] + [F^c(X), F(Y)], \\
&= [F(X) + F^c(X), F(Y) + F^c(Y)] \\
&\qquad - [F(X), F(Y)] - [F^c(X), F^c(Y)], \\
&= [F(X) + F^c(X), F(Y) + F^c(Y)], \\
&= [X, Y] = 0.
\end{aligned} \tag{4.1.1}$$

Thus $[dF \wedge dF^c] = 0$ and so $f := \exp(F)\mathfrak{p}^\perp, f^c := \exp(F^c)\mathfrak{q}^\perp$ are a pair of Christoffel dual isothermic submanifolds with $\eta := \exp(F)dF^c, \eta^c := \exp(F^c)dF$. In fact, these are maximal nondegenerate isothermic submanifolds. Indeed $q_f(X,Y) = (dF(X), dF^c(Y)) = (X,Y)/2$.

Assume R is self-dual and a Hermitian symmetric space. Now, we recall from Lemma 2.3.11 and Proposition 2.3.14 that with respect to some root system,

$$\mathfrak{a} = \langle E_i + E_{-i} | i = 1, \ldots, r \rangle, \tag{4.1.2}$$

where $r = \operatorname{rank} Z_R$ and $E_i \in \mathfrak{p}^\perp$, $E_{-i} \in \mathfrak{q}^\perp$ are root vectors for some roots β_i, β_{-i}. Moreover, $\{\beta_1, \ldots, \beta_r\}$ is a maximal set of strongly orthogonal long roots with root spaces in $\mathfrak{p}^\perp$. Let $\mathfrak{s}_i := \mathfrak{g}^{\beta_i} \oplus \mathfrak{g}^{\beta_{-i}} \oplus [\mathfrak{g}^{\beta_i}, \mathfrak{g}^{\beta_{-i}}]$ a copy of $\mathfrak{sl}_2$. The subalgebra $\mathfrak{s} := \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_r$, or more properly the direct sum of this subalgebra with its centraliser, stabilises a homogeneous space $C \subset R$ which contains $\mathfrak{p}, \mathfrak{q}$. Then since F, F^c take values in $\mathfrak{s}$ we see that f, f^c have image in C. Clearly C has the same tangent spaces as its submanifolds f, f^c and we shall see that all maximal non-degenerate isothermic submanifolds must be tangent at each point to such a submanifold or an appropriate real form.

4.2 Cyclides

Definition 4.2.1. *Let $\mathfrak{g}$ be a complex non-compact semisimple Lie algebra. We call $\mathfrak{s} \leq \mathfrak{g}$ a **cyclide algebra of** $\mathfrak{g}$ if:*

1. $\mathfrak{s} := \bigoplus_{i=1}^r \mathfrak{s}_i$ *where* $\mathfrak{s}_i \cong \mathfrak{sl}_2$ *and* $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$ *for* $i \neq j$.
2. $\mathfrak{g} = \mathfrak{s} \oplus U \oplus Z$ *is an* $\mathfrak{s}$*-module decomposition with:*
    (a) $[\mathfrak{s}, Z] = 0$.

(b) $U = \bigoplus_{1\le i<j\le r}\bigoplus_{k=1}^{l} U_{i,j}^{(k)}$ *such that each* $U_{i,j}^{(k)} \cong U_i \otimes U_j$, *with* $U_i \cong \mathbb{C}^2$ *the defining representation of* $\mathfrak{s}_i$.

Let $\mathfrak{g}_{\mathbb{R}}$ *be a (non-compact) real form of* $\mathfrak{g}$ *with real structure* σ. *Then we call* $\mathfrak{s}_{\mathbb{R}} := \mathfrak{g}_{\mathbb{R}} \cap \mathfrak{s}$ *a* ***cyclide algebra of*** $\mathfrak{g}_{\mathbb{R}}$ *if* σ *preserves* $\mathfrak{s}, U, Z$ *and for each* $\mathfrak{s}_i$ *either* $\mathrm{Fix}(\sigma) \cap \mathfrak{s}_i \cong \mathfrak{sl}(2,\mathbb{R})$ *or* $\sigma(\mathfrak{s}_i) = \mathfrak{s}_j$ *and* $\mathrm{Fix}(\sigma) \cap \mathfrak{s}_i \oplus \mathfrak{s}_j \cong \mathfrak{sl}(2,\mathbb{C})$.

Remark. Note first that Z is naturally a subalgebra itself since $[\mathfrak{s},[Z,Z]] = \{0\}$ and Z naturally forms the centraliser of $\mathfrak{s}$ in $\mathfrak{g}$. We shall see later[3] that the multiplicity l is independent of i,j and can be thought of as the dimension of a Z-module $V_{i,j}$ such that $U_{i,j} := \bigoplus_{k=1}^{l} U_{i,j}^{(k)} \cong U_{i,j}^{(1)} \otimes V_{i,j}$ as an $\mathfrak{s} \oplus Z$-module. Our definition for a non-complex cyclide algebra ensures that it is the sum of non-compact summands:

$$\mathfrak{s}_{\mathbb{R}} = \bigoplus_{i=1}^{k} \mathfrak{s}_i \oplus \bigoplus_{j=2k+1}^{r} \mathfrak{t}_j, \tag{4.2.1}$$

with each $\mathfrak{s}_i \cong \mathfrak{sl}(2,\mathbb{C})$, and each $\mathfrak{t}_j \cong \mathfrak{sl}(2,\mathbb{R})$.

A cyclide algebra is thus a real non-compact semisimple Lie algebra in its own right so we may consider its parabolic subalgebras and the symmetric R-spaces these form. In particular, as each simple summand is non-compact we may consider parabolic subalgebras whose intersection with each summand is proper and parabolic.

Let R_i denote the space of proper parabolic subalgebras of $\mathfrak{s}_i$ or $\mathfrak{t}_i$ for each i. There is only one conjugacy class of these and they have height 1. In fact, $R_i \cong S^2 \cong \mathbb{P}(\mathbb{C}^2)$ or $R_i \cong S^1 \cong \mathbb{P}(\mathbb{R}^2)$. Let $\mathfrak{p}_i \in R_i$. Then $\mathfrak{p}' = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r$ is a Borel subalgebra of $\mathfrak{s}$ and has height 1. In particular this shows that $R_{\mathfrak{s}} := R_1 \times \cdots \times R_r$ is a symmetric R-space. Parabolic subalgebras of a Lie algebra naturally induce filtrations of representations of that Lie algebra where the nilradical acts as lowering operators. In this case we will see that $\mathfrak{p}' \le \mathfrak{s}$ induces a filtration on $\mathfrak{g}$ which defines a "big" parabolic subalgebra $\mathfrak{p} \le \mathfrak{g}$. Let $\mathfrak{n}' \le \mathfrak{p}'$ denote the nilradical of $\mathfrak{p}' \le \mathfrak{s}$. Then we define:

$$\mathfrak{n}(\mathfrak{p}') := \mathrm{Ker}\,\mathrm{ad}_{\mathfrak{n}'} \cap (\mathfrak{s} \oplus U). \tag{4.2.2}$$

Note that $\mathfrak{n}(\mathfrak{p}') \cap \mathfrak{s} = \mathfrak{n}'$. We claim that this is the abelian nilradical of a parabolic subalgebra of $\mathfrak{g}$:

$$\Phi(\mathfrak{p}') := \mathfrak{n}(\mathfrak{p}')^{\perp}. \tag{4.2.3}$$

In fact, we could instead define these in terms of parabolic subalgebras of $\mathfrak{h}_{\mathfrak{s}} := \mathfrak{s} \oplus Z$ of the form $\mathfrak{p}' \oplus Z$ which also have nilradical $\mathfrak{n}'$.

[3] Corollary 4.4.4

Proposition 4.2.2. *The map Φ is an embedding of $R_{\mathfrak{s}}$ into a self-dual symmetric R-space such that $\Phi(\mathfrak{p}') \cap \mathfrak{s} = \mathfrak{p}'$.*

Proof. Firstly, we note that $\mathfrak{p}' \leq \mathfrak{s}$ is a Borel subalgebra of $\mathfrak{s}$. Therefore, it defines a highest weight space on each irreducible $\mathfrak{s}$-module. More precisely, for any choice of Cartan subalgebra of $\mathfrak{s}$ contained in $\mathfrak{p}'$, $\operatorname{Ker}\mathfrak{n}'$ is the highest weight space given by the ordering on weights induced by $\mathfrak{p}'$. Thus $\mathfrak{n}(\mathfrak{p}')$ is precisely the sum of the highest weight spaces for each non-trivial irreducible $\mathfrak{s}$-submodule of $\mathfrak{g}$ (according to the decomposition in Definition 4.2.1). Definition 4.2.1 gives us the exact forms of these modules so we can identify their weights explicitly. If the module is one of the $\mathfrak{s}_i$, the representation is just the adjoint representation of $\mathfrak{s}_i$ itself and the trivial action of each other $\mathfrak{s}_j$ since they commute. It has weights of the form $2\omega_i, 0, -2\omega_i$ where $2\omega_i$ is its highest weight. Indeed, $\pm 2\omega_i$ form a root system of $\mathfrak{s}$ since they are precisely the weights of its adjoint representation. If the module is $U_{i,j}^{(k)} \cong U_i \otimes U_j$ it is simply the tensor product of the "defining" 2-dimensional representations of $\mathfrak{s}_i$ and $\mathfrak{s}_j$. Again the other $\mathfrak{s}_k$ act trivially. Correspondingly, the weights of this representation are $\omega_i + \omega_j, \omega_i - \omega_j, \omega_j - \omega_i, -\omega_i - \omega_j$, with $\omega_i + \omega_j$ the highest weight. To summarise, if $\mathfrak{g}^\lambda$ denotes the weight space for any weight λ of the action of $\mathfrak{s}$ on $\mathfrak{g}$, then:

$$\mathfrak{n}(\mathfrak{p}') = \bigoplus_{1 \leq i \leq r} \mathfrak{g}^{2\omega_i} \oplus \bigoplus_{1 \leq i < j \leq r} \mathfrak{g}^{\omega_i + \omega_j}. \tag{4.2.4}$$

The Lie bracket of $\mathfrak{g}$ acts on weight spaces as follows. Let $\mathfrak{g}^\lambda, \mathfrak{g}^\mu \leq \mathfrak{g}$ be weight spaces for weights λ, μ. Then:

$$[\mathfrak{g}^\lambda, \mathfrak{g}^\mu] \subset \mathfrak{g}^{\lambda+\mu}. \tag{4.2.5}$$

Therefore, since any weight of $\mathfrak{g}$ is of one of the forms mentioned above, $\mathfrak{n}(\mathfrak{p}')$ must be abelian as no addition of its weights gives a valid weight of $\mathfrak{g}$. For example, $2\omega_i + (\omega_j + \omega_k)$ is not a weight of $\mathfrak{g}$. Furthermore, (4.2.5) implies that $\mathrm{ad}_{\mathfrak{g}^\lambda} \circ \mathrm{ad}_{\mathfrak{g}^\mu}$ sends each other weight space $\mathfrak{g}^\omega$ to $\mathfrak{g}^{\omega+\lambda+\mu}$ and so consists of trace-free endomorphisms unless $\lambda + \mu = 0$. In other words, $\mathfrak{g}^\lambda \perp \mathfrak{g}^\mu$ if $\lambda + \mu \neq 0$. Therefore $\Phi(\mathfrak{p}') = \mathfrak{n}(\mathfrak{p}')^\perp$ is contained in the sum of all weight spaces $\mathfrak{g}^\lambda$ such that $\lambda \neq -2\omega_i, -\omega_i - \omega_j$ for any i, j. By comparing dimensions this inclusion must be equality. Note that if λ, μ are two such weights then $\lambda + \mu$ is either such a weight itself or it is not a weight. As a result, $\Phi(\mathfrak{p}')$ is a subalgebra of $\mathfrak{g}$. Now, $\mathfrak{n}(\mathfrak{p}') = \Phi(\mathfrak{p}')^\perp \leq \Phi(\mathfrak{p}')$ and in fact $\Phi(\mathfrak{p}')$ is parabolic with abelian nilradical $\mathfrak{n}(\mathfrak{p}')$. It is also clear from this weight space decomposition that $\Phi(\mathfrak{p}') \cap \mathfrak{s} = \mathfrak{p}'$.

Let S, S_i denote the analytic subgroups of G corresponding to $\mathfrak{s}, \mathfrak{s}_i$ respectively. If $\mathfrak{q}' = \mathfrak{q}_1 \oplus \cdots \oplus \mathfrak{q}_r \leq \mathfrak{s}$ is another Borel subalgebra of $\mathfrak{s}$ then it must be conjugate to $\mathfrak{p}'$ by some element of S. To see this, note that $\mathfrak{q}_i$ is conjugate to $\mathfrak{p}_i$ by $g_i \in S_i$ and S_i, S_j commute so that for $g = g_1 \cdots g_r$, $g\mathfrak{p}' = \mathfrak{q}'$.

Now Φ is S-equivariant since G is orthogonal for the Killing form and:

$$\begin{aligned}\mathfrak{n}(\mathfrak{q}') &= \operatorname{Ker} \operatorname{ad}_{g\cdot\mathfrak{p}'^{\perp}} \cap (\mathfrak{s} \oplus U),\\ &= g \cdot (\operatorname{Ker} \operatorname{ad}_{\mathfrak{p}'^{\perp}}) \cap (\mathfrak{s} \oplus U),\\ &= g \cdot \operatorname{Ker} \operatorname{ad}_{\mathfrak{p}'^{\perp}} \cap (\mathfrak{s} \oplus U),\\ &= g \cdot \mathfrak{n}(\mathfrak{p}'). \end{aligned} \tag{4.2.6}$$

As a result, the image of Φ must be inside a single symmetric R-space.

Assume that $\mathfrak{q}' \leq \mathfrak{s}$ is complementary to $\mathfrak{p}'$. The pair $\mathfrak{p}', \mathfrak{q}'$ defines an explicit choice of weights since $\mathfrak{p}' \cap \mathfrak{q}' \leq \mathfrak{s}$ is a Cartan subalgebra. Under this choice, $\mathfrak{n}(\mathfrak{q}')$ consists of those weight spaces $\mathfrak{g}^{\lambda}$ for which $\mathfrak{g}^{-\lambda} \leq \mathfrak{n}(\mathfrak{p}')$. Consequently, $\Phi(\mathfrak{p}') \oplus \mathfrak{n}(\mathfrak{q}') = \mathfrak{g}$ and $\Phi(\mathfrak{p}'), \Phi(\mathfrak{q}')$ are complementary. Since they are also conjugate the image of Φ must be self-dual.

We noted earlier that $\Phi(\mathfrak{p}') \cap \mathfrak{s} = \mathfrak{p}'$. This implies that Φ is an injective immersion since $\Phi, d\Phi$ are invertible on their images. Finally, $R_{\mathfrak{s}}$ is compact so any injective immersion must be an embedding. □

Definition 4.2.3. *Let $\mathfrak{s}$ be a cyclide algebra and let $\Phi : R_{\mathfrak{s}} \to R$ be the corresponding embedding. We call $C_{\mathfrak{s}} := \operatorname{Im} \Phi$ the **cyclide** associated to $\mathfrak{s}$.*

Thus a cyclide is an embedded submanifold of R which is itself a symmetric R-space diffeomorphic to an r-dimensional product of spheres and circles.

4.3 Examples of cyclides

As we have seen cyclides are certain submanifolds of self-dual symmetric R-spaces. Here we list the examples for the classical irreducible self-dual symmetric R-spaces.

4.3.1 Quadric

Consider $\mathbb{C}^{n+2}$ equipped with a symmetric bilinear form. Let $W \leq \mathbb{C}^{n+2}$ be a non-degenerate subspace and $\dim W = 4$. We recall the identification (2.4.4), $\mathfrak{so}(V) \cong \Lambda^2 V$ and the exceptional isomorphism (2.5.23), $\mathfrak{so}(4, \mathbb{C}) = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. Let $\mathfrak{s} := \mathfrak{so}(W) = \Lambda^2 W$. Then:

$$\Lambda^2 \mathbb{C}^{n+2} = \Lambda^2 W \oplus W \wedge W^{\perp} \oplus \Lambda^2 W^{\perp}. \tag{4.3.1}$$

Then $Z := \bigwedge^2 W^\perp$ naturally commutes with $\mathfrak{s}$. Choosing any basis $\{w_1, \dots, w_k\}$ of $W^\perp$, we obtain irreducible $\mathfrak{s}$-modules $w_i \wedge W$ which span the second summand. Indeed, $U := W \wedge W^\perp = W \otimes W^\perp$ naturally realises U as the tensor product of a $\mathfrak{s}$-module and a Z-module.

Thus $\mathfrak{s}$ is a cyclide algebra and the associated cyclide is the intersection $\mathbb{P}(W \cap \mathcal{L}) \subset \mathbb{P}(\mathcal{L})$ which is a copy of the complex two dimensional quadric Q^2 inside the n-dimensional quadric Q^n. All of these are conjugate by the action of $G = \mathrm{Ad}SO(n+2, \mathbb{C})$ and so we recognise the space of cyclides as the symmetric space of non-degenerate subspaces of dimension 4. Similarly, the cyclides contained in the real quadric $S^{p,q}$ are given by a real subspace $W \leq \mathbb{R}^{p+1,q+1}$ of signature $(3,1)$, $(2,2)$ or $(1,3)$. Note that, for $p \leq 1$ or $q \leq 1$, some of these may not be possible. Then $\mathrm{Ad}SO(p+1, q+1) \cong SO_0(p+1, q+1)$ acts transitively on non-degenerate subspaces of a given signature so that the space of cyclides splits into orbits: $G_{(3,1)}(\mathbb{R}^{p+1,q+1}), G_{(2,2)}(\mathbb{R}^{p+1,q+1}), G_{(1,3)}(\mathbb{R}^{p+1,q+1})$ with corresponding cyclides of the form $S^{2,0}, S^{1,1}, S^{0,2}$ respectively.

4.3.2 Grassmannian

Let $V_1 \oplus \cdots \oplus V_n = \mathbb{C}^{2n}$ be a partition into 2 dimensional subspaces. Let $\mathfrak{s} := \bigoplus_{i=1}^r \mathfrak{sl}(V_i) \leq \mathfrak{sl}(2n, \mathbb{C})$, where $\mathfrak{sl}(V_i)$ is the set of endomorphisms which are trace-free on V_i and 0 on all other V_j, for each i. Then each $\mathfrak{s}_i, \mathfrak{s}_j$ commute and:

$$\mathfrak{sl}(2n, \mathbb{C}) = \mathfrak{s} \oplus \bigoplus_{1 \leq i < j \leq n} (\mathrm{Hom}(V_i, V_j) \oplus \mathrm{Hom}(V_j, V_i)) \oplus Z, \tag{4.3.2}$$

where $Z = \langle \mathrm{id}_{V_i} - \mathrm{id}_{V_j} | 1 \leq i, j \leq n \rangle$. Then Z is a trivial $\mathfrak{s}$ module and $\mathrm{Hom}(V_i, V_j)$, $\mathrm{Hom}(V_j, V_i)$ are each irreducible modules for $\mathfrak{sl}(V_i) \oplus \mathfrak{sl}(V_j)$. Thus $\mathfrak{s}$ is a cyclide algebra. The associated cyclide is:

$$\Phi : \mathbb{P}(V_1) \times \cdots \times \mathbb{P}(V_n) \to G_n(\mathbb{C}^{2n}); (L_1, \dots, L_n) \mapsto L_1 \oplus \cdots \oplus L_n. \tag{4.3.3}$$

Let σ be a real structure on $\mathbb{C}^{2n}$ that preserves $\{V_1, \dots, V_n\}$. There are two possibilities for each i either $\sigma(V_i) = V_i$ or $\sigma(V_i) = V_j$ for some $j \neq i$. Correspondingly, $\sigma(\mathfrak{sl}(V_i)) = \mathfrak{sl}(V_i)$ or $\sigma(\mathfrak{sl}(V_i)) = \mathfrak{sl}(V_j)$. Thus we obtain cyclides of $G_n(\mathbb{R}^{2n})$ of the form:

$$(\mathbb{P}(\mathbb{R}^2))^p \times (\mathbb{P}(\mathbb{C}^2))^q \cong (S^1)^p \times (S^2)^q, \tag{4.3.4}$$

where $p + 2q = n$. Clearly $G = PSL(2n, \mathbb{R})$ preserves these classes of cyclide. In general, however, these classes can split into finitely many orbits of this action. For example, in $G_2(\mathbb{R}^4) \cong S^{2,2}$ there are three orbits of cyclides: one of the form $(\mathbb{P}(\mathbb{R}^2))^2 \cong S^{1,1}$ and two of the form $\mathbb{P}(\mathbb{C}^2)$ which correspond to the cyclides of type $S^{2,0}, S^{0,2} \leq S^{2,2}$. If we instead take a quaternionic structure j then $j(V_i) \neq V_i$ as otherwise V_i is a single quaternionic line and $\mathfrak{sl}(1, \mathbb{H})$ is trivial. Thus we only obtain cyclides of $G_m(\mathbb{H}^{2m})$ isomorphic to:

$$(S^2)^m, \tag{4.3.5}$$

where $2m = n$ and there is precisely one orbit of these.

4.3.3 Lagrangian Grassmannian

Let $\mathbb{C}^{2n}$ be equipped with a symplectic form and let $V_1 \oplus \cdots \oplus V_n = \mathbb{C}^{2n}$ be a partition into non-degenerate subspaces of dimension 2. We recall that $\mathfrak{sp}(V) = \mathrm{S}^2 V$. Then:

$$\mathrm{S}^2 \mathbb{C}^{2n} = \bigoplus_{i=1}^{n} \mathrm{S}^2 V_i \oplus \bigoplus_{1 \leq i < j \leq n} V_i \odot V_j. \tag{4.3.6}$$

Since $\mathfrak{sp}(\mathbb{C}^2) \cong \mathfrak{sl}(\mathbb{C}^2)$, $\mathfrak{s} := \bigoplus_{i=1}^{n} \mathrm{S}^2 V_i$ is a cyclide algebra with exactly 1 irreducible 4-dimensional module $U_{i,j} := V_i \odot V_j$ for each (i,j), $i < j$. Then $Z = \{0\}$ here. The construction of the cyclide is then identical to the traditional Grassmannian, although we may think of $\mathbb{P}(V_i)$ as $\mathrm{Lag}(V_i)$:

$$\Phi : \mathrm{Lag}(V_1) \times \cdots \times \mathrm{Lag}(V_n) \to \mathrm{Lag}(\mathbb{C}^{2n}); (L_1, \ldots, L_n) \mapsto L_1 \oplus \cdots \oplus L_n. \tag{4.3.7}$$

We have a similar set of real forms by requiring each V_i to either be fixed (by a real structure) or exchanged with some V_j (by a real or quaternionic structure).

4.3.4 Isotropic Grassmannian

Let $\mathbb{C}^{4n}$ be equipped with a symmetric bilinear form and let $W_1 \oplus \cdots \oplus W_n = \mathbb{C}^{4n}$ be a decomposition into non-degenerate subspaces of dimension 4 which are pairwise orthogonal. As noted in the quadric example, $\Lambda^2 W_i$ splits into the sum $\mathfrak{s}_i^+ \oplus \mathfrak{s}_i^-$ where $\mathfrak{s}_i^\pm$ are commuting copies of $\mathfrak{sl}(2, \mathbb{C})$. Now:

$$\Lambda^2 \mathbb{C}^{4n} = \bigoplus_{i=1}^{n} \mathfrak{s}_i^+ \oplus \bigoplus_{1 \leq i < j \leq n} W_i \wedge W_j \oplus \bigoplus_{i=1}^{n} \mathfrak{s}_i^-. \tag{4.3.8}$$

Let $\mathfrak{s} := \bigoplus_{i=1}^{n} \mathfrak{s}_i^+$ and $U_{i,j} := W_i \wedge W_j$. Then each $U_{i,j}$ splits into 4 irreducible $\mathfrak{s}_i^+ \oplus \mathfrak{s}_j^+$-modules In fact, for $Z := \bigoplus_{i=1}^{n} \mathfrak{s}_i^-$, $[\mathfrak{s}, Z] = 0$ and each

$U_{i,j}$ is isomorphic as an $s \oplus Z$-module to the tensor product:

$$\mathbb{C}^2_{i,+} \otimes \mathbb{C}^2_{i,-} \otimes \mathbb{C}^2_{j,+} \otimes \mathbb{C}^2_{j,-}, \tag{4.3.9}$$

where $\mathbb{C}^2_{i,\pm}$ is the defining representation of $\mathfrak{s}_i^{\pm}$. Again, we have one irreducible $\mathfrak{s}_i^+ \oplus \mathfrak{s}_j^+$-module for each (i,j) and $Z := \bigoplus_{i=1}^n \mathfrak{s}_i^-$ commutes with $\mathfrak{s}$. Thus $\mathfrak{s}$ is a cyclide algebra with its associated cyclide:

$$\Phi : J^+(W_1) \times \cdots \times J^+(W_n) \to J^+(\mathbb{C}^{4n}); (V_1, \ldots, V_n) \mapsto V_1 \oplus \cdots \oplus V_n. \tag{4.3.10}$$

The choice of which orbit in $J^{\pm}(W_i), J^{\pm}(\mathbb{C}^{4n})$ to name $+$ or $-$ is arbitrary but we have a corresponding parity of cyclides. If we swap a single term from $J^+(W_i) = R_{\mathfrak{s}_i^+}$ to $J^-(W_i) = R_{\mathfrak{s}_i^-}$ we get a new cyclide inside $J^-(\mathbb{C}^{4n})$. The elements of this new cyclide intersect each element of the original cyclide with constant dimension $2n-1$. We can extend this further to produce a network of 2^n cyclides in $J^+(\mathbb{C}^{4n})$ and $J^-(\mathbb{C}^{4n})$ with various degrees of intersection. In particular, Z forms a cyclide algebra and the elements of the corresponding cyclide intersect those of $\mathfrak{s}$ with dimension n.

4.4 Geometry and existence of cyclides

In this section, we will see that every self-dual symmetric R-space contains cyclides and they all have the same dimension. In fact there are a finite number of orbits of cyclides in general and over $\mathbb{C}$ there is exactly 1 orbit.

4.4.1 Existence of cyclides

To demonstrate the existence of cyclides we can construct them explicitly out of root spaces for strongly orthogonal roots.

Proposition 4.4.1. *Let R be a self-dual symmetric R-space. Then R contains cyclides.*

Proof. Firstly, we assume $\mathfrak{g}$ is complex. Let $\mathfrak{p}, \mathfrak{q} \in R$ be complementary. Then, take a Cartan subalgebra $\mathfrak{h}$ containing $\xi^{\mathfrak{p}}_{\mathfrak{q}}$ so that $\mathfrak{h} \subset \mathfrak{p} \cap \mathfrak{q}$ (compare with Theorem 2.6.10). Choose a maximal set of strongly orthogonal long roots $\square_1 := \{\beta_1, \ldots, \beta_r\} \subset \Delta_1 = \{\alpha \in \Delta | \alpha(\xi^{\mathfrak{p}}_{\mathfrak{q}}) = 1\}$. Denote by $\mathfrak{sl}_{\beta_i}$ the subalgebra generated by β_i:

$$\mathfrak{sl}_{\beta_i} := \mathfrak{g}^{\beta_i} \oplus \mathfrak{g}^{-\beta_i} \oplus [\mathfrak{g}^{\beta_i}, \mathfrak{g}^{-\beta_i}]. \tag{4.4.1}$$

Then $[\mathfrak{sl}_{\beta_i}, \mathfrak{sl}_{\beta_j}] = 0$ since β_i, β_j are strongly orthogonal. From Proposition 2.6.11, each $\alpha \in \Delta_1 \setminus \square_1$ is strongly orthogonal to exactly 2 of the β_i.

Then denote by U^α the $\mathfrak{s}$-module generated by $\mathfrak{g}^\alpha$. Let $\alpha \in \Delta_0$. If α is not strongly orthogonal to β_i then either $\alpha+\beta_i \in \Delta_1 \setminus \Box_1$ or $-(\alpha-\beta_i) \in \Delta_1 \setminus \Box_1$. In the first case, $\mathfrak{g}^\alpha \leq U^{\alpha+\beta_i}$ and in the second, $-(\alpha-\beta_i)$ must not be strongly orthogonal to some β_j. Thus $-(\alpha-\beta_i)-(\beta_i+\beta_j) = -\alpha-\beta_j \in \Delta_{-1}$ and then $\mathfrak{g}^\alpha \leq U^{\alpha+\beta_j}$. If α is strongly orthogonal to each β_i then $[\mathfrak{s}, \mathfrak{g}^\alpha] = \{0\}$. Similarly, if α is not $-\beta_i$ for some i then $\mathfrak{g}^\alpha \leq U^{\alpha+\beta_i+\beta_j}$. Finally, we note that for $\mathfrak{h}_1 := (\mathfrak{h} \cap \mathfrak{s})^\perp$, $[\mathfrak{s}, \mathfrak{h}_1] = 0$. Thus we have a decomposition:

$$\mathfrak{g} = \mathfrak{s} \oplus \bigoplus_{\alpha \in \Delta_1 \setminus \Box_1} U^\alpha \oplus \left(\mathfrak{h}_1 \oplus \bigoplus_{\substack{\alpha \in \Delta_0, \\ \alpha \perp\!\!\!\perp \Box_1}} \mathfrak{g}^\alpha \right). \tag{4.4.2}$$

Each U^α is a 4-dimensional $\mathfrak{s}_i \oplus \mathfrak{s}_j$-module of the required form and the last summand is a trivial $\mathfrak{s}$-module. Thus $\mathfrak{s}$ is a cyclide algebra. Let $\mathfrak{g}$ be real and choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing $\xi^{\mathfrak{p}}_{\mathfrak{q}}$. Then $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$ containing $\xi^{\mathfrak{p}^{\mathbb{C}}}_{\mathfrak{q}^{\mathbb{C}}} = \xi^{\mathfrak{p}}_{\mathfrak{q}}$. We then proceed as in the complex case but now we require that $\Box_1 = \{\beta_1, \ldots, \beta_r\}$ is preserved by the real structure. Such a subset always exists (see, for example, [59, Lemma 4.3]). $\Box$

In fact this construction is completely general and all cyclide algebras can be constructed this way.

Proposition 4.4.2. *Let $\mathfrak{s} \leq \mathfrak{g}$ be a complex cyclide algebra. Let $\mathfrak{p}, \mathfrak{q} \in C_{\mathfrak{s}}$ and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{p}\cap\mathfrak{q}\cap\mathfrak{s}$. Let Δ be the corresponding root system and $\Delta_1 = \{\alpha \in \Delta | \alpha(\xi^{\mathfrak{p}}_{\mathfrak{q}}) = 1\}$. Then for some maximal subset of strongly orthogonal long roots $\Box = \{\beta_1, \ldots, \beta_r\} \subset \Delta_1$:*

$$\mathfrak{s} = \bigoplus_{i=1}^{r} \mathfrak{sl}_{\beta_i}. \tag{4.4.3}$$

If $\mathfrak{s}_{\mathbb{R}} \leq \mathfrak{g}_{\mathbb{R}}$ is a real cyclide algebra with complexification $\mathfrak{s}$ then:

$$\mathfrak{s}_{\mathbb{R}} = \bigoplus_{i=1}^{k} (\mathfrak{sl}_{\beta_{2i-1}} \oplus \mathfrak{sl}_{\beta_{2i}})_{\mathbb{R}} \oplus \bigoplus_{j=2k+1}^{r} (\mathfrak{sl}_{\beta_i})_{\mathbb{R}}, \tag{4.4.4}$$

for some $1 \leq k \leq \lfloor r/2 \rfloor$ and such that the roots β_{2i-1}, β_{2i} are complex conjugates for $i = 1, \ldots, k$ and β_j is real for $j = 2k+1, \ldots, r$.

Proof. Firstly we note $\mathfrak{h}_0 = \mathfrak{p}\cap\mathfrak{q}\cap\mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{s}$ since it is the direct sum of Cartan subspaces of each $\mathfrak{s}_i$. Extend $\mathfrak{h}_0$ to a Cartan subalgebra

$\mathfrak{h}$ of $\mathfrak{g}$. Recall the weight space decomposition we described in the previous section:

$$\begin{aligned} \mathfrak{s}_i &= \mathfrak{h}_0 \cap \mathfrak{s}_i \oplus \mathfrak{g}^{2\omega_i} \oplus \mathfrak{g}^{-2\omega_i}. \\ U_{i,j} &= \mathfrak{g}^{\omega_i+\omega_j} \oplus \mathfrak{g}^{-\omega_i+\omega_j} \oplus \mathfrak{g}^{\omega_i-\omega_j} \oplus \mathfrak{g}^{-\omega_i-\omega_j}. \\ Z &\subset \mathfrak{g}^0. \end{aligned} \tag{4.4.5}$$

Consequently the centraliser $C_{\mathfrak{g}}(\mathfrak{h}_0)$ is a subspace of $\mathfrak{h}_0 \oplus Z$, and thus $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$ for $\mathfrak{h}_1 \leq Z$. Let:

$$\beta_i := \begin{cases} 2\omega_i & \text{on } \mathfrak{h}_0, \\ 0 & \text{on } \mathfrak{h}_1. \end{cases} \tag{4.4.6}$$

Then, β_i is a root with root space $\mathfrak{g}^{2\omega_i} \leq \mathfrak{s}_i$. Similarly, $-\beta_i$ has root space $\mathfrak{g}^{-2\omega_i} \leq \mathfrak{s}_i$. Indeed (up to a choice of sign) we can assume $\mathfrak{g}^{2\omega_i} = \mathfrak{p}_i^{\perp} \cap \mathfrak{s}_i$, where $\mathfrak{p}_i := \mathfrak{p} \cap \mathfrak{s}_i$. Thus $\mathfrak{s}_i = \mathfrak{sl}_{\beta_i}$. Since $[\mathfrak{s}_i, \mathfrak{s}_j] = 0$, β_i, β_j are strongly orthogonal. If there is another root $\alpha \in \Delta_1$ which is strongly orthogonal to $\square$, then $[\mathfrak{s}, \mathfrak{g}^{\alpha}] = 0$. However, $\operatorname{Ker} \operatorname{ad}_{\mathfrak{s}} = Z$ by definition of cyclide algebra and $Z \cap \mathfrak{p}^{\perp} = 0$ by definition of the embedding $\Phi : R_{\mathfrak{s}} \to C_{\mathfrak{s}} \subset R$. Thus $\square$ is maximal.

If $\mathfrak{s}_{\mathbb{R}} \leq \mathfrak{s}$ is a real cyclide algebra of $\mathfrak{g}_{\mathbb{R}} \leq \mathfrak{g}$ then by definition, the real structure either preserves $\mathfrak{s}_i$ or swaps $\mathfrak{s}_i, \mathfrak{s}_j$ for some j. Consequently, β_i is either real, imaginary or β_i, β_j are complex conjugate. If β_i is imaginary then $(\mathfrak{s}_i)_{\mathbb{R}}$ is compact so by the definition of a real cyclide algebra we may assume β_i is real or β_i, β_j are complex conjugate. □

Remark. Proposition 4.4.2 tells us that cyclides in a self-dual Hermitian symmetric space are precisely the polyspheres of Harish-Chandra (c.f. [45]).

Then, in view of Proposition 2.3.14 Cartan subspaces of Z_R are also built from maximal subsets of strongly orthogonal long roots. Thus:

Corollary 4.4.3. *All cyclides in R have the same dimension which is the (real/complex) rank of Z_R.*

Recall $Z_R \subset R \times R^*$ here refers to the symmetric space of complementary pairs in R. Note by a result of Richardson et al. [53, Proposition 2.8] that all (ordered) subsets of $\square$ of a given length are conjugate by an element of the Weyl group. Consequently we have:

Corollary 4.4.4. *Each $U_{i,j}$ has the same dimension.*

4.4.2 The space of cyclides

Let $\mathfrak{s}$ be a cyclide algebra. Then $\mathfrak{h}_\mathfrak{s} := \mathrm{stab}(\mathfrak{s}) = \mathfrak{s} \oplus Z$ and $H_\mathfrak{s}$ the analytic Lie subgroup of G with Lie algebra $\mathfrak{h}_\mathfrak{s}$ Then $[\mathfrak{h}_\mathfrak{s}, U] \subset U$ so that U is an $\mathrm{ad}_{\mathfrak{h}_\mathfrak{s}}$-invariant complement. Hence, we can consider $\mathfrak{s}$ as a point in a reductive homogeneous space $G/H_\mathfrak{s}$. Indeed we shall see that the space of cyclides is composed of a disjoint union of reductive spaces. To do this we shall relate them to Cartan subspaces (c.f. Definition 2.3.10) of Z_R whose conjugacy is well understood:

Proposition 4.4.5. [50, Theorem 3] *Let N be a symmetric space with symmetric decomposition $\mathfrak{g} = \mathfrak{h}_x \oplus \mathfrak{m}_x$ at $x \in N$. There are a finite number of* $\mathrm{Stab}(x)$*-orbits of Cartan subspaces of* $\mathfrak{m}_x$.

Recall that for $(\mathfrak{p}, \mathfrak{q}) \in Z_R$ a Cartan subspace of $\mathfrak{p}^\perp \oplus \mathfrak{q}^\perp$ is a maximal abelian subspace which consists of only semisimple elements.

Theorem 4.4.6. *If $\mathfrak{g}$ is complex then any Cartan subspace of $\mathfrak{p}^\perp \oplus \mathfrak{q}^\perp \leq \mathfrak{g}$ is contained in a unique cyclide algebra. Conversely, every complex cyclide algebra such that $\mathfrak{p}, \mathfrak{q} \in C_\mathfrak{s}$ contains Cartan subspaces of $\mathfrak{p}^\perp \oplus \mathfrak{q}^\perp$.*

If $\mathfrak{g}_\mathbb{R}$ is a non-compact real form of $\mathfrak{g}$ then any Cartan subspace $\mathfrak{c}_\mathbb{R}$ of $\mathfrak{p}^\perp \oplus \mathfrak{q}^\perp \leq \mathfrak{g}_\mathbb{R}$ is contained in a unique cyclide algebra if it is positive definite for the Killing form. Conversely, every real cyclide algebra such that $\mathfrak{p}, \mathfrak{q} \in C_\mathfrak{s}$ contains Cartan subspaces of $\mathfrak{p}^\perp \oplus \mathfrak{q}^\perp$.

Proof. Assume $\mathfrak{g}$ is complex. Let $\mathfrak{c}$ be Cartan subspace of $\mathfrak{p}^\perp \oplus \mathfrak{q}^\perp$. Then by [35, Proposition 7.4]:

$$\mathfrak{c} = \langle E_i + E_{-i} | i = 1, \ldots, r \rangle, \tag{4.4.7}$$

where $\{\beta_1, \ldots, \beta_r\}$ is a maximal subset of strongly orthogonal long roots with root spaces in $\mathfrak{p}^\perp$ and $E_i \in \mathfrak{g}_i^\beta, E_{-i} \in \mathfrak{g}^{-\beta_i}$.

Let $\mathfrak{s}_i := \mathfrak{sl}_{\beta_i}$ and $\mathfrak{s} := \bigoplus_{i=1}^r \mathfrak{s}_i$. As we saw in the proof of Proposition 4.4.1 this is a cyclide algebra. Then $\mathfrak{c} \leq \mathfrak{s}$. Indeed it is a Cartan subalgebra of $\mathfrak{s}$ since it is abelian, contains only semisimple elements and $\dim \mathfrak{c} = r$ is the rank of $\mathfrak{s}$. Let $\mathfrak{s}'$ be another cyclide algebra containing $\mathfrak{c}$. Then by Proposition 4.4.2 the rank of $\mathfrak{s}$ is equal to the rank of $\mathfrak{s}'$. Thus $\mathfrak{c}$ must also be a Cartan subalgebra of $\mathfrak{s}'$ since again it is abelian, contains only semisimple elements and is maximal in $\mathfrak{s}'$ for these properties. Now we note by the definition of a cyclide algebra it is spanned by the weight spaces $\mathfrak{g}^{2\omega_i}, \mathfrak{g}^{-2\omega_i}$ (as defined in Section 4.2) together with a Cartan subalgebra. These weight spaces, up to ordering of i, depend only on the choice of Cartan subalgebra $\mathfrak{c}$ and its action on $\mathfrak{g}$. In particular, $\mathfrak{s}$ and $\mathfrak{s}'$ are the same.

Conversely, let $\mathfrak{s}$ be a cyclide algebra and let $\mathfrak{p}, \mathfrak{q} \in R$ be complementary parabolic subalgebras such that $\mathfrak{p}, \mathfrak{q} \in C_{\mathfrak{s}}$. Then $\mathfrak{p}\cap\mathfrak{s}, \mathfrak{q}\cap\mathfrak{s}$ are complementary Borel subalgebras of $\mathfrak{s}$. Take $E_i \in \mathfrak{p}^\perp \cap \mathfrak{s}_i, E_{-i} \in \mathfrak{q}^\perp \cap \mathfrak{s}_i$ and define $\mathfrak{c}$ as in (4.4.7). Note by Proposition 4.4.2 we note $E_{\pm i} \in \mathfrak{g}^{\pm\beta_i}$ for some choice of strongly orthogonal long roots $\{\beta_1, \ldots, \beta_r\}$. Thus, by [35, Proposition 7.4], $\mathfrak{c}$ is a Cartan subspace of $\mathfrak{p}^\perp \oplus \mathfrak{q}^\perp$.

Let $\mathfrak{g}_{\mathbb{R}}$ be a real form of $\mathfrak{g}$ and let σ denote conjugation over $\mathfrak{g}_{\mathbb{R}}$. Let $\mathfrak{c}_{\mathbb{R}}$ be a Cartan subspace of $\mathfrak{p}^\perp \oplus \mathfrak{q}^\perp$ with complexification $\mathfrak{c} \leq (\mathfrak{p}^\perp \oplus \mathfrak{q}^\perp)^{\mathbb{C}} \leq \mathfrak{g}$. Thus $\mathfrak{c}$ is the Cartan subalgebra of some cyclide algebra $\mathfrak{s}$. Then $\sigma(\mathfrak{c}) = \mathfrak{c}$ is contained in $\sigma(\mathfrak{s})$. Since $\sigma(\mathfrak{s})$ is also a cyclide algebra we see that $\sigma(\mathfrak{s}) = \mathfrak{s}$ by our earlier argument. As before, we write $\mathfrak{c} := \langle E_i + E_{-i} | i = 1, \ldots, r\rangle$ and $\mathfrak{s}_i = \langle E_i, E_{-i}, [E_i, E_{-i}]\rangle$. Since $\mathfrak{c}$ is σ-stable, the set of weight spaces of $\mathfrak{c}$ is σ-stable. Moreover, $\sigma(\mathfrak{g}^{\beta_i})$ is one of $\mathfrak{g}^{\beta_i}$, $\mathfrak{g}^{-\beta_i}$, $\mathfrak{g}^{\beta_j}$ or $\mathfrak{g}^{-\beta_j}$ $(j \neq i)$. If $\sigma(\mathfrak{g}^{\beta_i}) = \mathfrak{g}^{\beta_i}$ then $(\mathfrak{s}_i)_{\mathbb{R}} \cong \mathfrak{sl}(2, \mathbb{R})$ and if $\sigma(\mathfrak{g}^{\beta_i})$ is $\mathfrak{g}^{\beta_j}$ or $\mathfrak{g}^{-\beta_j}$ then $(\mathfrak{s}_i \oplus \mathfrak{s}_j)_{\mathbb{R}} \cong \mathfrak{sl}(2, \mathbb{C})$. If $\sigma(\mathfrak{g}^{\beta_i}) = \mathfrak{g}^{-\beta_i}$ then for some λ, $\sigma(E_i) = \lambda E_{-i}$ and $\sigma(E_i) = \frac{1}{\lambda}E_{-i}$. Thus if $H_i := [E_i, E_{-i}]$ then $\sigma(H_i) = -H_i$ so that $iH_i \in \mathfrak{c}$. However,S H_i is some (real) scale of the coroot of β_i and consequently, $(H_i, H_i) > 0$. So if $\mathfrak{c}_{\mathbb{R}}$ is positive definite $\mathfrak{s}_{\mathbb{R}}$ is a real cyclide algebra by Definition 4.2.1.

Conversely, let $\mathfrak{s}_{\mathbb{R}}$ is a real cyclide algebra with $\mathfrak{p}, \mathfrak{q} \in C_{\mathfrak{s}} \subset R$ complementary. Then using (4.4.4), we take $E_{\pm i} \in \mathfrak{g}^{\pm\beta_i}$ for $0 \leq i \leq r$ and set $X_j := E_j + \sigma(E_j), Y_j := i(E_j - \sigma(E_j))$. Then, let:

$$\mathfrak{c}_{\mathbb{R}} := \langle X_j, Y_j | j = 1, \ldots, k\rangle \oplus \langle E_i | i = 2k+1, \ldots, r\rangle \leq \mathfrak{s}_{\mathbb{R}}. \tag{4.4.8}$$

Clearly $\mathfrak{c}_{\mathbb{R}}$ is a real form of $\mathfrak{c}$ defined as in (4.4.7). Thus it is a Cartan subspace because $\mathfrak{c}$ is a Cartan subspace [42, Theorem 3.4]. □

As an immediate consequence we get a bound on the possible number of orbits of cyclide algebras.

Corollary 4.4.7. *The number of G-orbits of cyclide algebras and thus of cyclides is less than or equal to the number of orbits of Cartan subspaces of Z_R. In particular, for $\mathfrak{g}$ complex there is exactly one orbit.*

Proof. Let $\mathfrak{s}, \mathfrak{s}'$ be two cyclide algebras of R. If $\mathfrak{s}' = g\mathfrak{s}$ then for each Cartan subspace $\mathfrak{c}$ of Z_R such that $\mathfrak{c} \leq \mathfrak{s}$, $g\mathfrak{c}$ is a Cartan subspace of Z_R and $g\mathfrak{c} \leq \mathfrak{s}'$. Conversely, if $\mathfrak{c} \leq \mathfrak{s}$ and $\mathfrak{c}' \leq \mathfrak{s}'$ are Cartan subspaces such that $\mathfrak{c}' = g\mathfrak{c}$, then $\mathfrak{c}' \leq g\mathfrak{s}$ but $g\mathfrak{s}$ is a cyclide algebra so by Theorem 4.4.6 $g\mathfrak{s} = \mathfrak{s}'$. Also from Theorem 4.4.6 we see that $\mathfrak{s}$, $\mathfrak{s}'$ contains at least one Cartan subspace. Thus $\mathfrak{s}, \mathfrak{s}'$ are conjugate if, and only if, each Cartan subspace contained in $\mathfrak{s}$ is conjugate to one contained in $\mathfrak{s}'$. Thus the number of orbits of cyclide algebras (resp. cyclides) is bounded above by the number of Cartan subspace orbits. If $\mathfrak{g}$ is complex, then by Lemma 2.3.11 there is a single orbit of Cartan subspaces and so a single orbit of cyclide algebras (resp. cyclides). $\square$

In summary, we have:

Theorem 4.4.8. *The space of cyclides in R is a finite union of reductive homogeneous spaces for G.*

4.5 Splitting submanifolds

Definition 4.5.1. *A* ***cyclide congruence*** *C is a smooth map from Σ into the space of cyclides. A map $f : \Sigma \to R$* ***envelops*** *C if $f(x) \in C(x)$, and $\operatorname{Im} df_x = T_{f(x)}C(x)$.*

In S^n every surface envelops a S^2-congruence but this is not true in a general self-dual symmetric R-space.

Cyclides have a specific interaction with the generalised conformal structure. Recall that the generalised conformal structure on R at $\mathfrak{p} \in R$ is the G-invariant cone defined by $\mathscr{C}_\mathfrak{p} := \{X \in \mathfrak{g}/\mathfrak{p} | P(X) = \frac{1}{2}\mathrm{ad}_X^2 \text{ is invertible}\} \leq T_\mathfrak{p}R$ (c.f. Section 2.6.3). In this section, we shall see how this characterises cyclides and their envelopes. From this approach, we shall obtain a canonical choice of coordinates on such submanifolds as well as a canonical choice of enveloped cyclide congruence.

4.5.1 Splitting subspaces and splitting submanifolds

In this section we will define a splitting subspace and a splitting submanifold. The splitting subspaces form the tangent spaces to splitting submanifolds. We will also define a canonical complement to a splitting subspace and thus canonical dual subspaces. Thus these are then the normal, cotangent and conormal spaces to the submanifold. Our motivation for this is that splitting subspaces will form the tangent spaces to cyclides while the splitting submanifolds will be the envelopes of cyclide congruences. The possible enveloped cyclide congruences will then form an affine space modelled on the conormal bundle as can be seen in [15, Section 8.2].

Recall that the tangent space $T_\mathfrak{p}R \cong \mathfrak{g}/\mathfrak{p}$ splits into orbits:

$$\mathfrak{g}/\mathfrak{p} = \mathscr{C}_0 \sqcup \cdots \sqcup \mathscr{C}_r,$$

which can be distinguished by the rank of $P(X) = \frac{1}{2}\mathrm{ad}_X^2$. Here the complement of $\mathscr{C}_r$ is the generalised conformal structure and each $\overline{\mathscr{C}_k} = \mathscr{C}_0 \sqcup \cdots \sqcup \mathscr{C}_k$.

Definition 4.5.2. *Let* $\Pi \leq \mathfrak{g}/\mathfrak{p}$ *such that* $\Pi = \bigoplus_{i=1}^r L_i$ *where* $r = \operatorname{rank} Z_R$. *We call* Π ***splitting*** *if:*

$$\Pi \cap \overline{\mathscr{C}_k} = \bigcup_{\substack{S\subset\{1,\dots,r\}\\ |S|=k}} \bigoplus_{i\in S} L_i. \tag{4.5.1}$$

Thus, for example, $\Pi \cap \mathscr{C}$ is the union of the hyperplanes $H_i = \bigoplus_{j\neq i} L_j$, and $\Pi \cap \overline{\mathscr{C}_1} = \bigcup_{i=1}^r L_i$.

Definition 4.5.3. *We call a submanifold* $f : \Sigma \to R$ ***splitting*** *if at each point* $x \in \Sigma$*, its tangent space* $\operatorname{Im}(df_x)$ *is splitting.*

Then as motivation and example for this definition we note the following:

Proposition 4.5.4. *Cyclides are splitting submanifolds.*

Proof. Firstly, we note that $T_{\mathfrak{p}}C = \mathfrak{s}/\mathfrak{s}\cap\mathfrak{p} = \bigoplus_{i=1}^r L_i$ for $L_i = \mathfrak{s}_i/\mathfrak{s}_i \cap \mathfrak{p}$. Let $X \in T_{\mathfrak{p}}C$ then $X = \sum_{i=1}^r X_i$ for $X_i \in L_i$. Now:

$$P(X) = \sum_{i=1}^r P(X_i) + \sum_{1\leq i<j\leq r} \operatorname{ad}_{X_i} \circ \operatorname{ad}_{X_j}. \tag{4.5.2}$$

From Definition 4.2.1, we can see that $\operatorname{Im} P(X_i) = L_i$ whenever $X_i \neq 0$ and $\operatorname{Im} \operatorname{ad}_{X_i} \circ \operatorname{ad}_{X_j} = U_{i,j}/U_{i,j} \cap \mathfrak{p}$ if $X_i, X_j \neq 0$. So $\operatorname{rank} P(X) = l + al(l-1)/2$ where l is the number of X_i that are non-zero and $a = \dim U_{i,j}/4$. This is equivalent, by Proposition 2.6.12, to $X \in \mathscr{C}_l$. Moreover, Proposition 2.6.12 tells us that $\overline{\mathscr{C}}_l = \mathscr{C}_0 \sqcup \cdots \sqcup \mathscr{C}_l$. Therefore $X \in \overline{\mathscr{C}}_l$ if, and only if, it lies in the span of l of the L_i. $\square$

Other examples include any surface in a quadric whose tangent space is nondegenerate for the induced conformal structure.

We shall see that all splitting subspaces are the tangent space to some cyclide so that cyclides are, in some sense, the prototypical example of a splitting submanifold. To do this, however, we will need a few results first. Let $\Pi = \bigoplus_{i=1}^r L_i \leq \mathfrak{g}/\mathfrak{p}$ be splitting.

Lemma 4.5.5. *For each* L_i *we have* $[\mathfrak{p}/\mathfrak{p}^\perp, L_i] \cap \Pi = L_i$.

Proof. By Theorem 2.6.10, without loss of generality we may assume that L_i is the root space of a long root β. Then under the corresponding root decomposition we denote $D := \{\alpha \in \Delta_0 | \alpha - \beta \in \Delta\}$ and $\mathfrak{g}^D = \bigoplus_{\alpha\in D} \mathfrak{g}^\alpha$. Then by [47, Lemma 3.4], $[\mathfrak{g}^D, [\mathfrak{p}/\mathfrak{p}^\perp, L_i]] \subset L_i$ and $[\mathfrak{g}^D, L_i] = 0$. In particular, since $[\mathfrak{p}/\mathfrak{p}^\perp, L_i]$ is the sum of root spaces $\mathfrak{g}^\gamma$ such that $\gamma \not\perp \beta$ we can find A

such that $[A, X] = 0$ for any $X \in [\mathfrak{g}_0, L_i] \setminus L_i$. To see this, we note that according to the root space decomposition there is some non-zero component in some $\mathfrak{g}^\gamma$ and then we may take $A \in \mathfrak{g}^{\beta-\gamma} \leq \mathfrak{g}^D$. Let $X \in \mathscr{C}_k$. Now:

$$\exp(tA)(X) = X + t[A, X] \in L \oplus \langle X \rangle, \tag{4.5.3}$$

and thus $L \oplus \langle X \rangle \setminus L \subset \mathscr{C}_k$. However, if $X \in H_i = \bigoplus_{j \neq i} L_j \leq \Pi$ then, by (4.5.1), $X + X_i \in \mathscr{C}_{k+1}$ for $X_i \neq 0$. □

Remark. We may think of $[\mathfrak{p}/\mathfrak{p}^\perp, L_i]$ as the tangent space to $\mathscr{C}_1$ at $X_i \in L_i$. Then the proof of Lemma 4.5.5 tells us that the intersection of $T_{X_i}\mathscr{C}_1$ with $\overline{\mathscr{C}_1}$ is a the union of all linear subspaces in $\overline{\mathscr{C}_1}$ which contain L_i.

From this we can construct a complementary subspace to Π:

Definition 4.5.6. *Let $\Pi \leq \mathfrak{g}/\mathfrak{p}$ be splitting. Then we call:*

$$\Pi_\perp := \sum_{1 \leq i,j \leq r} \operatorname{Im} \operatorname{ad}_{L_i} \circ \operatorname{ad}_{L_j}, \tag{4.5.4}$$

*its **normal** subspace.*

Proposition 4.5.7. *Let $\Pi \leq \mathfrak{g}/\mathfrak{p}$ be splitting. Then $\mathfrak{g}/\mathfrak{p} = \Pi \oplus \Pi_\perp$. Moreover (4.5.4) is a direct sum.*

Proof. Firstly, note that $[\mathfrak{p}/\mathfrak{p}^\perp, L_i] \cap [\mathfrak{p}/\mathfrak{p}^\perp, L_j] \cap \Pi = \{0\}$ for each $i \neq j$. Take $X_i, X_j \neq 0$ in L_i, L_j respectively. Then $P(X_i + X_j) = P(X_i) + P(X_j) + \operatorname{ad}_{X_i} \circ \operatorname{ad}_{X_j}$ has rank $\kappa_2 = 2 + a$ by Proposition 2.6.12. Now, $\operatorname{Im}(P(X_i) + P(X_j)) = L_i \oplus L_j \leq \Pi$ and $\operatorname{Im} \operatorname{ad}_{X_i} \circ \operatorname{ad}_{X_j} \leq [\mathfrak{p}/\mathfrak{p}^\perp, L_i] \cap [\mathfrak{p}/\mathfrak{p}^\perp, L_i]$. Thus $\operatorname{Im} \operatorname{ad}_{X_i} \circ \operatorname{ad}_{X_j}$ is a subspace of dimension a which intersects trivially with Π.

Take $X \in \Pi_{reg}$ so that $P(X)$ has rank equal to $\dim R$ (c.f. Proposition 2.6.12). However:

$$P(X) = \sum_{i=1}^{r} P(X_i) + \sum_{1 \leq i < j \leq r} \operatorname{ad}_{X_i} \circ \operatorname{ad}_{X_j}. \tag{4.5.5}$$

This implies that $\operatorname{Im} P(X) \leq \Pi + \Pi_\perp$. Since $\dim R = r + ar(r-1)/2$ and X is regular we can see that $\dim \Pi_\perp \geq ar(r-1)/2$. However, since each $\operatorname{Im} \operatorname{ad}_{X_i} \circ \operatorname{ad}_{X_j}$ has dimension a, $\dim \Pi_\perp \leq ar(r-1)/2$. Thus it is complementary and indeed each $\operatorname{Im} \operatorname{ad}_{L_i} \circ \operatorname{ad}_{L_j}$ is linearly independent by comparing dimensions. □

Remark. When $R = Q^n, S^{p,q}$ the generalised conformal structure is simply the conformal structure and $\Pi_\perp$ is precisely the orthocomplement of Π.

When $R = G_n(\mathbb{C}^{2n})$ any splitting subspace of $T_V G_n(\mathbb{C}^{2n}) = \operatorname{Hom}(V, \mathbb{C}^{2n}/V)$ is given by $\Pi = \bigoplus_{i=1}^n \operatorname{Hom}(V_i, W_i)$ where $V = \bigoplus_{i=1}^n V_i$ and $\mathbb{C}^{2n}/V = \bigoplus_{i=1}^n W_i$ are decompositions into lines. Then $\Pi_\perp = \bigoplus_{i \neq j} \operatorname{Hom}(V_i, W_j)$.

Using this complementary subspace we may construct a canonical dual splitting subspace of $\mathfrak{p}^\perp$.

Proposition 4.5.8. *Let* $\Pi = \bigoplus_{i=1}^r L_i \leq \mathfrak{g}/\mathfrak{p}$ *be a splitting subspace. Let* $\Pi^* = \mathrm{ann}(\Pi_\perp) \leq \mathfrak{p}^\perp$. *Then* $\Pi^* = \bigoplus_{i=1}^r K_i$ *is splitting and:*

1. $P(X)\Pi^* = \Pi$ *for any* $X \in \Pi_{reg}$.
2. $I_\mathfrak{p}$ *restricts to a map* $\Pi_{reg} \to \Pi^*_{reg}$.

Proof. Let $X \in \Pi_{reg}$ and define $\Pi^* := P(X)^{-1}\Pi$. Then for $K_i := P(X)^{-1}L_i$ we have $\Pi^* = \bigoplus_{i=1}^r K_i$. Let $X = \sum_{i=1}^r X_i$ where $X_i \in L_i$. Firstly, we note that, from (4.5.5) and Proposition 4.5.7:

$$\mathrm{ad}_{X_i} \circ \mathrm{ad}_{X_j}|_{\Pi^*} = 0. \tag{4.5.6}$$

Let $\sum_{i=1}^r \lambda_i X_i \in \Pi_{reg}$ then:

$$P\left(\sum_{i=1}^r \lambda_i X_i\right) = \sum_{i=1}^r \lambda_i^2 P(X_i) + \sum_{1\leq i<j\leq r} \lambda_i\lambda_j \mathrm{ad}_{X_i} \circ \mathrm{ad}_{X_j}. \tag{4.5.7}$$

Therefore our definition is independent of our choice of $X \in \Pi_{reg}$. Immediately, we note that $I_\mathfrak{p}(X) \in P(X)^{-1}\Pi$ for every choice of X so $I_\mathfrak{p}(X) : \Pi_{reg} \to \Pi^*_{reg}$. Also for $Y \in \Pi^*, X_{k,l} \in \mathrm{Im}\,\mathrm{ad}_{X_k} \circ \mathrm{ad}_{X_l}$:

$$\begin{aligned}(Y, X_{i,j}) &= (Y, [X_i, [X_j, Z]]),\\ &= ([X_j, [X_i, Y]], Z),\\ &= 0.\end{aligned}$$

Thus $\Pi^* = \mathrm{ann}(\Pi_\perp)$. □

Definition 4.5.9. *We call* $\Pi^* \leq \mathfrak{p}^\perp \cong T^*_\mathfrak{p} R$ *the* ***dual*** *space to* $\Pi \leq \mathfrak{g}/\mathfrak{p} \cong T_\mathfrak{p} R$.

Symmetrically this also has a complementary subspace:

$$\Pi^*_\perp := \mathrm{ann}(\Pi) = \bigoplus_{1\leq i,j\leq r} \mathrm{Im}\,\mathrm{ad}_{K_i} \circ \mathrm{ad}_{K_j}. \tag{4.5.8}$$

Then for any $X = \sum_{i=1}^r \in \Pi_{reg}$ we define:

$$P(X)_\Pi := \sum_{i=1}^r P(X_i), \quad P(X)_\perp := \sum_{1\leq i<j\leq r} \mathrm{ad}_{X_i} \circ \mathrm{ad}_{X_j}. \tag{4.5.9}$$

Then $P(X)_\Pi : \Pi^* \cong \Pi$, $P(X)_\perp : \Pi^*_\perp \cong \Pi_\perp$ and $P(X)_\Pi(\Pi^*_\perp) = 0$, $P(X)_\perp((\Pi^*) = 0$.

Our motivation for defining this dual space is that it allows us to construct cyclides from splitting subspaces.

Proposition 4.5.10. *Let* $\Pi = \bigoplus_{i=1}^r L_i \leq \mathfrak{g}/\mathfrak{p}$ *be splitting with dual space* $\Pi^* = \bigoplus_{i=1}^r K_i \leq \mathfrak{p}^\perp$. *Then* $\mathfrak{s} := \Pi^* \oplus [\Pi, \Pi^*] \oplus \Pi$ *is a cyclide algebra of* $gr_{\mathfrak{p}}(\mathfrak{g}) = \mathfrak{p}^\perp \oplus \mathfrak{p}/\mathfrak{p}^\perp \oplus \mathfrak{g}/\mathfrak{p}$.

Proof. Let $X = \sum_{i=1}^r X_i \in \Pi_{reg}$. Each $P(X_i)|_{K_j} = 0$ for $i \neq j$ and $P(X_i)(K_i) = L_i$. Let $Y = I_{\mathfrak{p}}(X) = \sum_{i=1}^r Y_i$ for $Y_i \in K_i$ so that $P(Y) = P(X)^{-1}$ (c.f. Proposition 2.6.5). Then, symmetrically, $P(Y_i)|_{L_j} = 0$ and $P(Y_i)(L_i) = K_i$. Let $H_i := [X_i, Y_i]$. Thus, in particular $\mathfrak{s}_i := \langle X_i, Y_i, H_i\rangle$ is isomorphic to $\mathfrak{sl}_2$. To show that these $\mathfrak{s}_i$ commute, we first note that (4.5.6) implies that $[H_i, X_j] = 0$, and similarly we observe $[H_i, Y_j] = 0$ and $[H_i, H_j] = [[H_i, X_j], Y_j] - [[H_i, Y_j], X_j] = 0$. Then note that $P(X_i)(Y_i) = -X_i$ so that repeated application of the Jacobi identity gives us:

$$\begin{aligned}
[X_i, Y_j] &= -[P(X_i)(Y_i), Y_j], \\
&= -\frac{1}{2}\left([X_i, [H_i, Y_j]] - [H_i, [X_i, Y_j]]\right), \\
&= \frac{1}{2}\left([H_i, [X_i, Y_j]]\right), \\
&= \frac{1}{2}\left([X_i, [Y_i, [X_i, Y_j]]] - [Y_i, [X_i, [X_i, Y_j]]]\right), \\
&= \frac{1}{2}\left([X_i, [H_i, Y_j]] - [X_i, [Y_i, Y_j]]\right) - [Y_i, P(X_i)(Y_j)], \\
&= 0.
\end{aligned}$$

Therefore $[\Pi, \Pi^*] = \langle H_i | i = 1, \ldots, r\rangle$. Thus $\mathfrak{s} := \Pi^* \oplus [\Pi, \Pi^*] \oplus \Pi = \bigoplus_{i=1}^r \mathfrak{s}_i$ and we need only show that it has the correct module structure. Let $U := \mathfrak{s} \cdot \Pi_\perp$, that is:

$$U = \Pi_\perp + [\mathfrak{s}, \Pi_\perp] + [\mathfrak{s}, [\mathfrak{s}, \Pi_\perp]] + \cdots . \tag{4.5.10}$$

Note that $[\Pi, \Pi_\perp] = 0$ and $[[\Pi^*, \Pi], \Pi_\perp] = \Pi_\perp$ so:

$$[\mathfrak{s}, \Pi_\perp] = \Pi_\perp \oplus [\Pi^*, \Pi_\perp]. \tag{4.5.11}$$

Denote $U_0 := [\Pi^*, \Pi_\perp]$. Recall that $\Pi_\perp$ is the direct sum of spaces of the form $\mathrm{Im}(\mathrm{ad}_{L_i} \circ \mathrm{ad}_{L_j})$.

Then, $[\Pi, U_0] = \Pi_\perp$, $[[\Pi^*, \Pi], U_0] = U_0$ and $[\Pi^*, U_0] = \Pi^*_\perp$ so that:

$$[\mathfrak{s}, [\mathfrak{s}, \Pi_\perp]] = \Pi_\perp \oplus U_0 \oplus \Pi^*_\perp. \tag{4.5.12}$$

Finally, $[\Pi, \Pi^*_\perp] = U_0$, $[[\Pi^*, \Pi], \Pi^*_\perp] = \Pi^*_\perp$ and $[\Pi^*, \Pi^*_\perp] = 0$ so we conclude:

$$U = \Pi_\perp \oplus U_0 \oplus \Pi^*_\perp. \tag{4.5.13}$$

We split this into 4-dimensional submodules as follows: Let $X^1_{i,j}, \dots, X^a_{i,j}$ be a basis for $\operatorname{Im} \operatorname{ad}_{X_i} \circ \operatorname{ad}_{X_j}$. Then:

$$U^k_{i,j} := \mathfrak{s} \cdot X^k_{i,j} = \langle X^k_{i,j} \rangle \oplus [K_i, X^k_{i,j}] \oplus [K_j, X^k_{i,j}] \oplus [K_i, [K_j, X^k_{i,j}]], \quad (4.5.14)$$

is a 4-dimensional $\mathfrak{s}_i \oplus \mathfrak{s}_j$-module as in Definition 4.2.1. Then $\mathfrak{p}^\perp \oplus \mathfrak{g}/\mathfrak{p} \oplus [\mathfrak{p}^\perp \oplus \mathfrak{g}/\mathfrak{p}] = \mathfrak{s} \oplus U$. Let $Z := \operatorname{Ker} \operatorname{ad}_\Pi \cap \operatorname{Ker} \operatorname{ad}_{\Pi^*}$. Then clearly $[\mathfrak{s}, Z] = 0$. We can also see that $Z \oplus U_0 = \mathfrak{p}/\mathfrak{p}^\perp$ and thus $\mathfrak{g} = \mathfrak{s} \oplus U \oplus Z$ as required. $\square$

Corollary 4.5.11. *$\Pi^* = \bigoplus K_i$ is the unique splitting subspace such that $[K_i, L_j] \neq 0$ if, and only if, $i = j$. Consequently, $\mathfrak{s}$ as defined in Proposition 4.5.10 is the unique cyclide algebra of $gr_\mathfrak{p}(\mathfrak{g})$ which intersects $\mathfrak{p}^\perp \oplus \mathfrak{p}/\mathfrak{p}^\perp$ in a Borel subalgebra and which intersects $\mathfrak{g}/\mathfrak{p}$ in Π.*

Proof. Let $V = \bigoplus_{i=1}^r M_i \leq \mathfrak{p}^\perp$ be another splitting subspace. Let $X \in M_i$. We may write $X = \sum_{j=1}^r X_j + \sum_{1 \leq j < k \leq r} X_{j,k}$ for $X_j \in K_j, X_{j,k} \in \operatorname{Im} \operatorname{ad}_{K_j} \circ \operatorname{ad}_{K_k}$. Then $[X, L_j] = 0$ implies that $X_j = 0$ and $X_{j,k} = 0$, $X_{k,j} = 0$. Thus $X \in L_i$ so that $V = \Pi^*$. Take any cyclide algebra $\mathfrak{s}$ of $gr_\mathfrak{p}(\mathfrak{g})$ with $\mathfrak{s} \cap \mathfrak{g}/\mathfrak{p} = \Pi$. If $\mathfrak{s}$ intersects $\mathfrak{p}^\perp \oplus \mathfrak{p}/\mathfrak{p}^\perp$ in a Borel subalgebra, then it intersects $\mathfrak{p}^\perp$ in a splitting subspace $\mathfrak{s} \cap \mathfrak{p}^\perp = \bigoplus_{i=1}^r \mathfrak{s}_i \cap \mathfrak{p}^\perp$. Since $\mathfrak{s}_i, \mathfrak{s}_j$ commute we may conclude that $\mathfrak{s} \cap \mathfrak{p}^\perp = \Pi^*$ and so $\mathfrak{s}$ is precisely the cyclide algebra defined in Proposition 4.5.10. $\square$

Let $\mathfrak{q} \in \Omega_\mathfrak{p} \subset R$. Then $\mathfrak{q}$ defines an isomorphism $gr_\mathfrak{p}(\mathfrak{g}) \cong \mathfrak{g}$ as discussed in Section 2.1.2. Let $\mathfrak{s}_\mathfrak{q}$ be the image of $\mathfrak{s} = \Pi^* \oplus [\Pi, \Pi^*] \oplus \Pi$ under this isomorphism. Then $\mathfrak{s}_\mathfrak{q}$ is a cyclide algebra of $\mathfrak{g}$ with corresponding cyclide $C_\mathfrak{q} \subset R$. Naturally $\mathfrak{p} \in C_\mathfrak{q}$ and $T_\mathfrak{p} C_\mathfrak{q} = \Pi$. Indeed, Corollary 4.5.11 tells us that this is the unique such cyclide. The space $\Omega_\mathfrak{p}$ is an affine space modelled on $\mathfrak{p}^\perp$ (Proposition 2.1.4) and $\mathfrak{q}, \mathfrak{q}' \in \Omega_\mathfrak{p}$ (together with $\Pi \leq \mathfrak{g}/\mathfrak{p}$) define the same cyclide if $\mathfrak{q}' = \exp(Y)\mathfrak{q}$ for $Y \in \mathfrak{s}_\mathfrak{q} \cap \mathfrak{p}^\perp = \Pi^*$. In summary:

Theorem 4.5.12. *Let $\Pi \leq \mathfrak{g}/\mathfrak{p}$ be splitting. Then Π is tangent to a cyclide through $\mathfrak{p}$. Moreover, the space of cyclides containing $\mathfrak{p}$ with tangent space Π is an affine space modelled on $\Pi^*_\perp = \operatorname{ann}(\Pi)$.*

This pointwise result naturally extends to splitting submanifolds:

Corollary 4.5.13. *A submanifold $f : \Sigma \to R$ is splitting if, and only if, it envelopes a congruence of cyclides. The space of cyclide congruences enveloping it is then an affine space modelled on sections of the conormal bundle of f: $\Gamma N^* \Sigma = \Gamma(\operatorname{Im} df)^*_\perp$.*

4.5.2 Parabolic submanifold geometry

In this subsection, we incorporate the approach of [13] and [14]. In brief, an enveloping cyclide congruence allows us to view a splitting submanifold as a parabolic geometry (a Cartan geometry modelled on an R-space) [60]. More precisely, a splitting submanifold is a parabolic subgeometry of the 'flat' parabolic geometry R. We use this to provide a distinguished set of coordinates as well as a distinguished choice of enveloping cyclide congruence for a splitting submanifold. Let $f : \Sigma \to R$ be splitting and let C denote a cyclide congruence enveloped by f. Let $\mathfrak{s}$ be the corresponding bundle of cyclide algebras. Then $\mathfrak{g} = (\mathfrak{s} \oplus Z) \oplus U$ is a reductive decomposition:

$$[\mathfrak{s} \oplus Z, U] \subset U. \tag{4.5.15}$$

In particular, we get a corresponding reduction of the trivial connection Equation (2.3.5):

$$d = \mathscr{D} + \mathscr{N}, \tag{4.5.16}$$

where $\mathscr{D}$ is a connection on $\mathfrak{h} := \mathfrak{s} \oplus Z$ and U. Then $\mathscr{N} \in \Omega^1_\Sigma(U)$.

Proposition 4.5.14. *$\mathscr{N}$ takes values in $U \cap f$.*

Proof. Recall, *df* is characterised by:

$$d\sigma = \mathscr{D}\sigma + [\mathscr{N}, \sigma] \equiv [df, \sigma] \mod f, \tag{4.5.17}$$

for $\sigma \in \Gamma f$. Note that *df* takes values in $\mathfrak{h}/\mathfrak{h} \cap f = \mathfrak{s}/\mathfrak{s} \cap f$ since C envelops f. Thus *df* preserves $U, \mathfrak{h}$. Therefore if $\sigma \in \Gamma(\mathfrak{h} \cap f)$, $\mathscr{D}\sigma + [\mathscr{N}, \sigma] \in \mathfrak{h} + f$ and so $[\mathscr{N}, \sigma] \in f$. Likewise if $\sigma \in \Gamma(U \cap f)$, $\mathscr{D}\sigma + [\mathscr{N}, \sigma] \in U + f$ and $[\mathscr{N}, \sigma] \in f$. Then since parabolic subalgebras are self-normalising [17, Lemma 1.6], $\mathscr{N}$ takes values in f. □

Corollary 4.5.15. *The curvature of $\mathscr{D}$, $R^{\mathscr{D}} \in \Omega^2_\Sigma(\mathfrak{h})$ takes values in f. In other words, f, C form a torsion-free Cartan geometry (c.f.* [57]*).*

Proof. Since $d = \mathscr{D} + \mathscr{N}$ and d is flat we have:

$$0 = R^{\mathscr{D}} + d^{\mathscr{D}}\mathscr{N} + [\mathscr{N} \wedge \mathscr{N}]. \tag{4.5.18}$$

Splitting this by the decomposition $\mathfrak{g} = \mathfrak{h} \oplus U$ yields:

$$R^{\mathscr{D}} + [\mathscr{N} \wedge \mathscr{N}]_{\mathfrak{h}} = 0, \tag{4.5.19}$$

$$d^{\mathscr{D}}\mathscr{N} + [\mathscr{N} \wedge \mathscr{N}]_U = 0, \tag{4.5.20}$$

where $[\mathscr{N} \wedge \mathscr{N}]_{\mathfrak{h}}$, $[\mathscr{N} \wedge \mathscr{N}]_U$ denote the projections onto $\mathfrak{h}, U$ respectively. Then (4.5.19) together with Proposition 4.5.14 tells us $R^{\mathscr{D}} = -[\mathscr{N} \wedge \mathscr{N}]_{\mathfrak{h}} \in \Omega^2_\Sigma(\mathfrak{h} \cap f)$. □

As a consequence, we can equip our splitting submanifold with a distinguished set of coordinates in the direction of the L_i:

Theorem 4.5.16. *Let f be a splitting submanifold with $\operatorname{Im} df = \bigoplus_{i=1}^r L_i$. Then each $H_i := df^{-1}(\bigoplus_{j\neq i} L_j) \leq T\Sigma$ is a Frobenius integrable distribution. Indeed, each possible sum of the L_i forms an integrable distribution.*

In particular, we have a canonical (up to scale) set of coordinates $x_1, \dots, x_r$ on Σ such that x_i is constant on each integral hypersurface of Σ given by H_i.

Proof. Since f is splitting, there exists some enveloping cyclide congruence with cyclide algebra bundle $\mathfrak{s} = \bigoplus_{i=1}^r \mathfrak{s}_i$. Let $d = \mathscr{D} + \mathscr{N}$ be the splitting defined in (4.5.16). Then $\mathscr{D}$ preserves $\mathfrak{h} = \bigoplus_{i=1}^r \mathfrak{s}_i \oplus Z$. In fact, since the summands all commute, $\mathscr{D}$ preserves each $\mathfrak{s}_i$ and Z. We may write:

$$df = \sum_{i=1}^r \omega_i \otimes Y_i, \tag{4.5.21}$$

where $\omega_i \in \Omega^1_\Sigma$ and $Y_i \in \Gamma L_i$. Then $H_i = \operatorname{Ker} \omega_i$. Locally, the ω_i form a basis of Ω^1_Σ and we may take a locally dual basis of sections $X_1, \dots, X_r \in \Gamma T\Sigma$. Then $df_{X_i} = Y_i$ and $\mathscr{D}_{X_i}\sigma = (d + \mathscr{N})_{X_i}\sigma \equiv [Y_i, \sigma] \mod f$. Therefore $\mathscr{D}_{X_j}$ preserves $K_i := f^\perp \cap \mathfrak{s}_i$. Now, let $X, X' \in \Gamma H_i$, so that $\mathscr{D}_X, \mathscr{D}_{X'}$ preserve K_i. From Corollary 4.5.15, $R^{\mathscr{D}}$ takes values in $\mathfrak{h} \cap f$ and so it preserves K_i as well. Then:

$$R^{\mathscr{D}}_{X,X'} - \mathscr{D}_X \mathscr{D}_{X'} + \mathscr{D}_{X'} \mathscr{D}_X = -\mathscr{D}_{[X,X']}. \tag{4.5.22}$$

The left hand side preserves K_i and thus so does the right hand side. Since $\mathscr{D}_{X_i}$ does not preserve K_i this tells us that $[X, X'] \in \Gamma H_i$. Thus H_i is Frobenius integrable. Each sum of the L_i is a finite intersection of these hyperplanes and so is also integrable. Then we may find coordinates $x_1, \dots, x_r$ on Σ such that x_i is constant along H_i. Indeed, up to a choice of pointwise scale, we have $dx_i = \omega_i$. □

In the conformal geometry of surfaces in S^n every surface is splitting and the enveloped cyclide congruences are just the usual enveloped sphere congruences. As we have seen[4] the space of enveloped cyclide congruences is an affine space modelled on the conormal bundle. There is a way, however, to identify a special enveloped congruence due to Thomsen and Blaschke (c.f. [7]) which is as follows. The central sphere congruence of a surface in S^n is the unique enveloped sphere congruence with the same mean curvature (vector) as the surface. To expand this idea to cyclide congruence we employ a more modern description of this (c.f. Burstall–Calderbank [15, Section 10.1]) which makes use of Lie algebra homology:

[4] 4.5.13

Definition 4.5.17 (c.f [14, Section 4] or [15, Section 2A])**.** *The **Lie algebra homology** of f is the homology of the chain chain complex:*

$$0 \xleftarrow{\partial} \mathfrak{g} \xleftarrow{\partial} \Omega^1_\Sigma(\mathfrak{g}) \xleftarrow{\partial} \Omega^2_\Sigma(\mathfrak{g}) \xleftarrow{\partial} \cdots, \tag{4.5.23}$$

where $\partial : \Omega^i_\Sigma(\mathfrak{g}) \to \Omega^{i-1}_\Sigma(\mathfrak{g})$ is given by:

$$\partial(\omega \otimes X) = \sum_{i=1}^{\dim R} \omega_{e_i} \otimes [\epsilon^i, X], \tag{4.5.24}$$

for e_i, ϵ^i dual bases of $\mathfrak{g}/f, f^\perp$ respectively.

Remark. We have specialised this definition to the precise situation we are interested in, but this can be defined for any linear representation of $\mathfrak{g}$ in place of $\mathfrak{g}$. For a general R-space, ∂ has a slightly more complicated form.

Let $\omega \otimes X \in \Omega^1_\Sigma(\mathfrak{g})$. Then $\omega \in \Gamma T^*\Sigma \cong \Gamma f^\perp$ and under this identification (4.5.24) yields

$$\partial(\omega \otimes X) = [\omega, X]. \tag{4.5.25}$$

Definition 4.5.18. [14] *Let C be a cyclide congruence enveloped by f, and $d = \mathscr{D} + \mathscr{N}$ the corresponding reduction of d. Then we call C **normal** if $\partial\mathscr{N} = 0$.*

Example. Every surface $f : \Sigma \to S^n$ is splitting and the unique normal enveloping sphere congruence of f is the central sphere congruence.

Theorem 4.5.19. *Let $f : \Sigma \to R$ be splitting with $\Pi := \operatorname{Im} df$ and $\Pi^* = \bigoplus_{i=1}^r K_i$. There is a unique normal enveloped cyclide congruence $\hat{C}$. It has cyclide algebra:*

$$\hat{\mathfrak{s}} := \bigoplus_{i=1}^{r} \langle \sigma_i, d_{X_i}\sigma_i, d_{X_i}d_{X_i}\sigma_i \rangle, \tag{4.5.26}$$

where $\sigma_i \in \Gamma K_i$.

Proof. Firstly we note that this $\mathfrak{s}$ is indeed a bundle of cyclide algebras. This follows from the observation that $d\sigma_i \equiv [df, \sigma_i] \mod f^\perp$ and $d(d_{X_i}\sigma_i) \equiv [df, d_{X_i}\sigma_i] \mod f$. Thus:

$$\langle d_{X_i}\sigma_i | i = 1, \ldots, r \rangle \equiv [\Pi, \Pi^*] \mod f^\perp, \tag{4.5.27}$$

and:

$$\langle d_{X_i}d_{X_i}\sigma_i | i = 1, \ldots, r \rangle \equiv \Pi \mod f. \tag{4.5.28}$$

In particular $\hat{s}$ is a cyclide algebra. Now let $\mathfrak{s}$ be any cyclide algebra and let $d = \mathscr{D} + \mathscr{N}$ be the corresponding reduction of d. We may write:

$$\mathscr{N} = \sum_{i=1}^{r} \omega_i \otimes \mathscr{N}_i, \tag{4.5.29}$$

where $\omega_i \in \Omega^1_\Sigma$ are precisely the (exact) 1-forms in (4.5.21) which form a dual basis to X_i. Then f naturally identifies ω_i with a section σ_i of K_i. Under this identification:

$$\partial\mathscr{N} = \sum_{i=1}^{r} [\sigma_i, \mathscr{N}_i]. \tag{4.5.30}$$

Now $d_{X_i}\sigma_i = \mathscr{D}_{X_i}\sigma_i + [\mathscr{N}_{X_i}, \sigma_i]$. Thus $d_{X_i}\sigma_i \in \mathfrak{s}_i$ if, and only if, $[\mathscr{N}_{X_i}, \sigma_i] = 0$. Consequently, $\hat{C}$ is the unique normal enveloped cyclide congruence. □

By analogy with the central sphere congruence, we define:

Definition 4.5.20. *We call $\hat{C}$ the **central cyclide congruence** of f.*

The central cyclide congruence will be important for Chapter 6 and is also the first step towards defining a splitting submanifold in terms of intrinsic data as in [14][5].

4.6 Cyclides congruences with two envelopes

In Definition 3.3.2, we defined the Darboux transforms of an isothermic submanifold as certain bundles parallel for one of the isothermic connections. This, ultimately, is the generalisation of Darboux's linear system (see [20] for a modern account of this). However, Darboux's original description of these was as envelopes of a common sphere congruence [29]. More explicitly, a pair of (non-degenerate) isothermic surfaces in S^3 form a Darboux pair if they envelop a common sphere congruence that is conformal and Ribaucour (see for example [11]). This naturally extends to isothermic surfaces in S^n , [43] and in $S^{p,q}$ [61].

In this section, we shall describe the analogue of this for (maximal non-degenerate) isothermic submanifolds of self-dual symmetric R-spaces. Namely, we shall see that a pair of submanifolds are a Darboux pair if, and only if, they envelop a common congruence of cyclides with an appropriate condition on the induced generalised conformal structure.

[5]see [15, part II] for a full account of this in the conformal n-sphere

4.6.1 Curved flats

Firstly, we note that a Darboux pair of isothermic submanifolds form another type of integrable system, the curved flat of Ferus–Pedit [30]. Let N be a symmetric space with symmetric decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. This decomposition gives a reduction of flat differentiation on $\underline{\mathfrak{g}}$ as in (2.3.5):

$$d = \mathscr{D} + \mathscr{N}. \tag{4.6.1}$$

Definition 4.6.1. [30] *Let $\phi : \Sigma \to N$ be a map into a symmetric space N. Let $\mathscr{D}_\phi := \phi^*\mathscr{D}$ and $\mathscr{N}_\phi := \phi^*\mathscr{N}$. Then ϕ is a **curved flat** if $\mathscr{D}_\phi$ is flat, or equivalently if $[\mathscr{N}_\phi \wedge \mathscr{N}_\phi] = 0$.*

Proposition 4.6.2. [17, Theorem 5.8] *A pointwise complementary pair of submanifolds $f, \hat{f} : \Sigma \to R$ form a Darboux pair of isothermic submanifolds if, and only if, $\phi := (f, \hat{f}) : \Sigma \to Z_R$ is a curved flat.*

Furthermore we may then write $\mathscr{N}_\phi = m\eta + m\hat{\eta}$, where $\eta \in \Omega^1_\Sigma(f^\perp)$, $\hat{\eta} \in \Omega^1_\Sigma(\hat{f}^\perp)$. When ϕ is a curved flat these are the closed 1-forms of $f, \hat{f}$ respectively[6] [17, Section 5.2]. We note also that $\mathscr{D}_\phi$ preserves each of $f, \hat{f}, f^\perp, \hat{f}^\perp$.

Recall that an isothermic submanifold is non-degenerate if its quadratic form $q = (\eta, df)$ is non-degenerate. Then, specialising to the case that $f, \hat{f}$ are non-degenerate isothermic submanifolds of maximal dimension we obtain the following:

Proposition 4.6.3. [17, Theorem 6.4 and Proposition 6.5] *A pointwise complementary pair of submanifolds $f, \hat{f} : \Sigma \to R$ form a Darboux pair of maximal non-degenerate isothermic submanifolds if, and only if, $\phi := (f, \hat{f}) : \Sigma \to Z_R$ is a curved flat with $\mathscr{N}_\phi$ taking values in a bundle of Cartan subspaces of Z_R.*

Note that this implies that the dimension of Σ is the rank of Z_R.

4.6.2 The Darboux transform

Throughout this section we explicitly identify $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{q}^\perp$ and $\mathfrak{g}/\mathfrak{q} \cong \mathfrak{p}^\perp$ by projection whenever $\mathfrak{p}, \mathfrak{q} \in R$ are complementary. Let $C \subset R$ be a cyclide and let $\mathfrak{p}, \mathfrak{q} \in C$ be a complementary pair of parabolic subalgebras. As we have seen in Proposition 4.5.4 the tangent spaces at $\mathfrak{p}, \mathfrak{q}$ are splitting subspaces of $\mathfrak{g}/\mathfrak{p}$, $\mathfrak{g}/\mathfrak{q}$ respectively.

We draw attention to the fact that the lines (resp. hyperplanes) of these two splitting subspaces are naturally paired:

[6] Note that this differs from [17] by a sign but this shall not affect our result

Proposition 4.6.4. *We may write the splitting decompositions* $T_{\mathfrak{p}}C = \bigoplus_{i=1}^{r} L_i$ *and* $T_{\mathfrak{q}}C = \bigoplus_{i=1}^{r} K_i$ *such that* $[L_i, K_j] \neq 0$, $(L_i, K_j) \neq 0$ *if, and only if,* $i = j$.

Proof. This follows directly from defining $L_i = \mathfrak{s}_i \cap \mathfrak{q}^\perp$, $K_i = \mathfrak{s}_i \cap \mathfrak{q}^\perp$ where $\mathfrak{s} = \bigoplus_{i=1}^{r} \mathfrak{s}_i$ is the cyclide algebra of C. □

Let $f, \hat{f} : \Sigma \to R$ be two pointwise complementary envelopes of a common cyclide congruence C. Applying Proposition 4.6.4 then gives us paired line bundles $L_i \leq f^\perp \cong T_{\hat{f}}C$, $\hat{L}_i \leq \hat{f}^\perp \cong T_f C$ with $[L_i, \hat{L}_j] \neq 0$ if, and only if, $i = j$.

Theorem 4.6.5. *Let* $f, \hat{f} : \Sigma \to R$ *be a pair of pointwise complementary submanifolds. Then* $f, \hat{f}$ *are a Darboux pair of maximal non-degenerate isothermic submanifolds if, and only if, they envelop a common cyclide congruence such that they induce the same generalised conformal structure on* Σ *and moreover* $df \circ d\hat{f}^{-1}(\hat{L}_i) = L_i$.

Proof. Since $f, \hat{f}$ are pointwise complementary we may consider the map $\phi = (f, \hat{f}) : \Sigma \to R$ and the associated 1-form $\mathscr{N}_\phi$.

Let $\mathfrak{c} := \operatorname{Im} \mathscr{N}_\phi$. As noted above, $\mathscr{N}_\phi = m\eta + m\hat{\eta}$ for some $\eta \in \Omega^1_\Sigma(f^\perp)$, $\hat{\eta} \in \Omega^1_\Sigma(\hat{f}^\perp)$. For d the trivial connection on $\underline{\mathfrak{g}}_\Sigma$:

$$d = \mathscr{D}_\phi + \mathscr{N}_\phi = \mathscr{D}_\phi + m\eta + m\hat{\eta}. \tag{4.6.2}$$

Then $\mathscr{D}_\phi$, $m\eta$ preserve f and so for $\sigma \in \Gamma f$:

$$d\sigma \equiv [m\hat{\eta}, \sigma] \equiv [df, \sigma] \mod f. \tag{4.6.3}$$

Then $df \equiv m\eta \mod f$ and symmetrically $d\hat{f} \equiv m\hat{\eta} \mod \hat{f}$.

Recall from Proposition 4.6.3 that $f, \hat{f}$ are a Darboux pair of maximal non-degenerate isothermic submanifolds if, and only if, $\phi = (f, \hat{f})$ is a curved flat in Z_R and $\mathscr{N}_\phi$ takes values in a bundle of Cartan subspaces. If $\mathfrak{c}$ is a bundle of Cartan subspaces of $f^\perp \oplus \hat{f}^\perp$ then let $\mathfrak{s}$ be the cyclide algebra bundle containing it as per Theorem 4.4.6. Let $C_\mathfrak{s}$ be the corresponding cyclide congruence. Then $\mathfrak{s} = \operatorname{Im}\eta \oplus \operatorname{Im}\hat{\eta} \oplus [\operatorname{Im}\eta, \operatorname{Im}\hat{\eta}]$. Immediately, we note $f \cap \mathfrak{s} = \operatorname{Im}\eta \oplus [\operatorname{Im}\eta, \operatorname{Im}\hat{\eta}]$ and $\hat{f} \cap \mathfrak{s} = \operatorname{Im}\hat{\eta} \oplus [\operatorname{Im}\eta, \operatorname{Im}\hat{\eta}]$. These are both Borel subalgebras of $\mathfrak{s}$ so we see $f, \hat{f}$ are pointwise contained in $C_\mathfrak{s}$. Furthermore, $\operatorname{Im} df = \mathfrak{s}/f \cap \mathfrak{s}$ and $\operatorname{Im} d\hat{f} = \mathfrak{s}/\hat{f} \cap \mathfrak{s}$ so that $f, \hat{f}$ envelop $C_\mathfrak{s}$. Thus $f, \hat{f}$ are splitting. Let $x_i, \hat{x}_i$ denote the coordinates on Σ induced by $f, \hat{f}$ (c.f. Theorem 4.5.16). Then:

$$\eta = \sum_{i=1}^{r} dx_i \otimes \eta_i,$$

$$\hat{\eta} = \sum_{i=1}^{r} d\hat{x}_i \otimes \hat{\eta}_i,$$

for some $\eta_i \in \Gamma L_i$, $\hat{\eta}_i \in \Gamma \hat{L}_i$. Consequently:

$$\mathscr{N}_\phi = \sum_{i=1}^{r} dx_i \otimes m\eta_i + d\hat{x}_i \otimes m\hat{\eta}_i. \tag{4.6.4}$$

Then:

$$0 = [\mathscr{N}_\phi \wedge \mathscr{N}_\phi] = \sum_{i=1}^{r} dx_i \wedge d\hat{x}_i \otimes [m\eta_i, m\hat{\eta}_i]. \tag{4.6.5}$$

Since each $[L_i, \hat{L}_i] \neq \{0\}$ and they are linearly independent we have $dx_i \wedge d\hat{x}_i = 0$ for each i. Thus $d\hat{x}_i = \lambda_i dx_i$ and $d\hat{f}^{-1}(\hat{L}_i) = df^{-1}(L_i) = \bigcap_{j \neq i} \operatorname{Ker} dx_j$ or equivalently $df \circ d\hat{f}^{-1}(\hat{L}_i) = L_i$.

Conversely, let $f, \hat{f}$ be two envelopes of a cyclide congruence $C_{\mathfrak{s}}$ for a cyclide algebra $\mathfrak{s}$. Moreover, let $df \circ d\hat{f}^{-1}(\hat{L}_i) = L_i$. Thus:

$$\eta = \sum_{i=1}^{r} dx_i \otimes \eta_i,$$
$$\hat{\eta} = \sum_{i=1}^{r} dx_i \otimes \hat{\eta}_i,$$

for $\eta_i \in \Gamma L_i$ and $\hat{\eta}_i \in \Gamma \hat{L}_i$. Then $[\mathscr{N}_\phi \wedge \mathscr{N}_\phi] = 0$ and ϕ is a curved flat. Moreover, $\mathfrak{c} = \langle \eta_i + \hat{\eta}_i | i = 1, \ldots, r \rangle$ and $(\eta_i + \hat{\eta}_i, \eta_i + \hat{\eta}_i) = 2(\eta_i, \hat{\eta}_i) \neq 0$ by Proposition 4.6.4. Thus, $\mathfrak{c}$ is non-degenerate for the Killing form and abelian. Thus it is contained in a Cartan subspace by [17, Proposition 6.5] and since $\dim \mathfrak{c} = \operatorname{rank} Z_R$ it must be a Cartan subspace. Thus $f, \hat{f}$ are a Darboux pair of maximal non-degenerate isothermic submanifolds. $\square$

We conclude by noting the following:

Corollary 4.6.6. *Let $f : \Sigma \to R$ be a maximal non-degenerate isothermic submanifold. Then f is splitting and η takes values in* $(\operatorname{Im} df)^* \leq f^\perp$.

4.7 Fully worked example: The Grassmannian

Now we shall consider an example in full detail to illustrate the theory we have discussed. The theory is designed to generalise that of the conformal n-sphere so this is naturally a good example. This, however, is already well understood (see for example [36]) so we shall take instead the example of the self-dual Grassmannian $G_n(\mathbb{C}^{2n})$.

4.7.1 Cyclides

As seen in Section 4.3.2, a cyclide $C \subset G_n(\mathbb{C}^{2n})$ is given by a decomposition of $\mathbb{C}^{2n}$ into 2-dimensional subspaces:

$$V_1 \oplus \cdots \oplus V_n = \mathbb{C}^{2n}. \tag{4.7.1}$$

Let $V \in G_n(\mathbb{C}^{2n})$. Then C contains V if $\dim V \cap V_i = 1$ for each i. Thus for $L_i := V \cap V_i$ we have a decomposition:

$$L_1 \oplus \cdots \oplus L_n = V. \tag{4.7.2}$$

We also obtain a corresponding decomposition of the quotient $\mathbb{C}^{2n}/V$:

$$K_1 \oplus \cdots \oplus K_n = \mathbb{C}^{2n}/V, \tag{4.7.3}$$

where $K_i := V_i/L_i \leq \mathbb{C}^{2n}/V$. The tangent space $T_V C \leq T_V G_n(\mathbb{C}^{2n}) = \mathrm{Hom}(V, \mathbb{C}^{2n}/V)$ is then given by:

$$T_V C = \bigoplus_{i=1}^{n} \mathrm{Hom}(L_i, K_i). \tag{4.7.4}$$

This is naturally seen to be a splitting subspace. Indeed, the generalised conformal structure is precisely the determinantal variety [34, Lecture 9] of $\mathrm{Hom}(V, \mathbb{C}^{2n}/V)$ and the strata of this are given by:

$$\mathscr{C}_k = \{X \in \mathrm{Hom}(V, \mathbb{C}^{2n}/V) \mid \operatorname{rank} X = k\}. \tag{4.7.5}$$

The natural complement to $T_V C$ is:

$$(T_V C)_\perp = \bigoplus_{i \neq j} \mathrm{Hom}(L_i, K_j). \tag{4.7.6}$$

The dual space to $T_V C$ is then:

$$(T_V C)^* = \bigoplus_{i=1}^{n} \mathrm{Hom}(K_i, L_i). \tag{4.7.7}$$

4.7.2 Splitting submanifolds and the central cyclide congruence

Let $f = \mathrm{stab}(V) : \Sigma \to G_n(\mathbb{C}^{2n})$ be a submanifold. Then f is splitting if there are paired decompositions:

$$V = L_1 \oplus \cdots \oplus L_n,$$
$$\mathbb{C}^{2n} = K_1 \oplus \cdots \oplus K_n,$$

such that df takes values in $\bigoplus_{i=1}^n \operatorname{Hom}(L_i, K_i)$. Thus:

$$df = \sum_{i=1}^{r} \omega_i \otimes Y_i. \tag{4.7.8}$$

Then, from Theorem 4.5.16, we note that these ω_i are exact:

$$\omega_i = dx_i, \tag{4.7.9}$$

where x_i form a set of coordinates on Σ. Let $X_i \in \Gamma T\Sigma$ be the dual basis of sections to the ω_i. The central cyclide congruence of f is then given by:

$$V_i := \langle \sigma_i, d_{X_i}\sigma_i \rangle, \tag{4.7.10}$$

for $\sigma_i \in \Gamma K_i$. Indeed for any cyclide congruence $\mathscr{N}$ takes values in:

$$\bigoplus_{1\le i<j\le n} \operatorname{Hom}(V_i, V_j), \tag{4.7.11}$$

and we think of the $\operatorname{Hom}(V_i, V_j)$ component as the derivative of V_i in the direction of V_j. Then the normalisation condition $\partial\mathscr{N} = 0$ is precisely $\mathscr{N}_{X_i}\sigma_i = 0$ or equivalently $d_{X_i}\sigma_i \in V_i$.

4.7.3 Isothermic submanifolds

If $f = \operatorname{stab}(V), \hat{f} = \operatorname{stab}(W) : \Sigma \to R$ are pointwise complementary then Theorem 4.6.5 tells us that they are maximal non-degenerate isothermic submanifolds if there is a paired decomposition:

$$\begin{aligned} V &= L_1 \oplus \cdots \oplus L_n, \\ W &= K_1 \oplus \cdots \oplus K_n, \end{aligned}$$

such that, for some $E_i \in \Gamma \operatorname{Hom}(L_i, K_i), E_{-i} \in \Gamma \operatorname{Hom}(K_i, L_i)$:

$$\begin{aligned} df &\equiv \hat{\eta} := \omega_i \otimes E_i \mod f, \\ d\hat{f} &\equiv \eta := \omega_i \otimes E_{-i} \mod \hat{f}. \end{aligned}$$

4.8 Further directions

There are a number of natural directions to follow on from this work. Firstly, in S^n a pair of surfaces admitting a common conformal sphere congruence is one of three possibilities: a Darboux pair of isothermic surfaces, a dual pair of S-Willmore surfaces or a pair related by a Möbius transformation. This is due to Blaschke–Thomsen in S^3 [7] and Ma in S^n [43]. The analogous solution is also available for $S^{p,q}$ [61]. The natural question is then if there is a generalisation of S-Willmore surfaces into symmetric R-spaces and whether these also envelop cyclide congruences.

Secondly, a surface in S^n together with an enveloped sphere congruence can be described entirely in terms of intrinsic data on the surface. In this section we have only dealt with the extrinsic data, but an intrinsic approach following the ideas of [14] is eminently possible.

Thirdly, it would be enlightening to determine the conditions under which a cyclide congruence admits envelopes or even multiple envelopes. Considering S^2-congruences, in codimension 2 every congruence admits at least one envelope and in codimension 1 every congruence admits two envelopes. However, a more general answer to this question is less obvious.

Chapter 5

Semi-discrete isothermic submanifolds

Integrable geometric theories often lend themselves to discretisation. More specifically, maps from a discrete graph are considered where passage from one vertex to the next is akin to a transformation in the smooth theory. The permutability of the smooth transformations then ensures that this procedure is consistent. In particular, we note that a discrete version of isothermic surfaces was given by Bobenko–Pinkall [8] as maps from a discrete lattice where the image of an elementary quadrilateral is 4 concircular points with prescribed cross-ratio. Thus, this definition reflects the Bianchi permutability of the Darboux transformation as in Proposition 3.7.4. In Burstall et al. [17] this definition is naturally extended to maps into self-dual symmetric R-spaces.

Equally, the discrete and smooth theories are brought together in Burstall et al. [19] into a theory of semi-discrete isothermic surfaces. These are maps from the product of a discrete graph and a smooth curve where passage from one vertex to the next on the graph represents a Darboux transformation of the corresponding curves. In this chapter we aim to generalise this semi-discrete theory to maps into a self-dual symmetric R-space.

Throughout this section we shall assume R is a self-dual symmetric R-space and $\mathcal{I}$ a 1-dimensional real affine manifold.

5.1 Fanning curves

In Burstall et al. [19], semi-discrete isothermic surfaces are described as Darboux transformations of curves equipped with a holomorphic quadratic differential. To extend this naturally to any self dual-symmetric R-space we must consider instead a certain class of curves whose derivatives are always regular[1]. These *fanning* curves derive initially from the work of Ahdout [2] where they are used to study geodesic flows.

5.1.1 Fanning curves in the Grassmannian

In [3], Álvarez-Paiva–Durán define the notion of fanning curves in $G_n(\mathbb{R}^{2n})$. Here we recall the main points of that theory.

Let $\gamma = \mathrm{stab}(V) : \mathcal{I} \to G_n(\mathbb{R}^{2n})$ be a smooth curve. Then $\gamma' : \mathcal{I} \to T_\gamma G_n(\mathbb{R}^{2n}) \cong \mathrm{Hom}(V, \mathbb{R}^{2n}/V)$.

Definition 5.1.1. [3, Definition 1.1] *We call $\gamma : \mathcal{I} \to G_n(\mathbb{R}^{2n})$* ***fanning*** *if γ' is invertible everywhere. Its* ***fundamental endomorphism*** *is $F := (\gamma')^{-1} \in \Gamma\,\mathrm{Hom}(\mathbb{R}^{2n}/V, V)$ which we think of as a section of $\mathrm{End}(\mathbb{R}^{2n})$.*

Let $\sigma \in \Gamma V$ and note that $F(\sigma) = 0$. Then $\sigma' \equiv \gamma'(\sigma) \mod V$ so that $F(\sigma') = \sigma$. Differentiating $F(\sigma) = 0$ gives:

$$F'(\sigma) = -F(\sigma') = -\sigma. \tag{5.1.1}$$

Thus $F' \circ F = -F$. Differentiating $F(\sigma') = \sigma$ then yields $F'(\sigma') = \sigma' - F(\sigma'')$. Therefore,

$$\begin{aligned}
(F')^2(\sigma') &= F'(\sigma' - F(\sigma'')), && (5.1.2)\\
&= F'(\sigma') + F(\sigma''), && (5.1.3)\\
&= (F(\sigma'))', && (5.1.4)\\
&= \sigma'. && (5.1.5)
\end{aligned}$$

Since γ' is invertible, $\mathbb{R}^{2n}$ is spanned by $\{\sigma(t), \sigma'(t) | \sigma \in \Gamma(V)\}$ for each $t \in \mathcal{I}$, and thus $(F')^2$ is the identity. Since $F'(\sigma') = \sigma' - F(\sigma'') \equiv \sigma' \mod V$, we conclude:

Proposition 5.1.2. [3, Proposition 2.3] *F' is a reflection with -1 eigenspace equal to V.*

This leads to a natural complementary curve.

Definition 5.1.3. [3, Definition 2.4] *The* ***horizontal curve*** *$h = \mathrm{stab}(W) : \mathcal{I} \to G_n(\mathbb{R}^{2n})$ is the curve such that W is the $+1$ eigenspace of F'.*

[1] That is their derivatives do not lie in the generalised conformal structure

Note that the horizontal curve is not in general a regular curve and indeed if $\gamma = \exp(tX)\,\mathrm{stab}(V_0) : \mathbb{R} \to G_n(\mathbb{R}^{2n})$, where $V_0, V_\infty \in G_n(\mathbb{R}^{2n})$ and $X \in \mathrm{Hom}(V_0, V_\infty)$ then $h \equiv \mathrm{stab}(V_\infty)$ is constant [3, Proposition 5.8 and Theorem 5.1].

From Section 2.4.1 we may then write:

$$F' = 2\xi^h_\gamma. \tag{5.1.6}$$

To compute how h varies with respect to γ we differentiate further:

Definition 5.1.4. [3, Definition 3.1] *The **Jacobi endomorphism** is* $K := \frac{(F'')^2}{4} \in \Gamma\mathrm{End}(\mathbb{R}^{2n})$.

Since $2\xi^h_\gamma$ takes values in a single conjugacy class of $\mathfrak{g}$ we note that $F'' = 2(\xi^h_\gamma)'$ takes values in $\mathrm{Im}\,\mathrm{ad}_{\xi^h_\gamma} = h^\perp \oplus \gamma^\perp$ (c.f [17, Section 3.4]). Let π_γ, π_h be the projections from $\mathfrak{g}/h, \mathfrak{g}/\gamma$ onto $\gamma^\perp, h^\perp$ respectively. Then:

$$(\xi^h_\gamma)' = \frac{1}{2}F'' = \pi_\gamma(h') + \pi_h(\gamma'). \tag{5.1.7}$$

Consequently,

$$K = \frac{(F'')^2}{4} = \frac{1}{2}(\pi_\gamma(h')\circ\pi_h(\gamma') + \pi_h(\gamma')\circ\pi_\gamma(h')) = \xi^h_\gamma \circ [\pi_h(\gamma'), \pi_\gamma(h')]. \tag{5.1.8}$$

Fanning curves contained in $Lag(\mathbb{R}^{2n}) \subset G_n(\mathbb{R}^{2n})$ are also considered in [3, Section 6] and this suggests the possibility of generalising this to other symmetric R-spaces.

5.1.2 Fanning curves in self-dual symmetric R-spaces

Let $\mathfrak{p} \in R$. Recall that an element $X \in \mathfrak{g}/\mathfrak{p}$ is regular if $P(X) = \frac{1}{2}\mathrm{ad}^2_X : \mathfrak{p}^\perp \to \mathfrak{g}/\mathfrak{p}$ is an isomorphism.

Definition 5.1.5. *Let $\gamma : \mathcal{I} \to R$ be a curve in R. Then, γ is called **fanning** if $\gamma'(t)$ is regular for all $t \in \mathcal{I}$.*

Note immediately that this is a G-invariant property as $(g\gamma)' = g\gamma'$ and G preserves the set of regular elements.

Example. If $R = Q^n$ or $R = S^{p,q}$, a curve $\gamma : \mathcal{I} \to R$ is fanning if (γ', γ') is non-vanishing and, in particular, any smooth regular curve in S^n is fanning.

Example. Let $R = G_n(\mathbb{R}^{2n})$. Then γ' is regular precisely when it is invertible. Thus our definition for symmetric R-space generalises the definition for the Grassmannian.

Recall from Section 2.6.1 the isomorphism $I_{\mathfrak{p}} : (\mathfrak{g}/\mathfrak{p})_{reg} \to \mathfrak{p}^{\perp}_{reg} : X \mapsto P(X)^{-1}X$.

Definition 5.1.6. *Let $\gamma : \mathcal{I} \to R$ be a fanning curve. The* ***fundamental endomorphism*** *of γ is $F_\gamma := I_\gamma(\gamma') \in \Gamma\gamma^{\perp}$.*

Any curve $\gamma : \mathcal{I} \to R$ is vacuously isothermic for any $\eta \in \Omega^1_{\mathcal{I}}(\gamma^{\perp})$. However, in the case of fanning curves, the fundamental endomorphism provides a canonical choice of η (up to pointwise scale):

$$\eta := \frac{1}{\alpha} I_\gamma(\gamma')dt = \frac{1}{\alpha} F_\gamma dt. \tag{5.1.9}$$

Lemma 5.1.7. *Let γ be a fanning curve with $\eta := \frac{1}{\alpha} I_\gamma(\gamma')dt$. Then (γ, η) is a non-degenerate isothermic submanifold with quadratic form $q_\gamma = \frac{4 \dim R}{\alpha} dt^2$.*

Proof. This is a straightforward consequence of Proposition 2.6.5:

$$\begin{aligned} q_\gamma &= (d\gamma, \eta), \\ &= \frac{1}{\alpha}(\gamma', I_\gamma(\gamma'))dt^2, \\ &= \frac{4 \dim R}{\alpha} dt^2. \end{aligned}$$

□

In [19, Definition 2.4], this quadratic form is viewed as a polarisation[2] on $\mathcal{I}$. Then, any other choice of polarisation is just a pointwise scale of q_γ. In particular, it determines α and thus η.

Proposition 5.1.8. *The derivative of the fundamental endomorphism is a lift of the (scaled) grading element $2\xi_\gamma$.*

Proof. From Lemma 2.3.9 and Proposition 2.6.5 we obtain:

$$\begin{aligned} F' &\equiv [\gamma', F] \mod \gamma^{\perp}, \\ &\equiv [\gamma', I_\gamma(\gamma')] \mod \gamma^{\perp}, \\ &\equiv 2\xi_\gamma \mod \gamma^{\perp}. \end{aligned}$$

□

A lift of $\xi_{\mathfrak{p}}$ to $\mathfrak{g}$ is a grading element $\xi^{\mathfrak{q}}_{\mathfrak{p}}$ where $\mathfrak{q} \leq \mathfrak{g}$ is some complementary parabolic subalgebra (c.f. [25, Definition 2.4]). Thus F' defines a curve pointwise complementary to γ:

[2] holomorphic quadratic differential

Definition 5.1.9. *Let γ be a fanning curve with fundamental endomorphism F. The **horizontal curve** h of γ is the complementary curve to γ such that $F' = 2\xi^h_\gamma$.*

We note that, as with fanning curves in the Grassmannian, the definition of h depends only on the two-jet of γ. Also for $g \in G$, since $(g\cdot\gamma)' = g\cdot\gamma'$, the fundamental endomorphism of $g\cdot\gamma$ is $g\cdot F$ and the horizontal curve is $g\cdot h$.

Remark. The next step would be to define the Jacobian endomorphism but this is obtained by squaring $\frac{F''}{2}$. This is not naturally defined in the Lie algebra. However, we can calculate its square root $\frac{F''}{2}$. Note that ξ^h_γ takes values in a single G-orbit. Thus $(\xi^h_\gamma)'$ is a section of $\operatorname{Im}\operatorname{ad}_{\xi^h_\gamma} = \gamma^\perp \oplus h^\perp$. Therefore:

$$\frac{F''}{2} = (\xi^h_\gamma)' = K_\gamma + K_h, \tag{5.1.10}$$

for $K_\gamma \in \Gamma\gamma^\perp$, $K_h \in \Gamma h^\perp$. We can then see from Lemma 2.3.9:

$$\begin{aligned} K_\gamma &\equiv (\xi^h_\gamma)' \mod \gamma, \\ &\equiv [\gamma', \xi^h_\gamma] \mod \gamma, \\ &\equiv -\gamma' \mod \gamma. \end{aligned}$$

Similarly, $K_h \equiv h' \mod h$. Thus $\frac{F''}{2}$ is the sum of the derivatives of γ, h projected onto $h^\perp, \gamma^\perp$, respectively.

5.1.3 Fanning curves in S^n

As an example, we demonstrate all the constructions from the above section in the conformal n-sphere. Let $\gamma = \operatorname{stab}(\Lambda) : \mathcal{I} \to S^n$ be a curve. Then γ is automatically fanning. Let l be some lift of $\Lambda \leq \underline{\mathbb{R}^{n+1,1}}$. The derivative of γ is uniquely determined by the property that $\gamma'(l) \equiv l' \mod \Lambda$ for any such lift. In particular, $\gamma' \equiv \hat{l} \wedge l' \mod \gamma$, for some null section $\hat{l}$ such that $(\hat{l}, l) = -1$. Thus, the fundamental endomorphism is:

$$F = -2\frac{l \wedge l'}{(l', l')}. \tag{5.1.11}$$

Differentiating this yields:

$$\begin{aligned} F' &= \frac{-2}{(l',l')^2}\left((l',l')(l\wedge l')' - (l',l')'(l\wedge l')\right), \\ &= \frac{-2}{(l',l')^2}\left((l',l')l\wedge l'' - 2(l'',l')l\wedge l'\right), \\ &= l \wedge \left(\frac{4(l'',l')}{(l',l')^2}l' - \frac{2}{(l',l')}l''\right). \end{aligned}$$

We know that F' is 2 times some grading element. In particular, it can be written in the form $l \wedge m$ where $(m,m) = 0$ and $(l,m) = 2$. Let:

$$b := \frac{4(l'',l')}{(l',l')^2} l' - \frac{2}{(l',l')} l''.$$

Then $m = b + \lambda l$ for some $\lambda \in \mathbb{R}$. In fact since $0 = (m,m) = (b,b) + 2\lambda(b,l)$, we have $\lambda = \frac{-(b,b)}{2(b,l)}$. Firstly we note that since $(l,l') = 0$ and $(l',l') + (l'',l) = 0$:

$$\begin{aligned}(b,l) &= \frac{4(l'',l')}{(l',l')^2}(l',l) - \frac{2}{(l',l')}(l'',l),\\ &= 2.\end{aligned}$$

Secondly:

$$\begin{aligned}(b,b) &= \left(\frac{4(l'',l')}{(l',l')^2}\right)^2 (l',l') - 2\frac{4(l'',l')}{(l',l')^2}\frac{2}{(l',l')}(l'',l') + \left(\frac{2}{(l',l')}\right)^2 (l'',l'')\\ &= \frac{4(l'',l'')}{(l',l')^2}.\end{aligned}$$

Thus $\lambda = \frac{(b,b)}{4} = \frac{(l'',l'')}{(l',l')^2}$. Consequently:

$$m = \frac{4(l'',l')}{(l',l')^2} l' - \frac{2}{(l',l')} l'' - \frac{(l'',l'')}{(l',l')^2} l. \tag{5.1.12}$$

Then the horizontal curve of γ is given by the span of m. Note that $\langle l, l', l'' \rangle$ defines the congruence of osculating circles to γ and so (5.1.12) tells us that the horizontal curve is a curve through the osculating circles of γ.

Note that we can simplify this greatly by choosing a lift l such that $(l',l') = \pm 1$. In this case we obtain:

$$\begin{aligned}F &= \mp 2l \wedge l',\\ F' &= \mp 2l \wedge l'',\\ m &= \pm\left(\frac{(l'',l'')}{4} l - l''\right)\end{aligned}$$

5.2 Transformations of fanning curves

In this section, we explore how the transformations of isothermic submanifolds preserve the class of fanning curves.

Proposition 5.2.1. *Let $(\gamma, \eta) : \Sigma \to R$ be a fanning curve and let $(\hat{\gamma}, \hat{\eta})$ be a Darboux transform of γ with parameter m. Then, $\hat{\gamma}$ is also a fanning curve.*

Proof. From Proposition 3.3.3, we have:

$$m\eta \equiv d\hat{\gamma} \mod \gamma. \tag{5.2.1}$$

Consequently, $\hat{\gamma}' \equiv \frac{\alpha}{m} I_\gamma(\gamma')$, which must then be regular because γ' is regular. $\square$

We recall that the Darboux transform of isothermic surfaces manifests as the common envelope of a conformal Ribaucour sphere congruence. For isothermic submanifolds, we have a similar characterisation in terms of cyclides as described in Theorem 4.6.5. For fanning curves there is a natural analogue using the circles described in Section 2.6.2.

By analogy with the case of curves in S^n we define the Ribaucour transform.

Definition 5.2.2. *Let $\gamma, \hat{\gamma} : \mathcal{I} \to R$ be curves. Then, $\gamma, \hat{\gamma}$ is a Ribaucour pair of fanning curves if they envelop a common congruence of circles.*

We can characterise this in the following manner:

Proposition 5.2.3. *Let $(\gamma, \eta), (\hat{\gamma}, \hat{\eta}) : \mathcal{I} \to R$ be fanning curves. Then, $\gamma, \hat{\gamma}$ is a Ribaucour pair if, and only if, $\langle \hat{\gamma}' \rangle \equiv \langle \eta \rangle \mod \hat{\gamma}$ or equivalently $\langle \gamma' \rangle \equiv \langle \hat{\eta} \rangle \mod \gamma$.*

Proof. Let $(\gamma, \eta), (\hat{\gamma}, \hat{\eta}) : \mathcal{I} \to R$ be fanning curves. At each point $t \in \mathcal{I}$, there is a unique circle through $\gamma(t)$ and $\hat{\gamma}(t)$ that is tangent to γ. This is the circle defined by $\mathfrak{s} := \langle \xi^\gamma_{\hat{\gamma}}, \eta, I^\gamma_{\hat{\gamma}}(\eta) \rangle$. Then, $T_{\hat{\gamma}} C_{\mathfrak{s}} = \mathfrak{s}/\mathfrak{s} \cap \hat{\gamma}$ and $\hat{\gamma}' \in T_{\hat{\gamma}} C_{\mathfrak{s}}$ if, and only if, $\langle \hat{\gamma}' \rangle \equiv \langle \eta \rangle \mod \hat{\gamma}$. This is entirely symmetric in $\gamma, \hat{\gamma}$ and thus $\gamma, \hat{\gamma}$ is a Ribaucour pair if, and only if, $\langle \gamma' \rangle \equiv \langle \hat{\eta} \rangle \mod \gamma$. $\square$

Then, (5.2.1) characterises the Darboux transformation as particular Ribaucour transformations:

Proposition 5.2.4. *Let $(\gamma, \eta), (\hat{\gamma}, \hat{\eta}) : \mathcal{I} \to R$ be a Ribaucour pair of fanning curves. Then, they are a Darboux pair with parameter m if, and only if, $m\eta \equiv d\hat{\gamma} \mod \hat{\gamma}$ or equally, $m\hat{\eta} \equiv d\gamma \mod \gamma$. Equivalently, they are a Darboux pair with parameter m if, and only if, $(\eta, \hat{\eta}) = \frac{4 \dim R}{m\alpha} dt^2 = \frac{1}{m} q_f$.*

Proof. Firstly, (5.2.1) gives the first condition. Then $(\eta, \hat{\eta}) = \frac{1}{m}(\eta, d\gamma) = \frac{4 \dim R}{m\alpha} dt^2$ by Lemma 5.1.7. $\square$

Similarly, T-transforms and Christoffel duals are also fanning curves.

Proposition 5.2.5. *Let (γ, η) be a fanning curve and let γ_t be a T-transform of γ. Then, γ_t is a fanning curve.*

Proof. Recall that $\gamma_t = \Phi_t(\gamma)$, where Φ_t is some gauge transformation such that $\Phi_t(d + t\eta) = d$. Let $\sigma \in \Gamma\gamma$ and note that:

$$(d + t\eta)\sigma \equiv d\sigma \mod \gamma. \tag{5.2.2}$$

Then by pulling back (2.3.4) we can see that $d\gamma_t$ is defined by:

$$d\sigma_t \equiv [d\gamma_t, \sigma_t] \mod \gamma_t, \tag{5.2.3}$$

for any $\sigma_t \in \Gamma\gamma_t$. Now, $\sigma_t = \Phi_t\sigma$ for some $\sigma \in \Gamma\gamma$ and $d = \Phi_t \circ (d+t\eta) \circ \Phi_t^{-1}$. Therefore, $d\sigma_t = \Phi_t(d\sigma)$, and likewise $d\gamma_t = \Phi_t(d\gamma)$. Thus, $\gamma_t' = \Phi_t(\gamma')$, which must be regular. □

Proposition 5.2.6. *Let $\mathfrak{p}_0, \mathfrak{p}_\infty \in R$ be complementary and let $(\gamma, \eta) : \Sigma \to \Omega_{\mathfrak{p}_\infty} \subset R$ be a fanning curve. Let $\gamma^c : \Sigma \to \Omega_{\mathfrak{p}_0} \subset R$ be the Christoffel dual to γ with respect to $(\mathfrak{p}_0, \mathfrak{p}_\infty)$. Then, γ^c is a fanning curve.*

Proof. Let $F : \Sigma \to \mathfrak{p}_\infty^\perp$ and $F^c : \Sigma \to \mathfrak{p}_0^\perp$ be the stereoprojections of γ and γ^c respectively. Then $(\gamma^c)' = \exp(F^c)(F^c)' \equiv (F^c)' \mod \gamma^c$ and $(F^c)' = \exp(-F)\frac{\eta}{dt}$, which is regular. Therefore, γ^c is fanning. □

Proposition 5.2.7. *If F and F^c are the respective stereoprojections then $(F^c)' = \frac{1}{\alpha} I_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(F')$. Moreover, $\eta^c := \exp(F^c)dF$ is equal to $\frac{1}{\alpha} I_{\gamma^c}(\gamma^c)dt$.*

Proof. From the definition of η and since $g \circ I_{\mathfrak{p}}^{\mathfrak{q}} \circ g^{-1} = I_{g\mathfrak{q}}^{g\mathfrak{p}}$ (c.f. Proposition 2.6.5):

$$\begin{aligned}
(F^c)' &= \frac{1}{\alpha} \exp(-F) I_\gamma(\gamma'), \\
&= \frac{1}{\alpha} \exp(-F) I_\gamma^{\mathfrak{p}_\infty}(F'), \\
&= \frac{1}{\alpha} I_{\exp(-F)\gamma}^{\exp(-F)\mathfrak{p}_\infty}(\exp(-F)F'), \\
&= \frac{1}{\alpha} I_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(F').
\end{aligned}$$

Then, $(I^{\mathfrak{p}_\infty}_{\mathfrak{p}_0})^{-1} = I^{\mathfrak{p}_0}_{\mathfrak{p}_\infty}$ gives:

$$\begin{aligned}\eta^c &= \exp(F^c)F'dt,\\ &= \frac{1}{\alpha}\exp(F^c)I^{\mathfrak{p}_0}_{\mathfrak{p}_\infty}((F^c)')dt,\\ &= \frac{1}{\alpha}I^{\mathfrak{p}_0}_{\gamma^c}(\exp(F^c)(F^c)')dt,\\ &= \frac{1}{\alpha}I^{\mathfrak{p}_0}_{\gamma^c}((F^c)')dt,\\ &= \frac{1}{\alpha}I_{\gamma^c}(\gamma^c)dt.\end{aligned}$$

□

5.3 Semi-discrete isothermic submanifolds

Using the theory of fanning curves and their transformations, it is now possible to create a theory of semi-discrete isothermic submanifolds in self-dual symmetric R-spaces.

5.3.1 Semi-discrete isothermic submanifolds

Let N denote a subgraph of $\mathbb{Z}^n$, $E(N)$ its set of edges and $\vec{E}(N)$ the set of oriented edges.

Definition 5.3.1. *Let R be a self-dual symmetric R-space. Let $f : \mathcal{I} \times N \to R$. We say f is **semi-discrete isothermic** if:*

1. *$f_i := f|_{\mathcal{I}\times\{i\}}$ is a fanning curve for each $i \in N$.*
2. *There exists $m : E(N) \to \mathbb{R} \setminus \{0\}$, constant on opposite edges of each elementary quadrilateral, such that f_i, f_j are a Darboux pair with parameter $m(i, j)$.*

*We call m the **factorising function** of f.*

Example. Let $f_0 : \mathcal{I} \to R$ be a fanning curve. Iteratively define f_i as a Darboux transform of f_{i-1} with parameter $m(i-1, i) \in \mathbb{R} \setminus \{0\}$ for each $i \geq 1$. Similarly define f_{-i} as a Darboux transform of f_{-i+1} with parameter $m(-i, -i+1) \in \mathbb{R} \setminus \{0\}$ Then $f : \mathcal{I} \times \mathbb{Z}; (t, i) \mapsto f_i(t)$ is a semi-discrete isothermic submanifold with factorising function $m : E(\mathbb{Z}) \to \mathbb{R} \setminus \{0\}; (j, j+1) \mapsto m(j, j+1)$. Note in this example there are no quadrilaterals in the graph so there are no restrictions on the values of m apart from that it is non-vanishing.

Semi-discrete isothermic submanifolds naturally intertwine the smooth theory of isothermic submanifolds with the discrete theory:

Proposition 5.3.2. *Let $f : \mathcal{I} \times \mathbb{Z}^2 \to R$ be a semi-discrete isothermic submanifold. Assume that diagonal pairs are complementary on each elementary quadrilateral. Then $f|_{\{t\}\times\mathbb{Z}^2} : \mathbb{Z}^2 \to R$ is a discrete isothermic net in the sense of* [17, Definition 4.13]. *That is, for each elementary quadrilateral $(i,j,k,l) \subset \mathbb{Z}^2$, $f_i(t), f_j(t), f_k(t), f_l(t)$ are concircular with cross-ratio:*

$$cr(f_i(t), f_j(t), f_k(t), f_l(t)) = m(i,l)/m(i,j). \tag{5.3.1}$$

Proof. We must simply show that for each elementary quadrilateral $(i,j,k,l) \subset \mathbb{Z}^2$, (f_i, f_j, f_k, f_l) are pointwise concircular with cross-ratio

$$cr(f_i, f_j, f_k, f_l) = m(i,l)/m(i,j). \tag{5.3.2}$$

That is, f_j and f_l are pointwise complementary and both Darboux transforms of f_i with parameters $m(i,j)$ and $m(i,l)$ respectively. Then f_k is a Darboux transform of f_j and f_l with parameters $m(j,k) = m(i,l)$ and $m(l,k) = m(i,j)$ by assumption. Thus (f_i, f_j, f_k, f_l) form a Bianchi quadrilateral and are concircular with cross ratio (5.3.2) (c.f Proposition 3.7.4). $\square$

As with the smooth and discrete cases there is a natural gauge theory formalism for semi-discrete isothermic submanifolds.

Definition 5.3.3. *Let $N \times \mathcal{I}$ be a semi-discrete domain and let V be a vector bundle over $N \times \mathcal{I}$. That is V_i is a smooth bundle over $\{i\} \times \mathcal{I}$ for each $i \in N$. A **semi-discrete connection** on V is a pair (∇, Γ) such that*

1. *∇_i is a connection on V_i for $i \in N$.*
2. *$\Gamma_{(i,j)} : V_i \to V_j$ is a vector bundle isomorphism for $(i,j) \in \vec{E}(N)$ such that $\Gamma_{(j,i)} = \Gamma^{-1}_{(i,j)}$.*

*We say (∇, Γ) is **flat** if $\Gamma_{(i,j)} \cdot \nabla_i = \nabla_j$.*

These intertwine smooth and discrete connections. Consequently, we have an analogous idea of parallel subbundles (c.f Definition 3.3.1)

Definition 5.3.4. *Let $N \times \mathcal{I}$ be a semi-discrete domain and let V be a vector bundle over $N \times \mathcal{I}$. Let $L \leq V$. Then L is **parallel** for (∇, Γ) if:*

1. *L_i is parallel for ∇_i.*
2. *$L_j = \Gamma_{(i,j)} \cdot L_i$.*

Proposition 5.3.5. *Let $(f,\eta) : N \times \mathcal{I}$ be a semi-discrete surface with each f_i a fanning curve and $\eta_i := \frac{1}{\alpha} I_{f_i}(f_i')dt$. Let $\nabla_i^t := d + t\eta_i$ and $\Gamma_{(i,j)}^t := \Gamma_{f_i}^{f_j}(1 - t/m(i,j))$. Then f is isothermic if, and only if, (∇^t, Γ^t), a semi-discrete connection on the trivial bundle $\underline{\mathfrak{g}}_{N\times\mathcal{I}}$, is flat for all t.*

Proof. The condition that $\Gamma_{(i,j)}^t \cdot \nabla_i^t = \nabla_j^t$ is equivalent to the condition that f_i, f_j are a Darboux pair with parameter $m(i,j)$ (c.f. Proposition 3.3.3). □

5.3.2 Semi-discrete transformations

As with smooth isothermic submanifolds, there is an analogous theory of transformations of semi-discrete isothermic submanifolds.

Definition 5.3.6. *Let $(f,\eta) : N \times \mathcal{I} \to R$ be a semi-discrete isothermic submanifold. Then $\hat{f} : N \times \mathcal{I} \to R$ is a **Darboux transform** of f with parameter μ if:*

1. *f_i is pointwise complementary to $\hat{f}_i, \hat{f}_j$ for each $i \in N$ and each $(i,j) \in E(N)$.*
2. *$\hat{f}$ is parallel for (∇^μ, Γ^μ).*

Then each pair $f_i, \hat{f}_i$ is a Darboux pair of fanning curves with parameter μ and as such we can define $\hat{\eta}_i$ by the requirement that for $\hat{\nabla}_i := d + t\hat{\eta}_i$:

$$\hat{\nabla}_i^t = \Gamma_{f_i}^{\hat{f}_i}\left(1 - \frac{t}{\mu}\right) \cdot \nabla_i^t. \tag{5.3.3}$$

Proposition 5.3.7. *Let $(f,\eta) : N \times \mathcal{I} \to R$ be semi-discrete isothermic with factorising function m and $\hat{f}$ a Darboux transform with parameter μ. Then $(\hat{f}, \hat{\eta})$ is semi-discrete isothermic with factorising function m. Moreover, $f_i, f_j, \hat{f}_j, \hat{f}_i$ form a Bianchi quadrilateral (as in Proposition 3.7.4) of fanning curves:*

$$cr(f_i, f_j, \hat{f}_j, \hat{f}_i) = \mu/m(i,j). \tag{5.3.4}$$

Proof. By definition each $\hat{f}_i$ is a Darboux transform of f_i and is thus a fanning curve. Thus we need only show that $\hat{f}_i, \hat{f}_j$ are a Darboux pair for each $(i,j) \in E(N)$. From [17, Lemma 4.7] we have,

$$\Gamma_{f_j}^{\hat{f}_j}\left(1 - \frac{t}{\mu}\right)\Gamma_{f_i}^{f_j}\left(1 - \frac{t}{m(i,j)}\right) \cdot \nabla_i^t = \Gamma_{\hat{f}_i}^{\hat{f}_j}\left(1 - \frac{t}{m(i,j)}\right)\Gamma_{f_i}^{\hat{f}_i}\left(1 - \frac{t}{\mu}\right) \cdot \nabla_i^t.$$

$$\Gamma_{f_j}^{\hat{f}_j}\left(1 - \frac{t}{\mu}\right) \cdot \nabla_j^t = \Gamma_{\hat{f}_i}^{\hat{f}_j}\left(1 - \frac{t}{m(i,j)}\right) \cdot \hat{\nabla}_i^t.$$

$$\hat{\nabla}_j^t = \Gamma_{\hat{f}_i}^{\hat{f}_j}\left(1 - \frac{t}{m(i,j)}\right) \cdot \hat{\nabla}_i^t.$$

Therefore $\hat{f}_i, \hat{f}_j$ are a Darboux pair with parameter $m(i,j)$ so that $\hat{f}$ is semi-discrete isothermic with factorising function m. Then $\hat{f}_i, \hat{f}_j$ and f_i, f_j are Darboux pairs with the same parameter and equally, so are $f_i, \hat{f}_i$ and $f_j, \hat{f}_j$. Thus, we conclude by the same argument as in the proof of Proposition 5.3.2 that these form a Bianchi quadrilateral with cross-ratio (5.3.4). $\square$

Definition 5.3.8. *Let $f : N \times \mathcal{I} \to \Omega_{\mathfrak{p}_\infty} \subset R$ be semi-discrete isothermic for some $(\mathfrak{p}_0, \mathfrak{p}_\infty) \in Z_R$. Let $f^c : N \times \mathcal{I} \to \Omega_{\mathfrak{p}_0}$ be the map such that $f^c|_{\{i\}\times\mathcal{I}}$ is the Christoffel dual to $f|_{\{i\}\times\mathcal{I}}$ with respect to $(\mathfrak{p}_0, \mathfrak{p}_\infty) \in Z_R$. Then we say f^c is the* ***Christoffel dual*** *of f with respect to $(\mathfrak{p}_0, \mathfrak{p}_\infty)$.*

Proposition 5.3.9. *Let $f : N \times \mathcal{I} \to \Omega_{\mathfrak{p}_\infty} \subset R$ be semi-discrete isothermic and $f^c : N \times \mathcal{I} \to \Omega_{\mathfrak{p}_0} \subset R$ its Christoffel dual. Then f^c is semi-discrete isothermic with the same factorising function.*

Proof. As in Proposition 5.3.7 all we need to prove is that f_i^c, f_j^c is a Darboux pair with parameter $m(i,j)$. This is an immediate result of Theorem 3.6.3, since f_i, f_i^c and f_j, f_j^c are Christoffel dual and f_i, f_j is a Darboux pair. $\square$

Definition 5.3.10. *Let $(f,\eta) : N \times \mathcal{I} \to R$ be semi-discrete isothermic and let (∇^t, Γ^t) be the corresponding semi-discrete connections. Let Φ^t be a gauge transformation such that $\Phi^t \cdot (\nabla^t, \Gamma^t) = (d, \mathrm{id})$. Then we call $\Phi^t \cdot f$ a* ***T-transform*** *of f.*

Remark. In the above definition, $\Phi^t \in \Gamma\mathrm{Aut}(\mathfrak{g})$ and its action on smooth and discrete connections is:

$$(\Phi^t \cdot \nabla)_i = \Phi_i^t \circ \nabla_i \circ (\Phi_i^t)^{-1}, \tag{5.3.5}$$
$$(\Phi^t \cdot \Gamma)_{(i,j)} = \Phi_j^t \circ \Gamma_{(i,j)} \circ (\Phi_i^t)^{-1}. \tag{5.3.6}$$

In particular, if $\Phi_i^t \cdot \nabla_i^t = d$, then for $\Phi_j^t = \Phi_i^t \circ (\Gamma_{(i,j)}^t)^{-1}$:

$$\begin{aligned}(\Phi^t \cdot \nabla^t)_j &= \Phi_j^t \circ \nabla_j^t \circ (\Phi_j^t)^{-1}, && (5.3.7)\\ &= \Phi_j^t \circ \Gamma_{(i,j)}^t \circ \nabla_i^t \circ (\Gamma_{(i,j)}^t)^{-1} \circ (\Phi_j^t)^{-1} && (5.3.8)\\ &= \Phi_i^t \circ \nabla_i^t \circ (\Phi_i^t)^{-1} && (5.3.9)\\ &= d. && (5.3.10)\end{aligned}$$

Thus we can construct such a Φ^t from a smooth gauge transformation along one of the curves.

Proposition 5.3.11. *Let $(f,\eta) : N \times \mathcal{I} \to R$ be a semi-discrete isothermic submanifold and let $f^t = \Phi^t \cdot f$ be a T-transform of f. Then f^t is semi-discrete isothermic with factorising function $m - t$.*

Proof. Again we must show that f_i^t, f_j^t are a Darboux pair with parameter $m(i,j) - t$. This is simply the result of [17, Theorem 3.12]. □

The permutation theorems for smooth isothermic submanifolds can then be applied again in each case to see the analogous permutations theories of semi-discrete isothermic submanifolds.

5.4 Generalised curvature lines

In S^n, semi-discrete isothermic surfaces can be characterised as iterated Darboux transforms of a smooth isothermic surface restricted to a single curvature line (c.f. [19]). In this section, we shall recover a similar result for semi-discrete isothermic submanifolds. In particular, we shall describe them as iterated Darboux transforms of maximal non-degenerate isothermic submanifolds restricted to certain curves on the submanifold.

5.4.1 Curvature lines in S^n

Firstly, we describe the curvature lines in S^n in our notation. Let $(f, \eta) : \Sigma \to S^n$ be a non-degenerate isothermic surface. For ease of computation we fix some complementary null line bundle $\hat{\Lambda}$. Then for some $x \in ((\Lambda \oplus \hat{\Lambda})^\perp)^{\mathbb{C}}$ such that $(x, x) = 0 = (\bar{x}, \bar{x})$:

$$\eta = l \wedge (\omega \otimes x + \overline{\omega} \otimes \overline{x}), \tag{5.4.1}$$

$$df \equiv \hat{l} \wedge \left(\omega \otimes \overline{\lambda x} + \overline{\omega} \otimes \hat{l} \wedge \lambda x\right) \mod f, \tag{5.4.2}$$

where $\omega \in \Omega^1_\Sigma(\mathbb{C})$, and $l \in \Lambda, \hat{l} \in \hat{\Lambda}$ such that $(l, \hat{l}) = -1$. By appropriate choice of ω, we may assume that $(l \wedge x, \hat{l} \wedge \overline{\lambda x}) = 2\overline{\lambda}(x, \overline{x}) = 1$. In particular, this gives $\lambda = \overline{\lambda} = \frac{1}{2(x, \overline{x})}$ and we obtain:

$$df \equiv \frac{1}{2(x, \overline{x})} \hat{l} \wedge \left(\omega \otimes \overline{x} + \overline{\omega} \otimes \hat{l} \wedge x\right) \mod f. \tag{5.4.3}$$

Then the quadratic form of f is given simply by:

$$q_f = (\eta, df) = \omega^2 + \overline{\omega}^2. \tag{5.4.4}$$

Comparing with [55, Section 1.2.7], we see that, locally, $q^{2,0} := \omega^2 = dz^2$ is a holomorphic quadratic differential and z is a holomorphic coordinate. Then we may write $z = u + iv$ where u, v are conformal curvature line coordinates. Immediately, we see:

$$\frac{\partial}{\partial u} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}, \quad \frac{\partial}{\partial v} = i(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}). \tag{5.4.5}$$

Thus

$$\omega_{\frac{\partial}{\partial u}} = 1 = \overline{\omega}_{\frac{\partial}{\partial u}}. \tag{5.4.6}$$

$$\omega_{\frac{\partial}{\partial v}} = i = -\overline{\omega}_{\frac{\partial}{\partial v}}. \tag{5.4.7}$$

Let $p \in \Sigma$ and let $\mathcal{I}_u, \mathcal{I}_v$ be the corresponding coordinate curves through p. Let $\gamma_u := f|_{\mathcal{I}_u}$ and $\gamma_v := f|_{\mathcal{I}_v}$. The choice of coordinates then give a natural parametrisation. Naturally, these curves are fanning since $df_{\frac{\partial}{\partial u}}, df_{\frac{\partial}{\partial v}}$ are regular.

Then for $X \in \Gamma T\Sigma$:

$$\begin{aligned} I_f(df_X) &= \frac{2(x, \overline{x})}{(\omega_X \overline{x} + \overline{\omega}_X x, \omega_X \overline{x} + \overline{\omega}_X x)} l \wedge (\omega_X \overline{x} + \overline{\omega}_X x), \\ &= \frac{1}{\omega_X \overline{\omega}_X} l \wedge (\omega_X \overline{x} + \overline{\omega}_X x), \\ &= \left(l \wedge \frac{1}{\overline{\omega}_X} \overline{x} + \frac{1}{\omega_X} x \right). \end{aligned}$$

Thus, if X is equal to $\frac{\partial}{\partial u}$ or $\frac{\partial}{\partial v}$, (5.4.6) and (5.4.7) tell us that $\eta_X = I_f(df_X)$. As a result, for

$$\begin{aligned} \eta_u &:= \frac{1}{\alpha} I_{\gamma_u}(\gamma_u') dt, \\ \eta_v &:= \frac{1}{\alpha} I_{\gamma_v}(\gamma_v') dt, \end{aligned}$$

we have η_u proportional to $\eta|_{\mathcal{I}_u}$ and η_v proportional to $\eta|_{\mathcal{I}_v}$.

Finally, we note that for $\hat{f} : \Sigma \to S^n$ a Darboux transform of f then $\hat{f}|_{\mathcal{I}_u}, \hat{f}|_{\mathcal{I}_v}$ are Darboux transforms (as fanning curves) of $f|_{\mathcal{I}_u}, f|_{\mathcal{I}_v}$.

5.4.2 Generalised curvature lines

Let $(f, \eta) : \Sigma \to R$ be a maximal non-degenerate isothermic submanifold. Let $\mathcal{I} \subset \Sigma$ be some 1-dimensional submanifold and let $\gamma := f|_\gamma$. We denote the restriction of η to $\Omega^1_{\mathcal{I}}(\gamma^\perp)$ by $\eta|_{\mathcal{I}}$.

Definition 5.4.1. *Let $\mathcal{I} \subset \Sigma$ and let $\gamma := f|_{\mathcal{I}}$. Then, we call $\mathcal{I}$ a **generalised curvature line** of f if γ is a fanning curve and $\langle \eta_\gamma \rangle = \langle \eta|_{\mathcal{I}} \rangle$. That is, for each $X \in \Gamma T\mathcal{I}$, $\eta_X || I_\gamma(d\gamma_X)$.*

This definition ensures that any Darboux transform of f contains a specific Darboux transform of $f|_{\mathcal{I}}$ if $\mathcal{I}$ is a generalised curvature line:

Proposition 5.4.2. *Let $(f,\eta),(\hat{f},\hat{\eta}):\Sigma\to R$ be a Darboux pair of maximal non-degenerate isothermic submanifolds. Let $\mathcal{I}\subset\Sigma$ be a curve such that $f|_{\mathcal{I}}$ is a generalised curvature line of f. Then $f|_{\mathcal{I}},\hat{f}|_{\mathcal{I}}$ are a Darboux pair of fanning curves and $\hat{f}|_{\mathcal{I}}$ is a generalised curvature line of $\hat{f}$.*

Proof. Let $\gamma := f|_{\mathcal{I}},\hat{\gamma} := \hat{f}|_{\mathcal{I}}$. Since γ is a generalised curvature line of f we have $\eta_X||I_\gamma(d\gamma_X)$ for $X\in\Gamma T\mathcal{I}$. Indeed, we can choose X such that $\eta_X = I_\gamma(d\gamma_X)$. Then since $\hat{\gamma}$ is a submanifold of $\hat{f}$ it is parallel for $d+m\eta$ along $\mathcal{I}$. Consequently, it must be parallel for $d+mI_\gamma(d\gamma_X)dt$. Thus it is a Darboux transform of γ and is therefore a fanning curve Proposition 5.2.1. Moreover, $\hat{\eta}$ and $\eta_{\hat{\gamma}}$ are defined by the relations:

$$d+t\hat{\eta} = \Gamma_f^{\hat{f}}\left(1-\frac{t}{m}\right)d+t\eta.$$
$$d+t\eta_{\hat{\gamma}} = \Gamma_\gamma^{\hat{\gamma}}\left(1-\frac{t}{m}\right)d+t\eta_\gamma.$$

Thus, $\eta_{\hat{\gamma}} = \hat{\eta}|_{\mathcal{I}}$ so that $\hat{\gamma}$ is a generalised curvature line of $\hat{f}$. □

To find these lines we can use the explicit forms of η and df obtained in Section 4.6.2. Specifically, we have:

$$\eta = \sum_{i=1}^{r}\omega_i\otimes\eta_i, \tag{5.4.8}$$
$$df = \sum_{i=1}^{r}\omega_i\otimes\zeta_i. \tag{5.4.9}$$

Here, for some (smooth) pointwise choice of root system, η_i,ζ_i are pointwise root vectors of $\beta_i,-\beta_i$, where $\{\beta_1,\dots,\beta_r\}$ are a maximal set of strongly orthogonal roots with root spaces in $(f^\perp)^{\mathbb{C}}$. The ω_i are complex 1-forms and if we take η_i,ζ_i such that (η_i,ζ_i) is constant we can ensure these 1-forms are closed. Now, $[\zeta_i,\eta_j]=0$ if $j\neq i$ and otherwise, from Humphreys [40] we know that $[\zeta_i,\eta_i]=\frac{1}{2}(\beta_i,\beta_i)(\zeta_i,\eta_i)H_{\beta_i}$, where H_{β_i} is the coroot of β_i. Thus, $[\zeta_i,[\zeta_i,\eta_i]] = (\beta_i,\beta_i)(\zeta_i,\eta_i)\zeta_i$ and more generally:

$$\mathrm{ad}^2_{\sum_{i=1}^r\lambda_i\zeta_i}\left(\sum_{j=1}\mu_j\eta_j\right) = \sum_{i=1}^{r}(\beta_i,\beta_i)(\zeta_i,\eta_i)\lambda_i^2\mu_i\zeta_i. \tag{5.4.10}$$

By Theorem 2.6.10, $\sum_{i=1}^r \lambda_i\zeta_i$ is regular precisely when each λ_i is non-zero and in that case: Then ad_{df}^2 :, I_f restricts to a map $(\mathrm{Im}\, df)_{reg} \to (\mathrm{Im}\,\eta)_{reg}$. In fact, we see that:

$$I_f\left(\sum_{i=1}^r \lambda_i\zeta_i\right) = \sum_{i=1}^r \frac{-2}{(\beta_i,\beta_i)(\zeta_i,\eta_i)\lambda_i}\eta_i. \tag{5.4.11}$$

Without loss of generality we may assume $(\beta_i,\beta_i)(\zeta_i,\eta_i) = -2$ so that:

$$I_f\left(\sum_{i=1}^r \lambda_i\zeta_i\right) = \sum_{i=1}^r \frac{1}{\lambda_i}\eta_i. \tag{5.4.12}$$

Thus, if $\mathcal{I} \subset \Sigma$, $\gamma := f|_{\mathcal{I}}$ is fanning precisely when all ω_i are non vanishing on $T\mathcal{I}$.

Moreover, $\mathcal{I}$ is a generalised curvature line if $I_f(df_X) \in \langle \eta(X)\rangle$. From Proposition 2.6.5, scaling X inversely scales $I_f(df_X)$ and so using (5.4.12) we simply need to find $X \in \Gamma T\mathcal{I}$ such that:

$$\sum_{i=1}^r \frac{1}{\omega_i(X)}\eta_i = \sum_{i=1}^r \omega_i(X)\eta_i. \tag{5.4.13}$$

Since the η_i are independent and $\omega_i(X)$ are non-vanishing, this occurs precisely when for each i:

$$(\omega_i(X))^2 = 1. \tag{5.4.14}$$

Theorem 5.4.3. *There are 2^{r-1} generalised curvature lines through each point $p \in R$.*

Proof. The $\omega_i \in \Omega^1_\Sigma$ are pointwise a basis of $(T_p^*\Sigma)^{\mathbb{C}}$. Let $X_i \in \Gamma(T\Sigma)^{\mathbb{C}}$ be the dual basis. Indeed, since the ω_i are closed they are given locally by dz_i for some coordinates $z_i : \Sigma \supset U \to \mathbb{C}$ and then $X_i = \frac{\partial}{\partial z_i}$. Then by (5.4.14), $\eta(X) = \eta_\gamma(X)$ if, and only if:

$$X = \sum_{i=1}^r a_i X_i, \tag{5.4.15}$$

where each $a_i = \pm 1$. Therefore there are 2^{r-1} possibilities for X (up to an overall sign) and by extension there are 2^{r-1} generalised curvature lines passing through any given point. Note that for R a non-Hermitian symmetric R-space, the ω_i either come in complex conjugate pairs or they are real (up to multiplication by i) and therefore we have the same property for each X_i. Consequently, in the complex span of each X as defined in (5.4.15) we have a real line. Hence, this still defines a generalised curvature line on R. □

Chapter 6

Polynomial conserved quantities

Recall that isothermic submanifolds are characterised by a flat family of connections ∇^t (c.f Section 3.2). Locally these flat connections admit many parallel sections but conditions can be placed on the isothermic submanifold by requiring that it admits particular families of sections, polynomially dependent on t.

For isothermic surfaces in S^n, this approach has already identified certain subclasses as those admitting a polynomial conserved quantity of given degree [15], [24], [55]. In brief, these include those contained in some hypersphere, constant mean curvature surfaces and the special isothermic surfaces of Bianchi (for $n = 3$) [6].

Other integrable surface classes, such as Omega surfaces [21] and constrained Willmore surfaces [52], can also be investigated in a similar manner.

In this chapter we will see how this theory can be extended to isothermic submanifolds of symmetric R-spaces. For isothermic surfaces in S^n, the conserved quantities are thought of as sections in the trivial $\mathbb{R}^{n+1,1}$ bundle, but the family of flat connections can be constructed for any linear representation of $\mathfrak{g} = \mathfrak{so}(n+1,1)$. For an arbitrary symmetric R-space the choice of representation is less obvious. Thus, this chapter will demonstrate the basic theory of polynomial conserved quantities for isothermic submanifolds in any representation. We will then consider a specific example, that of the self-dual Grassmannian $G_n(\mathbb{C}^{2n})$ with polynomial conserved quantities in $\Lambda^n \mathbb{C}^{2n}$. Finally, we will explore some of the key features of a representation required to develop the theory further.

6.1 General theory

Let $(f,\eta):\Sigma\to R$ be an isothermic submanifold. Let V be a $\mathfrak{g}$-module and $\underline{V}=\Sigma\times V$ the trivial V bundle over Σ. As in the adjoint representation (c.f. Section 3.2) we may define a family of connections on $\underline{V}$:

$$\nabla_V^t = d + t\eta, \tag{6.1.1}$$

where d is the trivial flat connection on V and η is viewed as an End(V)-valued form.

Lemma 6.1.1. *The connections ∇_V^t are flat for all t.*

Proof. The curvature of ∇_V^t is:

$$R^{\nabla_V^t} = R^d + t[d,\eta] + \frac{t^2}{2}[\eta\wedge\eta]. \tag{6.1.2}$$

Since (f,η) is isothermic we know that $[\eta\wedge\eta]=0$ and η is closed as a $\mathfrak{g}$-valued form and thus $[d,\eta]=0$. Therefore $R^{\nabla_V^t}=0$. □

Remark. If the representation of $\mathfrak{g}$ on V is faithful then flatness of ∇_V^t conversely implies that (f,η) is isothermic. All non-trivial representations are faithful for $\mathfrak{g}$ simple.

Definition 6.1.2. *Let $p(t)=\sum_{i=0}^{\kappa} p_i t^i \in \Gamma\underline{V}[t]$. Then we say $p(t)$ is a **polynomial conserved quantity** of (f,η) of degree κ if $\nabla_V^t p(t)\equiv 0$. If (f,η) admits such a polynomial conserved quantity we shall call it a **special isothermic submanifold** of type (κ,V).*

We can split up this condition by collecting terms in t to obtain a more detailed understanding:

$$\begin{aligned} dp_0 &= 0; \\ dp_i + \eta p_{i-1} &= 0, \qquad\qquad i=1,\dots,\kappa; \\ \eta p_\kappa &= 0. \end{aligned} \tag{6.1.3}$$

Immediately, we note that p_0 is a constant section, and $p_\kappa \in \operatorname{Ker}\eta \leq \underline{\mathfrak{g}}$. We can put some more constraints on possibilities for $p(t)$ by considering invariant symmetric polynomial functions on V. Let $P\in S^kV^*$. Now P is invariant if:

$$P(Xv_1,v_2,\dots,v_k)+P(v_1,Xv_2,\dots,v_k)+\dots+P(v_1,v_2,\dots,Xv_k)=0, \tag{6.1.4}$$

for all $X \in \mathfrak{g}$, $v_1, \ldots, v_k \in V$. Thus, $P(Xv, v, \ldots, v) = 0$ for any $v \in V$. Then for $\sigma_i \in \Gamma V$,

$$dP(\sigma_1, \sigma_2, \ldots, \sigma_k) = P(d\sigma_1, \sigma_2, \ldots, \sigma_k) + \\ P(\sigma_1, d\sigma_2, \ldots, \sigma_k) + \cdots + P(\sigma_1, \sigma_2, \ldots, d\sigma_k). \quad (6.1.5)$$

Combining these facts we obtain:

$$\begin{aligned} dP(p(t), \ldots, p(t)) &= P(dp(t), \ldots, p(t)) + \cdots + P(p(t), \ldots, dp(t)), \\ &= kP(dp(t), \ldots, p(t)), \\ &= kP(dp(t), \ldots, p(t)) + tkP(\rho(\eta)p(t), \ldots, p(t)), \\ &= kP(\nabla^t p(t), \ldots, p(t)), \\ &= 0. \end{aligned}$$

Thus, $P(p(t), \cdots, p(t)) \in \Gamma\mathbb{F}[t]$ has constant coefficients.

To summarise:

Theorem 6.1.3. *Let $(f, \eta) : \Sigma \to R$ be a special isothermic submanifold of type (κ, V) and let $p(t) = \sum_{i=0}^{\kappa} p_i t^i$. Then:*

1. *p_0 is constant.*
2. *For any invariant polynomial function $P \in S^k V^*$, $P(p(t), \ldots, p(t)) \in \Gamma\underline{\mathbb{F}}[t]$ has constant coefficients.*
3. *p_κ is a section of* $\operatorname{Ker} \eta$.

6.1.1 T-transform

As we will see later, treatment of the Darboux and Christoffel transformations will depend on the choice of representation. However, we may describe the T-transform of a special isothermic submanifold without choosing a representation.

Proposition 6.1.4. *Let $(f, \eta) : \Sigma \to R$ be a special isothermic submanifold of type (κ, V) with conserved quantity $p(t)$. Let $f_s := \Phi_s f$ be a T-transform of f. Then $(f_s, \eta_{\mathfrak{s}} := \Phi_s \circ \eta \circ \Phi_s^{-1})$ is a special isothermic submanifold of type (κ, V).*

Proof. Firstly we note that we may treat Φ_s as a gauge transformation on $\underline{V}$. Recall that Φ_s is defined as a local gauge transformation on $\underline{\mathfrak{g}}$ such that $\Phi_s \cdot \nabla^s_{\mathfrak{g}} = d_{\mathfrak{g}}$, where $d_{\mathfrak{g}}$ denotes the trivial connection on $\underline{\mathfrak{g}}$. We can think of $\underline{\mathfrak{g}}$ as an associated bundle of the trivial principal G-bundle. Indeed we may replace G by the simply connected Lie group G' with Lie algebra $\mathfrak{g}$ so that any $\mathfrak{g}$-modules are also G'-modules. Then $d_{\mathfrak{g}}, \nabla^s_{\mathfrak{g}}$ correspond to principal G'-connections ω, ω^s on $\Sigma \times G'$. Locally, there exists a gauge transformation trivialising ω^s which acts as Φ_s on $\underline{\mathfrak{g}}$. We denote this also by Φ_s. The connections induced by ω, ω^t on $\underline{V}$ are precisely d_V, ∇^s_V, respectively. Then $\Phi_s \cdot \nabla^s_V = d_V$ under the appropriate representation of G' on V.

Note that:

$$\begin{aligned}\Phi_s \cdot \nabla^t_V &= \Phi_s \cdot (d_V + s\eta + (t-s)\eta),\\ &= d + (t-s)\eta_s.\end{aligned}$$

Thus if we define $q(t) := \Phi_s p(t+s)$ this is parallel for $d + t\eta_{\mathfrak{s}}$. Then $q(t)$ is polynomial of the same degree as $p(t)$ so that f_t is a special isothermic submanifold of type (κ, V). □

6.2 Example: $R = G_n(\mathbb{C}^{2n})$, $V = \Lambda^n \mathbb{C}^{2n}$

As motivation for this example we note that this coincides with the choice in [24] for $G_2(\mathbb{C}^4) \cong Q^4$. Also the span of a decomposable vector corresponds to a choice of point in R which we will use later.

Let $(f, \eta) : \Sigma \to G_n(\mathbb{C}^{2n})$ be a maximal non-degenerate isothermic submanifold. Let $p(t) := \sum_{i=1}^{\kappa} p_i t^i \in \Gamma \Lambda^n \mathbb{C}^{2n}$ be a polynomial conserved quantity of (f, η) of degree κ.

6.2.1 Invariant bilinear form

The wedge product $\wedge : \Lambda^n \mathbb{C}^{2n} \times \Lambda^n \mathbb{C}^{2n} \to \Lambda^{2n} \mathbb{C}^{2n}$ induces a non-degenerate bilinear form on $\Lambda^n \mathbb{C}^{2n}$ (up to scale). This form is invariant since, for $g \in \mathrm{SL}(2n, \mathbb{C})$:

$$\begin{aligned} g(v_1 \wedge \cdots \wedge v_n) \wedge g(w_1 \wedge \cdots \wedge w_n) &= gv_1 \wedge \cdots \wedge gv_n \wedge gw_1 \wedge \cdots \wedge gw_n,\\ &= \det(g) v_1 \wedge \cdots \wedge v_n \wedge w_1 \wedge \cdots \wedge w_n,\\ &= v_1 \wedge \cdots \wedge v_n \wedge w_1 \wedge \cdots \wedge w_n,\\ &= (v_1 \wedge \cdots \wedge v_n) \wedge (w_1 \wedge \cdots \wedge w_n).\end{aligned}$$

It is symmetric when n is even and symplectic when n is odd.

When n is even we obtain $p(t) \wedge p(t) \in \Gamma \underline{\Lambda^{2n} \mathbb{C}^{2n}}[t] \cong \Gamma \underline{\mathbb{C}}[t]$ has constant coefficients. Explicitly:

$$\begin{aligned} p(t) \wedge p(t) = (p_0 \wedge p_0) + (2p_0 \wedge p_1)t + (2p_0 \wedge p_2 + p_1 \wedge p_1)t^2 + \cdots \\ + (2p_{\kappa-1} \wedge p_\kappa)t^{2\kappa-1} + (p_\kappa \wedge p_\kappa)t^{2\kappa}. \quad (6.2.1)\end{aligned}$$

When n is odd we can construct an invariant symmetric polynomial function of degree 4. However, as we shall see shortly, there are more serious obstructions to a theory of polynomial conserved quantities when n is odd so we will not describe this in more detail.

6.2.2 Cyclide decomposition in $\Lambda^n \mathbb{C}^{2n}$

In this section, we obtain a result analogous to [24, Proposition 2.2], where the top term of a polynomial conserved quantity is seen to be orthogonal to the central sphere congruence. To achieve this we shall first explore the gradings and filtrations induced by f and an enveloped cyclide congruence.

Pointwise f defines a filtration on $\Lambda^n \mathbb{C}^{2n}$:

$$\{0\} = \mathcal{F}_{n+1} < \mathcal{F}_n < \cdots < \mathcal{F}_1 < \mathcal{F}_0 = \Lambda^n \mathbb{C}^{2n}. \tag{6.2.2}$$

where $\mathcal{F}_k = \Lambda^k f \wedge \Lambda^{n-k} \mathbb{C}^{2n}$. Thus:

$$\mathcal{F}_k/\mathcal{F}_{k+1} \cong \Lambda^k f \otimes \Lambda^{n-k}(\mathbb{C}^{2n}/f). \tag{6.2.3}$$

Note η acts as a lowering operator on this grading and similarly to Lemma 2.3.9 we have:

$$d\sigma \equiv df(\sigma) \mod \mathcal{F}_k, \tag{6.2.4}$$

for $\sigma \in \Gamma\mathcal{F}_k$.

Since f is splitting we have decompositions $f = \bigoplus_{i=1}^n f_i$, $\mathbb{C}^{2n} = \bigoplus_{i=1}^n g_i$. Correspondingly, we have:

$$\eta = \sum_{i=1}^n \omega_i \otimes \eta_i,$$
$$df = \sum_{i=1}^n \omega_i \otimes \beta_i,$$

where $\eta_i \in \mathrm{Hom}(g_i, f_i)$ and $\beta_i \in \mathrm{Hom}(f_i, g_i)$.

Let C be an enveloped cyclide congruence of f. Recall that C is given by a decomposition (c.f. Section 4.7):

$$\underline{\mathbb{C}^{2n}} = V_1 \oplus \cdots \oplus V_n, \tag{6.2.5}$$

where each V_i has dimension 2. This then naturally defines a corresponding decomposition of $\underline{\Lambda^n \mathbb{C}^{2n}}$. We shall demonstrate this in stages:

$$\underline{\Lambda^n \mathbb{C}^{2n}} = V_1 \wedge \cdots \wedge V_n \oplus \bigoplus_{i=1}^n \left(\Lambda^2 V_i \wedge \Lambda^{n-2} \left(\bigoplus_{j \neq i} V_j \right) \right). \tag{6.2.6}$$

Now $\bigwedge^{n-2}\left(\bigoplus_{j\neq i} V_j\right)$ can be further decomposed:

$$\bigwedge^{n-2}\left(\bigoplus_{j\neq i} V_j\right) = \bigoplus_{\{j_1,\dots,j_{n-2}\}\leq\{1,\dots,n\}\backslash\{i\}} V_{j_1}\wedge\dots\wedge V_{j_{n-2}} \oplus \bigoplus_{j\neq i}\left(\bigwedge^2 V_j \wedge \bigwedge^{n-4}\left(\bigoplus_{l\neq i,j} V_l\right)\right). \tag{6.2.7}$$

Applying this recursively, we obtain the decomposition:

$$\underline{\bigwedge^n \mathbb{C}^{2n}} = M_0 \oplus \dots \oplus M_{\lfloor n/2\rfloor}, \tag{6.2.8}$$

where:

$$M_l := \bigoplus_{\substack{\mathcal{I}=\{i_1,\dots,i_l\}\subset\{1,\dots,n\}\\ \mathcal{J}=\{j_1,\dots,j_{n-2l}\}\subset\mathcal{I}^c}} \left(\bigwedge^2 V_{i_1}\wedge\dots\wedge\bigwedge^2 V_{i_l}\right)\wedge V_{j_1}\wedge\dots\wedge V_{j_{n-2l}}. \tag{6.2.9}$$

We shall also denote the summands of this last equation by $M_l^{\mathcal{I},\mathcal{J}}$. Each $M_l^{\mathcal{I},\mathcal{J}}$ has dimension 2^{n-2l} so $\dim M_l = 2^{n-2l}\binom{n}{l}\binom{n-l}{n-2l}$.

Remark. This construction and grouping may seem arbitrary at first so we compare this with the quadric case via the exceptional isomorphism $Q^4 \cong G_2(\mathbb{C}^4)$ Section 2.5. Let $V_1\oplus V_2 = \mathbb{C}^4$ be a cyclide. From the quadric viewpoint $M_0 = V_1\wedge V_2 \leq \bigwedge^2\mathbb{C}^4$ is the 4-plane defining the cyclide $Q^2 := \mathbb{P}(M_0\cap\mathcal{L})\subset Q^4$ and $M_1 = \bigwedge^2 V_1\oplus\bigwedge^2 V_2$ is its orthocomplement. This is simply the complexification of the description of a sphere congruence given in [15, Definition 8.2] for S^4.

A quick calculation then gives:

Lemma 6.2.1. *With respect to $\wedge$, each M_l is non-degenerate and orthogonal to each M_k for $k\neq l$.*

The enveloping condition is that $f_i\leq V_i$ and $V_i/f_i = g_i$. Thus η preserves each $M_l^{\mathcal{I},\mathcal{J}}$. Indeed the filtration induced by f naturally induces one on $M_l^{\mathcal{I},\mathcal{J}}$:

$$0 = M_l^{\mathcal{I},\mathcal{J}}\cap\mathcal{F}_{n-l} < M_l^{\mathcal{I},\mathcal{J}}\cap\mathcal{F}_{n-l+1} < \dots < M_l^{\mathcal{I},\mathcal{J}}\cap\mathcal{F}_l = M_l^{\mathcal{I},\mathcal{J}}. \tag{6.2.10}$$

From Theorem 6.1.3, $p_d\in\operatorname{Ker}\eta = \bigcap_{i=1}^n\operatorname{Ker}\eta_i$ and we have:

$$\operatorname{Ker}\eta\cap M_l^{\mathcal{I},\mathcal{J}} = \left(\bigwedge^2 V_{i_1}\wedge\dots\wedge\bigwedge^2 V_{i_l}\right)\wedge f_{j_1}\wedge\dots\wedge f_{j_{n-2l}}. \tag{6.2.11}$$

At this point we specialise to the central cyclide congruence. Recall from Section 4.7, that this is the cyclide congruence $\hat{C}$ defined by

$$\hat{V}_i := \langle\sigma_i, d_{X_i}\sigma_i\rangle, \tag{6.2.12}$$

for $\sigma_i \in \Gamma f_i$ and $X_i \in \Gamma T\Sigma$ dual to $\omega_i \in \Omega^1_\Sigma$. Denote by $\hat{M}_l, \hat{M}_l^{\mathcal{I},\mathcal{J}}$ the corresponding summands of (6.2.9). Let $d = \mathscr{D} + \mathscr{N}$ be the corresponding reduction of connections. That is, $\mathscr{D}$ is a connection on each $\hat{V}_i$ (and thus on each $\hat{M}_l^{\mathcal{I},\mathcal{J}}$) while $\mathscr{N} \in \Omega^1_\Sigma\left(\bigoplus_{1\leq i<j\leq n} \mathrm{Hom}(\hat{V}_i, \hat{V}_j) \oplus \mathrm{Hom}(\hat{V}_i, \hat{V}_j)\right)$.

Thus we obtain the following analogue of [24, Proposition 2.2], [55, Proposition 2.12]:

Theorem 6.2.2. *Let $(f,\eta) : \Sigma \to G_n(\mathbb{C}^{2n})$ be a maximal non-degenerate isothermic submanifold. Let $p(t) = \sum_{i=0}^{\kappa} p_i t^i \in \Gamma\underline{\bigwedge^n \mathbb{C}^{2n}}[t]$ be a polynomial conserved quantity of f. Then if n is even $p_\kappa \in \Gamma\hat{M}_{n/2}$ and $\mathscr{D}p_\kappa = 0$. If n is odd $p(t) \equiv 0$.*

Proof. As we have seen:

$$\begin{aligned} p_\kappa \in \Gamma \operatorname{Ker}\eta &= \bigoplus_{l=0}^{\lfloor n/2\rfloor} \bigoplus_{\substack{\mathcal{I}=\{i_1,\dots,i_l\}\subset\{1,\dots,n\}\\ \mathcal{J}=\{j_1,\dots,j_{n-2l}\}}} \operatorname{Ker}\eta \cap \hat{M}_l^{\mathcal{I},\mathcal{J}}, \\ &= \bigoplus_{l=0}^{\lfloor n/2\rfloor} \bigoplus_{\substack{\mathcal{I}=\{i_1,\dots,i_l\}\subset\{1,\dots,n\}\\ \mathcal{J}=\{j_1,\dots,j_{n-2l}\}\subset\mathcal{I}^c}} \left(\textstyle\bigwedge^2 \hat{V}_{i_1} \wedge \dots \wedge \bigwedge^2 \hat{V}_{i_l}\right) \wedge f_{j_1} \wedge \dots \wedge f_{j_{n-2l}}. \end{aligned}$$

We denote the component of p_κ in $\hat{M}_l^{\mathcal{I},\mathcal{J}}$ by $p_\kappa^{\mathcal{I},\mathcal{J}}$ and we shall show that $p_\kappa^{\mathcal{I},\mathcal{J}} = 0$ unless $\mathcal{J} = \emptyset$. Firstly, we consider the case $\mathcal{I} = \emptyset$. Thus:

$$p_\kappa^{\emptyset,\{1,\dots,n\}} = \sigma_1 \wedge \dots \wedge \sigma_n \in \Gamma \textstyle\bigwedge^n f, \tag{6.2.13}$$

for some $\sigma_i \in \Gamma f_i$. Then, since $d_{X_i}\sigma_j \in \Gamma f$ whenever $i \neq j$:

$$d_{X_i} p_\kappa^{\emptyset,\{1,\dots,n\}} \equiv \sigma_1 \wedge \dots \wedge d_{X_i}\sigma_i \wedge \dots \wedge \sigma_n \mod \textstyle\bigwedge^n f. \tag{6.2.14}$$

Then $d_{X_i}p_\kappa = -\eta_i p_{\kappa-1}$ implies that this is in $\operatorname{Im}\eta_i = f_i \wedge \bigwedge^{n-1}\mathbb{C}^{2n}$. However, $d_{X_i}\sigma_i \in \hat{V}_i \setminus f_i$ unless $\sigma_i = 0$. Therefore $p_\kappa^{\emptyset,\{1,\dots,n\}} = 0$. Similarly we may write:

$$p_\kappa^{\mathcal{I},\mathcal{J}} = \sigma_{i_1} \wedge d_{X_{i_1}}\sigma_{i_1} \wedge \dots \wedge \sigma_{i_l} \wedge d_{X_{i_l}}\sigma_{i_l} \wedge \sigma_{j_1} \wedge \dots \wedge \sigma_{j_{n-2l}}, \tag{6.2.15}$$

for some $\sigma_i \in \Gamma f_i$. Then for $j \in \mathcal{J}$:

$$d_{X_j} p_\kappa^{\mathcal{I},\mathcal{J}} \equiv \sigma_{i_1} \wedge d_{X_{i_1}}\sigma_{i_1} \wedge \dots \wedge \sigma_{i_l} \wedge d_{X_{i_l}}\sigma_{i_l} \wedge \sigma_{j_1} \wedge \dots \wedge d_{X_j}\sigma_j \wedge \dots \wedge \sigma_{j_{n-2l}} \mod \mathcal{F}_{n-l}. \tag{6.2.16}$$

Again, this does not lie in the image of η_j unless $p_\kappa^{\mathcal{I},\mathcal{J}} = 0$. If n is odd $|\mathcal{J}| = n - 2l$ is never 0 and thus $p_\kappa = 0$. Inductively, $p(t) \equiv 0$. If n is even this implies $p_\kappa \in \Gamma\hat{M}_{n/2}$. Then $\hat{M}_{n/2} \cap \operatorname{Im}\eta = \{0\}$. Therefore since $\mathscr{D}$ preserves $\hat{M}_{n/2}$ and $dp_\kappa = -\eta p_{\kappa-1}$ we conclude that $\mathscr{D}p_\kappa = 0$. □

Remark. We compare this to the quadric case $Q^4 \cong G_2(\mathbb{C}^4)$ as before. There, $\hat{M}_0$ is the 4-plane congruence describing the central sphere congruence. Thus, Theorem 6.2.2 says that p_κ is a section of the orthocomplement of $\hat{M}_0$. The connection $\mathscr{D}$ restricted to $\hat{M}_1$ is precisely the normal connection $\nabla^\perp$ so that $\nabla^\perp p_\kappa = 0$ (c.f. [24, Proposition 2.2]).

We have some bounds on the nature of the lower terms of $p(t)$ as well:

Theorem 6.2.3. *Let $(f,\eta) : \Sigma \to G_n(\mathbb{C}^{2n})$ (n even) be a maximal non-degenerate isothermic submanifold. Let $p(t) = \sum_{i=0}^{\kappa} p_i t^i \in \Gamma \underline{\Lambda^n \mathbb{C}^{2n}[t]}$ be a polynomial conserved quantity of f. Then for $i \leq \min\{\kappa, n/2\}$:*

1. $p_{\kappa-i} \in \Gamma(\hat{M}_{n/2-i} \oplus \ldots \oplus \hat{M}_{n/2})$.
2. $p_{\kappa-i} \in \Gamma \mathcal{F}_{n/2-i}$.

Proof. Recall that $d = \mathscr{D} + \mathscr{N}$ where $\mathscr{D}$ preserves each $\hat{V}_i$ (as a connection on $\underline{\mathbb{C}^{2n}}$) and each $\hat{M}_l$ (as a connection on $\underline{\Lambda^n \mathbb{C}^{2n}}$). Equally, $\mathscr{N}(V_i) \subset \bigoplus_{j\neq i} V_j$ so that $\mathscr{N}(\hat{M}_l) \subset \hat{M}_{l-1} \oplus \hat{M}_l \oplus \hat{M}_{l+1}$. Thus the first point follows inductively from $p_\kappa \in \hat{M}_{n/2}$. The second is an immediate consequence of the first since $\hat{M}_{n/2-i} \oplus \ldots \oplus \hat{M}_{n/2} \leq \mathcal{F}_{n/2-i}$. □

6.2.3 Special isothermic submanifolds of type 0

Special isothermic surfaces of type $(0, \mathbb{R}^{n+1,1})$ in S^n are precisely those contained in some hypersphere $S^{n-1} \subset S^n$. There is an analogous result in our example, at least for p_0 decomposable (i.e. $p_0 \in \Lambda^n W$ for $W \in G_n(\mathbb{C}^2 n)$). This can be framed in the language of Schubert varieties. Fix a full flag:

$$0 \leq W_1 \leq \cdots \leq W_{2n} = \mathbb{C}^{2n}. \tag{6.2.17}$$

Definition 6.2.4. *Let $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ such that $1 \leq \alpha_1 < \cdots < \alpha_n \leq 2n$. The* ***Schubert variety*** *(relative to the flag) associated to α is:*

$$S_\alpha := \{V \in G_k(\mathbb{C}^{2n}) |\dim(V \cap W_{\alpha_i}) \geq i\} \subset G_n(\mathbb{C}^{2n}). \tag{6.2.18}$$

In particular, we are interested in the Schubert variety associated to

$$\alpha := (\frac{n}{2} + 1, \ldots, n, \frac{3n}{2} + 1, \ldots 2n). \tag{6.2.19}$$

In other words $\alpha_{\frac{n}{2}} = n$ and α splits into two sequences of consecutive numbers ending in n, $2n$ respectively. Thus:

$$\begin{aligned} S_\alpha := \{V \in G_n(\mathbb{C}^{2n}) |\dim(V \cap W_{n/2+i}) \geq i, 1 \leq i \leq n/2;& \\ \dim(V \cap W_{3n/2+i}) \geq i, n/2+1 \leq i \leq n\}.& \end{aligned} \tag{6.2.20}$$

Let $V \in G_n(\mathbb{C}^{2n})$. Then $\dim(V \cap W_{2n-i}) \geq n - j$, since W_{2n-j} has codimension i. Similarly if $\dim(V \cap W_n) \geq n/2$ then $\dim(V \cap W_{n-j}) \geq n/2 - j$. Consequently we can simplify:

$$S_\alpha = \{V \in G_n(\mathbb{C}^{2n}) | \dim(V \cap W_n) \geq n/2\}. \tag{6.2.21}$$

Note that S_α depends only on W_n and not on the full flag. Conversely, the set S_α determines W_n. One way to see this is that $W_n \in S_\alpha$ is a singular point and has the highest multiplicity among singular points. Therefore we will denote it S_{W_n}. From [41, Theorem 5.3.7] we see that $\dim S_{W_n} = \sum_{i=1}^n (\alpha_i - i) = n^2 - (\frac{n}{2})^2$.

Theorem 6.2.5. *Let $(f, \eta) : \Sigma \to G_n(\mathbb{C}^{2n})$, n even, be a maximal non-degenerate isothermic submanifold. Then f is special of type $(0, \Lambda^n \mathbb{C}^{2n})$ with decomposable conserved quantity $p_0 \in \Lambda^n W_n$ if, and only if, f is contained in S_{W_n}.*

Proof. If f admits a constant conserved quantity $p_0 \in \Lambda^n W_n$ then by Theorem 6.2.2, $p_0 \in \Gamma \hat{M}_{\frac{n}{2}} \leq \Gamma \mathcal{F}_{\frac{n}{2}}$. Thus $\dim(f \cap W_n) \geq \frac{n}{2}$ and so $\operatorname{Im} f \subset S_{W_n}$.

Conversely assume $\dim(f \cap W_n) \geq \frac{n}{2}$ so that $\Lambda^n W_n \leq \mathcal{F}_{\frac{n}{2}}$. Fix $p_0 \in \Lambda^n W_n$. Then $p_0 \perp \mathcal{F}_{\frac{n}{2}+1} = \mathcal{F}_{n/2}^\perp$. We will show that $p_0 \perp \operatorname{Im} \eta = \bigoplus_{l=0}^n \hat{M}_l \cap \mathcal{F}_{l-1}$ and thus $p_0 \in \operatorname{Ker} \eta = (\operatorname{Im} \eta)^\perp = \bigoplus_{l=0}^n \hat{M}_l \cap \mathcal{F}_{n-l+1}$.

Note that $\mathcal{F}_{\frac{n}{2}+1} = \bigoplus_{l=0}^{n/2-1} \hat{M}_l \cap \mathcal{F}_{\frac{n}{2}+1}$ so we may consider each $\hat{M}_l$ in turn. Firstly, $\hat{M}_{\frac{n}{2}-1} \cap \mathcal{F}_{\frac{n}{2}+1}$ is locally spanned by sections of the form:

$$s^{\mathcal{I},\mathcal{J}} := \sigma_{i_1} \wedge d_{X_{i_1}} \sigma_{i_1} \wedge \cdots \wedge \sigma_{i_{\frac{n}{2}-1}} \wedge d_{X_{i_{\frac{n}{2}-1}}} \sigma_{i_{\frac{n}{2}-1}} \wedge \sigma_{j_1} \wedge \sigma_{j_2}, \tag{6.2.22}$$

where $\sigma_i \in \Gamma f_i$, $\mathcal{I} = \{i_1, \dots i_{\frac{n}{2}-1}\}$, $\mathcal{J} = \{j_1, j_2\}$. Then $d_{X_j} s \equiv 0 \mod \mathcal{F}_{\frac{n}{2}+1}$ unless $j \in j_1, j_2$. Indeed, $\hat{M}^{\mathcal{I},\mathcal{J}}_{\frac{n}{2}-1} \cap \mathcal{F}_{\frac{n}{2}}$ is spanned by $s^{\mathcal{I},\mathcal{J}}, d_{X_{j_1}} s^{\mathcal{I},\mathcal{J}}, d_{X_{j_2}} s^{\mathcal{I},\mathcal{J}}$. Since $p_0 \perp s^{\mathcal{I},\mathcal{J}}$ and p_0 is constant, we have $p_0 \perp ds^{\mathcal{I},\mathcal{J}}$. Thus $p_0 \perp \hat{M}_{\frac{n}{2}-1} \cap \mathcal{F}_{\frac{n}{2}}$. Similarly, we see $p_0 \perp \hat{M}_l \cap \mathcal{F}_{\frac{n}{2}}$ for all $l < \frac{n}{2}$. By an almost identical argument we can then show that $p_0 \perp \hat{M}_l \cap \mathcal{F}_{\frac{n}{2}-1}$ for $l < \frac{n}{2} - 1$. Note that our maximum value for l drops since, for $s \in \Gamma \hat{M}_l \cap \mathcal{F}_{\frac{n}{2}}$, ds may have a non-zero $M_{\frac{n}{2}}$-component. Proceeding inductively, we obtain $p_0 \perp \hat{M}_l \cap \mathcal{F}_{l-1} = (\hat{M}_l \cap \mathcal{F}_{n-l+1})^\perp$ for each l. Thus $p_0 \in \bigoplus_{l=0}^{n/2} \hat{M}_l \cap \mathcal{F}_{n-l+1} = \operatorname{Ker} \eta$. □

Remark. When $n = 2$ this implies that f is contained in a (degenerate) quadric hypersurface of Q^4. Indeed the condition that $\dim f \cap W_2 \geq \frac{n}{2} = 1$ is precisely equivalent to $\Lambda^2 f \wedge \Lambda^2 W_2 = 0$.

6.2.4 Special isothermic submanifolds of type $\kappa \leq n/2$

Treating special isothermic submanifolds of higher type in our example proves to be more difficult than in [15], [24]. However, we can impose some restrictions which we discuss here.

Firstly, we note the difference between our theory and that of [24] for $\kappa = 1$. The key in their example is to use the constant term p_0 of the conserved quantity to define a space form inside S^n and a corresponding "tangent sphere congruence". Such a construction depends on the assumption that p_0 is not orthogonal to f. However, for $n > 2$, Theorem 6.2.3 tells us that $p_0 \in \mathcal{F}_{\frac{n}{2}-1}$ and so is orthogonal to f everywhere. Indeed, this holds true for all p_i, $i < n/2$.

We can, instead, recover a result similar to that of the case $\kappa = 0$. Again we restrict to the case that p_0 is decomposable.

Proposition 6.2.6. *Let $(f, \eta) : \Sigma \to G_n(\mathbb{C}^{2n})$ be a maximal non-degenerate isothermic submanifold. If f is special of type $(\kappa, \Lambda^n \mathbb{C}^{2n})$, $\kappa < n/2$ with decomposable constant term $p_0 \in \Lambda^n W_n$ then:*

$$\operatorname{Im} f \subset \{V \in G_n(\mathbb{C}^{2n}) |\, \dim(V \cap W_n) \geq \frac{n}{2} - \kappa\}. \tag{6.2.23}$$

Proof. This is a direct consequence of Theorem 6.2.3 which states that $p_0 \in \Gamma\mathcal{F}_{\frac{n}{2}-\kappa}$. □

Remark. The set $\{V \in G_n(\mathbb{C}^{2n}) |\, \dim(V \cap W_n) \geq \frac{n}{2} - \kappa\}$ is again a special kind of Schubert variety. Using the notation in the previous section it is S_α for:

$$\alpha = \left(\frac{n}{2} + \kappa + 1, \ldots, n, \frac{3n}{2} - \kappa + 1, \ldots, 2n\right). \tag{6.2.24}$$

That is, $\alpha_{\frac{n}{2}-\kappa} = n$ and α splits into two sequences of consecutive numbers ending in n and $2n$ respectively. It has dimension $n^2 - \left(\frac{n}{2} - \kappa\right)^2$.

6.2.5 Darboux transformations and the Christoffel dual

As with the T-transform we may use the corresponding gauge transformations. However, in general these may not have a well defined action on our representation. Let $V \oplus W = \mathbb{C}^{2n}$ for n even and define:

$$\Gamma_W^V(s) := \begin{cases} s^{\frac{n}{2}} & \text{on } \bigwedge^n V, \\ s^{\frac{n}{2}-1} & \text{on } \bigwedge^{n-1} V \wedge W, \\ \vdots & \vdots \\ s^{\frac{n}{2}-i} & \text{on } \bigwedge^{n-i} V \wedge \bigwedge^i W, \\ \vdots & \vdots \\ s^{-\frac{n}{2}} & \text{on } \bigwedge^n W. \end{cases} \tag{6.2.25}$$

This is precisely the action of $\Gamma_{\mathfrak{q}}^{\mathfrak{p}}(s)$ as defined in Section 3.3 where $\mathfrak{p} = \operatorname{stab}(V)$, $\mathfrak{q} = \operatorname{stab}(W)$. Note that if n is odd this is not well defined as $s^{\frac{n}{2}}$ is not well defined. As we noted before special isothermic submanifolds do not exist in this case anyway. Thus we consider only the case n is even.

6.2.6 Darboux transform

Proposition 6.2.7. *Let $(f, \eta) : \Sigma \to G_n(\mathbb{C}^{2n})$, n even, be a special isothermic submanifold of type $(\kappa, \bigwedge^n \mathbb{C}^{2n})$. Let $(\hat{f}, \hat{\eta}) : \Sigma \to G_n(\mathbb{C}^{2n})$ be a Darboux transform of f. Then $\hat{f}$ is a special isothermic submanifold of type $(\kappa + \frac{n}{2}, \bigwedge^n \mathbb{C}^{2n})$.*

Proof. Let:

$$q(t) := \Gamma_f^{\hat{f}}\left(1 - \frac{t}{m}\right) p(t). \tag{6.2.26}$$

and:

$$\hat{p}(t) := \left(1 - \frac{t}{m}\right)^{\frac{n}{2}} q(t). \tag{6.2.27}$$

Note that $\Gamma_f^{\hat{f}}\left(1 - \frac{t}{m}\right)$ acts on $\bigwedge^n f$ as $\left(1 - \frac{t}{m}\right)^{-\frac{n}{2}}$ and elsewhere as higher powers of $\left(1 - \frac{t}{m}\right)$. Therefore $\hat{p}(t)$ is polynomial. Then by Theorem 6.2.3 $p_{\kappa-i} \in \mathcal{F}_{\frac{n}{2}-i}$ so that the degree of $q(t)$ is at most κ and the degree of $\hat{p}(t)$ is at most $\kappa + \frac{n}{2}$. To see that it is conserved we note that:

$$\begin{aligned} \hat{\nabla}^t(q(t)) &= \Gamma_f^{\hat{f}}\left(1 - \frac{t}{m}\right) \circ \nabla_V^t \circ \Gamma_f^{\hat{f}}\left(1 - \frac{t}{m}\right)^{-1} \left(\Gamma_f^{\hat{f}}\left(1 - \frac{t}{m}\right) p(t)\right), \\ &= \Gamma_f^{\hat{f}}\left(1 - \frac{t}{m}\right) \circ \nabla_V^t(p(t)), \\ &= 0. \end{aligned}$$

Thus $\hat{\nabla}^t(\hat{p}(t)) = \left(1 - \frac{t}{m}\right)^{\frac{n}{2}} \hat{\nabla}^t(q(t)) = 0.$ □

We can also quantify when the Darboux transform is a special isothermic submanifold of degree between κ and $\kappa + \frac{n}{2}$. Let $\hat{\mathcal{F}}_k := \Lambda^k \hat{f} \wedge \Lambda^{n-k} \mathbb{C}^{2n}$.

Proposition 6.2.8. *Let $(f, \eta) : \Sigma \to G_n(\mathbb{C}^{2n})$, n even, be a special isothermic submanifold of type $(\kappa, \Lambda^n \mathbb{C}^{2n})$ with conserved quantity $p(t) = p(t) = \sum_{i=0}^{\kappa} p_i t^i$. Let $(\hat{f}, \hat{\eta}) : \Sigma \to G_n(\mathbb{C}^{2n})$ be a Darboux transform of f. Then $\hat{f}$ is a special isothermic submanifold of type $(\kappa + j, \Lambda^n \mathbb{C}^{2n})$, $1 \leq j \leq \frac{n}{2}$ if $p_i \in \Gamma \hat{\mathcal{F}}_{\frac{n}{2}-i-j}$.*

Proof. As in the previous proof we take $q(t) := \Gamma_f^{\hat{f}}\left(1 - \frac{t}{m}\right) p(t)$. Denote by p_i^k the component of p_i in $\Lambda^k f \wedge \Lambda^{n-k} \hat{f}$. Thus, by (6.2.25):

$$\Gamma_f^{\hat{f}}\left(1 - \frac{t}{m}\right) p_i^k = \left(1 - \frac{t}{m}\right)^{\frac{n}{2}-k} p_i^k. \tag{6.2.28}$$

The condition $p_i \in \Gamma \hat{\mathcal{F}}_{\frac{n}{2}-i-j}$ is equivalent to $p_i^k = 0$ for all $k > \frac{n}{2} + i + j$. Therefore the lowest degree term of $\Gamma_f^{\hat{f}}\left(1 - \frac{t}{m}\right) t^i p_i$ is $\Gamma_f^{\hat{f}}\left(1 - \frac{t}{m}\right) t^i p_i^{\frac{n}{2}+i+j} = \left(1 - \frac{t}{m}\right)^{-i-j} t^i p_i^{\frac{n}{2}+i+j}$ which has degree $-j$. Thus,

$$\hat{p}(t) := \left(1 - \frac{t}{m}\right)^j q(t), \tag{6.2.29}$$

is polynomial, has degree $\kappa + j$, and is conserved since $q(t)$ is. $\square$

If we let $n = 2$ we recover [24, Theorem 3.2] for Q^4. Then the condition for $\hat{f}$ to be special of degree κ is simply $p_0 \perp \Lambda^2 \hat{f}$.

6.2.7 Christoffel dual

The Christoffel transform follows a very similar argument with $\Gamma_f^{\hat{f}}\left(1 - \frac{t}{m}\right)$ replaced by $\Gamma^c(t) = \exp(F^c)\Gamma_{\mathfrak{p}_0}^{\mathfrak{p}_\infty}(t)\exp(-F)$.

Proposition 6.2.9. *Let $(f, \eta) : \Sigma \to G_n(\mathbb{C}^{2n})$, n even, be a special isothermic submanifold of type $(\kappa, \Lambda^n \mathbb{C}^{2n})$. Let $(f^c, \eta^c) : \Sigma \to G_n(\mathbb{C}^{2n})$ be the Christoffel dual of f with respect to $(V_0, V_\infty) \in Z_R$. Then f^c is a special isothermic submanifold of type $(\kappa + \frac{n}{2}, \Lambda^n \mathbb{C}^{2n})$.*

Proof. Let:

$$\begin{aligned} q(t) &:= \Gamma^c(t) p(t), \\ &= \exp(F^c)\Gamma_{V_0}^{V_\infty}(t)\exp(F)p(t), \end{aligned}$$

and

$$p^c(t) := t^{\frac{n}{2}} q(t), \tag{6.2.30}$$

where F, F^c are the stereoprojections of f, f^c respectively. Note that $\exp(-F)(f) \equiv V_0$ so:

$$\exp(-F)(\bigwedge^j f \otimes \bigwedge^{n-j} V_\infty) \equiv \bigwedge^j V_0 \otimes \bigwedge^{n-j} V_\infty. \tag{6.2.31}$$

Then $\Gamma_{V_0}^{V_\infty}(t)$ acts as $t^{\frac{n}{2}-j}$ on $\bigwedge^j V_0 \otimes \bigwedge^{n-j} V_\infty$. Thus the term of lowest degree in $q(t)$ has degree (at least) $-\frac{n}{2}$. Therefore $p^c(t)$ is polynomial and has degree (at most) $\kappa + \frac{n}{2}$.

Then it is conserved as:

$$\begin{aligned}(d + t\eta^c)(p^c(t)) &= t^{\frac{n}{2}} \Gamma^c(t) \circ \nabla^t \circ (\Gamma^c(t))^{-1}(\Gamma^c(t)p(t)),\\ &= t^{\frac{n}{2}} \Gamma^c(t) \circ \nabla^t p(t),\\ &= 0.\end{aligned}$$

□

Similarly we can deduce conditions for f^c to be special isothermic of lower degree with $\hat{\mathcal{F}}_k$ replaced by $\bigwedge^k V_\infty \wedge \bigwedge^{n-k} \mathbb{C}^{2n}$.

Proposition 6.2.10. *Let $(f, \eta) : \Sigma \to G_n(\mathbb{C}^{2n})$, n even, be a special isothermic submanifold of type $(\kappa, \bigwedge^n \mathbb{C}^{2n})$ with conserved quantity $p(t) = \sum_{i=0}^{\kappa} p_i t^i$. Let $(f^c, \eta^c) : \Sigma \to G_n(\mathbb{C}^{2n})$ be the Christoffel dual of f with respect to $(V_0, V_\infty) \in Z_{G_n(\mathbb{C}^{2n})}$. Then f^c is a special isothermic submanifold of type $(\kappa + j, \bigwedge^n \mathbb{C}^{2n})$, $1 \leq j \leq \frac{n}{2}$ if $p_i \in \Gamma \bigwedge^{\frac{n}{2}-i-j} V_\infty \wedge \bigwedge^{\frac{n}{2}+i+j} \mathbb{C}^{2n}$.*

Proof. Now we combine the approaches of Proposition 6.2.8 and Proposition 6.2.9. Let $q(t) = \Gamma^c(t)p(t)$.

Denote by p_i^k the component of p_i in $\bigwedge^k f \wedge \bigwedge^{n-k} V_\infty$. Thus, by (6.2.25):

$$\Gamma^c\,(t)\, p_i^k = t^{\frac{n}{2}-k} \exp(F^c - F) p_i^k. \tag{6.2.32}$$

The condition $p_i \in \Gamma \bigwedge^{\frac{n}{2}-i-j} V_\infty \wedge \bigwedge^{\frac{n}{2}+i+j} \mathbb{C}^{2n}$ is equivalent to $p_i^k = 0$ for all $k > \frac{n}{2}+i+j$. Therefore the lowest degree term of $\Gamma^c\,(t)\, t^i p_i$ is $\Gamma^c\,(t)\, t^i p_i^{\frac{n}{2}+i+j} = t^{-j} \exp(F^c - F) p_i^{\frac{n}{2}+i+j}$. Thus,

$$\hat{p}(t) := t^j q(t), \tag{6.2.33}$$

is polynomial, has degree $\kappa + j$, and is conserved since $q(t)$ is. □

Again, for $n = 2$ this recovers [24, Theorem 3.8] for Q^4. There it is assumed $p_0 \perp \bigwedge^2 V_\infty$ or equivalently that $p_0 \in \Gamma V_\infty \wedge \mathbb{C}^4$ so that the only Christoffel transforms considered are those of degree κ.

6.3 Choosing a representation

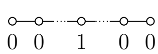
In this section, we identify a few of the key features that we used in the Grassmannian example in the more general setting. Thus we place some conditions on a general choice of representation to admit a useful theory of polynomial conserved quantities.

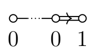
6.3.1 Classification of representations

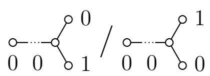
Firstly, we briefly recall the classification of irreducible representations of $\mathfrak{g}$. We can enumerate these by labelling the Dynkin or Satake diagram of $\mathfrak{g}$. The nodes of the diagram correspond to the simple roots $\alpha_1, \dots, \alpha_n$ (with respect to some Cartan subalgebra $\mathfrak{h}$). Their coroots are the elements H_α such that $\beta(H_\alpha) = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$ and these form a basis of $\mathfrak{h}$. Let $\omega_1, \dots, \omega_n$ be the dual basis of $\mathfrak{h}^*$.

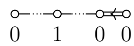
Definition 6.3.1. [40, Section 21.1] *The ω_i are called the **fundamental weights**. We say a weight $\omega = \sum_i \lambda_i \omega_i$ is **(algebraically) integral** if $\lambda_i \in \mathbb{Z}$ and **dominant** if $\lambda_i \geq 0$ for all i.*

Cartan's Theorem of the Highest Weight [33, Theorem 9.4] says that any representation has a unique highest weight which is a dominant integral weight. Moreover, a dominant integral weight determines a representation uniquely up to isomorphism. [33, Theorem 9.5] Thus we can represent an irreducible representation by annotating the Dynkin/Satake diagram as follows. Label each node on the diagram with the coefficient in the highest weight of the fundamental weight corresponding to that node. Some simple examples are:

$A_n :$  $\leftrightarrow \Lambda^k \mathbb{C}^{n+1}$,

$B_n :$  $\leftrightarrow \Sigma_{2^n}$,

$D_n :$  $\leftrightarrow \Sigma^{\pm}_{2^{n-1}}$,

$C_n :$  $\leftrightarrow \Lambda^k_o \mathbb{C}^{2n}$,

where Σ_{2^n} is the 2^n dimensional spin representation, $\Sigma^{\pm}_{2^{n-1}}$ are the 2^{n-1} dimensional half-spin representations and $\Lambda^k_o \mathbb{C}^{2n}$ is the kernel of the contraction of $\Lambda^k \mathbb{C}^{2n}$ by a symplectic form.

The dual representation is given by the permuting the labels by the duality isomorphism of the diagram. For A_n, E_6 this is obtained by flipping the diagram. For D_n (n odd) this is obtained by swapping the two nodes on the right. For the other diagrams this is just the identity (and so all representations are self-dual).

6.3.2 Low degree symmetric polynomial functions

Firstly we consider the existence of an invariant symmetric polynomial function. In general, we can do this by computing the symmetric powers $S^k V^*$ to see if they contain a copy of the trivial representation. However, there is a useful subset of representations that we can investigate more directly.

If V is self-dual, we have a $\mathfrak{g}$-isomorphism $\phi : V \to V^*$ and thus a non-degenerate $\mathfrak{g}$-invariant bilinear form $q \in \otimes^2 V^*$, $q(v, w) := \phi(v)(w)$. Since $\otimes^2 V^* = \Lambda^2 V^* \oplus \mathrm{S}^2 V^*$ and both summands are $\mathfrak{g}$-invariant we can see that q is symmetric or symplectic. In the latter case we can always then find a G-invariant symmetric form in $S^4(V^*)$. To see this we note that since q is $\mathfrak{g}$-invariant we have $\mathfrak{g} \leq \mathfrak{sp}(V) \cong \mathrm{S}^2(V)$ (indeed this is an irreducible submodule). Then $\mathfrak{g}$ has a symmetric bilinear form: the Killing form. Thus we define $q' \in S^4(V^*)$ by:

$$q'(v_1, v_2, v_3, v_4) := (\pi_{\mathfrak{g}}(v_1 \odot v_2), \pi_{\mathfrak{g}}(v_3 \odot v_4)) + (\pi_{\mathfrak{g}}(v_1 \odot v_3), \pi_{\mathfrak{g}}(v_2 \odot v_4)), \quad (6.3.1)$$

where $\pi_{\mathfrak{g}}$ is the projection onto $\mathfrak{g}$. In conclusion we have:

Proposition 6.3.2. *If V is a self-dual representation of $\mathfrak{g}$, then there exists a $\mathfrak{g}$-invariant symmetric polynomial function on V of degree 2 or 4.*

6.3.3 Darboux and Christoffel gauge transformations

As we noted in the Grassmannian example, the automorphism $\Gamma(s) := \Gamma^{\mathfrak{p}}_{\mathfrak{q}}(s) \in G^{\mathbb{C}}$ does not always have a well defined action on V. However, we can identify when this will be using the weight theory we described in Section 6.3.1.

Definition 6.3.3. [33, Definition 12.4] *Let V be a representation of $\mathfrak{g}$ and let ω be a weight of this representation. Let K be any Lie group with Lie algebra $\mathfrak{k} \leq \mathfrak{g}$. We say ω is **analytically integral for** K if $\omega(X) \in 2i\pi\mathbb{Z}$ for each $X \in \mathfrak{k}$ such that* $\exp(X) = 1 \in K$.

Proposition 6.3.4. [33, Section 12.1] *K has a well defined action on V if every weight of V is analytically integral for K.*

Since every weight of V differs from the highest weight by a sum of roots [40, Theorem 20.2] and roots are analytically integral weights [33, Proposition 12.7] every weight of V is analytically integral if the highest weight is.

Let K denote the one parameter subgroup $\{\Gamma(s)\} \subset G^{\mathbb{C}}$. Recall $\Gamma(s) = \exp(\ln(s)\xi)$ for $\xi := \xi^{\mathfrak{p}}_{\mathfrak{q}}$ the corresponding grading element. Then $\mathfrak{k} = \langle \xi \rangle$ and $\exp(\lambda\xi) = \Gamma^{\mathfrak{p}}_{\mathfrak{q}}(1) = 1$ if, and only if, $\lambda \in 2i\pi\mathbb{Z}$. Consequently a weight ω is analytically integral for K if $\omega(\xi) \in \mathbb{Z}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing ξ (so $\mathfrak{h} \subset \mathfrak{p} \cap \mathfrak{q}$) and let Δ be the corresponding root system. Let $\{\alpha_1, \dots, \alpha_n\} \subset \Delta$ be a choice of simple roots (such that $\mathfrak{p}^{\perp}$ is contained in the positive root spaces). Let $\{\omega_1, \dots, \omega_n\}$ be the corresponding fundamental weights. If α_s is the simple root defining $\mathfrak{p}$, then $\alpha_s(\xi) = 1$ and $\alpha_i(\xi) = 0$ for $i \neq s$. In other words ξ is an element of the dual basis to the simple roots. Then $\omega_i = \sum_{j=1}^{n} (\alpha_j, \omega_i)\alpha_j$ so that $\omega_i(\xi) = (\alpha_s, \omega_i)$. Thus for $\omega = \sum_{i=1}^{n} \lambda_i \omega_i$:

$$\omega(\xi) = \sum_{i=1}^{n} \lambda_i (\alpha_s, \omega_i). \tag{6.3.2}$$

Note that the matrix $A_{ij} = ((\alpha_j, \omega_i))$ is precisely the inverse of the Cartan matrix and we can find these in [49, Reference Chapter 2, Table 2].

6.3.4 Example: adjoint representation

A natural choice of representation is the adjoint representation $\mathfrak{g}$. It is always self-dual and its symmetric bilinear form is the Killing form. Its weights are the roots of $\mathfrak{g}$ which are analytically integral. Indeed, we originally defined $\Gamma(s)$ as an automorphism of $\mathfrak{g}$.

In general, however it is not clear what geometric conditions a polynomial conserved quantity of type $(\kappa, \mathfrak{g})$ would impose on f.

6.3.5 Example: the Kostant-Plücker representation

The choice of representation that we made in our example (and was used in [24]) also lends itself to a systematic choice. The Plücker embedding realises the Grassmannian as a projective variety in $\mathbb{P}(\Lambda^n \mathbb{C}^{2n})$. More generally we have:

Theorem 6.3.5 (c.f [51, Section 6.6])**.** *Any R-space can be embedded as a projective variety in some representation of $\mathfrak{g}$. It is the G-orbit of the highest weight space and it is given by the vanishing of a set of quadratic relations.*

Indeed, there are, in general, many such representations. Let Φ be a subset of the nodes of the Dynkin/Satake diagram of $\mathfrak{g}$ and let R_Φ denote the corresponding R-space. If Φ corresponds to the simple roots $\{\alpha_{i_1}, \dots, \alpha_{i_k}\}$ then R_Φ embeds as a projective variety in $\mathbb{P}(V)$ if, and only if, the highest weight of V is given by $\omega = \sum_{l=1}^{k} \lambda_{i_l} \omega_{i_l}$ for $\lambda_{i_l} \geq 1$.

R	V_ω	$\omega(\xi)$ integral	symmetric bilinear form
Q^n	$\mathbb{C}^{n+2}$	yes	yes
$G_n(\mathbb{C}^{2n})$	$\Lambda^n \mathbb{C}^{2n}$	n even	n even
$\mathrm{Lag}(\mathbb{C}^{2n})$	$\Lambda^n_o \mathbb{C}^{2n}$	n even	n even
$J^\pm(\mathbb{C}^{4n})$	$\Sigma^\pm_{2^{2n-1}}$	n even	n even

Table 6.1: Properties of the Kostant-Plücker relations

Definition 6.3.6. *The* ***Kostant-Plücker representation*** *of R_Φ is the representation V_ω with highest weight $\omega = \sum_{l=1}^k \omega_{i_l}$.*

Thus, this is just the simplest representation for which R_Φ embeds as a projective variety. It is naturally self-dual when R_Φ is. Notably $\omega(\xi)$ is simply one of the diagonal elements of A_{ij}. For the Grassmannian this is precisely the usual Plücker embedding.

Below we compute the Kostant-Plücker representation for some self-dual symmetric R-spaces. Furthermore, we note whether $\omega(\xi)$ is integral and whether the invariant bilinear form is symmetric.

6.3.6 Other choices

Lastly we note another possible series of choices for the Grassmannian in particular. Consider $f : \Sigma \to G_n(\mathbb{C}^{2n})$. Then a decomposable constant conserved quantity in $\Lambda^{2k} \mathbb{C}^{2n}$ defines a $2k$-plane $W_{2k} \leq \mathbb{C}^{2n}$ such that $\dim f \cap W_{2k} \geq k$. In general, $\Lambda^{2k} \mathbb{C}^{2n}$ does not have a symmetric bilinear form but we can compute that $\omega_k(\xi)$ is always integral for ω_k the highest weight of $\Lambda^{2k} \mathbb{C}^{2n}$. It is not clear whether there exist isothermic submanifolds admitting such conserved quantities. However, this suggests the possibility of a link between low-degree conserved quantities with decomposable constant term and other Schubert varieties. Indeed, by choosing more complicated representations of $\mathfrak{sl}(2n, \mathbb{C})$ it may be possible to find links to more complicated Schubert varieties.

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