PHD

Delta-Filtered Representations of Double Quivers and Seaweed Lie Algebras of Affine Type A

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Summary

By Richardson's Theorem, there exists a dense open adjoint orbit in the nilpotent radical of any parabolic subalgebra of a semisimple Lie algebra. Elements contained in this orbit are called Richardson elements. Jensen, Su and Yu generalised the study of Richardson elements to seaweed (biparabolic) subalgebras, using Δ-filtered modules of a quasi-hereditary algebra arising as a quotient of the path algebra of a double quiver.

In this thesis, we extend the study of these quiver representations to affine type $\mathbf{A}$, providing an explicit construction for a general Δ-filtered representation and a classification of the existence of open orbits in terms of the Δ-dimension vector. This allows us to give a combinatorial classification for the existence of Richardson elements in proper standard seaweed subalgebras of non-twisted affine Lie algebras of type $\mathbf{A}$.

Contents

Chapter 1

Introduction

Throughout, we take the base field k to be the field of complex numbers.

Consider the following result, known as Richardson's Dense Orbit Theorem (or just Richardson's Theorem). The theorem involves the adjoint group of a Lie algebra, which we discuss in Chapter 11. We refer to Tauvel and Yu [27] for an approach from the perspective of Lie algebras. See Richardson [23] for the original result.

Theorem 1.1 (Theorem 33.5.6 in [27])**.** *Let $\mathfrak{g}$ be a semisimple Lie algebra, G its adjoint group, $\mathfrak{p}$ a parabolic subalgebra of $\mathfrak{g}$ with nilpotent radical $\mathfrak{n}$, and P a parabolic subgroup of G with Lie algebra $\operatorname{ad}_{\mathfrak{g}} \mathfrak{p}$.*

1. *There exists a unique nilpotent G-orbit $\mathcal{O}$ such that $\mathcal{O} \cap \mathfrak{n}$ is a dense open subset of $\mathfrak{n}$.*
2. *The set $\mathcal{O} \cap \mathfrak{n}$ is a P-orbit.*
3. *If $x \in \mathcal{O} \cap \mathfrak{n}$, then $[x, \mathfrak{p}] = \mathfrak{n}$.*

Remark 1.2. As $\mathfrak{g}$ is centreless, $\operatorname{ad}_{\mathfrak{g}}$ is injective, so $\operatorname{ad}_{\mathfrak{g}} \mathfrak{p} \cong \mathfrak{p}$.

Definition 1.3 (33.5.8 in [27])**.** An element $x \in \mathfrak{n}$ such that $[x, \mathfrak{p}] = \mathfrak{n}$ is called a **Richardson element**. This can be applied to any Lie algebra $\mathfrak{p}$ with a nilpotent radical $\mathfrak{n}$.

1.4. So Richardson's Theorem tells us that Richardson elements exist in any parabolic subalgebra $\mathfrak{p}$ of a semisimple Lie algebra $\mathfrak{g}$. There are two ways to generalise this:

- We can broaden the class of subalgebra $\mathfrak{p}$ beyond parabolic subalgebras. That is, we can also consider seaweed subalgebras (see Definition 16.4).
- We can broaden the class of Lie algebra $\mathfrak{g}$ beyond semisimple Lie algebras. That is, we can also consider affine Lie algebras (see Chapter 12).

The first point has been addressed by Jensen, Su and Yu [14–16], who showed that Richardson elements exist in any seaweed subalgebra of any semisimple Lie algebra $\mathfrak{g}$, except when $\mathfrak{g}$ is of type $\mathbf{E}_8$ (in which case there is a counterexample). The method used was to construct an equivalence between the adjoint action on the radical of the seaweed and a group action on a certain variety of quiver representations, while showing the existence of open orbits in the latter.

As for parabolic subalgebras of affine Lie algebras, the ideal which we expect to be the nilpotent radical is not in fact nilpotent. Even ignoring this, the parabolic subalgebra and its radical are infinite dimensional, which means the dimensional arguments used in Jensen, Su and Yu's quiver model cannot be used.

However, certain seaweed subalgebras of affine Lie algebras do not suffer from these problems, so are in fact a more promising candidate for study. In this work, we generalise the quiver model developed by Jensen, Su and Yu [15] to affine type **A** quivers. We then use this to give a combinatorial criterion for the existence of Richardson elements in seaweed subalgebras of affine type **A** Lie algebras.

1.5. The affine case for Lie algebras is somewhat more complicated than the affine case for quiver representations. Firstly, in types **A**, **D** and **E**, there are both twisted and non-twisted affine Lie algebras. The non-twisted affine Lie algebras are the most natural first step in generalising beyond finite type, so it is these which we restrict our attention to.

Additionally, parabolic subalgebras of affine Lie algebras come in different types: Type Ia (or standard type), Type Ib and Type II (or natural type). Seaweed subalgebras are therefore also separated into distinct types. The behaviour of these types is very different, with only standard type parabolics and seaweeds behaving in a comparable way to parabolics and seaweeds of finite type (in the sense that they are generated by simple roots; see Corollary 14.9 and Proposition 14.10).

We discuss all the types of parabolic and seaweed subalgebra in Part II, though our main results on the existence of Richardson elements apply only to seaweed subalgebras of standard type.

1.6. The class of quiver representations developed by Jensen, Su and Yu [15] are called A-projective D-modules, where A is the path algebra of a quiver Q, and D is the quotient of the path algebra of the double quiver $\overline{Q}$ by the ideal generated by a certain set of relations (Definition 2.2). D is a quasi-hereditary algebra and these A-projective D-modules are Δ-filtered modules of D, in the sense that they have a filtration by the indecomposable projective modules Δ_i of A (see Hille and Vossieck [10]). Δ-filtered modules are sometimes called good modules, while the Δ_i are sometimes called standard modules or Verma modules (see Ringel [24]).

The main result of Jensen, Su and Yu [15] is that a dense open $\operatorname{Aut} P$-orbit exists in $\operatorname{rad}\operatorname{End} P$ for all projective A-modules P if and only if Q is a Dynkin quiver. This gives a new characterisation of Dynkin quivers, mirroring Gabriel's Theorem [2]. The $\operatorname{Aut} P$-orbits turn out to be equivalent to isomorphism classes of A-projective D-modules.

The connection with Lie algebras and Richardson's Theorem comes from $\operatorname{Aut} P$ being an algebraic group and $\operatorname{rad}\operatorname{End} P$ the nilpotent radical of its Lie algebra (which is a seaweed Lie algebra).

1.7. In Part I, we study these A-projective D-modules / Δ-filtered representations, particularly in the new context of affine type **A** quivers. We develop a graph-based description of these representations which is effectively a generalisation of the description of Richardson elements using line diagrams by Baur [1]. This approach allows us to construct a representation of an affine type **A** quiver from a representation of a Dynkin type **A** quiver by using a graph operation, then directly compare the endomorphism algebras of the two representations.

In classifying when Δ-filtered representations have dense open orbits, we mirror the notions of general representations and canonical decomposition of dimension vectors developed by Kac [17,18]. For a concise summary of the relevant results, see the lecture notes by Holtmann in Bielefeld's Selected Topics in Representation Theory seminar.

We refer to Schiffler [26] for the fundamentals of quiver representation theory and to Rotman [25] for all things homological algebra. Throughout both Part I and II, for the theory of algebraic groups, we primarily refer to Tauvel and Yu [27], as well as to Procesi [22], Hall [7], Harris [8], Hartshorne [9], Humphreys [12] and Mumford [21].

In Part II, we use the quiver model developed in Part I to obtain the main result (Theorem 18.11). We refer to Kac [19] for the theory of affine Lie algebras and to Futorny [5] for the theory of parabolic subalgebras of affine Lie algebras. We also refer to Humphreys [11] for the theory of finite dimensional Lie algebras.

Part I

Quiver Representation Theory

Chapter 2

Delta-Filtered Representations of Double Quivers

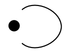
In this chapter, we discuss the objects we are concerned with in this thesis - modules of a double quiver (with certain relations) which are projective modules of the underlying quiver. These modules are referred to as A-projective D-modules (A being the path algebra of the underlying quiver and D the path algebra of the double quiver with relations). They are sometimes also referred to as Δ-filtered representations or Δ-filtered modules, as they have a filtration by the Δ_i (the indecomposable projectives of the underlying quiver). We also write Δ-filtered as "Delta-filtered".

For a quiver Q we write Q_0 for the set of vertices and Q_1 for the set of arrows. For $i \in Q_0$, we write e_i for the trivial path at i. The e_i are a complete set of primitive idempotents of the path algebra. For $\alpha \in Q_1$, we write $s(\alpha)$ for the source of α and $t(\alpha)$ for the target of α.

We call a vertex a **source** if it is not the target of any arrows, or a **sink** if it is not the source of any arrows. We call a vertex **admissible** if it is either a sink or a source.

Representations of a quiver Q and modules of the path algebra $A = kQ$ are equivalent as categories, hence we use the terms representation and module interchangeably. For an A-module M, we write $M_i = e_i M$ for the space at vertex i and write $M_\alpha : M_{s(\alpha)} \to M_{t(\alpha)}$ for the map associated to the arrow α.

Definition 2.1. We say that a quiver Q is **of affine type A** if the underlying graph of Q is one of the following.

$\mathbf{A}_1^{(1)}$ 

$\mathbf{A}_2^{(1)}$ 

$\mathbf{A}_n^{(1)}$ $(n \geq 3)$ 

Definition 2.2. Let Q be a quiver and $A = kQ$ the path algebra. For $i \in Q_0$, write $\Delta_i = Ae_i$ for the indecomposable projective A-module associated to the vertex i. For $d \in \mathbb{N}^{Q_0}$, we write

$$P(d) = \bigoplus_{i \in Q_0} \Delta_i^{d_i}.$$

Write $\overline{Q}$ for the **double** of Q, which is the quiver with the same vertex set as Q and contains the arrows $\alpha \in Q_1$ as well as the opposite arrows $\overline{\alpha}$, defined by $s(\overline{\alpha}) = t(\alpha)$ and $t(\overline{\alpha}) = s(\alpha)$.

Write $D = k\overline{Q}/\mathcal{I}$ for the quotient of the path algebra of the double quiver $\overline{Q}$ by the ideal $\mathcal{I}$ generated by

$$\overline{\alpha}\alpha - \sum_{\beta \in Q_1 : t(\beta) = s(\alpha)} \beta\overline{\beta} \tag{I1}$$

for $\alpha \in Q_1$ and

$$\overline{\alpha}\beta \tag{I2}$$

for distinct $\alpha, \beta \in Q_1$ such that $t(\alpha) = t(\beta)$. Note in particular that these relations – and hence D – depend upon the orientation of Q.

As A is a subalgebra of D, we may consider a D-module X to be an A-module, denoting this by ${}_AX$. If ${}_AX$ is projective, we say X is an **A-projective D-module**. If this is the case, we call the unique d for which ${}_AX \cong P(d)$ the **Δ-dimension vector of** X, writing $\underline{\dim}_\Delta X = d$. Additionally, we define the **Δ-support of** X to be the set

$$\operatorname{supp}_\Delta X = \{ i \in Q_0 : (\underline{\dim}_\Delta X)_i > 0 \}.$$

Recall $\operatorname{Rep}(Q, d)$, the **variety of representations** (or **space of representations**) of Q with dimension vector d (see for instance 1.3 in [3] or 8.1 in [26]). Recall also the group $\operatorname{Gl}(d)$, which acts on $\operatorname{Rep}(Q, d)$ by conjugation, and that the orbits under the $\operatorname{Gl}(d)$-action correspond to isomorphism classes of representations.

We write $\operatorname{Rep}(\overline{Q}, \mathcal{I}, d)$ for the subspace of all $Y \in \operatorname{Rep}(\overline{Q}, d)$ which verify the relations $\mathcal{I}$. We may consider the points in $\operatorname{Rep}(Q,d)$ to be A-modules and the points in $\operatorname{Rep}(\overline{Q},\mathcal{I},d)$ to be D-modules. One should be careful with the converse – given a module, there is no unique point in the representation variety corresponding to that module (choosing a point amounts to choosing a basis). However, we abuse our notation and fix some point in $\operatorname{Rep}(Q,\underline{\dim}\,P(d))$ for each $d \in \mathbb{N}^{Q_0}$, writing $P(d) \in \operatorname{Rep}(Q,\underline{\dim}\,P(d))$.

Given some $Y \in \operatorname{Rep}(\overline{Q},\mathcal{I},d)$, we have that ${}_AY \in \operatorname{Rep}(Q,d)$. We can therefore consider the subset of $\operatorname{Rep}(\overline{Q},\mathcal{I},\underline{\dim}\,P(d))$ consisting of modules Y for which ${}_AY = P(d)$. We denote this by $\operatorname{Rep}_\Delta(Q,d)$, the space of A-projective D-modules with Δ-dimension vector d. This is an affine space (Lemma 4 in [14]). Again, note that $\operatorname{Rep}_\Delta(Q,d)$ depends not on $\overline{Q}$ but on Q.

The group $\operatorname{Aut}_A P(d)$ acts on $\operatorname{Rep}_\Delta(Q,d)$ by conjugation. The $\operatorname{Aut}_A P(d)$-orbits correspond to isomorphism classes of A-projective d-modules.

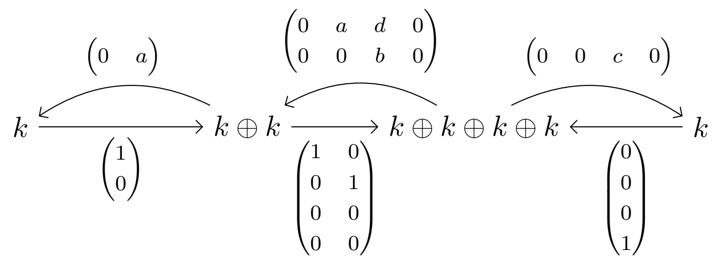
Example 2.3. Let Q be the following quiver.

$$1 \xrightarrow[\alpha]{} 2 \xrightarrow[\beta]{} 3 \xleftarrow[\gamma]{} 4$$

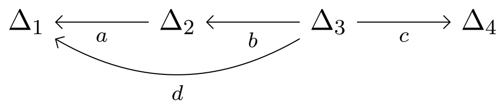
Then the Δ_i can be written as follows.

$$\begin{aligned}
\Delta_1 &\cong k \xrightarrow[1]{} k \xrightarrow[1]{} k \xleftarrow[0]{} 0\\
\Delta_2 &\cong 0 \xrightarrow[0]{} k \xrightarrow[1]{} k \xleftarrow[0]{} 0\\
\Delta_3 &\cong 0 \xrightarrow[0]{} 0 \xrightarrow[0]{} k \xleftarrow[0]{} 0\\
\Delta_4 &\cong 0 \xrightarrow[0]{} 0 \xrightarrow[0]{} k \xleftarrow[1]{} k
\end{aligned}$$

Let $d = (1,1,1,1)$ and $M \in \operatorname{Rep}_\Delta(Q,d)$. Then M can be written as follows for some $a,b,c,d \in k$.



Note that a occurs in two places - both mapping from Δ_2 to Δ_1. In fact, each coordinate is the coefficient of the maps from Δ_i to Δ_j for some i and j. All we need to define M is this data, which we can present in a diagram as follows.



Observe that there is a coordinate for each nontrivial path ρ in the quiver, mapping from $\Delta_{t(\rho)}$ to $\Delta_{s(\rho)}$. We spend the remainder of the chapter making this notation rigorous.

Remark 2.4. The Δ_i are the simple objects in the category of A-projective D-modules. Extensions can be formed between the Δ_i along nontrivial paths in the quiver. Contrast this with the category of ordinary modules of the same quiver, in which there is also a simple object for each vertex, but extensions can only be made across arrows.

Definition 2.5. Throughout, we use the following notation. For $i, j \in Q_0$,

- denote by $P(i, j)$ the set of paths in Q from i to j,
- denote by $P(i, -)$ the set of paths in Q starting at i,
- denote by $P(-, j)$ the set of paths in Q ending at j,
- denote by $P(-, -)$ the set of all paths in Q,
- denote by $P^*(i, j)$, $P^*(i, -)$, $P^*(-, j)$ and $P^*(-, -)$ the respective subsets of the above consisting of nontrivial paths.

Remark 2.6. Given some path ρ, we write $\rho = \rho_r \cdots \rho_1$ for the decomposition of ρ into arrows $\rho_1, \cdots, \rho_r \in Q_1$, unless we explicitly state that $\rho_1, \cdots, \rho_r$ are not arrows.

Remark 2.7. Observe that we can also express $P(d)$ as follows.

$$P(d) = \bigoplus_{i \in Q_0} k^{d_i} \otimes_k \Delta_i = \bigoplus_{\rho \in P(-,-)} k^{d_{s(\rho)}} \otimes \rho,$$

where the space at vertex i is given by

$$P(d)_i = \bigoplus_{\rho \in P(-,i)} k^{d_{s(\rho)}} \otimes \rho.$$

Consequently, for $M \in \text{Rep}_\Delta(Q,d)$, the maps $M_{\overline{\alpha}}$ $(\alpha \in Q_1)$ split into components according to the above decomposition of $P(d)_i$. The relations generating $\mathcal{I}$ annihilate many of these components and identify many others. The maps M_α simply send $v \otimes \rho$ to $v \otimes \alpha\rho$.

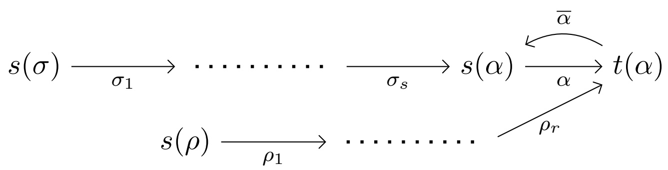
Lemma 2.8. *For $\alpha \in Q_1$, $\rho \in P(-,t(\alpha))$ and $\sigma \in P(-,s(\alpha))$, the $k^{d_{s(\rho)}} \otimes \rho \to k^{d_{s(\sigma)}} \otimes \sigma$ component of $M_{\overline{\alpha}}$ is zero unless*

- *ρ is trivial, or*
- *writing $\rho = \rho_r \cdots \rho_1$ for $\rho_i \in Q_1$, we have $\rho_r = \alpha$ and $\sigma = \rho_{r-1} \cdots \rho_1 \tau$ for some $\tau \in P^*(-,s(\rho))$.*

Proof. Firstly, as $M_{\overline{\alpha}} : M_{t(\alpha)} \to M_{s(\alpha)}$, we have

$$M_{\overline{\alpha}} : \bigoplus_{\rho \in P(-,t(\alpha))} k^{d_{s(\rho)}} \otimes \rho \to \bigoplus_{\sigma \in P(-,s(\alpha))} k^{d_{s(\sigma)}} \otimes \sigma.$$

Fix $\rho \in P(-,t(\alpha))$, $v \in k^{d_{s(\rho)}}$ and $\sigma \in P(-,s(\alpha))$. Suppose ρ is nontrivial and write $\sigma = \sigma_s \cdots \sigma_s$ for $\sigma_i \in Q_1$.



Consider the $k^{d_{s(\sigma)}} \otimes \sigma$ component of $M_{\overline{\alpha}}(v \otimes \rho)$. We have that

$$M_{\overline{\alpha}}(v \otimes \rho) = M_{\overline{\alpha}} M_{\rho_r} \cdots M_{\rho_1}(v \otimes e_{s(\rho)}),$$

which (I2) implies is zero unless $\rho_r = \alpha$. In that case, we can apply (I1) and (I2) repeatedly to see that

$$\begin{aligned}
M_{\overline{\alpha}}(v \otimes \rho) &= (M_{\overline{\alpha}} M_\alpha) M_{\rho_{r-1}} \cdots M_{\rho_1}(v \otimes e_{s(\rho)}) \\
&= \sum_{\beta \in Q_1 : t(\beta) = s(\alpha)} M_\beta M_{\overline{\beta}} M_{\rho_{r-1}} \cdots M_{\rho_1}(v \otimes e_{s(\rho)}) \\
&= M_{\rho_{r-1}} (M_{\overline{\rho}_{r-1}} M_{\rho_{r-1}}) M_{\rho_{r-2}} \cdots M_{\rho_1}(v \otimes e_{s(\rho)}) \\
&= M_{\rho_{r-1}} \sum_{\beta \in Q_1 : t(\beta) = s(\rho_{r-1})} M_\beta M_{\overline{\beta}} M_{\rho_{r-2}} \cdots M_{\rho_1}(v \otimes e_{s(\rho)})
\end{aligned}$$

$$
\begin{aligned}
&= M_{\rho_{r-1}} M_{\rho_{r-2}} M_{\overline{\rho}_{r-2}} M_{\rho_{r-2}} \cdots M_{\rho_1}(v \otimes e_{s(\rho)}) \\
&\vdots \\
&= M_{\rho_{r-1}} \cdots M_{\rho_1} M_{\overline{\rho}_1} M_{\rho_1}(v \otimes e_{s(\rho)}) \\
&= M_{\rho_{r-1}} \cdots M_{\rho_1} \sum_{\beta \in Q_1 : t(\beta) = s(\rho_1)} M_\beta M_{\overline{\beta}}(v \otimes e_{s(\rho)}).
\end{aligned}
$$

For each β, we have

$$
\operatorname{im} M_{\overline{\beta}} \subseteq \bigoplus_{\tau \in P(-, s(\beta))} k^{d_{s(\tau)}} \otimes \tau,
$$

hence

$$
\operatorname{im}(M_{\rho_{r-1}} \cdots M_{\rho_1} M_\beta M_{\overline{\beta}}) \subseteq \bigoplus_{\tau \in P(-, s(\beta))} k^{d_{s(\tau)}} \otimes \rho_{r-1} \cdots \rho_1 \beta \tau.
$$

The $k^{d_{s(\sigma)}} \otimes \sigma$ component of this is zero unless $\rho_{r-1} \cdots \rho_1 \beta \tau = \sigma$ for some β and τ, as in the diagram below.

$$
s(\tau) \xrightarrow[\tau_1]{\sigma_1} \cdots\cdots\cdot \xrightarrow[\tau_t]{\sigma_t} s(\beta) \xrightarrow[\beta]{\sigma_{t+1}} s(\rho) \xrightarrow[\rho_1]{\sigma_{t+2}} \cdots\cdots\cdot \xrightarrow[\rho_{r-1}]{\sigma_s} s(\alpha) \underset{\alpha}{\overset{\overline{\alpha}}{\rightleftarrows}} t(\alpha)
$$

If no such β and τ exist, then the $k^{d_{s(\sigma)}} \otimes \sigma$ component of $M_{\overline{\alpha}}$ is zero. Otherwise, the β and τ are unique for each σ. □

Remark 2.9. From Lemma 2.8, we conclude that the only possible nonzero components are the

$$
k^{d_{t(\tau)}} \otimes e_{t(\tau_t)} \to k^{d_{s(\tau)}} \otimes \tau_{t-1} \cdots \tau_1
$$

component of $M_{\overline{\tau}_t}$ and the

$$
k^{d_{t(\tau)}} \otimes \rho_r \cdots \rho_1 \to k^{d_{s(\tau)}} \otimes \rho_{r-1} \cdots \rho_1 \tau_t \cdots \tau_1
$$

component of $M_{\overline{\rho}_r}$, for nontrivial paths $\rho = \rho_r \cdots \rho_1$ and $\tau = \tau_t \cdots \tau_1$. Each such component is given by some linear map in $\operatorname{Hom}(k^{d_{t(\tau)}}, k^{d_{s(\tau)}})$.

Lemma 2.10. *Let $\tau = \tau_t \cdots \tau_1$ be a nontrivial path in Q. Suppose the*

$$
k^{d_{t(\tau)}} \otimes e_{t(\tau_t)} \to k^{d_{s(\tau)}} \otimes \tau_{t-1} \cdots \tau_1
$$

component of $M_{\overline{\tau}_t}$ is given by $\varphi \in \operatorname{Hom}(k^{d_{t(\tau)}}, k^{d_{s(\tau)}})$. Then the

$$
k^{d_{t(\tau)}} \otimes \rho_r \cdots \rho_1 \to k^{d_{s(\tau)}} \otimes \rho_{r-1} \cdots \rho_1 \tau_t \cdots \tau_1
$$

component of $M_{\overline{\rho}_r}$ is also given by φ for all nontrivial paths $\rho = \rho_r \cdots \rho_1$.

Proof. We use an induction on r. Let $v \in k^{d_{t(\tau)}}$. Firstly, by (I1), we have

$$\begin{aligned} M_{\overline{\rho}_1}(v \otimes \rho_1) &= M_{\overline{\rho}_1} M_{\rho_1}(v \otimes e_{t(\tau)}) \\ &= \sum_{\beta \in Q_1 : t(\beta) = s(\rho_1)} M_\beta M_{\overline{\beta}}(v \otimes e_{t(\tau)}). \end{aligned}$$

The $k^{d_{s(\tau)}} \otimes \tau$ component of this is just that of $M_{\tau_t} M_{\overline{\tau}_t}(v \otimes e_{t(\tau)})$, which by assumption is $M_{\tau_t}(\varphi(v) \otimes \tau_{r-1} \cdots \tau_1) = \varphi(v) \otimes \tau$.

Suppose the $k^{d_{t(\tau)}} \otimes \rho_r \cdots \rho_1 \to k^{d_{s(\tau)}} \otimes \rho_{r-1} \cdots \rho_1 \tau$ component of $M_{\overline{\rho}_r}$ is given by φ. Then

$$\begin{aligned} M_{\overline{\rho}_{r+1}}(v \otimes \rho_{r+1} \cdots \rho_1) &= M_{\overline{\rho}_{r+1}} M_{\rho_{r+1}}(v \otimes \rho_r \cdots \rho_1) \\ &= \sum_{\beta \in Q_1 : t(\beta) = s(\rho_{r+1})} M_\beta M_{\overline{\beta}}(v \otimes \rho_r \cdots \rho_1). \end{aligned}$$

The $k^{d_{s(\tau)}} \otimes \rho_r \cdots \rho_1$ component of this is just that of $M_{\rho_r} M_{\overline{\rho}_r}(v \otimes \rho_r \cdots \rho_1)$, which by the supposition is $M_{\rho_r}(\varphi(v) \otimes \rho_{r-1} \cdots \rho_1 \tau) = \varphi(v) \otimes \rho_r \cdots \rho_1 \tau$. □

Remark 2.11. The above lemmas tell us that the representation M is defined by a choice of linear maps $\varphi_\rho : k^{d_{t(\rho)}} \to k^{d_{s(\rho)}}$ for each nontrivial path ρ in Q.

Lemma 2.12. *Any choice of φ_ρ gives a well-defined representation M.*

Proof. Firstly, we check (I1). Let $\alpha \in Q_1$, $\rho \in P(-, s(\alpha))$ and $v \in k^{d_{s(\rho)}}$. The only components of $M_{\overline{\alpha}}(v \otimes \alpha\rho)$ which can be nonzero are those in $k^{d_{s(\sigma)}} \otimes \sigma$ for $\sigma = \rho\tau$, where $\tau \in P^*(-,-)$. Therefore,

$$\begin{aligned} M_{\overline{\alpha}} M_\alpha(v \otimes \rho) &= M_{\overline{\alpha}}(v \otimes \alpha\rho) \\ &= \sum_{\tau \in P^*(-,-)} \varphi_\tau(v) \otimes \rho\tau. \end{aligned}$$

On the other hand, if $\rho = \rho_r \cdots \rho_1$ is nontrivial and $\beta \neq \rho_r$, then $M_{\overline{\beta}}(v \otimes \rho) = 0$, hence

$$\begin{aligned} \sum_{\beta \in Q_1 : t(\beta) = s(\alpha)} M_\beta M_{\overline{\beta}}(v \otimes \rho) &= M_{\rho_r} M_{\overline{\rho}_r}(v \otimes \rho). \\ &= M_{\rho_r} \sum_{\tau \in P^*(-,-)} \varphi_\tau(v) \otimes \rho_{r-1} \cdots \rho_1 \tau \end{aligned}$$

$$= \sum_{\tau \in P^*(-,-)} \varphi_\tau(v) \otimes \rho\tau.$$

If ρ is trivial, then $\rho = e_{s(\alpha)}$ and

$$M_{\overline{\beta}}(v \otimes e_{s(\alpha)}) = \sum_{\sigma \in P(-,s(\beta))} w_\sigma \otimes \sigma$$

for some $w_\sigma \in k^{d_{s(\sigma)}}$. Writing $\tau = \beta\sigma$, Lemma 2.10 gives us that each $w_\sigma = \varphi_{\beta\sigma}(v)$. Therefore,

$$\begin{aligned}
\sum_{\beta \in Q_1 : t(\beta) = s(\alpha)} M_\beta M_{\overline{\beta}}(v \otimes \rho) &= \sum_{\beta \in Q_1 : t(\beta) = s(\alpha)} M_\beta \left(\sum_{\sigma \in P(-,s(\beta))} \varphi_{\beta\sigma}(v) \otimes \sigma \right) \\
&= \sum_{\beta \in Q_1 : t(\beta) = s(\alpha)} \left(\sum_{\sigma \in P(-,s(\beta))} \varphi_{\beta\sigma}(v) \otimes \beta\sigma \right) \\
&= \sum_{\tau \in P^*(-,s(\alpha))} \varphi_\tau(v) \otimes \rho\tau.
\end{aligned}$$

So in either case (ρ trivial or nontrivial), we have

$$M_{\overline{\alpha}} M_\alpha (v \otimes \rho) = \sum_{\beta \in Q_1 : t(\beta) = s(\alpha)} M_\beta M_{\overline{\beta}}(v \otimes \rho).$$

Next, we check (I2). Let $\alpha, \beta \in Q_1$ such that $\alpha \neq \beta$ and $t(\alpha) = t(\beta)$. Let $\rho \in P(-, s(\beta))$ and $v \in k^{d_{s(\rho)}}$. Then $M_{\overline{\alpha}} M_\beta(v \otimes \rho) = M_{\overline{\alpha}}(v \otimes \beta\rho)$, which by Lemma 2.8 is zero, as required. □

Definition 2.13. For each choice of linear maps $\varphi = (\varphi_\rho)_{\rho \in P^*(-,-)}$, denote the corresponding representation by $M(d, \varphi)$. The data (d, φ) can be expressed as a graph, referred to from this point on as a **Δ-graph**, in the following way. Fix bases $u_{i,1}, \cdots, u_{i,d_i}$ of k^{d_i} for each $i \in Q_0$. For each pair $i \in Q_0$ and $j = 1, \cdots, d_i$, draw a vertex with label Δ_i associated to $u_{i,j} \otimes \Delta_i$. For each nontrivial path ρ in Q, we draw arrows as follows. Let λ be the coefficient of $u_{s(\rho),j}$ in $\varphi_\rho(u_{t(\rho),i})$. If $\lambda \neq 0$, draw an arrow from the vertex associated to $u_{t(\rho),i} \otimes \Delta_{t(\rho)}$ to the vertex associated to $u_{s(\rho),j} \otimes \Delta_{s(\rho)}$ with label ρ, λ. If $\lambda = 1$, we may omit the λ from this label. If ρ is the only path from $s(\rho)$ to $t(\rho)$, then we may omit the ρ from this label.

Given a Δ-graph, we can recover the module (up to isomorphism) by reversing the above process.

Proposition 2.14. $\mathrm{Rep}_\Delta(Q, d)$ *is isomorphic (as an affine space) to the space of all* $M(d, \varphi)$.

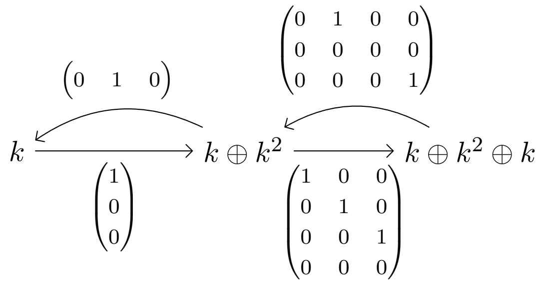
Proof. Follows immediately from Lemma 2.8, Lemma 2.10 and Lemma 2.12. $\square$

Definition 2.15. In graph theory, two vertices of an undirected graph are called **connected** if there is a path between them. A **connected component** is a subgraph in which any two vertices are connected and no vertices are connected to any vertex outside the subgraph. We apply these terms to the directed Δ-graphs by simply ignoring the orientation of the graph.

Example 2.16. Let Q be the following quiver.

$$1 \xrightarrow[\alpha]{} 2 \xrightarrow[\beta]{} 3$$

Let M be the following D-module.



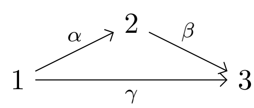
This is isomorphic to $M(d, \varphi)$ for $d = (1, 2, 1)$ and φ given by

$$\varphi_\alpha = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad \varphi_\beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \varphi_{\beta\alpha} = 0.$$

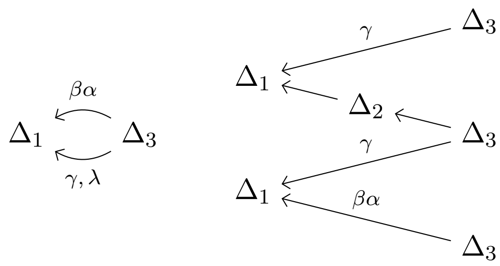
With respect to the standard bases, this has the following Δ-graph.

$$\begin{array}{lll} \Delta_1 \longleftarrow \Delta_2 & & \\ & \Delta_2 \longleftarrow \Delta_3 & \end{array}$$

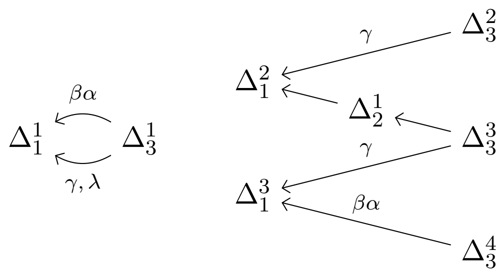
Example 2.17. Let Q be the following quiver.



Let $M(d, \varphi)$ have the following Δ-graph.



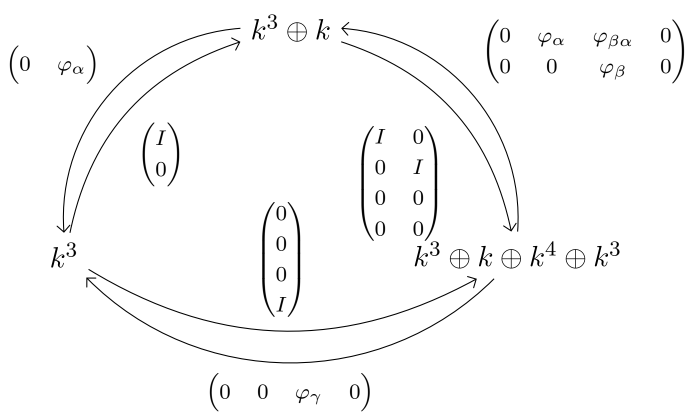
From this we immediately see that $d = (3, 1, 4)$. For each $i \in Q_0$, fix orders on the vertices labelled Δ_i.



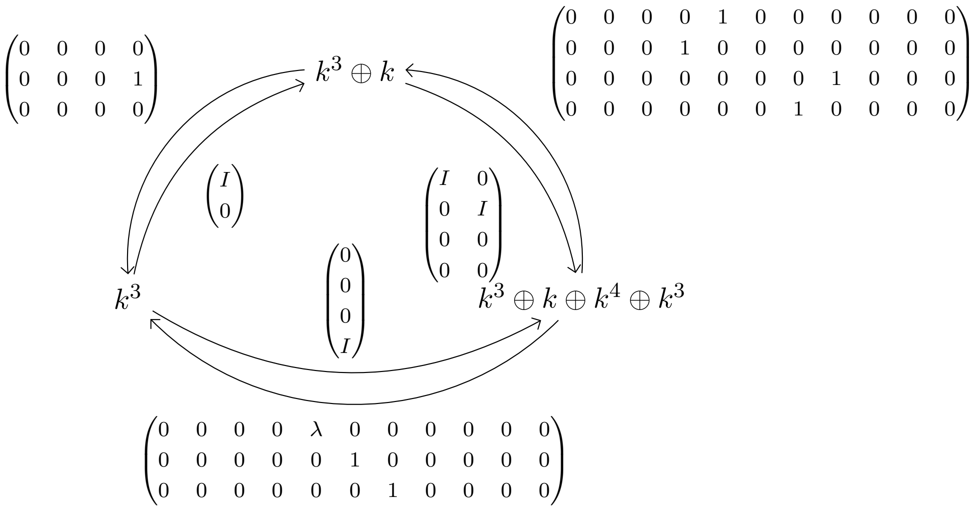
Then the maps φ_ρ can be read off for each $\rho \in P^*(-,-)$. Here, we get the following.

$$\varphi_\alpha = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \varphi_\beta = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \quad \varphi_\gamma = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \varphi_{\beta\alpha} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consequently, $M(d, \varphi)$ is isomorphic to the following representation.



Expressing the maps on the arrows in Q^{op} in full, this becomes the following.


$\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$
$k^3 \oplus k$
$\begin{pmatrix}0&0&0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0&0&0\end{pmatrix}$
$\begin{pmatrix}I\\0\end{pmatrix}$
$\begin{pmatrix}I&0\\0&I\\0&0\\0&0\end{pmatrix}$
$\begin{pmatrix}0\\0\\0\\I\end{pmatrix}$
k^3
$k^3 \oplus k \oplus k^4 \oplus k^3$
$\begin{pmatrix}0&0&0&0&\lambda&0&0&0&0&0&0\\0&0&0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&1&0&0&0&0\end{pmatrix}$

Chapter 3

General Representations and Canonical Decomposition

We recall the results of Kac [17, 18] on canonical decomposition of dimension vectors and general representations of quivers. We include some additional definitions in order to simplify the statements of these results. We then present the main results of Part I, which are analogous results for A-projective D-modules in the case where the quiver is of affine type $\mathbf{A}$ (without an oriented cycle). Throughout this chapter, Q is a quiver and $d \in \mathbb{N}^{Q_0}$.

Definition 3.1. Let $M \in \mathrm{Rep}(Q, d)$ and suppose $d = \sum_{i=1}^{n} d_i$ is a decomposition of d. We say M **decomposes according to** $\sum_{i=1}^{n} d_i$ if M is expressible as a direct sum $\bigoplus_{i=1}^{n} M_i$, where each M_i is indecomposable and has dimension vector d_i.

Theorem 3.2 (2.8(a) in [17])**.** *There exists a (unique) decomposition $d = \sum_{i=1}^{n} d_i$ such that the set of all $M \in \mathrm{Rep}(Q, d)$ which decompose according to $\sum_{i=1}^{n} d_i$ contains a dense open subset of* $\mathrm{Rep}(Q, d)$.

Definition 3.3 (2.8(a) in [17]; 1(b) and 4 in [18]; Section 4 in [18])**.** The decomposition $d = \sum_{i=1}^{n} d_i$ in the theorem is called the **canonical decomposition** of d. If $n = 1$, we say the canonical decomposition of d is trivial. Denote by $\mathrm{Can}(Q, d)$ the set of all representations in $\mathrm{Rep}(Q, d)$ which decompose according to the canonical decomposition of d.

Denote by $\mathrm{Rep}^0(Q, d)$ the subset of $\mathrm{Rep}(Q, d)$ consisting of representations with $\mathrm{Gl}(d)$-orbits of maximal dimension. This is a dense open subset, sometimes called the **open sheet** of the action of $\mathrm{Gl}(d)$ on $\mathrm{Rep}(Q, d)$.

Denote by $\mathrm{Gen}(Q,d)$ the intersection of $\mathrm{Rep}^0(Q,d)$ and $\mathrm{Can}(Q,d)$. A representation in $\mathrm{Gen}(Q,d)$ is called a **general representation** (or **representation in general position**).

We call d a **Schur root** if there exists an $M \in \mathrm{Rep}(Q,d)$ with $\dim \mathrm{End}\, M = 1$ (such an M is indecomposable).

Theorem 3.4 (Proposition 3 in [18]). *The canonical decomposition of d is $\sum_{i=1}^n d_i$ if and only if*

- *all the d_i are Schur roots, and*
- *there exist $U_i \in \mathrm{Rep}^0(Q,d_i)$ for $i = 1,\cdots,n$ such that $\mathrm{Ext}^1(U_i,U_j) = 0$ if $i \neq j$.*

Lemma 3.5. *d is a Schur root if and only if $\mathrm{Rep}(Q,d)$ contains a dense open subset consisting of indecomposable representations.*

Proof. If d is a Schur root, then $\mathrm{Rep}^0(Q,d)$ consists of indecomposables, so is the dense open subset required.

On the other hand, if $\mathrm{Rep}(Q,d)$ contains a dense open subset of indecomposables, this must intersect $\mathrm{Can}(Q,d)$, which implies that the canonical decomposition of d is trivial. Now d is a Schur root by Theorem 3.4. □

Proposition 3.6. *Kac's Theorem (Theorem 3.4) is equivalent to the following:*

The canonical decomposition of d is $\sum_{i=1}^n d_i$ if and only if

- *all the d_i are Schur roots, and*
- *there exist $U_i \in \mathrm{Gen}(Q,d_i)$ such that $\mathrm{Ext}^1(U_i,U_j) = 0$ if $i \neq j$.*

Proof. We need to prove that if d_i is a Schur root, then $\mathrm{Rep}^0(Q,d_i) = \mathrm{Gen}(Q,d_i)$. We have $\mathrm{Rep}^0(Q,d_i) \supseteq \mathrm{Gen}(Q,d_i)$ by definition of the latter. For the reverse inclusion, observe that $\mathrm{Rep}^0(Q,d_i)$ consists of representations M with $\dim \mathrm{End}\, M = 1$, hence they are indecomposable. Consequently, $\mathrm{Gen}(Q,d_i)$ consists of indecomposable representations, so canonical decomposition of d is trivial. Thus any $M \in \mathrm{Rep}^0(Q,d_i)$ is also contained in $\mathrm{Gen}(Q,d_i)$. □

Definition 3.7. Let $M \in \mathrm{Rep}_\Delta(Q,d)$ and suppose $d = \sum_{i=1}^n d_i$ is a decomposition of d. We say M **decomposes according to** $\sum_{i=1}^n d_i$ if M is expressible as a direct sum $\bigoplus_{i=1}^n M_i$, where each M_i is indecomposable and has Δ-dimension vector d_i.

This does not conflict with Definition 3.1, as the two agree when the dimension vector and Δ-dimension vector of a representation are the same.

We say d is **Δ-Schurian** if $\operatorname{Rep}_\Delta(Q,d)$ contains a dense open subset consisting of indecomposable representations. This definition makes sense in light of Lemma 3.5.

Remark 3.8. The main results of Part I are

- Theorem 3.9, which is an analogue of Theorem 3.2 for A-projective D-modules.
- Theorem 3.11, which is an analogue of Theorem 3.4 in light of Proposition 3.6.

For the rest of this chapter, Q is of affine type $\mathbf{A}$ without an oriented cycle.

Theorem 3.9 (Corollary 8.22)**.** *There exists a decomposition $d = \sum_{i=1}^{n} d_i$ such that the set of $M \in \operatorname{Rep}_\Delta(Q,d)$ which decompose according to $\sum_{i=1}^{n} d_i$ contains a dense open subset of $\operatorname{Rep}_\Delta(Q,d)$.*

Definition 3.10. The decomposition $d = \sum_{i=1}^{n} d_i$ in Theorem 3.9 is called the **canonical Δ-decomposition of** d. Denote by $\operatorname{Can}_\Delta(Q,d)$ the set of all $M \in \operatorname{Rep}_\Delta(Q,d)$ which decompose according to the canonical Δ-decomposition of d.

Denote by $\operatorname{Rep}^0_\Delta(Q,d)$ the dense open subset of $\operatorname{Rep}_\Delta(Q,d)$ consisting of representations with $\operatorname{Aut}_A P(d)$-orbits of maximal dimension.

In Chapter 8, we define by construction a subset $\operatorname{Gen}_\Delta(Q,d)$ of $\operatorname{Rep}_\Delta(Q,d)$. We show that $\operatorname{Gen}_\Delta(Q,d)$ contains a dense open subset of $\operatorname{Rep}_\Delta(Q,d)$, and is contained in the intersection of $\operatorname{Rep}^0_\Delta(Q,d)$ and $\operatorname{Can}_\Delta(Q,d)$.

Theorem 3.11 (Theorem 8.32)**.** *$\sum_{i=1}^{n} d_i$ is the canonical Δ-decomposition of d if and only if*

- *all the d_i are Δ-Schurian, and*
- *there exist $U_i \in \operatorname{Gen}_\Delta(Q,d)$ for $i = 1, \cdots, n$ such that $\operatorname{Ext}^1(U_i, U_j) = 0$ if $i \neq j$.*

Chapter 4

Morphisms of Delta-Filtered Representations of Double Quivers

Let $M = M(d, \varphi)$ and $M' = M(d', \varphi')$ for some d, d' and some φ, φ'. For $i \in Q_0$, we have that $M_i = e_i M$, where

$$\begin{aligned} e_i M &= \bigoplus_{j \in Q_0} k^{d_j} \otimes_k e_i \Delta_j \\ &= \bigoplus_{j \in Q_0} k^{d_j} \otimes_k \operatorname{span} P(j, i) \\ &= \bigoplus_{\rho \in P(-,i)} k^{d_{s(\rho)}} \otimes_k \rho. \end{aligned}$$

Similarly for M'_i.

Let $f : M \to M'$ be a D-module homomorphism. That is, $f = (f_i)_{i \in Q_0}$ where each f_i is a linear map $M_i \to M'_i$ such that

$$f_{t(\alpha)} \circ M_\alpha = M'_\alpha \circ f_{s(\alpha)}, \tag{1}$$

$$f_{t(\overline{\alpha})} \circ M_{\overline{\alpha}} = M'_{\overline{\alpha}} \circ f_{s(\overline{\alpha})}, \tag{2}$$

for all $\alpha \in Q_1$.

Lemma 4.1. *There exist some linear maps $f_\sigma : k^{d_{t(\sigma)}} \to k^{d'_{s(\sigma)}}$ for $\sigma \in P(-,-)$ such* *that*

$$f_i(v \otimes \rho) = \sum_{\sigma \in P(-,-)} f_\sigma(v) \otimes \rho\sigma$$

for each $i \in Q_0$*,* $\rho \in P(-,i)$ *and* $v \in k^{d_{s(\rho)}}$*.*

Proof. For $i \in Q_0$ and $v \in k^{d_i}$, we have $f_i(v \otimes e_i) \in M'_i$, so that

$$f_i(v \otimes e_i) = \sum_{\sigma \in P(-,i)} w_{f,\sigma,v} \otimes \sigma$$

for some values $w_{f,\sigma,v} \in k^{d_{s(\sigma)}}$. We can write each $w_{f,\sigma,v} = f_\sigma(v)$ for some function $f_\sigma : k^{d_{t(\sigma)}} \to k^{d_{s(\sigma)}}$. That is,

$$f_i(v \otimes e_i) = \sum_{\sigma \in P(-,i)} f_\sigma(v) \otimes \sigma. \qquad (*)$$

As the f_i are linear maps, these functions $f_\sigma : k^{d_{t(\sigma)}} \to k^{d_{s(\sigma)}}$ are necessarily linear maps also.

Now for a nontrivial path $\rho = \rho_n \cdots \rho_1$ in Q and $v \in k^{d_{s(\rho)}}$, we have (by repeated application of relation (1))

$$\begin{aligned} f_{t(\rho)}(v \otimes \rho) &= f_{t(\rho)}(v \otimes \rho_n \cdots \rho_1) \\ &= (f_{t(\rho)} \circ M_{\rho_n} \circ \cdots \circ M_{\rho_1})(v \otimes e_{s(\rho)}) \\ &= (M'_{\rho_n} \circ \cdots \circ M'_{\rho_1} \circ f_{s(\rho)})(v \otimes e_{s(\rho)}) \\ &= (M'_{\rho_n} \circ \cdots \circ M'_{\rho_1}) \left(\sum_{\sigma \in P(-,s(\rho))} f_\sigma(v) \otimes \sigma \right) \\ &= \sum_{\sigma \in P(-,s(\rho))} f_\sigma(v) \otimes \rho\sigma \\ &= \sum_{\sigma \in P(-,-)} f_\sigma(v) \otimes \rho\sigma. \end{aligned}$$

The last equality holds, because $\rho\sigma = 0$ if $t(\sigma) \neq s(\rho)$.

Finally, note that we can write $(*)$ as

$$f_i(v \otimes e_i) = \sum_{\sigma \in P(-,-)} f_\sigma(v) \otimes e_i\sigma.$$

We can therefore combine this with the above to get a single expression for each f_i, as follows. For each $i \in Q_0$, $\rho \in P(-,i)$ and $v \in k^{d_{s(\rho)}}$, we have

$$f_i(v \otimes \rho) = \sum_{\sigma \in P(-,-)} f_\sigma(v) \otimes \rho\sigma.$$

□

Lemma 4.2. *The linear maps in Lemma 4.1 satisfy the following. For all $\rho \in P^*(-,-)$,*

$$\sum_{\sigma\sigma'=\rho} f_{\sigma'} \circ \varphi_\sigma = \sum_{\tau\tau'=\rho} \varphi'_{\tau'} \circ f_\tau,$$

where the sum on the left is over $\sigma \in P^*(-,-)$ and $\sigma' \in P(-,-)$ such that $\sigma\sigma' = \rho$, while the sum on the right is over $\tau \in P(-,-)$ and $\tau' \in P^*(-,-)$ such that $\tau\tau' = \rho$.

Proof. Let $\alpha \in Q_1$ and $v \in k^{d_{t(\alpha)}}$. By (2), we have that $(f_{t(\overline{\alpha})} \circ M_{\overline{\alpha}})(v \otimes e_{t(\alpha)}) = (M'_{\overline{\alpha}} \circ f_{s(\overline{\alpha})})(v \otimes e_{t(\alpha)})$. Firstly, we have that

$$\begin{aligned}(f_{t(\overline{\alpha})} \circ M_{\overline{\alpha}})(v \otimes e_{t(\alpha)}) &= f_{t(\overline{\alpha})}\left(\sum_{\sigma \in P(-,-)} \varphi_{\alpha\sigma}(v) \otimes \sigma\right) \\ &= \sum_{\sigma \in P(-,-)} \sum_{\sigma' \in P(-,-)} f_{\sigma'}(\varphi_{\alpha\sigma}(v)) \otimes \sigma\sigma'. \\ &= \sum_{\rho \in P(-,-)} \left(\sum_{\sigma,\sigma'} (f_{\sigma'} \circ \varphi_{\alpha\sigma})(v)\right) \otimes \rho,\end{aligned}$$

where the final sum is over $\sigma, \sigma' \in P(-,-)$ such that $\sigma\sigma' = \rho$. Note that we can also write this as

$$\sum_{\rho \in P(-,s(\alpha))} \left(\sum_{\sigma,\sigma'} (f_{\sigma'} \circ \varphi_\sigma)(v)\right) \otimes \rho,$$

where the unspecified sum is over $\sigma \in P^*(-,-)$ and $\sigma' \in P(-,-)$ such that $\sigma\sigma' = \alpha\rho$.

On the other hand, we have

$$(M'_{\overline{\alpha}} \circ f_{s(\overline{\alpha})})(v \otimes e_{t(\alpha)}) = M'_{\overline{\alpha}}\left(\sum_{\sigma \in P(-,-)} f_\sigma(v) \otimes e_{t(\alpha)}\sigma\right)$$

$$= M'_{\overline{\alpha}}\left(f_{e_{t(\alpha)}}(v) \otimes e_{t(\alpha)}\right) + \sum_{\sigma \in P^*(-,t(\alpha))} M'_{\overline{\alpha}}\left(f_\sigma(v) \otimes \sigma\right). \qquad (*)$$

Consider the term on the right. $M'_{\overline{\alpha}}$ annihilates $f_\sigma(v) \otimes \sigma$ unless $\sigma = \alpha\tau$ for some path τ, thus

$$\begin{aligned}
\sum_{\sigma \in P^*(-,t(\alpha))} M'_{\overline{\alpha}}\left(f_\sigma(v) \otimes \sigma\right) &= \sum_{\sigma \in P(-,s(\alpha))} M'_{\overline{\alpha}}\left(f_{\alpha\sigma}(v) \otimes \alpha\sigma\right) \\
&= \sum_{\sigma \in P(-,s(\alpha))} \sum_{\sigma' \in P^*(-,-)} \varphi'_{\sigma'}(f_{\alpha\sigma}(v)) \otimes \sigma\sigma' \\
&= \sum_{\rho \in P(-,s(\alpha))} \left(\sum_{\sigma,\sigma'} (\varphi'_{\sigma'} \circ f_{\alpha\sigma})(v) \right) \otimes \rho,
\end{aligned}$$

where the final sum is over $\sigma \in P(-,-)$ and $\sigma' \in P^*(-,-)$ such that $\sigma\sigma' = \rho$. Note we can also write this as

$$\sum_{\rho \in P(-,s(\alpha))} \left(\sum_{\sigma,\sigma'} (\varphi'_{\sigma'} \circ f_\sigma)(v) \right) \otimes \rho,$$

where $\sigma, \sigma' \in P^*(-,-)$ such that $\sigma\sigma' = \alpha\rho$. Now we return to $(*)$ and consider the term on the left.

$$\begin{aligned}
M'_{\overline{\alpha}}\left(f_{e_{t(\alpha)}}(v) \otimes e_{t(\alpha)}\right) &= \sum_{\rho \in P(-,s(\alpha))} \varphi'_{\alpha\rho}(f_{e_{t(\alpha)}}(v)) \otimes \rho \\
&= \sum_{\rho \in P(-,s(\alpha))} (\varphi'_{\sigma'} \circ f_\sigma)(v) \otimes \rho,
\end{aligned}$$

where $\sigma \in P(-,-) \setminus P^*(-,-)$ and $\sigma' \in P^*(-,-)$ such that $\sigma\sigma' = \alpha\rho$. Combining this with the above, we can now evaluate $(*)$ as follows.

$$(M'_{\overline{\alpha}} \circ f_{s(\overline{\alpha})})(v \otimes e_{t(\alpha)}) = \sum_{\rho \in P(-,s(\alpha))} \left(\sum_{\sigma,\sigma'} (\varphi'_{\sigma'} \circ f_\sigma)(v) \right) \otimes \rho,$$

where the sum is over $\sigma \in P(-,-)$ and $\sigma' \in P^*(-,-)$ such that $\sigma\sigma' = \alpha\rho$.

Consequently, we have the following equality

$$\sum_{\rho \in P(-,s(\alpha))} \left(\sum_{\sigma,\sigma'} (f_{\sigma'} \circ \varphi_\sigma)(v) \right) \otimes \rho = \sum_{\rho \in P(-,s(\alpha))} \left(\sum_{\sigma,\sigma'} (\varphi'_{\sigma'} \circ f_\sigma)(v) \right) \otimes \rho,$$

where the sum on the left is over $\sigma \in P^*(-,-)$ and $\sigma' \in P(-,-)$ such that $\sigma\sigma' = \alpha\rho$, while the sum on the right is over $\sigma \in P(-,-)$ and $\sigma' \in P^*(-,-)$ such that $\sigma\sigma' = \alpha\rho$.

We can equate coefficients to obtain

$$\sum_{\sigma,\sigma'}(f_{\sigma'} \circ \varphi_\sigma)(v) = \sum_{\sigma,\sigma'}(\varphi'_{\sigma'} \circ f_\sigma)(v)$$

for each $\rho \in P(-, s(\alpha))$. As this holds for all $v \in k^{d_{t(\alpha)}}$, we have

$$\sum_{\sigma\sigma'=\alpha\rho} f_{\sigma'} \circ \varphi_\sigma = \sum_{\sigma\sigma'=\alpha\rho} \varphi'_{\sigma'} \circ f_\sigma.$$

This holds for all $\alpha \in Q_1$ and $\rho \in P(-,-)$. Thus equivalently, we have

$$\sum_{\sigma\sigma'=\rho} f_{\sigma'} \circ \varphi_\sigma = \sum_{\sigma\sigma'=\rho} \varphi'_{\sigma'} \circ f_\sigma.$$

for all nontrivial paths ρ. □

Lemma 4.3. *All choices of linear maps $f_\sigma : k^{d_{t(\sigma)}} \to k^{d'_{s(\sigma)}}$ satisfying the relations in Lemma 4.2 form a morphism $M \to M'$ in the way described in Lemma 4.1.*

Proof. Define $f : M \to M'$ by

$$f_i(v \otimes \rho) = \sum_{\sigma \in P(-,-)} f_\sigma(v) \otimes \rho\sigma$$

for each $i \in Q_0$, $\rho \in P(-, i)$ and $v \in k^{d_{s(\rho)}}$. The f_σ are linear, which implies the f_i are linear. Further, the relations in Lemma 4.2 are sufficient to imply equations (1) and (2) hold. Thus f is a morphism of D-modules. □

Proposition 4.4. *$f : M \to M'$ is a D-module homomorphism if and only if there exists some set of linear maps $f_\sigma : k^{d_{t(\sigma)}} \to k^{d'_{s(\sigma)}}$ (indexed by $\sigma \in P(-,-)$) which satisfies*

$$\sum_{\sigma\sigma'=\rho} f_{\sigma'} \circ \varphi_\sigma = \sum_{\tau\tau'=\rho} \varphi'_{\tau'} \circ f_\tau$$

for all $\rho \in P^*(-,-)$, such that

$$f_i(v \otimes \rho) = \sum_{\sigma \in P(-,-)} f_\sigma(v) \otimes \rho\sigma$$

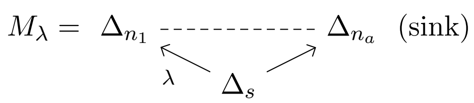
for each $i \in Q_0$, $\rho \in P(-, i)$ and $v \in k^{d_{s(\rho)}}$.

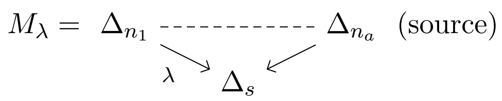
Proof. Consequence of Lemma 4.1, Lemma 4.2 and Lemma 4.3. □

Chapter 5

Representations with a Flat Cyclic Delta-Graph

In this chapter, Q is a quiver of affine type $\mathbf{A}$ without an oriented cycle. Choose a sink or source in Q and denote this vertex by s. For each $\lambda \in k$, let M_λ be given by the following Δ-graph (which also depends on the vertices $n_1, \cdots, n_a \in Q_0$).





The dashed line represents a connected sequence of arrows such that the corresponding paths in the quiver do not overlap. That is, each arrow in Q occurs in a path corresponding to an arrow in the Δ-graph precisely once. This implies that M_λ is Δ-flat in the sense that $(\underline{\dim}_\Delta M_\lambda)_i \leq 1$ for all $i \in Q_0$. Further, $(\underline{\dim}_\Delta M_\lambda)_i = 1$ whenever i is a sink or source.

For convenience, write $n_0 = n_{a+1} = s$. For each $i = 0, \cdots, n_a$, write σ_i for the path between n_i and n_{i+1}. That is, if $M_\lambda = M(d, \varphi)$, then $\varphi_{\sigma_0} = \lambda$, $\varphi_{\sigma_i} = 1$ for $i \neq 0$ and otherwise $\varphi_\sigma = 0$.

Lemma 5.1. *Let $f : M_\lambda \to M_\mu$ be a morphism of D-modules given by maps f_σ as in Proposition 4.4. Then $f_{e_s} = f_{e_{n_1}} = \cdots = f_{e_{n_a}}$. Further, if $\lambda \neq \mu$, then $f_{e_s} = 0$.*

Proof. Write $M_\lambda = M(d,\varphi)$ and $M_\mu = M(d,\varphi')$. The f_σ satisfy

$$\sum_{\sigma\sigma'=\rho} f_{\sigma'} \circ \varphi_\sigma = \sum_{\tau\tau'=\rho} \varphi'_{\tau'} \circ f_\tau,$$

for all $\rho \in P^*(-,-)$. In particular, for each $i = 0, \cdots, a$, we have

$$f_{e_{n_i}} \circ \varphi_{\sigma_i} = \varphi'_{\sigma_i} \circ f_{e_{n_{i+1}}},$$

as $\varphi_\sigma = 0$ and $\varphi'_\sigma = 0$ for all proper sub-paths σ of σ_i. For $i \neq 0$, we have $\varphi_{\sigma_i} = \varphi'_{\sigma_i} = 1$, hence $f_{e_{n_i}} = f_{e_{n_{i+1}}}$ for $i = 1, \cdots, a$. Thus $f_{e_{n_1}} = \cdots = f_{e_{n_a}} = f_{e_s}$. On the other hand, for $i = 0$, we have $f_{e_s} \circ \varphi_{\sigma_0} = \varphi'_{\sigma_0} \circ f_{e_{n_1}}$, which implies that $\lambda f_{e_s} = \mu f_{e_{n_1}}$. If $\lambda \neq \mu$, this in turn implies that $f_{e_s} = f_{e_{n_1}} = 0$. □

Lemma 5.2. *M_λ is indecomposable.*

Proof. We use the fact that a module is indecomposable if its endomorphism ring contains no idempotents besides 0 and the identity (Proposition 3.1 in [13]). Let $f : M_\lambda \to M_\lambda$ be a morphism of D-modules. Suppose f is idempotent, nonzero and not equal to the identity. Then $f_i = f_i \circ f_i$ for each $i \in Q_0$. We have

$$f_i(v \otimes \rho) = \sum_{\sigma \in P(-,-)} f_\sigma(v) \otimes \rho\sigma$$

and

$$\begin{aligned}(f_i \circ f_i)(v \otimes \rho) &= \sum_{\sigma \in P(-,-)} f_i\left(f_\sigma(v) \otimes \rho\sigma\right) \\ &= \sum_{\sigma \in P(-,-)} \left(\sum_{\sigma' \in P(-,-)} f_{\sigma'}(f_\sigma(v)) \otimes \rho\sigma\sigma' \right) \\ &= \sum_{\sigma,\sigma' \in P(-,-)} (f_{\sigma'} \circ f_\sigma)(v) \otimes \rho\sigma\sigma'\end{aligned}$$

for all $\rho \in P(-,i)$ and $v \in k^{d_{s(\rho)}}$. Equivalently,

$$f_\rho = \sum_{\sigma\sigma'=\rho} f_{\sigma'} \circ f_\sigma \qquad (*)$$

for each $\rho \in P(-,-)$.

The Δ-dimension is zero at all interior vertices of each path σ_i, so f_ρ can only be nonzero if ρ is a product of some σ_i or if $\rho = e_{n_i}$. Suppose all the $f_{e_i} = 0$. By $(*)$, this implies that each $f_{\sigma_i} = f_{e_{s(\sigma_i)}} \circ f_{\sigma_i} + f_{\sigma_i} \circ f_{e_{t(\sigma_i)}} = 0$. This in turn implies that all $f_{\sigma_i\sigma_j} = 0$ and so on until $f_\rho = 0$ for all $\rho \in P(-,-)$. That is, $f = 0$, which is a contradiction, therefore some $f_{e_i} \neq 0$.

But now Lemma 5.1 implies that $f_{e_{n_0}} = \cdots = f_{e_{n_a}} \neq 0$. Each $f_{e_{n_i}} = f_{e_{n_i}} \circ f_{e_{n_i}}$ by $(*)$. Thus each $f_{e_{n_i}} = 1$. By $(*)$, this implies that each $f_{\sigma_i} = f_{e_{s(\sigma_i)}} \circ f_{\sigma_i} + f_{\sigma_i} \circ f_{e_{t(\sigma_i)}} = f_{\sigma_i} + f_{\sigma_i}$, so that $f_{\sigma_i} = 0$. This in turn implies that all $f_{\sigma_i\sigma_j} = 0$ and so on until $f_\rho = 0$ for all $\rho \in P^*(-,-)$. But now

$$f_i(v \otimes \rho) = \sum_{\sigma \in P(-,-)} f_\sigma(v) \otimes \rho\sigma = f_{e_{s(\rho)}}(v) \otimes \rho e_{s(\rho)} = v \otimes \rho$$

for all $\rho \in P(-,i)$ and $v \in k^{d_{s(\rho)}}$. That is, f is the identity, which is a contradiction. Therefore, the supposition that there exists an idempotent different from 0 and the identity must be false. □

Lemma 5.3. *If $\lambda \neq \mu$, then M_λ is not isomorphic to M_μ.*

Proof. Write $M_\lambda = M(d,\varphi)$ and $M_\mu = M(d,\varphi')$. Suppose $f : M_\lambda \to M_\mu$ is an isomorphism of D-modules. Then f is given by maps f_σ as in Proposition 4.4. As f is surjective, for each $i = n_1, \cdots, n_a$, we have $1 \otimes e_i = f_i(x)$ for some $x \in M_i$. Specifically,

$$x = \sum_{\rho \in P(-,i)} x_\rho \otimes \rho$$

for some $x_\rho \in k$. We have that

$$f_i(x_\rho \otimes \rho) = \sum_{\sigma \in P(-,-)} f_\sigma(x_\rho) \otimes \rho\sigma.$$

But $\rho\sigma = e_i$ implies that $\rho = \sigma = e_i$, thus $x = x_{e_i} \otimes e_i$ and $f_i(x) = f_{e_i}(x_{e_i}) \otimes e_i = 1 \otimes e_i$. This implies that $f_{e_i} : k \to k$ is nonzero. However, as $\lambda \neq \mu$, this contradicts Lemma 5.1. Consequently, there does not exist an isomorphism of D-modules $f : M_\lambda \to M_\mu$. □

Lemma 5.4. *The dimension of* $\operatorname{End} M_\lambda$ *does not depend on λ.*

Proof. We assume that s is a source (if s is a sink, the proof is similar). By Proposition 4.4, $\operatorname{End} M_\lambda$ is isomorphic as a vector space to the subspace E of $P(-,-)$-tuples of linear maps $f_\sigma : k^{d_{t(\sigma)}} \to k^{d_{s(\sigma)}}$ which satisfy

$$\sum_{\sigma\sigma'=\rho} f_{\sigma'} \circ \varphi_\sigma = \sum_{\tau\tau'=\rho} \varphi_{\tau'} \circ f_\tau \qquad (*)$$

for all $\rho \in P^*(-,-)$. This space E is the intersection of the subspace E_λ satisfying the relations dependent on λ with the subspace $E_\emptyset$ satisfying the relations not dependent on λ. These relations are all linear, so E_λ and $E_\emptyset$ are the solution spaces of some subspaces X_λ and $X_\emptyset$ of the dual space E^*.

For ρ such that σ_0 is not a sub-path of ρ, the relation $(*)$ does not depend on λ. For ρ such that either $d_{t(\rho)} = 0$ or $d_{s(\rho)} = 0$, the relation $(*)$ is trivial. So the only cases when $(*)$ may depend on λ are when $\rho = \sigma_r \cdots \sigma_0$ $(r = 0, 1, \cdots)$. In these cases, $(*)$ reduces to the following.

$$\begin{aligned} f_{e_s} \circ \varphi_{\sigma_0} &= \varphi_{\sigma_0} \circ f_{e_{n_1}}, \\ f_{\sigma_0} \circ \varphi_{\sigma_1} &= \varphi_{\sigma_0} \circ f_{\sigma_1}, \\ &\vdots \\ f_{\sigma_{r-1}\cdots\sigma_0} \circ \varphi_{\sigma_r} &= \varphi_{\sigma_0} \circ f_{\sigma_r\cdots\sigma_1}. \end{aligned}$$

As $\varphi_{\sigma_0} = \lambda$ and each other $\varphi_{\sigma_i} = 1$, this is equivalent to

$$\begin{aligned} \lambda f_{e_s} &= \lambda f_{e_{n_1}}, \\ f_{\sigma_0} &= \lambda f_{\sigma_1}, \\ &\vdots \\ f_{\sigma_{r-1}\cdots\sigma_0} &= \lambda f_{\sigma_r\cdots\sigma_1}. \end{aligned}$$

The relations not dependent on λ imply that $f_{e_s} = f_{e_{n_1}}$ (Lemma 5.1), so the first equation can be disregarded (we may assume that X_λ does not include the first equation).

We can see that the dimension of X_λ does not depend on the value of λ. Further, the relations not dependent on λ do not involve $f_{\sigma_0}, \cdots, f_{\sigma_r\cdots\sigma_0}$, hence X_λ and $X_\emptyset$ are independent. Consequently, the dimension of $E = E_\lambda \cap E_\emptyset = \mathrm{sol}(X_\lambda + X_\emptyset)$ does not depend on the value of λ. □

Lemma 5.5. *If $\lambda \neq \mu$, then $\dim \mathrm{Hom}(M_\lambda, M_\mu)$ does not depend on λ and μ.*

Proof. Similar to the proof of Lemma 5.4. □

Definition 5.6. Splitting Q at vertex s to obtain a Dynkin type **A** quiver Q', the representation M_λ splits into a representation M'_λ with the following Δ-graph (n_0 and n_{a+1} are the two vertices which s splits into).

$$M'_\lambda = \ \Delta_{n_0} \underset{\lambda}{\longrightarrow} \Delta_{n_1} \text{ - - - - - - - - - - - - } \Delta_{n_a} \leftarrow \Delta_{n_{a+1}} \quad (s \text{ a sink})$$

$$M'_\lambda = \ \Delta_{n_0} \underset{\lambda}{\longleftarrow} \Delta_{n_1} \text{ - - - - - - - - - - - - } \Delta_{n_a} \rightarrow \Delta_{n_{a+1}} \quad (s \text{ a source})$$

Lemma 5.7. *The M'_λ belong to the same isomorphism class for all $\lambda \neq 0$.*

Proof. Assume that s is a sink (the proof is similar when s is a source). We show that $M'_\lambda \cong M'_1$ for all $\lambda \neq 0$.

Let $f : M'_\lambda \to M'_1$ be given (as in Proposition 4.4) by $f_{e_{n_0}} = \lambda$ and $f_{e_{n_i}} = 1$ for $i \neq 0$, and $f_\sigma = 0$ for all other $\sigma \in P(-,-)$. We can draw f as follows.

$$\begin{array}{ccccc}
\Delta_{n_0} & \underset{\lambda}{\longrightarrow} & \Delta_{n_1} & \text{- - - - - - - - - - - -} & \Delta_{n_a} \\
\downarrow \lambda & & \downarrow 1 & \cdots\cdots\cdots & \downarrow 1 \\
\Delta_{n_0} & \underset{1}{\longrightarrow} & \Delta_{n_1} & \text{- - - - - - - - - - - -} & \Delta_{n_a}
\end{array}$$

Referring to Proposition 4.4, we can see that f is a bijection and a morphism. $\square$

Chapter 6

Rigid Representations in Dynkin Type A

In this chapter, we give an algorithm for constructing Δ-filtered representations of Dynkin type **A** quivers (Definition 6.4). We show this algorithm follows the gluing procedure given in [16] and therefore produces rigid modules.

Until stated otherwise, we take Q to be a Dynkin type **A** quiver with linear orientation, as follows.

$$Q = \quad 1 \xrightarrow[\alpha_1]{} \cdots \xrightarrow[\alpha_{n-1}]{} n$$

For $i, j \in Q_0$ with $i \leq j$, there is a unique path $\rho(i,j)$ from i to j. When $i = j$, we have $\rho(i,j) = e_i$. When $i < j$, we have $\rho(i,j) = \alpha_{j-1} \cdots \alpha_i$. Let $\overline{\rho}(i,j)$ denote the corresponding path from j to i in Q^{op}. Let $P_D(n)$ denote the indecomposable projective D-module associated to vertex n.

The paths $\rho(i,j)\overline{\rho}(i,l)$ for $i \leq j, l$ form a basis for D (5.1 in [14]). Further, for each $i \in Q_0$, the paths $\rho(i,j)$ for $j \in Q_0$ form a basis for Δ_i.

Lemma 6.1. *Let $P_D(n)$ be the projective D-module associated to the vertex n. Then $P_D(n)$ has the following Δ-graph.*

$$\Delta_1 \longleftarrow \cdots \longleftarrow \Delta_n$$

Proof. The paths $\rho(i,j)\overline{\rho}(i,n)$ for $i,j \in Q_0$ form a basis for $P_D(n)$. That is,

$$\begin{aligned}
P_D(n) &= \operatorname{span}\{\rho(i,j)\overline{\rho}(i,n) : i,j \in Q_0\} \\
&= \bigoplus_{i\in Q_0} \operatorname{span}\{\rho(i,j)\overline{\rho}(i,n) : j \in Q_0\} \\
&= \bigoplus_{i\in Q_0} \operatorname{span}\{\rho(i,j) : j \in Q_0\}\,\overline{\rho}(i,n) \\
&= \bigoplus_{i\in Q_0} \Delta_i\overline{\rho}(i,n).
\end{aligned}$$

Now $\Delta_i \cong \Delta_i\overline{\rho}(i,n)$ as an A-module, so $P_D(n)$ has Δ-dimension 1 at each vertex.

Each $e_i \in \Delta_i$, hence $e_i\overline{\rho}(i,n) \in \Delta_i\overline{\rho}(i,n)$. The action by each opposite arrow $\overline{\alpha}_j$ on this element is

$$\overline{\alpha}_j \cdot e_i\overline{\rho}(i,n) = \overline{\alpha}_j e_i\overline{\rho}(i,n) = \begin{cases} 0, & \text{if } j \neq i-1, \\ \overline{\alpha}_{i-1}\overline{\rho}(i,n), & \text{if } j = i-1. \end{cases},$$

where $\overline{\alpha}_{i-1}\overline{\rho}(i,n) = \overline{\rho}(i-1,n) = e_{i-1}\overline{\rho}(i-1,n)$ which is a nonzero element of $\Delta_{i-1}\overline{\rho}(i-1,n)$. Therefore, each Δ_i is mapped into Δ_{i-1} by $\overline{\alpha}_{i-1}$, and these are the only nonzero maps between the Δ_i in $P_D(n)$. □

Definition 6.2 (Section 5.1 in [14])**.** Let $I \subseteq Q_0$. Denote by $X(I)$ the unique submodule of $P_D(n)$ with Δ-support I.

Lemma 6.3. *Let $I = \{n_1, \cdots, n_r\} \subseteq Q_0$, non-empty, labelled such that each $n_i < n_{i+1}$. Then $X(I)$ has the following Δ-graph.*

$$\Delta_{n_1} \longleftarrow \cdots \longleftarrow \Delta_{n_r}$$

Proof. Firstly, we must have $1 \leq r \leq n$ and each $n_i \geq i$. By Lemma 6.1, $P_D(n)$ has a (unique) submodule with the following Δ-graph.

$$\Delta_1 \longleftarrow \cdots \longleftarrow \Delta_r$$

Specifically, this submodule is

$$\bigoplus_{i=1}^{r} \Delta_i\overline{\rho}(i,n).$$

For each $i = 1, \cdots, r$, we have

$$\begin{aligned}
\Delta_i &= \operatorname{span}\{\rho(i,j) : j = i, \cdots, n\} \\
&\supseteq \operatorname{span}\{\rho(i,j) : j = n_i, \cdots, n\} \\
&= \operatorname{span}\{\rho(n_i,j)\rho(i,n_i) : j = n_i, \cdots, n\} \\
&= \operatorname{span}\{\rho(n_i,j) : j = n_i, \cdots, n\}\, \rho(i,n_i) \\
&= \Delta_{n_i}\rho(i,n_i).
\end{aligned}$$

Consequently, the following X is a subset of $P_D(n)$.

$$X = \bigoplus_{i=1}^{r} \Delta_{n_i}\rho(i,n_i)\overline{\rho}(i,n)$$

Each $\Delta_{n_i}\rho(i,n_i)\overline{\rho}(i,n) \cong \Delta_{n_i}$ as an A-module, so X has Δ-dimension 1 at each n_i and 0 elsewhere. Further, for X to be a submodule of $P_D(n)$, we need only check the action of each opposite arrow $\overline{\alpha}_j$ on each $e_{n_i}\rho(i,n_i)\overline{\rho}(i,n)$.

$$\overline{\alpha}_j \cdot e_{n_i}\rho(i,n_i)\overline{\rho}(i,n) = \overline{\alpha}_j e_{n_i}\rho(i,n_i)\overline{\rho}(i,n) = \begin{cases} 0, & \text{if } j \neq n_i - 1, \\ \overline{\alpha}_{n_i-1}\rho(i,n_i)\overline{\rho}(i,n), & \text{if } j = n_i - 1, \end{cases}$$

where, by application of the relations $\mathcal{I}$,

$$\begin{aligned}
\overline{\alpha}_{n_i-1}\rho(i,n_i)\overline{\rho}(i,n) &= \overline{\alpha}_{n_i-1}\alpha_{n_i-1}\cdots\alpha_i\overline{\rho}(i,n), \\
&= \alpha_{n_i-2}\overline{\alpha}_{n_i-2}\alpha_{n_i-2}\cdots\alpha_i\overline{\rho}(i,n), \\
&\ \ \vdots \\
&= \alpha_{n_i-2}\cdots\alpha_i\overline{\alpha}_i\alpha_i\overline{\rho}(i,n), \\
&= \alpha_{n_i-2}\cdots\alpha_i\alpha_{i-1}\overline{\alpha}_{i-1}\overline{\rho}(i,n), \\
&= \rho(i-1,n_i-1)\overline{\rho}(i-1,n), \\
&= \rho(n_{i-1},n_i-1)\rho(i-1,n_{i-1})\overline{\rho}(i-1,n),
\end{aligned}$$

which is an element of $\Delta_{n_{i-1}}\rho(i-1,n_{i-1})\overline{\rho}(i-1,n) \subseteq X$ if $i > 1$, or zero if $i = 1$. Therefore, X is a submodule of $P_D(n)$ with a Δ-graph in which each Δ_{n_i} is mapped into $\Delta_{n_{i-1}}$. □

Definition 6.4. For the rest of this chapter (unless stated otherwise), let Q be a

Dynkin type **A** quiver with any orientation and with vertices labelled $1, \cdots, n$, such that each i is connected to $i-1$ and $i+1$. Let $a_1 = 1$ and for each $i > 1$ let a_i be the greatest admissible vertex such that $a_i < i$.

Let $d \in \mathbb{N}^{Q_0}$. Construct a Δ-graph as follows and write (d, φ^d) for the data of this Δ-graph.

- Create an empty grid of $\mathbb{Z}$-indexed rows and columns.
- In column 1, write Δ_1 in d_1 contiguous rows, starting at row 1 and working up.
- For each $i = 2, \cdots, n$:
  - If a_i is a sink:
    * In column i, write Δ_i in d_i contiguous rows, starting at the lowest row with a Δ_{a_i} and working up. If there are no Δ_{a_i}, choose an arbitrary row to start from.
    * For each row with a Δ_i: Let $j < i$ be the greatest column with a Δ_j in this row. If j exists and $a_i \leq j$, then draw an arrow from the Δ_j to the Δ_i in this row.
  - If a_i is a source:
    * In column i, write Δ_i in d_i contiguous rows, starting at the greatest row with a Δ_{a_i} and working down. If there are no Δ_{a_i}, choose an arbitrary row to start from.
    * For each row with a Δ_i: Let $j < i$ be the greatest column with a Δ_j in this row. If j exists and $a_i \leq j$, then draw an arrow from the Δ_i to the Δ_j in this row.

Example 6.5. Let Q be the following quiver, and $d = (2, 1, 3, 4, 2)$.

$$Q = \ 1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longleftarrow 5$$

We begin by writing d_1 copies of Δ_1 in a column.

$$\begin{array}{c} \Delta_1 \\ \Delta_1 \end{array}$$

Next, we deal with $i = 2$. The greatest admissible vertex less than 2 is 1, so we have $a_2 = 1$, which is a source. So we write d_2 copies of Δ_2 in the next column, starting at the highest row with a Δ_1 and working down. For each of these Δ_2, we draw an arrow starting at that Δ_2 and ending at the Δ_1 in the same row.

$$\begin{array}{lll} \Delta_1 & \longleftarrow & \Delta_2 \\ \Delta_1 & & \end{array}$$

Next, we deal with $i = 3$. We have $a_3 = 1$ again, so we write d_3 copies of Δ_3 in the next column, again starting at the highest row with a Δ_1 and working down. We draw an arrow from the first Δ_3 to the Δ_2 in its row, and from the second Δ_3 to the Δ_1 in its row. No arrow is drawn from the final Δ_3, as there is no Δ_j in its row with $j \geq 1 = a_3$.

$$\begin{array}{lllll} \Delta_1 & \longleftarrow & \Delta_2 & \longleftarrow & \Delta_3 \\ \Delta_1 & \longleftarrow & \longleftarrow & \longleftarrow & \Delta_3 \\ & & & & \Delta_3 \end{array}$$

Next, we deal with $i = 4$. This time, we have $a_4 = 3$, which is a sink, so we work upwards. Write d_4 copies of Δ_4 in the next column, starting at the highest row with a Δ_3 and working upwards. From each of the first three Δ_4, we draw an arrow to the Δ_3 in the same row. Again, no arrow is drawn from the final Δ_4, as there is no Δ_j in its row with $j \geq 3 = a_4$.

$$\begin{array}{lllllll} & & & & & & \Delta_4 \\ \Delta_1 & \longleftarrow & \Delta_2 & \longleftarrow & \Delta_3 & \longrightarrow & \Delta_4 \\ \Delta_1 & \longleftarrow & \longleftarrow & \longleftarrow & \Delta_3 & \longrightarrow & \Delta_4 \\ & & & & \Delta_3 & \longrightarrow & \Delta_4 \end{array}$$

Lastly, we deal with $i = 5$. We have $a_5 = 3$ again, so we write d_5 copies of Δ_5 in the next column, starting at the highest row with a Δ_3 and working upwards. For each of these Δ_5, we draw an arrow starting at that Δ_5 and ending at the Δ_3 in the same row.

$$
\begin{array}{ccccccccc}
 & & & & & & \Delta_4 & & \\
\Delta_1 & \longleftarrow & \Delta_2 & \longleftarrow & \Delta_3 & \longrightarrow & \Delta_4 & & \\
\Delta_1 & \longleftarrow & \multicolumn{2}{c}{\longleftarrow} & \Delta_3 & \longrightarrow & \Delta_4 & \longrightarrow & \Delta_5 \\
 & & & & \Delta_3 & \longrightarrow & \Delta_4 & \longrightarrow & \Delta_5
\end{array}
$$

The above is the completed Δ-graph for $X(d)$.

Proposition 6.6. *(d, φ^d) is a Δ-graph for the rigid module $X(d) \in \mathrm{Rep}_\Delta(Q, d)$.*

Proof. We begin by recalling the construction of the rigid A-projective D-module $X(d)$ from 5.3 in [14].

Write i_s, where $s = 0, \cdots, t$, for the admissible vertices of Q, so that $i_s, i_s + 1, \cdots, i_{s+1}$ are the vertices of the sth linear component of the quiver. We write d^s for the vector with $(d^s)_j = d_j$ when j is contained in the sth linear component, and $(d^s)_j = 0$ otherwise.

We define a module $Y(d^s)$ for the restriction of Q to the sth linear component. The indecomposable summands of $Y(d^s)$ are the modules $X(I_1), \cdots, X(I_r)$ for some $I_1 \subseteq \cdots \subseteq I_r \subseteq \{i_s, \cdots, i_{s+1}\}$. These indecomposable summands are all rigid.

Consider the Δ-graph (d, φ^d) defined above. Restrict this to the sth linear component of Q. In this restriction, the nonempty rows are Δ-graphs for the modules $X(I_1), \cdots, X(I_r)$ for some $I_1 \subseteq \cdots \subseteq I_r \subseteq \{i_s, \cdots, i_{s+1}\}$. Further, the Δ-dimension vector of this restriction is d^s (restricted to the sth linear component). Therefore, this restriction is in fact a Δ-graph for $Y(d^s)$.

We extend $Y(d^s)$ to a module $X(d^s) \in \mathrm{Rep}_\Delta(Q, d^s)$. Note that this does not alter the Δ-graph. Finally, observe that in the construction of $X(d)$, the summands of each $X(d^s)$ are glued in an order which results in $X(d)$ having (d, φ^d) as a Δ-graph. $\square$

Remark 6.7. The arrows of (d, φ^d) each start and end in the same row, so the connected components of (d, φ^d) are each contained in a single row. Further, these connected components are precisely the Δ-graphs of the indecomposable summands of $M(d, \varphi^d) \cong X(d)$. We can therefore assign a row index in $\mathbb{Z}$ to each indecomposable summand of $X(d)$. Note that some rows may have multiple indecomposable summands.

6.8. Write i_s, where $s = 0, \cdots, t$, for the admissible vertices of Q, so that $i_s, i_s + 1, \cdots, i_{s+1}$ are the vertices of the sth linear section of the quiver.

Let X be an indecomposable A-projective D-module and let $I_s = \operatorname{supp}_\Delta X \cap \{i_s, \cdots, i_{s+1}\}$ be the Δ-support of X on each sth linear component of Q. Similarly, let X' be an indecomposable A-projective D-module and J_s be the Δ-support of X' on each sth linear component.

Write $X \leq X'$ when

- $I_s \supseteq J_s$ if s is a sink
- $I_s \subseteq J_s$ if s is a source

for all $s = 0, \cdots, t$ such that $I_s, J_s \neq \emptyset$.

Note that this is functionally the same as the orders $\leq_i$ on summands of $X(d)$ as in Definition 2 in [14].

The rows of the Δ-graph (d, φ^d) respect this order in the following sense. If X and X' are indecomposable summands of $X(d)$ in rows r and r' respectively, and both have Δ-support at some vertex $i \in Q_0$, then $r \leq r'$ implies $X \leq X'$. This is because, by construction, if a_i is a sink, then lower rows have larger Δ-support in the linear section following a_i, whereas if a_i is a source, then higher rows have larger Δ-support.

Lemma 6.9. *Suppose the vertices* 1 *and* n *are either both sinks or both sources. Let* X *and* X' *be indecomposable summands of* $X(d)$*.*

- *Suppose* X *and* X' *both have* Δ*-support at vertex* 1 *and that* $X \leq X'$*. Then there exists a morphism* $f : X \to X'$ *such that* $f_{e_1} \neq 0$*. Further, if* $f_{e_n} \neq 0$*, then* $X \cong X'$*.*
- *Suppose* X *and* X' *both have* Δ*-support at vertex* n *and that* $X' \leq X$*. Then there exists a morphism* $f : X \to X'$ *such that* $f_{e_n} \neq 0$*. Further, if* $f_{e_1} \neq 0$*, then* $X \cong X'$*.*

Proof. We prove the first point. If 1 and n are sinks, then a_n is a source; if 1 and n are sources, then a_n is a sink. Therefore, the second point is a reflection of the first.

Write $X = M(d, \varphi)$ and $X' = M(d', \varphi')$. By Proposition 4.4, f is a morphism if

$$\sum_{\sigma\sigma'=\rho} f_{\sigma'} \circ \varphi_\sigma = \sum_{\tau\tau'=\rho} \varphi'_{\tau'} \circ f_\tau \qquad (*)$$

for all $\rho \in P^*(-,-)$.

Recall $i_0, \cdots, i_t$ denote the admissible vertices of Q. For $s = 0, \cdots, t-1$, let P_s be the set of sub-paths of the unique path between i_s and i_{s+1}. Each trivial path e_{i_s} belongs to both P_{s-1} and P_s. Any other path belongs to precisely one P_s. Let P^s be the union of the P_i for $i = 0, \cdots, s$.

Let $u = i_s$ be some admissible vertex. Suppose for an induction that $f_{e_u} = 1$ and that $(*)$ is satisfied for all $\rho \in P^{s-1}$. Note that the base case of $u = i_0 = 1$ is satisfied trivially by defining $f_{e_1} = 1$.

Recall I_s and J_s are the Δ-support of X and X' respectively on the sth linear component of the quiver. Write $I_s = \{x_0, \cdots, x_a\}$ and $J_s = \left\{x'_0, \cdots, x'_{a'}\right\}$ such that each $x_i < x_{i+1}$ and each $x'_i < x'_{i+1}$.

Suppose u is a sink. As $X \leq X'$, this implies that $I_s \supseteq J_s$. Thus $a \geq a'$ and each $x_i \leq x'_i$. For each $\sigma \in P_s$, define $f_\sigma = 1$ if σ is the path from x'_i to x_i for some $i = 0, \cdots, a'$, otherwise define $f_\sigma = 0$. Note that the path from $x'_0 = u$ to $x_0 = u$ is e_u, so this is consistent with the assumption that $f_{e_u} = 1$.

Let $\rho \in P_s$. Let $\sigma\sigma' = \rho$ be a decomposition of ρ (so $\sigma, \sigma' \in P_s$ also). Then $f_{\sigma'} \circ \varphi_\sigma$ is nonzero only if σ' is the path from x'_i to x_i and σ is the path from x_i to x_{i-1} for some i. Similarly, if $\tau\tau' = \rho$ is a decomposition of ρ, then $\varphi'_{\tau'} \circ f_\tau$ is nonzero only if τ' is the path from x'_{i+1} to x'_i and τ is the path from x'_i to x_i for some i. Therefore, the set of equations $(*)$ for $\rho \in P_t$ is equivalent to

$$\forall i = 1, \cdots, a' : f_{\rho(x'_i, x_i)} \circ \varphi_{\rho(x_i, x_{i-1})} = \varphi'_{\rho(x'_i, x'_{i-1})} \circ f_{\rho(x'_{i-1}, x_{i-1})},$$

where $\rho(i,j)$ denotes the path from i to j ($i_s \leq i, j \leq i_{s+1}$). These maps are all equal to 1, so the above holds. Similarly, if u is a source, then the equations $(*)$ are satisfied for $\rho \in P_t$.

Let $v = i_{s+1}$. If $f_{e_v} = 0$, define $f_\rho = 0$ for all $\rho \in P_j$ for all $j = s+1, \cdots, t$. Then $(*)$ is satisfied for all $\rho \in P^*(-,-)$ and we are done.

Otherwise, $f_{e_v} = 1$. Then e_v is the path from $x'_{a'}$ to x_a, so $x'_{a'} = v = x_a$ and $a' = a$. Therefore the inductive assumption is satisfied for $s+1$. If $s+1 = t$, then $(*)$ is satisfied for all $\rho \in P^*(-,-)$ and we are done. In this case, $f_{e_n} \neq 0$ and we have $I_s = J_s$ for all s, which implies $X \cong X'$. $\square$

Chapter 7

Orbits and their Dimensions

In this chapter, we obtain equalities involving Δ-dimension vectors and the dimension of orbits, endomorphism algebras and Ext- and Hom-spaces (Theorems 7.10 and 7.11). These generalise existing results by Jensen and Su (Lemma 7 and Theorem 4 in [14]) which relate to $\operatorname{End}_A X$ of an A-projective D-module X for a quiver without any cycles (oriented or non-oriented). We expand upon the existing proofs to get similar results for $\operatorname{Hom}_A(X, Y)$ for A-projective D-modules X and Y, allowing the quiver to have non-oriented cycles.

Throughout this chapter, Q is a quiver without oriented cycles, X and Y are A-projective D-modules, ${}_AX = P(d)$ and ${}_AY = P(d')$. Additionally, denote by J the ideal in D generated by the arrows in Q^{op}. Whenever dimension is referred to, we mean dimension as a k-vector space. We begin with two results for which the existing proofs hold for such a Q.

Theorem 7.1 (Theorem 3 in [14])**.** *The varieties* $\operatorname{Rep}_\Delta(Q, d)$ *and* $\operatorname{rad}\operatorname{End}_A P(d)$ *are* $\operatorname{Aut}_A P(d)$*-equivariantly isomorphic.*

Lemma 7.2 (Lemma 6 in [14])**.** *We have the following exact sequence, which is a projective resolution of* X *as a* D*-module.*

$$0 \longrightarrow J \otimes_A X \longrightarrow D \otimes_A X \longrightarrow X \longrightarrow 0$$

Definition 7.3. We denote the **Jacobson radical** of A by $\operatorname{rad} A$. This is the ideal generated by all arrows in Q (Corollary 4.5 in [26]) or, equivalently, the span of the nontrivial paths in Q.

Remark 7.4. Consequently, $\operatorname{rad}\operatorname{Hom}_A(A,A) \cong \operatorname{rad} A = \operatorname{span} P^*(-,-)$. For $i,j \in Q_0$, write $\operatorname{rad}\operatorname{Hom}_A(Ae_i, Ae_j)$ for the intersection of $\operatorname{rad}\operatorname{Hom}_A(A,A)$ with $\operatorname{Hom}_A(Ae_i, Ae_j)$. Thus

$$\operatorname{rad}\operatorname{Hom}_A(Ae_i, Ae_j) \cong \operatorname{rad} A \cap e_i A e_j \cong \operatorname{span} P^*(-,-) \cap \operatorname{span} P(j,i) \cong \operatorname{span} P^*(j,i).$$

As $\operatorname{span} P^*(-,-)$ is the direct sum of $\operatorname{span} P^*(j,i)$ for $i,j \in Q_0$, we have

$$\operatorname{rad}\operatorname{Hom}_A(A,A) \cong \bigoplus_{i,j\in Q_0} \operatorname{rad}\operatorname{Hom}_A(Ae_i, Ae_j).$$

For $X \in \operatorname{Rep}_\Delta(Q,d)$ and $Y \in \operatorname{Rep}_\Delta(Q,d')$, we write

$$\operatorname{rad}\operatorname{Hom}_A(X,Y) = \bigoplus_{i,j\in Q_0} \operatorname{rad}\operatorname{Hom}_A(Ae_i, Ae_j)^{d_i d'_j}.$$

Lemma 7.5. *$\operatorname{Hom}_D(D \otimes_A X, Y) \cong \operatorname{Hom}_A(X,Y)$.*

Proof. By the adjoint property of Hom and $\otimes$ (Theorem 2.76 in [25]), we have that

$$\operatorname{Hom}_D(D \otimes X, Y) \cong \operatorname{Hom}_A(X, \operatorname{Hom}_D(D,Y)).$$

This is then isomorphic to $\operatorname{Hom}_A(X,Y)$. □

Lemma 7.6. *$\operatorname{Hom}_D(J \otimes_A X, Y) \cong \operatorname{rad}\operatorname{Hom}_A(X,Y)$.*

Proof. As an A-module, $X = P(d) = \bigoplus (Ae_i)^{d_i}$, hence

$$J \otimes_A X \cong \bigoplus_{i\in Q_0} (J \otimes_A Ae_i)^{d_i} \cong \bigoplus_{i\in Q_0} (Je_i)^{d_i}.$$

Consequently,

$$\operatorname{Hom}_D(J \otimes_A X, Y) \cong \bigoplus_{i\in Q_0} \operatorname{Hom}_D(Je_i, Y)^{d_i}.$$

As the relations on Q are sums of terms of the form $\alpha\overline{\beta}$ or $\overline{\alpha}\beta$, we can express J as a direct sum of $D\overline{\alpha}$ for $\alpha \in Q_1$. So Je_i is a direct sum of $D\overline{\alpha}$ for $\alpha \in Q_1$ such that $t(\alpha) = i$. But $D\overline{\alpha} \cong De_{s(\alpha)}$, hence

$$\operatorname{Hom}_D(J \otimes_A X, Y) \cong \bigoplus_{i\in Q_0} \left(\bigoplus_{\alpha\in Q_1 : t(\alpha)=i} \operatorname{Hom}_D(De_{s(\alpha)}, Y) \right)^{d_i},$$

where we have

$$\mathrm{Hom}_D(De_{s(\alpha)}, Y) \cong e_{s(\alpha)}Y \cong \bigoplus_{j\in Q_0} (e_{s(\alpha)}Ae_j)^{d'_j},$$

therefore

$$\mathrm{Hom}_D(J \otimes_A X, Y) \cong \bigoplus_{i,j\in Q_0} \left(\bigoplus_{\alpha\in Q_1 : t(\alpha)=i} (e_{s(\alpha)}Ae_j) \right)^{d_i d'_j}.$$

We know that $e_{s(\alpha)}Ae_j \cong \operatorname{span} P(j, s(\alpha))$, hence

$$\begin{aligned} \bigoplus_{\alpha\in Q_1 : t(\alpha)=i} (e_{s(\alpha)}Ae_j) &\cong \bigoplus_{\alpha\in Q_1 : t(\alpha)=i} \operatorname{span} P(j, s(\alpha)) \\ &\cong \bigoplus_{\alpha\in Q_1 : t(\alpha)=i} \operatorname{span} \alpha P(j, s(\alpha)) \\ &\cong \operatorname{span} \{\alpha\sigma : \sigma \in P(j, -), \alpha \in Q_1, t(\alpha) = i\} \\ &= \operatorname{span} P^*(j, i) \\ &\cong \operatorname{rad} \mathrm{Hom}_A(Ae_i, Ae_j). \end{aligned}$$

Consequently,

$$\mathrm{Hom}_D(J \otimes_A X, Y) \cong \bigoplus_{i,j\in Q_0} \operatorname{rad} \mathrm{Hom}_A(Ae_i, Ae_j)^{d_i d'_j} = \operatorname{rad} \mathrm{Hom}_A(X, Y).$$

□

Lemma 7.7. *There exists an exact sequence of the form*

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_D(X, Y) \longrightarrow \mathrm{Hom}_A(X, Y) \\ \longrightarrow \operatorname{rad} \mathrm{Hom}_A(X, Y) \longrightarrow \mathrm{Ext}^1_D(X, Y) \longrightarrow 0. \end{aligned}$$

Consequently,

$$\dim \mathrm{Hom}_D(X, Y) + \dim \operatorname{rad} \mathrm{Hom}_A(X, Y) = \dim \mathrm{Hom}_A(X, Y) + \dim \mathrm{Ext}^1_D(X, Y).$$

In particular,

$$\dim \mathrm{End}_D X + \dim \operatorname{rad} \mathrm{End}_A X = \dim \mathrm{End}_A X + \dim \mathrm{Ext}^1_D(X, X).$$

Proof. We apply $\mathrm{Hom}_D(-, Y)$ to the short exact sequence from Lemma 7.2. Because the exact sequence from the lemma is a projective resolution, we obtain the following exact sequence.

$$0 \longrightarrow \operatorname{Hom}_D(X,Y) \longrightarrow \operatorname{Hom}_D(D\otimes_A X, Y)$$
$$\longrightarrow \operatorname{Hom}_D(J\otimes_A X, Y) \longrightarrow \operatorname{Ext}^1_D(X,Y) \longrightarrow 0.$$

Now by Lemma 7.5 and Lemma 7.6, we obtain the desired exact sequence. □

Definition 7.8. Denote the $\operatorname{Aut}_A P(d)$-orbit of X in $\operatorname{Rep}_\Delta(Q,d)$ by $\mathcal{O}_X$.

Remark 7.9. The $\operatorname{Aut}_A P(d)$-stabilizer of X is $\operatorname{Aut}_D X$, which has the same dimension as $\operatorname{End}_D X$. Consequently, $\dim \mathcal{O}_X = \dim \operatorname{Aut}_A P(d) - \dim \operatorname{End}_D X$.

Theorem 7.10. *Firstly,*

$$\sum_{i\in Q_0} d_i d_i' = \dim \operatorname{Hom}_D(X,Y) - \dim \operatorname{Ext}^1_D(X,Y).$$

In particular,

$$\sum_{i\in Q_0} d_i^2 = \dim \operatorname{End}_D X - \dim \operatorname{Ext}^1_D(X,X).$$

Proof. From Lemma 7.7, we have that

$$\dim \operatorname{Hom}_D(X,Y) - \dim \operatorname{Ext}^1_D(X,Y) = \dim \operatorname{Hom}_A(X,Y) - \dim \operatorname{rad} \operatorname{Hom}_A(X,Y). \quad (*)$$

We have that

$$\operatorname{Hom}_A(X,Y) = \operatorname{Hom}_A(P(d), P(d')) = \bigoplus_{i,j\in Q_0} \operatorname{Hom}_A(Ae_i, Ae_j)^{d_i d_j'},$$

while

$$\operatorname{rad} \operatorname{Hom}_A(X,Y) = \bigoplus_{i,j\in Q_0} \operatorname{rad} \operatorname{Hom}_A(Ae_i, Ae_j)^{d_i d_j'}.$$

We know $\operatorname{Hom}_A(Ae_i, Ae_j) \cong \operatorname{span} P(j,i)$, while $\operatorname{rad} \operatorname{Hom}(Ae_i, Ae_j) \cong \operatorname{span} P^*(j,i)$, hence $\dim \operatorname{Hom}_A(Ae_i, Ae_j) - \dim \operatorname{rad} \operatorname{Hom}(Ae_i, Ae_j)$ is equal to the number of trivial paths from j to i. That is, $\delta_{i,j}$ (the Kronecker delta). Consequently,

$$\dim \operatorname{Hom}_A(X,Y) - \dim \operatorname{rad} \operatorname{Hom}_A(X,Y) = \sum_{i,j\in Q_0} \delta_{i,j} d_i d_j' = \sum_{i\in Q_0} d_i d_i'.$$

Apply this to $(*)$ and we get the desired result. □

Theorem 7.11. *The codimension of $\mathcal{O}_X$ is equal to* $\dim \operatorname{Ext}^1_D(X,X)$*, which in turn equals*

$$\dim \operatorname{End}_D X - \sum_{i \in Q_0} d_i^2.$$

Proof. From Remark 7.9, we have $\dim \mathcal{O}_X = \dim \operatorname{Aut}_A P(d) - \dim \operatorname{End}_D X$, where $\operatorname{Aut}_A P(d) = \operatorname{Aut}_A X$ has dimension equal to that of $\operatorname{End}_A X$. Therefore

$$\dim \mathcal{O}_X = \dim \operatorname{End}_A X - \dim \operatorname{End}_D X.$$

So we can calculate the codimension of the orbit, as follows.

$$\begin{aligned} \operatorname{codim} \mathcal{O}_X &= \dim \operatorname{Rep}_\Delta(Q,d) - \dim \operatorname{End}_A X + \dim \operatorname{End}_D X \\ &= \dim \operatorname{rad} \operatorname{End}_A P(d) - \dim \operatorname{End}_A X + \dim \operatorname{End}_D X && \text{(Theorem 7.1)} \\ &= \dim \operatorname{rad} \operatorname{End}_A X - \dim \operatorname{End}_A X + \dim \operatorname{End}_D X \\ &= \dim \operatorname{Ext}^1_D(X,X) && \text{(Lemma 7.7)} \end{aligned}$$

□

Remark 7.12. In the following chapter, we construct a subset of $\operatorname{Rep}_\Delta(Q,d)$ which is a union of codimension p orbits over a p-dimensional parameter space. The remainder of this chapter is dedicated to showing that such a subset is a dense open subset.

Throughout, we restrict our attention to the Zariski topology and algebraic varieties. We begin by recalling some standard definitions and results from algebraic geometry.

Definition 7.13 (Lecture 5 in [8])**.** An algebraic variety $X \neq \emptyset$ is **irreducible** if for all closed subvarieties Y and Z such that $X = Y \cup Z$, we have $Y = X$ or $Z = X$. Equivalently, any non-empty open subset is dense.

Definition 7.14 (Exercise 3.17 in [9])**.** A morphism of algebraic varieties $\pi : X \to Y$ is **dominant** if $\overline{\pi(X)} = Y$.

Lemma 7.15 (Theorem 2, Section 8 in [21])**.** *If $\pi : X \to Y$ is a dominant morphism of irreducible algebraic varieties and $y \in \operatorname{im} \pi$, then any irreducible component of a fibre $\pi^{-1}(y)$ has dimension at least* $\dim X - \dim Y$*.*

Lemma 7.16 (Theorem 4.3 in [12])**.** *Let $\pi : X \to Y$ be a morphism of algebraic varieties. Then* $\operatorname{im} \pi$ *contains a dense open subset of* $\overline{\operatorname{im} \pi}$*.*

Lemma 7.17 (5.9 in [8])**.** *Let X and Y be irreducible algebraic varieties. Then the product variety $X \times Y$ under the Zariski topology (not the product topology) is also irreducible.*

Lemma 7.18. *Let X and Y be algebraic varieties and $f : X \to Y$ be a morphism of algebraic varieties. If X is irreducible, then $\overline{\operatorname{im} f}$ is irreducible.*

Proof. Let $Z_1, Z_2 \subseteq \overline{\operatorname{im} f}$ be closed. Then Z_1 and Z_2 are closed in Y also. Suppose $\overline{\operatorname{im} f} = Z_1 \cup Z_2$. As $X = f^{-1}\left(\overline{\operatorname{im} f}\right)$, this implies $X = f^{-1}(Z_1) \cup f^{-1}(Z_2)$. As f is a morphism, $f^{-1}(Z_1)$ and $f^{-1}(Z_2)$ are closed in X, so one of the $f^{-1}(Z_i) = X$ by irreducibility of X. This implies $\operatorname{im} f = f(f^{-1}(Z_i)) = Z_i \cap \operatorname{im} f$, hence $\overline{\operatorname{im} f} = \overline{Z_i} \cap \overline{\operatorname{im} f} = Z_i$. ☐

Proposition 7.19. *Let G be an algebraic group acting algebraically on an irreducible algebraic variety W. Let $v : \Lambda \to W : \lambda \mapsto v(\lambda)$ be an injective morphism into W from some p-dimensional affine space Λ, such that each G-orbit $\mathcal{O}_{v(\lambda)}$ intersects $\operatorname{im} v$ in a finite number of points. Let $\alpha : G \times \Lambda \to W$ be the morphism of varieties taking (g, λ) to $g \cdot v(\lambda)$. Then, for any dense open subset Λ_0 of Λ,*

1. $\operatorname{codim} \mathcal{O}_{v(\mu)} \geq p$ *for all* $\mu \in \Lambda_0$.
2. *if* $\operatorname{codim} \mathcal{O}_{v(\mu)} = p$ *for some* $\mu \in \Lambda_0$*, then* $\alpha(G \times \Lambda_0)$ *contains a dense open subset of* W.

Proof. Fix Λ_0 and $\mu \in \Lambda_0$. Let $\beta : G \times \Lambda_0 \to \overline{\operatorname{im} \alpha}$ be a restriction of α. This is a morphism of irreducible varieties (Lemma 7.18 and Lemma 7.17), and is dominant by definition. As the orbits of the $v(\lambda)$ are distinct, we have

$$\begin{aligned}\beta^{-1}(v(\mu)) &= \{(g, \lambda) \in G \times \Lambda : g \cdot v(\lambda) = v(\mu)\} \\ &= \{(g, \mu) : g \in G,\ g \cdot v(\mu) = v(\mu)\} \\ &= G_{v(\mu)} \times \{\mu\}.\end{aligned}$$

Thus

$$\dim \beta^{-1}(v(\mu)) = \dim G_{v(\mu)} = \dim G - \dim \mathcal{O}_{v(\mu)}.$$

We have

$$\dim \beta^{-1}(v(\mu)) \geq \dim G \times \Lambda_0 - \dim \overline{\operatorname{im} \alpha}$$

by Lemma 7.15, which now implies

$$\dim G - \dim \mathcal{O}_{v(\mu)} \geq \dim G + \dim \Lambda_0 - \dim \overline{\operatorname{im} \alpha},$$

hence

$$\dim \overline{\operatorname{im} \alpha} \geq p + \dim \mathcal{O}_{v(\mu)}.$$

So we have (1), as $\dim W \geq \dim \overline{\operatorname{im} \alpha}$.

If $\operatorname{codim} \mathcal{O}_{v(\mu)} = p$, then $\dim W = p + \dim \mathcal{O}_{v(\mu)}$, hence $\overline{\operatorname{im} \alpha} = W$ as W is irreducible. Now Lemma 7.16 implies that $\operatorname{im} \beta$ contains a dense open subset of W. But by definition, $\operatorname{im} \beta = \alpha(G \times \Lambda_0)$, so we have (2). □

Chapter 8

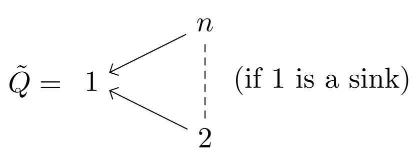
A Gluing Procedure for Affine Type A

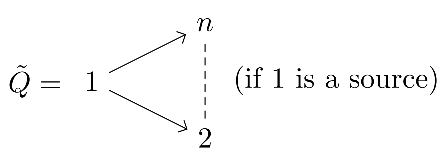
In this chapter, we present a construction for general Δ-filtered representations of affine type **A** quivers (without an oriented cycle). We do this in a 3 step process:

1. Split the quiver at an admissible vertex to obtain a Dynkin type **A** quiver.
2. Construct a general Δ-filtered representation of this Dynkin quiver.
3. Glue the ends of this representation to obtain a Δ-filtered representation of the affine quiver we started with.

The bulk of this chapter is dedicated to proving that the representation obtained is indeed a general representation (and in doing so, that canonical Δ-decomposition exists for all Δ-dimension vectors). Following this, we prove that canonical Δ-decomposition is characterised in the same way as canonical decomposition (discussed in Chapter 3).

Definition 8.1. Throughout this chapter, $\tilde{Q}$ is a quiver of affine type **A** and $\tilde{d} \in \mathbb{N}^{\tilde{Q}_0}$. Label the vertices of $\tilde{Q}$ by $1, \cdots, n$, with 1 an admissible vertex and each pair of vertices $i, i+1$ adjacent. The dashed lines represent a connected sequence of arrows with arbitrary orientation.





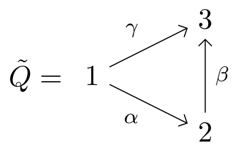
Construct a new quiver from $\tilde{Q}$, denoted by Q, where $Q_0 = \tilde{Q}_0 \cup \{n+1\}$ and $Q_1 = \tilde{Q}_1$. The arrow between 1 and n in $\tilde{Q}$ is instead between $n+1$ and n in Q (with the same orientation with respect to n).

$$Q = \; 1 \longleftarrow 2 \;\text{-------------}\; n \longrightarrow n+1 \quad \text{(if 1 is a sink)}$$

$$Q = \; 1 \longrightarrow 2 \;\text{-------------}\; n \longleftarrow n+1 \quad \text{(if 1 is a source)}$$

Let d be the Δ-dimension vector of Q corresponding to $\tilde{d}$. That is, $d_{n+1} = \tilde{d}_1$ and $d_i = \tilde{d}_i$ for $i \in \tilde{Q}_0$.

Example 8.2. Let $\tilde{Q}$ and $\tilde{d}$ be the following quiver and dimension vector respectively.



and $\tilde{d} = \begin{pmatrix} & 3 \\ 2 & \\ & 1 \end{pmatrix}$.

Then Q and d are as follows.

$$Q = \; 1 \xrightarrow[\alpha]{} 2 \xrightarrow[\beta]{} 3 \xleftarrow[\gamma]{} 4 \quad \text{and } d = (2\,1\,3\,2).$$

Definition 8.3. Recall from Chapter 6 that we can explicitly construct a Δ-graph (d, φ^d) for the rigid module $X(d) \in \mathrm{Rep}_\Delta(Q, d)$. Through this, we assign rows to each indecomposable summand of $X(d)$. Note that there can be multiple indecomposable summands in a single row. Specifically, each connected component gives an indecomposable summand.

$X(d)$ has d_1 indecomposable summands with Δ-support at 1. These summands are in rows $1, \cdots, d_1$ and shall be denoted by $X_1, \cdots, X_{d_1}$ respectively. $X(d)$ also has d_1 indecomposable summands with Δ-support at $n+1$. These summands are in rows $r, \cdots, r + d_1 - 1$ for some $r \in \mathbb{Z}$ and shall be denoted by $Y_1, \cdots, Y_{d_1}$ respectively. Note that if some X_i and Y_j are in the same row, it is possible that either $X_i = Y_j$ or $X_i \neq Y_j$ (the former happening when the row contains a single connected component).

We say **the $X_\bullet$ and $Y_\bullet$ are aligned** if $r = 1$, in which case X_i and Y_i are in the same row for each i.

Example 8.4 (continuation of 8.2)**.** We construct the following Δ-graph for the rigid module $X(d) \in \mathrm{Rep}_\Delta(Q, d)$. The picture indicates the row-indexing on the left, as well as the $X_\bullet$ and $Y_\bullet$ (note that $X_1 = Y_2$ here).

$$
\begin{array}{llll}
(2) & X_2 = & \Delta_1 \longleftarrow \Delta_2 \longleftarrow \Delta_3 & \\
(1) & X_1 = & \Delta_1 \longleftarrow\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\, \Delta_3 \longrightarrow \Delta_{n+1} & = Y_2 \\
(0) & & \qquad\qquad\quad \Delta_3 \longrightarrow \Delta_{n+1} & = Y_1
\end{array}
$$

Definition 8.5. Construct a new Δ-graph $(\tilde{d}, \psi)$ of $\tilde{Q}$ from (d, φ^d) by identifying the Δ_1 in X_i with the Δ_{n+1} in Y_i into a single vertex, labelled Δ_1, for each $i = 1, \cdots, d_1$. Note that the representation Δ_1 of $\tilde{Q}$ is different from the representation Δ_1 of Q, though this ambiguity is harmless.

Let $I_0 = \{i = 1, \cdots, d_1 : X_i = Y_i\}$ and $I_1 = \{i = 1, \cdots, d_1 : X_i \neq Y_i\}$. For each $i \in I_0$, the summand $X_i = Y_i$ has the following Δ-graph, for some vertices $n_1, \cdots, n_a \in Q_0$ $(a \geq 1)$. Dashed lines represent a connected sequence of arrows with an arbitrary orientation.

$$\Delta_1 \longrightarrow \Delta_{n_1} \text{ - - - - - - - - - - - - } \Delta_{n_a} \longleftarrow \Delta_{n+1} \quad \text{(if 1 is a sink)}$$

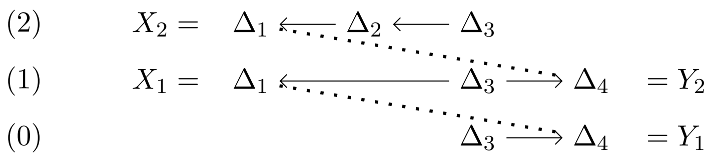
$$\Delta_1 \longleftarrow \Delta_{n_1} \text{ - - - - - - - - - - - - } \Delta_{n_a} \longrightarrow \Delta_{n+1} \quad \text{(if 1 is a source)}$$

The identifications result in X_i being glued to itself, giving a cyclic component of $(\tilde{d}, \psi)$. We set one of the maps in this component to equal a parameter $\lambda_i \in k$, as shown below. Doing this gives us a set of Δ-graphs $(\tilde{d}, \psi^\lambda)$ for $\lambda \in k^{I_0}$.

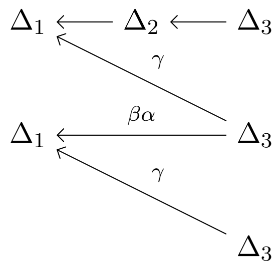
Denote the summand corresponding to this component by $Z_i(\lambda_i)$. We call each $Z_i(\lambda_i)$ a **self-glued summand of** $M(\tilde{d}, \psi^\lambda)$ and call the $X_i = Y_i$ **self-glued summands of** $X(d)$. Let $p = |I_0|$ denote the **number of self-glued summands**.

Lastly, denote by Λ_0 the set of $\lambda \in k^{I_0}$ with all λ_i distinct and nonzero, and define $\mathrm{Gen}_\Delta(\tilde{Q}, \tilde{d})$ to be the union over all $\lambda \in \Lambda_0$ of the orbits of $M(\tilde{d}, \psi^\lambda)$ in $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$.

Example 8.6 (continuation of 8.4)**.** We perform the following identifications, denoted by dotted lines.

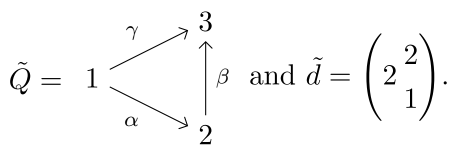


This gives us the following Δ-graph $(\tilde{d}, \psi^\lambda)$. Note that there are two paths from 1 to 3 (namely γ and $\beta\alpha$), so we need to specify which path each arrow from Δ_3 to Δ_1 is associated to, in order to avoid ambiguity.



Because the $X_\bullet$ and the $Y_\bullet$ are not aligned, there are no self-glued summands.

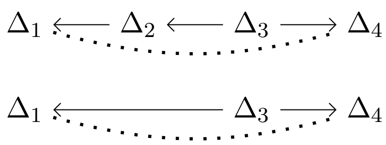
Example 8.7. We go through an example which demonstrates self-glued summands. Let $\tilde{Q}$ and $\tilde{d}$ be as follows.



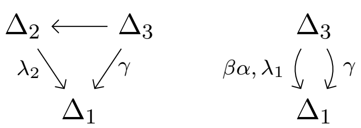
Then Q and d are the following.

$$Q = \; 1 \xrightarrow[\alpha]{} 2 \xrightarrow[\beta]{} 3 \xleftarrow[\gamma]{} 4 \quad \text{and } d = (2\,1\,2\,2).$$

We construct the following Δ-graph for $X(d)$.



Performing the indicated identifications, we obtain $(\tilde{d}, \psi^\lambda)$.



Lemma 8.8. *There exists a decomposition $\tilde{d} = \sum_{i=1}^{m} c_i$ such that all $N \in \mathrm{Gen}_\Delta(\tilde{Q}, \tilde{d})$ decompose according to $\sum_{i=1}^{m} c_i$ (see Definition 3.7).*

Proof. The $Z_\bullet(\lambda_\bullet)$ are all indecomposable by Lemma 5.2 and their Δ-dimension vectors do not vary with λ. This implies the $M(\tilde{d}, \psi^\lambda)$ all decompose according to the same decomposition of $\tilde{d}$. Each $N \in \mathrm{Gen}_\Delta(\tilde{Q}, \tilde{d})$ is isomorphic to $M(\tilde{d}, \psi^\lambda)$ for some $\lambda \in \Lambda_0$, so also decomposes according to that decomposition of $\tilde{d}$. □

Lemma 8.9. *For $\lambda \in \Lambda_0$, we have*

$$\dim \mathrm{End}\, M(\tilde{d}, \psi^\lambda) \geq \dim \mathrm{End}\, X(d) - (d_1^2 - p),$$

with equality implying that $\mathrm{Gen}_\Delta(\tilde{Q}, \tilde{d})$ contains a dense open subset of $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$ and that the orbit of $M(\tilde{d}, \psi^\lambda)$ has codimension p.

Proof. By construction, the set of all $M(\tilde{d}, \psi^\lambda)$ for $\lambda \in k^{I_0}$ is a p-dimensional affine subspace of $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$ consisting of points with all but p coordinates fixed (the remaining p coordinates are those with a parameter λ_i for $i \in I_0$).

Therefore, k^{I_0} injects into $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$ via the morphism of varieties taking λ to $M(\tilde{d}, \psi^\lambda)$. By Lemma 5.3, the orbits of each $M(\tilde{d}, \psi^\lambda)$ are distinct for distinct λ, up to permutation of coordinates. That is, only a finite number of these orbits coincide.

Λ_0 is a dense open subset of k^{I_0}. Let $\alpha : G \times k^{I_0} \to \mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$ be the morphism of varieties taking (g, λ) to $g \cdot M(\tilde{d}, \psi^\lambda)$, so that $\alpha(G \times \Lambda_0) = \mathrm{Gen}_\Delta(\tilde{Q}, \tilde{d})$.

Let $\lambda \in \Lambda_0$ and $N = M(\tilde{d}, \psi^\lambda)$. We can apply Proposition 7.19 to get that

1. $\operatorname{codim} \mathcal{O}_N \geq p$.
2. if $\operatorname{codim} \mathcal{O}_N = p$, then $\mathrm{Gen}_\Delta(\tilde{Q}, \tilde{d})$ contains a dense open subset of $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$.

By Theorem 7.11, we have that

$$\operatorname{codim} \mathcal{O}_N = \dim \mathrm{End}\, N - \sum_{i \in \tilde{Q}_0} \tilde{d}_i^2,$$

and that

$$\dim \mathrm{End}\, X(d) = \sum_{i \in Q_0} d_i^2 = d_1^2 + \sum_{i \in \tilde{Q}_0} \tilde{d}_i^2,$$

the latter because $X(d)$ is rigid. Combining these two equations gives

$$\operatorname{codim} \mathcal{O}_N + \dim \operatorname{End} X(d) = \dim \operatorname{End} N + d_1^2.$$

By (1), this implies

$$p + \dim \operatorname{End} X(d) - d_1^2 \leq \dim \operatorname{End} N,$$

with equality implying $\operatorname{codim} \mathcal{O}_N = p$. Now the result follows from (2). □

Remark 8.10. It turns out that the set of maps ψ^λ which defines the Δ-graph $(\tilde{d}, \psi^\lambda)$ can also be used directly to define a Δ-graph (d, ψ^λ) in Q. This is only well-defined because of how d was constructed from $\tilde{d}$.

Lemma 8.11. *Let $\lambda \in \Lambda_0$. Then (d, ψ^λ) is a well-defined Δ-graph in Q. Further, $M(d, \psi^\lambda)$ and $X(d)$ are isomorphic.*

Proof. As the vertex 1 is admissible, the set of nontrivial paths in Q is equal to the set of nontrivial paths in $\tilde{Q}$. Denote the source and target vertex in Q of a path ρ by $s(\rho)$ and $t(\rho)$ respectively, and denote the source and target vertex in $\tilde{Q}$ by $\tilde{s}(\rho)$ and $\tilde{t}(\rho)$.

For any path ρ such that $\tilde{t}(\rho) \neq t(\rho)$, we must have $\tilde{t}(\rho) = 1$ and $t(\rho) = n + 1$. Similarly, for $\tilde{s}(\rho)$. But $d_{n+1} = \tilde{d}_1$. Thus for any path ρ, we have $\tilde{d}_{\tilde{s}(\rho)} = d_{s(\rho)}$ and $\tilde{d}_{\tilde{t}(\rho)} = d_{t(\rho)}$. Therefore, (d, ψ^λ) is a well-defined Δ-graph in Q.

In particular, (d, ψ^μ) is a Δ-graph in Q, where $\mu \in k^{I_0}$ is the vector with $\mu_i = 1$ for all $i \in I_0$. Notice that (d, ψ^μ) is precisely the Δ-graph we constructed for $X(d)$ earlier! That is, $X(d) \cong M(d, \psi^\mu)$.

Further, the only differences between the Δ-graphs (d, ψ^μ) and (d, ψ^λ) are the p parameters. As these are all nonzero for both λ and μ, we have that $M(d, \psi^\mu)$ and $M(d, \psi^\lambda)$ are isomorphic (Lemma 5.7). □

Definition 8.12. From now onwards, we fix a specific value for the parameter $\lambda \in \Lambda_0$. Write $\psi = \psi^\lambda$ for the set of maps corresponding to this choice of parameter. Write $N = M(\tilde{d}, \psi)$ for the module in $\operatorname{Rep}_\Delta(\tilde{Q}, \tilde{d})$ corresponding to the Δ-graph in $\tilde{Q}$ with this set of maps. Write $M = M(d, \psi)$ for the module in $\operatorname{Rep}_\Delta(Q, d)$ corresponding to the Δ-graph in Q with this set of maps.

Lemma 8.13. *There exists an injective linear map $\varepsilon : \operatorname{End} N \to \operatorname{End} M$ such that all $g \in \operatorname{im} \varepsilon$ satisfy $g_{e_1} = g_{e_{n+1}}$.*

Proof. We use the characterisation of endomorphisms given in Proposition 4.4. Define a map

$$\varepsilon : \operatorname{End} N \to \operatorname{End} M : f \mapsto \varepsilon(f),$$

where $\varepsilon(f)_\sigma = f_\sigma$ for $\sigma \in P(-,-)$ and $\varepsilon(f)_{e_{n+1}} = f_{e_1}$. In order to be well-defined, we must have

$$\sum_{\sigma\sigma'=\rho} \varepsilon(f)_{\sigma'} \circ \psi_\sigma = \sum_{\sigma\sigma'=\rho} \psi_{\sigma'} \circ \varepsilon(f)_\sigma \qquad (*)$$

for all $\rho \in P^*(-,-)$. As $f \in \operatorname{End} N$, we have that

$$\sum_{\sigma\sigma'=\rho} f_{\sigma'} \circ \psi_\sigma = \sum_{\sigma\sigma'=\rho} \psi_{\sigma'} \circ f_\sigma \qquad (**)$$

for all $\rho \in P^*(-,-)$. The only difference between $(*)$ and $(**)$ occurs when $\sigma' = e_{n+1}$ in the left sum in $(*)$ or when $\sigma = e_{n+1}$ in the right sum in $(*)$. But as $\varepsilon(f)_{e_{n+1}} = f_{e_1}$, we have that $(*)$ and $(**)$ are equivalent. Therefore $\varepsilon(f) \in \operatorname{End} M$, hence ε is well-defined. Further, ε is linear and injective by definition. □

8.14. By Lemma 8.11, we have $M \cong X(d)$. Fix some such isomorphism and, for each $i = 1, \cdots, d_1$ write X'_i and Y'_i for the summands of M corresponding to X_i and Y_i respectively under that isomorphism.

Lemma 8.15. *For each pair of integers $i, j = 1, \cdots, d_1$, there exists an endomorphism $g_{i,j}$ of M such that if $i \leq j$, then*

- $g_{i,j}(X) = 0$ *for all indecomposable summands $X \neq X'_i$ of M,*
- $g_{i,j}(X'_i) \subseteq X'_j$,
- $(g_{i,j})_{e_1} \neq 0$,

and if $i > j$, then

- $g_{i,j}(X) = 0$ *for all indecomposable summands $X \neq Y'_i$ of M,*
- $g_{i,j}(Y'_i) \subseteq Y'_j$,
- $(g_{i,j})_{e_{n+1}} \neq 0$.

Proof. If $i \leq j$, then $X_i \leq X_j$, hence there exists a morphism $f_{i,j} : X_i \to X_j$ with $(f_{i,j})_{e_1} \neq 0$ by Lemma 6.9. Thus there is also a morphism $f'_{i,j} : X'_i \to X'_j$ with

$(f'_{i,j})_{e_1} \neq 0$. Extend $f'_{i,j}$ to an endomorphism $g_{i,j}$ of M by mapping all other summands to zero.

If $i > j$, then $Y_j \leq Y_i$, hence there exists a morphism $f_{i,j} : Y_i \to Y_j$ with $(f_{i,j})_{e_{n+1}} \neq 0$. Thus there is also a morphism $f'_{i,j} : Y'_i \to Y'_j$ with $(f'_{i,j})_{e_{n+1}} \neq 0$. Extend $f'_{i,j}$ to an endomorphism $g_{i,j}$ of M by mapping all other summands to zero. □

Lemma 8.16. *The $g_{i,j}$ are linearly independent.*

Proof. The $X'_\bullet$ are all distinct, as are the $Y'_\bullet$, so the only possible linear dependence between the $g_{i,j}$ is if we have some pair $i \leq j$ with $g_{i,j} : X'_i \to X'_j$ and some pair $k > l$ with $g_{k,l} : Y'_k \to Y'_l$, where $X'_i = Y'_k$ and $X'_j = Y'_l$. But then $i \leq j$ implies $k \leq l$, which is a contradiction. □

8.17. Let $J = \{(i,j) : i,j = 1, \cdots, d_1\} \setminus \{(i,i) : i \in I_0\}$. We show the span of $g_{i,j}$ for $(i,j) \in J$ is independent of $\operatorname{im} \varepsilon$. This will ultimately give us a bound on the dimension of $\operatorname{End} N$ opposite to that of Lemma 8.9.

Lemma 8.18. *Let $V = \operatorname{span}\{g_{i,j} : (i,j) \in J\}$. Then $V \cap \operatorname{im} \varepsilon = 0$.*

Proof. Let $g \in V$ be arbitrary. So for some $\mu_{i,j} \in k$ we have

$$g = \sum_{(i,j) \in J} \mu_{i,j} g_{i,j}.$$

Suppose that $g \in \operatorname{im} \varepsilon$, so that $g_{e_1} = g_{e_{n+1}}$ and $g = \varepsilon(h)$ for some $h \in \operatorname{End} N$. Consider the decomposition of g into a sum of homomorphisms between the indecomposable summands of M.

Let $(i,j) \in J$ such that $i \leq j$ and $\mu_{i,j} \neq 0$. By Lemma 8.15, the $\operatorname{Hom}(X'_i, X'_j)$-component of g is $\mu_{i,j} g_{i,j}$, where $(g_{i,j})_{e_1} \neq 0$. As $g_{e_1} = g_{e_{n+1}}$, this implies that the $\operatorname{Hom}(Y'_i, Y'_j)$-component of g is nonzero at e_{n+1}. The only way this can happen is if $Y'_i = X'_k$ and $Y'_j = X'_l$ for some $(k,l) \in J$ with $k \leq l$ and $\mu_{k,l} \neq 0$.

If $(k,l) = (i,j)$, then we must have $i \neq j$, else $(i,j) \notin J$. But then $(\mu_{i,j} g_{i,j})_{e_1} = f_{e_1}$, where f is the $\operatorname{Hom}(Z_i(\lambda_i), Z_j(\lambda_j))$-component of h. Now Lemma 5.1 implies $f_{e_1} = 0$, hence $\mu_{i,j} = 0$. This is a contradiction, therefore $(k,l) \neq (i,j)$.

We can apply the above argument to (k,l) to obtain some $(k',l') \in J$, distinct from (k,l), for which $\mu_{k',l'} \neq 0$ and $Y'_k = X'_{k'}$ and $Y'_l = X'_{l'}$ and $k' \leq l'$. We have at least one of $i < k$ or $i > k$ or $j < l$ or $j > l$, so due to the ordering, we have that $(k',l') \neq (i,j)$.

By repeated application of the above argument, we can find an arbitrarily large set of distinct $(i, j) \in J$ such that $\mu_{i,j} \neq 0$. But J is a finite set, so this is a contradiction. Therefore the supposition that there exists some $(i, j) \in J$ with $i \leq j$ and $\mu_{i,j} \neq 0$ must be false. That is, if $(i, j) \in J$ with $i \leq j$, then $\mu_{i,j} = 0$.

A similar argument shows that if $(i, j) \in J$ with $i > j$, then $\mu_{i,j} = 0$. Therefore, $\mu_{i,j} = 0$ for all $(i, j) \in J$, so $g = 0$. □

Lemma 8.19. *The dimension of* $\operatorname{coker} \varepsilon$ *is at least* $d_1^2 - p$.

Proof. The $g_{i,j}$, for $(i, j) \in J$, are linearly independent (Lemma 8.16). By definition of J and p, $|J| = d_1^2 - |I_0| = d_1^2 - p$. Therefore, $\dim V = d_1^2 - p$.

As $V \cap \operatorname{im} \varepsilon = 0$ (Lemma 8.18), we have $\dim \operatorname{im} \varepsilon + \dim V \leq \dim \operatorname{End} M$. Therefore, by the above, $\dim \operatorname{coker} \varepsilon \geq d_1^2 - p$. □

Theorem 8.20. $\operatorname{Gen}_\Delta(\tilde{Q}, \tilde{d})$ *contains a dense open subset of* $\operatorname{Rep}_\Delta(\tilde{Q}, \tilde{d})$. *The codimension of* $\mathcal{O}_N$ *in* $\operatorname{Rep}_\Delta(\tilde{Q}, \tilde{d})$ *is* p.

Proof. By Lemma 8.13, the map $\varepsilon : \operatorname{End} N \to \operatorname{End} M$ is injective, so we have

$$\dim \operatorname{End} N = \dim \operatorname{End} M - \dim \operatorname{coker} \varepsilon.$$

Thus Lemma 8.19 implies

$$\dim \operatorname{End} N \leq \dim \operatorname{End} M - (d_1^2 - p).$$

By Lemma 8.11, $M \cong X(d)$, hence

$$\dim \operatorname{End} N \leq \dim \operatorname{End} X(d) - (d_1^2 - p).$$

The result now follows from Lemma 8.9 (recall that $N = M(\tilde{d}, \psi^\lambda)$). □

Corollary 8.21. *The dimension of* $\mathcal{O}_X$ *(equivalently, the dimension of* $\operatorname{End} X$*) is the same for all* $X \in \operatorname{Gen}_\Delta(\tilde{Q}, \tilde{d})$. *Specifically,*

$$\dim \operatorname{End} X = p + \sum_{i \in \tilde{Q}_0} \tilde{d}_i^2.$$

Proof. As $\lambda \in \Lambda_0$ was arbitrary, Theorem 8.20 implies the codimension of the orbit of $M(\tilde{d}, \psi^\mu)$ is p for all $\mu \in \Lambda_0$. This in turn implies $\operatorname{codim} \mathcal{O}_X = p$ for all $X \in \operatorname{Gen}_\Delta(\tilde{Q}, \tilde{d})$.

Finally, $\operatorname{codim} \mathcal{O}_X = \dim \operatorname{End} X - \sum \tilde{d}_i^2$ by Theorem 7.11. □

Corollary 8.22. *There exists a decomposition $\tilde{d} = \sum_{i=1}^m \tilde{d}_i$ such that the set of modules in $\operatorname{Rep}_\Delta(\tilde{Q}, \tilde{d})$ which decompose according to $\sum_{i=1}^m \tilde{d}_i$ contains a dense open subset of $\operatorname{Rep}_\Delta(\tilde{Q}, \tilde{d})$.*

Proof. By Lemma 8.8, all modules in $\operatorname{Gen}_\Delta(\tilde{Q}, \tilde{d})$ decompose according to the same decomposition of $\tilde{d}$. Now Theorem 8.20 implies the result. □

Remark 8.23. With the above, we have the first main result of Part I: that canonical Δ-decomposition exists (see Definition 3.10). We now show that canonical Δ-decomposition has a description which mirrors that of ordinary canonical decomposition (see Theorem 3.4 and Proposition 3.6). We utilize the following result from Brion.

Lemma 8.24. *Let G be an algebraic group and X a G-variety. Then for any integer n, the set $\{x \in X : \dim(G_x) \geq n\}$ is closed in X, where G_x denotes the G-stabilizer of x.*

Proof. We repeat the proof given by Brion. Consider the morphism

$$\beta : G \times X \to X \times X : (g, x) \mapsto (x, g \cdot x).$$

Then the fibre of β at any point (g, x) is (gG_x, x); thus, all irreducible components of this fibre have the same dimension, $\dim(G_x)$. Now the assertion follows from semicontinuity of the dimension of fibres of a morphism (see e.g. [27]). □

Lemma 8.25. *For any integer n, the set*

$$\left\{X \in \operatorname{Rep}_\Delta(\tilde{Q}, \tilde{d}) : \dim \operatorname{End} X \leq n\right\}$$

is open in $\operatorname{Rep}_\Delta(\tilde{Q}, \tilde{d})$.

Proof. The $\operatorname{Aut}_A P(d)$- stabilizer of $X \in \operatorname{Rep}_\Delta(\tilde{Q}, \tilde{d})$ has dimension $\dim \operatorname{End} X$ (Remark 7.9). Appling Lemma 8.24 then gives us that

$$\left\{X \in \operatorname{Rep}_\Delta(\tilde{Q}, \tilde{d}) : \dim \operatorname{End} X \geq n\right\}$$

is closed, hence its complement is open. □

Lemma 8.26. $\mathrm{Gen}_\Delta(\tilde{Q}, \tilde{d}) \subseteq \mathrm{Rep}^0_\Delta(\tilde{Q}, \tilde{d})$. *That is, for all* $X \in \mathrm{Gen}_\Delta(\tilde{Q}, \tilde{d})$, *the dimension of* $\mathrm{End}\, X$ *is minimal amongst modules in* $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$.

Proof. By Corollary 8.21, we need only show $\dim \mathrm{End}\, N$ is minimal. Suppose there exists some $N' \in \mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$ such that $\dim \mathrm{End}\, N' < \dim \mathrm{End}\, N$. Consider the set

$$\left\{ Y \in \mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d}) : \dim \mathrm{End}\, Y \leq \dim \mathrm{End}\, N' \right\}.$$

This is an open subset of $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$ by Lemma 8.25. As it contains N', it is non-empty, so it is a dense open subset. $\mathrm{Gen}_\Delta(\tilde{Q}, \tilde{d})$ is also a dense open subset (Theorem 8.20), so the two intersect. But Corollary 8.21 implies the two sets do not intersect, thus we have a contradiction. □

8.27. Write $N_1 \oplus \cdots \oplus N_r$ for the decomposition of N into indecomposable summands, labelled such that for $i = 1, \cdots, p$ we have $N_i = Z_j(\lambda_j)$ for some $j \in I_0$. For each i, write $d_i = \underline{\dim}_\Delta N_i$.

Lemma 8.28. *For* $i = 1, \cdots, p$,

1. *the orbit of* N_i *has codimension* 1.
2. d_i *is* Δ*-Schurian.*
3. $N_i \in \mathrm{Gen}_\Delta(\tilde{Q}, d_i)$.

Proof. Let $i = 1, \cdots, p$, so that $N_i = Z_j(\lambda_j)$ and $d_i = \underline{\dim}_\Delta Z_j(\lambda_j)$ for some $j \in I_0$. Observe that if we perform the gluing procedure using the Δ-dimension vector d_i, then $X(d_i)$ has a single indecomposable summand with Δ-support at both 1 and $n+1$. This is therefore a self-glued summand, and the resulting module (up to a parameter) is the N_i we started with – in particular, the value of p is 1 in this process. That is, $N_i \in \mathrm{Gen}_\Delta(\tilde{Q}, d_j)$. This implies that $\mathrm{Gen}_\Delta(\tilde{Q}, d_i)$ consists of indecomposables.

Now by Theorem 8.20, we have that $\mathrm{codim}\, \mathcal{O}_{N_i} = 1$ and $\mathrm{Gen}_\Delta(\tilde{Q}, d_i)$ contains a dense open subset of $\mathrm{Rep}_\Delta(\tilde{Q}, d_i)$, hence d_i is Δ-Schurian. □

Lemma 8.29. *For* $i = p+1, \cdots, r$,

1. d_i *is* Δ*-Schurian.*
2. $N_i \in \mathrm{Gen}_\Delta(\tilde{Q}, d_i)$.

Additionally, $\mathrm{Ext}^1(N_i, N_j) = 0$ *for all* $i, j = 1, \cdots, r$ *such that* $i \neq j$.

Proof. We have that $\mathrm{codim}\, \mathcal{O}_N = p$ (Theorem 8.20) and that $\mathrm{codim}\, \mathcal{O}_{N_i} = 1$ for $i = 1, \cdots, p$ (Lemma 8.28). That is, $\dim \mathrm{Ext}^1(N, N) = p$ and $\dim \mathrm{Ext}^1(N_i, N_i) = 1$ for $i = 1, \cdots, p$ (Theorem 7.11). As the N_i are summands of N, this implies $\mathrm{Ext}^1(N_i, N_i) = 0$ for all $i > p$, and $\mathrm{Ext}^1(N_i, N_j) = 0$ for all $i \neq j$.

Therefore, for $i > p$, the orbit of N_i is dense and open in $\mathrm{Rep}_\Delta(\tilde{Q}, d_i)$, hence d_i is Δ-Schurian. Further, this orbit intersects (and thus is contained in) $\mathrm{Gen}_\Delta(\tilde{Q}, d_i)$. □

Lemma 8.30. *Suppose* $\tilde{d}$ *is* Δ*-Schurian. Then* $\mathrm{Gen}_\Delta(\tilde{Q}, \tilde{d})$ *is either a single open orbit or a* 1*-parameter family of orbits of codimension* 1.

Proof. The dense open subset of indecomposables must intersect $\mathrm{Gen}_\Delta(\tilde{Q}, \tilde{d})$ by Theorem 8.20, hence $\mathrm{Gen}_\Delta(\tilde{Q}, \tilde{d})$ consists of indecomposables. Therefore, considering how Definition 8.5 applies to an indecomposable, we have either $p = 1$ and $N = Z_1(\lambda_1)$, or $p = 0$. Applying Theorem 8.20 again, we have that

- if $p = 0$, then $\mathrm{Gen}_\Delta(\tilde{Q}, \tilde{d})$ is a single codimension 0 orbit (that is, an open orbit) in $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$.
- if $p = 1$, then $\mathrm{Gen}_\Delta(\tilde{Q}, \tilde{d})$ is a union over a 1-dimensional parameter space of orbits of codimension 1.

□

Lemma 8.31. *Let* $\tilde{d}_1, \cdots, \tilde{d}_r \in \mathbb{N}^{\tilde{Q}_0}$ *be* Δ*-Schurian. Let* $N_i \in \mathrm{Gen}_\Delta(\tilde{Q}, \tilde{d}_i)$ *for each* i*, such that* $\mathrm{Ext}^1(N_i, N_j) = 0$ *if* $i \neq j$. *Then* $\tilde{d} = \sum_{i=1}^r \tilde{d}_i$ *is the canonical* Δ*-decomposition of* $\tilde{d}$.

Proof. In light of Lemma 8.30, we may label the N_i such that (for some $p \in \mathbb{N}$)

- N_i belongs to a 1-parameter family of modules $Z_i(\lambda_i)$ with codimension 1 orbits in $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d}_i)$ for $i = 1, \cdots, p$.
- N_i has an open orbit in $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d}_i)$ for $i > p$.

Theorem 7.11 now implies that

- $\dim \operatorname{Ext}^1(N_i, N_i) = 1$ for $i = 1, \cdots, p$,
- $\operatorname{Ext}^1(N_i, N_i) = 0$ for $i > p$.

Therefore, $\dim \operatorname{Ext}^1(N, N) = p$, where $N = \bigoplus_{i=1}^r N_i$. For each $\lambda \in k^p$, let

$$N(\lambda) = \left(\bigoplus_{i=1}^{p} Z_i(\lambda_i) \right) \oplus \left(\bigoplus_{i=p+1}^{r} N_i \right).$$

So $N = N(\mu)$ for some $\mu \in k^p$. We now proceed similarly to the proof of Lemma 8.9.

Fix Δ-graphs for each indecomposable summand of $N(\mu)$. By varying a single coordinate in the Δ-graph of $Z_i(\mu_i)$, we obtain a Δ-graph for $Z_i(\lambda_i)$ for all $\lambda_i \in k$. Now varying the p coordinates from the self-glued summands gives us Δ-graphs for $N(\lambda)$ for all $\lambda \in k^p$. Specifically, this gives us a p-dimensional affine subspace of $\operatorname{Rep}_\Delta(\tilde{Q}, \tilde{d})$.

Therefore, k^p injects into $\operatorname{Rep}_\Delta(\tilde{Q}, \tilde{d})$ via the morphism of varieties taking λ to $N(\lambda)$. By Lemma 5.3, the orbits $\mathcal{O}_{N(\lambda)}$ are distinct for distinct λ, up to permutation of coordinates (that is, only a finite number of these orbits coincide). The codimension of $\mathcal{O}_{N(\mu)}$ in $\operatorname{Rep}_\Delta(\tilde{Q}, \tilde{d})$ is p (Theorem 7.11). We can therefore apply Proposition 7.19 (taking the dense open subset of k^p to be k^p itself) to get that

$$\bigcup_{\lambda \in k^p} \mathcal{O}_{N(\lambda)}$$

contains a dense open subset of $\operatorname{Rep}_\Delta(\tilde{Q}, \tilde{d})$. This intersects $\operatorname{Gen}_\Delta(\tilde{Q}, \tilde{d})$, as the latter also contains a dense open subset (Theorem 8.20). Therefore $\tilde{d} = \sum_{i=1}^r d_i$ is the canonical Δ-decomposition of $\tilde{d}$. □

Theorem 8.32. *$\sum_{i=1}^n \tilde{d}_i$ is the canonical Δ-decomposition of $\tilde{d}$ if and only if*

- *all the $\tilde{d}_i$ are Δ-Schurian, and*
- *there exist $U_i \in \operatorname{Gen}_\Delta(\tilde{Q}, \tilde{d})$ such that $\operatorname{Ext}^1(U_i, U_j) = 0$ if $i \neq j$.*

Proof. Lemma 8.28 and Lemma 8.29 together give us the forwards implication; Lemma 8.31 gives the reverse implication. □

Chapter 9

When is a General Delta-Filtered Representation Rigid?

In this chapter, we give a combinatorial condition for whether a rigid Δ-filtered representation exists with a given Δ-dimension vector (see Theorem 9.19 for the result). We observe that the Δ-dimension at non-admissible vertices plays no role in this.

Lemma 9.1. *If there exists an admissible vertex $i \in \tilde{Q}_0$ with $\tilde{d}_i = 0$, then there exists a rigid module in $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$.*

Proof. Without loss of generality, we may choose to label $\tilde{Q}$ such that $i = 1$. Then $X(d)$ has no self-glued summands. Therefore, by Theorem 8.20, there exists a $N \in \mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$ with a codimension zero orbit. That is, there exists a rigid module in $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$. □

Remark 9.2. In light of the above result, for the rest of this chapter, we assume that all Δ-dimension vectors $\tilde{d}$ satisfy $\tilde{d}_i \neq 0$ for all admissible vertices $i \in \tilde{Q}_0$.

Definition 9.3. For $i, j \in Q_0$, we say that **a Δ_i and a Δ_j are connected** in the Δ-graph (d, φ^d), when there is a connected component containing both a vertex labelled Δ_i and a vertex labelled Δ_j.

Lemma 9.4. *There does not exist a rigid module in $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$ if and only if*

1. *the $X_\bullet$ and $Y_\bullet$ are aligned, and*
2. *a Δ_1 and a Δ_{n+1} are connected.*

Proof. A rigid module has an open orbit in its representation variety. Thus by Theorem 8.20 and Corollary 8.21, there exists a rigid module in $\operatorname{Rep}_\Delta(\tilde{Q}, \tilde{d})$ if and only if the number of self glued summands, p, is zero.

To see whether $X(d)$ has a self-glued summand, consider the components of the Δ-graph (d, φ^d) which correspond to the $X_\bullet$ and $Y_\bullet$. There is a self glued summand precisely when $X_i = Y_i$ for some $i = 1, \cdots, d_1$. This implies the $X_\bullet$ and $Y_\bullet$ are aligned, as otherwise X_j and Y_j would be in different rows for all j. Additionally, it implies that the connected component corresponding to X_i (hence containing a Δ_1) is the same connected component which corresponds to Y_i (hence containing a Δ_{n+1}).

Conversely, if there is a connected component containing a Δ_1 and a Δ_{n+1}, this is the component corresponding to X_i for some i, and to Y_j for some j. If additionally the $X_\bullet$ and $Y_\bullet$ are aligned, then $i = j$, hence there is a self-glued summand. $\square$

Definition 9.5. For $\tilde{d} \in \mathbb{N}^{\tilde{Q}_0}$, we define $\tilde{d}^0 \in \mathbb{N}^{\tilde{Q}_0}$ to be given by $\tilde{d}^0_i = \tilde{d}_i$ for admissible vertices i, and $\tilde{d}^0_i = 0$ otherwise.

Example 9.6. Let $\tilde{Q}$ and $\tilde{d}$ be as follows.

$$\tilde{Q} = \begin{array}{ccccc} & & 4 & & \\ & \nearrow & & \searrow & \\ 1 & & & & 3 \\ & \searrow & & \nearrow & \\ & & 2 & & \end{array} \qquad \tilde{d} = \begin{pmatrix} & 1 & \\ 2 & & 3 \\ & 4 & \end{pmatrix}.$$

Then the split quiver Q and corresponding Δ-dimension vector d are the following.

$$Q = \ 1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longleftarrow 5 \ \text{ and } d = \begin{pmatrix} 2 & 4 & 3 & 1 & 2 \end{pmatrix}.$$

The Δ-graph (d, φ^d) is then the following.

$$\begin{array}{ccccccccc}
\Delta_1 & \longleftarrow & \Delta_2 & \longleftarrow & \Delta_3 & & & & \\
\Delta_1 & \longleftarrow & \Delta_2 & \longleftarrow & \Delta_3 & \longrightarrow & \longrightarrow & \longrightarrow & \Delta_5 \\
 & & \Delta_2 & \longleftarrow & \Delta_3 & \longrightarrow & \Delta_4 & \longrightarrow & \Delta_5 \\
 & & \Delta_2 & & & & & &
\end{array}$$

Setting the Δ-dimension at each non-admissible vertex to zero, we obtain the Δ-

dimension vectors $\tilde{d}^0$ of $\tilde{Q}$ and d^0 of Q.

$$\tilde{d}^0 = \begin{pmatrix} & 0 & \\ 2 & & 3 \\ & 0 & \end{pmatrix} \qquad d^0 = \begin{pmatrix} 2 & 0 & 3 & 0 & 2 \end{pmatrix}$$

The Δ-graph (d^0, φ^{d^0}) is then as follows.

$$\begin{array}{ccccc} \Delta_1 & \longleftarrow & \Delta_3 & & \\ \Delta_1 & \longleftarrow & \Delta_3 & \longrightarrow & \Delta_5 \\ & & \Delta_3 & \longrightarrow & \Delta_5 \end{array}$$

Note how the locations of the Δ_i, and whether there are connections between them, are unaffected by this change. That is, whether there is a self-glued summand only depends on the Δ-dimension at admissible vertices. The following lemma tells us that this is always the case.

Lemma 9.7. *There exists a rigid module in* $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$ *if and only if there exists a rigid module in* $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d}^0)$.

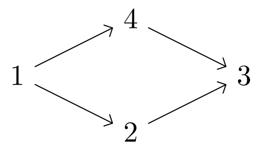
Proof. Consider the construction of $X(d)$ and $X(d^0)$. Observe that for each admissible vertex i, the rows in which Δ_i occur are the same in both $X(d)$ and $X(d^0)$, because the placement only depends on the Δ-dimension at admissible vertices. That is, the X_i and Y_i are aligned in $X(d)$ if and only if the X_i and Y_i are aligned in $X(d^0)$.

Observe also that for a row containing a Δ_1 and a Δ_{n+1}, they are connected in $X(d)$ if and only if they are connected in $X(d^0)$. That is, a Δ_1 and a Δ_{n+1} are connected in $X(d)$ if and only if a Δ_1 and a Δ_{n+1} are connected in $X(d^0)$.

The result now follows from Lemma 9.4. $\square$

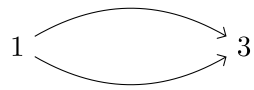
Definition 9.8. Let $\tilde{Q}'$ be a quiver such that $\tilde{Q}'_0$ is the set of admissible vertices in $\tilde{Q}$ and $\tilde{Q}'_1$ is the set of paths between admissible vertices in $\tilde{Q}$, with the same orientation as arrows in $\tilde{Q}'$ and as paths in $\tilde{Q}$. Let $\tilde{d}' \in \mathbb{N}^{\tilde{Q}'_0}$ be given by $\tilde{d}'_i = \tilde{d}_i$ for all $i \in \tilde{Q}'_0$. Thus $\tilde{Q}'$ and Q' are alternating quivers (no paths of length 2). Lastly, let (d', φ') be a Δ-graph in Q' given by the maps $\varphi'_\rho = \varphi^{d^0}_\rho$ for each $\rho \in Q'{}_1$.

Example 9.9. In Example 9.6, we had the following $\tilde{Q}$ and (d, φ^d).

$\tilde{Q} =$ 

$$\begin{array}{ccccccccc}
\Delta_1 & \longleftarrow & \Delta_2 & \longleftarrow & \Delta_3 & & & & \\
\Delta_1 & \longleftarrow & \Delta_2 & \longleftarrow & \Delta_3 & \longrightarrow & & \longrightarrow & \Delta_5 \\
 & & \Delta_2 & \longleftarrow & \Delta_3 & \longrightarrow & \Delta_4 & \longrightarrow & \Delta_5 \\
 & & \Delta_2 & & & & & &
\end{array}$$

From these, we obtain the following $\tilde{Q}'$ and $(\tilde{d}', \varphi')$.

$\tilde{Q} =$ 

$$\begin{array}{ccccc}
\Delta_1 & \longleftarrow & \Delta_3 & & \\
\Delta_1 & \longleftarrow & \Delta_3 & \longrightarrow & \Delta_5 \\
 & & \Delta_3 & \longrightarrow & \Delta_5
\end{array}$$

Remark 9.10. If we draw the Δ-graph (d', φ'), we obtain precisely the same picture as that of (d^0, φ^{d^0}). The only difference between these two Δ-graphs is that (d^0, φ^{d^0}) has a zero map associated to each non-maximal path in Q, which are left implied when drawing the picture.

Lemma 9.11. *There exists a rigid module in* $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$ *if and only if there exists a rigid module in* $\mathrm{Rep}_\Delta(\tilde{Q}', \tilde{d}')$.

Proof. As noted in Remark 9.10, the Δ-graphs (d', φ') and (d^0, φ^{d^0}) have the same picture. Thus, applying Lemma 9.4 to both of these, we get that there exists a rigid module in $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d}^0)$ if and only if there exists a rigid module in $\mathrm{Rep}_\Delta(\tilde{Q}', \tilde{d}')$. Now the result follows from Lemma 9.7. □

Remark 9.12. The consequence of Lemma 9.11 is that, to classify which dimension vectors have rigid modules, we need only consider alternating quivers (it should be noted that this only applies to type **A** quivers).

Lemma 9.13. *Suppose* $\tilde{Q}$ *is alternating.*

1. *The* $X_\bullet$ *and* $Y_\bullet$ *are aligned if and only if*

$$\sum_{i \in \tilde{Q}_0 \text{ sink}} \tilde{d}_i = \sum_{i \in \tilde{Q}_0 \text{ source}} \tilde{d}_i.$$

2. *Let* $i \in Q_0$ *and* $a \in \mathbb{N}$ *such that* $i + 2a \leq n + 1$. *There exists a row containing both a* Δ_i *and a* Δ_{i+2a} *if and only if*

$$d_{i+2a} > \sum_{s=0}^{a-1} (d_{i+2s+1} - d_{i+2s}).$$

Proof. Suppose i is a source. Let r be the lowest row with a Δ_i. Then the highest row with a Δ_i is $r + d_i - 1$. As the Δ_{i+1} are written from this row downwards, this is also the highest row with a Δ_{i+1}, while the lowest row with a Δ_{i+1} is $r + d_i - d_{i+1}$. As the Δ_{i+2} are written from this row upwards, this is also the lowest row with a Δ_{i+2}. So the lowest row with a Δ_{i+2} is $r - (r + d_i - d_{i+1}) = d_{i+1} - d_i$ rows below the lowest row with a Δ_i. By repeatedly applying this result, we see that the lowest row with a Δ_{i+2a} is

$$d_{i+2a-1} - d_{i+2a-2} + \cdots + d_{i+1} - d_i = \sum_{s=0}^{a-1} (d_{i+2s+1} - d_{i+2s})$$

rows below the lowest row with a Δ_i.

By the same argument, if i is a sink and r is the highest row with a Δ_i, the highest row with a Δ_{i+2a} is

$$\sum_{s=0}^{a-1} (d_{i+2s+1} - d_{i+2s})$$

rows above the highest row with a Δ_i. Therefore, point (2) holds in both cases.

Further, the $X_\bullet$ and $Y_\bullet$ are aligned if and only if this difference is zero for $i = 1$ and $2a = n$. That is, when

$$\sum_{s=0}^{a-1} (d_{i+2s+1} - d_{i+2s}) = \sum_{i=1}^{n} (-1)^i d_i$$

is zero. Equivalently, when the sum of d_i over even $i \in \tilde{Q}_0$ equals the sum of d_i over odd $i \in \tilde{Q}_0$. As $\tilde{Q}$ is alternating, either we have all even $i \in \tilde{Q}_0$ sinks and all odd $i \in \tilde{Q}_0$ sources, or we have all odd $i \in \tilde{Q}_0$ sinks and all even $i \in \tilde{Q}_0$ sources. □

9.14. For each $i \in Q_0$, let R_i be the set of rows in (d, φ^d) containing a Δ_i.

Lemma 9.15. *Suppose $\tilde{Q}$ is alternating. A Δ_1 and a Δ_{n+1} are connected if and only if $R_i \cap R_j \neq \emptyset$ for all $i, j \in Q_0$.*

Proof. As $\tilde{Q}$ (and hence Q) is alternating, a Δ_1 and a Δ_{n+1} are connected if and only if there exists a single row containing a Δ_i for all $i \in Q$. This is equivalent to $R_1 \cap \cdots \cap R_{n+1} \neq \emptyset$.

If $R_i \cap R_j = \emptyset$ for some $i, j \in Q_0$, then $R_1 \cap \cdots \cap R_{n+1} = \emptyset$. On the other hand, suppose $R_1 \cap \cdots \cap R_{n+1} = \emptyset$. Let $i \in Q$ be the highest for which $R_1 \cap \cdots \cap R_i \neq \emptyset$.

The $R_\bullet$ are sets of consecutive integers, as is $R_1 \cap \cdots \cap R_i$. Specifically,

$$\min R_1 \cap \cdots \cap R_i = \max \{\min R_1, \cdots, \min R_i\} = \min R_s$$

for some $s \in \{1, \cdots, i\}$, and

$$\max R_1 \cap \cdots \cap R_i = \min \{\max R_1, \cdots, \max R_i\} = \max R_t$$

for some $t \in \{1, \cdots, i\}$. As $R_{i+1} \cap (R_1 \cap \cdots \cap R_i) = \emptyset$, we have either $\min R_{i+1} > \max R_t$ or $\max R_{j+1} < \min R_s$. Thus either $R_{j+1} \cap R_t = \emptyset$ or $R_{j+1} \cap R_s = \emptyset$. □

Lemma 9.16. *Suppose $\tilde{Q}$ is alternating. A Δ_1 and a Δ_{n+1} are connected if and only if*

$$R_i \cap R_{i+2a} \neq \emptyset$$

for all $i \in Q_0$ and nonzero $a \in \mathbb{N}$ such that $i + 2a \leq n + 1$.

Proof. By Lemma 9.15, if $R_i \cap R_{i+2a+1} = \emptyset$ for some i and a, then a connected Δ_1 and a Δ_{n+1} do not exist. Further, for the reverse implication, we need only show that the following three statements are contradictory.

1. $R_i \cap R_{i+2a} \neq \emptyset$,
2. $R_{i+1} \cap R_{i+2a+1} \neq \emptyset$,
3. $R_i \cap R_{i+2a+1} = \emptyset$.

Suppose the above statements hold. By (3), we have either

$$\min R_i > \max R_{i+2a+1} \qquad \text{or} \qquad \min R_{i+2a+1} > \max R_i.$$

Suppose i is a source. Then $\max R_i = \max R_{i+1}$ and $\max R_{i+2a} = \max R_{i+2a+1}$. Thus $\min R_i > \max R_{i+2a}$, which contradicts (1), or $\min R_{i+2a+1} > \max R_{i+1}$, which contradicts (2).

Now suppose i is a sink. Then $\min R_i = \min R_{i+1}$ and $\min R_{i+2a} = \min R_{i+2a+1}$. Thus $\min R_{i+1} > \max R_{i+2a+1}$, which contradicts (2), or $\min R_{i+2a} > \max R_i$, which contradicts (1). □

Example 9.17. It may be tempting to think that we can reduce the condition in Lemma 9.16 to simply $R_i \cap R_{i+2} \neq \emptyset$ for all $i = 1, \cdots, n-1$. However, this is not the case. Consider the following example.

$$\tilde{Q} = \begin{array}{ccccc} & & 4 & & \\ & \nearrow & & \nwarrow & \\ 1 & & & & 3 \\ & \searrow & & \swarrow & \\ & & 2 & & \end{array} \qquad \tilde{d} = \begin{pmatrix} & 3 & \\ 2 & & 2 \\ & 3 & \end{pmatrix}$$

$$Q = \ 1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5 \qquad d = \begin{pmatrix} 2 & 3 & 2 & 3 & 2 \end{pmatrix}$$

The Δ-graph (d, φ^d) is then the following.

$$\begin{array}{ccccccccc} \Delta_1 & \longleftarrow & \Delta_2 & & & & & & \\ \Delta_1 & \longleftarrow & \Delta_2 & \longrightarrow & \Delta_3 & \longleftarrow & \Delta_4 & & \\ & & \Delta_2 & \longrightarrow & \Delta_3 & \longleftarrow & \Delta_4 & \longrightarrow & \Delta_5 \\ & & & & & & \Delta_4 & \longrightarrow & \Delta_5 \end{array}$$

Observe that for $i = 1, 2, 3$ we have $R_i \cap R_{i+2} \neq \emptyset$, but $R_1 \cap R_5 = \emptyset$, so there is no row connecting a Δ_1 and a Δ_5.

Definition 9.18. We say vertices i and j are **adjacent admissible vertices** when i and j are both admissible and there exists a path between them.

Theorem 9.19. *Let the admissible vertices of $\tilde{Q}$ be labelled $1, \cdots, n$ such that each pair $i, i+1$ are adjacent admissible vertices. There exists a rigid module in $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$ if and only if*

1. *the sum of $\tilde{d}_i$ over all sinks i is not equal to the sum of $\tilde{d}_j$ over all sources j in $\tilde{Q}$, or*
2. *for some admissible vertex $i \in \tilde{Q}_0$ and $a \in \mathbb{N}$, we have*

$$d_{i+2a} \leq \sum_{s=0}^{a-1} d_{i+2s+1} - d_{i+2s}.$$

Proof. Consider the alternating quiver $\tilde{Q}'$ obtained from $\tilde{Q}$ (Definition 9.8). Note that, by construction, the admissible vertices of $\tilde{Q}'$ are also labelled $1, \cdots, n$ such that each pair $i, i+1$ are adjacent.

By Lemma 9.13(1), the $X_\bullet$ and $Y_\bullet$ are aligned in $X(\tilde{d}')$ if and only if the sum of $\tilde{d}'_i$ over all sinks i is equal to the sum of $\tilde{d}'_j$ over all sources j in $\tilde{Q}'$. By construction, this holds if and only if the sum of $\tilde{d}_i$ over all sinks i is equal to the sum of $\tilde{d}_j$ over all sources j in $\tilde{Q}$. That is, if and only if (1) does not hold.

Applying Lemma 9.16 to Lemma 9.13(2), we get that a Δ_1 and a Δ_{n+1} are connected in $X(\tilde{d}')$ if and only if

$$d'_{i+2a} > \sum_{s=0}^{a-1} d'_{i+2s+1} - d'_{i+2s}$$

for all $i \in \tilde{Q}'$ and nonzero $a \in \mathbb{N}$ such that $i + 2a \leq n + 1$. By construction, this holds if and only if

$$d_{i+2a} > \sum_{s=0}^{a-1} d_{i+2s+1} - d_{i+2s}$$

for all admissible $i \in \tilde{Q}$ and nonzero $a \in \mathbb{N}$ such that $i + 2a \leq n + 1$. That is, if and only if (2) does not hold.

Now Lemma 9.4 implies that there exists a rigid module in $\mathrm{Rep}_\Delta(\tilde{Q}', \tilde{d}')$ if and only if (1) or (2) hold. The result now follows from Lemma 9.11. □

Corollary 9.20. *The codimension of the orbit in* $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$ *of any general representation is* p*, where* $p = 0$ *if the sum of* $\tilde{d}_i$ *over all sinks* i *is not equal to the sum of* $\tilde{d}_j$ *over all sources* j *in* $\tilde{Q}$*. Otherwise,* p *is the minimum of*

$$c_{i,a}(d) = d_{i+2a} - \sum_{s=0}^{a-1} (d_{i+2s+1} - d_{i+2s})$$

amongst all admissible vertices $i \in \tilde{Q}$ *and* $a \in \mathbb{N}$ *(or* $p = 0$ *if this is negative).*

Proof. Note that p is the number of self-glued summands. Suppose points (1) and (2) of Theorem 9.19 are false. Then $p \geq 1$ and $c_{i,a} > 0$ for all i and a. Consider the Δ-graph obtained by deleting a row with a self-glued summand. The new Δ-dimension vector is d', where each admissible $d'_i = d_i - 1$. This new Δ-graph is precisely the Δ-graph obtained by applying the algorithm to d'. Point (1) of the Theorem holds for d'. Therefore, $p \geq 2$ if and only if there exists a self-glued summand in this new Δ-graph, if and only if $c_{i,a}(d') > 0$ for all i and a. But $c_{i,a}(d') = c_{i,a}(d) - 1$, hence by induction, p is equal to the minimum of all the $c_{i,a}(d)$. □

Example 9.21. Consider the following quiver $\tilde{Q}$ and its associated Dynkin quiver Q.

$$\tilde{Q} = \begin{array}{ccccccc} & & 4 & \longleftarrow & 3 & & \\ & \nearrow & & & & \searrow & \\ 1 & & & & & & b \\ & \searrow & & & & \swarrow & \\ & & a & \longrightarrow & 2 & & \end{array}$$

$$Q = \; 1 \longrightarrow a \longrightarrow 2 \longleftarrow b \longleftarrow 3 \longrightarrow 4 \longleftarrow 5$$

Let $\tilde{d}$ be as follows.

$$\tilde{d} = \begin{pmatrix} & 2 & 1 & \\ 3 & & & 1 \\ & 1 & 2 & \end{pmatrix} \qquad\qquad d = \begin{pmatrix} 3 & 1 & 2 & 1 & 1 & 2 & 3 \end{pmatrix}$$

Checking the conditions of Theorem 9.19, we see that

1. does not hold, as $\tilde{d}_1 + \tilde{d}_3 = 4 = \tilde{d}_2 + \tilde{d}_4$.
2. does not hold, because
   - $d_3 > d_2 - d_1$ $(1 > 2 - 3)$,
   - $d_4 > d_3 - d_2$ $(2 > 1 - 2)$,
   - $d_5 > d_4 - d_3$ $(3 > 2 - 1)$,
   - $d_5 > d_4 - d_3 + d_2 - d_1$ $(3 > 2 - 1 + 2 - 3)$.

Therefore, there does not exist a rigid module in $\text{Rep}_\Delta(\tilde{Q}, \tilde{d})$. We can also see this by constructing $X(d)$ and observing the self-glued summand.

$$\begin{array}{ccccccccccccc} \Delta_1 & \longleftarrow & \Delta_a & \longleftarrow & \Delta_2 & & & & & & & & \Delta_5 \\ \Delta_1 & \longleftarrow & & \longleftarrow & \Delta_2 & \longrightarrow & \Delta_b & \longrightarrow & \Delta_3 & \longleftarrow & \Delta_4 & \longrightarrow & \Delta_5 \\ \Delta_1 & & & & & & & & & & \Delta_4 & \longrightarrow & \Delta_5 \end{array}$$

Example 9.22. Let $\tilde{Q}$ be as in the previous example, but this time take the following $\tilde{d}$.

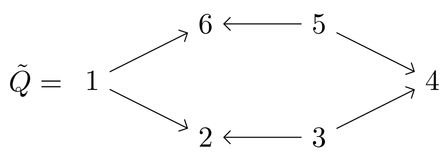
$$\tilde{d} = \begin{pmatrix} & 3 & 1 & \\ 3 & & & 1 \\ & 1 & 2 & \end{pmatrix} \qquad\qquad d = \begin{pmatrix} 3 & 1 & 2 & 1 & 1 & 3 & 3 \end{pmatrix}$$

If we check the conditions of Theorem 9.19 now, we see that (1) holds, because $\tilde{d}_1 + \tilde{d}_3 = 3 + 1 = 4$, whereas $\tilde{d}_2 + \tilde{d}_4 = 2 + 3 = 5$. This indicates that the $X_\bullet$ and $Y_\bullet$ are not aligned. Drawing out the Δ-graph for $X(d)$, we see this is indeed the case.

$$
\begin{array}{ccccccccccccc}
\Delta_1 & \longleftarrow & \Delta_a & \longleftarrow & \Delta_2 & & & & & & & & \\
\Delta_1 & \longleftarrow & \longleftarrow & \longleftarrow & \Delta_2 & \longrightarrow & \Delta_b & \longrightarrow & \Delta_3 & \longleftarrow & \Delta_4 & \longrightarrow & \Delta_5 \\
\Delta_1 & & & & & & & & & & \Delta_4 & \longrightarrow & \Delta_5 \\
 & & & & & & & & & & \Delta_4 & \longrightarrow & \Delta_5
\end{array}
$$

Even though there is a row with a Δ_1 and a Δ_{n+1} connected, the $X_\bullet$ and $Y_\bullet$ are not aligned, so there is no self glued summand.

Example 9.23. Consider the following quiver $\tilde{Q}$ and its associated Dynkin quiver Q.



$$
Q = \ 1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4 \longleftarrow 5 \longrightarrow 6 \longleftarrow 7
$$

Take the following $\tilde{d}$.

$$
\tilde{d} = \begin{pmatrix} & 4 & 1 & \\ 3 & & & 1 \\ & 1 & 2 & \end{pmatrix} \qquad d = \begin{pmatrix} 3 & 1 & 2 & 1 & 1 & 4 & 3 \end{pmatrix}
$$

If we check the conditions of Theorem 9.19, we see that (1) does not hold, because $\tilde{d}_1 + \tilde{d}_3 + \tilde{d}_5 = 3 + 2 + 1 = 6$ and $\tilde{d}_2 + \tilde{d}_4 + \tilde{d}_6 = 1 + 1 + 4 = 6$. However, when we check (2), we see that

- $d_3 > d_2 - d_1$ $(2 > 1 - 3)$, but
- $d_4 \leq d_3 - d_2$ $(1 \leq 2 - 1)$,

therefore (2) holds. Consequently, there are no rows with a connected Δ_1 and Δ_{n+1}, hence no self-glued summands.

$$
\begin{array}{ccccccccccccc}
 & & & & \Delta_3 & \longleftarrow & \Delta_4 & \longrightarrow & \Delta_5 & \longleftarrow & \Delta_6 & & \\
\Delta_1 & \longleftarrow & \Delta_2 & \longrightarrow & \Delta_3 & & & & & & \Delta_6 & \longrightarrow & \Delta_7 \\
\Delta_1 & & & & & & & & & & \Delta_6 & \longrightarrow & \Delta_7 \\
\Delta_1 & & & & & & & & & & \Delta_6 & \longrightarrow & \Delta_7
\end{array}
$$

Example 9.24. Consider the following quiver, with one sink and source.

$$\tilde{Q} = \begin{array}{ccccc} & & \cdots & & \\ & \nearrow & & \searrow & \\ 1 & & & & 2 \\ & \searrow & & \nearrow & \\ & & \cdots & & \end{array}$$

$$Q = \ 1 \longrightarrow \cdots \longrightarrow 2 \longleftarrow \cdots \longleftarrow 3$$

By the theorem, we know a rigid module exists in $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$ if and only if

- $\tilde{d}_1 \neq \tilde{d}_2$, or
- $d_3 \leq d_2 - d_1$.

But the second point is equivalent to $\tilde{d}_1 \leq \tilde{d}_2 - \tilde{d}_1$, which implies the first point, as $\tilde{d}_1 > 0$. Therefore, there exists a rigid module in $\mathrm{Rep}_\Delta(\tilde{Q}, \tilde{d})$ if and only if $\tilde{d}_1 \neq \tilde{d}_2$.

Chapter 10

Representations of Alternating Quivers

In this chapter, Q is any type of quiver (not just affine type $\mathbf{A}$), but we assume Q is alternating - that is, Q has no paths of length 2 (and hence no paths of length greater than 2 either). We show that, for such a Q, the existence of rigid Δ-filtered representations in $\mathrm{Rep}_\Delta(Q, d)$ is equivalent to the existence of rigid representations in $\mathrm{Rep}(Q^{\mathrm{op}}, d)$.

Definition 10.1. Recall that for $i, j \in Q_0$, the space $\mathrm{Hom}_A(P(i), P(j))$ has basis $\{f_\rho : \rho \in P(j,i)\}$, where the morphism $f_\rho : x \mapsto x\rho$. Consequently, $f_\rho \circ f_\tau = f_{\tau\rho}$ for paths $\rho, \tau \in P(-,-)$, where we define $f_0 = 0$ in order to include the cases when $\tau\rho = 0$. In particular, $f_\rho \circ f_\tau = 0$ if $t(\rho) \neq s(\tau)$.

Lemma 10.2. *Firstly,*

$$\mathrm{Aut}_A P(d) = \left\{ \sum_{i \in Q_0} A_i f_{e_i} + \sum_{\alpha \in Q_1} A_\alpha f_\alpha : A_i \in \mathrm{Aut}_k(k^{d_i}),\ A_\alpha \in \mathrm{Hom}_k(k^{d_{t(\alpha)}}, k^{d_{s(\alpha)}}) \right\}.$$

Secondly,

$$\mathrm{rad}\,\mathrm{End}_A P(d) = \left\{ \sum_{\alpha \in Q_1} X_\alpha f_\alpha : X_\alpha \in \mathrm{Hom}_k(k^{d_{t(\alpha)}}, k^{d_{s(\alpha)}}) \right\}.$$

Lastly, the action of $g \in \operatorname{Aut}_A P(d)$ on $X \in \operatorname{rad}\operatorname{End}_A P(d)$ is given by

$$g \cdot X = \sum_{\alpha \in Q_1} A_{s(\alpha)} X_\alpha A_{t(\alpha)}^{-1} f_\alpha.$$

Proof. As $\operatorname{Hom}_A(P(i), P(j))$ is the span of f_ρ for $\rho \in P(j,i)$, we have that

$$\operatorname{Hom}_A(P(i)^{d_i}, P(j)^{d_j}) = \left\{ \sum_{\rho \in P(j,i)} A_\rho f_\rho : A_\rho \in \operatorname{Hom}_k(k^{d_i}, k^{d_j}) \right\},$$

where $(A_\rho f_\rho) \circ (A_\sigma f_\sigma) = (A_\rho \circ A_\sigma)(f_\rho \circ f_\sigma) = A_\sigma A_\rho f_{\sigma\rho}$. Therefore, $\operatorname{End}_A P(d)$ is the set of

$$\sum_{i,j \in Q_0} \sum_{\rho \in P(j,i)} A_\rho f_\rho = \sum_{\rho \in P(-,-)} A_\rho f_\rho$$

for $A_\rho \in \operatorname{Hom}_A(k^{d_{t(\rho)}}, k^{d_{s(\rho)}})$.

As Q is alternating, the set of nontrivial paths in Q is just the set of arrows Q_1. Thus the set $P(-,-)$ of all paths in Q is the union of the set $E = \{e_i : i \in Q_0\}$ of trivial paths with the set Q_1 of arrows. Consequently, $\operatorname{End}_A P(d)$ is the set of

$$\sum_{i \in Q_0} A_i f_{e_i} + \sum_{\alpha \in Q_1} A_\alpha f_\alpha$$

for $A_i \in \operatorname{End}_k k^{d_i}$ and $A_\alpha \in \operatorname{Hom}_k(k^{d_{t(\alpha)}}, k^{d_{s(\alpha)}})$. The subset consisting of invertible morphisms is $\operatorname{Aut}_A P(d)$. That is, $\operatorname{Aut}_A P(d)$ is the subset where $A_i \in \operatorname{Aut}_A(k^{d_i})$ for each $i \in Q_0$. The radical $\operatorname{rad}\operatorname{End}_A P(d)$ is the subset of

$$\sum_{\alpha \in Q_1} X_\alpha f_\alpha$$

for $X_\alpha \in \operatorname{Hom}_k(k^{d_{t(\alpha)}}, k^{d_{s(\alpha)}})$.

Let $g \in \operatorname{Aut}_A P(d)$ and $X \in \operatorname{rad}\operatorname{End}_A P(d)$. Then

$$g^{-1} = \sum_{i \in Q_0} A_i^{-1} f_{e_i} + \sum_{\alpha \in Q_1} B_\alpha f_\alpha$$

for some $B_\alpha \in \operatorname{Hom}_k(k^{d_{t(\alpha)}}, k^{d_{s(\alpha)}})$. The action of $\operatorname{Aut}_A P(d)$ on $\operatorname{rad}\operatorname{End}_A P(d)$ is

$g \cdot X = gXg^{-1}$. As Q is alternating, $\alpha\beta = 0$ for all $\alpha, \beta \in Q_1$, hence

$$\begin{aligned} gXg^{-1} &= \left(\sum_{i\in Q_0} A_i f_{e_i}\right) \circ \left(\sum_{\alpha\in Q_1} X_\alpha f_\alpha\right) \circ \left(\sum_{i\in Q_0} A_i^{-1} f_{e_i}\right) \\ &= \sum_{i\in Q_0} \sum_{\alpha\in Q_1} \sum_{j\in Q_0} (A_i f_{e_i}) \circ (X_\alpha f_\alpha) \circ (A_{e_j}^{-1} f_{e_j}) \\ &= \sum_{i,j\in Q_0} \sum_{\alpha\in Q_1} A_i X_\alpha A_j^{-1} (f_{e_i} \circ f_\alpha \circ f_{e_j}). \end{aligned}$$

But $f_{e_i} \circ f_\alpha \circ f_{e_j} = f_{e_j \alpha e_i}$, which is zero if $i \neq s(\alpha)$ or $j \neq t(\alpha)$. Therefore,

$$gXg^{-1} = \sum_{\alpha\in Q_1} (A_{s(\alpha)} X_\alpha A_{t(\alpha)}^{-1}) f_\alpha.$$

$\square$

Remark 10.3. Note that the representation variety for the opposite quiver Q^{op} is

$$\mathrm{Rep}(Q^{\mathrm{op}}, d) = \bigoplus_{\alpha\in Q_1} \mathrm{Hom}_k(k^{d_{t(\alpha)}}, k^{d_{s(\alpha)}}).$$

Lemma 10.4. *Let $g \in \mathrm{Gl}(d)$ and $X \in \mathrm{Rep}(Q^{\mathrm{op}}, d)$. Writing*

$$g = (A_i)_{i\in Q_0} \qquad \text{and} \qquad X = \sum_{\alpha\in Q_1} X_\alpha$$

for some $X_\alpha \in \mathrm{Hom}_k(k^{d_{t(\alpha)}}, k^{d_{s(\alpha)}})$ and some $A_i \in \mathrm{Aut}_k(k^{d_i})$, the action of g on X is given by

$$g \cdot X = \sum_{\alpha\in Q_1} A_{s(\alpha)} X_\alpha A_{t(\alpha)}^{-1}.$$

Proof. The action of $\mathrm{Gl}(d)$ on $\mathrm{Rep}(Q^{\mathrm{op}}, d)$ is given by the following commutative diagram.

$$\begin{array}{ccc} k^{d_{t(\alpha)}} & \xrightarrow{X_\alpha} & k^{d_{s(\alpha)}} \\ {\scriptstyle A_{t(\alpha)}}\downarrow & & \downarrow{\scriptstyle A_{s(\alpha)}} \\ k^{d_{t(\alpha)}} & \xrightarrow{(g\cdot X)_\alpha} & k^{d_{s(\alpha)}} \end{array}$$

That is, $(g \cdot X)_\alpha = A_{s(\alpha)} X_\alpha A_{t(\alpha)}^{-1}$, from which the result follows. $\square$

Proposition 10.5. *The following are equivalent.*

- *There exists a rigid module in* $\mathrm{Rep}_\Delta(Q,d)$.
- *There exists a rigid module in* $\mathrm{Rep}(Q^{\mathrm{op}},d)$.

Proof. We know that $\mathrm{Rep}_\Delta(Q,d)$ and $\operatorname{rad}\operatorname{End}_A P(d)$ are $\operatorname{Aut}_A P(d)$-equivariantly isomorphic (Theorem 7.1). So we need only show that there exists an open $\mathrm{Gl}(d)$-orbit in $\mathrm{Rep}(Q^{\mathrm{op}},d)$ if and only if there exists an open $\operatorname{Aut}_A P(d)$ orbit in $\operatorname{rad}\operatorname{End}_A P(d)$.

Let $\psi : \operatorname{Aut}_A P(d) \to \mathrm{Gl}(d)$ be given by

$$\sum_{i\in Q_0} A_i f_{e_i} + \sum_{\alpha\in Q_1} B_\alpha f_\alpha \mapsto (A_i)_{i\in Q_0}\,.$$

This is a surjective morphism of algebraic groups.

Let $\varphi : \operatorname{rad}\operatorname{End}_A P(d) \to \mathrm{Rep}(Q^{\mathrm{op}},d)$ be given by

$$\sum_{\alpha\in Q_1} X_\alpha f_\alpha \mapsto \sum_{\alpha\in Q_1} X_\alpha.$$

This is an isomorphism of affine varieties.

For $g \in \operatorname{Aut}_A P(d)$ and $X \in \operatorname{rad}\operatorname{End}_A P(d)$, we have

$$g\cdot X = \sum_{\alpha\in Q_1} A_{s(\alpha)} X_\alpha A_{t(\alpha)}^{-1} f_\alpha$$

by Lemma 10.2, hence

$$\varphi(g\cdot X) = \sum_{\alpha\in Q_1} A_{s(\alpha)} X_\alpha A_{t(\alpha)}^{-1}.$$

On the other hand,

$$\psi(g) = (A_i)_{i\in Q_0} \qquad \text{and} \qquad \varphi(X) = \sum_{\alpha\in Q_1} X_\alpha,$$

hence by Lemma 10.4, we have that $\varphi(g\cdot X) = \psi(g)\cdot\varphi(X)$. □

Part II

Lie Theory

Chapter 11

The Adjoint Action

Recall that for a Lie algebra $\mathfrak{p}$ with nilpotent radical $\mathfrak{n}$, a Richardson element is an element $x \in \mathfrak{n}$ such that $[x, \mathfrak{p}] = \mathfrak{n}$. In the following chapters, we study seaweed subalgebras of affine Lie algebras, ultimately using the theory developed in Part I to give a criterion for the existence of Richardson elements. In this chapter, we discuss the mechanisms behind why we are able to use quiver representations in this way.

Let Q be a quiver and $d \in \mathbb{N}^{Q_0}$, writing $A = kQ$ for the path algebra as usual. Recall from Part I that $P(d)$ is the A-projective representation with Δ-dimension vector d and that the spaces $\operatorname{Rep}_\Delta(Q, d)$ and $\operatorname{rad}\operatorname{End} P(d)$ are $\operatorname{Aut} P(d)$-equivariantly isomorphic (Theorem 7.1).

In this chapter, we see that $\operatorname{Aut} P(d)$ is an algebraic group whose Lie algebra is $\operatorname{End} P(d)$. Later, we see that the nilpotent radical of $\operatorname{End} P(d)$ is $\operatorname{rad}\operatorname{End} P(d)$ (Corollary 17.14). This means that we can consider elements in the nilpotent radical of certain Lie algebras to be representations in $\operatorname{Rep}_\Delta(Q, d)$.

We also see in this chapter that the $\operatorname{Aut} P(d)$-action on $\operatorname{End} P(d)$ is the adjoint action and that Richardson elements are those with dense open adjoint orbits in the nilpotent radical. This means that we can use the classification of the codimension of $\operatorname{Aut} P(d)$-orbits in $\operatorname{Rep}_\Delta(Q, d)$ from Corollary 9.20 to obtain a criterion for the existence of Richardson elements in a Lie algebra, as long as we can show that the Lie algebra is isomorphic to $\operatorname{End} P(d)$ for some Q and d.

We refer to Tauvel and Yu [27], to Procesi [22] and to Hall [7] for various results on algebraic groups.

11.1. Write

$$\mathfrak{gl}(d) = \bigoplus_{i \in Q_0} \mathfrak{gl}_{d_i}(k).$$

For each $\mathfrak{gl}_{d_i}(k)$, we have an exponential map

$$\exp : \mathfrak{gl}_{d_i}(k) \to \mathrm{GL}_{d_i}(k).$$

The exponential map

$$\exp : \mathfrak{gl}(d) \to \mathrm{GL}(d)$$

is then the map sending $(f_i)_{i \in Q_0} \mapsto (\exp(f_i))_{i \in Q_0}$. Each $\mathrm{GL}_{d_i}(k)$ has Lie algebra $\mathfrak{gl}_{d_i}(k)$ (Proposition 3.23 in [7]), so $\mathrm{GL}(d)$ has Lie algebra $\mathfrak{gl}(d)$.

Let $X \in \mathrm{Rep}(Q, d)$. Then $\mathrm{Aut}\, X$ is an algebraic subgroup of $\mathrm{GL}(d)$ and $\mathrm{End}\, X$ is the Lie subalgebra of $\mathfrak{gl}(d)$ satisfying the commutativity relations for X.

Proposition 11.2. *The Lie algebra of* $\mathrm{Aut}\, X$ *is* $\mathrm{End}\, X$.

Proof. Firstly, note that $\mathrm{Aut}\, X = \mathrm{GL}(d) \cap \mathrm{End}\, X$. The Lie algebra of $\mathrm{Aut}\, X$ is the set of $f \in \mathfrak{gl}(d)$ such that $\exp(\lambda f) \in \mathrm{Aut}\, X$ for all $\lambda \in \mathbb{R}$ (3.18 in [7]). If $f \in \mathrm{End}\, X$, then $\exp(f) \in \mathrm{End}\, X$, hence $\exp(f) \in \mathrm{Aut}\, X$. As $\lambda f \in \mathrm{End}\, X$ for all $\lambda \in \mathbb{R}$, this implies $\mathrm{End}\, X$ is contained within the Lie algebra of $\mathrm{Aut}\, X$. This Lie algebra has the same dimension as $\mathrm{Aut}\, X$ (Corollary 3.45 in [7]), so $\mathrm{End}\, X$ is the Lie algebra of $\mathrm{Aut}\, X$ by the fact that $\dim \mathrm{End}\, X = \dim \mathrm{Aut}\, X$. □

Definition 11.3 (23.5.2 in [27])**.** Let G be an algebraic group and $\mathfrak{g}$ its Lie algebra. There is a representation of G,

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}) : g \mapsto \mathrm{Ad}(g),$$

called the **adjoint representation** of G. See Tauvel and Yu [27] for the full definition of $\mathrm{Ad}(g)$. For our purposes, we treat Lemma 11.4 as the definition, though the relationship between Ad and the adjoint representation ad of $\mathfrak{g}$,

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) : x \mapsto \mathrm{ad}(x), \qquad \mathrm{ad}(x)(y) = [x, y],$$

given in Theorem 11.6, should also be noted.

The action $\mathrm{Ad}(g)(x)$ of $g \in G$ on $x \in \mathfrak{g}$ is called the **adjoint action** and the orbit $\mathrm{Ad}(G)(x)$ is called the **adjoint orbit** of x. It turns out that this action does not depend upon the choice of G, but only on $\mathfrak{g}$.

Lemma 11.4 (Lemma 23.5.4 in [27])**.** *When G is a closed subgroup of* $\mathrm{GL}_n(k)$ *and* $\mathfrak{g}$ *is a subalgebra of* $\mathfrak{gl}_n(k)$*, we have* $\mathrm{Ad}(g)(x) = gxg^{-1}$ *for* $g \in G$ *and* $x \in \mathfrak{g}$*.*

Lemma 11.5. *The adjoint action of* $\operatorname{Aut} X$ *on* $\operatorname{End} X$ *is given by conjugation. That is,*

$$\mathrm{Ad}(g)(f) = (g_i f_i g_i^{-1})_{i \in Q_0} = gfg^{-1}$$

for $g \in \operatorname{Aut} X$ *and* $f \in \operatorname{End} X$*.*

Proof. For each $i \in Q_0$, the ith component of g and f lie in $\mathrm{GL}(d_i)$ and $\mathfrak{gl}_{d_i}(k)$ respectively, so we have $\mathrm{Ad}(g_i)(f_i) = g_i f_i g_i^{-1}$. The result follows. □

Theorem 11.6 (Theorem 2, Section 3.1, Chapter 4 in [22])**.** $\mathrm{Ad}(\exp(x)) = \exp(\mathrm{ad}(x))$ *for all* $x \in \mathfrak{g}$*.*

Definition 11.7 (24.8 in [27])**.** The **algebraic adjoint group** of $\mathfrak{g}$ is the smallest algebraic subgroup of $\mathrm{GL}(\mathfrak{g})$ whose Lie algebra contains $\mathrm{ad}(\mathfrak{g})$; if its Lie algebra is equal to $\mathrm{ad}(\mathfrak{g})$, then this is called the **adjoint group** of $\mathfrak{g}$. We denote the adjoint group of $\mathfrak{g}$ by $\mathrm{Ad}(\mathfrak{g})$.

Note that the adjoint group is sometimes simply defined as the group generated by $\exp(\mathrm{ad}\,\mathfrak{g})$, for example by Procesi [22].

Lemma 11.8 (Section 3.1, Chapter 4 in [22])**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If G is connected, then G is generated by* $\exp(\mathfrak{g})$*.*

Lemma 11.9. *Let G be a connected algebraic group with Lie algebra $\mathfrak{g}$. Then* $\mathrm{Ad}(G) = \mathrm{Ad}(\mathfrak{g})$*.*

Proof. Observe that

$$\mathrm{Ad}(G) = \mathrm{Ad}\,\langle \exp(\mathfrak{g}) \rangle = \langle \mathrm{Ad}(\exp(\mathfrak{g})) \rangle = \langle \exp(\mathrm{ad}(\mathfrak{g})) \rangle\,,$$

the first equality by Lemma 11.8 and the third by Theorem 11.6. So $\mathrm{Ad}(G)$ has Lie algebra $\mathrm{ad}(\mathfrak{g})$. Further, $\mathrm{Ad}(G)$ is the image of a morphism of algebraic groups, so is an algebraic subgroup (that is, a closed subgroup) of $\mathrm{GL}(\mathfrak{g})$ (Proposition 21.2.4 in [27]). □

Remark 11.10. Lemma 11.9 tells us that the adjoint orbit is the same as the orbit of the adjoint group. The rest of this chapter is dedicated to showing that Richardson elements are those with a dense open adjoint orbit.

Definition 11.11 (21.4.1 in [27]). Let G be an algebraic group and X a variety with a G-action. X is called a G**-variety** if the map $G \times X \to X : (g, x) \mapsto g \cdot x$ is a morphism of algebraic varieties.

Remark 11.12. $\mathrm{Ad}(\mathfrak{g})$ is a G-variety.

Proposition 11.13 (Proposition 21.4.3(iii) in [27]). *Let X be a G-variety and $x \in X$. Then $\dim(G \cdot x) = \dim G - \dim G_x$, where G_x denotes the G-stabilizer of x.*

Proposition 11.14 (Proposition 24.3.6(ii) in [27]). *Let G be an algebraic group with Lie algebra $\mathfrak{g}$ and $x \in \mathfrak{g}$. Write*

$$C_G(x) = \{g \in G : \mathrm{Ad}(g)(x) = x\}, \qquad \mathfrak{c}_{\mathfrak{g}}(x) = \{y \in \mathfrak{g} : \mathrm{ad}(x)(y) = 0\}.$$

Then the Lie algebra of $C_G(x)$ is $\mathfrak{c}_{\mathfrak{g}}(x)$.

Lemma 11.15 (24.8.3 in [27]). *Let $\mathfrak{g}$ be a Lie algebra and G its algebraic adjoint group. Then any ideal of $\mathfrak{g}$ is G-stable. Conversely, if a subspace of $\mathfrak{g}$ is G-stable, then it is also $\mathrm{ad}(\mathfrak{g})$-stable. Thus G-stable subspaces of $\mathfrak{g}$ are exactly the ideals of $\mathfrak{g}$.*

Lemma 11.16. *Let $\mathfrak{g}$ be a Lie algebra. Then $\dim[x, \mathfrak{g}] = \dim \mathrm{Ad}(\mathfrak{g})(x)$ for all $x \in \mathfrak{g}$. If $\mathfrak{n}$ is an ideal of $\mathfrak{g}$ and $x \in \mathfrak{n}$, then $[\mathfrak{g}, x] = \mathfrak{n}$ if and only if $\mathrm{Ad}(\mathfrak{g})(x)$ is a dense open subset of $\mathfrak{n}$.*

Proof. Let G be a connected algebraic group with Lie algebra $\mathfrak{g}$. Then $\mathrm{Ad}(\mathfrak{g}) = \mathrm{Ad}(G)$, by Lemma 11.9. The $\mathrm{Ad}(G)$-stabilizer of x is

$$\{g \in G : \mathrm{Ad}(g)(x) = x\} = C_G(x).$$

By Proposition 11.14, this has Lie algebra

$$\mathfrak{c}_{\mathfrak{g}}(x) = \{y \in \mathfrak{g} : \mathrm{ad}(x)(y) = 0\} = \ker \mathrm{ad}(x),$$

hence $\dim C_G(x) = \dim \mathfrak{c}_{\mathfrak{g}}(x)$. As $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ is a linear map, we have

$$\dim \mathrm{im}\, \mathrm{ad}(x) = \dim \mathfrak{g} - \dim \ker \mathrm{ad}(x),$$

where $\mathrm{im}\, \mathrm{ad}(x) = \mathrm{ad}(x)(\mathfrak{g}) = [x, \mathfrak{g}]$. That is,

$$\dim[x, \mathfrak{g}] = \dim G - \dim C_G(x),$$

so Proposition 11.13 implies that $\dim [x, \mathfrak{g}] = \dim \mathrm{Ad}(G)(x)$.

Lastly, $\mathrm{Ad}(\mathfrak{g})(x) \subseteq \mathfrak{n}$ by Lemma 11.15, so $\mathrm{Ad}(\mathfrak{g})(x)$ is a dense open subset of $\mathfrak{n}$ if and only if $\dim \mathrm{Ad}(\mathfrak{g})(x) = \dim \mathfrak{n}$. □

Chapter 12

Affine Lie Algebras

Recall the classification of finite dimensional simple Lie algebras by Cartan matrices and Dynkin diagrams (see for example Humphreys [11]). Given a Cartan matrix or Dynkin diagram, we recover a list of simple roots α_i and Cartan integers $\langle \alpha_i, \alpha_j \rangle \in \mathbb{Z}$. This data then determines the Lie algebra associated to that Cartan matrix and Dynkin diagram.

The same procedure can be done with wider class of Dynkin diagrams to obtain Lie algebras associated to generalised Cartan matrices, known as Kac-Moody algebras. These include the finite dimensional simple Lie algebras, as well Lie algebras of affine type and of indefinite type (the latter is also referred to as wild type).

Kac [19] provides a general construction which works for all types, but the structure theory developed for a subclass of affine Lie algebras known as the non-twisted affine Lie algebras is of particular interest, as such Lie algebras turn out to be the affinization of finite dimensional simple Lie algebras.

In this chapter, we introduce non-twisted affine Lie algebras, primarily in terms of this affinization. See Kac [19] for a full account of the theory of affine Lie algebras.

Remark 12.1. It should be noted that we deviate from Kac [19] in using $\langle \alpha_i, \alpha_j \rangle$ for the Cartan integers, as opposed to $\langle \alpha_i^\vee, \alpha_j \rangle$, as we do not need to use the coroots $\alpha_i^\vee$ for anything else. This is harmless, as both take the same values. Specifically

$$\langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i|\alpha_j)}{(\alpha_j|\alpha_j)},$$

where $(-|-)$ is the invariant bilinear form $(-|-)$ on $\mathfrak{h}^*$ defined in 2.1 to 2.3 in [19].

This coincides with the finite dimensional case (see 9.1 in [11]), where the form is the Killing form.

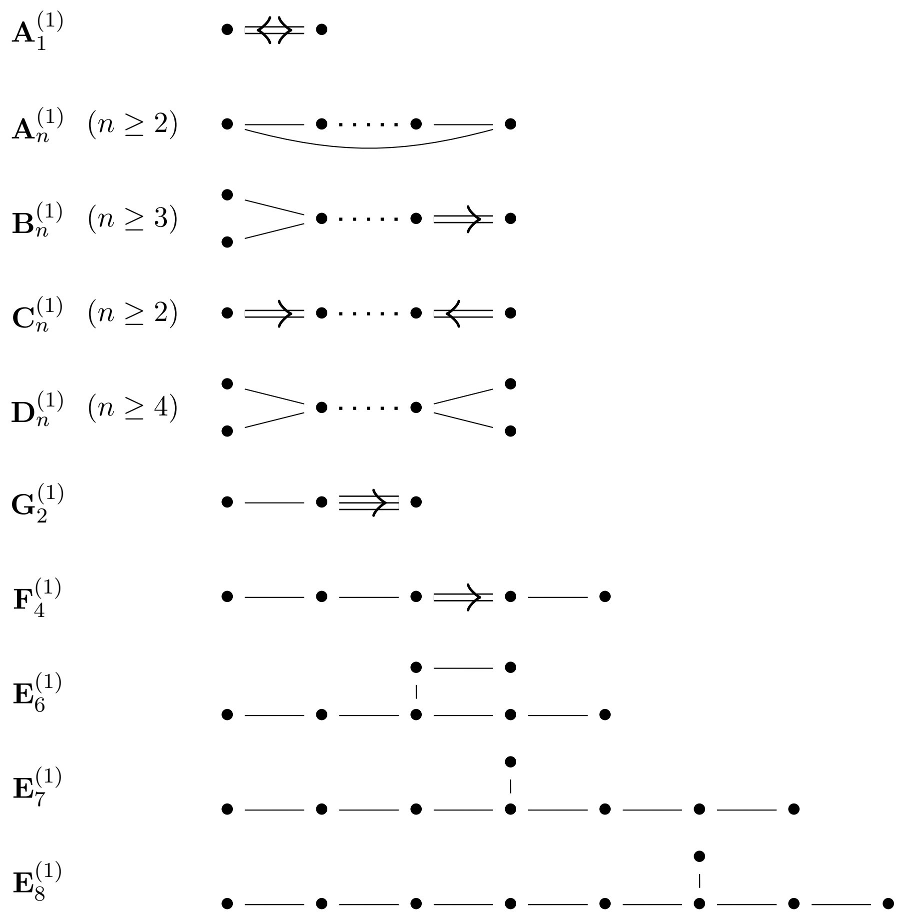


Figure 12-1: The non-twisted affine diagrams

Definition 12.2 (Section 7.2 and Table Aff 1 in [19])**.** A **non-twisted affine Lie algebra** is a Lie algebra $\tilde{\mathfrak{g}}$ associated to one of the diagrams in Figure 12-1, each of which has $n+1$ vertices. We say that $\tilde{\mathfrak{g}}$ is **of affine type $\mathbf{A}_n$, $\mathbf{B}_n$, etc.** when $\tilde{\mathfrak{g}}$ is a Lie algebra associated to the diagram $\mathbf{A}_n^{(1)}$, $\mathbf{B}_n^{(1)}$, etc. respectively.

Given a diagram, label the vertices $0, \cdots, n$ such that removing vertex 0 from the diagram leaves the corresponding finite type Dynkin diagram. The root system $\tilde{\Delta}$ of $\tilde{\mathfrak{g}}$ has simple roots $\alpha_0, \cdots, \alpha_n$ with Cartan integers are determined as follows.

- $\langle \alpha_i, \alpha_i \rangle = 2$ for all i.
- If there are no edges connecting i and j, then $\langle \alpha_i, \alpha_j \rangle = 0$.

- If there are $t > 0$ edges connecting i and j, then either
  - $\langle \alpha_i, \alpha_j \rangle = -t$ if there is an arrow pointing to i, or
  - $\langle \alpha_i, \alpha_j \rangle = -1$ otherwise.

This allows the whole root system $\tilde{\Delta}$, then the whole of $\tilde{\mathfrak{g}}$, to be constructed. For the sake of brevity, we skip the construction and go straight to the results of the structure theory.

Remark 12.3. Note that the diagram with a single vertex and one looping edge is excluded from the list. Such a diagram is generally considered to be an affine diagram, but it is not the diagram of any affine Lie algebra (it is a degenerate case in this context).

12.4. Due to the special choice of α_0, we have that $\alpha_1, \cdots, \alpha_n$ are the simple roots of the corresponding finite type root system Δ. Denote the highest root in Δ by θ. Then $\delta = \alpha_0 + \theta$ is a root in $\tilde{\Delta}$, called the **null root**. We have that $m\delta \in \tilde{\Delta}$ for all $m \neq 0$ and $\langle \delta, \gamma \rangle = 0$ for all $\gamma \in \tilde{\Delta}$.

Definition 12.5. For a finite dimensional Lie algebra $\mathfrak{g}$, the **affinization of** $\mathfrak{g}$, denoted $\tilde{\mathfrak{g}}$, is the Lie algebra

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes k[t,\, t^{-1}] \oplus kc \oplus kd$$

with Lie bracket

$$[x \otimes t^a, y \otimes t^b] = [x, y]_{\mathfrak{g}} \otimes t^{a+b} + a\delta_{a,-b}(x|y)c,$$

$$[d, x \otimes t^a] = ax \otimes t^a, \qquad [c, \tilde{\mathfrak{g}}] = 0,$$

for $x, y \in \mathfrak{g}$ and $a, b \in \mathbb{Z}$, where $(x|y)$ denotes the Killing form on $\mathfrak{g}$. For our purposes, all we need to know about the Killing form is given in Theorem 12.9. The element d is sometimes referred to as the **scaling element**.

The affinization of $\mathfrak{sl}_n(k)$ is denoted $\tilde{\mathfrak{sl}}_n(k)$ and the affinization of $\mathfrak{gl}_n(k)$ is denoted $\tilde{\mathfrak{gl}}_n(k)$.

Theorem 12.6 (Chapter 7 in [19])**.** *A non-twisted affine Lie algebra is the affinization of a simple, finite dimensional Lie algebra.*

Remark 12.7 (Chapter 1 in [19])**.** Note that the concept of Cartan subalgebra can carried over to the affine context, however in that context it is simply defined to be the complex vector space $\mathfrak{h}$ whose dual contains the simple roots (both $\mathfrak{h}$ and the simple roots are given in the realisation of a generalised Cartan matrix).

12.8. Fix a non-twisted affine Lie algebra $\tilde{\mathfrak{g}}$ and corresponding finite dimensional simple Lie algebra $\mathfrak{g}$ (that is, $\tilde{\mathfrak{g}}$ is the affinization of $\mathfrak{g}$). Additionally, fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ with corresponding set of roots Δ and simple roots $\alpha_1, \cdots, \alpha_n$. Then there is a Cartan subalgebra $\tilde{\mathfrak{h}}$ of $\tilde{\mathfrak{g}}$ given by $\tilde{\mathfrak{h}} = (\mathfrak{h} \otimes t^0) \oplus kc \oplus kd$. The set of roots $\tilde{\Delta}$ of $\tilde{\mathfrak{g}}$ with respect to $\tilde{\mathfrak{h}}$ is given by

$$\tilde{\Delta} = \{\alpha + m\delta : \alpha \in \Delta,\, m \in \mathbb{Z}\} \cup \{m\delta : m \in \mathbb{Z},\, m \neq 0\}.$$

The root spaces $\tilde{\mathfrak{g}}_\gamma$ of $\tilde{\mathfrak{g}}$, where $\gamma \in \tilde{\Delta}$, are related to the root spaces $\mathfrak{g}_\alpha$ of $\mathfrak{g}$, where $\alpha \in \Delta$, as follows.

- $\tilde{\mathfrak{g}}_{\alpha+m\delta} = \mathfrak{g}_\alpha \otimes t^m$ for $\alpha \in \Delta$ and $m \in \mathbb{Z}$.
- $\tilde{\mathfrak{g}}_{m\delta} = \mathfrak{h} \otimes t^m$ for nonzero $m \in \mathbb{Z}$.

Theorem 12.9 (Theorem 2.2 in [19])**.** *The form* $(-|-)$ *on* $\mathfrak{g}$ *satisfies*

1. $([x,\, y]|z) = (x|[y,\, z])$ *for all* $x, y, z \in \mathfrak{g}$.
2. $(\mathfrak{g}_\alpha|\mathfrak{g}_\beta) = 0$ *for all* $\alpha, \beta \in \Delta$ *such that* $\beta \neq -\alpha$.

Corollary 12.10. $(\mathfrak{g}_\alpha|\mathfrak{h}) = 0$ *for all* $\alpha \in \Delta$.

Proof. From the standard structure theory of finite dimensional Lie algebras, we know there are $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ for which the elements $[e_i,\, f_i]$ span $\mathfrak{h}$. For $x \in \mathfrak{g}_\alpha$, Theorem 12.9(1) implies $(x|\,[e_i,\, f_i]) = ([x,\, e_i]\,|f_i)$, where $[x,\, e_i] \in \mathfrak{g}_{\alpha+\alpha_i}$. Theorem 12.9(2) now implies $(x|\,[e_i,\, f_i]) = 0$. □

12.11. It is helpful to draw the following diagram, in which $\tilde{\Delta}$ is the disjoint union of all the components except zero.

	$\Delta^- + \mathbb{Z}^+\delta$	$\mathbb{Z}^+\delta$	$\Delta^+ + \mathbb{Z}^+\delta$
$\tilde{\Delta} =$	Δ^-	0	Δ^+
	$\Delta^- + \mathbb{Z}^-\delta$	$\mathbb{Z}^-\delta$	$\Delta^+ + \mathbb{Z}^-\delta$

Definition 12.12 (3.7 in [19])**.** The **Weyl group** of $\tilde{\mathfrak{g}}$, denoted $\mathcal{W}$, is the group generated by the **fundamental reflections** $r_0, \cdots, r_n$ of $\tilde{\mathfrak{h}}^*$, given by

$$r_i : \gamma \mapsto \gamma - \langle \gamma, \alpha_i \rangle \alpha_i.$$

Lemma 12.13 (5.1 in [19])**.** *For every root $\gamma \in \tilde{\Delta}$ of the form $\gamma = \alpha + m\delta$ for $\alpha \in \Delta$ and $m \in \mathbb{Z}$, the reflection*

$$r_\gamma : \gamma' \mapsto \gamma' - \left\langle \gamma', \gamma \right\rangle \gamma$$

lies in $\mathcal{W}$.

Lemma 12.14. *For all $\gamma \in \tilde{\Delta}$ and $m \in \mathbb{Z}$, the reflection r_γ fixes $m\delta$ and the reflection $r_{m\delta}$ fixes γ.*

Proof. This follows immediately the fact that $\langle \gamma, \delta \rangle = 0$ for all $\gamma \in \tilde{\Delta}$. □

Lemma 12.15. *Let $\alpha + m\delta \in \tilde{\Delta}$ for $\alpha \in \Delta$ and $m \in \mathbb{Z}$. Then*

$$r_{\alpha+m\delta} : \alpha' + m'\delta \mapsto r_\alpha(\alpha') + (m' - m \left\langle \alpha', \alpha \right\rangle)\delta, \qquad (\alpha' \in \Delta, m' \in \mathbb{Z}).$$

Proof. As $\langle \gamma, \delta \rangle = 0$ for all $\gamma \in \tilde{\Delta}$, we have

$$\begin{aligned}(\alpha' + m'\delta) - \left\langle \alpha' + m'\delta, \alpha + m\delta \right\rangle (\alpha + m\delta) &= (\alpha' + m'\delta) - \left\langle \alpha', \alpha \right\rangle (\alpha + m\delta) \\ &= (\alpha' - \left\langle \alpha', \alpha \right\rangle \alpha) + (m' + m \left\langle \alpha', \alpha \right\rangle)\delta \\ &= r_\alpha(\alpha') + (m' + m \left\langle \alpha', \alpha \right\rangle)\delta,\end{aligned}$$

from which the result follows. □

Chapter 13

Partitions and Borel Subalgebras

We begin with the necessary discussion of how to define Borel and parabolic subalgebras of affine Lie algebras. We then recall Futorny's classification of partitions of affine root systems [5], which gives the classification of affine Borel subalgebras.

Definition 13.1 (Section 2 in [6])**.** For convenience of notation, we define the sum of two sets of roots $R, R' \subseteq \Delta$ to be $R + R' = \{\alpha + \alpha' : \alpha \in R,\ \alpha' \in R'\} \cap \Delta$.

A subset $R \subseteq \Delta$ is called **additively closed** (or just **closed**) if $R + R \subseteq R$. The **additive closure** of a subset $R \subseteq \Delta$, denoted $\mathrm{add}(R)$ is the smallest additively closed subset of Δ containing R. If R is additively closed, the **subalgebra generated by** R is

$$\mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

The requirement that R is additively closed implies that this is indeed a subalgebra.

A closed subset $P \subseteq \Delta$ such that $P \cup -P = \Delta$ is called **parabolic**. If additionally $P \cap -P = \emptyset$, then P is called a **partition** (or sometimes **parabolic partition**).

The above definitions also apply to $\tilde{\Delta}$ and $\tilde{\mathfrak{g}}$. In particular, as Δ is a root subsystem of $\tilde{\Delta}$, any subset of Δ is closed in Δ if and only if it is closed in $\tilde{\Delta}$.

Definition 13.2 (Section 2 in [6])**.** A **Borel subalgebra** of a Lie algebra is a subalgebra generated by a partition of the root system. A **parabolic subalgebra** is a subalgebra generated by a parabolic set of roots.

13.3. The traditional definition of a Borel subalgebra $\mathfrak{b}$ of a finite dimensional Lie algebra is a maximal solvable subalgebra (see [11] and [27], for example). This turns out to be equivalent to $\mathfrak{b}$ being a subalgebra generated by a partition (29.4.6 in [27]), so it is justified to use the above definition of Borel subalgebra instead of the traditional one. In fact, it is necessary to do so, because the notion of solvability becomes somewhat pathological when applied to infinite dimensional Lie algebras.

The traditional definition of a parabolic subalgebra $\mathfrak{p}$ of a finite dimensional Lie algebra is one which contains a Borel subalgebra. If this is the case, then $\mathfrak{p}$ corresponds to a closed set of roots P which contains the partition B corresponding to the Borel. Thus $P \cup -P$ contains $B \cup -B = \Delta$, which means P is a parabolic set of roots. The reverse – that a parabolic set of roots contains a partition – also turns out to be true (Proposition 18.10.5 in [27]). Therefore, the traditional definition of parabolic subalgebra is equivalent to the above. We shall see that it is also equivalent in the affine case (Proposition 15.5).

See [4] for further discussion on the definition of parabolic subalgebra and parabolic set of roots.

Definition 13.4. The **nilpotent radical** of a parabolic subalgebra $\mathfrak{p}$ generated by a parabolic set of roots P is

$$\mathfrak{n} = \bigoplus_{\alpha \in N} \mathfrak{g}_\alpha,$$

where $N = P \setminus -P$.

13.5. Similarly to Borel and parabolic subalgebras, the traditional definition of the nilpotent radical of a Lie algebra (see 19.6.1 in [27]) is pathological when applied to infinite dimensional Lie algebras. The traditional definition leads to a description in terms of sets of roots (see Theorem 20.8.6(iii) in [27]), which can be applied equally well in both finite dimensional and affine case. We therefore take this description as our definition.

This does create the unfortunate situation that *the nilpotent radical is not always nilpotent*, though we only look at cases where the nilpotent radical is nilpotent in this thesis. One could perhaps consider generalising the notion of nilpotency of a subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ from

$$\exists n \in \mathbb{N} : \mathrm{ad}_{\mathfrak{g}}(\mathfrak{n})^n(\mathfrak{n}) = 0$$

to

$$\forall x \in \mathfrak{n} \setminus \{0\} : \exists n \in \mathbb{N} : x \notin \mathrm{ad}_{\mathfrak{g}}(\mathfrak{n})^n(\mathfrak{n}),$$

which is equivalent to nilpotency if $\mathfrak{n}$ is finite dimensional.

Definition 13.6 (28.0 in [27])**.** Let G be a connected algebraic group with Lie algebra

$\mathfrak{g}$. Two elements $x, y \in \mathfrak{g}$ are **conjugate** if $\mathrm{Ad}(g)(x) = y$ for some $g \in G$. Note that $\mathrm{Ad}(g)$ is an automorphism of $\mathfrak{g}$.

Definition 13.7 (Section 2 in [5])**.** Two subsets of $\tilde{\Delta}$ will be called **equivalent** if they belong to the same orbit of $\mathcal{W} \times \{\pm 1\}$. Note that the ± 1 is required for a sensible notion of equivalence, as no element of $\mathcal{W}$ can send $+\delta$ to $-\delta$.

Remark 13.8. Presently, we are interested in subalgebras which are generated by closed sets of roots and classifying them up to equivalence of those sets of roots.

We would like this to be equivalent to classifying the corresponding subalgebras up to conjugation. This requires two things. Firstly, as there is an implicit choice of Cartan subalgebra when dealing with roots, we need all Cartan subalgebras to be conjugate. Secondly, we need the action of each Weyl group element on the root system to correspond to the automorphism action of some element of the adjoint group which stabilizes the Cartan subalgebra.

The first point is known to be true for both finite dimensional semisimple Lie algebras and affine Lie algebras (Theorem 2 in [20]). The second point is known to be true for finite dimensional semisimple Lie algebras (Lemma 29.1.3 in [27]), however we do not know whether it also holds for affine Lie algebras.

Remark 13.9 (29.4.3 in [27])**.** The Borel subalgebras of a finite dimensional Lie algebra are all conjugate. Consequently, we need only consider the Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ generated by the partition Δ^+. For an affine Lie algebra, this is not the case.

Theorem 13.10 (Proposition 2.3 and Theorem 2.4 in [5])**.** *The partitions of $\tilde{\Delta}$ are precisely the sets equivalent to*

$$P(X) = \left\{\gamma \in \tilde{\Delta} : \varphi_X(\gamma) > 0\right\} \cup \left\{\gamma \in \tilde{\Delta} : \varphi_X(\gamma) = 0,\, \varphi_I(\gamma) > 0\right\},$$

for some $X \subseteq I = \{1, \cdots, n\}$, where

$$\varphi_X = \sum_{i \in I \setminus X} \alpha_i^* - \left(\sum_{i \in I \setminus X} k_i\right) \alpha_0^* \qquad \text{if } X \neq I$$

and

$$\varphi_I = \sum_{i=0}^{n} \alpha_i^*,$$

where $\alpha_i^*(\alpha_j) = \delta_{i,j}$ and k_i is the coefficient of α_i in δ.

Corollary 13.11. *Let $X \subseteq \{1, \cdots, n\}$. Then $P(X) = \mathcal{B}(P)$, where*

$$\mathcal{B}(P) = \{\alpha + m\delta : \alpha \in N,\, m < 0\} \cup \Delta^+ \cup \{\alpha + m\delta : \alpha \in P,\, m > 0\} \cup \{m\delta : m > 0\}\,,$$

for some $P = \Delta^+ \cup \operatorname{add}(\{-\alpha_i : i \in X\})$ and $N = P \setminus -P$.

Consequently, the partitions of $\tilde{\Delta}$ are precisely the sets equivalent to $\mathcal{B}(P)$ for some standard parabolic subset P of Δ.

Proof. Write $R = \{\alpha_i : i \in X\}$. Firstly, $\alpha_i^*(\delta) = k_i$ and $\alpha_0^*(\delta) = k_0 = 1$, so $\varphi_X(\delta) = 0$. Further, $\varphi_I(\delta) = \sum_{i=0}^n k_i > 0$, as each $k_i > 0$. So $m\delta \in P(X)$ if and only if $m > 0$.

For $\alpha \in \Delta$, we have $\alpha_0^*(\alpha) = 0$, hence

$$\varphi_X(\alpha) = \sum_{i \in I \setminus X} \alpha_i^*(\alpha).$$

So $\varphi_X(\alpha) = 0$ if $\alpha \in \operatorname{add}(R) \cup \operatorname{add}(-R)$. Otherwise, $\varphi_X(\alpha) > 0$ if $\alpha \in \Delta^+$ and $\varphi_X(\alpha) < 0$ if $\alpha \in \Delta^-$. Further, $\varphi_I(\alpha) = \sum_{i=0}^n \alpha_i^*(\alpha) > 0$ if and only if $\alpha \in \Delta^+$. Therefore, $\alpha \in P(X)$ if and only if $\alpha \in \Delta^+$.

As $\varphi_X(\delta) = 0$, we have $\varphi_X(\alpha + m\delta) = \varphi(\alpha)$. On the other hand,

$$\varphi_I(\alpha + m\delta) = \sum_{i=0}^n (\alpha_i^*(\alpha) + mk_i)\,.$$

We have that $k_i \geq \alpha_i^*(\alpha) \geq -k_i$ for $i = 1, \cdots, n$ and $k_0 > \alpha_0^*(\alpha) > -k_0$, hence $\varphi_I(\alpha + m\delta)$ is positive if $m > 0$ or negative if $m < 0$. Consequently,

- if $m > 0$, then $\alpha + m\delta \in P(X)$ if and only if $\varphi_X(\alpha + m\delta) \geq 0$,
- if $m < 0$, then $\alpha + m\delta \in P(X)$ if and only if $\varphi_X(\alpha + m\delta) > 0$.

That is, $\alpha + m\delta \in P(X)$ if and only if

- $m > 0$ and $\alpha \in \Delta^+ \cup \operatorname{add}(-R) = P$, or
- $m < 0$ and $\alpha \in \Delta^+ \setminus \operatorname{add}(R) = P \setminus -P$.

Therefore, $P(X) = \mathcal{B}(P)$. □

13.12. The expression for the partitions of $\tilde{\Delta}$ in Corollary 13.11 translates directly into the following expression for Borel subalgebras of $\tilde{\mathfrak{g}}$. The case $P = \Delta$ will turn out to be the case which behaves most like a partition of a finite dimensional Lie algebra - it is this case which we will focus on in the upcoming chapters.

Proposition 13.13 (Section 2 in [6])**.** *Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. For any Borel subalgebra $\mathfrak{B}$ of $\tilde{\mathfrak{g}}$, there exists a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$, containing $\mathfrak{b}$, such that $\mathfrak{B}$ is conjugate to*

$$\mathfrak{B}(\mathfrak{p}) = \left(\bigoplus_{n<0} \mathfrak{n} \otimes t^n\right) \oplus (\mathfrak{b} \otimes 1) \oplus \left(\bigoplus_{n>0} \mathfrak{p} \otimes t^n\right) \oplus kc \oplus kd,$$

where $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ is the Levi decomposition of $\mathfrak{p}$.

Definition 13.14 (Section 2 in [6])**.** Any Borel conjugate to $\mathfrak{B}(\mathfrak{g})$ is called **standard**. Any Borel conjugate to $\mathfrak{B}(\mathfrak{b})$ is called **natural**.

Considered as a parabolic subalgebra, $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{n}$ for $\mathfrak{l} = \mathfrak{g}$ and $\mathfrak{n} = 0$. Therefore, we can express the standard Borel of $\tilde{\mathfrak{g}}$ as

$$\mathfrak{B}(\mathfrak{g}) = (\mathfrak{b} \otimes 1) \oplus \left(\bigoplus_{n>0} \mathfrak{g} \otimes t^n\right) \oplus kc \oplus kd.$$

This corresponds to the partition

$$\mathcal{B}(\Delta) = \Delta^+ \cup \{\alpha + n\delta : \alpha \in \Delta,\, n > 0\} \cup \{n\delta : n > 0\},$$

which we can relate to our diagram for $\tilde{\Delta}$ as follows.

	$\Delta^- + \mathbb{Z}^+\delta$	$\mathbb{Z}^+\delta$	$\Delta^+ + \mathbb{Z}^+\delta$
$\mathcal{B}(\Delta) =$			Δ^+

Now considering $\mathfrak{b}$ as a parabolic subalgebra, we have $\mathfrak{b} = \mathfrak{l} \oplus \mathfrak{n}$ for $\mathfrak{l} = \mathfrak{h}$ and $\mathfrak{n}$ the sum of all $\mathfrak{g}_\alpha$ for $\alpha \in \Delta^+$. We can therefore express the natural Borel of $\tilde{\mathfrak{g}}$ as

$$\begin{aligned}\mathfrak{B}(\mathfrak{b}) &= \left(\bigoplus_{n<0} \mathfrak{n} \otimes t^n\right) \oplus (\mathfrak{b} \otimes 1) \oplus \left(\bigoplus_{n>0} \mathfrak{b} \otimes t^n\right) \oplus kc \oplus kd. \\ &= \left(\bigoplus_{n\in\mathbb{Z}} \mathfrak{n} \otimes t^n\right) \oplus \left(\bigoplus_{n\geq 0} \mathfrak{h} \otimes t^n\right) \oplus kc \oplus kd.\end{aligned}$$

This corresponds to the partition

$$\mathcal{B}(\Delta^+) = \left\{\alpha + n\delta : \alpha \in \Delta^+,\, n \in \mathbb{Z}\right\} \cup \{n\delta : n > 0\},$$

which we can relate to our diagram for $\tilde{\Delta}$ as follows.

$\mathcal{B}(\Delta^+) =$

	$\mathbb{Z}^+\delta$	$\Delta^+ + \mathbb{Z}^+\delta$
		Δ^+
		$\Delta^+ + \mathbb{Z}^-\delta$

Chapter 14

Standard Parabolic Sets of Roots

We show that the standard partition $\mathcal{B}(\Delta)$ is the additive closure of the positive simple roots (Corollary 14.9), mirroring the behaviour of partitions in the finite dimensional case. We also see that a non-standard partition cannot be expressed as the additive closure of a finite set (Proposition 14.10).

We then show that any parabolic set of roots containing $\mathcal{B}(\Delta)$ is the additive closure of the positive simple roots and some subset of the negative simple roots (Proposition 14.13), again mirroring the behaviour in the finite dimensional case.

On the way, we need some results about both the affine type root system $\tilde{\Delta}$ and the root system Δ of the simple Lie algebra $\mathfrak{g}$. In particular, we observe that the highest root θ in Δ is the only locally highest root, in the sense that all other roots $\alpha \in \Delta^+$ can be extended to a higher root $\alpha + \alpha_i$ for some i (Proposition 14.6). We also see that the standard result that each $\alpha \in \Delta^+$ is a sum of simple roots for which each partial sum is itself a root (Lemma 14.3) also holds for each $\gamma \in \mathcal{B}(\Delta)$ (Proposition 14.8).

We begin by recalling some standard results about the root system Δ.

Lemma 14.1 (Lemma 9.4 in [11])**.** *Let $\alpha, \beta \in \Delta$ such that $\beta \neq \pm\alpha$. Then*

- $(\alpha, \beta) > 0 \implies \alpha - \beta \in \Delta$,
- $(\alpha, \beta) < 0 \implies \alpha + \beta \in \Delta$.

Lemma 14.2 (Lemma 10.1 in [11])**.** *If $i \neq j$, then $(\alpha_i, \alpha_j) \leq 0$ and $\alpha_i - \alpha_j \notin \Delta$.*

Lemma 14.3 (Corollary 10.2 in [11])**.** *Any root $\alpha \in \Delta^+$ can be written as $\alpha = \beta_1 +$*

$\cdots + \beta_r$ *for some positive simple roots* β_i*, where each partial sum* $\beta_1 + \cdots + \beta_i$ *is a positive root.*

Lemma 14.4 (Proposition 14.1 in [11])**.** Δ *is irreducible.*

Lemma 14.5 (Compare to Lemma 10.4 A in [11])**.** *Let* $\alpha \in \Delta^+$ *such that* $\alpha + \alpha_i \notin \Delta$ *for all* $i = 1, \cdots, n$*. Write*

$$\alpha = \sum_{i=1}^{n} k_i \alpha_i.$$

Then $k_i > 0$ *and* $(\alpha, \alpha_i) \geq 0$ *for all* i*, and* $(\alpha, \alpha_j) > 0$ *for some* j*.*

Proof. Firstly, Lemma 14.1 implies that $(\alpha, \alpha_i) \geq 0$ for all i.

Suppose that there exists an s for which $k_s = 0$. Then, without loss of generality, we may assume that $k_i > 0$ for $i = 1, \cdots, s-1$ and $k_i = 0$ for $i = s, \cdots, n$. Then we have

$$0 \leq (\alpha, \alpha_i) = \sum_{j=1}^{s-1} k_j(\alpha_j, \alpha_i) \leq 0$$

for all $i = s, \cdots, n$, as $(\alpha_j, \alpha_i) \leq 0$ for $j \neq i$ by Lemma 14.2. This implies $(\alpha_j, \alpha_i) = 0$ for all $j = 1, \cdots, s-1$ and $i = s, \cdots, n$, which contradicts the irreducibility of Δ (Lemma 14.4). Therefore, no s exists such that $k_s = 0$. That is, $k_i > 0$ for all $i = 1, \cdots, n$.

Finally, suppose $(\alpha, \alpha_i) = 0$ for all $i = 1, \cdots, n$. Then

$$(\alpha, \alpha) = \sum_{i=1}^{n} k_i(\alpha, \alpha_i) = 0,$$

but $\alpha \neq 0$, so this is a contradiction. Therefore, $(\alpha, \alpha_i) > 0$ for some i. □

Proposition 14.6. $\alpha \in \Delta^+$ *satisfies* $\alpha + \alpha_i \notin \Delta$ *for all* i *if and only if* $\alpha = \theta$*.*

Proof. As θ is the highest root, it satisfies the condition. Suppose $\alpha \in \Delta^+$, $\alpha \neq \theta$, also satisfies the condition. We can write $\alpha = \theta - \beta_1 - \cdots - \beta_r$ for some simple roots $\beta_1, \cdots, \beta_r$ and $r \geq 2$ (we can exclude $r = 1$ as then $\alpha + \beta_1 = \theta$). By the supposition, $\theta - (\beta_1 + \cdots + \beta_{r-1}) = \alpha + \beta_r$ is not a root; neither is $\theta - (\beta_2 + \cdots + \beta_r) = \alpha + \beta_1$. Lemma 14.1 therefore implies that

$$(\theta, \beta_1) + \cdots + (\theta, \beta_{r-1}) = (\theta, \beta_1 + \cdots + \beta_{r-1}) \leq 0,$$

$$(\theta, \beta_2) + \cdots + (\theta, \beta_r) = (\theta, \beta_2, \cdots, \beta_r) \leq 0.$$

But $(\theta, \beta_i) \geq 0$ for all $i = 1, \cdots, r$ by Lemma 14.5, so the above implies all $(\theta, \beta_i) = 0$. Let $\gamma = \beta_1 + \cdots + \beta_r$. Then we have $(\theta, \gamma) = (\theta, \beta_1) + \cdots + (\theta, \beta_r) = 0$, which implies that $\sigma_\gamma(\alpha) = \sigma_\gamma(\theta - \gamma) = \theta + \gamma$. But this implies $\theta + \gamma$ is a root, which is a contradiction. □

Corollary 14.7. *Let $\alpha \in \Delta^+$. Then there exist some positive simple roots $\beta_1, \cdots, \beta_r$ such that $\alpha + \beta_1 + \cdots + \beta_i \in \Delta^+$ for all $i = 1, \cdots, r$, where $\alpha + \beta_1 + \cdots + \beta_r = \theta$.*

Equivalently, for $-\alpha \in \Delta^-$, there exist positive simple roots $\beta_1, \cdots, \beta_r$ such that $-\theta + \beta_1 + \cdots + \beta_i \in \Delta^-$ for all $i = 1, \cdots, r$, where $-\theta + \beta_1 + \cdots + \beta_r = -\alpha$.

Proof. Proposition 14.6 implies each subsequent β_i exists until $\alpha + \beta_1 + \cdots + \beta_r = \theta$. □

Proposition 14.8. *Let $\gamma \in \mathcal{B}(\Delta)$. Then there exist $\beta_1, \cdots, \beta_r \in \{\alpha_0, \cdots, \alpha_n\}$ such that $\gamma = \beta_1 + \cdots + \beta_r$ and each partial sum $\beta_1 + \cdots + \beta_i$ lies in $\mathcal{B}(\Delta)$.*

Proof. If the result holds for some $\gamma' \in \mathcal{B}(\Delta)$, then the result also holds for $\gamma \in \mathcal{B}(\Delta)$ if there exist $\beta_1, \cdots, \beta_r \in \{\alpha_0, \cdots, \alpha_n\}$ such that $\gamma = \gamma' + \beta_1 + \cdots + \beta_r$ and each partial sum $\gamma' + \beta_1 + \cdots + \beta_i$ lies in $\mathcal{B}(\Delta)$.

Recall from Definition 13.14 that

$$\mathcal{B}(\Delta) = \Delta^+ \cup \{\alpha + m\delta : \alpha \in \Delta, \, m > 0\} \cup \{m\delta : m > 0\}.$$

The result holds for all $\alpha \in \Delta^+$ by Lemma 14.3. In particular, the result holds for θ. As $\alpha_0 = -\theta + \delta$, the result also holds for $\theta + \alpha_0 = \delta$.

Suppose the result holds for $m\delta$ and $-\theta + m\delta$ for some $m > 0$. Then Lemma 14.3 implies it holds for $\alpha + m\delta$ for all $\alpha \in \Delta^+$, hence also for $(\theta + m\delta) + \alpha_0 = (m+1)\delta$. Additionally, Corollary 14.7 implies it holds for $-\alpha + m\delta$ for all $-\alpha \in \Delta^-$.

Therefore, by induction on m, the result holds for $\alpha + m\delta$ and $m\delta$ for all $\alpha \in \Delta$ and $m > 0$. □

Corollary 14.9. $\mathcal{B}(\Delta) = \operatorname{add}\{\alpha_0, \cdots, \alpha_n\}$.

Proof. Follows immediately from Proposition 14.8. □

Proposition 14.10. *Let $\mathcal{B}$ be a non-standard partition. Then $\mathcal{B}$ cannot be expressed as the additive closure of a finite set.*

Proof. Firstly, $\mathcal{B} = \mathcal{B}(P)$ for some parabolic subset $P \subset \Delta$. Write $\mathfrak{p}$ for the corresponding parabolic Lie algebra, $\mathfrak{n}$ for the radical of $\mathfrak{p}$ and $N = P \setminus -P$ for the set of roots corresponding to $\mathfrak{n}$. By Corollary 13.11, we can write

$$\mathcal{B} = \{\alpha + m\delta : \alpha \in N, \, m < 0\} \cup \Delta^+ \cup \{\alpha + m\delta : \alpha \in P, \, m > 0\} \cup \{m\delta : m > 0\} .$$

Suppose $\mathcal{B} = \operatorname{add} R$ for some finite subset $R \subseteq \mathcal{B}$. As $P \neq \Delta$, we have $N \neq \emptyset$, hence the set $X = \{\alpha + m\delta : \alpha \in N, \, m < 0\} \subset \mathcal{B}$ is nonempty. As X contains all the roots in $\mathcal{B}$ with a negative multiple of δ, we must have that R intersects X nontrivially (otherwise $\operatorname{add} R$ cannot contain any roots of that form). Let r be some integer greater than $\max\{m \in \mathbb{Z} : \exists \alpha \in N : \alpha - m\delta \in R\}$, which exists because R is finite.

As $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{p}$, there exists some $s > 0$ such that $\operatorname{ad}(\mathfrak{n})^s(\mathfrak{n}) = 0$. That is, $N + \cdots (s \text{ times}) \cdots + N = \emptyset$. Therefore, $\operatorname{add} R$ does not contain any roots $\alpha - m\delta$ for $m > sr$. But such roots exist in $\mathcal{B}$, so this implies $\operatorname{add} R \neq \mathcal{B}$. ☐

Definition 14.11. Let $\gamma \in \mathcal{B}(\Delta)$. We can write

$$\gamma = \sum_{i=0}^{n} k_i \alpha_i,$$

for some $k_i \in \mathbb{Z}$. Call the simple roots α_i for which $k_i \neq 0$ the **simple composition factors** of α. The **simple composition factors** of $-\gamma$ are the negative simple roots $-\alpha_i$ for which $k_i \neq 0$.

Lemma 14.12. *Let C be the set of simple composition factors of some root $\gamma \in \mathcal{B}(\Delta)$. Then $-C$ is contained in the additive closure of $C \cup \{-\gamma\}$.*

Proof. Write $\gamma = \beta_1 + \cdots + \beta_r$ as in Proposition 14.8, so that each $\beta_1 + \cdots + \beta_i \in \mathcal{B}(\Delta)$. Then $C = \{\beta_1, \cdots, \beta_r\}$, so $\operatorname{add}(C \cup \{-\gamma\})$ contains

$$-\gamma + \beta_r = -(\beta_1 + \cdots + \beta_{r-1}),$$

which implies $\operatorname{add}(C \cup \{-\gamma\})$ also contains

$$-\gamma + \beta_r + \beta_{r-1} = -(\beta_1 + \cdots + \beta_{r-2}),$$

and so on, until

$$-\gamma + \beta_r + \cdots + \beta_2 = -(\beta_1).$$

That is, $\text{add}(C \cup \{-\gamma\})$ contains

$$-\gamma + \beta_r + \cdots + \beta_{i+1} = -(\beta_1 + \cdots + \beta_i)$$

for all $i = r, \cdots, 1$. Additionally, each $\beta_1 + \cdots + \beta_i \in \text{add}(C)$. Therefore,

$$-(\beta_1 + \cdots + \beta_i) + (\beta_1 + \cdots + \beta_{i-1}) = -\beta_i.$$

is contained in $\text{add}(C \cup \{-\gamma\})$ for all $i = r, \cdots, 1$. □

Proposition 14.13. *The parabolic sets of roots containing $\mathcal{B}(\Delta)$ are precisely those of the form*

$$\mathcal{P} = \mathcal{B}(\Delta) \cup \text{add}(-R)$$

for some $R \subseteq \{\alpha_0, \cdots, \alpha_n\}$.

Proof. As $\tilde{\Delta}$ is the disjoint union of $\mathcal{B}(\Delta)$ and $-\mathcal{B}(\Delta)$, we have that $\mathcal{P}$ is a disjoint union $\mathcal{P} = \mathcal{B}(\Delta) \cup (-X)$ for some $X \subseteq \mathcal{B}(\Delta)$. For each $\gamma \in X$, let C_γ be the set of simple composition factors of γ. Let R be the union of all the C_γ. By Lemma 14.12, each $-C_\gamma$ is contained in the additive closure of $\mathcal{B}(\Delta) \cup \{-\gamma\}$. Therefore, $-R$ is contained in the additive closure of $\mathcal{B}(\Delta) \cup (-X) = \mathcal{P}$. As $\mathcal{P}$ is additively closed, this implies firstly that $-R \subseteq \mathcal{P}$ and secondly that $\text{add}(-R) \subseteq \mathcal{P}$. As $-R$ contains only negative roots, $-R \cap \mathcal{B}(\Delta) = \emptyset$, so we have $\text{add}(-R) \subseteq -X$ (equivalently, $\text{add}\, R \subseteq X$). But R contains all the simple composition factors of all the roots in X, hence $X \subseteq \text{add}\, R$. Therefore, $X = \text{add}\, R$, so $\mathcal{P}$ is of the desired form.

Finally, to show all such sets are parabolic, we need only show they are additively closed, as they contain $\mathcal{B}(\Delta)$ by definition. Let $R \subseteq \{\alpha_0, \cdots, \alpha_n\}$ and $\mathcal{P} = \mathcal{B}(\Delta) \cup \text{add}(-R)$. Suppose $\mathcal{P}$ is not additively closed. Then there exists a root of the form $\alpha - \beta \notin \mathcal{P}$ for $\alpha \in \mathcal{B}(\Delta)$ and $\beta \in \text{add}(R)$. This must lie in $-\mathcal{B}(\Delta)$, which implies its simple composition factors are a subset of the simple composition factors of $-\beta$, which themselves are contained in $-R$, hence $\alpha - \beta \in \text{add}(-R)$. This is a contradiction, hence $\mathcal{P}$ is additively closed. □

Chapter 15

Parabolic Sets of Roots

We recall Futorny's classification of parabolic sets of roots [5], using what we have learned in the previous chapter to express this in a simpler form. We discuss the different types of parabolic subset, and the relationship between parabolic sets of roots and partitions.

Theorem 15.1 (Theorem 2.5 and 2.6 in [5]). *Any parabolic subset of $\tilde{\Delta}$ not equal to a partition is equivalent to either*

$$P(X,Y) = P(X) \cup \mathrm{add}(\{-\alpha_i : i \in Y\})$$

for some $\emptyset \neq Y \subseteq X \subseteq I = \{1, \cdots, n\}$, or to

$$P_X(I) = \mathrm{add}(P(I) \cup \{-\alpha_i : i \in X\} \cup \{-\alpha_0\})$$

for some $X \subseteq I$, or to

$$P_X = \mathrm{add}(P(\emptyset) \cup \{-\delta\} \cup \{-\alpha_i : i \in X\})$$

for some $X \subseteq I$.

Example 15.2. $P(X, \emptyset) = P(X)$ and $P_I(I) = \tilde{\Delta}$. Note also that for $X \neq I$, we have $-\alpha_0 \in P(X)$.

Corollary 15.3. *Any parabolic subset of $\tilde{\Delta}$ is equivalent to one of the following.*

1. *$\mathcal{B}(\Delta) \cup \mathrm{add}(-R)$ for $R \subseteq \{\alpha_0, \cdots, \alpha_n\}$.*

2. $\mathcal{B}(P) \cup \operatorname{add}(-R)$ *for* $P = \Delta^+ \cup \operatorname{add}(-R')$ *and* $R \subseteq R' \subseteq \{\alpha_1, \cdots, \alpha_n\}$.

3. $(P + \mathbb{Z}\delta) \cup \{m\delta : m \in \mathbb{Z},\, m \neq 0\}$, *where* $P = \Delta^+ \cup \operatorname{add}(-R)$ *for some* $R \subseteq \{\alpha_1, \cdots, \alpha_n\}$. *This contains* $\mathcal{B}(\Delta^+)$.

Proof. For $Y \subseteq X \subseteq I = \{1, \cdots, n\}$, let $R = \{\alpha_i : i \in Y\}$ and $P = \Delta^+ \cup \operatorname{add}(-R')$ for $R' = \{\alpha_i : i \in X\}$. Then $P(X) = \mathcal{B}(P)$ by Corollary 13.11. Therefore,

$$P(X, Y) = \mathcal{B}(P) \cup \operatorname{add}(-R).$$

In the case when $X = I$, then $P = \Delta$, hence

$$P(I, Y) = \mathcal{B}(\Delta) \cup \operatorname{add}(-R).$$

Next, for $X \subseteq I$, let $R = \{\alpha_i : i \in X\} \cup \{\alpha_0\}$. Corollary 13.11 implies $P(I) = \mathcal{B}(\Delta)$. Now Proposition 14.13 implies

$$P_X(I) = \operatorname{add}(P(I) \cup \{-\alpha_i : i \in X\} \cup \{-\alpha_0\}) = \mathcal{B}(\Delta) \cup \operatorname{add}(-R).$$

Next, for $X \subseteq I$, let $R = \{\alpha_i : i \in X\}$. We have $m\delta \in P_X$ for all nonzero $m \in \mathbb{Z}$, so whenever $\alpha + m\delta \in P_X$ for some $m \in \mathbb{Z}$, we have $\alpha + m\delta \in P_X$ for all $m \in \mathbb{Z}$. Therefore, both $\operatorname{add}(-R) \subseteq P_X$ and $\Delta^+ \subseteq P_X$. Consequently,

$$P_X = (P + \mathbb{Z}\delta) \cup \{m\delta : m \in \mathbb{Z},\, m \neq 0\}.$$

□

Definition 15.4 (Section 2 in [6])**.** Parabolic sets of form 1 in Corollary 15.3 are known as **type Ia** or as **standard** type. Parabolic sets of form 2 are known as **type Ib**. Those of form 3 are known as **type II**; we refer to them as **natural** type.

Proposition 15.5. *Parabolic subsets are precisely those subsets which are closed and contain a partition.*

Proof. Suppose $\mathcal{P}$ is a closed subset containing a partition $\mathcal{B}$. As $\mathcal{B} \cup -\mathcal{B} = \tilde{\Delta}$, we have $\mathcal{P} \cup -\mathcal{P} = \tilde{\Delta}$, hence $\mathcal{P}$ is parabolic. On the other hand, Corollary 15.3 implies that any parabolic subset contains a partition. □

Remark 15.6. The connection between the types of parabolic subsets and the types of partitions is not completely straightforward. Firstly, there is the trivial example of $\mathcal{P} = \tilde{\Delta}$, which contains every partition and can be expressed as both standard type and natural type. In addition to this, any natural type parabolic subset $\mathcal{P} = (P + \mathbb{Z}\delta) \cup \{m\delta : m \in \mathbb{Z},\, m \neq 0\}$ contains both the partitions $\mathcal{B}(P)$ and $\mathcal{B}(\Delta^+)$. The partition $\mathcal{B}(\Delta^+)$ itself - which corresponds to the natural Borel - is a type Ib parabolic subset, not a natural parabolic subset.

Example 15.7. Let $\mathfrak{g} = \mathfrak{sl}_3$, so that $\Delta^+ = \{\alpha_1, \alpha_2, \alpha_{12}\}$, $\alpha_{12} = \alpha_1 + \alpha_2$. Consider the set

$$\mathcal{P} = P(\{1\}, \emptyset) \cup \{\alpha_1 - \delta\} = \mathcal{B}(P) \cup \{\alpha_1 - \delta\},$$

for $P = \Delta^+ \cup \{-\alpha_1\}$ (so $N = P \setminus -P = \{\alpha_2, \alpha_{12}\}$). Recall that

$$\mathcal{B}(P) = \{\alpha + m\delta : \alpha \in N,\, m < 0\} \cup \Delta^+ \cup \{\alpha + m\delta : \alpha \in P,\, m > 0\} \cup \{m\delta : m > 0\}.$$

As $\mathcal{B}(P)$ is additively closed and $(\alpha_1 - \delta) + (\alpha_1 - \delta)$ is not a root, we need only show $\mathcal{B}(P) + (\alpha_1 - \delta) \subseteq \mathcal{B}(P) \cup \{\alpha_1 - \delta\}$ for $\mathcal{P}$ to be a parabolic subset.

- For $m < 0$ and $\alpha \in N = \{\alpha_2, \alpha_{12}\}$, we have that $(\alpha_1 - \delta) + \alpha + m\delta = (\alpha_1 + \alpha) + (m-1)\delta$, which lies in $\mathcal{B}(P)$ because $m - 1 < 0$ and $\alpha_1 + \alpha$ is either α_{12} or not a root (and is never zero).
- For $\alpha \in \Delta^+$, we have that $(\alpha_1 - \delta) + \alpha = (\alpha_1 + \alpha) - \delta$, which lies in $\mathcal{B}(P)$ because $\alpha_1 + \alpha$ is either α_{12} or not a root (and is never zero).
- For $\alpha \in P$ and $m > 0$, we have that $(\alpha_1 - \delta) + (\alpha + m\delta) = (\alpha_1 + \alpha) + (m-1)\delta$, which lies in $\mathcal{B}(P)$ because $\alpha_1 + \alpha$ is either α_{12}, not a root, or zero (which is OK, as $m - 1 \geq 0$ in this case).
- For $m > 0$, we have that $(\alpha_1 - \delta) + m\delta = \alpha_1 + (m-1)\delta$, which lies in $\mathcal{B}(P)$ because $m - 1 \geq 0$ and $\alpha_1 \in \Delta^+ \subseteq P$.

So $\mathcal{P}$ is a parabolic subset which contains the partition $\mathcal{B}(P)$, but is not of a form given in Corollary 15.3. We can, however, show that $\mathcal{P}$ is equivalent to one of these forms.

Recall that the Weyl group $\mathcal{W}$ contains the reflections

$$r_\gamma : \gamma' \mapsto \gamma' - \langle \gamma', \gamma \rangle \gamma,$$

for $\gamma = \alpha + m\delta \in \tilde{\Delta}$ (Lemma 12.13). We have that $\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = 2$ and $\langle \alpha_1, \alpha_2 \rangle = \langle \alpha_2, \alpha_1 \rangle = -1$, while $\langle \gamma, \delta \rangle = 0$ for all $\gamma \in \tilde{\Delta}$. (An immediate consequence of this is that all reflections fix $m\delta$). Using the above, we can calculate

$$\begin{aligned}
r_{\alpha_{12}-\delta}(\alpha_1) &= -\alpha_2 + \delta, \\
r_{\alpha_{12}-\delta}(\alpha_2) &= -\alpha_1 + \delta, \\
r_{\alpha_{12}}(\alpha_1) &= -\alpha_2, \\
r_{\alpha_{12}}(\alpha_2) &= -\alpha_1.
\end{aligned}$$

Therefore, $r = r_{\alpha_{12}} r_{\alpha_{12}-\delta}$ acts in the following way.

$$\begin{aligned}
\alpha_1 + m\delta &\mapsto \alpha_1 + (m+1)\delta, \\
\alpha_2 + m\delta &\mapsto \alpha_2 + (m+1)\delta, \\
\alpha_{12} + m\delta &\mapsto \alpha_{12} + (m+2)\delta, \\
-\alpha_1 + m\delta &\mapsto \alpha_1 + (m-1)\delta, \\
-\alpha_2 + m\delta &\mapsto \alpha_2 + (m-1)\delta, \\
-\alpha_{12} + m\delta &\mapsto \alpha_{12} + (m-2)\delta.
\end{aligned}$$

Applying this to $\mathcal{P}$, we then see that

- $\alpha_2 + m\delta$ and $\alpha_{12} + m\delta$ lie in $\mathcal{P}$ for all m, so lie in $r(\mathcal{P})$.
- $\alpha_1 + m\delta$ lies in $\mathcal{P}$ for $m \geq -1$, so lies in $r(\mathcal{P})$ for $m \geq 0$.
- $-\alpha_1 + m\delta$ lies in $\mathcal{P}$ for $m \geq 1$, so lies in $r(\mathcal{P})$ for $m \geq 0$.
- $-\alpha_2 + m\delta$ and $-\alpha_{12}$ do not lie in $\mathcal{P}$, so do not lie in $r(\mathcal{P})$ for any m.

Consequently, $r(\mathcal{P}) = \mathcal{B}(P) \cup \{-\alpha_1\} = \mathcal{B}(P) \cup \mathrm{add}(- \{\alpha_1\})$.

Chapter 16

Seaweed Sets of Roots

We discuss seaweed sets of roots and seaweed Lie algebras. We use what we have learned about parabolic sets of roots to give a simple expression for standard type seaweed sets of roots (Lemma 16.6). We also show that multiples of the null root δ do not occur in proper standard seaweed sets of roots.

Definition 16.1. Let $\mathcal{P}$ and $\mathcal{P}'$ be parabolic sets of roots in Δ (respectively $\tilde{\Delta}$). If $\mathcal{P} \cup \mathcal{P}' = \Delta$ (respectively $\mathcal{P} \cup \mathcal{P}' = \tilde{\Delta}$), then we say $\mathcal{P}$ and $\mathcal{P}'$ are **weakly opposite**. A set $\mathcal{S} = \mathcal{P} \cap \mathcal{P}'$ is called a **seaweed set of roots** when $\mathcal{P}$ and $\mathcal{P}'$ are weakly opposite. If both $\mathcal{P}$ and $\mathcal{P}'$ are proper subsets, then $\mathcal{S}$ is not a parabolic subset, in which case we call $\mathcal{S}$ a **proper** seaweed set of roots.

For $\mathcal{P}, \mathcal{P}' \subseteq \tilde{\Delta}$, if both $\mathcal{P}$ and $\mathcal{P}'$ are standard type, we say $\mathcal{S}$ is of **standard type**. Similarly, if both $\mathcal{P}$ and $\mathcal{P}'$ are natural type, we say $\mathcal{S}$ is of **natural type**.

Lemma 16.2. *Let $\mathcal{P}$ and $\mathcal{P}'$ be weakly opposite parabolic set of roots in Δ (respectively $\tilde{\Delta}$). Then there exists a partition $\mathcal{B}$ of Δ (respectively $\tilde{\Delta}$) such that $\mathcal{B} \subseteq \mathcal{P}$ and $-\mathcal{B} \subseteq \mathcal{P}'$.*

Proof. The proof of Proposition 40.7.3 in Tauvel-Yu [27] provides an argument which applies equally well to the finite type Δ and the affine type $\tilde{\Delta}$, as follows.

Let $\mathcal{T} = \mathcal{P} \cap -\mathcal{P}'$. We have the following.

1. $\mathcal{P} \cup -\mathcal{P} = \Delta$.
2. $\mathcal{P}' \cup -\mathcal{P}' = \Delta$.
3. $\mathcal{P} \cup \mathcal{P}' = \Delta$.
4. $-\mathcal{P} \cup -\mathcal{P}' = \Delta$.

Let $\alpha \in \Delta \setminus \mathcal{T}$. Then $\alpha \notin \mathcal{P}$ or $\alpha \notin -\mathcal{P}'$. The former implies $\alpha \in -\mathcal{P}$ by (1) and $\alpha \in \mathcal{P}'$ by (3), while the latter implies $\alpha \in -\mathcal{P}$ by (4) and $\alpha \in \mathcal{P}'$ by (2). Thus $\alpha \in -\mathcal{T}$. That is, $\mathcal{T} \cup -\mathcal{T} = \Delta$. Additionally, $\mathcal{T}$ is closed, therefore it is a parabolic set of roots. Now there exists a partition $\mathcal{B} \subseteq \mathcal{T}$ (Proposition 15.5). As $\mathcal{T} \subseteq \mathcal{P}$ and $-\mathcal{T} \subseteq \mathcal{P}'$, this $\mathcal{B}$ is the desired partition. □

Lemma 16.3. *Let $\mathcal{P}$ be a proper standard parabolic set of roots containing a partition $\mathcal{B}$. Then $\mathcal{B}$ is equivalent to the standard partition $\mathcal{B}(\Delta)$.*

Proof. Without loss of generality, take $\mathcal{B} = \mathcal{B}(P)$ for some $P = \Delta^+ \cup \mathrm{add}(-S)$, where $S \subseteq \{\alpha_1, \cdots, \alpha_n\}$ (Corollary 13.11). Suppose $\mathcal{B}$ is a non-standard partition, so that $P \neq \Delta$. Recall that

$$\mathcal{B}(P) = \{\alpha + m\delta : \alpha \in N,\, m < 0\} \cup \Delta^+ \cup \{\alpha + m\delta : \alpha \in P,\, m > 0\} \cup \{m\delta : m > 0\},$$

where $N = P \setminus -P$. As $P \neq \Delta$, we have that N is nonempty, so there is some $\beta \in \Delta$ such that $\beta + \mathbb{Z}\delta \subseteq \mathcal{B}(P)$.

On the other hand, $\mathcal{P}$ is a standard parabolic, so contains a partition equivalent to the standard partition $\mathcal{B}(\Delta)$. That is, $\mathcal{P}$ contains either $r_{\alpha+m\delta}(\mathcal{B}(\Delta))$ or $-r_{\alpha+m\delta}(\mathcal{B}(\Delta))$ for some $\alpha \in \Delta$ and $m \in \mathbb{Z}$. Recall that

$$\mathcal{B}(\Delta) = \Delta^+ \cup \{\alpha + n\delta : \alpha \in \Delta,\, n > 0\} \cup \{n\delta : n > 0\},$$

and (from Lemma 12.15) that for $\alpha' \in \Delta$ and $m' \in \mathbb{Z}$,

$$r_{\alpha+m\delta} : \alpha' + m'\delta \mapsto r_\alpha(\alpha') + (m' - m\langle \alpha', \alpha \rangle)\delta.$$

Note that r_α permutes Δ and the possible values of $\langle \alpha', \alpha \rangle$ are bounded. So there is some $n \in \mathbb{Z}$ such that $\alpha + n\delta \in \mathcal{P}$ for all $\alpha \in \Delta$. Therefore, $\mathcal{P}$ contains

$$(-\beta + n\delta) + (\beta + \mathbb{Z}\delta) = \mathbb{Z}\delta.$$

This implies that $\mathcal{P} = \tilde{\Delta}$, which is a contradiction. Consequently, the supposition that $\mathcal{B}$ is a non-standard partition must be false. □

Definition 16.4. A **seaweed subalgebra** of a Lie algebra is a subalgebra generated by a seaweed set of roots. In an affine type Lie algebra, we say a seaweed subalgebra is **standard type** or **natural type** if the set of roots is standard type or natural type, respectively.

If $\mathfrak{p}$ and $\mathfrak{p}'$ are parabolic subalgebras of $\mathfrak{g}$ (respectively $\tilde{\mathfrak{g}}$) such that $\mathfrak{p} + \mathfrak{p}' = \mathfrak{g}$ (respectively $\mathfrak{p} + \mathfrak{p}' = \tilde{\mathfrak{g}}$), they are called **weakly opposite**.

Remark 16.5. If $\mathcal{S} = \mathcal{P} \cap \mathcal{P}'$ is a seaweed set of roots in Δ and $\mathfrak{s}$ the corresponding seaweed subalgebra of $\mathfrak{g}$, then $\mathfrak{s} = \mathfrak{p} \cap \mathfrak{p}'$, where $\mathfrak{p}$ and $\mathfrak{p}'$ are the parabolic subalgebras generated by $\mathcal{P}$ and $\mathcal{P}'$ respectively. Further, $\mathfrak{p}$ and $\mathfrak{p}'$ are weakly opposite.

Now suppose $\mathfrak{p}$ and $\mathfrak{p}'$ are arbitrary weakly opposite parabolic subalgebras of $\mathfrak{g}$ and let $\mathfrak{s} = \mathfrak{p} \cap \mathfrak{p}'$. Then $\mathfrak{p}$ and $\mathfrak{p}'$ contain Borel subalgebras $\mathfrak{b}$ and $\mathfrak{b}'$ respectively. The intersection of $\mathfrak{b}$ and $\mathfrak{b}'$ contains a Cartan subalgebra (Proposition 29.4.9 in [27]). Consequently, the two parabolics can be expressed as the subalgebras generated by parabolic subsets P and P' of the same root system. This implies $\mathfrak{s}$ is the subalgebra generated by the set of roots $\mathcal{P} \cap \mathcal{P}'$. The fact that $\mathfrak{p} + \mathfrak{p}' = \mathfrak{g}$ implies that $\mathcal{P} \cup \mathcal{P}'$ is the whole root system. That is to say, $\mathfrak{s}$ is generated by a seaweed set of roots.

We would like to be able to do the same for arbitrary weakly opposite parabolic subalgebras of the affine Lie algebra $\tilde{\mathfrak{g}}$. As in the finite dimensional case, this relies on the existence of a Cartan subalgebra in $\mathfrak{b} \cap \mathfrak{b}'$ for pairs of Borel subalgebras $\mathfrak{b}$ and $\mathfrak{b}'$. Such a result is not presently available in the literature, and is beyond the scope of this thesis.

Lemma 16.6. *Standard seaweed sets of roots are precisely those equivalent to*

$$\mathcal{S} = \operatorname{add}(-R) \cup \operatorname{add}(R')$$

for some $R, R' \subseteq \{\alpha_0, \cdots, \alpha_n\}$.

Proof. If $\mathcal{S}$ is a standard seaweed set of roots, then by definition $\mathcal{S} = \mathcal{P} \cap \mathcal{P}$ for weakly opposite standard parabolic sets of roots $\mathcal{P}$ and $\mathcal{P}'$. First we show that $\mathcal{B} \subseteq \mathcal{P}'$ and $-\mathcal{B} \subseteq \mathcal{P}'$ for some partition $\mathcal{B}$ which is equivalent to $\mathcal{B}(\Delta)$.

If $\mathcal{P}$ or $\mathcal{P}'$ is not proper, then this is trivially true, so suppose both are proper. Then Lemma 16.2 implies there exists some partition $\mathcal{B}$ such that $\mathcal{B} \subseteq \mathcal{P}'$ and $-\mathcal{B} \subseteq \mathcal{P}'$. Now Lemma 16.3 implies this partition is equivalent to $\mathcal{B}(\Delta)$.

Now Proposition 14.13 implies that $\mathcal{S}$ is equivalent to

$$(\mathcal{B}(\Delta) \cup \operatorname{add}(-R)) \cap (-\mathcal{B}(\Delta) \cup \operatorname{add}(R')) = \operatorname{add}(-R) \cup \operatorname{add}(R')$$

for some $R, R' \subseteq \{\alpha_0, \cdots, \alpha_n\}$. Finally, observe that all such sets $\operatorname{add}(-R) \cup \operatorname{add}(R')$ can be obtained in this way. □

Lemma 16.7. *Let $\mathcal{S}$ be a proper standard seaweed set of roots. Then no multiples of δ lie in $\mathcal{S}$.*

Proof. As $\mathcal{S}$ is a standard seaweed, $\mathcal{S} = \text{add}(-R) \cup \text{add}(R')$ by Lemma 16.6. As $\mathcal{S}$ is proper, there is some i for which $\alpha_i \notin R$. Now $\text{add}(-R)$ is contained in a root subsystem of $\tilde{\Delta}$ with Dynkin diagram obtained from that of $\tilde{\Delta}$ by removing α_i. This is therefore a finite type root system, hence $\text{add}(-R)$ is a finite set. If $-n\delta \in \text{add}(-R)$ for some $n > 0$, then $-mn\delta \in \text{add}(-R)$ for all $m > 0$, which contradicts this. Similarly, $n\delta \notin \text{add}(R')$. □

Definition 16.8. The **nilpotent radical** of a seaweed subalgebra $\mathfrak{s}$ generated by a seaweed set of roots $\mathcal{S}$ is

$$\mathfrak{n} = \bigoplus_{\alpha \in \mathcal{N}} \mathfrak{g}_\alpha,$$

where $\mathcal{N} = \mathcal{S} \setminus -\mathcal{S}$.

Remark 16.9. If $\mathfrak{s}$ is a proper seaweed of affine type, then Lemma 16.6 and Lemma 16.7 imply that $\mathfrak{s}$ is finite dimensional. The nilpotent radical of $\mathfrak{s}$ is therefore nilpotent.

We only consider the nilpotent radical in this case, so the abuse of naming convention does not create ambiguity here - though this probably should be addressed in future work.

Chapter 17

A Quiver Model for Proper Standard Seaweeds of Type A

Let $\tilde{\mathfrak{g}}$ be the affine Lie algebra whose corresponding finite dimensional Lie algebra is $\mathfrak{g} = \mathfrak{gl}_{n+1}(k)$. Let $\mathfrak{h}$ be the standard Cartan subalgebra of $\mathfrak{g}$, the subalgebra of diagonal matrices. Then $\tilde{\mathfrak{h}} = (\mathfrak{h} \otimes t^0) \oplus kc \oplus kd$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}$. Write Δ and $\tilde{\Delta}$ for the root systems of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ with respect to these Cartan subalgebras.

Δ is a Dynkin type $\mathbf{A}_n$ root system and $\tilde{\Delta}$ is an affine type $\mathbf{A}_n$ root system. We write $\alpha_1, \cdots, \alpha_n$ for the simple roots of Δ, indexed such that $\alpha_i + \alpha_{i+1} \in \Delta$ for $i = 1, \cdots, n-1$. Then $\alpha_0, \alpha_1, \cdots, \alpha_n$ are the simple roots of $\tilde{\Delta}$ and $\alpha_i + \alpha_{i+1} \in \tilde{\Delta}$ for $i = 0, \cdots, n$, taking indices modulo $n+1$ so that $\alpha_{n+1} = \alpha_0$.

Let $\mathcal{S}$ be a proper standard seaweed set of roots in $\tilde{\Delta}$. Write $\mathcal{S}_0 = \mathcal{S} \cap -\mathcal{S}$ and $\mathcal{N} = \mathcal{S} \setminus -\mathcal{S} = \mathcal{S} \setminus \mathcal{S}_0$. The seaweed subalgebra of $\tilde{\mathfrak{g}}$ corresponding to $\mathcal{S}$ is

$$\tilde{\mathfrak{s}} = (\mathfrak{h} \otimes t^0) \oplus kc \oplus kd \oplus \bigoplus_{\gamma \in \mathcal{S}} \tilde{\mathfrak{g}}_\gamma.$$

Let

$$\hat{\mathfrak{s}} = (\mathfrak{h} \otimes t^0) \oplus \bigoplus_{\gamma \in \mathcal{S}} \tilde{\mathfrak{g}}_\gamma$$

be a Lie algebra with bracket given by

$$[x \otimes t^a,\, y \otimes t^b] = [x,\, y] \otimes t^{a+b},$$

where the bracket on the right hand side is that of $\mathfrak{g}$. Note that $\hat{\mathfrak{s}}$ is a vector subspace of $\tilde{\mathfrak{g}}$, but is not a Lie subalgebra of $\tilde{\mathfrak{g}}$. Finally, let

$$\mathfrak{l} = (\mathfrak{h} \otimes t^0) \oplus \bigoplus_{\gamma \in \mathcal{S}_0} \tilde{\mathfrak{g}}_\gamma \qquad \text{and} \qquad \mathfrak{n} = \bigoplus_{\gamma \in \mathcal{N}} \tilde{\mathfrak{g}}_\gamma,$$

so that we have the decomposition $\hat{\mathfrak{s}} = \mathfrak{l} \oplus \mathfrak{n}$.

We construct a projective quiver representation whose endomorphism ring is isomorphic (as a linear Lie algebra) to $\hat{\mathfrak{s}}$ by indexing the blocks in $\hat{\mathfrak{s}}$ with roots, which can then be associated with paths in the quiver, which index the blocks in the endomorphism ring. This allows us to equate the dimension of $[\hat{\mathfrak{s}}, x]_{\hat{\mathfrak{s}}}$ for $x \in \mathfrak{n}$ with the dimension of the orbit of a representation in $\mathrm{Rep}_\Delta(Q, d)$. Lemma 17.1 then allows us to find the dimension of $[\tilde{\mathfrak{s}}, x]_{\tilde{\mathfrak{s}}}$, once we have dealt with the action of the scaling element d.

Lemma 17.1. *Let $x \in \mathfrak{n}$ and $y \in \hat{\mathfrak{s}}$. Then $[x, y]_{\hat{\mathfrak{s}}} = [x, y]_{\tilde{\mathfrak{s}}}$. Further,*

$$[\tilde{\mathfrak{s}}, x]_{\tilde{\mathfrak{s}}} = [\hat{\mathfrak{s}}, x]_{\hat{\mathfrak{s}}} + k[d, x]_{\tilde{\mathfrak{s}}}.$$

Proof. We may assume that $x \in \tilde{\mathfrak{g}}_\gamma$ for some $\gamma \in \mathcal{S} \setminus -\mathcal{S}$ and that either $y \in \mathfrak{h} \otimes t^0$ or $y \in \tilde{\mathfrak{g}}_{\gamma'}$ for some $\gamma' \in \mathcal{S}$. Suppose the latter.

By Lemma 16.7, we can write $\gamma = \alpha + a\delta$ and $\gamma' = \beta + b\delta$ for $\alpha, \beta \in \Delta$. So $x = x' \otimes t^a$ and $y = y' \otimes t^b$ for $x' \in \mathfrak{g}_\alpha$ and $y' \in \mathfrak{g}_\beta$. Recall that

$$\begin{aligned} [x, y]_{\tilde{\mathfrak{s}}} &= [x', y'] \otimes t^{a+b} + a\delta_{a,-b}(x'|y')c \\ &= [x, y]_{\hat{\mathfrak{s}}} + a\delta_{a,-b}(x'|y')c. \end{aligned}$$

As $\gamma \in \mathcal{S} \setminus -\mathcal{S}$, we know $\gamma' \neq -\gamma$, so $b = -a$ implies $\beta \neq -\alpha$, which in turn implies $(\mathfrak{g}_\alpha|\mathfrak{g}_\beta) = 0$ by Theorem 12.9. That is, $\delta_{a,-b} \neq 0$ implies $(x'|y') = 0$.

Suppose instead that $y = y' \otimes t^0$ for $y' \in \mathfrak{h}$. By Corollary 12.10, $(\mathfrak{g}_\alpha|\mathfrak{h}) = 0$, hence $(x'|y') = 0$. Therefore, in both cases, we have $[x, y]_{\tilde{\mathfrak{s}}} = [x, y]_{\hat{\mathfrak{s}}}$. This implies that

$$\begin{aligned} [\tilde{\mathfrak{s}}, x]_{\tilde{\mathfrak{s}}} &= [\hat{\mathfrak{s}} \oplus kc \oplus kd, x]_{\tilde{\mathfrak{s}}} \\ &= [\hat{\mathfrak{s}}, x]_{\tilde{\mathfrak{s}}} + k[c, x]_{\tilde{\mathfrak{s}}} + k[d, x]_{\tilde{\mathfrak{s}}} \\ &= [\hat{\mathfrak{s}}, x]_{\hat{\mathfrak{s}}} + k[d, x]_{\tilde{\mathfrak{s}}}, \end{aligned}$$

as c is central. ☐

Remark 17.2. We assume, without loss of generality, that $-\alpha_0 \notin \mathcal{S}$. We may do this because the Dynkin diagram we constructed $\tilde{\mathfrak{g}}$ with has rotational symmetry and there is at least one negative simple root not in $\mathcal{S}$. One immediate implication of this is that $\mathcal{S}_0 \subseteq \Delta$.

Definition 17.3. First, we describe the blocking of $\hat{\mathfrak{s}}$. Split $I = \{1, \cdots, n+1\}$ into a disjoint union $I = I_1 \cup \cdots \cup I_r$ where i and $i+1$ are in the same component whenever $\alpha_i \in \mathcal{S}_0$. For each $I_i = \{m, m+1, \cdots, m+t\}$, let $\mathcal{S}_i = \operatorname{add}\{\pm\alpha_m, \cdots, \pm\alpha_{m+t-1}\}$ be the corresponding set of roots.

Let

$$\mathfrak{l}_i = \left(\bigoplus_{j \in I_i} k e_{j,j} \otimes t^0 \right) \oplus \left(\bigoplus_{\alpha \in \mathcal{S}_i} \tilde{\mathfrak{g}}_\alpha \right),$$

so that $\mathfrak{l} = \mathfrak{l}_1 \oplus \cdots \oplus \mathfrak{l}_r$. For each $i = 1, \cdots, r$, let V_i be the space spanned by the e_j for $j \in I_i$. Then each $\mathfrak{l}_i = \operatorname{End}_k(V_i) \otimes t^0$.

For each $\gamma \in \mathcal{N}$, we have $\tilde{\mathfrak{g}}_\gamma = k e_{b,a} \otimes t^m$ for some $a, b \in I$, where $a \in I_i$ and $b \in I_j$ for some $i, j = 1, \cdots, r$ (and $m \in \{0, 1\}$, but this is not needed). Let $\mathfrak{n}_\gamma = \operatorname{Hom}_k(V_i, V_j) \otimes t^m$ denote the block containing the root space $\tilde{\mathfrak{g}}_\gamma$. This is generated by $\mathfrak{l}_i$, $\tilde{\mathfrak{g}}_\gamma$, and $\mathfrak{l}_j$. Note that different γ can give the same $\mathfrak{n}_\gamma$.

Remark 17.4. Let $\gamma \in \mathcal{N}$. Depending on whether γ lies in $\tilde{\Delta}^+$ or $\tilde{\Delta}^-$, we have either $\gamma = \alpha_i + \cdots + \alpha_{i+t}$ or $\gamma = -\alpha_i - \cdots - \alpha_{i+t}$ for some i, t (indices modulo $n+1$). Each root belongs to either $\mathcal{S}_0$ or to $\mathcal{N}$ and sums of consecutive simple roots in $\mathcal{S}_0$ also lie in $\mathcal{S}_0$. We can therefore rewrite this expression for γ uniquely as $\gamma = \beta_0 + \gamma_1 + \beta_1 + \cdots + \gamma_s + \beta_s$ for $\beta_i \in \mathcal{S}_0 \cup \{0\}$ and $\gamma_i \in \{\pm\alpha_0, \cdots, \pm\alpha_n\} \cap \mathcal{N}$ such that the ordering $\beta_0, \gamma_1, \cdots, \beta_s$ corresponds to the ordering $\alpha_i \cdots, \alpha_{i+t}$. It follows that $\tilde{\mathfrak{g}}_\gamma$ is contained in the subalgebra generated by $\mathfrak{n}_{\gamma_1}, \cdots, \mathfrak{n}_{\gamma_s}$.

Consequently, $\hat{\mathfrak{s}}$ is generated by the blocks

- $\mathfrak{l}_i$ for $i = 1, \cdots, r$, and
- $\mathfrak{n}_\gamma$ for $\gamma \in \{\pm\alpha_0, \cdots, \pm\alpha_n\} \cap \mathcal{N}$.

Definition 17.5. Next, we describe the quiver model. Let Q be a quiver with vertex set $Q_0 = \{1, \cdots, r\}$ and set of arrows $Q_1 = \{\pm\alpha_0, \cdots, \pm\alpha_n\} \cap \mathcal{N}$. For each $\gamma \in Q_1$, we have $\tilde{\mathfrak{g}}_\gamma = k e_{b,a} \otimes t^m$, where $a \in I_i$ and $b \in I_j$ for some $i, j = 1, \cdots, r$. Set $t(\gamma) = i$ and $s(\gamma) = j$. As usual, write $A = kQ$ for the path algebra. Let $d \in \mathbb{N}^{Q_0}$ be given by $d_i = |I_i| = \dim V_i$.

Recall that

$$P(d) = \bigoplus_{i \in Q_0} \Delta_i^{d_i},$$

where $\Delta_i = Ae_i$ is the indecomposable projective A-module associated to vertex i. The space $\mathrm{Hom}_A(\Delta_i, \Delta_j)$ has a basis f_ρ indexed by paths ρ from j to i, where $f_\rho \circ f_\sigma = f_{\sigma\rho}$. As $\mathrm{Hom}_A(\Delta_i^{d_i}, \Delta_j^{d_j}) \cong \mathrm{Hom}_A(\Delta_i, \Delta_j)^{d_i d_j}$ is the space of d_j by d_i matrices with entries in $\mathrm{Hom}_A(\Delta_i, \Delta_j)$, this implies that

$$\mathrm{Hom}_A(\Delta_i^{d_i}, \Delta_j^{d_j}) \cong \bigoplus_{\rho \in P(j,i)} \mathrm{Hom}_k(V_i, V_j) \otimes_k f_\rho.$$

For $M \in \mathrm{Hom}_k(V_i, V_j)$ and $\rho \in P(j,i)$, the tensor $M \otimes f_\rho$ is just the matrix with (a,b)th entry $M_{a,b} f_\rho$.

Lemma 17.6. $\mathrm{End}_A P(d)$ *is generated by*

- $\mathrm{End}_k(V_i) \otimes f_{e_i}$ *for* $i \in Q_0$*, and*
- $\mathrm{Hom}_k(V_{t(\gamma)}, V_{s(\gamma)}) \otimes f_\gamma$ *for* $\gamma \in Q_1$.

Proof. We have $f_\rho \circ f_\sigma = f_{\sigma\rho}$ for paths ρ and σ. This is nonzero only if $\sigma\rho \neq 0$, in which case $t(\rho) = s(\sigma)$, $s(\rho) = s(\sigma\rho)$ and $t(\sigma) = t(\sigma\rho)$. Therefore,

$$\mathrm{Hom}_k(V_{t(\rho)}, V_{s(\rho)}) \otimes f_\rho \circ \mathrm{Hom}_k(V_{t(\sigma)}, V_{s(\sigma)}) \otimes f_\sigma = \mathrm{Hom}_k(V_{t(\sigma\rho)}, V_{s(\sigma\rho)}) \otimes f_{\sigma\rho},$$

from which the result follows. □

Definition 17.7. As in Remark 17.4, we can write each root $\gamma \in \mathcal{N}$ uniquely as

$$\gamma = \beta_0 + \gamma_1 + \beta_1 + \cdots + \gamma_s + \beta_s$$

for some $\beta_i \in \mathcal{S}_0 \cup \{0\}$ and $\gamma_i \in Q_1$. This implies either $\gamma_1 \cdots \gamma_s$ or $\gamma_s \cdots \gamma_1$ is a path in Q (otherwise γ would not be a root). That is, each root $\gamma \in \mathcal{N}$ can be associated to a path $\rho(\gamma)$ in Q in this way.

Define a map $\varphi : \hat{\mathfrak{s}} \to \mathrm{End}_A P(d)$ as follows.

- For $x \otimes t^0 \in \mathfrak{l}_i$ $(i = 1, \cdots, r)$, let $\varphi(x \otimes t^0) = x \otimes f_{e_i}$.
- For $x \otimes t^m \in \mathfrak{n}_\gamma$ $(\gamma \in \mathcal{N})$, let $\varphi(x \otimes t^m) = x \otimes f_{\rho(\gamma)}$.

This is well-defined, because if $\mathfrak{n}_\gamma = \mathfrak{n}_{\gamma'}$, then $\rho(\gamma) = \rho(\gamma')$. It is now a simple matter to check that φ is an isomorphism.

Lemma 17.8. *φ is an isomorphism of Lie algebras.*

Proof. We show φ is an isomorphism of linear algebras, from which the result follows. As φ is a bijection on the generators, it is bijective if it is a morphism.

First, let $x \otimes t^0 \in \mathfrak{l}_i$ and $y \otimes t^0 \in \mathfrak{l}_j$, so $x \in \operatorname{End}_k V_i$ and $y \in \operatorname{End}_k V_j$. Then $x \circ y = 0$ if $i \neq j$, and $x \circ y \in \operatorname{End} V_i$ otherwise. Therefore,

$$\varphi(x \otimes t^0 \circ y \otimes t^0) = \varphi(x \circ y \otimes t^0) = \delta_{i,j} x \circ y \otimes f_{e_i}.$$

On the other hand, $e_i e_j = \delta_{i,j} e_i$ is the product of trivial paths in Q, which implies

$$\varphi(x \otimes t^0) \circ \varphi(y \otimes t^0) = x \otimes f_{e_i} \circ y \otimes f_{e_j} = x \circ y \otimes f_{e_j e_i} = \delta_{i,j} x \circ y \otimes f_{e_i}.$$

Now let $x \otimes t^m \in \mathfrak{n}_\gamma = \operatorname{Hom}(V_i, V_j) \otimes t^m$ and $y \otimes t^0 \in \mathfrak{l}_k = \operatorname{End} V_k \otimes t^0$. Then $x \circ y = 0$ if $k \neq i$ and $x \circ y \in \operatorname{Hom}(V_i, V_j)$ otherwise. Therefore,

$$\varphi(x \otimes t^m \circ y \otimes t^0) = \varphi(x \circ y \otimes t^m) = \delta_{i,k} x \circ y \otimes f_{\rho(\gamma)}.$$

On the other hand,

$$\varphi(x \otimes t^m) \circ \varphi(y \otimes t^0) = x \otimes f_{\rho(\gamma)} \circ y \otimes f_{e_k} = x \circ y \otimes f_{e_k \rho(\gamma)} = \delta_{i,k} x \circ y \otimes f_{\rho(\gamma)},$$

as $t(\rho(\gamma)) = i$. Similarly, $\varphi(y \otimes t^0) \circ \varphi(x \otimes t^m) = \varphi(y \otimes t^0 \circ x \otimes t^m)$.

Finally, let $x \otimes t^m \in \mathfrak{n}_\gamma = \operatorname{Hom}(V_i, V_j) \otimes t^m$ and $y \otimes t^r \in \mathfrak{n}_\sigma$. If $i \neq b$, then $x \circ y = 0$. Otherwise, $\rho(\sigma)\rho(\gamma) = 0$, as $t(\rho(\gamma)) = i$ and $s(\rho(\sigma)) = b$. Then $\mathfrak{n}_\gamma \circ \mathfrak{n}_\sigma = \operatorname{Hom}(V_a, V_j) \otimes t^{m+r} = \mathfrak{n}_{\gamma+\beta+\sigma}$ for some $\beta \in \mathcal{S}_0 \cup \{0\}$ where $\rho(\gamma + \beta + \sigma) = \rho(\sigma)\rho(\gamma)$. Therefore,

$$\varphi(x \otimes t^m \circ y \otimes t^r) = \varphi(x \circ y \otimes t^{m+r}) = \delta_{i,b} x \circ y \otimes f_{\rho(\sigma)\rho(\gamma)}.$$

On the other hand,

$$\varphi(x \otimes t^m) \circ \varphi(y \otimes t^r) = x \otimes f_{\rho(\gamma)} \circ y \otimes f_{\rho(\sigma)} = \delta_{i,b} x \circ y \otimes f_{\rho(\sigma)\rho(\gamma)},$$

as $x \circ y = 0$ if $i \neq b$. □

Lemma 17.9. *Q is a Dynkin or affine type* $\mathbf{A}$ *quiver with no oriented cycles.*

Proof. The Dynkin diagram of $\tilde{\Delta}$ is a cycle. Q is obtained from this graph by applying the following operations for each $i = 0, \cdots, n$:

1. If both $\pm\alpha_i \in \mathcal{S}$, identify the vertices connected by the edge α_i.
2. If precisely one of $\pm\alpha_i \in \mathcal{S}$, replace the edge α_i with an arrow, oriented clockwise if $+\alpha_i \in \mathcal{S}$ or anticlockwise if $-\alpha_i \in \mathcal{S}$.
3. If both $\pm\alpha_i \notin \mathcal{S}$, remove the edge α_i.

This implies Q is either quiver whose connected components are all of Dynkin type $\mathbf{A}$ (if $\pm\alpha_i \notin \mathcal{S}$ for some i) or an affine type $\mathbf{A}$ quiver (if at least one of $\pm\alpha_i \in \mathcal{S}$ for all i). Further, if Q is affine, there is an oriented cycle only if the simple roots in $\mathcal{N}$ are either all positive or all negative. But then either all positive simple roots or all negative simple roots are contained in $\mathcal{S}$, which contradicts the assumption that $\mathcal{S}$ is a proper seaweed set of roots. Therefore, $\mathcal{S}$ has no oriented cycles. □

Remark 17.10. Q is a Dynkin quiver if both $\pm\alpha_i \notin \mathcal{S}$ for some i, which implies that the seaweed set of roots $\mathcal{S}$ is a subset of the finite-type root system generated by the remaining simple roots. Finite-type seaweeds have already been dealt with by Jensen, Su and Yu in [15, 16], so we will assume that Q is an affine type quiver.

Definition 17.11 (Definition 4.3 in [26]; Definition, Section 4.6 in [25])**.** For a ring R, denote by $\operatorname{rad} R$ the **Jacobson radical** of R, which defined as the intersection of all maximal right (equivalently left) ideals of R.

Lemma 17.12. *Let A be a k-algebra and I a two-sided nilpotent ideal of A. Suppose that* $\operatorname{rad}(A/I) = 0$*. Then* $I = \operatorname{rad} A$*.*

Proof. Schiffler's proof for Proposition 4.4 in [26] is sufficient to prove this result. □

Lemma 17.13.

$$\operatorname{rad} \operatorname{End} P(d) = \bigoplus_{\rho \in P^*(-,-)} \operatorname{Hom}_k(V_{t(\rho)}, V_{s(\rho)}) \otimes f_\rho.$$

Proof. Let R be the desired expression for $\operatorname{rad} \operatorname{End} P(d)$. As $x \otimes f_\rho \circ y \otimes f_\sigma = x \circ y \otimes f_{\sigma\rho}$ lies in R if either ρ or σ is a nontrivial path, R is a two-sided ideal.

As Q has no oriented cycles (Lemma 17.9), there exists a maximum path length m. Nontrivial paths have length at least 1, so any product of $m+1$ nontrivial paths is zero. Thus R is nilpotent.

Finally,

$$\operatorname{End} P(d)/R \cong \bigoplus_{i \in Q_0} \operatorname{End}_k(V_i),$$

where $\operatorname{rad} \operatorname{End}_k(V_i) = 0$ for each $i \in Q_0$, hence $\operatorname{rad}(\operatorname{End} P(d)/R) = 0$. The result now follows from Lemma 17.12. □

Corollary 17.14. $\varphi(\mathfrak{n}) = \operatorname{rad} \operatorname{End} P(d)$. *Consequently,* $\operatorname{rad} \operatorname{End} P(d)$ *is the nilpotent radical of* $\operatorname{End} P(d)$*.*

Proof. φ is an isomorphism (Lemma 17.8) and maps the generators of $\mathfrak{n}$ to the generators of $\operatorname{rad} \operatorname{End} P(d)$. □

Lemma 17.15. *Let* $x \in \mathfrak{n}$*. Let* $M \in \operatorname{Rep}_\Delta(Q, d)$ *be the representation corresponding to* $\varphi(x)$*. Then* $\dim[\hat{\mathfrak{s}}, x]$ *is equal to the dimension of the* $\operatorname{Aut} P(d)$*-orbit of* M *in* $\operatorname{Rep}_\Delta(Q, d)$*.*

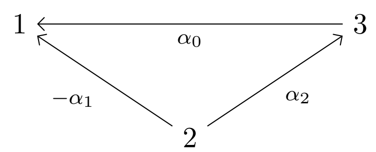
Proof. Write $f = \varphi(x)$. Then $f \in \operatorname{rad} \operatorname{End} P(d)$ by Lemma 17.13. Further, $\dim[\hat{\mathfrak{s}}, x] = \dim[\operatorname{End} P(d), f]$ by Lemma 17.8. Now $\dim[\operatorname{End} P(d), f]$ equals the dimension of the $\operatorname{Aut} P(d)$-orbit of f (Lemma 11.16), which equals the dimension of the $\operatorname{Aut} P(d)$-orbit of M in $\operatorname{Rep}_\Delta(Q, d)$ by Theorem 7.1. □

Example 17.16. Let $\mathfrak{g} = \mathfrak{gl}_4(k)$ and $\mathcal{S} = \operatorname{add}\{+\alpha_0, -\alpha_1, +\alpha_2, \pm\alpha_3\}$. We can draw $\hat{\mathfrak{s}}$ as follows: The coordinates generating $\hat{\mathfrak{s}}$ are labelled with the corresponding root; the other nonzero coordinates are labelled with a $*$; and the dotted lines highlight $\mathfrak{l}$.

0	0	$*$	0	0	0	0	0
0	0	$-\alpha_1$	$*$	α_2	$*$	$*$	0
0	0	0	0	$*$	α_3	$*$	0
0	0	0	0	$-\alpha_3$	$*$	α_0	0

Then $\mathcal{S}_0 = \{\pm\alpha_3\}$, so that $I_1 = \{1\}$, $I_2 = \{2\}$ and $I_3 = \{3, 4\}$, while $\mathcal{S}_1 = \emptyset$, $\mathcal{S}_2 = \emptyset$ and $\mathcal{S}_3 = \{\pm\alpha_3\}$. Writing $V_1 = ke_1$, $V_2 = ke_2$ and $V_3 = ke_3 \oplus ke_4$, we can see that each $\mathfrak{l}_i = \mathrm{End}_k(V_i) \otimes t^0$, while $\mathfrak{n}_{\alpha_0} = \mathrm{Hom}_k(V_1, V_3) \otimes t^1$, $\mathfrak{n}_{-\alpha_1} = \mathrm{Hom}_k(V_1, V_2) \otimes t^0$ and $\mathfrak{n}_{\alpha_2} = \mathrm{Hom}_k(V_3, V_2) \otimes t^0$.

For the quiver model, we have $Q_0 = \{1, 2, 3\}$ and $Q_1 = \{\alpha_0, -\alpha_1, \alpha_2\}$, where $s(\alpha_0) = 3$, $t(\alpha_0) = 1$, $s(-\alpha_1) = 2$, $t(-\alpha_1) = 1$, $s(\alpha_2) = 2$ and $t(\alpha_2) = 3$. That is, we can draw Q as follows.



The dimension vector is simply $d_i = \dim V_i$, so $d = (1, 1, 2)$. By Proposition 2.14, we know that $\mathrm{Rep}_\Delta(Q, d)$ is just the space of all $M(d, \varphi)$, where φ is a set of maps $\varphi_\rho : k^{d_{t(\rho)}} \to k^{d_{s(\rho)}}$ indexed by nontrivial paths $\rho \in P^*(-,-)$. As each $V_i \cong k^{d_i}$, this is just

$$\begin{aligned} \varphi_{\alpha_0} &: V_1 \to V_3, \\ \varphi_{-\alpha_1} &: V_1 \to V_2, \\ \varphi_{\alpha_2} &: V_3 \to V_2, \\ \varphi_{\alpha_0\alpha_2} &: V_1 \to V_2. \end{aligned}$$

The blocks in $\mathfrak{n}$ are precisely the $\varphi_\rho \otimes t^m$ for the appropriate m.

Chapter 18

Implications For Affine Seaweeds

We begin by setting up the notation we will use throughout the rest of this chapter.

18.1. Let $M \in \mathrm{Rep}_\Delta(Q, d)$ be a general representation of the form constructed in Chapter 8. Suppose M is non-rigid. Then $M = M_\lambda \oplus N$, where M_λ is a self-glued summand with parameter λ. Let $x \in \mathfrak{n}$ be the element corresponding to M and write $x = x_M + x_N$ for the decomposition of x corresponding to the decomposition $M = M_\lambda \oplus N$.

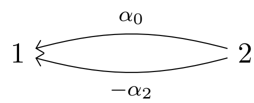
Each vertex of the Δ-graph of M corresponds to some unique basis element e_i of k^n. An arrow from the vertex of e_a to the vertex of e_b corresponds to the basis element $e_{b,a} \otimes t^p$ of $\mathfrak{n}$ (for some p), in the sense that the label of the arrow is the coefficient of $e_{b,a} \otimes t^p$ in x.

The arrows of the M_λ component of M form a (non-oriented) cycle. We write $v_1 \otimes t^{p_1}, \cdots, v_r \otimes t^{p_r}$ for the basis elements of $\mathfrak{n}$ corresponding to these arrows. There is a sequence $n_1, \cdots, n_r$ such that either $v_i = e_{n_i,n_{i+1}}$ or $v_i = e_{n_{i+1},n_i}$ for $i = 1, \cdots, r-1$ and either $v_r = e_{n_1,n_r}$ or $v_r = e_{n_r,n_1}$.

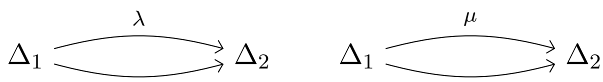
Example 18.2. Let $\mathfrak{g} = \mathfrak{gl}_4(k)$ with $\tilde{\mathfrak{g}}$ the corresponding affine Lie algebra and $\mathcal{S} =$ add $\{+\alpha_0, \pm\alpha_1, -\alpha_2, \pm\alpha_3\}$ a seaweed set of roots in $\tilde{\Delta}$. We can draw $\hat{\mathfrak{s}}$ as follows.

0	0	*	$+\alpha_1$	0	0	0	0	0
0	0	$-\alpha_1$	*	0	0	0	0	0
0	0	*	*	*	$+\alpha_3$	*	*	0
0	0	*	*	$-\alpha_3$	*	$+\alpha_0$	*	0

This gives us the following Q.



We construct the following Δ-graph for M. Take the component on the left to be M_λ and the component on the right to be N.



Choosing some pairing of e_1, e_2 with the two Δ_1 vertices and of e_3, e_4 with the two Δ_2 vertices, we can write down x.

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	1	0	0	0	λ	0	0
0	0	0	1	0	0	0	μ	0

Observe that x_M has nonzero coefficients of $e_{3,1} \otimes t^0$ and $e_{3,1} \otimes t^1$.

18.3. Each root γ in $\mathcal{S}$ is in the additive closure of either the positive simple roots or negative simple roots of $\tilde{\Delta}$. As $\mathfrak{g}_{\pm\alpha_0} \subset \mathfrak{g} \otimes t^{\pm 1}$ and all other $\mathfrak{g}_{\pm\alpha_i} \subset \mathfrak{g} \otimes t^0$, we have that $\mathfrak{g}_\gamma \subset \mathfrak{g} \otimes t^p$, where p is the multiplicity of α_0 in γ. Specifically, either $p = 0$ or $p = 1$ for each γ.

As the arrows of the M_λ component of the Δ-graph form a cycle, precisely one arrow in the Δ-graph of M_λ corresponds to a path involving α_0. So there is precisely one i such that $p_i = 1$. We may choose our indexing such that this i is r. That is, $p_r = 1$ and all other $p_i = 0$. We may also assume that the coefficient of $v_r \otimes t^1$ in M_λ is the parameter λ (and consequently that each other $v_i \otimes t^0$ has a coefficient of 1).

Definition 18.4. Let $\pi : \hat{\mathfrak{s}} \to \hat{\mathfrak{s}}$ be the projection map acting on each $v_i \otimes t^{p_i}$ as the identity and annihilating all other $e_{a,b} \otimes t^p$. That is, π is the projection onto the space spanned by the coordinates with nonzero coefficients in x_M, so that we have $\pi(x_M) = x_M$.

Lemma 18.5. *$\pi([y, x_N]_{\hat{\mathfrak{s}}}) = 0$ for all $y \in \hat{\mathfrak{s}}$.*

Proof. There are no arrows between the M_λ component and the N component. Therefore, if some $e_{a,b}$ has a nonzero coefficient in x_N, then both a and b are distinct from all the n_i. Thus, for any $e_{c,d}$, the bracket

$$[e_{c,d}, e_{a,b}]_{\mathfrak{g}} = \delta_{d,a} e_{c,b} - \delta_{b,c} e_{a,d}$$

has no nonzero coefficients of any v_i. □

Definition 18.6. Let $\pi_{\mathfrak{h}}$ be the projection map acting on each $e_{a,a} \otimes t^0$ as the identity and annihilating all other $e_{a,b} \otimes t^p$.

Lemma 18.7. *$\pi([y, x_M]_{\hat{\mathfrak{s}}}) = [\pi_{\mathfrak{h}}(y), x_M]_{\hat{\mathfrak{s}}}$ for all $y \in \hat{\mathfrak{s}}$.*

Proof. Firstly, each $[e_{a,a} \otimes t^0, v_i \otimes t^{p_i}]_{\hat{\mathfrak{s}}}$ is either $\pm v_i \otimes t^{p_i}$ or zero. Recall also that $\hat{\mathfrak{s}}$ does not contain any $e_{a,a} \otimes t^p$ for $p \neq 0$ (Lemma 16.7). We need only consider the following cases.

1. $[e_{a,b} \otimes t^0, v_i \otimes t^0] = \pm v_j \otimes t^0$ for some a, b, i and j.
2. $[e_{a,b} \otimes t^1, v_i \otimes t^0] = \pm v_r \otimes t^1$ for some a, b and i.

Case (1) can be narrowed down to the following.

- If $v_i = e_{n_i,n_{i+1}}$ and $v_{i+1} = e_{n_{i+2},n_{i+1}}$, then $[e_{n_{i+2},n_i}, v_i] = v_{i+1}$.

- If $v_i = e_{n_i,n_{i+1}}$ and $v_{i-1} = e_{n_i,n_{i-1}}$, then $[e_{n_{i+1},n_{i-1}}, v_i] = -v_{i-1}$.
- If $v_i = e_{n_{i+1},n_i}$ and $v_{i+1} = e_{n_{i+1},n_{i+2}}$, then $[e_{n_i,n_{i+2}}, v_i] = -v_{i+1}$.
- If $v_i = e_{n_{i+1},n_i}$ and $v_{i-1} = e_{n_{i-1},n_i}$, then $[e_{n_{i-1},n_{i+1}}, v_i] = v_{i-1}$.

For each, it can be shown that $e_{a,b} \otimes t^0 \notin \hat{\mathfrak{s}}$ using the same argument: Consider the arrows in M_λ corresponding to $v_i \otimes t^0$ and $v_j \otimes t^0$. The arrows correspond to paths ρ and σ in Q which, by the above points, either both start or both end at the same vertex.

- $n_i \leftarrow\text{-----}\ n_{i+1}\ \text{----}\rightarrow n_{i+2}$
- $n_{i-1}\ \text{-----}\rightarrow n_i \leftarrow\text{-----}\ n_{i+1}$
- $n_i\ \text{-----}\rightarrow n_{i+1} \leftarrow\text{----}\ n_{i+2}$
- $n_{i-1} \leftarrow\text{-----}\ n_i\ \text{-----}\rightarrow n_{i+1}$

If the appropriate $e_{a,b} \otimes t^0$ lies in $\hat{\mathfrak{s}}$, then the other ends are connected by the corresponding path τ. This path cannot pass through the vertex connecting ρ and σ, as that vertex is either a sink or a source. Therefore, the three paths ρ, σ and τ form a non-oriented cycle, which is the whole of Q. This implies that one of them contains the arrow α_0, which contradicts the fact that they all correspond to some $- \otimes t^0$.

Case (2) can be narrowed down to the following.

- If $v_1 = e_{n_1,n_2}$ and $v_r = e_{n_1,n_r}$, then $[e_{n_2,n_r}, v_1] = -v_r$.
- If $v_1 = e_{n_2,n_1}$ and $v_r = e_{n_r,n_1}$, then $[e_{n_r,n_2}, v_1] = v_r$.
- If $v_{r-1} = e_{n_{r-1},n_r}$ and $v_r = e_{n_1,n_r}$, then $[e_{n_1,n_{r-1}}, v_{r-1}] = v_r$.
- If $v_{r-1} = e_{n_r,n_{r-1}}$ and $v_r = e_{n_r,n_1}$, then $[e_{n_{r-1},n_1}, v_{r-1}] = -v_r$.

These also all imply that $e_{a,b} \otimes t^1 \notin \hat{\mathfrak{s}}$ by a similar argument to that of case (1). □

Proposition 18.8. *There is no $y \in \hat{\mathfrak{s}}$ such that $[y, x] = [d, x]$.*

Proof. Suppose there is such a y. Then $\pi([y, x]) = \pi([d, x]) = \lambda v_r \otimes t^1$. On the other hand,

$$\pi([y, x]) = \pi([y, x_M]) + \pi([y, x_N]) = \pi([y, x_M]) = [\pi_{\mathfrak{h}}(y), x_M]$$

by Lemma 18.5 and Lemma 18.7 in turn. We can write

$$\pi_{\mathfrak{h}}(y) = \sum_{i=1}^{n} y_i e_{i,i} \otimes t^0$$

for some $y_i \in k$ and we have

$$x_M = \lambda v_r \otimes t^1 + \sum_{j=1}^{r-1} v_j \otimes t^0.$$

For each $j = 1, \cdots, r-1$, either $v_j = e_{n_j, n_{j+1}}$ or $v_j = e_{n_{j+1}, n_j}$, so

$$[\pi_{\mathfrak{h}}(y),\, v_j \otimes t^0] = \pm(y_{n_j} - y_{n_{j+1}}) v_j \otimes t^0.$$

Additionally, either $v_r = e_{n_1, n_r}$ or $v_r = e_{n_r, n_1}$, so

$$[\pi_{\mathfrak{h}}(y),\, v_r \otimes t^1] = \pm(y_{n_1} - y_{n_r}) v_r \otimes t^1.$$

Therefore,

$$[\pi_{\mathfrak{h}}(y),\, x_M] = \pm\lambda(y_{n_1} - y_{n_r}) v_r \otimes t^1 + \sum_{j=1}^{r-1} \pm(y_{n_j} - y_{n_{j+1}}) v_j \otimes t^0.$$

As this is equal to $\lambda v_r \otimes t^1$, we have both

- $(y_{n_1} - y_{n_r}) = \pm 1$, and
- $y_{n_j} = y_{n_{j+1}}$ for $j = 1, \cdots, r-1$.

But these two points contradict one another, therefore the supposition that y exists must be false. □

Proposition 18.9. $\dim[\tilde{\mathfrak{s}},\, x]_{\tilde{\mathfrak{s}}} = \dim[\hat{\mathfrak{s}},\, x] + 1$.

Proof. By Lemma 17.1, $[\tilde{\mathfrak{s}},\, x] = [\hat{\mathfrak{s}},\, x] + k[d,\, x]_{\tilde{\mathfrak{s}}}$. By Proposition 18.8, the intersection of $[\hat{\mathfrak{s}},\, x]$ and $[d,\, x]$ is trivial. Lastly, $[d,\, x]_{\tilde{\mathfrak{s}}}$ is nonzero, as x has a nonzero coefficient of $v_r \otimes t^1$. The result follows. □

18.10. We now lift the assumption that the representation corresponding to x is non-rigid, obtaining the following general result.

Theorem 18.11. *Let p be the codimension of the* $\operatorname{Aut} P(d)$*-orbit of a general representation M in* $\operatorname{Rep}_\Delta(Q,d)$*. If $p=0$ or $p=1$, then there exists an $x\in\mathfrak{n}$ with an open $\tilde{\mathfrak{s}}$-adjoint orbit in $\mathfrak{n}$. Otherwise, the adjoint orbits in $\mathfrak{n}$ have codimension at least $p-1$.*

Proof. Take x to be the element of $\mathfrak{n}$ corresponding to M under φ (Definition 17.7). By Lemma 17.15 and Corollary 17.14 in turn, $\dim[\hat{\mathfrak{s}},\, x] = \dim \operatorname{rad} \operatorname{End} P(d) - p = \dim \mathfrak{n} - p$. This implies $[\tilde{\mathfrak{s}},\, x] = \mathfrak{n}$ if $p=0$, because $\mathfrak{n} \supseteq [\tilde{\mathfrak{s}},\, x] \supseteq [\hat{\mathfrak{s}},\, x]$ by Lemma 17.1. If $p>0$, then by Proposition 18.9, we get $\dim[\tilde{\mathfrak{s}},\, x] = \dim\mathfrak{n} - p + 1$.

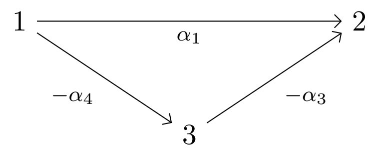
As M is a general representation, any other $N \in \operatorname{Rep}_\Delta(Q,d)$ has an $\operatorname{Aut} P(d)$-orbit with codimension at least that of M. Therefore, any other $y\in\mathfrak{n}$ must satisfy $\dim[\hat{\mathfrak{s}},\, y] \leq \dim[\hat{\mathfrak{s}},\, x]$ and thus $\dim[\tilde{\mathfrak{s}},\, y] \leq \dim[\tilde{\mathfrak{s}},\, x]$.

The result now follows from the fact that the dimension of the $\tilde{\mathfrak{s}}$-adjoint orbit of x in $\mathfrak{n}$ is equal to $\dim[\tilde{\mathfrak{s}},\, x]$ by Lemma 11.16. □

Remark 18.12. Given a proper seaweed set of roots $\mathcal{S}$ in the affine type **A** root system $\tilde{\Delta}$, we can now use Corollary 9.20 from Part I and Theorem 18.11 to determine, in a combinatorial manner, whether a Richardson element exists in the corresponding seaweed Lie algebra.

Example 18.13. Let $\mathfrak{g} = \mathfrak{gl}_4(k)$ with $\tilde{\mathfrak{g}}$ the corresponding affine Lie algebra and $\mathcal{S} = \operatorname{add}\{\pm\alpha_0, +\alpha_1, -\alpha_3\}$ the set of roots for the seaweed Lie subalgebra $\tilde{\mathfrak{s}}$ of $\tilde{\mathfrak{g}}$. As both $+\alpha_2$ and $-\alpha_2$ are not in $\mathcal{S}$, this is in fact a Dynkin type **A** seaweed, so we know immediately that a Richardson element exists.

Example 18.14. Let $\mathfrak{g} = \mathfrak{gl}_5(k)$ with $\tilde{\mathfrak{g}}$ the corresponding affine Lie algebra and $\mathcal{S} = \operatorname{add}\{\pm\alpha_0, +\alpha_1, \pm\alpha_2, -\alpha_3, -\alpha_4\}$ the set of roots for the seaweed Lie subalgebra $\tilde{\mathfrak{s}}$ of $\tilde{\mathfrak{g}}$. Then we can read off that $I_1 = \{5,1\}$, $I_2 = \{2,3\}$ and $I_3 = \{4\}$, hence $d_1 = 2$, $d_2 = 2$ and $d_3 = 1$. We know that $\tilde{\mathfrak{g}}_{\alpha_1} = ke_{1,2}\otimes t^0$, $\tilde{\mathfrak{g}}_{-\alpha_3} = ke_{4,3}\otimes k^0$ and $\tilde{\mathfrak{g}}_{-\alpha_4} = ke_{5,4}\otimes t^0$, so Q has arrows α_1 from 1 to 2; $-\alpha_3$ from 3 to 2; and $-\alpha_4$ from 1 to 3. So Q can be drawn as follows.

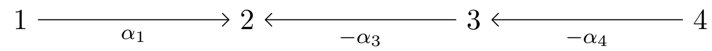


The sum of the dimensions at the sinks is equal to the sum of dimensions at the sources. Therefore, the codimension of the orbit of general Δ-filtered representations of this quiver is the minimum of

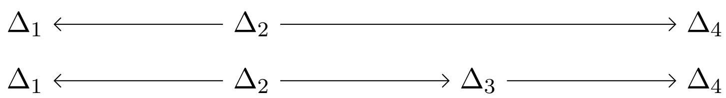
- $d_1 = 2$,
- $d_2 = 2$,
- $d_4 - d_2 + d_1 = d_1 - d_2 + d_1 = 2$,

which is 2. Consequently, there is not a Richardson element in this seaweed Lie algebra.

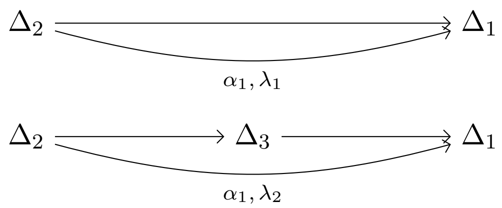
We can use the quiver model to find elements of the seaweed whose adjoint orbit is of maximal dimension. Splitting Q at the source 1, we obtain the following Dynkin quiver.



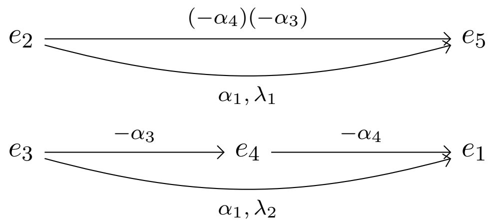
We construct the rigid representation of the appropriate Δ-dimension vector.



We then glue this representation to itself at vertices 1 and 4 to obtain a general representation of Q with Δ-dimension vector d.



We fix a choice of basis vector for each vertex in the Δ-graph. As $I_1 = \{5, 1\}$, we assign e_5 and e_1 to the two vertices labelled Δ_1; as $I_2 = \{2, 3\}$, we assign e_2 and e_3 to the two vertices labelled Δ_2; and as $I_3 = \{4\}$, we assign e_4 to the vertex labelled Δ_3.



Each arrow in the Δ-graph is associated to a path in Q, which is a product of simple roots (as the arrows in Q are simple roots). Specifying basis vectors at each vertex then takes each of these associated paths to the unique root which is a sum of all the simple roots in the path and some roots in $\mathcal{S}_0$ such that the root space corresponds to the basis vectors at the source and target of the arrow. Here, we have $\mathcal{S}_0 = \{\pm\alpha_0, \pm\alpha_2\}$.

- The arrow $e_2 \mapsto e_5$ associated to the path $(-\alpha_4)(-\alpha_3)$ becomes associated to the root $-\alpha_4 - \alpha_3 - \alpha_2$, as $\tilde{\mathfrak{g}}_{-\alpha_4-\alpha_3-\alpha_2} = ke_{5,2} \otimes t^0$.
- The arrow $e_2 \mapsto \lambda_1 e_5$ associated to the path α_1 becomes associated to the root $\alpha_0 + \alpha_1$, as $\tilde{\mathfrak{g}}_{\alpha_0+\alpha_1} = ke_{5,2} \otimes t^1$.
- The arrow $e_3 \mapsto e_4$ associated to $-\alpha_3$ becomes associated to the root $-\alpha_3$, as $\tilde{\mathfrak{g}}_{-\alpha_3} = ke_{4,3} \otimes t^0$.
- The arrow $e_4 \mapsto e_1$ associated to the path $-\alpha_4$ becomes associated to the root $-\alpha_4 - \alpha_0$, as $\tilde{\mathfrak{g}}_{-\alpha_4-\alpha_0} = ke_{1,4} \otimes t^{-1}$.
- The arrow $e_3 \mapsto \lambda_2 e_1$ associated to the path α_1 becomes associated to the root $\alpha_1 + \alpha_2$, as $\tilde{\mathfrak{g}}_{\alpha_1+\alpha_2} = ke_{1,3} \otimes t^0$.

We can then write this as an element of $\mathfrak{n}$, as follows.

$$e_{5,2} \otimes t^0 + \lambda_1 e_{5,2} \otimes t^1 + e_{4,3} \otimes t^0 + e_{1,4} \otimes t^{-1} + \lambda_2 e_{1,3} \otimes t^0$$

$$= e_{1,4} \otimes t^{-1} + (e_{5,2} + e_{4,3} + \lambda_2 e_{1,3}) \otimes t^0 + \lambda_1 e_{5,2} \otimes t^1.$$

18.15. We use $\tilde{\mathfrak{gl}}_n(k)$ and $\mathfrak{gl}_n(k)$ rather than $\tilde{\mathfrak{sl}}_n(k)$ and $\mathfrak{sl}_n(k)$, because $\mathfrak{gl}_n(k)$ matches the quiver model more closely. However, Theorem 18.11 also holds when $\tilde{\mathfrak{g}} = \tilde{\mathfrak{sl}}_n(k)$. This is because $\mathfrak{gl}_n(k) = \mathfrak{sl}_n(k) \oplus kz$ for a central element $z \in \mathfrak{h}$, which implies that:

1. As there are no roots $m\delta$ in a proper standard seaweed set of roots (Lemma 16.7), there are no root spaces $\mathfrak{g}_{m\delta} = \mathfrak{h} \otimes t^m$ in the seaweed Lie algebra, so adding a dimension to $\mathfrak{h}$ does not affect the nilpotent radical. Therefore, the nilpotent radical is the same regardless of whether the seaweed is a subalgebra of $\tilde{\mathfrak{gl}}_n(k)$ or $\tilde{\mathfrak{sl}}_n(k)$.
2. For any $x \in \mathfrak{n}$, we have $[z, x] = 0$. So if $\mathcal{S} \subseteq \tilde{\Delta}$ is a seaweed set of roots, $\tilde{\mathfrak{s}}$ and $\tilde{\mathfrak{s}}'$ the corresponding seaweed subalgebras of $\tilde{\mathfrak{gl}}_n(k)$ and $\tilde{\mathfrak{sl}}_n(k)$ respectively, then $[\tilde{\mathfrak{s}}', x] = [\tilde{\mathfrak{s}}, x]$.

18.16. There are various paths to take from here. The most obvious next step is to show that pairs of affine Borel subalgebras contain a common Cartan. With this, the present results would then hold in the generality which we naturally expect them to - currently, we cannot assume that the intersection of two weakly opposite parabolic subalgebras is a seaweed subalgebra in the sense of Definition 16.4.

The restriction to proper seaweed subalgebras - that is, the omission of parabolic subalgebras - is another obvious gap to address, though there are also clear difficulties with this case: Firstly, the ideal $\mathfrak{n}$ which which we expect to be the nilpotent radical is not in fact nilpotent. This issue can simply be ignored as unfortunate terminology, though there are instances where one would need to tread carefully in adapting the quiver model presented here (for example, the use of Lemma 17.12 and the assumption of no oriented cycles in Lemma 17.13). Secondly, the parabolic subalgebra and its radical are infinite dimensional, so alternatives would have to be found for the dimensional arguments used throughout. The adjoint group is therefore not finite dimensional either, so extra care would need to be taken when considering the group action - it is possible not all results used here would apply.

Considering the projection onto the superdiagonal blocks (the $\mathfrak{n}_\gamma$ for $\gamma \in Q_1$) allows us to show the non-existence of open orbits in the loop algebra $\hat{\mathfrak{s}}$, though the scaling element d in the actual affine Lie algebra $\tilde{\mathfrak{s}}$ means this is not enough to conclude Richardson elements do not exist in $\tilde{\mathfrak{s}}$ - we would need to calculate the codimension of the orbit in $\hat{\mathfrak{s}}$.

Additionally, we have not looked into the existence of Richardson elements in non-standard seaweeds. Natural parabolic sets of roots (and hence also natural seaweed sets of roots) have a particularly simple classification, which could potentially allow a proof of the existence of Richardson elements to be arrived at easily. Seaweeds involving type Ib parabolic subalgebras could also be studied, though their behaviour is likely to be much less pleasant.

The largest unexplored area is of course affine Lie algebras of types other than type **A**. Jensen and Su generalised their results on Richardson elements in seaweed subalgebras of finite dimensional Lie algebras to types **B**, **C** and **D** [16], so it would be natural to utilize their techniques in generalizing the present results to affine types **B**, **C** and **D**. Additionally, their analysis of types **E**, **F** and **G** was done computationally, using the program GAP4, so there has yet to be representation-theoretic analysis of these types in either the finite or affine case.

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