A New Frequency-uniform Coercive Boundary Integral Equation for Acoustic Scattering

E. A. Spence*†, S. N. Chandler-Wilde‡, I. G. Graham*, and V. P. Smyshlyaev*§

Abstract

A new boundary integral operator is introduced for the solution of the sound-soft acoustic scattering problem, i.e. for the exterior problem for Helmholtz equation with Dirichlet boundary conditions. We prove that this integral operator is coercive in $L^2(\Gamma)$ (where Γ is the surface of the scatterer) for all Lipschitz star-shaped domains. Moreover, the coercivity is uniform in the wavenumber $k = \omega/c$, where ω is the frequency and c is the speed of sound. The new boundary integral operator, which we call the "star-combined" potential operator, is a slight modification of the standard combined potential operator, and is shown to be as easy to implement as the standard one. Additionally, to the authors' knowledge, it is the only second-kind integral operator for which convergence of the Galerkin method in $L^2(\Gamma)$ is proved without smoothness assumptions on Γ except that it is Lipschitz. The coercivity of the star-combined operator implies frequency-explicit error bounds for the Galerkin method for any approximation space. In particular these error estimates apply to several hybrid asymptotic-numerical methods developed recently which provide robust approximations in the high frequency case. The proof of coercivity of the star-combined operator critically relies on an identity first introduced by Morawetz and Ludwig in 1968, supplemented further by more recent harmonic analysis techniques for Lipschitz domains.

1 Introduction

The aim of this paper is to introduce a new boundary integral operator (the so-called "star-combined" operator) for the Helmholtz equation describing acoustic scattering in two and three dimensions with Dirichlet boundary conditions. This formulation constitutes new advances in two different directions. On the one hand, for a wide class of scattering geometries, it yields a method for numerically evaluating the scattered field with rigorous frequency-explicit error estimates. On the other hand, even for moderate and small frequencies, it is the only second-kind integral operator known to the authors for which L^2-convergence of the Galerkin method is proved without smoothness assumptions on the boundary except that it is Lipschitz.

Geometrical optics (GO) and Keller's Geometrical Theory of Diffraction (GTD) [37] provide a set of general recipes for constructing the asymptotics of the scattered field for large frequencies. Over recent decades considerable effort has been devoted both to constructing and to justifying these asymptotics, i.e. proving error bounds with respect to large wavenumber k. (We do not attempt here to review this vast subject, referring the interested reader to e.g. [16], [47], [5], [23], [22], [6] and further references therein.) One of the first works justifying GO asymptotics was by Morawetz and Ludwig in 1968 [52]. This paper introduced a new identity which establishes continuous dependence (in a suitable norm) of the solution to a scattering problem upon the boundary data, and is thus capable of justifying the GO approximation for smooth convex scatterers with Dirichlet boundary conditions. Along with the more widely-known simultaneous work of Morawetz [50], this laid the foundation of the method of so-called "Morawetz-multipliers", which

*Department of Mathematical Sciences, University of Bath, Bath, BA2 7AY, UK
†Corresponding author: E.A.Spence@bath.ac.uk
‡Department of Mathematics and Statistics, University of Reading, Whiteknights, PO Box 220, Reading, RG6 6AX, UK
§Current address: Department of Mathematics, University College London, Gower Street, London, WC1E 6BT, UK

have since been intensively used and further developed for both linear and non-linear PDEs (e.g. [53], [57], [35], [27]).

As a separate development, "hybrid asymptotic-numerical" boundary integral methods have attracted considerable recent attention e.g. [1, 34, 10, 20, 28, 36, 18]. These methods seek to incorporate the oscillatory components from GO and the GTD explicitly into the numerical method, and thus efficiently compute the highly oscillatory solutions. The convergence analysis for these hybrid methods requires two key ingredients. One ingredient uses the results that justify the GO and GTD approximations to ensure that the specific oscillatory approximation space (constructed by considering the asymptotics) has small "best approximation error". This first ingredient, sufficient from the asymptotic point of view, is *not* sufficient in this numerical analysis context, and an additional second ingredient is essential. This other key ingredient is a "quasi-optimality" property, guaranteeing that the error in the computed numerical solution is close to the best approximation error, where "closeness" should be quantified explicitly with respect to k. This quasi-optimality may be obtained by proving that the boundary integral operator is *coercive*, with explicitly known k-dependence of the corresponding coercivity constant. (Note that coercivity is a stronger property than boundedness of the inverse of the operator.) The first main contribution of this paper is that the newly proposed "star-combined" boundary integral operator *is* coercive, uniformly in k, for all star-shaped domains, leading to the first frequency-explicit error bounds for hybrid methods in domains other than the circle and sphere. We prove this coercivity (and indeed construct this new operator) by a novel application of the Morawetz and Ludwig identity. Thus, the classical arguments of Morawetz and Ludwig, initially used to justify GO (and thus best approximation properties of the new hybrid numerical methods), are now also used to provide a novel formulation of the boundary value problem and a proof of its numerical stability.

The Morawetz and Ludwig identity is related to an identity introduced by Rellich [58] and generalised by Payne and Weinberger [56]. A Rellich-type identity was a key tool in Verchota's proof in 1984 that the standard boundary integral operators for the Laplace equation are invertible on Lipschitz domains [60]. This present paper appears to be the first one to realise the additional potential of the Morawetz and Ludwig identity in this context, and make the extensions required to these arguments (albeit with slight modifications to the integral operator) to prove the much stronger property of coercivity for a class of Lipschitz domains. Thus, we expect this to be of additional interest in its own right, i.e. independently of the original motivation in high-frequency scattering.

1.1 Formulation of the problem

Consider the problem of scattering of a time-harmonic ($e^{-i\omega t}$ time dependence) acoustic wave by a bounded, sound soft, obstacle occupying a compact set $\Omega_i \subset \mathbb{R}^d$ ($d = 2$ or 3) with Lipschitz boundary Γ, such that the set $\Omega_e := \mathbb{R}^d \setminus \Omega_i$ is connected. The medium of propagation, occupying Ω_e, is assumed to be homogeneous, isotropic, and at rest. Under the assumption that u^I is an entire solution of the Helmholtz (or reduced wave) equation with wave number $k = \omega/c > 0$ (where $c > 0$ denotes the speed of sound) we seek the resulting time-harmonic acoustic pressure field u that satisfies the Helmholtz equation

$$\mathcal{L}u := \Delta u + k^2 u = 0 \qquad \text{in} \qquad \Omega_e, \tag{1.1}$$

the sound soft (Dirichlet) boundary condition

$$u = 0 \qquad \text{on} \qquad \Gamma := \partial\Omega_e, \tag{1.2}$$

and the Sommerfeld radiation condition

$$\frac{\partial u^S}{\partial r} - iku^S = o(r^{-(d-1)/2}) \tag{1.3}$$

as $r := |x| \to \infty$, uniformly in $\hat{x} := x/r$, where $u^S := u - u^I$ is the scattered part of the field (see e.g. [25]). This problem has exactly one solution under the constraint that u and ∇u are locally square integrable.

This boundary value problem can be reformulated as an integral equation on the surface of the scatterer, Γ, using Green's integral representation for the solution u, that is

$$u(x) = u^I(x) - \int_\Gamma \Phi_k(x,y)\frac{\partial u}{\partial n}(y)ds(y), \quad x \in \Omega_e, \tag{1.4}$$

where $\partial/\partial n$ is the derivative in the normal direction, with the unit normal n directed into Ω_e, and $\Phi_k(x,y)$ the fundamental solution of the Helmholtz equation given by

$$\Phi_k(x,y) = \frac{i}{4}H_0^{(1)}(k|x-y|),\ d=2, \qquad \Phi_k(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|},\ d=3.$$

Taking the Dirichlet and Neumann traces of (1.4) on Γ one obtains two integral equations for the unknown Neumann boundary value $\partial u/\partial n$:

$$S_k\frac{\partial u}{\partial n} = u^I, \tag{1.5}$$

$$\left(\frac{1}{2}I + D'_k\right)\frac{\partial u}{\partial n} = \frac{\partial u^I}{\partial n}, \tag{1.6}$$

where the integral operators S_k and D'_k, the single layer potential and its normal derivative, are defined for $\psi \in L^2(\Gamma)$ by

$$S_k\psi(x) = \int_\Gamma \Phi_k(x,y)\psi(y)ds(y)\ , \tag{1.7}$$

$$D'_k\psi(x) = \int_\Gamma \frac{\partial \Phi_k(x,y)}{\partial n(x)}\psi(y)ds(y), \quad x \in \Gamma. \tag{1.8}$$

Both integral equations (1.5) and (1.6) fail to be uniquely solvable for certain values of k (those such that k^2 is a Dirichlet or Neumann eigenvalue of the Laplacian in Ω_i respectively), and the standard way to resolve this difficulty is to take a linear combination of the two equations. This yields the integral equation

$$A_{k,\eta}\frac{\partial u}{\partial n} = f \tag{1.9}$$

where

$$A_{k,\eta} := \frac{1}{2}I + D'_k - i\eta S_k \tag{1.10}$$

is the standard combined potential operator, with $\eta \in \mathbb{R}\setminus\{0\}$ the so-called coupling parameter, and

$$f(x) = \frac{\partial u^I}{\partial n}(x) - i\eta u^I(x), \quad x \in \Gamma.$$

Standard trace results imply that the unknown Neumann boundary value $\partial u/\partial n$ is in $H^{-1/2}(\Gamma)$, and a regularity result due to Nečas [54] (stated as Theorem 3.2 below) implies that $\partial u/\partial n$ is in fact in $L^2(\Gamma)$. Thus we can consider the integral equation (1.9) as an operator equation in $L^2(\Gamma)$, which is a natural space for the practical solution of second kind integral equations since it is self-dual. It is well known that, for $\eta \neq 0$, $A_{k,\eta}$ is a bounded and invertible operator on $L^2(\Gamma)$ (see [20] for details, particularly regarding how classical results can be adapted to the general Lipschitz case).

Commonly recommended choices for the coupling parameter η are to take η proportional to k for k large, and η constant (when $d=3$) or proportional to $(\log k)^{-1}$ (when $d=2$) for k small. These choices have been justified by theoretical studies for the case of Γ a circle or sphere [41, 40, 2, 3], and also on the basis of computational experience [11]. Recently these choices have been shown to be near optimal in terms of minimising the condition number of $A_{k,\eta}$ for more general domains by the analysis and numerical experiments of [19] and [8].

A well-known method for solving the integral equation (1.9) is the Galerkin method, namely, given an approximation space (of dimension N) $\mathcal{S}_N \subset L^2(\Gamma)$, find $v_N \in \mathcal{S}_N$ such that

$$(A_{k,\eta}v_N, \phi)_{L^2(\Gamma)} = (f,\phi)_{L^2(\Gamma)}, \quad \forall \phi \in \mathcal{S}_N.$$

Denoting the unknown Neumann boundary value $\partial u/\partial n$ on Γ by v, one would then like to prove an error estimate of the form

$$\|v - v_N\|_{L^2(\Gamma)} \leq C \inf_{\phi_N \in \mathcal{S}_N} \|v - \phi_N\|_{L^2(\Gamma)}, \tag{1.11}$$

and if such an estimate holds the Galerkin scheme is said to be "quasi-optimal". In the high frequency context, one would also like to know how the constant C depends on k.

When the boundary Γ is C^1 the integral operators D_k' and S_k are compact on $L^2(\Gamma)$ so that $A_{k,\eta}$ is a compact perturbation of the identity (see e.g. [33]). Classical arguments based on this property can be used to show that the numerical solution v_N obtained by the Galerkin method with an approximation space of piecewise polynomials satisfies the error estimate (1.11) once the dimension N of the approximation space is sufficiently high, i.e. $N \geq N_0$ for some $N_0 \in \mathbb{N}$ (see e.g. [4]). However these arguments have the following severe limitations: (a) since the perturbation depends on k in a complicated nonlinear way, the classical compact perturbation argument gives no information about how the constants C and N_0 depend on k, rendering the bound (1.11) that is obtained useless in the high frequency case; and (b) they do not apply in the case when Γ is non-smooth, in particular Lipschitz. Recently, using a sophisticated k-explicit version of the classical arguments, Melenk has overcome the limitation (a) in the case when Γ is C^∞ and an hp-Galerkin boundary element method is used on a quasi-uniform mesh. Indeed, he has shown that given $\varepsilon > 0$, (1.11) holds with $C = 1 + \varepsilon$ provided that firstly $N \geq k^{d-1} N_0(\varepsilon)$, where $N_0(\varepsilon)$ depends only on ε, and secondly that the polynomial degree is carefully chosen to depend logarithmically on k [42]. This result was obtained using a novel splitting of the operator $A_{k,\eta}$ [45] motivated by a related numerical analysis for a domain-based (finite element) formulation [46].

1.2 The main result of this paper

This paper introduces a new boundary integral operator closely related to $A_{k,\eta}$. Note that with u given by (1.4), the integral equation (1.9) involving $A_{k,\eta}$ arises from

$$(n.\nabla u)|_\Gamma - i\,\eta\, u|_\Gamma \tag{1.12}$$

expressed in terms of boundary integral operators. In fact, with u given by (1.4), recalling the boundary condition (1.2), we find that $u|_\Gamma$ yields the integral equation (1.5) and $\nabla u|_\Gamma$ yields the integral equation

$$\left(n\left(\frac{1}{2}I + D_k'\right) + \nabla_\Gamma S_k\right)\frac{\partial u}{\partial n} = \nabla u^I,$$

where the vector valued boundary integral operator $\nabla_\Gamma S_k$ is defined on $L^2(\Gamma)$ by

$$\nabla_\Gamma S_k \psi(x) = \int_\Gamma \nabla_{\Gamma,x} \Phi_k(x,y)\psi(y)ds(y) \tag{1.13}$$

for almost every $x \in \Gamma$, and where $\nabla_{\Gamma,x}\Phi_k(x,y) = \nabla_x \Phi_k(x,y) - n(x)\frac{\partial \Phi_k}{\partial n(x)}(x,y)$ is the surface gradient on Γ of the fundamental solution. The integral in (1.13) must be understood in the principal value sense.

The boundary integral operator that we focus on in this paper is obtained by replacing the normal vector in (1.12) with the position vector x with respect to a suitable origin, i.e.

$$(x.\nabla u)|_\Gamma - i\,\eta\, u|_\Gamma,$$

and this choice is motivated by the Morawetz and Ludwig identity, as clarified below. The main result of this paper is that this resulting new boundary integral operator is coercive, uniformly in k, for star-shaped Lipschitz domains with a particular choice of η. Because of this property on star-shaped domains, we call this integral operator, denoted by $\mathscr{A}_k$ and defined by (1.15) below, the "star-combined" potential operator. The main result is the following:

Theorem 1.1 (Coercivity of the star-combined operator) *Suppose that Ω_i is a bounded star-shaped Lipschitz domain, and x is the position vector relative to an origin from which Ω_i is star-shaped. Then for all $\phi \in L^2(\Gamma)$*

$$\Re(\mathscr{A}_k\phi, \phi)_{L^2(\Gamma)} \geq \gamma\|\phi\|^2_{L^2(\Gamma)}, \tag{1.14}$$

where the "star-combined" operator $\mathscr{A}_k$ is given by

$$\mathscr{A}_k = (x \cdot n)\left(\frac{1}{2}I + D'_k\right) + x \cdot \nabla_\Gamma S_k - i\eta S_k \tag{1.15}$$

with the function η chosen as

$$\eta = kr + i\frac{d-1}{2}, \tag{1.16}$$

and the k-independent coercivity constant γ is given by

$$\gamma = \frac{1}{2}\operatorname*{ess\,inf}_{x\in\Gamma}(x \cdot n(x)) > 0. \tag{1.17}$$

This is an interesting result for the following reasons:

Firstly, it is perhaps surprising that this formulation of the Helmholtz equation is coercive at all, let alone for a large class of non-smooth domains and uniformly in k. Indeed, Helmholtz problems are usually thought to be sign-indefinite, and the standard analysis for both the domain-based weak formulation and Galerkin boundary integral equation methods is to attempt to prove a Gårding inequality, i.e. to attempt to show that the operator is a compact perturbation of a coercive operator. Moreover, in the boundary integral context, for general Lipschitz domains, not even a Gårding inequality is known for the operator (1.10). The only rigorous coercivity result known until now is that $A_{k,k}$ is coercive uniformly in k on the circle and sphere in the limit $k \to \infty$. Indeed, it was proved in [28] that for the circle there exists a k_0 such that

$$\Re(A_{k,k}\phi, \phi)_{L^2(\Gamma)} \geq \gamma\|\phi\|^2_{L^2(\Gamma)}, \quad \forall k \geq k_0,$$

with $\gamma = 1/2$. (For the sphere, it was proved that coercivity holds for any $\gamma < 1/2$ [28].) These proofs relied on Fourier analysis on the circle/sphere and involved bounding combinations of Bessel functions uniformly in argument and order. Numerical computations indicate that the coercivity of $A_{k,k}$ in the high frequency limit holds for much more general domains [9], however this has yet to be proved. When Γ is the unit circle or sphere, the star-combined operator $\mathscr{A}_k$ reduces to $A_{k,\eta}$, with the choice of η given by (1.16), since in this case $x \cdot n = 1$ and $x \cdot \nabla_\Gamma S_k = 0$ as $\nabla_\Gamma S_k$ is a vector-valued operator in the tangent space of Γ. Thus Theorem 1.1 provides alternative (and, as we shall see, much simpler) proofs of the coercivity results as $k \to \infty$ of [28], and also shows that coercivity holds uniformly for all k on the circle and sphere provided we make the choice of coupling constant (1.16).

Returning to numerical methods for the Helmholtz equation (1.1), the unknown Neumann boundary value $\partial u/\partial n$ on Γ satisfies the following boundary integral equation involving the star-combined operator,

$$\mathscr{A}_k\frac{\partial u}{\partial n} = x.\nabla u^I - i\eta u^I. \tag{1.18}$$

Supposing the coercivity (1.14) holds, if (1.18) is solved by the Galerkin method using *any* approximation space $\mathcal{S}_N$ then, by the Lax-Milgram Theorem and Céa's Lemma (see e.g. [4]), the error estimate (1.11) holds with

$$C = \frac{\|\mathscr{A}_k\|_{L^2(\Gamma)}}{\gamma}. \tag{1.19}$$

Thus, once the k-dependence of $\|\mathscr{A}_k\|_{L^2(\Gamma)}$ is established (see Theorem 4.2 below), the k-dependence of the constant C in (1.11) is then explicitly known. Note also that no "threshold" requirement of a minimum value of N is needed in contrast to error estimates obtained via a Gårding inequality.

Recently there has been much research interest in designing "hybrid asymptotic-numerical" boundary integral methods for the solution of the Helmholtz equation (1.1) when k is very large. The reason for this is that in conventional methods, using approximation spaces comprising of piecewise polynomials, the dimension of the approximation space N must grow like k^{d-1} as $k \to \infty$ to maintain accuracy, putting very high frequency problems out of reach of standard methods. One popular approach to overcome this difficultly is to incorporate the oscillation of the solution into the approximation space, often using asymptotic results from GO and the GTD to identify the rapidly oscillating part of the solution. Some of the pioneering work in this area was carried out in [1], [34], and [10] – see e.g. the review [18] for a survey. The goal of these methods is to design approximation spaces $\mathcal{S}_{N,k}$ such that the best approximation error $\inf_{\phi_N \in \mathcal{S}_{N,k}} \|v - \phi_N\|_{L^2(\Gamma)}$ is bounded, or grows mildly, as $k \to \infty$ for fixed N. Since these approximation spaces depend on k, both the standard and the novel (due to Melenk) perturbation arguments, where the perturbation is k-dependent, apparently cannot be used to prove useful estimates for the stability and convergence of these hybrid methods. However, if the star-combined operator $\mathscr{A}_k$ is used instead of $A_{k,\eta}$, then Theorem 1.1 gives the first stability and convergence proofs of the hybrid Galerkin methods of e.g. [20, 28] in domains other than the circle/sphere. A natural question is then, how much more difficult is the star-combined operator $\mathscr{A}_k$ to implement than the standard combined operator $A_{k,\eta}$? In Section §4.3 we show that in principle $\mathscr{A}_k$ is no more difficult to implement that $A_{k,\eta}$. Indeed, the only substantial difference between the two is the presence of the Cauchy singular operator $\nabla_\Gamma S_k$ in $\mathscr{A}_k$. However, this integral operator is equal to the surface gradient, ∇_Γ, of the single layer potential, i.e. $\nabla_\Gamma(S_k)$ (which is the reason for this notation), and in the Galerkin method the surface gradient in this term can be moved onto the test function by integration by parts. This means that the relevant integrals only require evaluations of the single layer potential S_k.

The theory of boundary integral equations in $L^2(\Gamma)$ on Lipschitz domains relies on the harmonic analysis results of (among others) Calderón, Coifman, McIntosh, Meyer, and Verchota [17], [24], [60] (and is summarised in e.g. [48], [38]). However, a proof has yet to be found that the Galerkin method for the boundary integral equation (1.9) converges in $L^2(\Gamma)$ on a general Lipschitz domain. (A summary of related results, including the notable work of Elschner [30], is given in [61].) To compensate for this lack of theory there have been several recent investigations proposing modified or "stabilized" boundary integral equation formulations of the Helmholtz and Maxwell equations e.g. [14], [13], [31], [32], [15]. These investigations are all based on the fact that the single layer potential S_k satisfies a Gårding inequality when viewed as a mapping from $H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$. Thus suitable operators can be constructed which act on $\frac{1}{2}I + D'_k$ in the combined potential operator $A_{k,\eta}$ so that the resulting modified combined potential operator maps $H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and satisfies a Gårding inequality on general Lipschitz domains. To the authors' knowledge, the star-combined operator $\mathscr{A}_k$ is unique in being the only second kind boundary integral formulation of the Helmholtz equation (1.1) that is coercive in $L^2(\Gamma)$ on a general Lipschitz domain (albeit a star-shaped one).

1.3 The motivation for the proof of Theorem 1.1

In the rest of the introduction we give an outline of the main ideas in the proof of Theorem 1.1. The proof is completely dependent on an identity introduced by Morawetz and Ludwig in [52]. Morawetz-type estimates have formed the basis of many investigations of well-posedness of PDEs following the pioneering work [50], and the related Rellich-type identities for linear elliptic equations have been used extensively in numerical analysis of the Helmholtz equation. However it appears that the natural role of the identity in [52] has been overlooked in a numerical analysis context until now.

The proof of Theorem 1.1 is motivated by a simple result about the single layer potential S_k, namely that $\Re(-i(S_k v, v)_{L^2(\Gamma)}) \geq 0$ for all $v \in L^2(\Gamma)$. This can be proved using Green's identity, as we shall now show. (A different proof of this result appears in [55, §3.4.4], while essentially the same proof as that given here also appears in [31].)

Let D be any bounded Lipschitz domain with outward normal ν. If u is sufficiently regular up to the boundary, namely $u \in C^2(\overline{D})$, then, by Green's identity and the divergence theorem,

$$\int_D \left(\bar{u}\mathcal{L}u - k^2|u|^2 + |\nabla u|^2\right) dx = \int_{\partial D} \bar{u} \frac{\partial u}{\partial \nu} ds. \tag{1.20}$$

Let Ω_i be as described at the beginning of the paper, and let u be the single layer potential $\mathcal{S}_k$

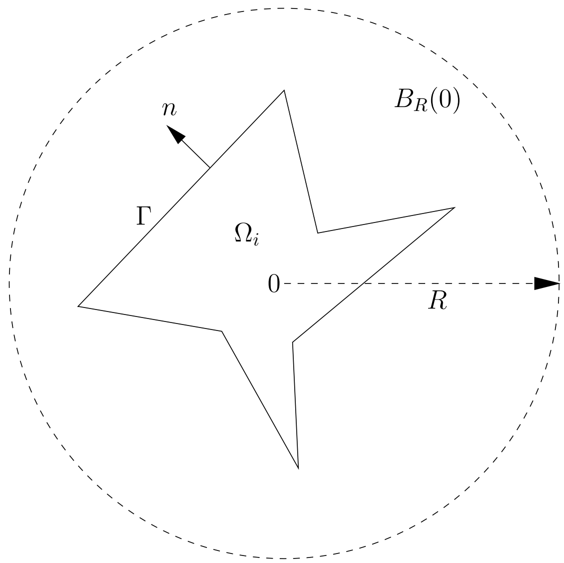

Figure 1: The domain Ω_i, with boundary Γ and outward pointing normal n, and the ball of radius R denoted by $B_R(0)$

with density $\phi \in L^2(\Gamma)$, that is,

$$u(x) = \mathcal{S}_k\phi(x) := \int_\Gamma \Phi_k(x,y)\phi(y)ds(y), \quad x \in \mathbb{R}^d \setminus \Gamma.$$

Then $\mathcal{L}u = 0$ in $\Omega_i \cup \Omega_e$, u is continuous across Γ, but $\partial u/\partial n$ has a jump across Γ. Assume Ω_i contains the origin and let $R > 0$ be such that $\Omega_i \subset B_R(0)$, see Figure 1. Apply the identity (1.20) first with $D = \Omega_i$ and then with $D = \Omega_e \cap B_R(0)$ (for a general Lipschitz domain this involves some technicalities, see Remark 4.7 below). Adding the two resulting equations yields

$$0 = \int_{(\Omega\cup\Omega_e)\cap B_R} \left(k^2|u|^2 - |\nabla u|^2\right) dx + \int_\Gamma \bar{u}\left(\frac{\partial u_-}{\partial n} - \frac{\partial u_+}{\partial n}\right) ds + \int_{\partial B_R(0)} \bar{u}\frac{\partial u}{\partial r} ds \tag{1.21}$$

where $\partial u_\pm/\partial n$ denote the limits of $\partial u/\partial n$ on Γ from within Ω_e and Ω_i respectively (and recall that n points into Ω_e). The term involving $k^2|u|^2 - |\nabla u|^2$ is real, but, in general, sign-indefinite, so we take the imaginary part of the expression (1.21). After using the jump relation for the normal derivative of $\mathcal{S}_k$ on Γ, this yields

$$0 = \Im \int_\Gamma \overline{S_k\phi}\,\phi\, ds + \Im \int_{\partial B_R} \bar{u}\frac{\partial u}{\partial r} ds.$$

When $R \to \infty$ the last term can be expressed in terms of the far-field pattern of u, which we denote by $f_1(\hat{x})$ where $\hat{x} := x/r$, see (2.12) below, and the result is

$$\Re(-i(S_k\phi, \phi)_{L^2(\Gamma)}) = k \int_{S^{d-1}} |f_1(\hat{x})|^2 ds \geq 0, \tag{1.22}$$

where S^{d-1} is the d-dimensional unit sphere. Note that (1.22) implies non-negativity of the operator $-iS_k$, but that this operator is not invertible, let alone coercive. Indeed, if $k^2 = \lambda_j$ and $\phi = \partial u_j/\partial n$, where λ_j and u_j are an eigenvalue and a corresponding eigenfunction of the Dirichlet Laplacian in Ω_i respectively, then $u = \mathcal{S}_k\phi \equiv 0$ in Ω_e and hence $f_1 \equiv 0$.

The motivation for the proof of Theorem 1.1 is the following question: Given that the above argument involving Green's identity yields information about part of the combined potential operator $A_{k,\eta}$ (1.10), namely the part involving S_k, can we repeat the argument using a different identity to obtain information about more, or even all, of $A_{k,\eta}$? This leads us to consider other identities involving solutions of the Helmholtz equation.

One way of obtaining identities for solutions of the Helmholtz equation is to multiply $\mathcal{L}u = 0$ by $\overline{\mathcal{N}u}$, where the "multiplier" $\mathcal{N}$ is some suitable operator, and integrate by parts. For example, in this framework (1.20) arises from the choice $\mathcal{N}u = u$. Rellich-type identities are obtained by choosing $\mathcal{N}u$ to be a derivative of u. Common choices of derivatives include a derivative in the radial direction in the case of star-shaped obstacles (so $\mathcal{N}u = x \cdot \nabla u$) and a derivative along the vertical co-ordinate axis in the case of scattering by a rough surface. Originally introduced by Rellich in [58], these identities have been used extensively in analysis, for example to prove elliptic regularity results [54] (which we use below as Theorem 3.2), to prove the invertibility of the boundary integral operator $A_{0,0}$ on Lipschitz domains [60], and in the famous work on elliptic problems on non-smooth domains by Jerison and Kenig, see e.g. [38]. In a numerical analysis context Rellich identities have recently been used to prove regularity results for the Helmholtz equation in interior domains with impedence boundary conditions [44, Prop. 8.1.4], [26], and to prove k-explicit bounds in the exterior of star-shaped domains for both $\|A_{k,\eta}^{-1}\|$ and the inf-sup constant for the domain based formulation of the Helmholtz equation [21].

For simplicity, let $d = 2$ (the three dimensional case, $d = 3$, is slightly more complicated). The multiplier $\mathcal{N}u = x \cdot \nabla u$ leads to the following identity for solutions of $\mathcal{L}u = 0$:

$$\nabla \cdot \bigg(2\Re\left(x \cdot \overline{\nabla u}\nabla u\right) + \left(k^2|u|^2 - |\nabla u|^2\right) x\bigg) = 2k^2|u|^2 \tag{1.23}$$

(see Lemma 2.1 below). The reason this identity can be used to obtain estimates is that the non-divergence terms are sign-definite. However, when u satisfies the radiation condition (1.3) and this identity is integrated over $\Omega_e \cap B_R(0)$, the contribution from the surface integral is unbounded as $R \to \infty$, meaning that repeating the argument leading to (1.22) is not possible. This drawback of the Rellich identity was encountered in [21]. There the authors avoided this difficulty by keeping R fixed, expanding u on $\partial B_R(0)$ as a Fourier series and using properties of Bessel functions to relate the integral over $\partial B_R(0)$ to an integral on Γ [21, Lemma 2.1].

In [52] Morawetz and Ludwig introduced the multiplier $\mathcal{N}u = r\mathcal{M}u$ where

$$\mathcal{M}u = \frac{x}{r} \cdot \nabla u - iku + \frac{d-1}{2r}u. \tag{1.24}$$

This leads to an identity very similar to (1.23), namely that for solutions of $\mathcal{L}u = 0$,

$$\nabla \cdot \bigg(2\Re\left(r\overline{\mathcal{M}u}\nabla u\right) + \left(k^2|u|^2 - |\nabla u|^2\right) x\bigg) = \Big(|\nabla u|^2 - |u_r|^2\Big) + |u_r - iku|^2 \tag{1.25}$$

where $ru_r = x \cdot \nabla u$. As before the non-divergence terms are all positive. However, the choice of terms subtracted from $x \cdot \nabla u$ in the multiplier (1.24) means that the integral on $\partial B_R(0)$ tends to zero as $R \to \infty$, making it perfectly suited for repeating the argument leading to (1.22). In this way we ultimately obtain the inequality

$$\Re\left(\left(x \cdot n\, D_k' + x \cdot \nabla_\Gamma S_k - i\eta S\right)\phi, \phi\right)_{L^2(\Gamma)} \geq 0,$$

where η is given by (1.16), which gives the coercivity result of Theorem 1.1. We also note that using the Morawetz-Ludwig identity (1.25) instead of the Rellich one (1.23) yields the main results of [21] without the use of the result [21, Lemma 2.1] described above. Thus for Helmholtz problems in unbounded domains the Morawetz-Ludwig identity has a distinct advantage over the standard Rellich one.

1.4 Outline of paper

For completeness, in Section 2 we briefly derive the Rellich and Morawetz-Ludwig identities, and emphasise the advantages the latter has over the former in this context. Section 3 states precisely the acoustic scattering problem in Lipschitz domains and recalls standard results we shall use later. Section 4 introduces the star-combined operator, with §4.2 containing the proof of Theorem 1.1, and §4.3 demonstrating that the star-combined operator is as easy to implement as the standard combined operator in a Galerkin context. We conclude with some remarks in Section 5.

2 The Rellich and Morawetz-Ludwig identities

Lemma 2.1 (Rellich identity) *Let $v \in C^2(D)$ where $D \subset \mathbb{R}^d$, and $\mathcal{L}v = \Delta v + k^2 v$ where $k \in \mathbb{R}$. Then*

$$2\Re(x \cdot \overline{\nabla v}\mathcal{L}v) = \nabla \cdot \left[2\Re\left(x \cdot \overline{\nabla v}\nabla v\right) + (k^2|v|^2 - |\nabla v|^2)x\right] + (d-2)|\nabla v|^2 - dk^2|v|^2. \tag{2.1}$$

Proof. The basic building block of the Rellich identity is

$$(x.\overline{\nabla v})\Delta v = \nabla \cdot \left[\left(x \cdot \overline{\nabla v}\right)\nabla v\right] - |\nabla v|^2 - \nabla v \cdot \left((x \cdot \nabla)\,\overline{\nabla v}\right) \tag{2.2}$$

which can be proved by expanding the divergence term on the right hand side. We would like each term on the right hand side of (2.2) to either be sign-definite, or be the divergence of something, and the only term that is not one of these is the final term. However, the real part of this final term can be expressed as the sum of a divergence and a quadratic term using

$$\nabla \cdot \left(|\nabla v|^2 x\right) = d|\nabla v|^2 + 2\Re\left[\nabla v \cdot \left((x \cdot \nabla)\,\overline{\nabla v}\right)\right]. \tag{2.3}$$

Thus, by taking twice the real part of (2.2) and using (2.3), we obtain

$$2\Re(x \cdot \overline{\nabla v}\Delta v) = \nabla \cdot \left[2\Re\left(x \cdot \overline{\nabla v}\nabla v\right)\right] - 2|\nabla v|^2 - \nabla \cdot \left(|\nabla v|^2 x\right) + d|\nabla v|^2. \tag{2.4}$$

Finally we add k^2 times the identity

$$2\Re(x \cdot \overline{\nabla v}\, v) = \nabla \cdot \left(|v|^2 x\right) - d|v|^2$$

to (2.4) to obtain (2.1). ■

The identity (2.1) with $k = 0$ (which appears in [38, Lemma 2.1.13]) is a special case of a general identity for second order strongly elliptic operators introduced by Payne and Weinberger [56], [43, Lemma 4.22].

To obtain the Morawetz-Ludwig identity from the Rellich one, we seek to add more terms to the instances of $x \cdot \overline{\nabla v}$ appearing on the left and right hand sides of (2.1).

Lemma 2.2 (Morawetz-Ludwig identity) *[52, Equation 1.2] Let v and $\mathcal{L}v$ be as in Lemma 2.1 and define the operator $\mathcal{M}_\alpha$ by*

$$\mathcal{M}_\alpha v = v_r - ikv + \frac{\alpha}{r}v, \tag{2.5}$$

where $\alpha \in \mathbb{R}$ and $v_r = x \cdot \nabla v/r$. Then

$$\begin{aligned}2\Re(r\overline{\mathcal{M}_\alpha v}\mathcal{L}v) =& \nabla \cdot \left[2\Re\left(r\overline{\mathcal{M}_\alpha v}\nabla v\right) + \left(k^2|v|^2 - |\nabla v|^2\right)x\right]\\ &+ (2\alpha - (d-1))(k^2|v|^2 - |\nabla v|^2) - (|\nabla v|^2 - |v_r|^2) - \left|\mathcal{M}_\alpha v - \frac{\alpha}{r}v\right|^2.\end{aligned} \tag{2.6}$$

Proof. By expanding the divergences on the right hand sides we have both

$$2\Re(ikr\bar{v}\mathcal{L}v) = \nabla \cdot \left[2\Re\left(ikr\bar{v}\nabla v\right)\right] - 2\Re\left(ikv_r\bar{v}\right) \tag{2.7}$$

and

$$2\Re\left(\bar{v}\mathcal{L}v\right) = \nabla \cdot \left[2\Re\left(\bar{v}\nabla v\right)\right] - 2|\nabla v|^2 + 2k^2|v|^2. \tag{2.8}$$

Thus adding (2.1), (2.7) and α times (2.8) we obtain

$$\begin{aligned}2\Re(r\overline{\mathcal{M}_\alpha v}\mathcal{L}v) =& \nabla \cdot \left[2\Re\left(r\overline{\mathcal{M}_\alpha v}\nabla v\right) + \left(k^2|v|^2 - |\nabla v|^2\right)x\right]\\ &+ (d-2-2\alpha)|\nabla v|^2 + (2\alpha - d)k^2|v|^2 + 2\Re(ik\bar{v}_r v),\end{aligned} \tag{2.9}$$

(this is [52, Equation (A.3)]). As before, we would like each term on the right hand side to either be sign-definite or be a divergence; the only term not of this form is $2\Re(ik\bar{v}_r v)$. By expanding $|\mathcal{M}_\alpha v|^2$, $2\Re(ik\bar{v}_r v)$ can be written as

$$|v_r|^2 + k^2|v|^2 - |\mathcal{M}_\alpha v|^2 + \frac{\alpha^2}{r^2}|v|^2 + \frac{2\alpha}{r}\Re(v_r\bar{v}),$$

and thus the non-divergence terms in (2.9) become

$$(d-2-2\alpha)|\nabla v|^2 + (2\alpha-(d-1))k^2|v|^2 + |v_r|^2 - |\mathcal{M}_\alpha v|^2 + \frac{\alpha^2}{r^2}|v|^2 + \frac{2\alpha}{r}\Re(v_r\bar{v}). \tag{2.10}$$

Using

$$\frac{2\alpha}{r}\Re(v_r\bar{v}) = \frac{2\alpha}{r}\Re(v\,(\bar{v}_r + ik\bar{v})) = \frac{2\alpha}{r}\Re\left(v\left(\overline{\mathcal{M}_\alpha v} - \frac{\alpha}{r}\bar{v}\right)\right),$$

upon straightforward rearrangement (2.10) becomes

$$(2\alpha-(d-1))(k^2|v|^2 - |\nabla v|^2) - (|\nabla v|^2 - |v_r|^2) - |\mathcal{M}_\alpha v|^2 - \frac{\alpha^2}{r^2}|v|^2 + \frac{2\alpha}{r}\Re\left(v\overline{\mathcal{M}_\alpha v}\right)$$

and thus, factorising the last three terms of the above expression, (2.9) yields (2.6). ∎

The Rellich and Morawetz-Ludwig identities, derived above for an arbitrary function $v \in C^2(D)$, are designed to be used for solution of the Helmholtz equation $\mathcal{L}u = 0$ (or its inhomogeneous form). Both the next remark and the next lemma concern properties of these identities in this situation.

Remark 2.3 (The choice of α in $\mathcal{M}_\alpha$ and connection with the radiation condition) *Noting firstly that $|\nabla v|^2 - |v_r|^2 \geq 0$, and secondly that $(k^2|v|^2 - |\nabla v|^2)$ is, in general, sign-indefinite, the choice*

$$\alpha = \frac{(d-1)}{2} \tag{2.11}$$

ensures that all the non-divergence terms on the right hand side of (2.6) have the same sign (i.e. all non-positive). It is perhaps surprising that this choice (2.11) is connected to the far-field behaviour of solutions of the Helmholtz equation. Indeed, if $\mathcal{L}u = 0$ and u itself satisfies the radiation condition (1.3) (i.e. (1.3) holds with u^S replaced by u) then

$$u(x) = \frac{e^{ikr}}{r^{(d-1)/2}}\left(f_1(\hat{x}) + \frac{f_2(\hat{x})}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right) \quad \text{as } r \to \infty, \tag{2.12}$$

where f_1 and f_2 are functions of the angular variable $\hat{x} = x/r$, and the asymptotics admits differentiation in r and $\hat{x}$, see e.g. [25, Theorem 3.6]. A simple calculation shows that

$$\mathcal{M}_\alpha u = u_r - iku + \frac{\alpha}{r}u = \frac{e^{ikr}}{r^{(d+1)/2}}\left(\alpha - \left(\frac{d-1}{2}\right)\right)f_1(\hat{r}) + \mathcal{O}\left(\frac{1}{r^{(d+3)/2}}\right) \quad \text{as } r \to \infty. \tag{2.13}$$

Thus, for general α, $\mathcal{M}_\alpha u$ is $\mathcal{O}(r^{-(d+1)/2})$, but the choice (2.11) makes the coefficient of the $\mathcal{O}(r^{-(d+1)/2})$ term in (2.13) zero, so

$$\mathcal{M}_{\frac{d-1}{2}}u = \mathcal{O}\left(\frac{1}{r^{(d+3)/2}}\right) \quad \text{as } r \to \infty.$$

Lemma 2.4 (The key difference between the Rellich and Morawetz-Ludwig identities) *Let u satisfy $\mathcal{L}u = 0$ in the domain $\{|x| \geq R_0\}$ for some $R_0 > 0$, and suppose that u satisfies the radiation condition (1.3). Then, when the Rellich (2.1) and Morawetz-Ludwig (2.6) identities, with v replaced by u, are integrated in $\{R_0 \leq |x| \leq R\}$ using the divergence theorem, the surface integral on $|x| = R$ is $\mathcal{O}(R)$ as $R \to \infty$ in the Rellich case and $\mathcal{O}(R^{-1})$ in the Morawetz-Ludwig case (independent of the value of α in $\mathcal{M}_\alpha$).*

Proof. Using the fact that the outward normal to the surface $|x| = R$ is x, which equals $R\hat{x}$, the relevant surface integral is

$$\int_{|x|=R} R\Big(|u_r|^2 + k^2|u|^2 - (|\nabla u|^2 - |u_r|^2)\Big)ds$$

in the Rellich case, and

$$\int_{|x|=R} R\left(|u_r|^2 + k^2|u|^2 + 2\Re\left[\left(ik + \frac{\alpha}{R}\right)\bar{u}u_r\right] - (|\nabla u|^2 - |u_r|^2)\right)ds$$

$$= \int_{|x|=R} R \left(|\mathcal{M}_\alpha u|^2 - \alpha^2 \frac{|u|^2}{R^2} - (|\nabla u|^2 - |u_r|^2) \right) ds$$

in the Morawetz-Ludwig case. The asymptotics of u above, equation (2.12), imply that $|u|^2$ and $|u_r|^2$ are both $\mathcal{O}(R^{1-d})$ on $|x| = R$ as $R \to \infty$, and $|\mathcal{M}_\alpha u|^2$ is $\mathcal{O}(R^{-1-d})$ (for any α). The quantity $|\nabla u|^2 - |u_r|^2$ equals $|\nabla_S u|^2$ where ∇_S is the surface gradient on $|x| = R$, which satisfies $\nabla_S u = \nabla u - \hat{x} u_r$. This differential operator is equal to $1/R$ multiplied by an operator acting only on $\hat{x}$, i.e. the angular variables, thus $|\nabla_S u|^2$ is $\mathcal{O}(R^{-1-d})$. Since $\int_{|x|=R} ds = \mathcal{O}(R^{d-1})$ the conclusions follow. ■

Later we will need the Morawetz-Ludwig identity (2.6) integrated over a Lipschitz domain, which is given by the next lemma. Following Remark 2.3, from now on we shall only consider the Morawetz-Ludwig identity (2.6) with the particular choice of α given by (2.11).

Lemma 2.5 *Let $D \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain, let $v \in C^2(\overline{D})$ and let ν denote the outward pointing unit normal to D. Let*

$$\mathcal{M}v := \mathcal{M}_{\frac{d-1}{2}} v = v_r - ikv + \frac{(d-1)}{2r} v. \tag{2.14}$$

Then,

$$\begin{aligned} &\int_D \left(2\Re(r\overline{\mathcal{M}v}\mathcal{L}v) + |v_r - ikv|^2 + (|\nabla v|^2 - |v_r|^2) \right) dx \\ &= \int_{\partial D} \left(2\Re \left(r\overline{\mathcal{M}v} \frac{\partial v}{\partial \nu} \right) + (k^2|v|^2 - |\nabla v|^2) x \cdot \nu \right) ds. \end{aligned} \tag{2.15}$$

Proof. This is a consequence of applying the divergence theorem to the identity (2.6) with α given by (2.11). The divergence theorem

$$\int_D \nabla \cdot F dx = \int_{\partial D} F \cdot \nu \, ds$$

is valid when D is Lipschitz and $F \in C^1(\overline{D})$. [43, Theorem 3.34]. (Note that by the density of $C^1(\overline{D})$ in $H^1(D)$ and the continuity of the trace operator $\gamma : H^1(D) \to H^{1/2}(\partial D)$ (2.15) holds for $v \in H^2(D)$, however we will not need this in what follows.) ■

Remark 2.6 (The second Morawetz and Ludwig identity) *In addition to the identity* (2.6), *Morawetz and Ludwig obtained a second identity, namely*

$$\begin{aligned} 2\Re(r\overline{\mathcal{M}_\alpha v}\mathcal{L}v) =& \nabla \cdot \left[2\Re \left(r\overline{\mathcal{M}_\alpha v} \nabla v \right) + \left(k^2|v|^2 - |\nabla v|^2 + \alpha \frac{|v|^2}{r^2} \right) x \right] \\ &+ (2\alpha - (d-1))(k^2|v|^2 - |\nabla v|^2) - |\mathcal{M}_\alpha v|^2 - \frac{\alpha(d-2-\alpha)}{r^2}|v|^2 - (|\nabla v|^2 - |v_r|^2). \end{aligned} \tag{2.16}$$

This is obtained by using

$$\frac{2\alpha}{r} \Re(v_r \bar{v}) = \alpha \nabla \cdot \left(\frac{|v|^2}{r^2} x \right) - \frac{\alpha(d-2)}{r^2} |v|^2$$

in (2.10), *and combined with* (2.9) *this yields* (2.16). *In this identity two conditions need to be met for the non-divergence terms to be the same sign, namely* (2.11) *and* $0 \leq \alpha \leq (d-2)$. *These two conditions hold if and only if* $d \geq 3$, *but not for* $d = 2$. *Morawetz and Ludwig used the identity* (2.16) *for* $d = 3$ *and* (2.6) *for* $d = 2$. *The reason for this is that* $|\mathcal{M}_\alpha v|^2$ *appears in the non-divergence terms of* (2.16) *instead of* $|\mathcal{M}_\alpha v - \alpha v/r|^2$ *in* (2.6), *and this fact simplifies the proof of the main result in [52] for* $d = 3$. *For our purposes this difference does not matter, and so will we use* (2.6) *for both* $d = 2$ *and* 3.

3 The acoustic scattering problem in Lipschitz domains

In this section we formulate the boundary value problem in a standard Sobolev setting. (Formulations in other function spaces are possible, see e.g. [20, Remark 2.2] for an overview.) Recall that we assume that the domain corresponding to the scatterer, Ω_i, is Lipschitz (and hence so is Ω_e), and we denote the outward pointing normal to Ω_i (i.e. into Ω_e) by n. We now summarise some standard facts about Lipschitz domains, see e.g. [43].

Given a Lipschitz domain $D \subset \mathbb{R}^d$ with outward pointing normal ν, recall that there is a well-defined trace operator $\gamma : H^1(D) \to H^{1/2}(\partial D)$ which satisfies $\gamma v = v|_{\partial D}$ when $v \in \mathcal{D}(D) := \{w|_{\bar{D}} : w \in C^\infty(\mathbb{R}^d)\}$. Let $H^1(D, \Delta) := \{v \in H^1(D) : \Delta v \in L^2(D)\}$ (where Δ is the Laplacian in a weak sense). There is also a well-defined normal derivative operator, which is the unique bounded linear operator $\partial_\nu : H^1(D, \Delta) \to H^{-1/2}(\partial D)$ such that

$$\partial_\nu v = \frac{\partial v}{\partial \nu} := \nu \cdot \nabla v$$

almost everywhere on ∂D, when $v \in \mathcal{D}(D)$. Let $H^1_{\mathrm{loc}}(D)$ denote the space of measurable $v : D \to \mathbb{C}$ for which $\chi v \in H^1(D)$ for every compactly supported $\chi \in \mathcal{D}(D)$. Finally, there exists a unique operator ∇_Γ, the surface (or tangential) gradient, such that the mapping $\nabla_\Gamma : H^1(\partial D) \to (L^2(\partial D))^d$ is bounded, $\nu \cdot \nabla_\Gamma v = 0$ for all $v \in H^1(\partial D)$, and if w is C^1 in a neighbourhood of ∂D then

$$\nabla w(x) = \nabla_\Gamma w(x) + \nu \frac{\partial w}{\partial \nu}(x), \quad x \in \partial D. \tag{3.1}$$

An explicit formula for ∇_Γ in terms of a parametrisation of the boundary is given by Definition 4.10 in §4.3.

Definition 3.1 (The plane-wave time harmonic acoustic scattering problem) *Given $k > 0$ and u^I an entire solution of the Helmholtz equation* (1.1) *(such as a plane wave), find $u \in C^2(\Omega_e) \cap H^1_{\mathrm{loc}}(\Omega_e)$ such that u satisfies the Helmholtz equation* (1.1), *$\gamma u = 0$ on Γ, and $u^S = u - u^I$ satisfies the radiation condition* (1.3).

The boundary integral equation method reformulates the problem of finding u in Ω_e to finding the normal derivative of u, $\partial u/\partial n$, on Γ. Since $u \in H^1_{\mathrm{loc}}(\Omega_e)$, $\partial u/\partial n \in H^{-1/2}(\Gamma)$. However, since $u = 0$ on Γ, we actually have $\partial u/\partial n \in L^2(\Gamma)$ by the following theorem of Nečas.

Theorem 3.2 *[54, Chapter 5], [43, Theorem 4.24] Let D be a bounded Lipschitz domain with outward pointing normal ν and let $u \in H^1(D)$ satisfy $\mathcal{L}u = 0$ (in a distributional sense). If $\gamma u \in H^1(\partial D)$ then $\partial u/\partial \nu \in L^2(\partial D)$.*

When Ω_e is Lipschitz and $\phi \in L^2(\Gamma)$, the boundary integral operators S_k, D'_k and $\nabla_\Gamma S_k$ are defined by (1.7), (1.8), and (1.13) respectively, where the first integral is well-defined in a Lebesgue sense and the last two are understood in the Cauchy principal value sense, see e.g. [48]. All three operators are bounded operators on $L^2(\Gamma)$. In fact, S_k is a bounded operator from $L^2(\Gamma)$ to $H^1(\Gamma)$ and the surface gradient of S_k is $\nabla_\Gamma S_k$, i.e. $\nabla_\Gamma(S_k) = \nabla_\Gamma S_k$ [48]. Note that this last fact implies that $n \cdot \nabla_\Gamma S_k \phi = 0$ for all $\phi \in L^2(\Gamma)$.

4 The star-combined boundary integral equation

4.1 Derivation of star-combined operator

In this section we obtain the integral equation involving the star-combined operator (1.18) from Green's integral representation for the solution u, and investigate the properties of the star-combined operator as an operator on $L^2(\Gamma)$. We actually consider a more general integral operator than the star-combined, replacing x in (1.15) by a suitable vector field Z.

Lemma 4.1 *Suppose u solves the scattering problem of Definition 3.1 and suppose $Z \in (L^\infty(\Gamma))^d$, $\eta \in L^\infty(\Gamma)$. Then $\partial u/\partial n$ satisfies the integral equation*

$$A_{k,\eta,Z} \frac{\partial u}{\partial n} = f \tag{4.1}$$

where the integral operator $A_{k,\eta,Z}$ is defined by

$$A_{k,\eta,Z} = (Z \cdot n)\left(\frac{1}{2}I + D'_k\right) + Z \cdot \nabla_\Gamma S_k - i\eta S_k, \tag{4.2}$$

and the known function f is defined in terms of u^I by

$$f = Z \cdot \nabla u^I - i\eta u^I.$$

Proof. Green's integral representation (1.4) holds, e.g. by combining Theorems 9.6 and 7.15 of [43]. Apply the Dirichlet and Neumann traces on Ω_e, γ and ∂_n respectively, use the standard jump relations for the single layer potential $\mathcal{S}_k$, and rearrange the resulting equations to yield the integral equations (1.5) and (1.6). Take the surface gradient of (1.5) (valid since $S_k : L^2(\Gamma) \to H^1(\Gamma)$ and $\partial u/\partial n \in L^2(\Gamma)$ by Theorem 3.2) to obtain

$$\nabla_\Gamma S_k \frac{\partial u}{\partial n} = \nabla_\Gamma u^I.$$

Equation (4.2) follows by taking the scalar product of this last equation with Z, adding $(Z \cdot n)$ times by (1.6), and subtracting $i\eta$ times (1.5). ∎

Theorem 4.2 *If Γ is Lipschitz, then $A_{k,\eta,Z}$ is a bounded operator on $L^2(\Gamma)$ and, for every $k_0 > 0$,*

$$\|A_{k,\eta,Z}\| \lesssim k^{(d-1)/2}\left(1 + \frac{\|\eta\|_\infty}{k}\right) \tag{4.3}$$

for all $k \geq k_0$, where $\|.\|$ denotes the $L^2(\Gamma)$ norm, and the notation $D \lesssim E$ means $D \leq c\,E$ where c is independent of k and η.

Proof. The mapping property of $A_{k,\eta,Z}$ follows from mapping properties of S_k, D', and $\nabla_\Gamma S_k$. The bounds

$$\|S_k\| \lesssim k^{(d-3)/2}, \quad \|D'_k\| \lesssim k^{(d-1)/2} \tag{4.4}$$

are proved using the Riesz-Thorin interpolation theorem in [19]. Mimicking the proof of the bound for $\|D'_k\|$ it is straightforward to obtain

$$\|\nabla_\Gamma S_k\| \lesssim k^{(d-1)/2}. \tag{4.5}$$

The bound (4.3) then follows via the triangle inequality. ∎

The standard combined potential operator $A_{k,\eta}$ is equal to the operator $A_{k,\eta,Z}$ (4.2) with $Z = n$. The star-combined operator $\mathscr{A}_k$ (1.15) is equal to $A_{k,\eta,Z}$ with $Z = x$ and the particular choice of η given by (1.16), and thus the integral equation (4.1) reduces to (1.18) in this case.

4.2 Coercivity of the star-combined operator

In this section we prove the main theorem, Theorem 1.1, which we restate here in slightly more detail.

Assumption 4.3 ***(Γ Lipschitz and star-shaped)*** *Let $S^{d-1} := \{x \in \mathbb{R}^d : |x| = 1\}$. For some $f \in C^{0,1}(S^{d-1}, \mathbb{R})$ with $f_- := min_{\hat{x}\in S^{d-1}} f(\hat{x}) > 0$, we have*

$$\Gamma = \{f(\hat{x})\hat{x} : \hat{x} \in S^{d-1}\}.$$

Recall that $f \in C^{0,1}(S^{d-1}, \mathbb{R})$ means that there exists $L > 0$ such that

$$|f(\hat{x}) - f(\hat{y})| \leq L|\hat{x} - \hat{y}|$$

for all $\hat{x}, \hat{y} \in S^{d-1}$, and that, by Rademacher's theorem, f is differentiable a.e. with $\nabla_{S^{d-1}} f \in L^\infty$ where $\nabla_{S^{d-1}}$ is the surface gradient on S^{d-1}. The unit outward normal and the surface measure on Γ are given by

$$n(x) = n_\Gamma(x) := \frac{f(\hat{x})\hat{x} - \nabla_{S^{d-1}} f(\hat{x})}{\sqrt{(f(\hat{x}))^2 + |\nabla_{S^{d-1}} f(\hat{x})|^2}}$$

and

$$ds_\Gamma(x) = (f(\hat{x}))^{d-2}\sqrt{(f(\hat{x}))^2 + |\nabla_{S^{d-1}} f(\hat{x})|^2}\, ds(\hat{x})$$

where $ds(\hat{x})$ is the surface measure on S^{d-1}.

Theorem 4.4 (Coercivity for star-shaped Lipschitz domains) *Suppose that* $\Gamma := \partial\Omega_i$ *satisfies Assumption 4.3. Then, for all* $\phi \in L^2(\Gamma)$,

$$\Re(\mathscr{A}_k\phi, \phi)_{L^2(\Gamma)} \geq \gamma\|\phi\|^2_{L^2(\Gamma)}$$

where the star-combined operator $\mathscr{A}_k$ *is given by* (1.15) *and the coercivity constant* γ *is given by*

$$\gamma = \frac{1}{2}\operatorname*{ess\,inf}_{x\in\Gamma}(x \cdot n(x)) > 0. \tag{4.6}$$

A lower bound on γ *in terms of the function* f *defining* Γ *is given by*

$$\gamma \geq \frac{f_-^2}{2\sqrt{L^2 + f_+^2}}$$

where $f_+ := max_{\hat{x}\in S} f(\hat{x})$, *and both* f_- *and* L *are defined in terms of* f *in Assumption 4.3.*

Coercivity of $\mathscr{A}_k$ implies that $\mathscr{A}_k$ is invertible, and thus the solution of the integral equation (1.18) is unique. Theorem 4.4 will follow immediately from the following key lemma:

Lemma 4.5 *Suppose that* Γ *satisfies Assumption 4.3. Then for all* $\phi \in L^2(\Gamma)$

$$\Re\int_\Gamma \left(x \cdot n\ D'_k\phi + x \cdot \nabla_\Gamma S_k\phi + \left(-ikr + \frac{d-1}{2}\right) S_k\phi\right)\bar{\phi}\ ds \geq 0. \tag{4.7}$$

Remark 4.6 *The inequality* (4.7) *actually holds when* Γ *is Lipschitz, not just star-shaped. However we will only need it for the case when Assumption 4.3 holds, and this assumption also minimises technicalities in the proof.*

Proof of Theorem 4.4. By Lemma 4.5, using equations (1.15), (1.16), and (4.6),

$$\Re(\mathscr{A}_k\phi, \phi)_{L^2(\Gamma)} \geq \frac{1}{2}\Re\int_\Gamma (x \cdot n)|\phi|^2 ds \geq \gamma\|\phi\|^2_{L^2(\Gamma)}.$$

The lower bound for the coercivity constant γ in (1.17) follows from the definition of the normal vector n in Assumption 4.3. ■

It now remains to prove Lemma 4.5.

Proof of Lemma 4.5. Our strategy is to mimic the proof of $\Re(-i\int_\Gamma S_k\phi\,\bar{\phi}ds) \geq 0$ discussed in §1.3 with Green's identity replaced by the Morawetz-Ludwig identity. That is, apply the identity (2.15) with v replaced by $u = \mathcal{S}_k\phi$ with $\phi \in L^2(\Gamma)$, and D first equal to Ω_i, and then equal to $\Omega_e \cap B_R(0)$. This formally results in

$$\int_\Gamma Q_- ds = \int_{\Omega_i} \left(|\nabla u|^2 - |u_r|^2 + |u_r - iku|^2\right) dx, \tag{4.8a}$$

$$-\int_\Gamma Q_+ ds + \int_{\partial B_R(0)} Q_R ds = \int_{\Omega_e\cap B_R(0)} \left(|\nabla u|^2 - |u_r|^2 + |u_r - iku|^2\right) dx, \tag{4.8b}$$

where

$$Q_\pm(x) = 2\Re\left(r\overline{\mathcal{M}u_\pm}\frac{\partial u_\pm}{\partial n}\right) + (k^2|u_\pm|^2 - |\nabla u_\pm|^2)(x \cdot n), \quad x \in \Gamma,$$

$$Q_R(x) = 2\Re\left(R\overline{\mathcal{M}u}\,u_r\right) + (k^2|u|^2 - |\nabla u|^2)R, \quad x \in \partial B_R(0).$$

and the subscripts $\pm$ denote limits on Γ from Ω_e and Ω_i respectively. However, u does not satisfy the conditions of Lemma 2.5 since it is only in $C^2(\Omega_i)$ and not necessarily in $C^2(\overline{\Omega_i})$ (and similarly for Ω_e). A careful limiting argument, making explicit use of Lipschitz domain results from harmonic analysis, shows that nevertheless the equations (4.8) do hold, with $u_\pm, \nabla u_\pm$ given almost everywhere on Γ by

$$u_\pm(x) = S_k\phi(x), \tag{4.9a}$$

$$\nabla u_\pm(x) = n(x)\left(\mp\frac{1}{2}I + D'_k\right)\phi(x) + \nabla_\Gamma S_k\phi(x). \tag{4.9b}$$

We postpone this argument, proceed with the proof, and then return to it at the end.

Adding (4.8a) and (4.8b) yields

$$\int_\Gamma (Q_- - Q_+)\, ds + \int_{\partial B_R(0)} Q_R ds = \int_{B_R(0)} \left(|\nabla u|^2 - |u_r|^2 + |u_r - iku|^2\right) dx. \tag{4.10}$$

Note that, using the definition of $\mathcal{M}u$ (2.14) and the fact that, on Γ, $r(u_r)_\pm = (x\cdot n)\frac{\partial u_\pm}{\partial n} + x\cdot\nabla_\Gamma u_\pm$,

$$Q_\pm = (x\cdot n)\left|\frac{\partial u_\pm}{\partial n}\right|^2 + 2\Re\left[x\cdot\overline{\nabla_\Gamma u_\pm}\frac{\partial u_\pm}{\partial n} + \left(ikr + \frac{d-1}{2}\right)\overline{u_\pm}\frac{\partial u_\pm}{\partial n}\right] + (k^2|u_\pm|^2 - |\nabla_\Gamma u_\pm|^2)(x\cdot n). \tag{4.11}$$

Now let $R\to\infty$ in (4.10). Since u is a solution of the Helmholtz equation in Ω_e satisfying the radiation condition (1.3), $\int_{\partial B_R(0)} Q_R ds \to 0$ as $R\to\infty$ by Lemma 2.4, and the volume integral over $B_R(0)$ tends to the integral over $\mathbb{R}^d$. Next, combine the expression (4.11) for $Q_\pm$ with the expressions (4.9) for $u_\pm$ and $\nabla u_\pm$ (noting that (4.9b) implies $\nabla_\Gamma u$ is continuous across Γ) and substitute into equation (4.10) to obtain

$$\begin{aligned}\int_\Gamma \Bigg[(x\cdot n)\left(\left|\frac{\partial u_-}{\partial n}\right|^2 - \left|\frac{\partial u_+}{\partial n}\right|^2\right) &+ 2\Re\left(\left(x\cdot\overline{\nabla_\Gamma u} + \left(ikr + \frac{d-1}{2}\right)\overline{u}\right)\left(\frac{\partial u_-}{\partial n} - \frac{\partial u_+}{\partial n}\right)\right)\Bigg] ds \\ &= \int_{\mathbb{R}^d}\left(|\nabla u|^2 - |u_r|^2 + |u_r - iku|^2\right) dx.\end{aligned} \tag{4.12}$$

Finally, note that (4.9b) implies that

$$\frac{\partial u_-}{\partial n}(x) - \frac{\partial u_+}{\partial n}(x) = \phi(x), \quad x\in\Gamma, \tag{4.13}$$

and

$$\frac{\partial u_-}{\partial n}(x) + \frac{\partial u_+}{\partial n}(x) = 2D_k'\phi(x), \quad x\in\Gamma,$$

so that

$$\begin{aligned}\left|\frac{\partial u_-}{\partial n}(x)\right|^2 - \left|\frac{\partial u_+}{\partial n}(x)\right|^2 &= \Re\left[\left(\frac{\partial u_-}{\partial n}(x) + \frac{\partial u_+}{\partial n}(x)\right)\overline{\left(\frac{\partial u_-}{\partial n}(x) - \frac{\partial u_+}{\partial n}(x)\right)}\right] \\ &= 2\Re(D_k'\phi(x)\bar\phi(x)), \quad x\in\Gamma.\end{aligned} \tag{4.14}$$

Substitute (4.13) and (4.14) into (4.12), and note that the right hand side of (4.12) is positive, to obtain (4.7).

We now need to justify the claim made earlier that (4.8) holds with $u_\pm, \nabla u_\pm$ given by (4.9). Consider (4.8a). The standard strategy for proving an identity involving the single layer potential such as this is to approximate Ω_i by a sequence of domains inside Ω_i, apply the identity in each of these domains, and then take the limit. In the general Lipschitz case, both a sequence of approximating domains and the limiting process are described in [60, Theorem 1.12, Remark 1.14]. Since we are dealing with a *star-shaped* domain a convenient sequence is given by

$$\Omega^t = t\Omega_i, \quad t\in(0,1), \quad t\to 1^-.$$

In order to justify the limiting process we need the following two facts, consequences of the famous results about the single layer potential on Lipschitz domains [48], [60] (see Remark 4.7 below): if $u = \mathcal{S}_k\phi$ for $\phi\in L^2(\Gamma)$ with Γ satisfying Assumption 4.3 and $x\in\Gamma$ then

1. the limits

$$\lim_{t\to 1^-} u(tx) \text{ and } \lim_{t\to 1^-}\nabla u(tx)$$

exist for almost every $x\in\Gamma$ and are given by the right hand sides of (4.9).

2. u^* and $(\nabla u)^* \in L^2(\Gamma)$ where u^* is defined by

$$u^*(x) = \sup_{0<t<1}|u(tx)|,$$

and $(\nabla u)^*$ is defined similarly.

To obtain (4.8a) apply Lemma 2.5 with $v = u$ and $D = t\Omega_i$, for $t \in (0, 1)$; this is allowed since $u \in C^2(\Omega_i)$. Note that the volume term in (2.15) involving $\mathcal{L}u$ is zero as u is a solution of the Helmholtz equation. The integrals over $\partial(t\Omega_i)$ are of the following form

$$\int_{\partial(t\Omega_i)} \mathcal{Q}\big(u(x), \nabla u(x), n_{\partial(t\Omega_i)}(x)\big) ds_{\partial(t\Omega_i)}(x)$$

where $\mathcal{Q}$ is a continuous function which is quadratic in the first two variables. Making the change of variable $x = ty$, $y \in \Gamma$, this becomes

$$t^{d-1} \int_{\Gamma} \mathcal{Q}\big(u(ty), \nabla u(ty), n_\Gamma(y)\big) ds_\Gamma(y)$$

where we have used the fact that $n_{\partial(t\Omega_i)}(x) = n_\Gamma(y)$. This expression tends to

$$\int_{\Gamma} \mathcal{Q}\big(u_-(y), \nabla u_-(y), n_\Gamma(y)\big) ds_\Gamma(y),$$

where $u_-, \nabla u_-$ are given by (4.9), as $t \to 1^-$ by the dominated convergence theorem. Indeed, the integrand converges pointwise almost everywhere due to point 1 above, and the integral is dominated by a multiple of $\|u^*\|^2_{L^2(\Gamma)} + \|\nabla u^*\|^2_{L^2(\Gamma)}$ which is finite due to point 2. The equation (4.8b) follows in an almost identical way. ∎

Remark 4.7 (The single layer potential on Lipschitz domains) *This material is summarised in e.g. [38, Chapter 2 §2], [60], with a particularly accessible account found in [48, Chapter 15]. Given a Lipschitz domain D, a key concept in formulating boundary conditions for potential problems with L^2 boundary data is the notion of non-tangential limit. This is defined by assigning to every point $x \in \partial D$ a "non-tangential approach cone", $\Theta(x)$. The important point about these cones is that if $y \in \Theta(x)$, then there exists an $\alpha > 1$ such that $|y - x| \le \alpha \,\mathrm{dist}(y, \partial D)$. Thus when y tends to x whilst remaining in $\Theta(x)$, y is "relatively far" from the other points on ∂D. If $u = \mathcal{S}_k\phi$ then the non-tangential limits of u and ∇u exist almost everywhere on ∂D and are given by the right hand sides of the expressions* (4.9)*, and the "non-tangential maximal functions" of u and ∇u, defined as the supremums in the approach cone of $|u|$ and $|\nabla u|$ respectively, are in $L^2(\Gamma)$. Thus, the statements 1 and 2 in the proof of Lemma 4.5 follow from these results since for any $L^* > L$, and given $x \in \Gamma$, there exists a neighbourhood of x such that the surface Γ in this neighbourhood is the graph of a Lipschitz function with Lipschitz constant L^* where the vertical co-ordinate lies in the direction of $\hat{x}$; thus the limit $tx \to x$ is contained inside the approach cone. (Strictly speaking the references cited above contain these results only for the Laplace case, but the results are in fact true for the Helmholtz case as well. For a little more detail on this see [59].)*

Corollary 4.8 (Coercivity for circle ($d = 2$) and sphere ($d = 3$)) *Let $\Gamma = S^{d-1}$, that is $f \equiv 1$ in Assumption 4.3. Then the standard combined potential operator $A_{k,\eta}$ given by* (1.10) *is coercive uniformly in k, that is*

$$\Re(A_{k,\eta}\phi, \phi)_{L^2(\Gamma)} \ge \frac{1}{2}\|\phi\|^2_{L^2(\Gamma)}$$

for all $k > 0$ if

$$\eta = k + \frac{d-1}{2} i. \tag{4.15}$$

If the choice $\eta = k$ is made then given $\delta > 0$ there exists k_0 such that

$$\Re(A_{k,k}\phi, \phi)_{L^2(\Gamma)} \ge \left(\frac{1}{2} - \delta\right) \|\phi\|^2_{L^2(\Gamma)}, \quad \forall k \ge k_0, \tag{4.16}$$

i.e. coercivity holds with any constant less than $1/2$ for large enough k.

Proof. The first part follows immediately from the fact that on the unit circle and sphere the star-combined potential operator $\mathscr{A}_k$ is the standard one $A_{k,\eta}$ with η given by (4.15). The second part (when $\eta = k$) follows from the first if we can show that

$$\|S_k\| \to 0 \text{ as } k \to \infty.$$

When $d = 2$ this follows from the bound given by (4.4), however when $d = 3$ this is too crude and we must use the bound

$$\|S_k\| \lesssim k^{-2/3}, \quad \forall k > 0, \tag{4.17}$$

obtained in [7] via the explicit expressions for the eigenvalues of S_k. ■

Remark 4.9 (Comparison of the different proofs of coercivity for the circle and sphere) *For smooth domains the coercivity result of Theorem 4.4 does not require any of the deep harmonic analysis results of Remark 4.7. Thus, Corollary 4.8 gives a much simpler proof of coercivity for the circle and sphere than the proof by Fourier analysis given in [28] (although this latter proof shows that δ can be taken to be zero in (4.16)). Note that the proof in Corollary 4.8 for the sphere does require one result obtained by Fourier analysis, namely the bound (4.17), however this upper bound is much easier to prove than the lower bound required for coercivity itself.*

4.3 Implementing the star-combined operator

In this section we show that implementing the Galerkin approximation of the new star-combined operator (1.15) is in principle no more difficult than implementing the standard combined potential operator (1.10). We present this just for three dimensions; the demonstration for two dimensions is even simpler.

Comparing the star-combined operator with the standard one, the only new term is the one involving the surface gradient of the single layer potential, namely $x.\nabla_\Gamma S_k$. The Galerkin approximation of this operator requires computing surface integrals of the form

$$\int_\Gamma x.\nabla_\Gamma (S_k\phi)\, \psi(x)\, ds(x) \;=\; \int_\Gamma (\psi(x)x)\;.\nabla_\Gamma (S_k\phi)(x)\, ds(x) \tag{4.18}$$

where ϕ and ψ belong to the approximation space $\mathcal{S}_N$ used in the method.

The integration by parts formula which we shall give in Proposition 4.11 below shows that integrals of the form (4.18) are easily computable in terms of integrals of certain derivatives of ψ and values (but not derivatives) of $S_k\phi$.

We first set up some notation: Consider a surface patch $\Gamma^0 \subset \Gamma$ (not necessarily all of Γ) which is parametrised by a Lipschitz map $\eta : \widehat{\Gamma^0} \to \Gamma^0$, where $\widehat{\Gamma^0} \subset \mathbb{R}^2$ is a reference plane polygonal domain. For $\widehat{x} \in \widehat{\Gamma^0}$, η is given by

$$\eta(\widehat{x}) = \begin{bmatrix} \widehat{x} \\ \xi(\widehat{x}) \end{bmatrix}$$

where $\xi : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz function, see e.g. [43, 12]. (A typical situation is where $\widehat{\Gamma^0}$ is a unit planar triangle or a square.) Note that since ξ is Lipschitz its gradient exists almost everywhere on Γ, and hence so does the gradient of η. All expressions below involving derivatives of η are to be understood as holding almost everywhere on $\widehat{\Gamma^0}$. A point $x \in \Gamma^0$ then corresponds to $\widehat{x} \in \widehat{\Gamma^0}$ via $x = \eta(\widehat{x})$. The map η provides a parametrization of Γ^0 such that the columns of the 3×2 Jacobian matrix

$$J(\widehat{x}) \;=\; \left[\frac{\partial \eta}{\partial \widehat{x}_1}(\widehat{x})\,,\, \frac{\partial \eta}{\partial \widehat{x}_2}(\widehat{x})\right]$$

are linearly independent and form a basis for the tangent plane at $x = \eta(\widehat{x})$. The unit normal $n(x)$ is orthogonal to this plane at $x = \eta(\widehat{x})$. The Gram determinant of η is defined as

$$g(\widehat{x}) \;=\; (\det\; G(\widehat{x}))^{1/2} \quad \text{where} \quad G(\widehat{x}) \;=\; J(\widehat{x})^T J(\widehat{x})\;.$$

Definition 4.10 (The surface gradient ∇_Γ) *On the patch Γ_0 defined above, the surface gradient operator ∇_Γ is defined by*

$$(\nabla_\Gamma v)(x) \;=\; J(\widehat{x}) G(\widehat{x})^{-1} \widehat{\nabla} \widehat{v}(\widehat{x}), \quad \widehat{x} \in \widehat{\Gamma_0}, \tag{4.19}$$

where $\widehat{v}(\widehat{x}) := v(\eta(\widehat{x}))$ and $\widehat{\nabla}$ denotes the (two dimensional) gradient with respect to the vector $\widehat{x}$.

This is a standard formula for the surface gradient in terms of the parametrization η and coincides with those given for general Lipschitz domains in [12, Definition 3.1] and [60, Definition 1.9] and for smooth domains in [49, §3.4] [25, §2.1]. (See [55, §2.5.6] for an alternative point of view.) To see that the property (3.1) holds, let w be C^1 in a neighbourhood of Γ_0. The chain rule implies that

$$\widehat{\nabla}\widehat{w}(\widehat{x}) \ = \ J(\widehat{x})^T(\nabla w)(\eta(\widehat{x})) \qquad \widehat{x} \in \widehat{\Gamma}^0.$$

This expression can be used to resolve $\nabla w(\eta(\widehat{x}))$ in terms of the basis

$$\left\{\frac{\partial\eta(\widehat{x})}{\partial\widehat{x}_1}, \frac{\partial\eta(\widehat{x})}{\partial\widehat{x}_2}, n(\eta(\widehat{x}))\right\}$$

to obtain

$$\nabla w(x) \ = \ J(\widehat{x})G(\widehat{x})^{-1}\widehat{\nabla}\widehat{w}(\widehat{x}) + \frac{\partial w}{\partial n}(x)n(x)$$

which is equal to (3.1) using the definition (4.19).

Any vector field $w : \Gamma^0 \to \mathbb{R}^3$ can be resolved in the tangent and normal directions via the formula

$$w(x) \ = \ w(\eta(\widehat{x})) \ = \ J(\widehat{x})\widehat{\omega}(\widehat{x}) \ + \ (w(x).n(x))n(x),$$

for some field $\widehat{\omega} : \widehat{\Gamma}^0 \to \mathbb{R}^2$. Since $\nabla_\Gamma v$ is in the tangent plane (by (4.19)), we have

$$w(x).\nabla_\Gamma v(x) \ = \ \widehat{\omega}(\widehat{x}).\left(J(\widehat{x})^T J(\widehat{x})G(\widehat{x})^{-1}\right)\widehat{\nabla}\widehat{v}(\widehat{x}) \ = \ \widehat{\omega}(\widehat{x}).\widehat{\nabla}\widehat{v}(\widehat{x}) \ . \tag{4.20}$$

We now derive the integration by parts formula for dealing with integrals of the form (4.18). For simplicity we will assume that ξ (and hence also η) is C^2, as is the case in many applications. Note that this does not imply that Γ has to be globally smooth; Γ could be a Lipschitz polyhedron, for example, but the edges of the polyhedron are required to coincide with element edges. This assumption avoids difficulties in taking the derivative of the Gram determinant $g(\widehat{x})$.

The formula obtained is given on the reference domain $\widehat{\Gamma}^0$, since this is where practical boundary integral computation would be done.

Proposition 4.11 *Suppose $w \in (\mathcal{C}^1(\Gamma^0))^3$ and $v \in \mathcal{C}^1(\Gamma^0)$. Then*

$$\int_{\Gamma^0} w(x).\nabla_\Gamma v(x)\mathrm{d}s(x) \ = \ \int_{\partial\hat{\Gamma}^0} g(\widehat{x})\,(\widehat{\omega}(\widehat{x}).\widehat{\nu}(\widehat{x}))\ \widehat{v}(\widehat{x})\,\mathrm{d}\gamma(\widehat{x}) - \int_{\Gamma^0} \widehat{\nabla}.\left[g(\widehat{x})\widehat{\omega}(\widehat{x})\right]\ \widehat{v}(\widehat{x})\,\mathrm{d}\hat{x} \ ,$$

where $\widehat{\nu}(\widehat{x})$ is the outward normal from $\widehat{\Gamma}^0$ at $\widehat{x} \in \partial\hat{\Gamma}^0$.

Proof. By (4.20), we have

$$\int_{\Gamma^0} (w(x).\nabla_\Gamma v(x))\,\mathrm{d}s(x) \ = \ \int_{\widehat{\Gamma}^0} (\widehat{\omega}(\widehat{x}).\widehat{\nabla}\widehat{v}(\widehat{x}))\,g(\widehat{x})\,\mathrm{d}s(\widehat{x})$$

and the result follows from the divergence theorem on $\widehat{\Gamma}^0$. Note that all the integrals make sense classically because of the assumed smoothness of w, v and η, η^{-1}. ∎

Finally we note that if Proposition 4.11 is used to compute the integral (4.18) with ϕ supported on Γ_0 then the functions $\widehat{v}$ and $\widehat{\omega}$ are given by

$$\widehat{v}(\widehat{x}) = (S_k\phi)(\eta(\widehat{x}))$$

and

$$\begin{aligned} \widehat{\omega}(\widehat{x}) \quad &= \quad G(\widehat{x})^{-1}J(\widehat{x})^T\,(\widehat{x}\psi(\widehat{x})) \\ &= \quad \psi(\widehat{x})\,G(\widehat{x})^{-1}\begin{bmatrix} (\partial\eta(\widehat{x})/\partial\widehat{x}_1).\eta(\widehat{x}) \\ (\partial\eta(\widehat{x})/\partial\widehat{x}_2).\eta(\widehat{x}) \end{bmatrix}. \end{aligned}$$

In the high frequency case, the resulting integrals will be highly oscillatory. The efficient calculation of these type of integrals is an active area of research (see e.g. [10], [36], [29] and the references therein).

5 Concluding remarks

One of the attractions of the coercivity result of Theorem 1.1 is that it proves that if the new integral equation (1.18) is solved by *any* Galerkin method, then the error estimate (1.11) holds. Moreover, it follows from Theorems 1.1 and 4.2 and the expression (1.19) that given $k_0 > 0$, the constant C in the estimate (1.11) satisfies

$$C \leq C_0 k^{(d-1)/2}, \tag{5.1}$$

for all $k \geq k_0$, where C_0 is independent of k. In particular, this result applies to the non-standard Galerkin methods of [28] and [20], for high frequency scattering by smooth convex two-dimensional obstacles and convex polygons respectively (if the star-combined formulation (1.18) is used instead of the standard combined potential formulation (1.9)).

The method of [28] designs a k-dependent approximation space, $\mathcal{S}_{N,k}$, based on knowledge of the high frequency asymptotics (e.g. [47]), for scattering by a smooth convex obstacle in two dimensions. The space $\mathcal{S}_{N,k}$ approximates $v := \partial u/\partial n$ on Γ as an oscillatory factor multiplied by a polynomial of degree N in the illuminated zone and in the two shadow boundary zones; and is designed so that the best approximation error $\inf_{\phi_N \in \mathcal{S}_{N,k}} \|v - \phi_N\|_{L^2(\Gamma)}$ grows slowly with k for fixed N. If the method is implemented using the star-combined formulation (1.18), then the estimate (1.11), with C given by the right hand side of (5.1), combined with the bound on the best approximation error from [28] yields the following theorem:

Theorem 5.1 (k-explicit quasi-optimality for the method of [28]) *Let N denote the degree of the polynomials used in each of the three zones (so N is proportional to the total number of degrees of freedom of the method), and let p be an integer with $6 \leq p \leq N+1$. Then for every $k_0 > 0$ there exist δ, C_1, and C_p all greater than zero such that*

$$\frac{\|v - v_N\|_{L^2(\Gamma)}}{k} \leq C_p k^{1/18} \left\{ \left(\frac{k^{1/9}}{N} \right)^p + k^{4/9} \exp(-C_1 k^{\delta}) \right\}, \tag{5.2}$$

for all $k \geq k_0$, where C_p only depends on p and Γ, and δ and C_1 only depend on Γ.

Since $\|v\|_{L^2(\Gamma)}$ is proportional to k as k increases, the left hand side of (5.2) measures the relative error. This bound shows that the number of degrees of freedom only needs to grow slighter faster than $k^{1/9}$ in order to maintain accuracy as $k \to \infty$; this is to be contrasted with the linear growth required in conventional boundary element methods in two dimensions as proved in [42]. Preliminary results on implementing the star-combined formulation show that this property is realised in practice [39].

The method of [20], which concerns high frequency scattering by convex polygons, is slightly more complicated to explain. However similarly, if this method is implemented using the star-combined formulation, then combining (1.11) and (5.1) with results about the best approximation error in [20] proves k-explicit quasi-optimality of the method. In this case, the number of degrees of freedom only needs to grow like $(\log k)^{3/2}$ in order to maintain accuracy as $k \to \infty$. This rigorous convergence analysis has been made possible by the coercivity result of this paper.

Finally we note that the proof of Theorem 1.1 required only the most basic Morawetz-type identity for the Helmholtz equation from [52]. The application of more sophisticated identities, such as those appearing in [51] and [53], to these type of problems (in particular for more general "non-trapping" scattering geometries) is underway.

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