Principal-Agent VCG Contracts*

Ron Lavi[†] Elisheva S. Shamash[‡]

March 16, 2022

Abstract

We study a complete information game with multiple principals and multiple common agents. Each agent takes an action that can affect the payoffs of all principals. Prat and Rustichini (Econometrica, 2003) who introduce this model assume classic contracts: each principal offers monetary transfers to each agent conditional on the action taken by the agent. We define VCG contracts in which the monetary transfers to each agent additionally depend on all principals' offers, and study its effect on the existence of efficient pure subgame perfect equilibrium outcomes. Using a necessary and sufficient condition for the existence of a pure subgame perfect equilibrium (pure SPE) with VCG contracts, which we develop, we show that the class of instances that admit an efficient pure SPE with VCG contracts strictly contains the class of instances that admit an efficient pure SPE with classic contracts. In addition, the difference between the former class and the class of instances that admit a 'weakly truthful' SPE with classic contracts has positive measure. Although VCG contracts broaden the existence of pure subgame perfect equilibria, we show that the worst case welfare loss in a pure SPE outcome, over all games with any fixed $M \geq 2$ number of principals, is the same for both VCG contracts and classic contracts.

Keywords: principal-agent; games played through agents; contractible contracts; VCG

*A one-page abstract of a preliminary version of this paper has appeared in the proceedings of the 2019 ACM Conference on Economics and Computation (Lavi and Shamash, 2019). This research was supported by the ISF-NSFC joint research program (grant No. 2560/17).

†University of Bath, UK, and Technion – Israel Institute of Technology. Email: ron.lavi.ac@gmail.com.

‡Technion – Israel Institute of Technology. Email: elisheva@gmail.com.

1 Introduction

Most of the existing principal-agent literature consider classic contracts where the payment of a principal to an agent can be fully determined given the agent's action and/or the resulting outcome. For example, Bernheim and Whinston (1986) consider a complete information common agency model where multiple principals offer such contracts to a single common agency. Prat and Rustichini (2003) generalize this model to the case of multiple principals and multiple agents. One of the implications of their elegant characterizations is that efficient equilibrium outcomes may not necessarily exist with multiple agents.

In fact, it is now well-known that simple classic contracts limit the existence of desirable equilibrium outcomes and more elaborate contract types have been proposed. For example, Peters and Szentes (2012) and Szentes (2015) define and study contractible contracts – contracts that can depend on the contracts proposed by other principals. These types of contracts can significantly expand the set of implementable outcomes (in particular the efficient ones). On the other hand, these papers assume a broad space of contracts that allows for various rather strong collusive contractual statements. There could be settings in which such contracts may not be acceptable or legal. In addition, the non-simplicity and the complicated nature of these contracts could be a disadvantage, see for example the discussion in Peters (2014).

The study of various types of contracts that can depend on other contracts has an additional practical appeal as a growing number of settings, especially online, either explicitly or implicitly employ such a practice. For example, Hilton's website (as well as many others) states a Price Match Guarantee, which adds a VCG-like aspect to their seemingly standard posted-pricing policy. Amazon and eBay provide opportunities for various types of contractible contracts like the click stream pricing technique that Peters (2014) describes.

The current paper proposes a specific contract type that employs the idea that the contract of one principal can depend on other principals' contracts. Our proposed contract, on the one hand, seems simple, realistic and legally acceptable and on the other hand it broadens the existence of efficient equilibrium outcomes.

1.1 VCG contracts: intuition

We study our proposed contract in the context of the complete-information setting of "Games Played Through Agents" (GPTA) introduced by Prat and Rustichini (2003). There are M principals and N agents, each agent n chooses an action $s_n \in S_n$, each principal m obtains a value $G^m : S_1 \times \cdots \times S_N \to \mathbb{R}_{\geq 0}$ from the tuple of agent actions, and each agent n attains a value $F_n : S_n \to \mathbb{R}$ from her chosen action. The extensive-form game has two steps: First, principals simultaneously offer contracts to agents that describe the transfer that the agent will receive from the principal. Second, based on these contracts, agents choose actions. Utilities are then realized:

a principal's utility is her value from the tuple of chosen actions minus the sum of transfers that she makes; an agent's utility is her value from her chosen action plus the sum of transfers she receives.

Table 1 shows an example with two agents and two principals. Agents' actions correspond to the rows and columns of the table while principal gains are given in the table cells (note that this is not a standard normal-form game as the final utilities are not given). E.g., according to the table, when agent 1 plays T and agent 2 plays L then principal 1 obtains a value of $G^1(T, L) = 1$ and principal 2 obtains a value of $G^2(T, L) = 2$. All our examples assume $F \equiv 0$, unless otherwise stated.

I , II	L	R
T	1 , 2	2 , 1
B	2 , 1	1 , 2

Table 1: A GPTA with 2 principals and 2 agents

Prat and Rustichini (2003) study classic contracts: For each agent n, this is a mapping from the agent's action s to a payment for that action $b_n^m(s)$ that principal m pays the agent, termed principal's bid. We refer to the resulting game as classic-GPTA. Unfortunately, in the classic-GPTA based on table 1 there is no pure subgame-perfect equilibrium (pure SPE). For intuition, if a pure SPE were to exist, suppose without loss of generality that its second stage outcome is (T, L). Then:

- First, we must have $b_1^2(T) + b_2^2(L) > 0$, otherwise $b_1^2(T) = b_2^2(L) = 0$ and in this case principal 1 has a profitable deviation $\tilde{b}$ where $1 > \tilde{b}_1^1(B) > 0$ and zero otherwise.

- We must also have $b_1^2(T) + b_2^2(L) \leq 1$, otherwise a profitable deviation for principal 2 would be to bid all zeroes and obtain 1.

- Now there are two cases. If $b_1^2(T) \leq b_2^2(L)$, a profitable deviation of principal 1 is then any $1 > \tilde{b}_1^1(B) > b_1^2(T)$ and zero otherwise. If $b_1^2(T) > b_2^2(L)$, a profitable deviation of principal 1 is then any $1 > \tilde{b}_2^1(R) > b_2^2(L)$ and zero otherwise.

Section 4 further analyzes the commonality of this non-existence of pure SPE in classic-GPTA, and shows that the example in table 1 is not a rare exception.

To alleviate this inexistence problem, we propose a new type of contract – VCG contract – in which a principal's payment depends on the bids of the other principals as well as on her own bids. The name choice is due to a conceptual connection to the VCG framework which we describe in section 3. In a VCG contract, if agent n chooses action s, principal m pays agent n her bid $b_n^m(s)$ minus the "potential social gain" accredited to principal m for agent n: This is defined to be the maximal sum of bids for an action of agent n minus the maximal sum of bids of all other principals for an action of agent n. This description assumes that the agent has no direct preferences over actions, see section 3 for a fully formal definition. We term the resulting game VCG-GPTA.

Consider again the GPTA in table 1. In contrast to the classic-GPTA which does not have a pure SPE, the VCG-GPTA does have a pure SPE. For example, $\hat{b}_1^2(T) = \hat{b}_2^2(L) = 1$, with all other principal bids equal to zero is a pure SPE. With these bids, agents 1,2 maximize utility by playing T,L, respectively. The potential social gain accredited to principal 2 for agent 1 (or agent 2) is $1 - 0 = 1$. Hence, her payment to the agent is $1 - 1 = 0$. Similarly, the other principal pays $0 - 0 = 0$. For intuition that this is a pure SPE, consider a possible deviation where principal 1 tries to incentivize the row agent to deviate and play B, say by offering $b_1^1(B) = 1.5$ and keeping all her other bids zero. In this case, the row agent will indeed play B and the column agent will play L resulting in a value of 2 for principal 1. However, principal 1 will now pay $1.5 - 0.5 = 1$ to agent 1, since the potential social gain accredited to this principal will now be $1.5 - 1 = 0.5$. This deviation therefore does not increase the utility of principal 1.

More generally, viewing principals' bids as their maximal willingness to pay for specific actions, in section 3 we show that if the agent chooses an action that maximizes her utility, the principal's payment is equal to the difference between the maximal aggregate willingness to pay of the other principals for some action and their aggregate willingness to pay for the chosen action. This is reminiscent of the way each player in VCG internalizes her external effect on the social welfare and is a key property in our proofs. In figure 1, principal 1's payment to agent 1 in case her bid incentivizes the agent to play B will be the difference between the maximal willingness to pay of the other principals for some action (which in this case is principal 2's bid for T, i.e., 1) and for the chosen action (which in this case is principal 2's bid for B, i.e., 0). Thus, principal 1 faces a trade off between paying nothing for the outcome (T, L) and paying 1 for the outcome (B, L). Principal 1 is indifferent between these two options hence will not gain from any deviation that leads to (B, L).

VCG contracts can be used by Internet "platforms" to reduce principals' payments by an amount equal to the potential social gain. This reduction can be viewed as a kind of tax determined irrespective of the agents actions and thus appears constant to the agents. This tax claws away at the agents' profits and enables the emergence of efficient pure SPE even in cases where it does not exist in the classic GPTA. Note though that the potential social gain is not literally a tax since it is returned to the principals and no external entity benefits from it.

1.2 Overview of Results

As in Prat and Rustichini (2003), our main focus is on the resulting equilibrium welfare (i.e., the sum of agents' and principals' payoffs) and in particular on the existence of efficient equilibria. While there exist both classic-GPTA as well as VCG-GPTA that do not admit an efficient pure subgame perfect equilibrium (pure SPE), the main message of this paper is that VCG contracts improve the existence of efficient pure SPE relative to classic contracts. Specifically, we prove that the set of GPTA settings that admit an efficient pure SPE with VCG contracts strictly contains the set of GPTA settings that admit an efficient pure SPE with classic contracts. Prat and Rustichini

(2003) focus in their paper on a subset of efficient pure sub-game perfect equilibria with classic contracts which they term 'weakly truthful'. We additionally show that there are infinitely many GPTA settings that admit an efficient pure SPE with VCG contracts and do not admit a weakly truthful SPE with classic contracts (in fact the Lebesgue measure of such GPTAs, in the set of all GPTAs, is positive).[1] We further show that principals prefer VCG contracts over classic contracts thus establishing robustness when principals choose between these two contracts endogenously.[2]

Since our VCG contracts expand the set of subgame perfect equilibria relative to classic contracts, one might wonder whether additional low welfare subgame perfect equilibria are added. We study this issue using the "Price of Anarchy (PoA)" (Roughgarden, 2005) of classic-GPTA and VCG-GPTA over all games with M principals, where the PoA is the ratio of the lowest pure SPE welfare to the maximal welfare (not necessarily in equilibrium). We show that the PoA of any VCG-GPTA as well as classic-GPTA is at least $\frac{1}{M}$, and that this bound is tight in the worst-case for both games. Therefore, the worst-case welfare loss (over all games with M principals) in a VCG-GPTA is not worse than in a classic-GPTA while the best-case welfare guarantee strictly improves due to the improved existence of efficient equilibria.[3]

The remainder of the paper is organized as follows. Section 2 reviews the GPTA setting and some relevant results from Prat and Rustichini (2003). Section 3 defines the notion of VCG contracts along with some of their fundamental properties. Section 3.2 presents a sufficient and necessary condition for the existence of a pure SPE in VCG-GPTA using a notion of generalized balancedness. In section 4 we compare the set of principal and agent valuations that admit an efficient pure SPE in the corresponding VCG-GPTA to the set of principal and agent valuations that admit a weakly truthful pure SPE in the corresponding classic-GPTA. We show that the latter is strictly contained in the former, and that their difference has positive measure. Section 5 analyzes the price of anarchy of the two games, section 6 studies the relation to contractible contracts and complex mechanisms, and section 7 concludes.

2 Games Played Through Agents (GPTA)

Prat and Rustichini (2003) introduced the following complete-information principal-agent setting $\mathcal{G} = (M, N, S, G, F)$. There is a set M of principals and a set N of agents (abusing notation, M and N are also sometimes used to denote the respective cardinalities of these sets). Each agent

[1] When each agent has exactly two possible actions, a pure SPE with classic contracts is efficient if and only if it is weakly truthful. With more than two actions this does not hold, and we do not know if there exist infinitely many GPTA settings that admit an efficient pure SPE with VCG contracts but do not admit an efficient pure SPE with classic contracts. See the discussion in section 4.

[2] An endogenous contract choice is assumed in most contractible contract models. For example, Han (2007) considers a class of complex mechanisms that includes classic contracts (but not VCG contracts) and shows that any equilibrium in classic-GPTA is robust to the possibility of a principal's deviation to any complex mechanisms.

[3] Prat and Rustichini (2003) do not discuss equilibrium selection issues and the possible welfare loss in inefficient equilibria, focusing mainly on efficient equilibria. We feel that this issue deserves attention.

$n \in N$ has a finite set of *actions* S_n, and let $S \equiv \Pi_{n\in N} S_n$. A tuple of actions $s = (s_1, s_2, ..., s_N) \in S$ is termed an *outcome*. Each agent $n \in N$ obtains an intrinsic benefit (or cost, if negative) $F_n(s_n)$ from action s_n, where $F_n : S_n \to \mathbb{R}$. Each principal $m \in M$ obtains different gains from different outcomes, as represented by the function $G^m : S_1 \times S_2 \times ... \times S_N \to \mathbb{R}_{\geq 0}$.[4]

The principal-agent extensive-form game is composed of two steps. First, principals offer contracts to agents. Second, given these contracts, agents choose actions. The contract that principal $m \in M$ offers to agent $n \in N$ specifies a vector of "principal bids" $b_n^m \in \mathbb{R}_{\geq 0}^{|S_n|}$ where $b_n^m(s_n)$ is the maximal willingness of principal m to pay to agent n if she will choose action s_n. Let $b_n^m \in B_n^m \subseteq \mathbb{R}_{\geq 0}^{|S_n|}$, let $B^m = \prod_{n\in N} B_n^m$ and let $B = \prod_{m\in M} B^m$. The actual payment principal m will pay agent n is an exogenously-determined payment function $t_{m,n} : S_n \times \mathbb{R}_{\geq 0}^{|M|\cdot|S_n|} \to \mathbb{R}$, where $t_{m,n}(s_n, b_n^1, ..., b_n^M)$ is the payment that agent n receives from principal m when her action was s_n and the principal bids were $(b_n^1, ..., b_n^M)$. The game therefore proceeds as follows:

1. Each principal $m \in M$ chooses $b^m = (b_1^m, ..., b_N^m)$. Let $b = (b^1, ..., b^M) \in B$.

2. Given these principal bids, each agent $n \in N$ chooses an action $s_n \in S_n$.

3. Agent n's utility is $u_n(s_n, b) = F_n(s_n) + \sum_{m\in M} t_{m,n}(s_n, b_n^1, ..., b_n^M)$; Principal m's utility is:
$$u^m(s, b) = G^m(s) - \sum_{n\in N} t_{m,n}(s_n, b_n^1, ..., b_n^M). \tag{1}$$

Prat and Rustichini (2003) use the classic payment function $t_{m,n}^{classic}(s_n, b) = b_n^m(s_n)$. We refer to this game as classic-GPTA. Clearly, agent strategies $\hat{s}(\cdot)$ that are part of a SPE in the classic-GPTA must satisfy the following condition:

- Agent Maximization (AM): $\forall n, \forall b : \hat{s}_n(b) \in \text{argmax}_{s_n\in S_n} \sum_{m\in M} b_n^m(s_n) + F_n(s_n)$. That is, the agent chooses an action that maximizes her utility given the principal bids.

Throughout (for brevity and slightly abusing notation), when we write that $(\hat{b}, \hat{s})$ is a pure SPE we mean that the agent strategies in this pure SPE (as in all pure SPEs) satisfy AM and that $\hat{s}$ is the tuple of actions that the agents choose according to AM when the principals play $\hat{b}$.

Lemma 1. *[Prat and Rustichini (2003)] Given a classic-GPTA, $(\hat{b}, \hat{s})$ is a pure SPE if and only if the following three conditions are satisfied:*

1. *Agent Maximization (AM): As was defined above.*

2. *Incentive Compatibility (IC): $\forall m \in M, s \in S$:*
$$G^m(\hat{s}) + \sum_{n\in N} F_n(\hat{s}_n) + \sum_{j\in M\setminus m} \sum_{n\in N} \hat{b}_n^j(\hat{s}_n) \geq G^m(s) + \sum_{n\in N} F_n(s_n) + \sum_{j\in M\setminus m} \sum_{n\in N} \hat{b}_n^j(s_n).$$

[4] Assuming that $G(\cdot)$ is non-negative can be viewed as a normalization of G. Without this normalization, our equilibrium refinement is slightly more complicated to define, which unnecessarily complicates subsequent proofs.

That is, the added value for principal m if agents choose the tuple of actions s instead of the tuple of actions $\hat{s}$ (i.e., $G^m(s) - G^m(\hat{s})$) is not larger than the additional payment required in order to incentivize the agents to move from $\hat{s}$ to s.

3. *Cost Minimization (CM):* $\forall m \in M, n \in N:$

$$\sum_{j \in M} \hat{b}_n^j(\hat{s}_n) + F_n(\hat{s}_n) = Max_{s_n \in S_n}[\sum_{j \in M \setminus m} \hat{b}_n^j(s_n) + F_n(s_n)].$$

That is, each principal submits the minimal necessary bid for the agents to choose $\hat{s}$.

For example, in the classic-GPTA based on table 2 below, one pure SPE is: $\hat{b}_1^1(T) = 2, \hat{b}_2^1(R) = 3, \hat{b}_1^1(B) = \hat{b}_2^1(L) = 0, \hat{b}_1^2(B) = 2, \hat{b}_2^2(L) = 3, \hat{b}_1^2(T) = \hat{b}_2^2(R) = 0$, with agent actions $\hat{s} = (T, L)$. Note that AM, IC and CM are satisfied.

I , II	L	R
T	4 , 4	5 , 0
B	0 , 5	1 , 1

Table 2: A classic-GPTA with a pure SPE.

3 VCG Contracts

We consider a different payment rule and refer to the resulting game as VCG-GPTA:

$$t_{m,n}^{VCG}(s_n, b) = Max_{x \in S_n} \left\{ \sum_{m' \in M \setminus m} b_n^{m'}(x) + F_n(x) \right\} - Max_{y \in S_n} \left\{ \sum_{m' \in M} b_n^{m'}(y) + F_n(y) \right\} + b_n^m(s_n) \quad (2)$$

In words, a principal pays each agent her bid for the agent's chosen action minus the 'potential social gain' accredited to this principal. The 'potential social gain' is the maximal sum of bids and the agent's value, for some action of this agent, minus the maximal sum of bids of all other principals plus the agent's value, for some action of this agent. We term the resulting game VCG-GPTA.

For intuition, consider the extensively studied model of multiple simultaneous single-item auctions (e.g., Bikhchandani (1999); Syrgkanis and Tardos (2013); Feldman, Fu, Gravin, and Lucier (2013); Roughgarden, Syrgkanis, and Tardos (2017)). Each buyer has a value for subsets of items and submits separate bids in the different auctions. With classic single-item auctions this becomes a special case of classic-GPTA: the sellers are the agents, the agent's actions are the choice of which buyer will receive the item, the buyers are the principals, the valuation function of buyer m is $G^m(\cdot)$, and the valuation function of seller n is $F_n(\cdot)$. The bid that buyer m submits specifies the price that will be paid to seller n if her action is to assign the item to m. This is exactly the classic contract of Prat and Rustichini (2003). Second-price auctions cannot be represented in this model since the contract of a buyer in this case depends on the other buyers' contracts. Namely, principal m submits her willingness to pay but her actual payment is the second highest bid which is stated in another principal's contract. Thus, to capture second-price auctions one needs to submit contracts that depend on other contracts.

VCG contracts follow this intuition. Conceptually one can consider a setting in which each agent n is a social designer, the set of alternatives to her choice is S_n, the principals are the players of the VCG framework, and alternative $s_n \in S_n$ is considered to have a private value of $b_n^m(s_n)$ to principal m. Principal m submits a willingness to pay $b_n^m(s_n)$ for every action s_n of every agent n. The payment in Eq. (2) of principal m to agent n is dependent on agent n's chosen action, and may be different than the payment in the VCG mechanism. Nevertheless, lemma 3 below shows that if the agent chooses a welfare-maximizing alternative $\hat{s}_n \in \text{argmax}_{x \in S_n} \left\{F_n(x) + \sum_{m \in M} b_n^m(x)\right\}$ then the payment in Eq. (2) becomes exactly equal to Clarke's payment rule, viewing principal m as the player who pays the social designer (the agent). Interestingly, although agent n is not a social designer but a strategic player herself, lemma 2 below shows that her best response to any b is to choose an alternative/action $\hat{s}_n \in S_n$ that maximizes the social welfare.

Although VCG contracts are conceptually motivated by the VCG mechanism, there are three significant differences between the two settings. First, agent n is not a social designer but a strategic player. Second, principals pay multiple agents rather than a single social designer. Third, a principal's value is determined by a combination of all agents' actions and not only by a specific action of a single agent. In particular, the principals often do not have a dominant strategy and a multiplicity of subgame perfect equilibria exist in our setting.

3.1 Main Properties

Interestingly, although VCG contracts are different than classic contracts, agents' optimal actions continue to satisfy AM:

Lemma 2. *For any b and n: $argmax_{x \in S_n} \{u_n(x,b)\} = argmax_{x \in S_n} \left\{F_n(x) + \sum_{m \in M} b_n^m(x)\right\}$. Therefore, AM holds in any pure SPE of the VCG-GPTA game.*

Proof. The utility an agent obtains from an action $x \in S_n$ is $F_n(x) + \sum_{m \in M} t_{m,n}^{VCG}(x,b)$. For each m, $t_{m,n}^{VCG}(x,b)$ is equal to $b_n^m(x)$ plus a constant that does not depend on x. Thus, the total utility is equal to a constant plus $F_n(x) + \sum_{m \in M} b_n^m(x)$, implying that the agent will choose an action x that maximizes this term. $\square$

Lemma 3. *For any b, m, n, and for any chosen $s_n \in argmax_{x \in S_n} u_n(x,b)$:*

$$t_{m,n}^{VCG}(s_n,b) = Max_{x \in S_n}\left\{\sum_{m' \in M \setminus m} b_n^{m'}(x) + F_n(x)\right\} - \left\{\sum_{m' \in M \setminus m} b_n^{m'}(s_n) + F_n(s_n)\right\}$$

As a result, three useful properties of t^{VCG} are:

1. *For any b, m, n and chosen s_n, we have $t^{VCG}_{m,n}(s_n, b) \geq 0$.* [5]

2. *For all $m \in M, b^{-m} \in B^{-m}, \tilde{b}^m, b^m \in B^m$, and for all $n \in N$, if $s_n \in argmax_{x \in S_n} u_n(x, (b^m, b^{-m}))$ and $s_n \in argmax_{x \in S_n} u_n(x, (\tilde{b}^m, b^{-m}))$, then $t^{VCG}_{m,n}(s_n, (b^m, b^{-m})) = t^{VCG}_{m,n}(s_n, (\tilde{b}^m, b^{-m}))$.*

3. *For any b, m, n and s_n, we have $t^{VCG}_{m,n}(s_n, b) \leq b^m_n(s_n)$.*

Proof. According to equation (2):

$$t^{VCG}_{m,n}(s_n, b) = Max_{x \in S_n} \left\{ \sum_{m' \in M \setminus m} b^{m'}_n(x) + F_n(x) \right\} - Max_{y \in S_n} \left\{ \sum_{m' \in M} b^{m'}_n(y) + F_n(y) \right\} + b^m_n(s_n).$$

Therefore, due to lemma 2,

$$t^{VCG}_{m,n}(s_n, b) = Max_{x \in S_n} \left\{ \sum_{m' \in M \setminus m} b^{m'}_n(x) + F_n(x) \right\} - \left\{ \sum_{m' \in M} b^{m'}_n(s_n) + F_n(s_n) \right\} + b^m_n(s_n).$$

Hence, $t^{VCG}_{m,n}(s_n, b) = Max_{x \in S_n} \left\{ \sum_{m' \in M \setminus m} b^{m'}_n(x) + F_n(x) \right\} - \left\{ \sum_{m' \in M \setminus m} b^{m'}_n(s_n) + F_n(s_n) \right\}$. □

With classic contracts, pure SPE outcomes are characterized using the three conditions AM, IC, and CM. We next show that with VCG contracts, pure SPE outcomes are characterized using just AM and IC. Intuitively, in VCG contracts, CM is made redundant since principal m's payment is 'automatically' minimized via its dependence on the other principals' bids. This is similar to the difference between first-price and second-price auction rules, where, in first-price each bidder aims to submit a minimal bid while in second-price the auction rule takes care of that.

Theorem 1. *Given a VCG-GPTA, $(\hat{b}, \hat{s})$ is a pure SPE if and only if AM and IC are satisfied.*

Proof. Lemma 2 shows that AM is necessary and for action $\hat{s}$ to be agent n's best response to $\hat{b}$. In order to show that AM+IC is sufficient, recall that IC states that $\forall m \in M$ and $\forall y \in S$:

$$G^m(\hat{s}) + \sum_{n \in N} F_n(\hat{s}_n) + \sum_{j \in M \setminus m} \sum_{n \in N} \hat{b}^j_n(\hat{s}_n) \geq G^m(y) + \sum_{n \in N} F_n(y_n) + \sum_{j \in M \setminus m} \sum_{n \in N} \hat{b}^j_n(y_n) \qquad (3)$$

Fix any principal bid b^m of m, that results in the agents choosing some action vector $y \in S$. By subtracting $\sum_{n \in N} Max_{x \in S_n}[\sum_{m' \in M \setminus m} \hat{b}^{m'}_n(x) + F_n(x)]$ from both sides of inequality (3), due to

[5] If an agent uses the strategy suggested by lemma 2 her resulting payment will be non-negative regardless of the actions taken by the other players. This is true even off the equilibrium path, as long as the agent best responds to the principal contracts stated in the first phase of the game. Classic "limited liability" (see for example Carroll (2015)) requires the agent to *never* pay the principal. Here, the agent is guaranteed not to pay any principal as long as she best responds to the contracts presented to her, regardless of the actions of the other agents, and even if principals submit non-equilibrium contracts.

lemma 3, we attain $G^m(\hat{s}) - \sum_{n\in N} t^{VCG}_{m,n}(\hat{s}_n, \hat{b}) \geq G^m(y) - \sum_{n\in N} t^{VCG}_{m,n}(y_n, (b^m, \hat{b}^{-m}))$ which means that the utility of principal m decreases when deviating from $\hat{b}^m$ to b^m.

In order to show that IC is necessary, let us assume negatively that IC does not hold. That is $\exists \tilde{s} \in S, \exists m \in M$:

$$G^m(\hat{s}) + \sum_{n\in N} F_n(\hat{s}_n) + \sum_{j\in M\setminus m} \sum_{n\in N} \hat{b}^j_n(\hat{s}_n) < G^m(\tilde{s}) + \sum_{n\in N} F_n(\tilde{s}_n) + \sum_{j\in M\setminus m} \sum_{n\in N} \hat{b}^j_n(\tilde{s}_n) \tag{4}$$

By announcing $\forall n \in N, \forall s_n \neq \tilde{s}_n : b^m_n(s_n) = 0$ and $\forall \tilde{s}_n \in \tilde{s} : b^m_n(\tilde{s}_n) = \sum_{m'\in M\setminus m} \hat{b}^{m'}_n(\hat{s}_n) + F_n(\hat{s}_n) - \sum_{m'\in M\setminus m} \hat{b}^{m'}_n(\tilde{s}_n) - F_n(\tilde{s}_n)$, principal m can cause agents $n \in N$ to deviate to $\tilde{s}$, and thus increase her utility by at least: $G^m(\tilde{s}) - G^m(\hat{s}) - \sum_{n\in N}[\sum_{m'\in M\setminus m} \hat{b}^{m'}_n(\hat{s}_n) + F_n(\hat{s}_n) - \sum_{m'\in M\setminus m} \hat{b}^{m'}_n(\tilde{s}_n) - F_n(\tilde{s}_n)]$, which is positive, due to equation (4). Therefore $(\hat{s}, \hat{b})$ is not a pure SPE, which is a contradiction. $\square$

Notice that a pure SPE for VCG-GPTA always exists, because $\forall m \in M$, *and* $\forall \hat{s} \in \text{argmax}_{x\in S} G^m(x)$, principal m can bid $\forall n \in N : b^m_n(\hat{s}_n)$ a sufficiently large number and $\forall s_n \neq \hat{s}_n : b^m_n(s_n) = 0$, while all other principals bid zero, on all agent actions. This is the same issue which arises in the 2^{nd} price auction setting. To rule out these equilibria, which are not so interesting, Fu, Kleinberg, Lavi, and Smorodinsky (2017) use the refinement of *Nash equilibria with no overbidding.*[6]

Definition 1. *No-over-bidding: A pure SPE $(\hat{s}, \hat{b})$ satisfies no-over-bidding if*

$$\forall m \in M : \sum_{n\in N} \hat{b}^m_n(\hat{s}) \leq G^m(\hat{s}).$$

We emphasize that no-over-bidding applies only for $(\hat{s}, \hat{b})$, i.e., only on the equilibrium path. No -over-bidding does not impose any restriction off the equilibrium path. An additional discussion is given in appendix A.

In table 2, the VCG-GPTA has the same pure SPE with no-over-bidding $(\hat{s}, \hat{b})$ as the classic-GPTA with $\forall m \in M, \forall n \in N : t^{classic}_{m,n}(\hat{s}, \hat{b}) = t^{VCG}_{m,n}(\hat{s}, \hat{b})$. On the other hand, in contrast to the classic-GPTA which does not have a pure SPE in table 1, the VCG-GPTA does have a pure SPE satisfying no-over-bidding. For example, $(\hat{s}, \hat{b})$ where $\hat{s} = TL$, $\hat{b}^2_1(T) = \hat{b}^2_2(L) = 1$, with all other principal bids equal to zero. Essentially, substituting CM with the weaker condition no-over-bidding enables a pure SPE in this case.

Theorem 2. *Every pair $(\hat{s}, \hat{b})$ that satisfies AM, IC and CM, also satisfies no-over-bidding. As a result, every pure SPE $(\hat{s}, \hat{b})$ in a classic-GPTA is also a pure SPE with no-over-bidding in a VCG-GPTA.*

Proof. Prat and Rustichini introduced the following corollary:

[6] See also Christodoulou, Kovács, and Schapira (2008).

Observation 3. *[Based on Prat and Rustichini (2003) - corollary 1] Given a pure SPE $(\hat{s}, \hat{b})$ in a classic-GPTA, $\forall n \in N$: $\exists \tilde{s}_n \in S_n, \tilde{s}_n \neq \hat{s}_n$ s.t. $\sum_{m\in M} \hat{b}_n^m(\hat{s}_n) + \sum_{n\in N} F_n(\hat{s}_n) = \sum_{m\in M} \hat{b}_n^m(\tilde{s}_n) + \sum_{n\in N} F_n(\tilde{s}_n)$. Moreover, $\forall m \in M, \forall n \in N$, $\exists a_n \in S_n$ s.t.*

$$F_n(\hat{s}_n) + \sum_{m'\in M} \hat{b}_n^{m'}(\hat{s}_n) = F_n(a_n) + \sum_{m'\in M} \hat{b}_n^{m'}(a_n) \quad \textit{and} \quad \hat{b}_n^m(a_n) = 0.$$

Consider a pure SPE $(\hat{s}, \hat{b})$ in a classic-GPTA, and fix $m \in M$. Because $(\hat{s}, \hat{b})$ is a pure SPE in the classic-GPTA, IC is satisfied, so $\forall a \in S$:

$$G^m(\hat{s}) + \sum_{n\in N} \sum_{m'\in M} \hat{b}_n^{m'}(\hat{s}_n) + \sum_{n\in N} F_n(\hat{s}_n) - \sum_{n\in N} \hat{b}_n^m(\hat{s}_n) \geq$$

$$\geq G^m(a) + \sum_{n\in N} \sum_{m'\in M} \hat{b}_n^{m'}(a_n) + \sum_{n\in N} F_n(a_n) - \sum_{n\in N} \hat{b}_n^m(a_n)$$

Due to observation 3, there exists $a = (a_1, ..., a_n)$ such that $G^m(\hat{s}) - \sum_{n\in N} \hat{b}_n^m(\hat{s}_n) \geq G^m(a) \geq 0$. This shows that no-over-bidding holds. □

Observation 4. *Every pair $(\hat{s}, \hat{b})$ which is a pure SPE in a classic-GPTA and in a VCG-GPTA, has the same transfers. That is: $\forall m \in M, \forall n \in N : t_{m,n}^{classic}(\hat{s}_n, \hat{b}) = t_{m,n}^{VCG}(\hat{s}_n, \hat{b})$.*

Proof. Consider $(\hat{s}, \hat{b})$ which is a pure SPE in a classic-GPTA. Due to CM, we have

$$\forall m \in M, \forall n \in N : \hat{b}_n^m(\hat{s}_n) = Max_{x\in S_n} \left[\sum_{m'\in M\setminus m} \hat{b}_n^{m'}(x) + F_n(x) \right] - \left[\sum_{m'\in M\setminus m} \hat{b}_n^{m'}(\hat{s}_n) + F_n(\hat{s}_n) \right].$$

The observation follows from lemma 3. □

Theorem 2 and table 1 together imply:

Corollary 1. *The class of instances that admit an efficient pure SPE with no-over-bidding with VCG contracts strictly contains the class of instances that admit an efficient pure SPE with classic contracts.*

3.2 Equilibrium existence via balancedness

We give a technical characterization of pure SPEs in the VCG-GPTA via a notion of generalized balancedness. This characterization serves as a building block for the analysis in section 4. Prat and Rustichini (2003) define a similar notion of balancedness and it is useful to start with their notion.

Definition 2. *[Prat and Rustichini (2003) - Definition 7]*

- *In a classic-GPTA, the matrix ω and vector z with respective dimensions $(M \times S)$ and $(\sum_{n\in N}|S_n|)$ are balanced weights with respect to $\hat{s}$ if all their elements are non-negative and $\forall m \in M, \forall n \in N, \forall a_n \in S_n \backslash \hat{s}_n : \sum_{s:s_n=a_n} \omega^m(s) = z_n(a_n)$.*

- *A classic-GPTA is balanced with respect to $\hat{s}$ if, for every collection of balanced weights ω, z,*

$$\sum_{m\in M}\sum_{s\in S}\omega^m(s)[G^m(\hat{s}) - G^m(s)] + \sum_{n\in N}\sum_{s_n\in S_n} z_n(s_n)(F_n(\hat{s}_n) - F_n(s_n)) \geq 0.$$

The matrix ω assigns a weight to every principal and every tuple of actions different than $\hat{s}$. It thus assigns per-principal weights to all deviations from $\hat{s}$. The condition balanced weights requires that for every agent n the sum of weights on a deviation involving a switch on the part of n, from $\hat{s}_n$ to a_n, is constant across principals. This sum of weights may depend on a_n and is denoted by $z_n(a_n)$. A game is balanced with respect to a given outcome $\hat{s}$ if, for any vector of balanced weights, the weighted sum of payoffs for principals and agents is maximal for $\hat{s}$. Prat and Rustichini (2003) define a notion of a "weakly truthful equilibrium" and show that a classic-GPTA is balanced w.r.t. $\hat{s}$ if and only if there exists such an equilibrium.[7] A similar characterization holds for VCG-GPTA:

Definition 3. *The triplet of vectors $\omega \in \mathbb{R}^{M\times S}, v \in \mathbb{R}^M$, and $z \in \mathbb{R}^N$ are generalized balanced weights with respect to $\hat{s}$, if $\forall m \in M, \forall n \in N, \forall s \in S : \omega^m(s) \geq 0, v_m \geq 0, z_n \geq 0$, and if*

$$v_j = \sum_{\{m'\in M\backslash j\}}\sum_{\{s:s_n\neq\hat{s}_n\}} \omega^{m'}(s) + z_n \qquad \forall j \in M, \forall n \in N; \tag{5}$$

Definition 4. *A VCG-GPTA is said to be Generalized-Balanced (GB) with respect to $\hat{s}$ if for every collection of generalized balanced weights ω, v, z*

$$\sum_{m\in M}\sum_{s\in S}\omega^m(s)[G^m(\hat{s}) + \sum_{n\in N} F_n(\hat{s}_n) - G^m(s) - \sum_{n\in N} F_n(s_n)] +$$

$$+ \sum_{m\in M} v_m \cdot G^m(\hat{s}) + \sum_{n\in N} z_n(F_n(\hat{s}) - Max_{y\in S_n} F_n(y)) \geq 0 \tag{6}$$

Theorem 5. *A VCG-GPTA has a pure SPE that satisfies no-over-bidding with outcome $\hat{s} \in S$ if and only if it is generalized balanced with respect to $\hat{s}$.*

This theorem fully characterizes existence of pure SPE while Prat and Rustichini (2003) only characterize the case where weakly truthful strategies form a pure SPE (and in fact explicitly mention the full characterization of sub-game equilibria as an open problem).

[7] Specifically, $\hat{b}^m$ is weakly truthful relative to $\hat{s}$ if $\forall s \in S$, $G^m(s) - \sum_{n\in N}\hat{b}_n^m(s) \leq G^m(\hat{s}) - \sum_{n\in N}\hat{b}_n^m(\hat{s})$. Weakly truthful equilibria generalize truthful equilibria (Bernheim and Whinston, 1986). Weak truthfulness is imposed for every $s \in S$ whereas no-over-bidding imposes a requirement only for action vectors on the equilibrium path.

The proof is in appendix B. At a high-level, the proof is composed of two parts. First, lemma 8 encodes the conditions AM, IC, and no-over-bidding using a certain polytope: every point in this polytope is associated with agents' actions and principal bids $(\hat{s}, \hat{b})$ that satisfy AM, IC, and no-over-bidding (notice that all these conditions are linear inequalities). Second, lemma 9 uses the Farkas Lemma to show that this polytope is non-empty if and only if the weighted sum in the generalized-balancedness condition is non-negative.

4 Existence of Efficient Pure Equilibria with VCG Contracts

In this section, we show that VCG contracts expand the set of efficient equilibria relative to the set of "weakly truthful" (WT) equilibria which is the focus of Prat and Rustichini (2003) – recall the discussion prior to theorem 5 and additionally note that all weakly truthful equilibria are efficient (as shown by Prat and Rustichini (2003)). We restrict attention to games in which $|M| \geq 2$, $|N| \geq 2$, where $\forall n \in N : |S_n| \geq 2$.[8] Let us define:

Definition 5.

$$\begin{aligned} WT^{classic}(M,N,S) = \{(G,F) \in (\mathbb{R}^{|M|\cdot|S|} \times \mathbb{R}^{\sum_{n\in N}|S_n|}) | &\textit{ The classic-GPTA that corresponds to} \\ &(M,N,S,G,F) \textit{ has a weakly truthful SPE } \}, \\ SPEe^{VCG}(M,N,S) = \{(G,F) \in (\mathbb{R}^{|M|\cdot|S|} \times \mathbb{R}^{\sum_{n\in N}|S_n|}) | &\textit{ The VCG-GPTA that corresponds to} \\ &(M,N,S,G,F) \textit{ has an efficient pure SPE that satisfies no-over-bidding } \}. \end{aligned}$$

Theorem 6. *The set of VCG-GPTA with an efficient pure SPE that satisfies no-over-bidding strictly contains the set of classic-GPTA with a weakly truthful SPE. Moreover,* $\forall |M| \geq 2, |N| \geq 2, S = S_1 \times \cdots \times S_N$ *such that* $\forall n \in N : |S_n| \geq 2$*,* $SPEe^{VCG}(M,N,S) \setminus WT^{classic}(M,N,S)$ *has a positive Lebesgue measure in* $\mathbb{R}^{|M|\cdot|S|} \times \mathbb{R}^{\sum_{n\in N}|S_n|}$.

Proof. Theorem 2 implies that $WT^{classic}(M,N,S) \subseteq SPEe^{VCG}(M,N,S)$. To prove strict containment, we next show a game (G,F) that is generalized balanced with respect to the efficient $\hat{s} \in S$, where G is described in table 3 and F is all zeros. Furthermore we show that (G,F) is not balanced with respect to the efficient $\hat{s} \in S$. Thus, $(G,F) \in SPEe^{VCG}(\cdot) \setminus WT^{classic}(\cdot)$. We consider the set of games in the "close neighbourhood" of (G,F), and show that these games also belong to $SPEe^{VCG}(\cdot) \setminus WT^{classic}(\cdot)$.

For general M, N, S we extend this (G,F) as follows. First, designate a specific action $\hat{s}_n \in S_n$

[8] Obviously, we can ignore agents with $|S_n| = 1$, and the case of $|M| = 1$ is straightforward. The case of $|N| = 1$ (common agency) was handled by Bernheim and Whinston (1986). They showed that an efficient pure SPE always exists in the classic-GPTA, and therefore, due to theorem 2, in the VCG-GPTA.

I , II	L	R
T	5 , 0	0 , 4
B	0 , 4	3 , 3

Table 3: $G \in \mathbb{R}^{|M|\cdot|S|}$, where $|M| = 2$ and $|N| = 2$

for every agent $n \geq 3$. Second, let $G^m(s)$ (for any $m \in M, s \in S$) be:

$$\begin{aligned}
G^1(T, L, \hat{s}_3, ..., \hat{s}_N) &= 5 \\
G^1(B, R, \hat{s}_3, ..., \hat{s}_N) &= 3 \\
G^2(T, R, \hat{s}_3, ..., \hat{s}_N) &= 4 \\
G^2(B, L, \hat{s}_3, ..., \hat{s}_N) &= 4 \\
G^2(B, R, \hat{s}_3, ..., \hat{s}_N) &= 3 \\
\text{Otherwise: } G^m(s) &= 0 \\
\text{and let} \qquad \forall n \in N, \forall s_n \in S_n : F_n(s_n) &= 0
\end{aligned}$$

All games in the close neighbourhood of (G, F) are GB: In order to show that all games in the "close neighbourhood" of (G, F) (including (G, F)) are generalized balanced with respect to $BR \in S$, we perturbate the values of (G, F) by a small number $\epsilon : 0 \leq \epsilon \leq \overline{\epsilon}$, where $\overline{\epsilon} > 0$, and show that the generalized balanced requirement is still positive. Specifically, let $\overline{\epsilon} = \frac{1}{(2\cdot|N|^2+4\cdot|N|+1)\cdot|M|}$ and consider the "close neighbourhood" of (G, F) as all games $(G', F') \in \mathbb{R}^{|M|\cdot|S|} \times \mathbb{R}^{\sum_{n\in N}|S_n|}$:

$$\forall m \in M, \forall n \in N, \forall s_n \in S_n : |G'^m(s) - G^m(s)| \leq \overline{\epsilon} \;\; and \;\; |F'_n(s_n) - F_n(s_n)| \leq \overline{\epsilon} \tag{7}$$

Note that even though $F \equiv 0$, F' may differ from zero.

Due to definition 4, if the following equation (8) is non-negative $\forall \omega, v, z \in generalized - balanced - weights$ and $\forall \epsilon : 0 \leq \epsilon \leq \overline{\epsilon}$, then $\forall (G', F')$ that satisfies equation (7) is generalized balanced with respect to $\hat{s} = BR$.

$$\begin{aligned}
&\sum_{m\in M}\sum_{s\in S\setminus\hat{s}} \omega^m(s)[G'^m(\hat{s}) + \sum_{n\in N} F'_n(\hat{s}_n) - G'^m(s) - \sum_{n\in N} F'_n(s_n)]+ \\
&\quad + \sum_{m\in M} v_m \cdot G'^m(\hat{s}) + \sum_{n\in N} z_n(F'_n(\hat{s}_n) - Max_{y\in S_n} F'_n(y)) \geq \\
&\geq \sum_{m\in M}\sum_{s\in S\setminus\hat{s}} \omega^m(s)[(G^m(\hat{s}) - \epsilon) + \sum_{n\in N}(F_n(\hat{s}_n) - \epsilon) - (G^m(s) + \epsilon) - \sum_{n\in N}(F_n(s_n) + \epsilon)]+ \\
&\quad + \sum_{m\in M} v_m \cdot (G^m(\hat{s}) - \epsilon) + \sum_{n\in N} z_n((F_n(\hat{s}_n) - \epsilon) - Max_{y\in S_n}(F_n(y) + \epsilon)) =
\end{aligned} \tag{8}$$

$$= \sum_{m\in M}\sum_{s\in S\setminus\hat{s}} \omega^m(s)[G^m(\hat{s}) + \sum_{n\in N} F_n(\hat{s}_n) - G^m(s) - \sum_{n\in N} F_n(s_n)]+$$

$$+ \sum_{m\in M} v_m \cdot G^m(\hat{s}) + \sum_{n\in N} z_n(F_n(\hat{s}_n) - Max_{y\in S_n} F_n(y))-$$

$$-\epsilon[\sum_{m\in M}\sum_{s\in S\setminus\hat{s}} (2+2\cdot|N|)\omega^m(s) + \sum_{m\in M} v_m + 2\sum_{n\in N} z_n] \geq$$

$$\geq \omega^1(TL)[3-5] + \omega^1(TR)[3-0] + \omega^1(BL)[3-0] + \omega^2(TL)[3-0] + \omega^2(TR)[3-4] + \omega^2(BL)[3-4]+$$

$$+3\cdot(v^1) + 3\cdot(v^2) - \epsilon[\sum_{m\in M}\sum_{s\in S\setminus\hat{s}} (2+2\cdot|N|)\omega^m(s) + \sum_{m\in M} v_m + 2\sum_{n\in N} z_n]$$

Replacing v^1, v^2 according to definition 3:

$$\geq \omega^1(TL)[3-5] + \omega^1(TR)[3-0] + \omega^1(BL)[3-0] + \omega^2(TL)[3-0] + \omega^2(TR)[3-4] + \omega^2(BL)[3-4]+$$

$$v^1 + 2\cdot(\omega^2(TL) + Max\{\omega^2(TR), \omega^2(BL)\}) + v^2 + 2\cdot(\omega^1(TL) + Max\{\omega^1(TR), \omega^1(BL)\})-$$

$$-\epsilon[\sum_{m\in M}\sum_{s\in S\setminus\hat{s}} (2+2\cdot|N|)\omega^m(s) + \sum_{m\in M} v_m + 2\sum_{n\in N} z_n] \geq$$

$$v^1 + v^2 - \epsilon[\sum_{m\in M}\sum_{s\in S\setminus\hat{s}} (2+2\cdot|N|)\omega^m(s) + \sum_{m\in M} v_m + 2\sum_{n\in N} z_n] \geq$$

Due to definition 3 and lemmas 4 and 5 below:

$$\geq \frac{1}{|M|}\sum_{m\in M} v_m - \epsilon\left[(2+2\cdot|N|)\cdot|N|\cdot\sum_{m\in M} v_m + \sum_{m\in M} v_m + 2\sum_{n\in N}\sum_{m\in M} v_m\right] \geq$$

$$\geq \frac{1}{|M|}\sum_{m\in M} v_m - \epsilon\sum_{m\in M} v_m \cdot \left[(2\cdot|N| + 2\cdot|N|^2 + 1 + 2\cdot|N|\right] \geq 0.$$

Therefore, (G, F) satisfies GB.

Lemma 4. *Given a VCG-GPTA with $\omega, v, z \in generalized-balanced-weights$ we have*

$$\sum_{m\in M}\sum_{s\in S\setminus\hat{s}} \omega^m(s) \leq N\sum_{m\in M} v_m.$$

Proof. Due to definition 3:

$$\sum_{m\in M}\sum_{s\in S\setminus\hat{s}} \omega^m(s) \leq \sum_{m\in M}\sum_{\{m'\in M\setminus m\}}\sum_{s\in S\setminus\hat{s}} \omega^{m'}(s) \leq \sum_{m\in M}\sum_{\{m'\in M\setminus m\}}\sum_{n\in N}\sum_{\{s:s_n\neq\hat{s}_n\}} \omega^{m'}(s) \leq N\sum_{m\in M} v_m.$$

$\square$

Lemma 5. *Given a VCG-GPTA with $\omega, v, z \in generalized - balanced - weights$ we have*

$$v^1 + v^2 \geq \frac{1}{M} \sum_{m \in M} v_m.$$

Proof. Due to definition 3:

$$\forall n \in N : v^1 + v^2 \geq [\sum_{\{m' \in M \setminus 1\}} \sum_{\{s : s_n \neq \hat{s}_n\}} \omega^{m'}(s) + z_n] + [\sum_{\{m' \in M \setminus 2\}} \sum_{\{s : s_n \neq \hat{s}_n\}} \omega^{m'}(s) + z_n] =$$

$$= \sum_{\{m' \in M \setminus \{1,2\}\}} \sum_{\{s : s_n \neq \hat{s}_n\}} \omega^{m'}(s) + \sum_{\{m' \in M\}} \sum_{\{s : s_n \neq \hat{s}_n\}} \omega^{m'}(s) + 2z_n \geq$$

$$\geq Max_{m \in M} \{ \sum_{\{m' \in M \setminus m\}} \sum_{\{s : s_n \neq \hat{s}_n\}} \omega^{m'}(s) + z_n \} \geq \frac{1}{M} \sum_{m \in M} v_m.$$

□

(G, F) is not balanced: In order to show that (G, F) is not balanced with respect to $\hat{s} = BR$, we show there exist balanced weights ω, z which violate definition 2. Specifically, $z_2(L) = z_1(T) = 1$, otherwise $z(\cdot) = 0$, and $\forall m \geq 3 : \omega^m(TL) = \omega^2(BL) = \omega^2(TR) = \omega^1(TL) = 1$, and otherwise $\omega(\cdot) = 0$. We show that for these weights,

$$\sum_{m \in M} \sum_{s \in S} \omega^m(s)[G^m(\hat{s}) - G^m(s)] + \sum_{n \in N} \sum_{s_n \in S_n} z_n(s_n)(F_n(\hat{s}_n) - F_n(s_n)) = -4$$

This follows due to the following calculation:

$$\omega^1(TL)[3-5] + \omega^1(TR)[3-0] + \omega^1(BL)[3-0] + \omega^2(TL)[3-0] + \omega^2(TR)[3-4] + \omega^2(BL)[3-4] = \quad (9)$$

$$1 \cdot [3-5] + 0 \cdot [3-0] + 0 \cdot [3-0] + 0 \cdot [3-0] + 1 \cdot [3-4] + 1 \cdot [3-4] = -4.$$

Thus, (G, F) is not balanced with respect to BR (note that BR is the unique efficient outcome hence there cannot be any weakly truthful SPE with any other outcome $s \in S$), therefore $(G, F) \notin WT^{classic}(M, N, S)$. Furthermore, once again, one can verify that for these weights, all games in the "close neighbourhood" of (G, F) are not balanced. In fact, this holds for the same neighbourhood $\bar{\epsilon}$.

To conclude, we have shown a subset of games which are generalized balanced with respect to $BR \in S$ but are not balanced with respect to BR. Since this subset has positive Lebesgue measure in the set of all GPTAs, the theorem follows. □

Remark: When $|S_n| = 2$ for all $n \in N$ and there are no agent preferences, there is no loss of generality in considering weakly truthful equilibria instead of efficient pure SPE in classic-GPTA, since Prat and Rustichini (2003) show that all pure SPE in such a classic-GPTA must be weakly truthful. In contrast, when considering games with $|S_n| > 2$, Prat and Rustichini (2003) provide an example in which there exist efficient equilibria in the classic-GPTA which are not weakly truthful, and leave the characterization of such equilibria as an open question.

We do not know whether theorem 6 is correct in cases where $|S_n| > 2$ and where we consider all efficient equilibria instead of only those that are weakly truthful. For example, the game in table 4 is an instantiation of the game used in the proof of theorem 6 for the case of $|M| = 3, |N| = 2, |S_1| = |S_2| = 3$. In this case, efficiency is obtained (only) when agents play (MV,MH). This outcome cannot be achieved in any weakly truthful equilibrium (as the proof of theorem 6 shows) but can be achieved by an efficient pure SPE $(\hat{s}, \hat{b})$ in the classic-GPTA which is not weakly truthful: $\hat{b}_2^3(R) = \hat{b}_1^3(B) = 3$, $\hat{b}_1^1(MV) = \hat{b}_1^2(MV) = \hat{b}_2^1(MH) = \hat{b}_2^2(MH) = 1.5$, and otherwise zero, with $\hat{s} = (MV, MH)$. Thus, the proof of theorem 6 is not correct if we change the weakly truthful requirement in definition 5 to a general efficiency requirement.

	L	MH	R
T	5 , 0 , 0	0 , 4 , 0	0 , 0 , 0
MV	0 , 4 , 0	3 , 3 , 0	0 , 0 , 0
B	0 , 0 , 0	0 , 0 , 0	0 , 0 , 0

Table 4: $(\tilde{G}, \tilde{F}) \in \mathbb{R}^{|M|\cdot|S|} \times \mathbb{R}^{\sum_{n\in N}|S_n|}$, where $|M| = 3$, $|N| = 2$, and $\forall n \in N : |S_n| = 3$

5 A Price of Anarchy Comparison

We have shown that VCG-GPTA expands the set of subgame perfect equilibria of classic-GPTA, hence it could add both efficient as well as inefficient equilibria. This section evaluates the resulting worst-case inefficiency and shows two main insights:

1. The social welfare in the worst possible pure SPE of a VCG-GPTA is no less than a fraction of $\frac{1}{M}$ of the maximal social welfare that can be obtained by any combination of agents actions (whether in an equilibrium or not). This is true in any arbitrary GPTA setting.

2. There exists a classic-GPTA with a pure SPE whose welfare is exactly a fraction of $\frac{1}{M}$ of the maximal social welfare that could be obtained in that game. Thus, the worst-case fraction of the optimal social welfare (worst-case over all GPTAs with any fixed number of principals $|M|$) that is obtained in equilibrium is the same for both VCG-GPTA and classic-GPTA.

We obtain these conclusions using the formal framework of Price of Anarchy (see, e.g., Roughgarden (2005)). Fix a principal agent setting $\mathcal{G} = (M, N, S, G, F)$. Let $G(s) = \sum_{m\in M} G^m(s)$. The social welfare of an outcome s is $W(s) = G(s) + \sum_{n\in N} F_n(s_n)$. Let $s^* \in \text{argmax}_{s\in S}[W(s)]$ be an efficient outcome of $\mathcal{G}$. We discuss separately the case of positive optimal social welfare, i.e., $W(s^*) > 0$, and the case of non-positive optimal social welfare.[9]

Definition 6 (Price of Anarchy for positive optimal social welfare (PoA^+))**.** *Let $\mathcal{G}$ be a principal agent setting with $W(s^*) > 0$ and with a pure SPE in the corresponding VCG-GPTA. The Price of Anarchy of $\mathcal{G}$ with VCG contracts, $PoA^+_{VCG}(\mathcal{G})$, is defined as follows:*

$$PoA^+_{VCG}(\mathcal{G}) = \inf_{\substack{\hat{b} \in B \\ s.t.\ (\hat{s},\hat{b}) \text{ is a pure SPE with no-over-bidding in the VCG-GPTA of } \mathcal{G}}} \frac{W(\hat{s})}{W(s^*)}.$$

The Price of Anarchy with classic contracts, $PoA^+_{classic}(\mathcal{G})$, is defined in a similar manner.

If a principal agent setting $\mathcal{G}$ satisfies the conditions of definition 6 and admits a pure SPE in classic-GPTA then due to theorem 2 $PoA^+_{classic}(\mathcal{G}) \geq PoA^+_{VCG}(\mathcal{G})$. On the other hand, as we have seen, there exist many prinicipal agent settings in which classic-GPTA does not admit a pure SPE while the VCG-GPTA does admit a pure SPE. This section gives an explicit bound on $PoA^+_{VCG}(\mathcal{G})$ and shows that the worst-case bound on $PoA^+_{classic}(\mathcal{G})$ is the same. Formally:

Theorem 7. *For any $\mathcal{G}$ that satisfies the conditions of definition 6,*

$$PoA^+_{VCG}(\mathcal{G}) \geq \frac{1}{M} + \frac{(M-1)}{M} \cdot \frac{\sum_{n \in N} F_n(s^*_n)}{[G(s^*) + \sum_{n \in N} F_n(s^*_n)]}.$$

Furthermore, for any M, there exists a principal-agent setting $\mathcal{G}$ with M principals, for which

$$PoA^+_{VCG}(\mathcal{G}) = PoA^+_{classic}(\mathcal{G}) = \frac{1}{M} + \frac{(M-1)}{M} \cdot \frac{\sum_{n \in N} F_n(s^*_n)}{[G(s^*) + \sum_{n \in N} F_n(s^*_n)]}.$$

Remarks:

1. When $W(s^*) < 0$, any outcome s has $W(s) < 0$. In this case, the ratio $\frac{W(\hat{s})}{W(s^*)}$ is positive and not smaller than 1; a larger ratio represents a larger inefficiency. Thus, one should define $PoA^-_{VCG}(\mathcal{G})$ as the supremum of $\frac{W(\hat{s})}{W(s^*)}$ over all pure SPE outcomes $\hat{s}$. Practically the same proof shows that $PoA^-_{VCG}(\mathcal{G})$ is at most the bound stated in theorem 7.

2. When $W(s^*) \geq 0$, the proof shows that, for any pure SPE $(\hat{s}, \hat{b})$,

$$W(\hat{s}) \geq \frac{1}{M} \cdot W(s^*) + \frac{M-1}{M} \cdot \sum_{n \in N} F_n(s^*_n).$$

   When $W(s^*) < 0$, the inequality is reversed. When $W(s^*) \neq 0$, dividing by $W(s^*)$ gives the bound on the price of anarchy stated in theorem 7.

[9]Note that normalizing the setting so that $W(s^*) > 0$ would change the price of anarchy as this is a ratio term.

3. The same price-of-anarchy bounds hold for mixed strategy equilibria (the proof is similar to the above and is not included given our focus on pure strategy equilibria).

The two parts of theorem 7 are proved using the following two lemmas.

Lemma 6. *For any $\mathcal{G}$ that satisfies the conditions of definition 6,*

$$PoA^{+}_{VCG}(\mathcal{G}) \geq \frac{1}{M} + \frac{(M-1)}{M} \cdot \frac{\sum_{n\in N} F_n(s^*_n)}{W(s^*)}$$

Proof. This proof is based on Roughgarden (2015)'s smoothness technique. Let $(\hat{s}(\hat{b}), \hat{b}))$ be a pure SPE, and let s^* be an efficient outcome. Consider the following principal bids:

$$\forall n \in N : b^{*m}_n(s_n) = \begin{cases} Max_{x\in S_n} \sum_{m'\in M\setminus m} \hat{b}^{m'}_n(x) - \sum_{m'\in M\setminus m} \hat{b}^{m'}_n(s^*_n) & if \quad s_n = s^*{}_n \\ 0 & Otherwise \end{cases} \tag{10}$$

Due to AM, if the principal bids are $(b^{*m}, \hat{b}^{-m})$, all agents $n \in N$ will play s^*_n (i.e., $\hat{s}_n(b^{*m}, \hat{b}^{-m}) = s^*_n$) and the outcome will be s^*. Principal m's payoff function can be bounded above and below as follows, where the first inequality below follows from the definition of pure SPE:

$$u^m[\hat{s}, \hat{b}] = G^m(\hat{s}) - \sum_{n\in N} t^{VCG}_{m,n}(\hat{s}_n, (\hat{b}^m, \hat{b}^{-m})) \geq u^m[\hat{s}(b^{*m}, \hat{b}^{-m}), (b^{*m}, \hat{b}^{-m})] =$$
$$= G^m(s^*) - \sum_{n\in N} t^{VCG}_{m,n}(s^*_n, (b^{*m}, \hat{b}^{-m}))$$

Due to the definition of $t(\cdot)$ (see lemma (3)):

$$G^m(\hat{s}) - \sum_{n\in N}\left[Max_{x\in S_n}\left\{\sum_{m'\in M\setminus m} \hat{b}^{m'}_n(x) + F_n(x)\right\} - \left\{\sum_{m'\in M\setminus m} \hat{b}^{m'}_n(\hat{s}) + F_n(\hat{s}_n)\right\}\right] \geq$$
$$\geq G^m(s^*) - \sum_{n\in N}\left[Max_{x\in S_n}\left\{\sum_{m'\in M\setminus m} \hat{b}^{m'}_n(x) + F_n(x)\right\} - \left\{\sum_{m'\in M\setminus m} \hat{b}^{m'}_n(s^*) + F_n(s^*_n)\right\}\right] \geq$$
$$\geq [G^m(s^*) + \sum_{n\in N} F_n(s^*_n)] - \sum_{n\in N}\left[Max_{x\in S_n}\left\{\sum_{m'\in M\setminus m} \hat{b}^{m'}_n(x) + F_n(x)\right\}\right] \tag{11}$$

So, we have

$$G^m(\hat{s}) + \sum_{n\in N}\left\{\sum_{m'\in M\setminus m} \hat{b}^{m'}_n(\hat{s}) + F_n(\hat{s}_n)\right\} \geq [G^m(s^*) + \sum_{n\in N} F_n(s^*_n)]$$

Due to no-over-bidding :

$$G^m(\hat{s}) + \sum_{m' \in M \setminus m} [G^{m'}(\hat{s})] + \sum_{n \in N} F_n(\hat{s}) \geq [G^m(s^*) + \sum_{n \in N} F_n(s_n^*)]$$

That is,

$$G(\hat{s}) + \sum_{n \in N} F_n(\hat{s}) \geq [G^m(s^*) + \sum_{n \in N} F_n(s_n^*)]$$

Summing over $m \in M$:

$$M \cdot G(\hat{s}) + M \cdot \sum_{n \in N} F_n(\hat{s}) \geq G(s^*) + M \cdot \sum_{n \in N} F_n(s_n^*)$$

Re-organizing, we have,

$$M \cdot W(\hat{s}) \geq W(s^*) + (M-1) \cdot \sum_{n \in N} F_n(s_n^*)$$

Dividing by $W(s^*) \cdot M$ concludes the claim.

□

Lemma 7. *There exists $\mathcal{G}$ such that*

$$PoA^+_{VCG}(\mathcal{G}) = PoA^+_{classic}(\mathcal{G}) = \frac{1}{M} + \frac{(M-1)}{M} \cdot \frac{\sum_{n \in N} F_n(s_n^*)}{W(s^*)}$$

Proof. $\mathcal{G}$ is given in table 5, where $\forall n \in N, \forall s_n \in S_n : F_n(s_n) = 0$, hence the terms $F(\cdot)$ are omitted. Consider the following pure SPE of classic-GPTA, with outcome $\hat{s}(\hat{b}) = TL$ where for odd

	L	MV	R
T	1 , 1 , ... , 1	2 , 0 , 2, 0 , 2 , ...	0 , ... , 0
MH	0 , 2 , 0 , 2, 0 , ...	0 , ... , 0	0 , ... , 0
B	0 , ... , 0	0 , ... , 0	M , M , ... , M

Table 5: A principal-agent setting $\mathcal{G}$ with M principals.

$m_{odd} \in M$: $\hat{b}_1^{m_{odd}}(T) = \hat{b}_2^{m_{odd}}(MV) = 1$, and for even $m_{even} \in M$: $\hat{b}_1^{m_{even}}(MH) = \hat{b}_2^{m_{even}}(L) = 1$, otherwise, $\hat{b}_{(\cdot)}^{(\cdot)}(\cdot) = 0$. Notice that the efficient outcome is $s^* = BR$. Therefore, $\frac{1}{M} = \frac{G(TL)}{G(BR)} \geq PoA^+_{classic}(\mathcal{G}) \geq PoA^+_{VCG}(\mathcal{G}) \geq \frac{1}{M}$, where the last inequality follows from lemma 6. This implies the claim.

To see that $(\hat{b}, \hat{s})$ is a pure SPE, consider an even principal, m. Given the bids of the other principals, m is able to incentivize the agents to collectively choose either (T,L), (T,MV), or (MH,MV). This principal is not able to incentivize the agents to play (B,R) which is the socially preferred outcome, nor to play (MH,L) which is the second best outcome for m. Thus, intuitively, we see here two problematic aspects that together decrease the price of anarchy. There is a coordination problem since all principals would prefer to collectively deviate to (B,R), and there is a problem of conflicting interests that prevents them from reaching their second-best outcome (on which they disagree). □

Remark. In the above example, taking any constant $F_n(s_n) = F$ for all n, s_n (instead of zero) will give the same result. In addition, the example is for an even M. The case where M is odd is a simple adaptation, as follows:

- in cell (MH, L), instead of "2" in the even spaces have $G^{m_{even}}(MH, L) = \frac{M}{(M \text{ div } 2)}$,
- in cell (T, MV), instead of "2" in the odd spaces have $G^{m_{odd}}(T, MV) = \frac{M}{(M \text{ div } 2)+1}$,
- in cell (T, L), instead of "1" in the even spaces have $G^{m_{even}}(TL) = \frac{M}{(M \text{ div } 2) \cdot 2}$, and
- in cell (T, L), instead of "1" in the odd spaces have $G^{m_{odd}}(TL) = \frac{M}{((M \text{ div } 2)+1) \cdot 2}$.

We have shown that there exist principal-agent settings $\mathcal{G}$ for which $PoA^+_{VCG}(\mathcal{G}) = PoA^+_{classic}(\mathcal{G})$. Of-course, there exist games $\mathcal{G}$ for which $PoA^+_{VCG}(\mathcal{G}) < PoA^+_{classic}(\mathcal{G})$. For example, this is the case in the setting given in Table 2. Prat and Rustichini (2003) prove that when there are two actions for every agent, if a pure SPE exists then it is efficient. Indeed, an efficient pure SPE for the classic-GPTA of Table 2 is described in Section 2. Thus, the PoA of this game with classic contracts is 1. On the other hand, one can verify that with VCG contracts the following is an inefficient pure SPE: $\hat{b}^2_1(B) = 4$, $\hat{b}^2_2(L) = 1$, and zero otherwise. Note that Table 1 shows a game with two actions for each agent for which no pure SPE exists with classic contracts but for which an efficient pure SPE exists with VCG contracts. The comparison between the two types of contracts, in terms of efficiency versus existence is therefore inconclusive. However, if one is willing to accept a bounded welfare loss for wider equilibrium existence, then VCG contracts present a clear improvement over classic contracts.

6 The Relation to Contractible Contracts and Complex Mechanisms

The term contractible contracts as referred to in previous literature differs from classic contracts in two main aspects: First, the contract specifies payments contingent on agent actions and other principals' contracts. Second, these models endogenize the choice of these contracts, meaning that, in a first-phase of the game, each principal proposes a contract. Our model as stated above does not fully capture the second aspect, as the payment rule t^{VCG} is set exogenously. This begs a question of whether principals will prefer to participate in a "platform" that uses VCG contracts over other platforms that use classic contracts.

To answer this question, theorem 8 below shows that all our results continue to hold when considering the following model: First, each principal m chooses both a contract type $t^m \in \{t^{classic}, t^{VCG}\}$ and a bid vector $b^m \in B^m$. Second, based on these choices, each agent chooses an action. Finally, utilities are realized, where each principal pays according to the contract type chosen by her in the first stage. Let us term this game classic-VCG-GPTA. We show that pure SPE in VCG-GPTA are robust even if each principal has the option to deviate to a classic contract, regardless of whether the setting at hand admits pure SPE in classic-GPTA or not. In cases where there does not exist a pure SPE in classic-GPTA but there does exist a pure SPE in VCG-GPTA, classic-VCG-GPTA as well as VCG-GPTA therefore offer a clear advantage over classic-GPTA.

Theorem 8. *Given any setting $\mathcal{G}$, if $(\hat{s}, \hat{b})$ is a pure SPE in VCG-GPTA then $(\hat{s}, \hat{t}, \hat{b})$ is a pure SPE in classic-VCG-GPTA where $\hat{t}^m = t^{VCG}$ for all m.*[10]

Proof. Consider some principal m and a possible deviation to $\tilde{b}^m, \tilde{t}^m = \tilde{t}^{classic}$. Let $\tilde{s} = \hat{s}(\hat{b}^{-m}, \tilde{b}^m, \hat{t}^{-m}, \tilde{t}^m)$. Then,

$$u^m(\hat{b}^{-m}, \tilde{b}^m, \hat{t}^{-m}, \tilde{t}^m, \tilde{s}) \leq u^m(\hat{b}^{-m}, \tilde{b}^m, \hat{t}^{-m}, \hat{t}^m, \tilde{s}) \leq u^m(\hat{b}^{-m}, \hat{b}^m, \hat{t}^{-m}, \hat{t}^m, \hat{s}),$$

where the first inequality follows from the third item in Lemma 3 (note that both sides of the first inequality have $\tilde{s}$) and the second inequality is due to the assumption that $(\hat{s}, \hat{b})$ is a pure SPE in VCG-GPTA. $\square$

What about more complex contracts? Han (2007) defines a class of complex mechanisms and shows that any pure SPE in classic-GPTA is robust to the possibility of a principal's deviation to any complex mechanism. We note that the definition of complex mechanisms does not include VCG contracts. This raises the interesting question of whether the results in Han (2007) can be generalized to a class of contracts that contains VCG contracts, and if so, whether the above mentioned robustness – for either classic and/or VCG contracts – still applies. We leave this for future work. Another related direction is examining settings where agents are the ones choosing contracts, see for example Peters (1997) and Han (2015).

7 Conclusions

This paper studies a type of contractible contract with a structure more limiting relative to the previously studied general class. This has the advantage that many possible collusive agreements, that are possible under the general class, are eliminated due to the new structure. We study this in the context of the framework of Games Played Through Agents (GPTA) of Prat and Rustichini (2003), who assumed classic contracts. The new contract, termed the VCG contract, allows

[10]The same statement also holds for pure subgame-perfect equilibria of classic-GPTA (see observation 4).

principals to submit bids per agents' actions. These bids are interpreted as a principal's maximal willingness to pay (the agent) for any given action. The actual payment a principal pays an agent then depends on all principal bids in a way that is reminiscent of the VCG mechanism. We show three main results:

1. Every pure SPE in a GPTA with a classic contract (classic-GPTA) is also a pure SPE that satisfies no-over-bidding in a GPTA with a VCG contract (VCG-GPTA). Hence, VCG contracts do not eliminate any efficient pure SPE in classic-GPTA.

2. Moreover, the set of GPTAs that admit an efficient pure SPE with no-over-bidding in the corresponding VCG-GPTA but do not admit a weakly truthful SPE in the corresponding classic-GPTA has positive measure in the space of all possible GPTAs.

3. Even though the VCG-GPTA expands the set of pure subgame-perfect equilibria that satisfy no-over-bidding relative to the classic-GPTA, the worst-case welfare loss of the VCG-GPTA over the class of all games with M principals, for any fixed $M > 2$, is equal to that of the classic-GPTA.

The conceptually interesting aspect in VCG contracts is the agent's perception versus the principal's perception. The agent's payment is according to equation (2) which is different from the payment of a classic contract. Still, as lemma 2 shows, when the payment rule is according to equation (2), the agents' incentives are the same as in classic contracts, i.e., the action that maximizes an agent's utility is the action that maximizes the sum of bids. Thus, the agent perceives the VCG-contract as a classic contract. On the other hand, principals perceive the VCG-contract as multiple VCG mechanisms.

Our analysis of pure sub-game perfect equilibria in the VCG-GPTA relies on a necessary and sufficient condition for their existence. We have compared our condition, "generalized balancedness", to a similar balancedness condition given in Prat and Rustichini (2003), which, as explained there, is conceptually reminiscent of the cooperative concept of balancedness (Scarf, 1967). The satisfiability of both conditions can be checked by (two different) linear programs. The comparison between the two conditions makes it clear why some VCG-GPTA admit efficient equilibria while the corresponding classic-GPTA do not.

This characterization approach is conceptually different from the approach taken by most of the literature on multiple simultaneous single-item auctions. The necessary and sufficient conditions identified both here and in Prat and Rustichini (2003) examine tuples of valuations (i.e., the *combination* of all principal valuations) that characterize the existence of a desirable outcome. In contrast, the auction literature characterizes valuation classes (e.g., gross substitutes) for each *single* buyer that guarantee existence of a desirable outcome. For example, if all valuations are gross substitutes then a pure equilibrium is guaranteed to exist in both classic and second-price multiple simultaneous auctions Bikhchandani (1999); Fu, Kleinberg, and Lavi (2012). The connections between these two approaches and their implications are an interesting topic for future investigation, especially since the GPTA setting is much more general (for example, it allows for auctions with externalities).

Many additional interesting open questions present themselves. It could be interesting to further investigate the measure of the set of GPTAs that admit a weakly truthful equilibrium in classic-GPTA relative to the set of GPTAs that admit an efficient pure SPE with no-over-bidding in VCG-GPTA. Similarly, we do not know what the measure is of VCG-GPTAs that satisfy the balancedness condition relative to the entire set of GPTAs (i.e., we do not have a complete understanding of how restrictive this condition is). It also seems desirable to further characterize the relationship between the price of anarchy of VCG contracts and classic contracts for various classes of GPTAs. Furthermore, we believe that VCG contracts may improve efficiency in incomplete-information settings as well. Finally, it could be interesting to identify other simple and natural types of contractible contracts.

A No-over-bidding on and off the Equilibrium Path

A question that arises regarding no-over-bidding is whether it is a requirement concerning only the final SPE outcome $\hat{s}$ of the game (if a pure SPE $(\hat{s}, \hat{b})$ exists) or whether it is imposed on all outcomes $s \in S$ given $\hat{b}$. Our no-over-bidding definition is based on the former option. We briefly consider the latter option and explain why it was not chosen.

Definition 7. *A pure SPE $(\hat{s}, \hat{b})$ satisfies* ***no-over-bidding on and off the equilibrium path*** *if*

$$\forall m \in M, \forall s \in S : \sum_{n \in N} \hat{b}_n^m(s_n) \leq G^m(s).$$

Clearly, this condition implies no-over-bidding. As we have shown, no-over-bidding is satisfied in any pure SPE of the classic-GPTA. The following example demonstrates that this is not the case for no-over-bidding on and off the equilibrium path:

	L	R
T	2 , 0	ϵ , ϵ
B	0 , 0	0 , 2

Table 6: A GPTA with two agents and two principals

Consider the following principal bids: $\hat{b}_1^1(T) = \hat{b}_2^1(L) = 1$, $\hat{b}_2^2(B) = \hat{b}_2^2(R) = 1$ and otherwise $\hat{b}(\cdot) = 0$. These principal bids form the pure SPE $(\hat{b}, TL)$ or the pure SPE $(\hat{b}, BR)$ in the classic-GPTA and in the VCG-GPTA, and satisfy no-over-bidding . However, these two sub-game perfect equilibria do not satisfy no-over-bidding on and off the equilibrium path since, for example, for outcome $s = BL$ we have $\hat{b}_1^1(B) + \hat{b}_2^1(L) > G^1(BL)$. In fact, neither the outcome TL nor the outcome BR can be achieved in a pure SPE with no-over-bidding on and off the equilibrium path: Due to outcomes TR and BL, principal bids that satisfy no-over-bidding on and off the equilibrium path must satisfy $\hat{b}_1^1(T) \le \epsilon$, $\hat{b}_2^2(R) \le \epsilon$, and $\hat{b}(\cdot) = 0$ otherwise. Thus, the only outcome attainable in a pure SPE satisfying no-over-bidding on and off the equilibrium path, AM and IC, is TR.[11]

In conclusion, one of the useful properties of VCG contracts is that it maintains all efficient equilibria of the more basic classic contracts, and expands their existence. The above example shows that this property cannot be achieved with no-over-bidding on and off the equilibrium path, since the latter would eliminate some of the efficient equilibria that classic-GPTA exhibits. Therefore, we base our analysis on the no-over-bidding requirement, which seems a natural requirement that is satisfied in all subgame-perfect equilibria of classic-GPTA.

B Proof of Theorem 5

We prove that a VCG-GPTA has a pure SPE that satisfies no-over-bidding with outcome $\hat{s} \in S$ if and only if it is generalized balanced with respect to $\hat{s}$. For the proof, consider the following matrices and vectors (using the indices $m, j \in [1, |M|]$, $i, n \in [1, N]$, $s \in [1, |S|]$): Matrix $A \in \mathbb{R}^{(|S|\cdot|M|)\times(|N|\cdot|M|)}$ such that[12]

$$A_{(sm,nj)} = \{ {}^{-1}_{\ 0} \quad {}^{if \quad j \neq m, s_n \neq \hat{s}_n,}_{otherwise} \tag{12}$$

We demonstrate Matrix A, where $S = \{s^1, s^2, ..., s^{|S|}\}$:

$S \cdot M \backslash N \cdot M$	$n=1, j=1$	$n=1, j=2$	...	$n=1, j=\lvert M\rvert$	$n=2, j=1$	...
$s^1, m=1$	0	-1	...	-1	0	$\cdot$
$s^1, m=2$	-1	0	...	-1	-1	...
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$
$s^1, m=\lvert M\rvert$	-1	-1	...	0	-1	...
$s^2, m=1$	0	0	0 ... 0	0	0	0 ... 0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$

Table 7: The matrix A, where $\hat{s} = s^2$, $\forall i \in \{1, 2\} : \hat{s}_i \neq s_i^1$

Matrix $B \in \mathbb{R}^{|M|\times(|N|\cdot|M|)}$, such that:

$$B_{(m,nj)} = \{ {}^{1}_{0} \quad {}^{if \quad j=m,}_{otherwise} \tag{13}$$

[11] In particular, the principal bids $b_n^m(s_n) = 0$ for all m, n, s_n is a pure SPE only if the agents use the tie-breaking $\hat{s}_1(b) = T$ and $\hat{s}_2(b) = R$. Any other s is not a pure SPE as one of the principals will then want to deviate. For example, if $\hat{s}_1(b) = B$, principal 1 can increase utility by offering $b_1^1(T) = \epsilon/2$.

[12] Formally, if we denote the $A_{(sm,nj)}$ element as $A_{(k,h)}$, then: s=(k div $|M|$)+1, and m=(k mod $|M|$)+1. n=(h div $|M|$)+1, and j = (h mod $|M|$)+1.

Matrix $C \in \mathbb{R}^{|N| \times |N| \cdot |M|}$, such that:

$$C_{(i,nj)} = \{ {}^{-1}_{0} \quad {}^{if \quad i=n,}_{otherwise} \tag{14}$$

Vector a is of length $|S| \cdot |M|$, such that:[13]

$$a_{sm} = G^m(\hat{s}) + \sum_{n \in N} F_n(\hat{s}_n) - G^m(s) - \sum_{n \in N} F_n(s_n) \tag{15}$$

Vector b is of length $|M|$ such that:

$$b_m = G^m(\hat{s}) \tag{16}$$

Vector c is of length $|N|$ such that:

$$c_n = F_n(\hat{s}_n) - Max_{y \in S_n}\{F_n(y)\} \tag{17}$$

Consider the following two lemmas:

Lemma 8. *There exists a vector $x \in \mathbb{R}^{N \cdot M}$ such that:*

$$A \cdot x \leq a \tag{18}$$

$$B \cdot x \leq b \tag{19}$$

$$C \cdot x \leq c \tag{20}$$

$$x \geq 0 \tag{21}$$

iff there exists a pure SPE $(\hat{s}, \hat{b})$ in the VCG-GPTA that satisfies no-over-bidding .

Proof. Initially assume that there exists $x \in \mathbb{R}^{N \cdot M}$ that satisfies conditions (18) - (21). We show that there exists a pure SPE that satisfies no-over-bidding. In particular, construct the following $\hat{b} \in B$, such that $\forall m \in M, \forall n \in N, \forall s_n \in S_n$:

$$\hat{b}^m_n(s_n) = \{ {}^{x_{|M| \cdot (n-1)+m}}_{0} \quad {}^{if \quad s_n = \hat{s}_n}_{otherwise} \tag{22}$$

We shall show that these b's satisfy IC, no-over-bidding and AM, which in turn will prove, due to theorem 1, that they maintain a pure SPE in the VCG-GPTA that satisfies no-over-bidding.

Let us prove IC. For every principal $\forall m \in M$ and for every outcome $\forall s \in S$, due to condition (18) when multiplying the row in matrix A that corresponds to (s, m) by the vector x, we have

[13] Formally, if we denote the a_{sm} element as a_k, then, s=(k div $|M|$)+1, and m=(k mod $|M|$)+1.

$$-\sum_{\{m'\in M\setminus m\}}\sum_{\{n:s_n\neq\hat{s}_n\}} x_{M\cdot(n-1)+m'} \leq a_{ms} = G^m(\hat{s}) + \sum_{n\in N} F_n(\hat{s}_n) - G^m(s) - \sum_{n\in N} F_n(s_n). \tag{23}$$

Due to equation (22):

$$-\sum_{\{m'\in M\setminus m\}}\sum_{\{n:s_n\neq\hat{s}_n\}} \hat{b}_n^{m'}(\hat{s}_n) \leq G^m(\hat{s}) + \sum_{n\in N} F_n(\hat{s}_n) - G^m(s) - \sum_{n\in N} F_n(s_n) - \sum_{\{m'\in M\setminus m\}}\sum_{\{n:s_n\neq\hat{s}_n\}} \hat{b}_n^{m'}(s_n).$$

By subtracting $\sum_{\{m'\in M\setminus m\}}\sum_{\{n:s_n=\hat{s}_n\}} \hat{b}_n^{m'}(\hat{s}_n)$ from both sides, we attain:

$$-\sum_{\{m'\in M\setminus m\}}\sum_{\{n:s_n\neq\hat{s}_n\}} \hat{b}_n^{m'}(\hat{s}_n) - \sum_{\{m'\in M\setminus m\}}\sum_{\{n:s_n=\hat{s}_n\}} \hat{b}_n^{m'}(\hat{s}_n) \leq$$

$$\leq G^m(\hat{s}) + \sum_{n\in N} F_n(\hat{s}_n) - G^m(s) - \sum_{n\in N} F_n(s_n) - \sum_{\{m'\in M\setminus m\}}\sum_{\{n:s_n\neq\hat{s}_n\}} \hat{b}_n^{m'}(s_n) - \sum_{\{m'\in M\setminus m\}}\sum_{\{n:s_n=\hat{s}_n\}} \hat{b}_n^{m'}(\hat{s}_n)$$

and so we have IC.

Let us prove no-over-bidding . Due to condition (19), $\forall m \in M : \sum_{n\in N} x_{M\cdot(n-1)+m} \leq G^m(\hat{s})$. Due to equation (22) we have $\forall m \in M : \sum_{n\in N} b_n^m(\hat{s}) = \sum_{n\in N} x_{M\cdot(n-1)+m} \leq G^m(\hat{s})$, hence we have no-over-bidding .

Let us prove AM. Due to condition (20), $\forall n \in N : -\sum_{m\in M} x_{M\cdot(n-1)+m} \leq F_n(\hat{s}_n) - Max_{y\in S_n} F_n(y)$. Due to equation (22) we have $\forall n \in N : -\sum_{m\in M} b_n^m(\hat{s}_n) \leq F_n(\hat{s}_n) - Max_{y\in S_n} F_n(y)$. Due to equation (22) $\forall n \in N : s_n \neq \hat{s}_n$ the principal bids are equal to zero, therefore we have AM.

Now we will show the other direction. That is, if a pure SPE $(\hat{s}, \hat{b})$ exists then there exists a vector $x \in \mathbb{R}^{N\cdot M}$ that satisfies conditions (18)- (21).

Consider a pure SPE with no-over-bidding $(\hat{s}, \hat{b})$. Due to theorem 1, we have AM and IC. We will construct another pure SPE with no-over-bidding $(\tilde{s}, \tilde{b})$, such that $\tilde{s} = \hat{s}$, and,

$$\forall m \in M, \forall n \in N : \tilde{b}_n^m(s_n) = \begin{cases} \hat{b}_n^m(\hat{s}_n) + \frac{G^m(\hat{s}) - \sum_{n\in N} \hat{b}_n^m(\hat{s}_n)}{N} & if \quad s_n = \hat{s}_n \\ 0 & otherwise \end{cases} \tag{24}$$

This pair $(\tilde{s}, \tilde{b})$ still satisfies AM, IC and no-over-bidding (because IC, AM and no-over-bidding inequalities are maintained). And so, by denoting a vector $\tilde{x} \in \mathbb{R}^{N\cdot M} : \tilde{x}_{|M|\cdot(n-1)+m} = \tilde{b}_n^m(\hat{s})$, we can see that $\tilde{x}$ satisfies condition (18) (because of IC), condition (19) (because of no-over-bidding), condition (20) (because of AM), and condition (21) because $\forall m \in M, \forall s \in S, \forall n \in N : \tilde{b}_n^m(s_n) \geq \hat{b}_n^m(s_n)$. □

Lemma 9. *There exists a vector $x \in \mathbb{R}^{N\cdot M}$ which satisfies conditions (18) - (21), if and only if the VCG-GPTA is generalized balanced with respect to $\hat{s}$.*

Proof. Initially assume there does not exists a vector $x \in \mathbb{R}^{N\cdot M}$ that satisfies conditions (18) - (21).

We shall use the following Farkas lemma to prove that the game is not GB.

Lemma (The Farkas Lemma, as in Prat and Rustichini (2003))**.** *Given a matrix H and a vector h, either (i) there exits an x such that $H \cdot x \leq h$; or (ii) there exists a y such that $y \cdot H = 0, y \cdot h < 0$ and $y \geq 0$.*

Let $H = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ and let $h = [a, b, c]$. According to the Farkas Lemma, a solution $x \in \mathbb{R}^{N \cdot M}$ for conditions (18) - (21) exists if and only if there does not exist $y = (\omega^m(s)_{m \in M, s \in S}, (v_m)_{m \in M}, (z^n)_{n \in N})$, where ω is a vector of length of $|S| \cdot |M|$, v is a vector of length $|M|$ and z is a vector of length $|N|$ for the following system (25)-(27), such that,[14]

$$\omega \cdot A + v \cdot B + z \cdot C = \vec{0}; \tag{25}$$

$$\omega \cdot a + v \cdot b + z \cdot c < 0 \ ; \tag{26}$$

$$\omega, v, z \geq 0. \tag{27}$$

The system (25) may be rewritten as: $\forall j \in M, \forall n \in N:$

$$-\sum_{\{m' \in M \setminus j\}} \sum_{\{s : s_n \neq \hat{s}_n\}} \omega^{m'}(s) + v_j - z_n = 0 \qquad \forall j \in M, \forall n \in N; \tag{28}$$

That is,

$$v_j = \sum_{\{m' \in M \setminus j\}} \sum_{\{s : s_n \neq \hat{s}_n\}} \omega^{m'}(s) + z_n \qquad \forall j \in M, \forall n \in N; \tag{29}$$

Equation (26) can be re-written as:

$$\sum_{m \in M} \sum_{s \in S} \omega^m(s)[G^m(\hat{s}) + \sum_{n \in N} F_n(\hat{s}_n) - G^m(s) - \sum_{n \in N} F_n(s_n)] + \\ + \sum_{m \in M} v_m \cdot G^m(\hat{s}) + \sum_{n \in N} z_n(F_n(\hat{s}) - Max_{y \in S_n} F_n(y)) < 0 \tag{30}$$

which is a contradiction to GB.

The proof in the opposite direction is similar. □

Theorem 5 is a direct consequence of lemma 8 and lemma 9.

References

B Douglas Bernheim and Michael D Whinston. 1986. Menu auctions, resource allocation, and economic influence. *The quarterly journal of economics* 101, 1 (1986), 1–31.

[14] Formally, if we denote the $\omega_{(sm)}$ as $\omega_{(k)}$, then: s=(k div $|M|$)+1, and m=(k mod $|M|$) +1.

Sushil Bikhchandani. 1999. Auctions of heterogeneous objects. *Games and Economic Behavior* 26, 2 (1999), 193–220.

Gabriel Carroll. 2015. Robustness and linear contracts. *American Economic Review* 105, 2 (2015), 536–63.

George Christodoulou, Annamária Kovács, and Michael Schapira. 2008. Bayesian combinatorial auctions. In *International Colloquium on Automata, Languages, and Programming*. Springer, 820–832.

Michal Feldman, Hu Fu, Nick Gravin, and Brendan Lucier. 2013. Simultaneous auctions are (almost) efficient. In *Proceedings of the forty-fifth annual ACM symposium on Theory of computing*. ACM, 201–210.

Hu Fu, Robert Kleinberg, and Ron Lavi. 2012. Conditional equilibrium outcomes via ascending price processes with applications to combinatorial auctions with item bidding.. In *The 13th ACM conference on Economics and Computation (EC)*.

Hu Fu, Robert Kleinberg, Ron Lavi, and Rann Smorodinsky. 2017. Stability and auctions in labor markets with job security. *Economics Letters* 154 (2017), 55–58.

Seungjin Han. 2007. Strongly robust equilibrium and competing-mechanism games. *Journal of Economic Theory* 137, 1 (2007), 610–626.

Seungjin Han. 2015. Robust competitive auctions. *Economics Letters* 136 (2015), 207–210.

Ron Lavi and Elisheva Shamash. 2019. Principal-Agent VCG Contracts. In *Proceedings of the 2019 ACM Conference on Economics and Computation*. ACM, 783–783.

Michael Peters. 1997. A competitive distribution of auctions. *The Review of Economic Studies* 64, 1 (1997), 97–123.

Michael Peters. 2014. Competing mechanisms. *Canadian Journal of Economics/Revue canadienne d'économique* 47, 2 (2014), 373–397.

Michael Peters and Balázs Szentes. 2012. Definable and contractible contracts. *Econometrica* 80, 1 (2012), 363–411.

Andrea Prat and Aldo Rustichini. 2003. Games played through agents. *Econometrica* 71, 4 (2003), 989–1026.

Tim Roughgarden. 2005. *Selfish routing and the price of anarchy*. Vol. 174. MIT press Cambridge.

Tim Roughgarden. 2015. Intrinsic robustness of the price of anarchy. *Journal of the ACM (JACM)* 62, 5 (2015), 32.

Tim Roughgarden, Vasilis Syrgkanis, and Eva Tardos. 2017. The price of anarchy in auctions. *Journal of Artificial Intelligence Research* 59 (2017), 59–101.

Herbert E Scarf. 1967. The core of an N person game. *Econometrica: Journal of the Econometric Society* (1967), 50–69.

Vasilis Syrgkanis and Eva Tardos. 2013. Composable and efficient mechanisms. In *Proceedings of the forty-fifth annual ACM symposium on Theory of computing*. ACM, 211–220.

Balázs Szentes. 2015. Contractible contracts in common agency problems. *The Review of Economic Studies* 82, 1 (2015), 391–422.