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FINITE-RANK ADI ITERATION FOR OPERATOR LYAPUNOV EQUATIONS*

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Abstract. We give an algorithmic approach to the approximative solution of operator Lyapunov equations for controllability. Motivated by the successfully applied *alternating direction implicit (ADI)* iteration for matrix Lyapunov equations, we consider this method for the determination of Gramian operators of infinite-dimensional control systems. In the case where the input space is finite-dimensional, this method provides approximative solutions of finite rank. Under the assumption of infinite-time admissibility and boundedness of the semigroup, we analyze convergence in several operator norms. We show that under a mild assumption on the shift parameters, convergence to the Gramian is obtained. Particular emphasis is placed on systems governed by a heat equation with boundary control. We present that ADI iteration for the heat equation consists of solving a sequence of Helmholtz equations. Two numerical examples are presented; the first showing the benefit of adaptive finite elements and the second illustrating convergence to something other than the Gramian in a case where our condition on the shift parameters is not satisfied.

Key words. Lyapunov equation, ADI iteration, numerical method in control theory, infinite-dimensional linear systems theory, heat equation

AMS subject classifications. 39B42, 47J25, 65J10, 93B28, 93C20, 93B05

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1. Introduction.

A fundamental concept in linear finite-dimensional systems theory is the *Gramian matrix*, i.e., the solution $P \in \mathbb{R}^{n\times n}$ of the *Lyapunov equation*

$$AP + PA^T + BB^T = 0 \tag{1.1}$$

associated to a linear control system $\dot{x}(t) = Ax(t) + Bu(t)$ with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. These equations, for instance, arise in stability and controllability analysis [51, sect. 3.8], model reduction by balanced truncation [5, Chap. 7], and the Newton–Kleinman method for solving Riccati equations which arise, e.g., in optimal control [13, 32].

Due to their importance, a variety of numerical methods have been developed for Lyapunov equations, such as alternating direction implicit (ADI) iteration [29, 30, 35], Bartels–Stewart method [10], Smith's method [49], Krylov subspace method [27, 46], sign function method [41], Hammarling's method [24], and proper orthogonal decomposition (POD) based methods [43, 56]. (See [5, Chap. 6] for an overview.) Especially the ADI iteration, Smith method, matrix sign function method, and POD based methods have in common that, in case of $m \ll n$, they typically provide so-called low-rank approximative solutions. That is, instead of the full Gramian matrix, a factor $S \in \mathbb{R}^{n\times k}$ with $k \ll n$ and $P \approx SS^T$ is computed iteratively. This feature makes these methods suitable for problems of large state space dimension $n \in \mathbb{N}$, especially since memory effort is saved. An important class of large-scale systems are those emerging from fine spatial discretization of controlled systems which are

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governed by linear partial differential equations [11]. The latter, however, actually has (before discretization) infinite state space dimension; in the Lyapunov equation, the variables A, B, and the to-be solved P are actually operators acting on infinite-dimensional spaces. It is hence natural to wonder about the following questions:

(a) Can iterative algorithms be formulated for operator Lyapunov equations, and (when) do they converge?

(b) What are the (computational) consequences for systems governed by PDEs?

These questions have been considered for POD based methods in [47, 48]. In this work we consider the ADI iteration for operator Lyapunov equations. This iteration provides an operator S, acting on some space that will be finite-dimensional if the input space is finite-dimensional, such that $P \approx SS^*$. In this article, convergence of SS^* to P will be shown under the assumption (apart from the necessary assumption that the Gramian operator exists) of boundedness of the semigroup generated by A. In contrast to the POD based methods considered in [47, 48], we allow the input operator to be "unbounded," which is motivated by partial differential equations with boundary control. To prove convergence we also need to impose a rather mild restriction on the shift parameters used in the ADI algorithm. In addition, we give a numerical example which indicates that if the shift parameters do not satisfy this restriction, then SS^* may converge to something else than the Gramian.

Particular emphasis is placed on systems governed by a boundary controlled heat equation, where it will turn out that the ADI method consists of the solution of a sequence of Helmholtz equations. Since the latter can be (approximatively) solved by using adaptive finite element methods, we will also discuss the impact of approximative solution in each step of the ADI iteration.

This possibility of using adaptive finite element methods is a distinct advantage of our approach. At each iteration step of ADI, an appropriate discretization is chosen (rather than this being fixed a priori). This leads to significant computational savings.

The paper is organized as follows. The subsequent section 2 reviews the basic notational and functional analytic framework. Section 3 contains basic facts about semigroups and operator Lyapunov equations. In section 4 we introduce the ADI iteration for the solution of operator Lyapunov equations and present results about convergence. In section 5 we expand our analysis to an inexact ADI iteration that one would have to do in practical computations due to the necessity of discretization in the infinite-dimensional context. In section 6 the developed theory is applied to a heat equation with Robin boundary control on an L-shaped domain. We give two numerical examples. The first illustrates the benefit of using adaptive finite elements. In the second numerical example we give a shift parameter choice for which convergence of SS^* to something other than the Gramian occurs.

2. Basic notation and functional analytic prerequisites. Throughout the paper, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, $\mathbb{C}_+$, $\mathbb{C}_-$, and $\mathbb{C}^{n\times m}$, respectively, denote the sets of positive real numbers, nonnegative real numbers, complex numbers with positive real part, complex numbers with negative real part, and the space of $n \times m$ complex matrices. $\mathbb{N}$ stands for the set of positive integers, and by $\overline{z}$ we mean the complex conjugate of $z \in \mathbb{C}$.

For $p \geq 1$, ℓ_p stands for the p-summable complex sequences. We use the notation from [1] for Lebesgue and Sobolev spaces $L_p(\Omega)$ and $H^k(\Omega)$, and the one from [17] for the Hardy spaces $\mathcal{H}_2(U)$ and $\mathcal{H}_2$.

Throughout this work, integrals of functions with values in Hilbert space are understood in the sense of Bochner. For a brief overview on abstract integration theory we refer to [17, p. 621] and the bibliography therein. For $p \in [1, \infty]$, some interval I and some separable Hilbert space X, $L_p(I,X)$ denotes the Lebesgue space of measurable functions $f(\cdot): I \to X$ with the property that $\|f\|_X \in L_p(I)$.

Let Z and X be Hilbert spaces, such that $Z \subset X$ and the canonical injection $Z \to X$, $x \mapsto x$ is continuous and dense. By calling a Hilbert space X *pivot space*, we mean that X is identified with its own topological dual X' (which is possible by the Riesz representation theorem [38, p. 48]), and the dual of Z is defined in a way such that the dual pairing $\langle \cdot,\cdot \rangle_{Z',Z}$ continuously (w.r.t. the norm in Z) extends the inner product in X. It follows that $X \subset Z'$ and the canonical injection $X \to Z'$, $x \mapsto x$ is continuous and dense.

$\mathcal{B}(X,Y)$ and $\mathcal{K}(X,Y)$ are the spaces of bounded, respectively, compact, linear operators $T: X \to Y$, and we abbreviate $\mathcal{B}(X) := \mathcal{B}(X,X)$, $\mathcal{K}(X) := \mathcal{K}(X,X)$. For a densely defined operator $T: D(T) \subset X \to X$, the symbols $\rho(T)$ and $\sigma(T)$ indicate its resolvent set and spectrum, respectively. The identity mapping on X is denoted by I_X and the zero operator from X to Y by $0_{X,Y}$. Given an operator $T: D(T) \subset X \to Y$, the *graph norm* is defined via $\|x\|^2_{D(T)} = \|x\|^2_X + \|Tx\|^2_Y$. If $D(T)$ associated with the graph norm $\|\cdot\|_{D(T)}$ is complete, then T is called *closed*.

A vector $v \in X$ is in a canonical way identified as an operator $v \in \mathcal{B}(\mathbb{C},X)$ via $\lambda \mapsto \lambda v$. For a Hilbert space X and $m \in \mathbb{N}$, the product space X^m is equipped with the canonical inner product. For another Hilbert space Y and operators $T_1,\dots,T_m \in \mathcal{B}(X,Y)$, the operator column matrix

$$T = \begin{bmatrix} T_1 & \cdots & T_m \end{bmatrix}$$

defines an operator $T \in \mathcal{B}(X^m, Y)$ in a straightforward manner.

The *adjoint* of $T \in \mathcal{B}(X,Y)$ is denoted by $T^* \in \mathcal{B}(Y,X)$ and the *dual* by $T' \in \mathcal{B}(Y',X')$. The adjoint of a densely defined operator $T: D(T) \subset X \to Y$ is defined on $T^*: D(T^*) \subset Y \to X$, where $D(T^*)$ consists of all $y \in Y$ with the property that there exists some $z \in X$ with $\langle Tx, y\rangle_X = \langle x, z\rangle_X$ for all $x \in D(T)$. (In this case, we define $T^*y = z$.) The dual of a densely defined operator $T: D(T) \subset X \to Y$ is defined on $T': D(T') \subset Y' \to X'$, where $D(T')$ consists of all $y \in Y$ with the property that the mapping $D(T) \to \mathbb{C}$, $z \mapsto \langle Tz, x\rangle_X$ has an extension to an element in X'. For $y \in D(T')$, the element $T'y$ is defined via $\langle T'y, x\rangle_{X',X} = \langle y, Tx\rangle_{Y',Y}$ for all $x \in D(T)$. Note that T^* and T' coincide if both X and Y are considered to be pivot spaces. For further details concerning duals and adjoints, we refer to [6, p. 49].

A densely defined operator $P: D(P) \subset X \to X$ is called *self-adjoint* if $P = P^*$. (This also includes that $D(P) = D(P^*)$.) A self-adjoint operator P is *nonnegative* if $\langle x, Px\rangle_X \geq 0$ for all $x \in D(P)$. The notions of negativity, positivity, and nonpositivity of an operator can be defined in straightforward manner. This induces a partial order on the set of self-adjoint operators: For two self-adjoint operators $P_1: D(P_1) \subset X \to X$, $P_2: D(P_2) \subset X \to X$ we say that $P_1 \geq P_2$ if $P_1 - P_2 \geq 0$. The *square root* of a nonnegative operator $P: D(P) \subset X \to X$ is denoted by $P^{1/2}$; its domain $D(P^{1/2})$ is the completion of $D(P)$ with the norm $\|x\|^2_{D(P^{1/2})} = \|x\|^2_X + \langle x, Px\rangle_X$ [17, p. 606].

Compact operators are known to admit a singular value decomposition

$$Tx = \sum_{i=1}^{\infty} \sigma_i \langle x, u_i\rangle_X \cdot v_i, \tag{2.1}$$

where the sequence of singular values $(\sigma_i)_i$ is monotonically decreasing and tends to zero, and $(u_i)_i$, $(v_i)_i$ are orthonormal systems in X and Y, respectively [38, Thm. VI.17].

Subsequently, we introduce special classes and norms of operators which were originally introduced in [45].

Definition 2.1. *Let X, Y be separable Hilbert spaces and let $p \in [1, \infty[$. Then $T \in \mathcal{K}(X)$ is called a* pth Schatten class operator *if the sequence consisting of its singular values fulfill $(\sigma_i)_i \in \ell_p$. In this case we write $T \in \mathcal{S}_p(X, Y)$. Provided with the norm $\|T\|_{\mathcal{S}_p(X,Y)} = \|(\sigma_i)_i\|_{\ell_p}$, the space $\mathcal{S}_p(X, Y)$ becomes a Banach space. Operators of first Schatten class are called* nuclear *and those of second Schatten class are called* Hilbert–Schmidt.

We abbreviate $\mathcal{S}_p(X) := \mathcal{S}_p(X, X)$. For more details on the Schatten class, we refer to [36, p. 126]. The *trace* of $T \in \mathcal{S}_1(X)$ is well-defined by the expression

$$\operatorname{tr}(T) = \sum_{i=1}^{\infty} \langle e_i, Te_i \rangle, \tag{2.2}$$

where (e_i) is an (arbitrary) orthonormal basis of X [38, p. 206]. For self-adjoint and nonnegative $P \in \mathcal{S}_1(X)$, the spectral theorem [38, Thm. VI.16] implies that $\|P\|_{\mathcal{S}_1(X)} = \operatorname{tr}(P)$. Moreover, for $T \in \mathcal{S}_2(X, Y)$ it holds that $T^*T \in \mathcal{S}_1(X)$, $TT^* \in \mathcal{S}_1(Y)$ with $\|T\|^2_{\mathcal{S}_2(X,Y)} = \|T^*\|^2_{\mathcal{S}_2(Y,X)} = \operatorname{tr}(T^*T) = \operatorname{tr}(TT^*)$.

3. Operator Lyapunov equations. We review basic facts about solvability of operator Lyapunov equations. Consider the following setup throughout this article: For Hilbert spaces U, X (which are assumed to be pivot spaces), let $A : D(A) \subset X \to X$ and $B \in \mathcal{B}(U, D(A^*)')$ be given. The operator Lyapunov equation is given by

$$2 \operatorname{Re} \langle Px, A^*x \rangle_X + \|B'x\|^2_U = 0 \quad \text{for all } x \in D(A^*) \tag{3.1}$$

and has to be solved for the self-adjoint operator $P \in \mathcal{B}(X)$. Indeed, (3.1) is equivalent to (1.1) in the case where A and B are real matrices. The property of B to possibly map to a larger space $D(A^*)' \supset X$ is motivated by partial differential equations with boundary control; see [14] and [54, Chap. 10].

To analyze solvability of operator Lyapunov equations we first need to introduce the concept of strongly continuous semigroups, stability, and admissibility.

Definition 3.1 (strongly continuous semigroups, generators, exponential stability, contractivity, similarity). *An operator-valued function $T(\cdot) : \mathbb{R}_{\geq 0} \to \mathcal{B}(X)$ is called a* strongly continuous semigroup *if $T(0) = I_X$, $T(t + s) = T(t) \cdot T(s)$ for all $t, s \in \mathbb{R}_{\geq 0}$, and*

$$\lim_{t \to 0, t > 0} T(t)x = x \quad \textit{for all } x \in X.$$

A strongly continuous semigroup is called bounded *if there exists some $M \in \mathbb{R}_{\geq 0}$ such that*

$$\|T(t)\|_{\mathcal{B}(X)} \leq M \quad \textit{for all } t \in \mathbb{R}_{\geq 0},$$

and exponentially stable *if there exists some $M \in \mathbb{R}_{\geq 0}$, $\omega \in \mathbb{R}_{>0}$, such that*

$$\|T(t)\|_{\mathcal{B}(X)} \leq M \cdot e^{-\omega t} \quad \textit{for all } t \in \mathbb{R}_{\geq 0}.$$

If there holds $\|T(t)\| \leq 1$ for all $t \in \mathbb{R}_{\geq 0}$, then $T(\cdot)$ is called contractive. *Two semigroups $T_1(\cdot) : \mathbb{R}_{\geq 0} \to \mathcal{B}(X_1)$ and $T_2(\cdot) : \mathbb{R}_{\geq 0} \to \mathcal{B}(X_2)$ are called* similar *if there exists some bijective $H \in \mathcal{B}(X_1, X_2)$, such that $T_1(t) = H^{-1}T_2(t)H$ for all $t \in \mathbb{R}_{\geq 0}$.*

The operator $A : D(A) \subset X \to X$ *defined by*

$$Ax = \lim_{t\to 0, t>0} \tfrac{1}{t}(T(t)x - x),$$
$$D(A) = \left\{ x \in X \;\middle|\; \lim_{t\to 0, t>0} \tfrac{1}{t}(T(t)x - x) \in X \right\}$$

is called the generator *of the semigroup* $T(\cdot)$.

The domains of A and its adjoint are known to be dense in X; A is a closed operator [54, Cor. 2.1.8, Prop. 2.3.1, Prop. 2.8.1].

DEFINITION 3.2 (admissible control operator, Gramian operator). *Let* U, X *be Hilbert spaces, let* $A : D(A) \subset X \to X$ *be the generator of a strongly continuous semigroup* $T(\cdot)$ *on* X, *and let* $B \in \mathcal{B}(U, D(A^*)')$. *Then we call* B *an* admissible control operator *for* $T(\cdot)$ *if for some (and then also any)* $t \in \mathbb{R}_{>0}$, *there holds*

$$\Phi_t u := \int_0^t T(\tau)Bu(\tau)d\tau \in X \quad \text{for all } u \in L_2(\mathbb{R}_{\geq 0}, U). \tag{3.2}$$

The control operator B *is called* infinite-time admissible *if the* reachability map *fulfills*

$$\Phi u := \int_0^\infty T(\tau)Bu(\tau)d\tau \in X \quad \text{for all } u \in L_2(\mathbb{R}_{\geq 0}, U). \tag{3.3}$$

If B *is infinite-time admissible for* $T(\cdot)$, *then*

$$P = \Phi\Phi^* \in \mathcal{B}(X)$$

is called the Gramian *of* (A, B).

Some comments on facts about the above definition are stated below.

Remark 3.3.

(a) Expression 3.2 has to be understood in the following way: As $T(\cdot)$ extends to a strongly continuous semigroup on $D(A^*)'$ [54, Prop. 2.10.4], the function $T(\cdot)Bu(\cdot)$ is $D(A^*)'$-valued and measurable. Admissibility means that the integral is even in the smaller space X.

(b) Admissibility implies $\Phi_t \in \mathcal{B}(L_2(\mathbb{R}_{\geq 0}, U), X)$ [25, p. 6], and infinite-time admissibility implies the boundedness of the reachability map, i.e., $\Phi \in \mathcal{B}(L_2(\mathbb{R}_{\geq 0}, U),$
$X)$ [25, p. 5].

(c) If B is admissible for an exponentially stable semigroup $T(\cdot)$, then B is infinite-time admissible [25, p. 6].

(d) Any $B \in \mathcal{B}(U, X)$ is admissible.

Note that fully dual statements hold true for observability Gramians [54, p. 134]; all results in this article can be formulated for that case in a straightforward manner.

Next, recall that the Gramian indeed solves the operator Lyapunov equation (3.1) see [25].

THEOREM 3.4 (see [25, Thm. 3.1]). *Let* U, X *be Hilbert spaces and* $A : D(A) \subset X \to X$ *be the generator of a strongly continuous semigroup* $T(\cdot)$ *on* X. *Then there holds the following:*

(a) *If* $B \in \mathcal{B}(U, D(A^*)')$ *is an infinite-time admissible control operator for* $T(\cdot)$, *then the Gramian* P *of* (A, B) *solves the operator Lyapunov equation* (3.1).

(b) *If, additionally, the adjoint of* $T(\cdot)$ *is a strongly stable semigroup (that is,* $\lim_{t\to\infty} T^*(t)x = 0$ *for all* $x \in X$*), then the Gramian* P *of* (A, B) *is the unique solution of the operator Lyapunov equation* (3.1). *Moreover, all other solutions* $Q \geq 0$ *of* (3.1) *fulfill* $Q \geq P$.

(c) *On the other hand, if* $B \in \mathcal{B}(U, D(A^*)')$ *and there exists some nonnegative self-adjoint* $Q \in \mathcal{B}(X)$ *that satisfies*

$$2\operatorname{Re}\langle Qx, A^*x\rangle_X + \|B'x\|_U^2 \le 0 \quad \textit{for all } x \in D(A^*), \tag{3.4}$$

then B *is an admissible control operator for* $T(\cdot)$*. Moreover, all solutions* $Q \ge 0$ *of* (3.4) *fulfill* $Q \ge P$*.*

Remark 3.5. Being aware of $2\operatorname{Re}\langle Px, A^*x\rangle_X = \langle Px, A^*x\rangle_X + \langle A^*x, Px\rangle_X$, the substitution of x in (3.1) first by $x + y$ and then by $x + iy$ implies that the operator Lyapunov equation is equivalent to

$$\langle Px, A^*y\rangle_X + \langle A^*x, Py\rangle_X + \langle B'x, B'y\rangle_U = 0 \quad \text{for all } x, y \in D(A^*). \tag{3.5}$$

We briefly focus on the case where the generator A is self-adjoint and negative and has a compact inverse, and the control operator fulfills $B \in \mathcal{B}(U, D((-A)^{\frac{1}{2}})')$. (The heat equation considered in section 6 is of this type.) The latter property is equivalent to

$$(-A)^{-\frac{1}{2}}B \in \mathcal{B}(U, X). \tag{3.6}$$

PROPOSITION 3.6. *Assume that* $A : D(A) \subset X \to X$ *is self-adjoint and negative and has compact resolvent. Let* $B : U \to D(A)'$ *be such that* (3.6) *holds true. Then the following hold true:*

(a) A *generates an exponentially stable and contractive semigroup* $T(\cdot)$ *on* X*.*

(b) B *is admissible for* $T(\cdot)$*.*

Moreover, the Gramian P *of* (A, B) *is nuclear if and only if* $(-A)^{-1/2}B : U \to X$ *is Hilbert–Schmidt. In this case, there holds*

$$\|P\|_{\mathcal{S}_1(X)} = \operatorname{tr}(P) = \tfrac{1}{2} \cdot \|(-A)^{-1/2}B\|_{\mathcal{S}_2(U,X)}^2 = -\tfrac{1}{2} \cdot \operatorname{tr}(B'A^{-1}B). \tag{3.7}$$

Proof. By the assumptions on A, we may apply the spectral theorem [39, Thm. XIII.64] to infer the existence of an orthonormal basis $(e_i)_i$ of X and a decreasing and unbounded sequence of negative real numbers $(\lambda_i)_i$, such that

$$Ax = \sum_{i=1}^{\infty} \langle x, e_i\rangle_X \cdot e_i \quad \text{for all } x \in D(A). \tag{3.8}$$

In particular, we have $A - \lambda_1 I \le 0$, and the Lumer–Phillips theorem [34, Thm. 4.3] implies that $\|T(t)\|_{\mathcal{B}(X)} \le e^{\lambda_1 t}$ for all $t \ge 0$. The semigroup $T(\cdot)$ is therefore exponentially stable and contractive.

To prove that (b) holds true, we make use of (3.6) to see that there exists some $\mu \in \mathbb{R}_{>0}$ such that for all $z \in X$ holds

$$\|B'(-A)^{-\frac{1}{2}}z\|_U^2 \le \mu\|z\|_X^2.$$

Since this inequality holds clearly for all $z \in D((-A)^{\frac{1}{2}})$, we can perform the substitution $x = (-A)^{-\frac{1}{2}}z$ to see that for all $x \in D(A) = D(A^*)$ holds

$$\|B'x\|_U^2 \le \mu \cdot \|(-A)^{\frac{1}{2}}x\|_X^2 = -\mu \cdot \langle x, Ax\rangle_X .$$

That is, (3.4) holds true for $Q = \frac{\mu}{2}I_X$, and we may apply Theorem 3.4 (c) to conclude admissibility of B.

Since the vectors e_i in the spectral decomposition (3.8) fulfill $e_i \in D(A)$, we may replace x in (3.1) by e_i to see

$$\begin{aligned} 0 &= 2\operatorname{Re}\langle Ae_i, Pe_i\rangle_X + \|B'e_i\|_U^2 \\ &= 2\operatorname{Re}\langle \lambda_i e_i, Pe_i\rangle_X + \|B'e_i\|_U^2 = 2\lambda_i \langle e_i, Pe_i\rangle_X + \|B'e_i\|_U^2. \end{aligned}$$

Solving this equation for $\langle e_i, Pe_i\rangle_X$ and using that $B'(-A)^{-1/2} = ((-A)^{-1/2}B)^*$, we obtain

$$\begin{aligned} \langle e_i, Pe_i\rangle_X &= -\tfrac{1}{2\lambda_i}\cdot \|B'e_i\|_U^2 = \tfrac{1}{2}\cdot \|B'(-\lambda_i)^{-1/2}e_i\|_U^2 \\ &= \tfrac{1}{2}\cdot \|B'(-A)^{1/2}e_i\|_U^2 = \tfrac{1}{2}\cdot \left\langle e_i, (-A)^{-1/2}B\left((-A)^{-1/2}B\right)^* e_i\right\rangle_X. \end{aligned}$$

Since both P and $(-A)^{-1/2}B((-A)^{-1/2}B)^*$ are nonnegative, their nuclear norms are equal to their respective traces. Moreover, since we have that $(-A)^{-1/2}B \in \mathcal{S}_2(U,X)$ if and only if $(-A)^{-1/2}B((-A)^{-1/2}B)^* \in \mathcal{S}_1(X)$, equivalence between $P \in \mathcal{S}_1(X)$ and $(-A)^{-1/2}B \in \mathcal{S}_2(U,X)$ follows immediately. In this case, we have

$$\begin{aligned} \operatorname{tr}(P) &= \tfrac{1}{2}\cdot \operatorname{tr}((-A)^{-1/2}BB'(-A)^{-1/2}) = \tfrac{1}{2}\cdot \operatorname{tr}((-A)^{-1/2}B((-A)^{-1/2}B)^*) \\ &= \tfrac{1}{2}\cdot \|(-A)^{-1/2}B\|_{\mathcal{S}_2(U,X)}^2 = -\tfrac{1}{2}\cdot \operatorname{tr}(B'A^{-1}B). \qquad \square \end{aligned}$$

Remark 3.7. Note that, under the assumptions that $A : D(A) \subset X \to X$ is negative and has compact resolvent, the input operator B fulfills (3.6) and, additionally, the input space is finite-dimensional (i.e., without loss of generality $U = \mathbb{C}^m$), we can immediately infer from Proposition 3.6 that the Gramian P of (A, B) is nuclear. Namely,the expression $\|P\|_{\mathcal{S}(X)} = \operatorname{tr}(P)$ coincides with the trace of the matrix $-\frac{1}{2}B'A^{-1}B \in \mathbb{C}^{m\times m}$.

Remark 3.8. It is worthwhile to place particular emphasis on systems with diagonalizable A, the so-called Riesz-spectral operators [17, sect. 2.3] (which occur for the heat and wave equation as well as at time-delay systems), that is, there exists a Riesz basis of eigenvectors of A and, moreover, $\sigma(A)$ is the closure of a totally disconnected set. Roughly speaking, a Riesz basis is a family of vectors which is topologically equivalent to an orthonormal system. A Riesz-spectral operator A generates a strongly continuous $T(\cdot)$ if and only if $\sup \operatorname{Re}(\sigma(A)) < \infty$; exponential stability of $T(\cdot)$ is equivalent to $\sup(\operatorname{Re}\sigma(A)) < 0$ [17, Thm. 2.3.5]. Denoting $\sigma(A) = \{\lambda_i \mid i \in \mathbb{N}\}$, let $(\phi_i)_i$ be a Riesz basis of the eigenvectors of A^* (which exists due to [17, Lem. 2.3.2]) with $A^*\phi_i = \overline{\lambda_i}\phi_i$ for all $i \in \mathbb{N}$. Then the Lyapunov equation (3.5) gives rise to

$$\begin{aligned} 0 &= \langle P\phi_i, A^*\phi_j\rangle_X + \langle A^*\phi_i, P\phi_j\rangle_X + \langle B'\phi_i, B'\phi_j\rangle_U \\ &= (\overline{\lambda_j} + \lambda_i)\langle P\phi_i, \phi_j\rangle_X + \langle B'\phi_i, B'\phi_j\rangle_U \quad \text{for all } i, j \in \mathbb{N}. \end{aligned} \tag{3.9}$$

Then $\lambda_i, \lambda_j \in \mathbb{C}_-$ implies $\lambda_i + \overline{\lambda_j} \neq 0$, and thus

$$\langle P\phi_i, \phi_j\rangle_X = -\frac{1}{\lambda_i + \overline{\lambda_j}}\cdot \langle B'\phi_i, B'\phi_j\rangle_U \quad \text{for all } i, j \in \mathbb{N}.$$

Note that, by the property of $(\phi_i)_i$ being a Riesz basis, this relation uniquely determines P; it can be considered as a Cauchy matrix representation [4] of P.

4. ADI iteration for operator Lyapunov equations. We now present an algorithm, take a closer look at operator Lyapunov equations, and set up an iterative scheme for their solution; we consider the ADI iteration for the operator case and discuss convergence. Before presenting results about convergence, we first present the algorithm, which exactly reads as in the matrix case [44, p. 43]. Besides the pair (A, B) defining a control system, this algorithm involves so-called shift parameters $p_i \in \mathbb{C}$, which have to be chosen a priori. In the finite-dimensional case, they are known to determine the velocity of convergence [44, p. 43]. Their choice in the case of operator Lyapunov equations is discussed at the end of this section.

Algorithm 1. ADI iteration for operator Lyapunov equations.

Input: The generator A of a bounded strongly continuous semigroup $T(\cdot)$, an infinite-time admissible control operator $B \in \mathcal{B}(U, D(A^*)')$, and shift parameters $p_1, \ldots, p_{i_{\max}} \in \mathbb{C}_-$

Output: $S = S_{i_{\max}} \in \mathcal{B}(U^{i_{\max}}, X)$, such that $SS^* \approx P$, where P is the Gramian of (A, B).

$V_1 = (A + p_1 I)^{-1} B$
$S_1 = \sqrt{-2\,\mathrm{Re}(p_1)} \cdot V_1$
for $i = 2, 3, \ldots, i_{\max}$ **do**
    $V_i = V_{i-1} - (p_i + \overline{p_{i-1}}) \cdot (A + p_i I)^{-1} V_{i-1}$
    $S_i = [\, S_{i-1}\,,\ \sqrt{-2\,\mathrm{Re}(p_i)} \cdot V_i \,]$
end for

Remark 4.1.

(a) The operator $A\colon D(A) \subset X \to X$ uniquely extends to an operator in $\mathcal{B}(X, D(A^*)')$ [54, Cor. 2.10.3]. An immediate consequence is that, for $p \in \rho(-A)$, the operator $(pI + A)^{-1}$ uniquely extends to an element of $\mathcal{B}(D(A^*)', X)$. In particular, there holds $(pI + A)^{-1} B \in \mathcal{B}(U, X)$ for all $p \in \rho(-A)$.

(b) It follows from the Hille–Yosida theorem (see, e.g., [50, Thm. 3.4.1)]) that boundedness of the semigroup implies $\mathbb{C}_- \subset \rho(-A)$.

(c) Since the shift parameters are assumed to have negative real part, the findings in (a) and (b) guarantee the feasibility of the ADI iteration.

(d) In the case where (A, B) is in addition approximately controllable (that is, $\overline{\operatorname{im} \Phi} = X$), [54, Thm. 5.1.1.] guarantees weak stability of the semigroup $T(\cdot)$. That is, $\lim_{t\to\infty} \langle x_1, T(t)x_2 \rangle_X = 0$ for all $x_1, x_2 \in X$.

(e) In the case of a finite-dimensional input space, i.e., $U = \mathbb{C}^m$, we have $S_i \in \mathcal{B}(\mathbb{C}^{m\cdot i}, X)$. This means that $P_i = S_i^* S_i$ has finite rank and, in the block operator notation, S_i consists of an $m \cdot i$-tuple of elements of the state space X. These elements are obtained by solving equations of type $(p_i I + A)w = z$. In practice, A is usually a differential operator, and each step of ADI iteration consists of a (numerical) solution of the corresponding differential equation (see sect. 6).

(f) We note that the choice of $i_{\max}$ has of course not to be done a priori. Rather one might use a suitable *stopping criterion*. Due to $P_i - P_{i-1} = V_i V_i^*$, we have for each $\mathcal{N} \in \{\mathcal{B}, \mathcal{S}_p\}$ that

$$\|P_i - P_{i-1}\|_{\mathcal{N}(X)} = -2\,\mathrm{Re}(p_i) \cdot \|V_i V_i^*\|_{\mathcal{N}(X)} = -2\,\mathrm{Re}(p_i) \cdot \|V_i^* V_i\|_{\mathcal{N}(U)}.$$

A suitable criterion for termination of the ADI iteration is therefore to check whether the norm of the operator $V_i^* V_i \in \mathcal{B}(U)$ (which is a matrix, if $U = \mathbb{C}^m$) goes below a given absolute or relative threshold. For an overview on stopping criteria for the ADI iteration to solve matrix Lyapunov equations, we refer to [44, sect. 4.6].

Note that, for the class treated in Proposition 3.6, we will derive an explicit expression for the approximation error $P-P_i$ in the nuclear norm (see Proposition 4.12).

(g) An important class of infinite-dimensional systems are the so-called boundary control systems [15, 37]. They are, in an abstract setting, of the form

$$\begin{aligned} \dot{x}(t) &= \mathfrak{U}x(t), \\ \mathfrak{p}x(t) &= u(t), \end{aligned} \tag{4.1}$$

where $\mathfrak{U} : D(\mathfrak{U}) \subset X \to X$ is densely defined and closed, and $\mathfrak{p} \in \mathcal{B}(D(\mathfrak{U}), U)$ is onto. In case of well-posedness (that is, for all $t \in \mathbb{R}_{>0}$, $x(t) \in X$ continuously depends on $x(0) \in X$ and $u \in L_2([0,t],U)$), this system can be rewritten as $\dot{x}(t) = Ax(t) + Bu(t)$, where the operator A generates a strongly continuous semigroup $T(\cdot)$ on X, and B is admissible for $T(\cdot)$; see [37]. If A generates a bounded semigroup, then we may use [15, Thm. 2.9] to infer that for all $p \in \overline{\mathbb{C}_-}$, $z \in X$, and $u \in U$, the vectors $x_1 = (A+pI)^{-1}z$ and $x_2 = (A+pI)^{-1}Bu \in X$ are the unique solutions of the so-called abstract elliptic problems

$$\begin{aligned} px_1 + \mathfrak{U}x_1 &= z, & \qquad px_2 + \mathfrak{U}x_2 &= 0, \\ \mathfrak{p}x_1 &= 0, & \mathfrak{p}x_2 &= u. \end{aligned} \tag{4.2}$$

The Gramian P can therefore be computed by the ADI algorithm without explicit use of A and B.

Our convergence result is stated below as Theorem 4.7. Its proof is based on the (seemingly new) insight that the operator S_i computed by ADI equals the restriction of the reachability map Φ to a certain subspace $\mathcal{U}_i \subset L_2(\mathbb{R}_{\geq 0}, U)$. More precisely, we will derive a formula $S_i = \Phi \cdot \iota_i$, where $\iota_i : U^i \to \mathcal{U}_i \subset L_2(\mathbb{R}_{\geq 0}, U)$ is an isometry. Convergence of ADI will then be deduced from $\overline{\bigcup_{i\in\mathbb{N}} \mathcal{U}_i} = L_2(\mathbb{R}_{\geq 0}, U)$.

To derive this finding, denote $q_1, \ldots, q_J$ to be the pairwise different shift parameters which have been involved in the first i ADI iterations. Further, let k_j be the number of iterations in which q_j appears (thus $i = k_1 + \cdots + k_J$), and define

$$\begin{aligned} \mathcal{F}_i &= \bigoplus_{j=1}^{i} \operatorname{span}\left\{ \, t^l e^{q_j t} \;\middle|\; l = 0, \ldots, k_j - 1 \, \right\} \subset L_2(\mathbb{R}_{\geq 0}), \\ \mathcal{U}_i &= \mathcal{F}_i \otimes U := \left\{ \, f(\cdot) \cdot v \;\middle|\; f \in \mathcal{F}_j, \ v \in U \, \right\} \subset L_2(\mathbb{R}_{\geq 0}, U). \end{aligned} \tag{4.3}$$

The spaces defined in (4.3) under the Laplace transform (which is an isometric mapping from $L_2(\mathbb{R}_{\geq 0}, U)$ to $\mathcal{H}_2(U)$) become

$$\begin{aligned} \widehat{\mathcal{F}}_i &= \bigoplus_{j=1}^{i} \operatorname{span}\left\{ \, \frac{1}{(s-q_j)^l} \;\middle|\; l = 1, \ldots, k_j \, \right\} \subset \mathcal{H}_2, \\ \widehat{\mathcal{U}}_i &= \widehat{\mathcal{F}}_i \otimes U = \left\{ \, \widehat{f}(\cdot) \cdot v \;\middle|\; \widehat{f} \in \widehat{\mathcal{F}}_j, \ v \in U \, \right\} \subset \mathcal{H}_2(U). \end{aligned} \tag{4.4}$$

The *Takenaka–Malmquist system* [12] $(\widehat{\psi}_i)_i$ for $\widehat{F}_j$ is defined through the iteration

$$\begin{aligned} \widehat{\phi}_1(s) &= \frac{1}{-s+p_1}, & \widehat{\psi}_1(s) &= \sqrt{-2\operatorname{Re}(p_1)} \cdot \widehat{\phi}_1(s), \\ \widehat{\phi}_i(s) &= \widehat{\phi}_{i-1}(s) - \frac{p_i + \overline{p_{i-1}}}{-s+p_i} \cdot \widehat{\phi}_{i-1}(s), & \widehat{\psi}_i(s) &= \sqrt{-2\operatorname{Re}(p_i)} \cdot \widehat{\phi}_i(s). \end{aligned} \tag{4.5}$$

Note the similarity to the ADI iteration. Elementary properties of the Laplace transform show that the corresponding time-domain iteration is

$$\phi_1(t) = -\mathrm{e}^{p_1 t}, \qquad \psi_1(t) = \sqrt{-2\,\mathrm{Re}(p_1)}\cdot \phi_1(t),$$

$$\phi_i(t) = \phi_{i-1}(t) + (p_i + \overline{p_{i-1}})\cdot(\mathrm{e}^{p_i \cdot} * \phi_{i-1})(t), \qquad \psi_i(t) = \sqrt{-2\,\mathrm{Re}(p_i)}\cdot \phi_i(t), \tag{4.6}$$

where $*$ denotes convolution, that is, $(g * h)(t) = \int_0^t g(t-\tau)h(\tau)\,d\tau$. We will call the family $(\psi_i)_i$ as defined in (4.6) the *time-domain Takenaka–Malmquist system*. By using that the operator S_i obtained in the ADI iteration can be interpreted as a restriction of the reachability map to the space $\mathcal{U}_i = \mathrm{span}\{\psi_1, \dots, \psi_i\} \otimes U$, convergence of ADI will be related to the question whether the time-domain Takenaka–Malmquist system corresponding to the infinite sequence of shift parameters $(p_i)_i$ forms an orthonormal basis of $L_2(\mathbb{R}_{\geq 0})$. We will use classical results from complex analysis to deduce that this is true if and only if

$$\sum_{i=1}^{\infty} \frac{|\,\mathrm{Re}(p_i)|}{1+|p_i|^2} = \infty. \tag{4.7}$$

Remark 4.2. Note that analogous results regarding the Takenaka-Malmquist system have been mentioned for the discrete-time case (that is, one has to consider the complex unit disc instead of a half plane) in passing in [53, p. 309]. Density for the frequency domain version is known in more general spaces [12, Thm. 7.1.4].

Instead of translating the results mentioned in Remark 4.2 to our context, in the lemma below we show how, by elementary arguments using Laplace transforms, the property $\overline{\mathrm{span}\{\ \psi_i \ \mid i \in \mathbb{N}\ \}} = L_2(\mathbb{R}_{\geq 0})$ can be related to a standard result on the zero distribution of certain holomorphic functions.

Remark 4.3. The condition opposite to (4.7), namely, $\sum_{i=1}^{\infty} \frac{|\,\mathrm{Re}(p_i)|}{1+|p_i|^2} < \infty$, is called the *Blaschke condition* [20, sect. II.2] (since it guarantees the convergence of an infinite Blaschke product). It therefore seems reasonable to call condition (4.7) the *non-Blaschke condition*.

LEMMA 4.4. *The time-domain Takenaka–Malmquist system $(\psi_i)_i$ with parameters $(p_i)_i$ forms an orthonormal basis for $L_2(\mathbb{R}_{\geq 0})$ if and only if* (4.7) *is satisfied.*

Proof. It is shown in Appendix B that the Takenaka–Malmquist system $(\widehat{\psi}_i)_i$ is orthonormal in $\mathcal{H}_2$, whence, by the Paley–Wiener Theorem [50, Thm. 10.3.4], the time-domain system $(\psi_i)_i$ is orthonormal in $L_2(\mathbb{R}_{\geq 0})$. So it is an orthonormal basis if and only if its orthogonal complement is trivial.

Assume that $f \in L_2(\mathbb{R}_{\geq 0})$ is orthogonal to $\mathrm{e}^{p_1 t}$. By definition of Laplace transform, this is equivalent to $\widehat{f}(-p_1) = 0$. Differentiation under the integral sign implies that orthogonality of f to $t\mathrm{e}^{p_1 t}$ is equivalent to $\frac{d}{ds}\widehat{f}(s)|_{s=-p_1} = 0$. Continuing with this argumentation, we see that f is orthogonal to all the Takenaka–Malmquist functions if and only if $\widehat{f} \in \mathcal{H}_2$ has zeros in p_i (counting multiplicity).

From the above it is clear that our question is equivalent to a question on the zero distribution of $\mathcal{H}_2$ functions (and, in particular, what distribution of the zeros implies that the function is identically equaly to zero). To answer this equation, we utilize a classical result from complex analysis which generalizes the well-known fact that the zeros of a nonzero holomorphic function must be isolated and have finite order. Since the case of the disc is better studied, we translate to the disc. Let H_2 denote the Hardy space of the unit disc. Let $\mu \in \mathbb{C}_+$ and define

$$(MF)(z) := \frac{\sqrt{2\mu}}{1+z}F\left(\mu\frac{1-z}{1+z}\right), \quad (M^{-1}G)(s) := \frac{2\mu}{\mu+s}G\left(\frac{\mu-s}{\mu+s}\right),$$

and it is easily checked that these are indeed inverses and that $M : \mathcal{H}_2 \to H_2$, $M^{-1} : H_2 \to \mathcal{H}_2$ are unitary (see, e.g., [26, pp. 128–131] for the case $\mu = 1$). Define $z_k := \frac{\mu+p_k}{\mu-p_k}$. Then the non-Blaschke condition (4.7) is equivalent to the non-Blaschke condition for the disc (see, for example, [26, p. 132] or [20, p. 53]; the latter considers the upper half-plane rather than the right half-plane), which is

$$\sum_{k=1}^{\infty} 1 - |z_n| = \infty.$$

A classical result from complex analysis states that $G \in H^2$ is nonzero if and only if its zeros z_n satisfy the Blaschke condition $\sum_{k=1}^{\infty} 1-|z_n| < \infty$. See, for example, [3, Thms. 5.2.2 and 5.2.3], [18, Thms. 2.3 and 2.4], [20, Thms. II.2.1 and II.2.2], [42, Thms. 15.21 and 15.23]; it is important to note in interpreting these references that $H_\infty \subset H_2 \subset \mathcal{N}$, where $\mathcal{N}$ is the Nevalinna class; this result can also be found in the above references.

Combining the above, we have that f is orthogonal to all the Takenaka–Malmquist functions and $\widehat{f}$ has zeros in $-p_k$ (counting multiplicity) if and only if $G := M\widehat{f}$ has zeros in z_k (counting multiplicity). Since the non-Blaschke condition (4.7) is equivalent to $\sum_{k=1}^{\infty} 1 - |z_n| = \infty$, we see that f is orthogonal to all the Takenaka–Malmquist functions if and only $G = 0$, i.e., if and only if $f = 0$. ∎

Remark 4.5. The non-Blaschke condition is satisfied if all shift parameters are equal. (In this case the Takenaka–Malmquist system becomes a Laguerre basis.) More generally, the non-Blaschke condition is satisfied if for some $J \in \mathbb{N}$ the sequence $(p_i)_i$ is J-cyclic (that is, $p_{J+i} = p_i$ for all $i \in \mathbb{N}$).

An example of an unbounded sequence where the non-Blaschke condition is satisfied is $p_k = -\sqrt{k}$, and two examples where the non-Blaschke condition is not satisfied are given by $p_k = -k^2$ and $p_k = -1/k^2$.

LEMMA 4.6. *Let U, X be Hilbert spaces, let $A : D(A) \subset X \to X$ be the generator of a bounded semigroup $T(\cdot)$, let $B \in \mathcal{B}(U, D(A^*)')$ be an infinite-time admissible control operator for $T(\cdot)$, let $\Phi \in \mathcal{B}(L_2(\mathbb{R}_{\geq 0}, U), X)$ be the input map of (A, B), and let $(p_i)_i$ be a sequence in $\mathbb{C}_-$. Then Algorithm 1 is feasible. Further, for the time-domain Takenaka–Malmquist basis $(\psi_i)_i$ as in (4.6), the mapping*

$$\begin{aligned} \iota_i : \qquad & U^i \to \mathcal{U}_i \subset L_2(\mathbb{R}_{\geq 0}, U), \\ & (v_1, \dots, v_i) \mapsto \sum_{j=1}^{i} \psi_j \cdot v_j \end{aligned} \tag{4.8}$$

is an isometry, and there holds $S_i = \Phi \cdot \iota_i$.

Proof. Since the semigroup is bounded, there holds $\mathbb{C}_+ \subset \rho(A)$ by the Hille–Yosida theorem (see, e.g., [50, Thm. 3.4.1)]). We particularly have $-p_i \in \rho(A)$, which guarantees the feasibility of the ADI iteration.

The isometry property of ι_i follows from $(\psi_i)_i$ being an orthonormal basis. To show that $S_i = \Phi \cdot \iota_i$ holds true, we will make use of two formulas concerning the reachability map: Assuming that $u \in U$, it follows from [50, Thm. 4.2.1(iii)] that for $\mu \in \mathbb{C}_-$ holds

$$\Phi(\mathrm{e}^{\mu \cdot} v) = -(A + \mu I)^{-1} B v. \tag{4.9}$$

The product rule together with $\frac{d}{d\tau} T(\tau)x = AT(\tau)x$ for all $x \in X$ yields

$$\frac{d}{d\tau} e^{\mu\tau} (A + \mu I)^{-1} T(\tau) B = e^{\mu\tau} T(\tau) B \text{ for all } \mu \in \mathbb{C}_-.$$

Using this formula, we obtain by integration by parts that

$$
\begin{aligned}
\Phi((\mathrm{e}^{\mu\cdot} * f)v) &= \int_0^\infty T(\tau)B\int_0^\tau e^{\mu(\tau-\sigma)}f(\sigma)d\sigma\cdot v\,d\tau \\
&= \int_0^\infty e^{\mu\tau}T(\tau)B\int_0^\tau e^{-\mu\sigma}f(\sigma)d\sigma\cdot v\,d\tau \\
&= e^{\mu\tau}(\mu I+A)^{-1}T(\tau)B\int_0^\tau e^{-\mu\sigma}f(\sigma)d\sigma\cdot v\Big|_{\tau=0}^{\tau=\infty} \\
&\quad -\int_0^\infty e^{\mu\tau}(\mu I+A)^{-1}T(\tau)Be^{-\mu\tau}f(\tau)\cdot v\,d\tau \\
&= -(\mu I+A)^{-1}\int_0^\infty T(\tau)Bf(\tau)\cdot v\,d\tau \\
&= -(\mu I+A)^{-1}\Phi(fv).
\end{aligned}
\tag{4.10}
$$

Applying the reachability map to both sides of the time-domain Takenaka–Malmquist iteration equations (4.6) gives for $v\in U$

$$
\begin{aligned}
&\Phi(\phi_1 v) = -\Phi(\mathrm{e}^{p_1\cdot}v), && \Phi(\psi_1 v)=\sqrt{-2\,\mathrm{Re}(p_1)}\cdot\Phi(\phi_1 v),\\
&\Phi(\phi_i v) = \Phi(\phi_{i-1}v)+(p_i+\overline{p_{i-1}})\cdot\Phi((\mathrm{e}^{p_i\cdot}*\phi_{i-1})v), && \Phi(\psi_i v)=\sqrt{-2\,\mathrm{Re}(p_i)}\cdot\Phi(\phi_i v).
\end{aligned}
$$

Using (4.9) and (4.10), we obtain from this

$$
\begin{aligned}
&\Phi(\phi_1 v) = (A+p_1I)^{-1}Bv, && \Phi(\psi_1 v)=\sqrt{-2\,\mathrm{Re}(p_1)}\cdot\Phi(\phi_1 v),\\
&\Phi(\phi_i v) = \Phi(\phi_{i-1}v)-(p_i+\overline{p_{i-1}})(A+p_iI)^{-1}\Phi(\phi_{i-1}v), && \Phi(\psi_i v)=\sqrt{-2\,\mathrm{Re}(p_i)}\cdot\Phi(\phi_i v).
\end{aligned}
$$

Since this is exactly the ADI iteration from Algorithm 1, we conclude that for $j=1,\dots,i$ and $v_j\in U$ holds $\sqrt{-2\,\mathrm{Re}(p_j)}\cdot V_jv_j=\Phi(\psi_jv_j)$, whence

$$
\begin{aligned}
S_i\cdot(v_1,\dots,v_i) &= \sum_{j=1}^{i}\sqrt{-2\,\mathrm{Re}(p_i)}\cdot V_jv_j=\sum_{j=1}^{i}\Phi(\psi_iv)\\
&=\Phi\left(\sum_{j=1}^{i}\psi_iv\right)=\Phi\,\iota_i(v_1,\dots,v_i). \qquad \square
\end{aligned}
$$

We are now prepared to formulate the main result of this paper. We show that ADI iteration will be convergent to the Gramian if (A,B) is infinite-time admissible (i.e., the Gramian $P\in\mathcal{B}(X)$ exists), the semigroup generated by A is bounded, and the sequence of shift parameters fulfills the non-Blaschke condition.

THEOREM 4.7. *Let U,X be Hilbert spaces, let $A:D(A)\subset X\to X$ be the generator of a bounded strongly continuous semigroup $T(\cdot)$, let $B\in\mathcal{B}(U,D(A^*)')$ be an infinite-time admissible control operator for $T(\cdot)$, and let $P\in\mathcal{B}(X)$ be the Gramian of (A,B). Assume that the sequence of shifts $(p_i)_i$ in $\mathbb{C}_-$ satisfies the non-Blaschke condition*

$$
\sum_{i=1}^{\infty}\frac{|\,\mathrm{Re}(p_i)|}{1+|p_i|^2}=\infty.
$$

Then Algorithm 1 is feasible and the operator sequence $(P_i)_i=(S_iS_i^*)_i$ is monotone and is strongly convergent to P, i.e.,

$$
\lim_{i\to\infty}P_ix=Px\quad\text{for all } x\in X.
$$

Moreover, the following holds true:

(a) *If the Gramian P is compact, then*

$$\lim_{i\to\infty} \|P - P_i\|_{\mathcal{B}(X)} = 0.$$

(b) *If, for some $p \in [1,\infty)$, the Gramian P is of pth Schatten class, then*

$$\lim_{i\to\infty} \|P - P_i\|_{\mathcal{S}_p(X)} = 0.$$

Proof. That Algorithm 1 is feasible was already shown in Lemma 4.6. Further, using Lemma 4.6, we obtain

$$P_i = \Phi \iota_i \iota_i^* \Phi^*$$

with $\iota_i \in \mathcal{B}(U^i, L_2(\mathbb{R}_{\geq 0}, U))$ defined as in (4.8). The construction of ι_i, in particular its isometry property, implies that $\iota_i \iota_i^* \in \mathcal{B}(L_2(\mathbb{R}_{\geq 0}, U))$ is an orthogonal projection onto

$$\mathcal{U}_i = \{\ f \cdot v \ \mid f \in \operatorname{span}\{\psi_1, \ldots, \psi_i\},\, v \in U\ \},$$

where $\psi_1, \ldots, \psi_i$ are the first i functions in the time-domain Takenaka–Malmquist basis. Since, by Lemma 4.4, $(\psi_i)_i$ is an orthonormal basis of $L_2(\mathbb{R}_{\geq 0})$ provided that the non-Blaschke condition is fulfilled, it follows that the operator sequence $(\iota_i \iota_i^*)_i$ in $\mathcal{B}(L_2(\mathbb{R}_{\geq 0}, U))$ is monotone and strongly convergent to the identity operator on $L_2(\mathbb{R}_{\geq 0}, U)$. Hence, the sequence $(P_i) = (\Phi \iota_i \iota_i^* \Phi^*)$ is monotone and strongly convergent to the Gramian $P = \Phi\Phi^*$.

If we now assume that P is moreover pth Schatten class (compact), then Corollary A.3 will imply that even convergence towards P in the pth Schatten norm (operator norm) holds true. ∎

Remark 4.8.

(a) Since the operator sequence $(P_i)_i$ is monotone and bounded from above, it always converges strongly (Theorem A.1). However, it may converge to something other than the Gramian if the non-Blaschke condition (4.7) is not satisfied.

(b) Though the non-Blaschke condition (4.7) is a necessary and sufficient condition for the Takenaka–Malmquist system to form an orthonormal basis for $L_2(\mathbb{R}_{\geq 0})$, it is only sufficient for the convergence of ADI to the Gramian. For a trivial counterexample to necessity take $B = 0$, in which case $P = 0$ and ADI converges to P no matter what the shift parameters are. In the single input case $U = \mathbb{C}$, a necessary and sufficient condition for ADI to converge to the Gramian is that the Takenaka–Malmquist system forms an orthonormal basis for the orthogonal complement of the kernel of the input map. (Something similar is true in the general case, but this becomes difficult to formulate concisely.) Using that the orthogonal complement of the kernel of Φ equals the closure of the range of Φ^*, it follows for the single input case that if we choose the shifts such that the closure of the range of Φ^* is contained in $\overline{\bigcup_{i\in\mathbb{N}} \mathcal{U}_i}$, then ADI converges to the Gramian. An interesting case in which this happens is where A is a Riesz-spectral operator (see Remark 3.8) and the sequence of shifts equals the sequence of eigenvalues of A^*. Then ADI converges to the Gramian even if this sequence of shifts does not satisfy the non-Blaschke condition (4.7).

(c) Note that by Lemma 4.6 the ADI approximation has the same structure as the approximation obtained by truncating the singular value decomposition (which is an optimal approximation). The only difference is that the space which the input map is restricted to in the latter case consists of the eigenvectors of $\Phi^*\Phi$ corresponding to the largest eigenvalues. Since in the case where A is a Riesz-spectral operator (see Remark 3.8) the eigenvectors of $\Phi^*\Phi$ are (polynomials times) sums of exponentials where the exponents are eigenvalues of A^*, this gives another justification for the well-known fact that the shifts should somehow approximate the spectrum of A.

(d) With the interpretation of Lemma 4.6, column compression [44, sect. 4.4.1] in ADI means restricting the input map to a space spanned by (polynomials times) sums of exponentials rather than a space spanned by (polynomials times) exponentials. Given the above mentioned connection with the singular value decomposition, column compression indeed seems a logical thing to do.

Convergence to the Gramian follows from Theorem 4.7 for a wide range of shift choices (namely, those that satisfy the non-Blaschke condition (4.7)). In the matrix case some choices of shift parameters are know to give rise to fast convergence. Proposition 4.10 generalizes an equality that is important for this analysis of shift choices to the operator case. The following lemma is needed in the proof of Proposition 4.10.

LEMMA 4.9. *Let U, X be Hilbert spaces, let $A : D(A) \subset X \to X$ be the generator of a bounded semigroup $T(\cdot)$, let $B \in \mathcal{B}(U, D(A^*)')$ be an infinite-time admissible control operator for $T(\cdot)$, and let $P \in \mathcal{B}(X)$ be the Gramian of (A, B). Then for all $p \in \mathbb{C}_-$, there holds*

$$
\begin{aligned}
&(pI + A)^{-1}P + P(\overline{p}I + A^*)^{-1} \\
&-2\,\mathrm{Re}(p)(pI + A)^{-1}P(\overline{p}I + A^*)^{-1} + (pI + A)^{-1}BB'(\overline{p}I + A^*)^{-1} = 0.
\end{aligned}
\tag{4.11}
$$

Proof. The bounded invertibility of $pI + A$ follows from the boundedness of the semigroup $T(\cdot)$ together with the Hille–Yosida theorem [50, Thm. 3.4.1)]. Moreover, for all $z \in X$ holds

$$
A^*(\overline{p}I + A^*)^{-1}z = (\overline{p}I + A^* - \overline{p}I)(\overline{p}I + A^*)^{-1}z = z - \overline{p} \cdot (\overline{p}I + A^*)^{-1}z. \tag{4.12}
$$

Now let $z_1, z_2 \in X$ be arbitrary. Using that $(\overline{p}I + A^*)^{-1}z_1, (\overline{p}I + A^*)^{-1}z_2 \in D(A^*)$, we obtain from the Lyapunov equation (3.5) that

$$
\begin{aligned}
0 &= \left\langle P(\overline{p}I + A^*)^{-1}z_1, A^*(\overline{p}I + A^*)^{-1}z_2\right\rangle_X + \left\langle A^*(\overline{p}I + A^*)^{-1}z_1, P(\overline{p}I + A^*)^{-1}z_2\right\rangle_X \\
&\quad + \left\langle B'(\overline{p}I + A^*)^{-1}z_1, B'(\overline{p}I + A^*)^{-1}z_2\right\rangle_U \\
&= \left\langle P(\overline{p}I + A^*)^{-1}z_1, z_2\right\rangle_X - \left\langle P(\overline{p}I + A^*)^{-1}z_1, \overline{p}(\overline{p}I + A^*)^{-1}z_2\right\rangle_X \\
&\quad + \left\langle z_1, P(\overline{p}I + A^*)^{-1}z_2\right\rangle_X + \left\langle \overline{p}(\overline{p}I + A^*)^{-1}z_1, P(\overline{p}I + A^*)^{-1}z_2\right\rangle_X \\
&\quad + \left\langle B'(\overline{p}I + A^*)^{-1}z_1, B'(\overline{p}I + A^*)^{-1}z_2\right\rangle_U \\
&= \left\langle z_1, (pI + A)^{-1}Pz_2\right\rangle_X - \left\langle z_1, \overline{p}(pI + A)^{-1}P(\overline{p}I + A^*)^{-1}z_2\right\rangle_X \\
&\quad + \left\langle z_1, P(\overline{p}I + A^*)^{-1}z_2\right\rangle_X - \left\langle z_1, p(pI + A)^{-1}P(\overline{p}I + A^*)^{-1}z_2\right\rangle_X \\
&\quad + \left\langle z_1, (pI + A)^{-1}BB'(\overline{p}I + A^*)^{-1}z_2\right\rangle_X .
\end{aligned}
$$

The linearity of the inner product in the second argument now gives rise to the desired result. ∎

PROPOSITION 4.10. *Let U, X be Hilbert spaces, A be the generator of a bounded semigroup $T(\cdot)$ on X, and $B \in \mathcal{B}(U, D(A^*)')$ be an infinite-time admissible control* *operator for* $T(\cdot)$*. Let* $P \in \mathcal{B}(X)$ *be the Gramian of* (A, B)*, let* $p_1, \dots, p_i \in \mathbb{C}_-$*, and let*

$$T_i := \prod_{j=1}^{i} (A - \overline{p_j} I)(A + p_j I)^{-1}. \tag{4.13}$$

Then the operator $P_i \in \mathcal{B}(X)$ *recursively defined as in Algorithm* 1 *fulfills*

$$P - P_i = T_i P T_i^*. \tag{4.14}$$

Proof. The statement is proved by induction on i. For $i = 0$, the assertion is fulfilled due to $T_0 = I_X$ and $P_0 = 0$. Now let $i \in \mathbb{N}$ and assume that (4.14) is fulfilled for $i - 1$. By an argument as in (4.12), we see that the operators $(A - \overline{p_1} I)(A + p_1 I)^{-1}, \dots, (A - \overline{p_i} I)(A + p_i I)^{-1}$ commute, and furthermore,

$$T_i = (A - \overline{p_i} I)(A + p_i I)^{-1} T_{i-1} = T_{i-1} - 2\,\mathrm{Re}(p_i)(A + p_i I)^{-1} T_{i-1}.$$

In the notation of Algorithm 1, there holds $V_i = (A + p_i I)^{-1} T_{i-1} B$. Now making use of the induction assumption, we can infer from Lemma 4.9 and the construction of P_i that

$$\begin{aligned} T_i P T_i^* = {} & T_{i-1} P T_{i-1}^* - 2\,\mathrm{Re}(p_i) \left(T_{i-1}(A + p_i I)^{-1} P T_{i-1}^* + T_{i-1} P (A^* + \overline{p_i} I)^{-1} T_{i-1}^* \right) \\ & + 4\,\mathrm{Re}(p_i)^2 T_{i-1} (A + p_i I)^{-1} P (A^* + \overline{p_i} I)^{-1} T_{i-1}^* \\ = {} & P - P_{i-1} + 2\,\mathrm{Re}(p_i)(A + p_i I)^{-1} T_{i-1} B B' T_{i-1}^* (A^* + \overline{p_i} I)^{-1} \\ = {} & P - S_{i-1} S_{i-1}^* - (\sqrt{2\,\mathrm{Re}(-p_i)} \cdot V_i)(\sqrt{2\,\mathrm{Re}(-p_i)} \cdot V_i)^* \\ = {} & P - S_i S_i^* = P - P_i. \qquad \square \end{aligned}$$

Subsequently, we present some remarks about the shift parameter choice.

Remark 4.11 (shift parameters).

(a) Equation (4.14) gives rise to the fact that the approximation $P_i \approx P$ is better the smaller T_i is. In practice, one is interested in fast convergence, since this gives rise to good approximations of low rank.

(b) In the finite-dimensional case, the shift parameters are chosen in a way that the spectral radius of T_J is minimized. With $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$, this leads to the optimization problem

$$\min_{p_1, \dots, p_J \in \mathbb{C}_-} \max_{\lambda \in \sigma(A)} \prod_{j=1}^{J} \left| \frac{\overline{p_j} - \lambda}{p_j + \lambda} \right|.$$

This optimization problem can be solved by using advanced techniques of complex analysis, in particular the theory of elliptic integrals [23]. Furthermore, several suboptimal choices of the shift parameters that do not require the full information of the spectrum have been proposed and successfully applied (see [44, p. 43] for an overview).

(c) In infinite dimensions and, in particular, in the case where A is unbounded, any choice of the shift parameters in $\mathbb{C}_-$ will lead to an iteration operator fulfilling $\rho(T_J) = 1$. In the numerical experiments, we will choose the shift parameters by applying the existing approaches to a sufficiently accurate discretization of A.

(d) We now consider the case of a Riesz-spectral operator as in Remark 3.8. With $\sigma(A) = \{\lambda_i \,|\, i \in \mathbb{N}\}$, let $(\psi_i)_i$ be a Riesz basis of eigenvectors of A. Assuming that, for $s \in \mathbb{N}$, $\Pi_{[s]} \in \mathcal{B}(X)$ is a projector onto the s-dimensional space $\operatorname{span}\{\psi_1, \ldots, \psi_s\}$ and along the complementary space $\operatorname{span}\{\psi_{s+1}, \psi_{s+2}, \ldots\}$, we obtain

$$\begin{aligned}(A + p_j)^{-1}\Pi_{[s]}x &= \Pi_{[s]}(A + p_j)^{-1}\Pi_{[s]}x \\ &= \Pi_{[s]}(A + p_j)^{-1}x \quad \text{for all } x \in X \text{ and } j \in \mathbb{N}.\end{aligned}$$

Thus $T_i\Pi_{[s]}x = \Pi_{[s]}T_i\Pi_{[s]}x = \Pi_{[s]}T_ix$ also holds true for all $x \in X$, $i \in \mathbb{N}$. Then (4.14) implies

$$\Pi_{[s]}(P - P_i)(\Pi_{[s]})^* = (\Pi_{[s]}T_i\Pi_{[s]})(\Pi_{[s]}P\Pi_{[s]}^*)(\Pi_{[s]}T_i\Pi_{[s]})^*. \tag{4.15}$$

On the other hand, the spectral radius of the projected iteration matrix is given by

$$\rho(\Pi_{[s]}T_i\Pi_{[s]}) = \max_{l=1,\ldots,s} \prod_{j=1}^{i} \left|\frac{\overline{p_j} - \lambda_l}{p_j + \lambda_l}\right| < 1.$$

As a consequence, we have linear convergence of $(\Pi_{[s]}P_i\Pi_{[s]}^*)_i$ to $\Pi_{[s]}P\Pi_{[s]}^*$ in the operator norm for any $s \in \mathbb{N}$. If the Gramian P of (A, B) is compact (or even of Schatten class), then, by using

$$\Pi_{[s]}P\Pi_{[s]}^* - P = \Pi_{[s]}(P\Pi_{[s]}^* - P) + (\Pi_{[s]}P - P),$$

Theorem A.2 implies that there exists some $s \in \mathbb{N}$ such that $\|P - \Pi_{[s]}P_i\Pi_{[s]}^*\|_{\mathcal{B}(X)}$ ($\|P - \Pi_{[s]}P_i\Pi_{[s]}^*\|_{\mathcal{S}_p(X)}$) is arbitrarily small. One has therefore to find shift parameters that guarantee fast convergence on the "dominant subspace of P." Anyway, various open questions in the (optimal) shift parameter selection are left; this is an interesting topic for further research.

We finally give an explicit representation of the ADI approximation error for the class of systems considered in Proposition 3.6. On the basis of this result, suitable stopping criteria, i.e., the determination of $i_{\max}$ in Algorithm 1, may be designed (see also Remark 4.1(f)).

Proposition 4.12. *Let $A : D(A) \subset X \to X$ be self-adjoint with $A \leq 0$ and $0 \in \rho(A)$. Further assume that A has compact resolvent and let $B : U \to D((-A)^{\frac{1}{2}})'$ such that $(-A)^{-1/2}B : U \to X$ is Hilbert–Schmidt. Let P be the Gramian of (A, B) and let shift parameters $p_1, \ldots, p_i \in \mathbb{C}_-$ be given. Then, in the notation of Algorithm 1, there holds*

$$\|P - P_i\|_{\mathcal{S}_1(X)} = -\frac{1}{2} \cdot \operatorname{tr}(B'A^{-1}B) + 2\sum_{k=1}^{i} \operatorname{Re}(p_i) \cdot \operatorname{tr}(V_k^*V_k). \tag{4.16}$$

In particular, if $U = \mathbb{C}^n$, then $B'A^{-1}B, V_k^*V_k \in \mathbb{C}^{m\times m}$ are Hermitian matrices.

Proof. By Lemma 4.10, we have $P - P_i \geq 0$, whence

$$\|P - P_i\|_{\mathcal{S}_1(X)} = \operatorname{tr}(P) - \operatorname{tr}(P_i).$$

Then the desired result follows from (3.7) and

$$\begin{aligned}\operatorname{tr}(P_i) &= \operatorname{tr}\left(\sum_{k=1}^{i} -2\operatorname{Re}(p_k)\cdot V_kV_k^*\right) = -2\sum_{k=1}^{i}\operatorname{Re}(p_k)\cdot\operatorname{tr}\left(V_kV_k^*\right)\\ &= -2\sum_{k=1}^{i}\operatorname{Re}(p_k)\cdot\operatorname{tr}\left(V_k^*V_k\right). \qquad \square\end{aligned}$$

5. Inexact ADI iteration. Algorithm 1 can, in general, not be implemented for practical purposes if the state space X is infinite-dimensional. Instead, one will have to work with suitable approximations of the equations arising in each step of the ADI iteration. That is, we approximate $(A + p_iI)^{-1}$ by some operator acting on a finite-dimensional subspace. This is in what follows referred to as *inexact ADI iteration*. For instance, if A is a differential operator, an approximation can be performed by using (adaptive) finite-element methods (see section 6).

It is the goal of this section to present an error analysis for inexact ADI: we will derive estimates for the error in the obtained approximation of the Gramian operator for the case where the semigroup is similar to a contractive one.

Remark 5.1. The assumption that the semigroup is similar to a contractive one is not very restrictive. Clearly a necessary condition is that the semigroup is bounded. As indicated below, in the finite-dimensional case and in the Riesz spectral case, this is also sufficient. There are, however, examples of bounded semigroups that are not similar to contractive ones [16, 33].

(a) If the state space X is finite-dimensional, then every bounded semigroup is similar to a contractive semigroup [28]. (In the notation of that reference the similarity transformation is $H^{1/2}$.)

(b) It follows immediately from the definition of Riesz bases in [17, Def. 2.3.1] that Riesz-spectral operators are diagonalizable by bounded and boundedly invertible transformations (see [17, p. 38]). That is, there exists some bijective $H \in \mathcal{B}(\ell_2, X)$ such that $H^{-1}AH = D_A$, where $D_A : D(D_A) \subset \ell_2 \to \ell_2$ fulfills $D_A(x_1, x_2, \ldots) = (\lambda_1x_1, \lambda_2x_2, \ldots)$ with

$$D(D_A) = H^{-1}D(A) = \{(x_1, x_2, \ldots) \in \ell_2 \,|\, (\lambda_1x_1, \lambda_2x_2, \ldots) \in \ell_2\}.$$

The Lumer–Phillips theorem [34, Thm. 4.3] implies that the semigroup generated by D_A is contractive if $\sup_{i\in\mathbb{N}} \operatorname{Re}(\lambda_i) \leq 0$. Hence, a Riesz-spectral operator A with spectrum contained in $\overline{\mathbb{C}_-}$ generates a semigroup which is similar to a contractive one.

(c) If A is nonpositive and has compact resolvent, then it is a Riesz-spectral operator generating a contractive semigroup. This follows from the spectral decomposition (3.8).

In this part we assume finite-dimensionality of the input space, i.e., $U = \mathbb{C}^m$ for some $m \in \mathbb{N}$. Note that this assumption is justified by practice: only finitely many actuating variables are available to control a given system. As a consequence, we have a representation $B = [b_1, \ldots, b_m] \in (D(A^*)')^m$; the operators in the ADI iteration may be written as $V_i = [v_{i1}, \ldots, v_m] \in X^m$. In the first step of the ADI iteration we therefore have to solve m equations $(A + p_1I)v_{1k} = b_k \in D(A^*)'$; the following steps consist of solving the equations $(A+p_iI)x_k = v_{i-1,k} \in X$. To suitably approximate the equations arising in the ADI iteration, let $(X^{(i)})$ be a sequence of (finite-dimensional) subspaces of X, let $\Pi^{(1)} \in \mathcal{B}(D(A^*)')$ be a projector onto $X^{(1)}$, and, for $i \geq 2$, let $\Pi^{(i)} \in \mathcal{B}(X)$ be a projector onto $X^{(i)}$. Further, assume that, for $i \in \mathbb{N}$, the operators $\widetilde{A}_{p_i}^{(i)} \in \mathcal{B}(X^{(i)})$ are approximations on $A + p_i I$ in the sense that the solution $x \in X$ of the equation $\widetilde{A}_{p_i}^{(i)} x = b$ is "close to $(\widetilde{A}_{p_i}^{(i)})^{-1}\Pi^{(i)} b$" for suitable right-hand side $b \in X$ or $b \in D(A^*)'$.

With these preparations we can formulate our inexact ADI iteration as follows.

Algorithm 2. Inexact ADI iteration for operator Lyapunov equations.

Input: A Hilbert space X, a sequence $(X^{(i)})$ of subspaces of X, operators $A : D(A) \subset X \to X$ and $B \in \mathcal{B}(U, D(A^*)')$; projectors $\Pi^{(i)}$ onto $X^{(i)}$ with $\Pi^{(1)} \in \mathcal{B}(D(A^*)')$ and $\Pi^{(j)} \in \mathcal{B}(X)$ for $i \geq 2$; operators $\widetilde{A}_{p_i}^{(i)} \in \mathcal{B}(X^{(i)})$; shift parameters $p_i \in \mathbb{C}_-$.
Output: $\widetilde{S} = \widetilde{S}_{i_{\max}} \in \mathcal{B}(U^{i_{\max}}, X)$, such that $\widetilde{S}\widetilde{S}^* \approx P$, where P is the Gramian of (A, B).

$\widetilde{V}_1 = (A_{p_1}^{(i)})^{-1}\Pi^{(i)} B$
$\widetilde{S}_1 = \sqrt{-2\operatorname{Re}(p_1)} \cdot \widetilde{V}_1$
for $i = 2, 3, \ldots, i_{\max}$ **do**
  $\widetilde{V}_i = \widetilde{V}_{i-1} - (p_i + \overline{p_{i-1}})(\widetilde{A}_{p_i}^{(i)})^{-1}\Pi^{(i)}\widetilde{V}_{i-1}$
  $\widetilde{S}_i = [\, \widetilde{S}_{i-1}\, ,\ \sqrt{-2\operatorname{Re}(p_i)} \cdot \widetilde{V}_i \,]$
end for

We will now derive expressions and estimates for the error between exact and inexact ADI iteration. We will first derive estimates for $V_i - \widetilde{V}_i$. Thereafter, we will provide upper bounds for the difference between S_i and $\widetilde{S}_i$, and, respectively, $P_i = S_i S_i^*$ and $\widetilde{P}_i = \widetilde{S}_i \widetilde{S}_i^*$.

Denoting

$$\begin{aligned} E_i &= V_i - \widetilde{V}_i, \\ G_i^E &= (A + p_i I)^{-1} - (\widetilde{A}_{p_i}^{(i-1)})^{-1}\Pi^{(i)}, \end{aligned}$$

the construction of V_i and $\widetilde{V}_i$ in Algorithm 1 and Algorithm 2 yields that the error recursively fulfills

$$E_i = (A + p_i I)^{-1}(A - \overline{p_i} I)E_{i-1} + (p_i + \overline{p_{i-1}}) \cdot G_i^E \widetilde{V}_{i-1}. \tag{5.1}$$

Using that the operators $(A + p_j)^{-1}$, $(A + p_k)^{-1}$, $(A - \overline{p_j} I)$, and $(A - \overline{p_k} I)$ commute for all $j, k \in \mathbb{N}$, we may inductively conclude from (5.1) that for all $i > 1$, there holds

$$\begin{aligned} E_i = {} & (A - \overline{p_1} I)(A + p_i I)^{-1} \left(\prod_{k=2}^{i-1} (A - \overline{p_k} I)(A + p_k I)^{-1} \right) \cdot G_1^E B + \sum_{j=2}^{i-1} (p_j + \overline{p_{j-1}}) \\ & \cdot (A - \overline{p_j} I)(A + p_i I)^{-1} \left(\prod_{k=j+1}^{i-1} (A - \overline{p_k} I)(A + p_k I)^{-1} \right) G_j^E \widetilde{V}_{j-1} \\ & + (p_i + \overline{p_{i-1}}) \cdot G_i^E \widetilde{V}_{i-1}. \end{aligned} \tag{5.2}$$

The error analysis will be performed for infinite-time admissible (A, B), where the semigroup generated by A is similar to a contractive semigroup. The following auxiliary result will be crucial for our estimates.

Lemma 5.2. *Let U, X be Hilbert spaces and A be the generator of a bounded semigroup $T(\cdot)$ on X, which is similar to a contractive semigroup $T_1(\cdot)$ on the Hilbert* *space X_1. Let $(p_i)_i$ be a sequence in $\mathbb{C}_-$. Then there exists some $m \in \mathbb{R}_{>0}$, such that for all $i, j \in \mathbb{N}$ holds*

$$\left\| \prod_{k=j+1}^{i} (A - \overline{p_k} I)(A + p_k I)^{-1} \right\|_{\mathcal{B}(X)} < m. \tag{5.3}$$

Proof. Let $A_1 : D(A_1) \subset X_1 \to X_1$ be the generator of $T_1(\cdot)$, and let $H \in \mathcal{B}(X_1, X)$ be bijective with $T_1(t) = H^{-1}T(t)H$ for all $t \in \mathbb{R}_{>0}$. By [50, Thm. 12.3.10], there holds

$$\left\| (A_1 - \overline{p_k} I)(A_1 + p_k I)^{-1} \right\|_{\mathcal{B}(X_1)} \leq 1.$$

Using that A and A_1 are related by $A_1 x = H^{-1}AHx$ for all $x \in D(A_1) = H^{-1}D(A)$ [19, p. 59], we obtain

$$\begin{aligned}
&\left\| \left(\prod_{k=j+1}^{i} (A - \overline{p_k} I)(A + p_k I)^{-1} \right) \right\|_{\mathcal{B}(X)} \\
&\quad = \left\| H \left(\prod_{k=j+1}^{i} (A_1 - \overline{p_k} I)(A_1 + p_k I)^{-1} \right) H^{-1} \right\|_{\mathcal{B}(X)} \\
&\quad \leq \|H\|_{\mathcal{B}(X_1,X)} \prod_{k=j+1}^{i} \left\| (A_1 - \overline{p_k} I)(A_1 + p_k I)^{-1} \right\|_{\mathcal{B}(X_1)} \cdot \|H^{-1}\|_{\mathcal{B}(X,X_1)} \\
&\quad \leq \|H\|_{\mathcal{B}(X_1,X)} \cdot \|H^{-1}\|_{\mathcal{B}(X,X_1)}.
\end{aligned}$$

That is, (5.3) holds true with $m = \|H\|_{\mathcal{B}(X_1,X)} \cdot \|H^{-1}\|_{\mathcal{B}(X,X_1)}$. ∎

PROPOSITION 5.3. *Let U, X be Hilbert spaces and A be the generator of a bounded semigroup $T(\cdot)$ on X, which is similar to a contractive semigroup $T_1(\cdot)$ on some Hilbert space X_1. Let $B \in \mathcal{B}(U, D(A^*)')$ be an infinite-time admissible control operator for $T(\cdot)$. Let $P \in \mathcal{B}(X)$ be the Gramian of (A, B) and let $(p_i)_i$ be a J-cyclic sequence in $\mathbb{C}_-$, i.e., $p_i = p_{i+J}$ for all $i \in \mathbb{N}$. Assume that the operators $V_i = [v_{i1}, \ldots, v_{im}] \in \mathcal{B}(\mathbb{C}^m, X)$ are obtained by Algorithm* 1.

Let $(X^{(i)})$ be a sequence of subspaces of X, let $\Pi^{(1)} \in \mathcal{B}(D(A^*)')$ be a projector onto $X^{(1)}$, and, for $i \geq 2$, let $\Pi^{(i)} \in \mathcal{B}(X)$ be a projector onto $X^{(i)}$. Further, let $\widetilde{A}_{p_i}^{(i)} \in \mathcal{B}(X^{(i)})$, and assume that the operators $\widetilde{V}_i = [\widetilde{v}_{i1}, \ldots, \widetilde{v}_{im}] \in \mathcal{B}(\mathbb{C}^m, X)$ are obtained by Algorithm 2.

Assume that

$$\begin{aligned}
\|(A + p_1 I)^{-1} b_l - (\widetilde{A}_{p_1}^{(1)})^{-1} \Pi^{(1)} b_l\|_X &\leq c^{(1l)} \text{ for } l = 1, \ldots, m, \text{ and} \\
\|(A + p_i I)^{-1} v_{i-1,l} - (\widetilde{A}_{p_i}^{(i)})^{-1} \Pi^{(i)} v_{i-1,l}\|_X &\leq c^{(il)} \text{ for } l = 1, \ldots, m, \quad i > 1.
\end{aligned} \tag{5.4}$$

Let either $\|\cdot\| = \|\cdot\|_{\mathcal{B}(X)}$ or $\|\cdot\| = \|\cdot\|_{\mathcal{S}_p(X)}$ for some $p \in [1, \infty)$. Then the following assertions hold true:

(a) *There exists some $M > 0$ such that for all $i \in \mathbb{N}$,*

$$\|E_i\| \leq M \cdot \sum_{k=1}^{i} \sum_{l=1}^{m} c^{(kl)}.$$

(b) *If A is a Riesz-spectral operator, then there exists some $M > 0$, an increasing sequence of finite-dimensional subspaces $X_{[s]}$ of X with $X = \overline{\bigcup_{s\in\mathbb{N}} X_{[s]}}$, a bounded sequence of projectors $\Pi_{[s]}$ with $\operatorname{im}\Pi_{[s]} = X_{[s]}$, and some $M > 0$, such that for all $s \in \mathbb{N}$ there exists some $\rho_{[s]} \in (0, 1)$, such that*

$$\|\Pi_{[s]}E_i\| \leq M \cdot \left(\sum_{l=1}^{m} c^{(il)} + \sum_{k=1}^{i-1} \rho_{[s]}^{i-k-1} \sum_{l=1}^{m} c^{(kl)} \right).$$

Proof. The following argumentation makes use of the fact that any $x \in X$ can be identified as an operator $x \in \mathcal{B}(\mathbb{C}, X)$ via scalar multiplication. It can be seen that this operator also belongs to any Schatten space with

$$\|x\|_X = \|x\|_{\mathcal{B}(\mathbb{C},X)} = \|x\|_{\mathcal{S}_p(\mathbb{C},X)}.$$

By the triangle inequality, we obtain that for any $x_1, \ldots, x_m \in X$, the operator $V = [x_1, \ldots, x_m] \in \mathcal{B}(\mathbb{C}^m, X)$ fulfills

$$\|V\|_{\mathcal{B}(\mathbb{C},X)} \leq \|V\|_{\mathcal{S}_p(\mathbb{C},X)} \leq \sum_{l=1}^{m} \|x_l\|_X.$$

(a) By Lemma 5.2, we know that there exists some $m \in \mathbb{R}_{>0}$, such that for all $i, j \in \mathbb{N}$, the inequality (5.3) holds true. Then, by setting

$$\begin{aligned} M = m \cdot \max\{\|(A - \overline{p_j}I)(A + p_iI)^{-1}\|_{\mathcal{B}(X)} \mid i, j \in \{1, \ldots, J-1\}\} \\ \cdot \max\left(\{|p_{j+1} - \overline{p_j}| \mid j \in \{1, \ldots, J-1\}\} \cup \{1\}\right), \end{aligned} \tag{5.5}$$

the desired result follows by a combination of (5.2) and (5.4).

(b) With the notation of Remark 5.1, define the projectors by $\Pi_{[s]} = H^{-1}\Pi_{[s]}^D H$, where $\Pi_{[s]}^D \in \mathcal{B}(\ell_2)$ truncates after the sth position, i.e.,

$$\Pi_{[s]}^D(x_1, \ldots, x_j, x_{j+1}, x_{j+2}, \ldots) = (x_1, \ldots, x_j, 0, 0, \ldots),$$

and set $X_{[s]} = \operatorname{im}\Pi_{[s]}$. Then we have $\dim X_{[s]} = s$ and $X = \overline{\bigcup_{s\in\mathbb{N}} X_{[s]}}$. Since $\|\Pi_{[s]}^D\|_{\mathcal{B}(\ell_2)} = 1$, the sequence $(\Pi_{[s]})_s$ is bounded by $C := \|\Pi_{[s]}\|_{\mathcal{B}(X)} \leq \|H\|_{\mathcal{B}(X)} \cdot \|H^{-1}\|_{\mathcal{B}(X)}$. The construction of $\Pi_{[s]}$ leads to

$$\Pi_{[s]}(A + p_kI)^{-1} = \Pi_{[s]}(A + p_kI)^{-1}\Pi_{[s]} = (A + p_kI)^{-1}\Pi_{[s]},$$

and we can make use of (5.2) to see that

$$\begin{aligned} \Pi_{[s]}E_i = (A - \overline{p_1}I)(A + p_iI)^{-1} \left(\prod_{k=2}^{i-1} \Pi_{[s]}(A - \overline{p_k}I)(A + p_kI)^{-1}\Pi_{[s]} \right) \cdot G_1^E B \\ + \sum_{j=1}^{i-2} (p_j + \overline{p_{j-1}}) \cdot (A - \overline{p_j}I)(A + p_iI)^{-1} \\ \cdot \left(\prod_{k=j+1}^{i-1} \Pi_{[s]}(A - \overline{p_k}I)(A + p_kI)^{-1}\Pi_{[s]} \right) G_j^E \widetilde{V}_{j-1} \\ + (p_i + \overline{p_{i-1}}) \cdot \Pi_{[s]} G_i^E \widetilde{V}_{i-1}. \end{aligned} \tag{5.6}$$

As in Remark 4.11(d), we can infer that the spectral radius of the operator products in the above expression are given by

$$\rho\left(\prod_{k=j+1}^{i-1}\Pi_{[s]}(A-\overline{p_k}I)(A+p_kI)^{-1}\Pi_{[s]}\right)=\max_{l=1,\dots,s}\prod_{k=j+1}^{i-1}\left|\frac{\overline{p_k}-\lambda_l}{p_k+\lambda_l}\right|. \tag{5.7}$$

The latter expression is below one, since the function $z\mapsto\frac{\overline{p_j}-z}{p_j+z}$ maps $\mathbb{C}_-$ onto the open complex unit circle. By J-cyclicity of the shift parameters, we can now infer that there exists some $\rho_{[s]}\in(0,1)$, such that for all $i,j\in\mathbb{N}$, there holds

$$\max_{l=1,\dots,s}\prod_{k=j+1}^{i-1}\left|\frac{\overline{p_k}-\lambda_l}{p_k+\lambda_l}\right|\le\rho_{[s]}^{i-j-1}.$$

Since the similarity transformation of the operator product in (5.7) with $H\in\mathcal{B}(X,\ell_2)$ gives a diagonal operator, we obtain

$$\begin{aligned}&\left\|\prod_{k=j+1}^{i-1}\Pi_{[s]}(A-\overline{p_k}I)(A+p_kI)^{-1}\Pi_{[s]}\right\|_{\mathcal{B}(X)}\\&\quad\le\underbrace{\|H\|_{\mathcal{B}(X,\ell_2)}\cdot\|H^{-1}\|_{\mathcal{B}(\ell_2,X)}}_{=:m}\cdot\rho_{[s]}^{i-j-1}.\end{aligned}$$

Now defining M as in (5.5), we obtain the desired result. □

As an immediate consequence of Proposition 5.3 and the fact

$$\|S_i-\widetilde{S}_i\|\le-\sum_{k=1}^{i}\mathrm{Re}(p_k)\cdot\|V_k-\widetilde{V}_k\|, \tag{5.8}$$

we may formulate the following estimates for $\|S_i-\widetilde{S}_i\|$.

Corollary 5.4. *Under the assumptions and notation of Proposition* 5.3 *and Algorithms* 1 *and* 2, *the following assertions hold true:*

(a) *There exists some $M>0$ such that for all $i\in\mathbb{N}$ holds*

$$\|S_i-\widetilde{S}_i\|\le M\cdot\sum_{k=1}^{i}\sum_{l=1}^{m}(i+1-k)\cdot c^{(kl)}.$$

(b) *If A is a Riesz-spectral operator, then there exists a sequence of finite-dimensional subspaces $X_{[s]}$ of X with $X=\overline{\bigcup_{j\in\mathbb{N}}X_{[s]}}$, and a sequence of projectors $\Pi_{[s]}$ with $\operatorname{im}\Pi_{[s]}=X_{[s]}$, such that for all $s\in\mathbb{N}$ there exists some $M_{[s]}>0$, such that*

$$\|\Pi_{[s]}(S_i-\widetilde{S}_i)\|\le M_{[s]}\cdot\left(\sum_{k=1}^{i}\sum_{l=1}^{m}c^{(kl)}\right).$$

Proof. Statement (a) follows from the triangle inequality in (5.8), and accordingly using the error bound from Proposition 5.3(a). To prove (b), we construct $X_{[s]}$ and $\Pi_{[s]}$ as in the proof of Proposition 5.3(b). Thereafter, making use of

$$\|\Pi_{[s]}(S_i-\widetilde{S}_i)\|\le-\sum_{k=1}^{i}\mathrm{Re}(p_k)\cdot\|\Pi_{[s]}(V_i-\widetilde{V}_i)\|,$$

the error bound in Proposition 5.3(b), and the formula for geometric sums gives rise to the result. ∎

Now we present estimates for the difference between approximative Gramians provided by exact and inexact ADI iteration.

COROLLARY 5.5. *Under the assumptions and notation of Proposition 5.3 and Algorithms 1 and 2, the following assertions hold true:*

(a) *There exists some $M > 0$ such that for all $i \in \mathbb{N}$ and*

$$L_i = \sum_{k=1}^{i} \sum_{l=1}^{m} (i+1-k) \cdot c^{(kl)},$$

there holds

$$\|P_i - \widetilde{P}_i\| \le M \cdot \left(L_i + L_i^2\right).$$

(b) *If A is a Riesz-spectral operator, then there exists a sequence of finite-dimensional subspaces $X_{[s]}$ of X with $X = \overline{\bigcup_{j\in\mathbb{N}} X_{[s]}}$ and a sequence of projectors $\Pi_{[s]}$ with $\operatorname{im}\Pi_{[s]} = X_{[s]}$, such that for all $s \in \mathbb{N}$ there exists some $M_{[s]} > 0$, such that for*

$$K_i := \sum_{k=1}^{i} \sum_{l=1}^{m} c^{(kl)},$$

there holds

$$\|\Pi_{[s]}(P_i - \widetilde{P}_i)\Pi_{[s]}^*\| \le M_{[s]} \cdot \left(K_i + K_i^2\right).$$

Proof. In the case of $\|\cdot\| = \|\cdot\|_{\mathcal{S}_p(X)}$, statement (a) follows from Corollary 5.5(a), together with

(5.9)
$$\begin{aligned}
&\|P_i - \widetilde{P}_i\|_{\mathcal{S}_p(X)} \\
&\quad= \|S_i S_i^* - \widetilde{S}_i \widetilde{S}_i^*\|_{\mathcal{S}_p(X)} \\
&\quad\le \|S_i (S_i^* - \widetilde{S}_i^*)\|_{\mathcal{S}_p(X)} + \|(S_i - \widetilde{S}_i)\widetilde{S}_i^*\|_{\mathcal{S}_p(X)} \\
&\quad\le \|S_i\|_{\mathcal{S}_{2p}(\mathbb{C}^{i\cdot m},X)} \cdot \|S_i^* - \widetilde{S}_i^*\|_{\mathcal{S}_{2p}(X,\mathbb{C}^{i\cdot m})} + \|S_i - \widetilde{S}_i\|_{\mathcal{S}_{2p}(\mathbb{C}^{i\cdot m},X)} \cdot \|\widetilde{S}_i^*\|_{\mathcal{S}_{2p}(X,\mathbb{C}^{i\cdot m})} \\
&\quad= \left(\|S_i\|_{\mathcal{S}_{2p}(\mathbb{C}^{i\cdot m},X)} + \|\widetilde{S}_i\|_{\mathcal{S}_{2p}(\mathbb{C}^{i\cdot m},X)}\right) \cdot \|S_i - \widetilde{S}_i\|_{\mathcal{S}_{2p}(\mathbb{C}^{i\cdot m},X)} \\
&\quad\le \left(2\|S_i\|_{\mathcal{S}_{2p}(\mathbb{C}^{i\cdot m},X)} + \|S_i - \widetilde{S}_i\|_{\mathcal{S}_{2p}(\mathbb{C}^{i\cdot m},X)}\right) \cdot \|S_i - \widetilde{S}_i\|_{\mathcal{S}_{2p}(\mathbb{C}^{i\cdot m},X)} \\
&\quad\le \left(2\|P_i\|_{\mathcal{S}_p(X)}^{1/2} + \|S_i - \widetilde{S}_i\|_{\mathcal{S}_{2p}(\mathbb{C}^{i\cdot m},X)}\right) \cdot \|S_i - \widetilde{S}_i\|_{\mathcal{S}_{2p}(\mathbb{C}^{i\cdot m},X)}.
\end{aligned}$$

If $\|\cdot\|$ is the standard operator norm, then we can analogously estimate

$$\|P_i - \widetilde{P}_i\|_{\mathcal{B}(X)} \le \left(2\|P_i\|_{\mathcal{B}(X)} + \|S_i - \widetilde{S}_i\|_{\mathcal{B}(X)}\right) \cdot \|S_i - \widetilde{S}_i\|_{\mathcal{B}(X)},$$

and we can argument as for Schatten norms.

To prove (b), we first construct $X_{[s]}$ and $\Pi_{[s]}$ as in the proof of Proposition 5.3(b). Then, by determining bounds for $\|\Pi_{[s]}(P_i - \widetilde{P}_i)\Pi_{[s]}^*\|$ analogous to (5.9), the desired result follows immediately from Corollary 5.4(b). ∎

Remark 5.6.

(a) If A is unbounded, then the construction of $\rho_{[s]}$ in Proposition 5.3 leads to $\sup_{s\in\mathbb{N}} \rho_{[s]} = 1$. According to their construction in Corollary 5.4, the constants $M_{[s]}$ in Corollary 5.5(b) are therefore not uniformly bounded.

(b) Under the assumptions of Corollary 5.5(b) and, additionally,

$$K := \lim_{i\to\infty} K_i = \sum_{k=1}^{\infty}\sum_{l=1}^{m} c^{(kl)} < \infty,$$

the error bound in (b) can be slightly reformulated to

$$\|\Pi_{[s]}(P_i - \widetilde{P}_i)\Pi_{[s]}^*\| \leq N_{[s]} \cdot K$$

for some $N_{[s]} > 0$.

Under the assumption that inexact ADI iteration converges, we now present estimates for the difference between the limit of inexact ADI iteration and the Gramian operator P.

THEOREM 5.7. *Let U, X be Hilbert spaces and A be a Riesz-spectral operator that generates a bounded semigroup $T(\cdot)$ on X. Let $B \in \mathcal{B}(U, D(A^*)')$ be an infinite-time admissible control operator for $T(\cdot)$. Let $P \in \mathcal{B}(X)$ be the Gramian of (A, B), let $(p_i)_i$ be a J-cyclic sequence in $\mathbb{C}_-$, let $(X^{(i)})$ be a sequence of subspaces of X, let $\Pi^{(1)} \in \mathcal{B}(D(A^*)')$ be a projector onto $X^{(1)}$, and, for $i \geq 2$, let $\Pi^{(i)} \in \mathcal{B}(X)$ be a projector onto $X^{(i)}$. Further, let $\widetilde{A}^{(i)}_{p_i} \in \mathcal{B}(X^{(i)})$, and assume that the operators $\widetilde{V}_i = [\widetilde{v}_{i1}, \ldots, \widetilde{v}_{im}] \in \mathcal{B}(\mathbb{C}^m, X)$ are obtained by Algorithm 2.*

(a) *Assume that the Gramian P of (A, B) is compact and let $\varepsilon \in \mathbb{R}_{>0}$. Then for all inexact ADI iterations with the property that the error bounds* (5.4) *are fulfilled in each step, and $(\widetilde{P}_i)_i$ converges to some $\widetilde{P} \in \mathcal{B}(X)$ in the operator norm, there exist $M \in \mathbb{R}_{>0}$, $j \in \mathbb{N}$ with the following property: For all $i \in \mathbb{N}$ with $i \geq j$, there holds*

$$\|P - \widetilde{P}_i\|_{\mathcal{B}(X)} \leq \varepsilon + M \cdot \sum_{k=1}^{\infty}\sum_{l=1}^{m} c^{(kl)}.$$

(b) *Assume that the Gramian P of (A, B) is of pth Schatten class and let $\varepsilon \in \mathbb{R}_{>0}$. Then for all inexact ADI iterations with the property that the error bounds* (5.4) *are fulfilled in each step, and $(\widetilde{P}_i)_i$ converges to some $\widetilde{P} \in \mathcal{S}_p(X)$ in the pth Schatten norm, there exist $M \in \mathbb{R}_{>0}$, $j \in \mathbb{N}$ with the following property: For all $i \in \mathbb{N}$ with $i \geq j$, there holds*

$$\|P - \widetilde{P}_i\|_{\mathcal{S}_p(X)} \leq \varepsilon + M \cdot \sum_{k=1}^{\infty}\sum_{l=1}^{m} c^{(kl)}.$$

Proof. We only prove (a), since (b) is analogous.

By construction we have $\widetilde{P}_i \leq \widetilde{P}_{i+1}$ for all $i \in \mathbb{N}$. Therefore, we have $\widetilde{P}_i \leq \widetilde{P}$ for all $i \in \mathbb{N}$. Let $\varepsilon > 0$. By Corollary A.3, there exists some $j \in \mathbb{N}$ such that for all $i \geq j$, there holds

$$\|P - P_i\|_{\mathcal{B}(X)} < \frac{\varepsilon}{5} \text{ and } \|\widetilde{P} - \widetilde{P}_i\|_{\mathcal{B}(X)} < \frac{\varepsilon}{5}.$$

Using a spectral decomposition of P, we can infer the existence of some orthogonal projector $\Pi \in \mathcal{B}(X)$ with

$$\|P - \widetilde{P}\|_{\mathcal{B}(X)} < \|\Pi^*(P - \widetilde{P})\Pi\|_{\mathcal{B}(X)} + \frac{\varepsilon}{5}.$$

Let $(\Pi^{[s]})_s$ be constructed as in the proof of Proposition 5.3, i.e., for the sequence $(\lambda_i)_i$ of eigenvalues of A, $\Pi^{[s]}$ projects onto the eigenspace corresponding to the eigenvalues $\lambda_i, \dots, \lambda_s$, and along the complementary eigenspace. By the property of A being a Riesz-spectral operator, the sequence $(\Pi^{[s]})_s$ converges in the strong operator topology towards the identity operator. In particular, the Banach–Steinhaus theorem [38, Thm. III.9] implies the existence of some $C > 0$ with $\|\Pi^{[s]}\| < C$ for all $s \in \mathbb{N}$. Another consequence of strong convergence of $(\Pi^{[s]})_s$ to I is that the sequence of complementary projectors $(I-\Pi^{[s]})_s$ converges to zero in the strong operator topology. Since $\operatorname{im}\Pi$ is finite-dimensional, the operator Π is compact. Using Theorem A.2, we obtain

$$\lim_{s\to\infty} \|(I-\Pi^{[s]})\Pi\|_{\mathcal{B}(X)} = 0.$$

Consequently, there exists some $s \in \mathbb{N}$ with

$$\|(I-\Pi^{[s]})\Pi\|_{\mathcal{B}(X)} \cdot \Big(\|P\|_{\mathcal{B}(X)} + \|\widetilde{P}\|_{\mathcal{B}(X)}\Big) \cdot (3C+1) < \frac{\varepsilon}{5}.$$

Further, by making use of the monotonicity of (inexact) ADI iteration, we have

$$\|P_i - \widetilde{P}_i\|_{\mathcal{B}(X)} \le \|P\|_{\mathcal{B}(X)} + \|\widetilde{P}\|_{\mathcal{B}(X)}.$$

Incorporating the above findings, we find that for all $i \ge j$, there holds

$$\begin{aligned}
&\|P-\widetilde{P}_i\|_{\mathcal{B}(X)} \\
&\quad\le \|P-\widetilde{P}\|_{\mathcal{B}(X)} + \underbrace{\|\widetilde{P}-\widetilde{P}_i\|_{\mathcal{B}(X)}}_{<\frac{\varepsilon}{5}} \\
&\quad< \|\Pi^*(P-\widetilde{P})\Pi\|_{\mathcal{B}(X)} + \frac{2\varepsilon}{5} \\
&\quad\le \underbrace{\|\Pi^*(P-P_i)\Pi\|_{\mathcal{B}(X)}}_{<\frac{\varepsilon}{5}} + \|\Pi^*(P_i-\widetilde{P}_i)\Pi\|_{\mathcal{B}(X)} + \underbrace{\|\Pi^*(\widetilde{P}_i-\widetilde{P})\Pi\|_{\mathcal{B}(X)}}_{<\frac{\varepsilon}{5}} + \frac{2\varepsilon}{5} \\
&\quad< \frac{4\varepsilon}{5} + \|\Pi^*(P_i-\widetilde{P}_i)\Pi\|_{\mathcal{B}(X)} \\
&\quad< \frac{4\varepsilon}{5} + \|\Pi^*(\Pi^{[s]})^*(P_i-\widetilde{P}_i)\Pi^{[s]}\Pi\|_{\mathcal{B}(X)} + 2\|\Pi^*(\Pi^{[s]})^*(P_i-\widetilde{P}_i)(I-\Pi^{[s]})\Pi\|_{\mathcal{B}(X)} \\
&\qquad + \|((I-\Pi^{[s]})\Pi)^*(P_i-\widetilde{P}_i)((I-\Pi^{[s]})\Pi)\|_{\mathcal{B}(X)} \\
&\quad\le \frac{4\varepsilon}{5} + \|(\Pi^{[s]})^*(P_i-\widetilde{P}_i)\Pi^{[s]}\|_{\mathcal{B}(X)} + 2\underbrace{\|\Pi^{[s]}\|_{\mathcal{B}(X)}}_{\le C}\ \underbrace{\|P_i-\widetilde{P}_i\|_{\mathcal{B}(X)}}_{\le\|P\|_{\mathcal{B}(X)}+\|\widetilde{P}\|_{\mathcal{B}(X)}}\ \|(I-\Pi^{[s]})\Pi\|_{\mathcal{B}(X)} \\
&\qquad + \underbrace{\|\Pi(I-\Pi^{[s]})\|_{\mathcal{B}(X)}}_{\le 1+C}\ \underbrace{\|P_i-\widetilde{P}_i\|_{\mathcal{B}(X)}}_{\le\|P\|_{\mathcal{B}(X)}+\|\widetilde{P}\|_{\mathcal{B}(X)}}\ \|(I-\Pi^{[s]})\Pi)\|_{\mathcal{B}(X)} \\
&\quad\le \frac{4\varepsilon}{5} + \|(\Pi^{[s]})^*(P_i-\widetilde{P}_i)\Pi^{[s]}\|_{\mathcal{B}(X)} \\
&\qquad + \underbrace{(3C+1)\cdot\Big(\|P\|_{\mathcal{B}(X)}+\|\widetilde{P}\|_{\mathcal{B}(X)}\Big)\cdot\|(I-\Pi^{[s]})\Pi)\|_{\mathcal{B}(X)}}_{<\frac{\varepsilon}{5}} \\
&\quad< \varepsilon + \|(\Pi^{[s]})^*(P_i-\widetilde{P}_i)\Pi^{[s]}\|_{\mathcal{B}(X)}.
\end{aligned}$$

Now using Corollary 5.5 (see also Remark 5.6(b)), there exists some $M > 0$ such that

$$\|\Pi_{[s]}(P_i - \widetilde{P}_i)\Pi_{[s]}^*\| \leq M \cdot \sum_{k=1}^{\infty}\sum_{l=1}^{m} c^{(kl)},$$

which implies (a). ∎

Remark 5.8. The above assumptions on inexact ADI iteration are, for instance, fulfilled if the projectors $\Pi^{(i)} = \widetilde{\Pi}$ and spaces $X^{(i)} = \widetilde{X}$ are finite-dimensional and constant, with, moreover

$$(\widetilde{A}_{p_i}^{(i)})^{-1} = (\widetilde{A} + p_i\widetilde{M})^{-1} \cdot \widetilde{M},$$

where $\widetilde{M}, \widetilde{A} \in \mathcal{B}(\widetilde{X})$ are invertible operators with the additional property that $\widetilde{A}+p\widetilde{M}$ is invertible for all $p \in \overline{\mathbb{C}_-}$. This follows from the results in [7].

6. Systems governed by the heat equation. To demonstrate the applicability of the presented operator theoretic results, we consider an infinite-dimensional system that is governed by the heat equation with spatially (but not time) constant Robin boundary conditions; the latter is assumed to be the input variable $u(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ of the system.

More precisely, for a bounded domain $\Omega \subset \mathbb{R}^n$ with piecewise C^2 boundary $\partial\Omega$ [1], we consider the heat equation

$$\frac{\partial x}{\partial t}(\xi, t) = \Delta x(\xi, t), \quad (\xi, t) \in \Omega \times \mathbb{R}_{\geq 0} \tag{6.1a}$$

with boundary condition

$$\nu(\xi)^T \nabla x(\xi, t) + \alpha x(\xi, t) = u(t), \quad (\xi, t) \in \partial\Omega \times \mathbb{R}_{\geq 0}, \tag{6.1b}$$

where $\nu(\xi)$ denotes the outward normal to $\partial\Omega$ in $\xi \in \partial\Omega$, $\alpha \in \mathbb{R}_{>0}$, and $u \in L_2(\mathbb{R}_{\geq 0})$ is the input of the system.

In the first step, we rewrite this as a system $\dot{x}(t) = Ax(t) + Bu(t)$, where the state is given by the spatial temperature function at time t, that is, $x(t) := x(\cdot, t) \in L_2(\Omega) := X$; the input is one-dimensional, i.e., $U = \mathbb{C}$. The construction of A and B has been performed in [14] in the case where $\alpha = 0$; our case can be treated analogously. (It can, for instance, be derived from the results in [14] by additionally employing the output feedback theory presented in [55].) The operators A and B are given by

$$\begin{aligned} D(A) &= \{x \in H^1(\Omega) \,|\, \Delta x \in L^2(\Omega),\ \nu^T \nabla x + \alpha x = 0 \text{ on } \partial\Omega\}, \\ Ax &= \Delta x \quad \text{for all } x \in D(A), \\ \langle Bu, z\rangle_{D(A^*)', D(A^*)} &= u \cdot \int_{\partial\Omega} \overline{z(\xi)}\, d\sigma_\xi, \end{aligned} \tag{6.2}$$

where by $d\sigma_\xi$ we denote the surface measure on $\partial\Omega$. It follows by the Gauß Theorem that A is self-adjoint with $A \leq 0$. Furthermore, $0 \in \rho(A)$, since the elliptic problem

$$\begin{aligned} -\Delta x(\xi) &= z(\xi), & \xi \in \Omega, \\ \nu(\xi)^T \nabla x(\xi) + \alpha x(\xi) &= 0, & \xi \in \partial\Omega \end{aligned}$$

has a unique solution for all $z \in L_2(\Omega)$. By the Rellich–Kondrachov Theorem [1, Thm. 6.3], $H^1(\Omega)$ is compactly embedded in $L_2(\Omega)$. This gives rise to $A^{-1} \in$

$\mathcal{K}(L_2(\Omega))$, whence, by the resolvent identity [54, Rem. 2.2.5], A has compact resolvent. The construction of B in (6.2) further leads to

$$B'z = \int_{\partial\Omega} z(\xi) d\sigma_\xi \quad \text{for all } z \in D(A^*) = D(A).$$

Now we can infer from the Gauß theorem and the Cauchy–Schwarz inequality that for all $z \in D(A)$ holds

$$\begin{aligned}\|z\|^2_{D((-A)^{\frac{1}{2}})} &= \|z\|^2_{L_2(\Omega)} - \langle z, Az\rangle_{L_2(\Omega)} = \|z\|^2_{L_2(\Omega)} - \int_\Omega z(\xi)\Delta\overline{z(\xi)} d\xi \\ &= \|z\|^2_{L_2(\Omega)} + \|\nabla z\|^2_{L_2(\Omega)} + \alpha \int_{\partial\Omega} |z(\xi)|^2 d\sigma_\xi \geq \alpha \int_{\partial\Omega} |z(\xi)|^2 d\sigma_\xi \\ &\geq \frac{\alpha}{\sigma(\partial\Omega)^2} \cdot \left| \int_{\partial\Omega} z(\xi) d\sigma_\xi \right|^2 = \frac{\alpha}{\sigma(\partial\Omega)^2} \cdot |B'z|^2,\end{aligned}$$

where $\sigma(\partial\Omega) = \int_{\partial\Omega} d\sigma_\xi$. The control operator therefore fulfills $B \in \mathcal{B}(\mathbb{C}, D((-A)^{\frac{1}{2}}))$, whence, due to the one-dimensionality of the input space, there further holds $B \in \mathcal{S}_2(\mathbb{C}, D((-A)^{\frac{1}{2}}))$. Altogether, we are now in the situation of Proposition 3.6 and are able formulate the following result.

COROLLARY 6.1. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise C^2 boundary $\partial\Omega$. Then the operator A as defined in* (6.2) *generates an exponentially stable semigroup $T(\cdot)$. The control operator B as in* (6.2) *is moreover admissible for $T(\cdot)$. Furthermore, the Gramian P of (A, B) is nuclear with*

$$\|P\|_{\mathcal{S}_1(X)} = \frac{1}{2} \cdot \int_{\partial\Omega} x(\xi) d\sigma_\xi, \tag{6.3a}$$

where $x \in H^1(\Omega)$ solves

$$\begin{aligned} -\Delta x(\xi) &= 0, \qquad \xi \in \Omega, \\ \nu(\xi)^T \nabla x(\xi) + \alpha x(\xi) &= 1, \qquad \xi \in \partial\Omega. \end{aligned} \tag{6.3b}$$

Proof. The considerations in front of this theorem imply that A is self-adjoint with compact resolvent, $A \leq 0$, $0 \in \rho(A)$ and $(-A)^{-\frac{1}{2}}B \in \mathcal{S}_2(\mathbb{C}, L_2(\Omega))$. Exponential stability of the semigroup $T(\cdot)$ generated by A, admissibility of B for $T(\cdot)$, and nuclearity of the Gramian P are then immediate consequences of Proposition 3.6.

Formula (6.3) follows by an application of the findings in Remark 4.1(g) to the expression (3.7) for the trace of the Gramian. ∎

We now consider the ADI iteration for the heat equation (6.1). Since A is self-adjoint in this case, it makes sense to only choose real shift parameters. A substitution $q_i = -p_i$ of the shift parameters leads, according to Remark 4.1(g), to the ADI algorithm in the following form.

Remark 6.2. Algorithm 3 requires the solution of a sequence of Helmholtz equations. These can be solved via a finite element method. Note that if the grid is chosen to be the same in all equations, then Algorithm 3 will be arithmetically equivalent to the approach of semidiscretizing the heat equation with respect to space, and an accordant application of the matrix version of the ADI method to the semidiscretized system.

Algorithm 3. ADI iteration for heat equation with one-dimensional Robin boundary control (6.1).

Input: Bounded domain $\Omega \subset \mathbb{R}^n$ with piecewise C^2 boundary $\partial\Omega$, negatives of the shift parameters $q_1, \ldots, q_{i_{\max}} \in \mathbb{R}_{>0}$
Output: $S = S_{i_{\max}} \in \mathcal{B}(\mathbb{R}^{i_{\max}}, X)$, such that $SS^* \approx P$, where P is the Gramian of (A, B) (with A, B as in (6.2)).

1: Solve

$$q_1 \cdot v_1(\xi) - \Delta v_1(\xi) = 0, \qquad \xi \in \Omega,$$
$$\nu(\xi)^T \nabla v_1(\xi) + \alpha v_1(\xi) = 1, \qquad \xi \in \partial\Omega$$

for $v_1 \in L_2(\Omega)$.

2: Define $S_1 = \sqrt{2q_1} \cdot v_1 \in \mathcal{B}(\mathbb{C}, L_2(\Omega))$.
3: **for** $i = 2, 3, \ldots, i_{\max}$ **do**
4: Solve

$$q_i \cdot \hat{v}(\xi) - \Delta \hat{v}(\xi) = v_{i-1}(\xi), \qquad \xi \in \Omega,$$
$$\nu(\xi)^T \nabla \hat{v}(\xi) + \alpha \hat{v}(\xi) = 0, \qquad \xi \in \partial\Omega$$

for $\hat{v} \in L_2(\Omega)$.

5: Set $v_i = v_{i-1} - (q_i + q_{i-1}) \cdot \hat{v}$.
6: $S_i = [\, S_{i-1} \,,\, \sqrt{2q_i} \cdot v_i \,] \in \mathcal{B}(\mathbb{R}^i, L_2(\Omega))$
7: **end for**

Applying Proposition 4.12 to the heat equation considered in this part, we can derive the following expression for the error of the ADI iteration.

Corollary 6.3. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise C^2 boundary $\partial\Omega$. Then, in the notation of Algorithm* 3, *there holds*

$$\|P - P_i\|_{\mathcal{S}_1(X)} = \frac{1}{2} \cdot \int_{\partial\Omega} x(\xi) d\sigma_\xi - 2 \sum_{k=1}^{i} q_k \cdot \int_\Omega |v_k(\xi)|^2 d\xi, \tag{6.4}$$

where $x \in H^1(\Omega)$ solves (6.3b).

6.1. Numerical results. Now, we illustrate our findings with a short numerical example. To this end we consider (6.1) on the L-shaped domain $(0,1)^2 \setminus (0.5,1)^2$. We fix $\alpha = 1$ and can evaluate (6.3) exactly as $\|P\|_{S_1(X)} = 2$ because the solution to (6.3b) is given by $x(\cdot) \equiv \frac{1}{\alpha} = 1$. Now, we can apply the inexact version of Algorithm 3, compare Algorithm 2 to calculate approximate values $\tilde{V}_i$ and $\tilde{S}_i$. To do so, we use a finite element discretization of the PDE's given in Algorithm 3. The discretization is done using a Cartesian mesh consisting of square elements with maximal diameter h. On this mesh we define a subspace $V_h \subset H^1(\Omega)$ using piecewise bilinear finite elements. The calculations are done using the toolkit `DOpElib` [22] based upon the C++-library `deal.II`; see [8, 9]. In order to assert that the approximation error during the solution of the discrete PDE is below a given tolerance, TOL > 0, we employ a standard residual based L^2-error estimator η; see, e.g., [2]. Thus we can allow for refinement of the discretization if the error is too large, i.e., $\eta >$ TOL and for optional coarsening of the discretization once the error is too small, i.e., $\eta < 0.1\,$TOL. Note that this means that the different approximations are not obtained with the same discretization and thus the software needs to work with solutions given on different meshes, which is done in the library `DOpElib`.

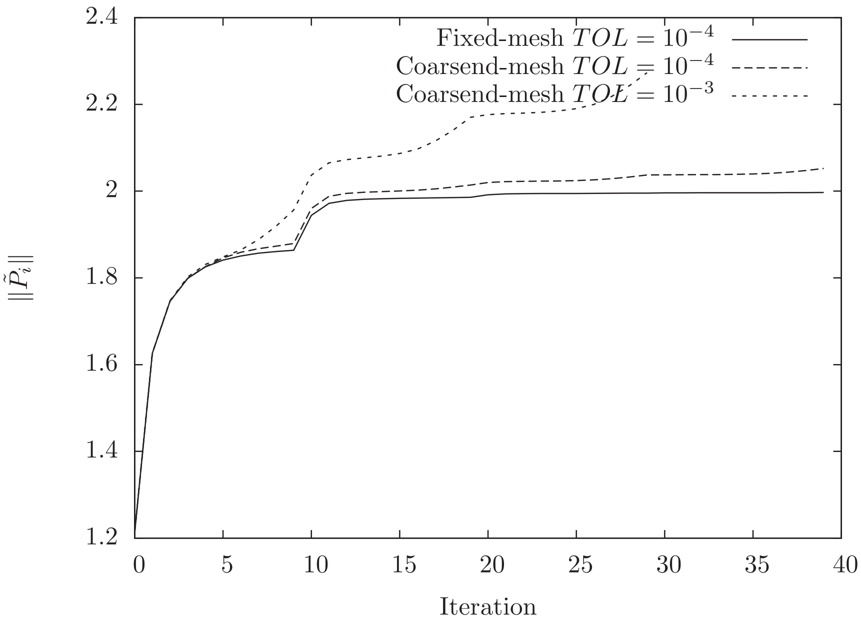

FIG. 6.1. *Behavior of the approximated Gramians for different algorithmic settings.*

As a first test case we consider the behavior of $\|\tilde{P}_i\|_{S_1(X)}$ for various settings in the algorithm. The results are depicted in Figure 6.1. The shift parameters where chosen by applying the method of WACHSPRESS [52] on the basis of the lowest hundred eigenvalues of the Robin Laplacian on the unit square $(0,1)^2$, which are given by $\pi^2(i^2+j^2)$, where $i,j=1,\ldots,10$. By this we generate a set of 10 shift parameters for the use in our example, i.e., $J=10$.

Here, the solid black line corresponds to the case when one (fixed) uniform mesh of mesh size h is used to discretize the PDE. Then from a priori error estimates we can conclude that the approximation errors in (5.4) satisfy

$$\begin{aligned} c^{(il)} &\le Ch^{2\alpha}\|\tilde{v}_{i-1}\|, && i>1,\\ c^{(1l)} &\le Ch^{2\alpha}. \end{aligned}$$

Here the exponent $\alpha = 2/3$ is given by singularities due to the reentrant corner; see, e.g., [31].

Thus by the fact that the elements $\|\tilde{v}_i\|$ are summable we deduce that the error $\|P-\tilde{P}_i\|_{S_1(X)}$ will be bounded, which can seen as well in the numerical result.

Further, to qualitatively test our error estimates, we in addition employed refinement and coarsening during the ADI-iteration. To this end, we employed a standard residual based L^2-error estimator η. With minor modifications to assert reliability of η on nonconvex domains, we can steer the meshes in such a way that the approximation errors $c^{(il)}$ $(i=1,\ldots;\, l=1)$ given in (5.4) satisfy

$$c^{(il)} \approx \mathrm{TOL}\,.$$

Since $\sum_{i=1}^{\infty} c^{(il)} = \infty$, we expect the error to grow as $i \to \infty$.

Finally, we have a more fine grained look onto the iteration. As is shown in Table 6.1, including the possibility to coarsen the mesh allows an almost identical approximation of the Gramian with severely fewer unknowns needed in the calculation.

Table 6.1
Convergence of the ADI-iterations with fixed mesh (left) and adjusted to tolerance 10^{-4} (right).

Iter. (i)	Unknowns	$\|\tilde{P}_i\|_{S_1(X)}$	$\|\tilde{v}_i\|$	Unknowns	$\|\tilde{P}_i\|_{S_1(X)}$	$\|\tilde{v}_i\|$
0	49665	1.21309	0.030728	49665	1.21309	0.030728
1	49665	1.62615	0.00910443	49665	1.62615	0.00910443
2	49665	1.74612	0.00121557	3201	1.74606	0.00121501
3	49665	1.79888	0.000267274	3201	1.79899	0.000268148
4	49665	1.82545	6.76124e-05	3201	1.82592	6.85014e-05
5	49665	1.841	2.18725e-05	225	1.8459	2.81147e-05
6	49665	1.85054	7.57344e-06	225	1.85887	1.02857e-05
7	49665	1.85676	2.61384e-06	225	1.86706	3.44507e-06
8	49665	1.86073	1.22589e-06	225	1.87312	1.87149e-06
9	49665	1.86353	7.09549e-07	225	1.87935	1.57927e-06
10	49665	1.94405	0.00203958	65	1.95994	0.00204132
11	49665	1.97195	0.000614905	833	1.98789	0.00061603
12	49665	1.97869	6.83044e-05	833	1.99473	6.93052e-05
13	49665	1.98125	1.29659e-05	833	1.99748	1.39308e-05
14	49665	1.98244	3.0429e-06	833	1.99903	3.94527e-06

Table 6.2
Convergence of the ADI-iterations with shift parameters not satisfying the non-Blaschke condition.

Iter. (i)	Unknowns	$\|\tilde{P}_i\|_{S_1(X)}$
0	49665	0.722646
1	49665	0.967484
2	49665	1.08674
3	49665	1.15745
4	49665	1.20112
5	49665	1.2323
6	49665	1.2544
7	49665	1.2707
8	49665	1.28382
9	49665	1.29425
10	49665	1.30297
11	49665	1.31023
12	49665	1.31644
13	49665	1.32178
14	49665	1.32643
15	49665	1.33048
16	49665	1.33405
17	49665	1.33722
18	49665	1.34007
19	49665	1.34263

To illustrate the non-Blaschke condition on the shifts, we now consider a choice of shift parameters that does not satisfy this condition. We again consider the eigenvalues of the Robin Laplacian on the unit square $(0,1)^2$, which are given by $\pi^2(i^2+j^2)$, where $i,j=1,2,\dots$, but instead of making a cyclic choice as we did above, we now consider an infinite subsequence which converges to infinity rapidly. Specifically, we order the values $\pi^2(i^2+j^2)$ by modulus to obtain a sequence $(\lambda_n)_{n=1}^{\infty}$ indexed by one natural number and define the shift parameters as $p_n := -\lambda_{n^2+1}$. Then $p_n = O(n^2)$ so that the non-Blaschke condition is not satisfied. By Remark 4.8(a) the sequence P_i obtained by ADI still converges, but it may not converge to the Gramian. Table 6.2 shows that with this choice of shift parameters ADI produces a bad approximation of the Gramian and strongly suggests that convergence to something other than the Gramian in fact occurs.

7. Conclusion. In this work, the ADI iteration has been generalized to operator Lyapunov equations for infinite-time admissible control systems with a bounded semigroup. This method provides approximations of the Gramian, which will be of finite rank if the input space is finite-dimensional. Under the assumption that the shift parameters fulfill the so-called non-Blaschke condition, strong convergence of the iteration to the Gramian has been proved. The non-Blaschke condition includes cyclic shift parameter choices. It has been furthermore shown that convergence in the operator and Schatten norm holds true if the Gramian is compact or, respectively, of Schatten class.

Motivated by the fact that computations in infinite-dimensional spaces (such as, the solution of partial differential equations) can usually only be done approximatively, we have also presented an error analysis for inexact ADI iteration.

The presented theory and methods have been applied to a heat equation with boundary control. It turned out that, for this class, the ADI iteration for determination of Gramians requires the numerical solution of a sequence of Helmholtz equations. This has been done by employing an adaptive finite-element solver.

Appendix A. Convergence of operator sequences. We state two classical convergence results for sequences of operators adapted slightly for our purposes.

THEOREM A.1 (see [40, p. 263]). *Let X be a Hilbert space and $(P_i)_i$ be a sequence of self-adjoint operators in $\mathcal{B}(X)$ with $P_{i+1} \geq P_i$ for all $i \in \mathbb{N}$. Moreover, assume that there exists some $Q \in \mathcal{B}(X)$ such that $P_i \leq Q$ for all $i \in \mathbb{N}$. Then there exists some self-adjoint $P \in \mathcal{B}(X)$ such that $(P_i)_i$ converges to P in the strong operator topology, that is,*

$$\lim_{i\to\infty} P_i x = Px \qquad \text{for all } x \in X.$$

The following is a classical result on compact and Schatten class operators.

THEOREM A.2. *Let X_1 and X_2 be Hilbert spaces, and let $\Pi_n \in \mathcal{B}(X_1)$ be a sequence of self-adjoint operators which converges in the strong operator topology to $\Pi \in \mathcal{B}(X_1)$.*

(a) *If $T \in \mathcal{K}(X_1, X_2)$, then*

$$\lim_{n\to\infty} \|\Pi_n T - \Pi T\|_{\mathcal{B}(X_1,X_2)} = 0.$$

(b) *If $T \in \mathcal{S}_p(X_1, X_2)$ with $p \in [1, \infty)$, then*

$$\lim_{n\to\infty} \|\Pi_n T - \Pi T\|_{\mathcal{S}_p(X_1,X_2)} = 0.$$

Proof. This follows from [21, Thm. III.6.3] (noting that by [21, Thm. III.7.1] the sets $\mathcal{K}(X_1, X_2)$, $\mathcal{S}_p(X_1, X_2)$ are separable symmetric norm ideals) by applying that theorem on the space $X_1 \times X_2$ with the operators

$$X_n = \begin{bmatrix} 0 & 0 \\ 0 & \Pi_n \end{bmatrix}, \quad A = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix}. \qquad \square$$

COROLLARY A.3. *Let X_1 and X_2 be Hilbert spaces, let $\Pi_n \in \mathcal{B}(X_1)$ be a sequence of self-adjoint operators which converges in the strong operator topology to the identity, and let $T \in \mathcal{B}(X_1, X_2)$.*

(a) *If* $T^*T \in \mathcal{K}(X_1)$*, then*

$$\lim_{n\to\infty} \|T^*\Pi_n T - T^*T\|_{\mathcal{B}(X_1)} = 0.$$

(b) *If* $T^*T \in \mathcal{S}_p(X_1)$ *with* $p \in [1,\infty)$*, then*

$$\lim_{n\to\infty} \|T^*\Pi_n T - T^*T\|_{\mathcal{S}_p(X_1)} = 0.$$

Proof. Since the singular values of T equal the square roots of the singular values of T^*T, the property $T^*T \in \mathcal{K}(X_1)$ ($T^*T \in \mathcal{S}_p(X_1)$) is equivalent to $T \in \mathcal{K}(X_1, X_2)$ ($T \in \mathcal{S}_{2p}(X_1, X_2)$). From [21, Prop. 2, p. 92] we then obtain

$$\|T^*\Pi_n T - T^*T\|_{\mathcal{B}(X_1)} \le \|T^*\|_{\mathcal{B}(X_2,X_1)} \, \|\Pi_n T - T\|_{\mathcal{B}(X_1,X_2)},$$
$$(\|T^*\Pi_n T - T^*T\|_{\mathcal{S}_p(X_1)} \le \|T^*\|_{\mathcal{S}_{2p}(X_2,X_1)} \, \|\Pi_n T - T\|_{\mathcal{S}_{2p}(X_1,X_2)}).$$

From Theorem A.2 we obtain that the right-hand side converges to zero, and the result follows. □

Appendix B. Orthonormality of the Takenaka–Malmquist system.

THEOREM B.1. *Let* (p_i) *be a sequence in* $\mathbb{C}_-$*. Then the Takenaka–Malmquist system* (4.5) *is an orthonormal system in* $\mathcal{H}_2$*.*

Proof. The frequency domain iteration can be rewritten as

$$\widehat{\phi}_1(s) = \frac{1}{-s+p_i} = -\frac{1}{s-p_i},$$
$$\widehat{\phi}_i(s) = \widehat{\phi}_{i-1}(s) - \frac{p_i + \overline{p_{i-1}}}{-s+p_i} \cdot \widehat{\phi}_{i-1}(s) = \frac{s+\overline{p_{i-1}}}{s-p_i}\widehat{\phi}_{i-1}(s),$$

from which it follows that

$$\widehat{\psi}_k(s) = \frac{-\sqrt{-2\,\mathrm{Re}(p_k)}}{s-p_k} \prod_{i=1}^{k-1} \frac{s+\overline{p_i}}{s-p_i}.$$

We define the functions (obtained from $\widehat{\psi}_k$ by replacing s by $-s$ and p_i by $\overline{p_i}$):

$$f_k(s) := \frac{\sqrt{-2\,\mathrm{Re}(p_k)}}{s+\overline{p_k}} \prod_{i=1}^{k-1} \frac{s-p_i}{s+\overline{p_i}}.$$

We then have for $n \ge k$

$$\left\langle \widehat{\psi_n}, \widehat{\psi_k} \right\rangle_{L^2(i\mathbb{R})} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\psi_n}(i\omega)\overline{\widehat{\psi_k}(i\omega)}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\psi_n}(i\omega) f_k(i\omega)\,d\omega.$$

Using the explicit expressions for $\widehat{\psi_n}$ and f_k, their product can be calculated as

$$\widehat{\psi_n}(s) f_k(s) = \frac{-\sqrt{-2\,\mathrm{Re}(p_n)}}{s-p_n} \frac{\sqrt{-2\,\mathrm{Re}(p_k)}}{s+\overline{p_k}} \prod_{i=k}^{n-1} \frac{s+\overline{p_i}}{s-p_i}.$$

With Γ_R the positively oriented curve consisting of the segment $[-iR, iR]$ on the imaginary axis together with the semi-circle in the right-half plane of radius R with center the origin, we have for R large enough that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{\psi_n}(i\omega) f_k(i\omega)\,d\omega = \frac{-1}{2\pi i}\int_{\Gamma_R} \widehat{\psi_n}(s) f_k(s)\,ds,$$

since $s = i\omega$ so that $ds = id\omega$ and the factor -1 is obtained from orientation reversion, and the integrand behaves like $O(R^{-2})$ on the semi-circle so that the integral over the semi-circle converges to zero as $R \to \infty$.

If $n > k$, then the above expression for the product further simplifies to

$$\widehat{\psi_n}(s) f_k(s) = \frac{-\sqrt{-2\,\mathrm{Re}(p_n)}}{s - p_n} \frac{\sqrt{-2\,\mathrm{Re}(p_k)}}{s - p_k} \prod_{i=k+1}^{n-1} \frac{s + \overline{p_i}}{s - p_i},$$

and if $n = k$, then it simplifies to

$$\widehat{\psi_n}(s) f_n(s) = \frac{2\,\mathrm{Re}(p_n)}{(s - p_n)(s + \overline{p_n})}.$$

We see that for $n > k$, the product is holomorphic on the right-half plane, so that by Cauchy's theorem we obtain $\langle \widehat{\psi_n}, \widehat{\psi_k} \rangle = 0$. For $n = k$, the curve Γ_R contains only the simple pole $-\overline{p_n}$ in its interior. The residue of $\widehat{\psi_n} f_n$ at this pole is -1, so that by the residue theorem, $\langle \widehat{\psi_n}, \widehat{\psi_n} \rangle = 1$. This shows that the Takenaka–Malmquist system is indeed orthonormal. ∎

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