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Analysis of FETI methods for multiscale PDEs – Part II: Interface variation

Abstract In this article, we give a new rigorous condition number estimate of the finite element tearing and interconnecting (FETI) method and a variant thereof, all-floating FETI. We consider a scalar elliptic equation in a two- or three-dimensional domain with a highly heterogeneous (multiscale) diffusion coefficient. This coefficient is allowed to have large jumps not only *across* but also *along* subdomain interfaces and in the interior of the subdomains. In other words, the subdomain partitioning does not need to resolve any jumps in the coefficient. Under suitable assumptions, we derive bounds for the condition numbers of one-level and all-floating FETI that are robust with respect to strong variations in the contrast in the coefficient, and that are explicit in some geometric parameters associated with the coefficient variation. In particular, robustness holds for face, edge, and vertex islands in high-contrast media. As a central tool we prove and use new weighted Poincaré and discrete Sobolev type inequalities that are explicit in the weight. Our theoretical findings are confirmed in a series of numerical experiments.

Keywords FETI · domain decomposition · finite element method · multiscale problems · preconditioning · varying coefficients · high contrast coefficients · weighted Poincaré inequalities

Mathematics Subject Classification (2000) 65F10, 65N22, 65N30, 65N55

1 Introduction

In recent years, the detailed and fast simulation of biological, physical or engineering processes has become an almost standard requirement. Often such problems are posed on complex geometries and involve highly heterogeneous (often non-linear) material parameters. As a consequence, the development of efficient and robust parallel solvers for heterogeneous media has been a very active area of research, specifically in the setting of multiscale solvers, and in the domain decomposition and multigrid communities [2,3,7,8,15–19,30,31,35–38,40,45–47].

In this paper, we are concerned with the convergence of two variants of the finite element tearing and interconnecting (FETI) domain decomposition method in the context of heterogeneous (multiscale) problems, for the general case that we have large jumps in the coefficient not only across, but also along the subdomain interfaces. As such this paper is a continuation of the work in [28,30,32], where we have shown the robustness of one-level FETI methods with respect to coefficient variation in the subdomain interiors, and of [31], where we have extended these results to some dual-primal FETI methods. In a wider context, the work follows some earlier work on the robustness of two-level

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Schwarz-type domain decomposition methods for heterogeneous media by Graham et al. [16], Graham and Scheichl [17, 18], as well as Scheichl and Vainikko [40].

FETI methods are robust domain decomposition methods for solving finite element discretisations of partial differential equations (PDEs) with excellent parallel scalability properties. They belong to the class of dual iterative substructuring methods and were introduced by Farhat and Roux [13]. For an extensive literature review on the analysis of FETI methods see our first paper [30].

Assuming that the coefficients of the PDE are constant in each subdomain, Klawonn and Widlund [21] proved (based on pioneering work by Mandel and Tezaur [25]) that the spectral condition number of the preconditioned FETI system is bounded by

$$C\,(1 + \log(H/h))^2, \tag{1.1}$$

where the constant C in (1.1) is independent of possible jumps in the coefficients across subdomain interfaces when a special scaling of the preconditioner is applied. Here, as usual, H and h denote the subdomain diameter and the mesh width, respectively, and C is independent of H and h, as well. This bound (1.1) was also shown to hold true for FETI-DP methods and for the related balancing Neumann-Neumann and BDDC methods [21, 22, 24]. An excellent account of all these results can be found in the monograph [44] by Toselli and Widlund. We also refer to [5] where it is shown that this bound is sharp. Finally, we mention also that until recently, certain regularity assumptions on the subdomains had to be made. Recently, Klawonn, Rheinbach, and Widlund [20] were able to weaken these assumptions significantly and to treat also quite irregular subdomains in two dimensions, as they appear when decomposing unstructured meshes with graph partitioners, see also [9].

However, all the above mentioned analyses assume that the coefficients of the PDE are piecewise constant with respect to the subdomain partitioning. The main focus of [30] and of the present work is the analysis of FETI methods for highly heterogeneous multiscale problems, i. e., in the case of coefficient jumps that are not aligned with the subdomain interfaces and/or vary strongly within a subdomain, particularly for the case that jumps occur *along* the interface. It has already been observed numerically by several authors (see e. g. [19, 23, 35, 36]) that a simple generalisation of the scaling employed by Klawonn and Widlund in [21] leads to robustness of the FETI method even in this case. In the following, we restrict ourselves to the model elliptic problem

$$-\nabla \cdot (\alpha\,\nabla u) \;=\; f, \tag{1.2}$$

in a bounded polygonal or polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, subject to suitable boundary conditions on the boundary $\partial\Omega$. The coefficient $\alpha(x)$ is assumed to be strictly positive, but it may vary over many orders of magnitude in an unstructured way on Ω. It is not surprising that (1.1) also holds in this case but in general with $C = C(\alpha)$, i. e., with some possible loss of α-robustness. In our recent article [30], we have shown that the dependence on α is restricted to the variation of $\alpha(x)$ in the vicinity of subdomain interfaces (within each subdomain). More precisely, if $\Omega_{i,\eta}$ denotes the boundary layer of width η of any of the subdomains Ω_i, and if $\alpha(x) \simeq \alpha(y)$ for all $x, y \in \Omega_{i,\eta}$, then $C(\alpha) \lesssim (H/\eta)^2$, independently of the variation of $\alpha(x)$ in the remainder $\Omega_i \backslash \Omega_{i,\eta}$ of each subdomain and independently of any jumps of $\alpha(x)$ across subdomain interfaces. The hidden constant in [30] depends on the local variation of $\alpha(x)$ in $\Omega_{i,\eta}$ and blows up when a large coefficient jump appears *along* a subdomain interface, but numerical experiments in [23, 30, 36] indicate that in practice the condition number of the preconditioned FETI system does not blow up, at least when α jumps only a few times along each subdomain interface.

In the present work, we extend the results in [30] and give a rigorous analysis showing that one-level FETI and a variant of one-level FETI, the *all-floating* (or *total*) FETI method [10, 26–28], are α-robust in the following sense:

(i) Assume that the boundary layer $\Omega_{i,\eta}$ of each subdomain can be subdivided into two connected subregions $\Omega_{i,\eta}^{(1)}$, $\Omega_{i,\eta}^{(2)}$ such that $\alpha(x) \simeq \alpha(y)$ for all x, $y \in \Omega_{i,\eta}^{(k)}$, and for each $k = 1, 2$. Then the constant in (1.1) is bounded by

$$C(\alpha) \;\lesssim\; (H/\eta)^\beta,$$

with $\beta \in \{1, 2, 3\}$ depending on the measure of the interface between $\Omega_{i,\eta}^{(1)}$ and $\Omega_{i,\eta}^{(2)}$ and on the particular choice of the components of the FETI preconditioner (more details later). The hidden constant is again completely independent of the values of $\alpha(x)$ in the subdomain interiors, but now also of the contrast of the value of $\alpha(x)$ in $\Omega_{i,\eta}^{(1)}$ and in $\Omega_{i,\eta}^{(2)}$.

(ii) Assume that the boundary layer $\Omega_{i,\eta}$ of each subdomain can be subdivided into M connected subregions $\Omega_{i,\eta}^{(1)}, \ldots, \Omega_{i,\eta}^{(M)}$ such that for each k and for all $x, y \in \Omega_{i,\eta}^{(k)}$, we have $\alpha(x) \simeq \alpha(y)$. In addition, we assume that each of the subregions $\Omega_{i,\eta}^{(k)}$ can be extended to the interior of Ω_i such that all extended subregions $\Omega_i^{(k)}$ have a *common* interface Γ_i^*, and $\min_{x\in\Omega_{i,\eta}^{(k)}} \alpha(x) \simeq \min_{x\in\Omega_i^{(k)}} \alpha(x)$, i. e., $\alpha(x)$ is not substantially smaller in the interior than in the boundary layer. Then

$$C(\alpha) \lesssim (H/\eta)^\beta \, \sigma(H/h).$$

with $\beta \in \{1, 2\}$ and $\sigma(H/h) = 1$, $(1 + \log(H/h))$, or H/h, respectively, depending on whether the interface Γ_i^* contains at least a coarse face, edge, or vertex (more details below).

Our central theoretical tools to prove this robustness are new *weighted* Poincaré and discrete Sobolev type inequalities that are explicit in the weight α and in the geometrical parameters H and η. To the best of our knowledge these inequalities have not appeared in the literature before. The difference to many existing inequalities in weighted norms is that our constants are explicit in the weight, and often even independent thereof. Note however that other weighted Poincaré type inequalities have been proved by Xu and Zhu [47] (based on [4]) and in a recent article by Galvis and Efendiev [14].

The remainder of this article is structured as follows. Section 2 starts with some preliminaries. Section 3 is devoted to weighted Poincaré and discrete Sobolev type inequalities. In Section 4, we describe our generalisation of the one-level and the all-floating FETI method to multiscale PDEs and give the statements of our key results. The proofs of these results are then given in Section 5 where we introduce some additional technical tools needed in our analysis. Section 6 contains a series of extensions of the key results to more general situations. We conclude with some numerical experiments that confirm our theoretical results in Section 7.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^d$ (with $d = 2$ or 3) be a connected, open, and bounded domain with Lipschitz boundary $\partial\Omega$. We consider the following model problem: Find $u \in H^1(\Omega)$, $u|_{\partial\Omega} = g_D$ such that

$$\int_\Omega \alpha(x)\, \nabla u(x) \cdot \nabla v(x)\, dx \;=\; \int_\Omega f(x)\, v(x)\, dx \qquad \forall v \in H_0^1(\Omega), \tag{2.1}$$

for given functions $f \in L^2(\Omega)$ and $g_D \in H^{1/2}(\Omega)$, and $\alpha(x) > 0$ uniformly on Ω. For simplicity, we choose Dirichlet boundary conditions on the whole of $\partial\Omega$. However, all of our work can easily be generalised to allow Neumann boundary conditions on a part of $\partial\Omega$. The domain Ω is decomposed into non-overlapping subdomains $\{\Omega_i\}_{i=1,\ldots,N}$ such that

$$\overline{\Omega} \;=\; \bigcup_{i=1}^N \overline{\Omega}_i\,. \tag{2.2}$$

The subdomain boundaries $\partial\Omega_i$ are assumed to be Lipschitz. We define the skeleton Γ_S, the interface Γ, and the subdomain interfaces Γ_{ij} by

$$\Gamma_S := \bigcup\nolimits_i \partial\Omega_i\,, \qquad \Gamma := \Gamma_S \setminus \partial\Omega, \qquad \Gamma_{ij} := (\partial\Omega_i \cap \partial\Omega_j) \setminus \partial\Omega. \tag{2.3}$$

The subdomain diameters are denoted by $H_i := \operatorname{diam} \Omega_i$. As in our previous work [30], we need some regularity assumptions on the subdomain partition.

Definition 2.1 (regular domain)**.** For $d = 2$ (or 3), let $D \subset \mathbb{R}^d$ be a bounded contractible domain with a simply-connected Lipschitz boundary. D is called a *regular domain*, if it can be decomposed into a conforming coarse mesh of shape-regular triangles (tetrahedra). Whenever considering a family of regular domains, such as partitions into subdomains, we implicitly assume that the number of simplices forming an individual subdomain is uniformly bounded.

Definition 2.2 (shape parameter)**.** For a simplex T, we define the *shape parameter* $\rho(T)$ to be the radius of the largest inscribed ball. The *shape parameter* of a regular domain D is defined as $\rho(D) := \min_{1\le i\le s} \rho(T_i)$, where $\{T_i\}_{1\le i\le s}$ are the simplices according to Definition 2.1.

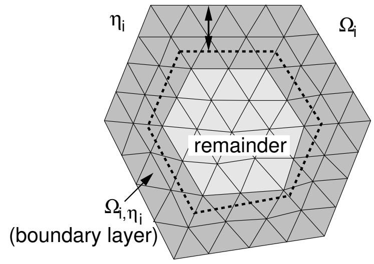

Fig. 1 Boundary layer Ω_{i,η_i} of the subdomain Ω_i, cf. Definition 2.5

Definition 2.3 (shape-regular partition)**.** Let D be an open domain in $\mathbb{R}^2$ or $\mathbb{R}^3$. A partitioning of D into regular subdomains $\{D_i\}_{i=1,\ldots,N}$, such that $\overline{D} = \bigcup_{i=1}^N \overline{D}_i$, is called *shape-regular*, if

$$\rho(D_i) \simeq \operatorname{diam} D_i\,, \qquad \text{and} \qquad \overline{D}_i \cap \overline{D}_j \neq \emptyset \implies \operatorname{diam} D_i \simeq \operatorname{diam} D_j \quad \forall i, j = 1, \ldots, N.$$

Assumption A1. The subdomains $\{\Omega_i\}$ form a non-overlapping shape-regular partition of Ω, and the underlying coarse mesh (cf. Definition 2.1) is conforming.

We introduce the following topological sets similar to [44, Definition 4.1].

Definition 2.4. The skeleton Γ_S is the disjoint union of

- *subdomain faces*, that are open and connected subsets of the interface, shared by two subdomains or by one subdomain and the outer boundary $\partial\Omega$,
- *subdomain edges*, that are open and connected subsets of the interface, shared by at least two subdomains, such that the closure of all edges forms the boundaries of the faces,
- *subdomain vertices*, single points where at least two subdomain edges meet.

We denote by

- $\mathcal{F}_i$ the set of subdomain faces,
- $\mathcal{E}_i$ the set of subdomain edges,
- $\mathcal{V}_i$ the set of subdomain vertices

on $\partial\Omega_i$. We denote by $\mathcal{F}_i^\Gamma$ and $\mathcal{F}_i^D$ the subset of subdomain faces on the interface $\partial\Omega_i \cap \Gamma$ and the outer (Dirichlet) boundary $\partial\Omega_i \cap \Omega$, respectively. Correspondingly, $\mathcal{E}_i^\Gamma$ and $\mathcal{E}_i^D$ ($\mathcal{V}_i^\Gamma$ and $\mathcal{V}_i^D$) denote the subset of subdomain edges (vertices) on the interface and the Dirichlet boundary, respectively.

We consider simplicial triangulations $\mathcal{T}_i$ on Ω_i which are quasi-uniform and shape-regular. The local mesh parameter is denoted by h_i. We require that the triangulations match on the subdomain interfaces. Note that these assumptions (together with Assumption A1) imply that $H_i \simeq H_j$ and $h_i \simeq h_j$ if $\Gamma_{i,j} \neq \emptyset$. The set of nodes of the mesh on the local boundary $\partial\Omega_i$ is denoted by $\partial\Omega_i^h$, and similarly, we define Γ^h and Γ_{ij}^h to be the sets of nodes on the interface Γ and on the subdomain interface Γ_{ij}, respectively. A typical node will be denoted by x^h. For the discretisation of (2.1) we use continuous piecewise linear finite elements. We denote by $V^h(\Omega)$, $V^h(\Omega_i)$ and $V^h(\partial\Omega_i)$ the spaces of continuous piecewise linear functions (with respect to the mesh) on the domain Ω, on a subdomain Ω_i and on the local boundary $\partial\Omega_i$, respectively. Note that these spaces do *not* incorporate the essential boundary conditions on $\partial\Omega$. Without loss of generality, we assume that the given Dirichlet trace g_D is in $V^h(\partial\Omega)$. Also without loss of generality, we assume that the coefficient α is piecewise constant on the elements of the triangulation.

Our analysis will require some notion of a boundary layer near subdomain interfaces. Therefore, we need the following definition which is closely related to the one in [16].

Definition 2.5 (discrete boundary layer)**.** Let $i \in \{1, \ldots, N\}$ and $\eta_i > 0$. We define the *discrete boundary layer* of Ω_i to be the open set Ω_{i,η_i} such that

$$\overline{\Omega}_{i,\eta_i} := \bigcup \bigl\{\overline{\tau} :\, \tau \in \mathcal{T}_i\,,\ \operatorname{dist}(\tau, \partial\Omega_i) \leq \eta_i\bigr\},$$

i. e., the set of all points within a distance η_i from the boundary $\partial\Omega_i$ extended to a union of elements. An illustration of this definition is given in Fig. 1.

Definition 2.6 (η_i-regular)**.** The discrete boundary layer Ω_{i,η_i} is called *η_i-regular* if there exists a shape-regular partition

$$\Xi_{i,\eta_i} := \{\omega_{i,1}, \ldots, \omega_{i,s_i}\}$$

of Ω_{i,η_i} into non-overlapping, regular *patches* $\omega_{i,j}$ (in the sense of Definition 2.1) with $\operatorname{diam} \omega_{i,j} \simeq \eta_i$, such that (i) the intersection of $\partial\omega_{i,j}$ with $\partial\Omega_i$ is non-empty and equal to the union of faces of the simplices forming the patch $\omega_{i,j}$ (in Definition 2.1), and (ii) the intersection of $\partial\omega_{i,j}$ with an edge $E \in \mathcal{E}_i$ is the union of edges of the simplices forming the patch $\omega_{i,j}$.

Assumption A2. For each $i \in \{1, \ldots, N\}$ the parameter $\eta_i > 0$ is chosen such that

(i) Ω_{i,η_i} is η_i–regular,
(ii) $\eta_i \simeq \eta_j$, if $\Gamma_{ij} \neq \emptyset$, and
(iii) the meshes induced by the patches match on the subdomain interfaces.

For the sake of simplicity, we make no difference between functions on discrete spaces and their vector representations with respect to the standard nodal basis, as well as between operators and their matrix representations with respect to the same basis. Similarly, we identify any discrete space X with its dual space X^*.

On the subdomain Ω_i, we can assemble the local finite element stiffness matrix K_i and group it with respect to the unknowns on the subdomain boundary (subscript B) and the interior (subscript I),

$$K_i \;=\; \begin{pmatrix} K_{i,BB} & K_{i,BI} \\ K_{i,IB} & K_{i,II} \end{pmatrix}. \tag{2.4}$$

Since none of the spaces $V^h(\Omega_i)$ incorporates essential boundary conditions, each of the local operators K_i is only positive semi-definite with $\ker K_i = \operatorname{span}\{\mathbf{1}_{\Omega_i}\}$, where $\mathbf{1}_{\Omega_i}$ denotes the constant function 1 on Ω_i. We define the Schur complement S_i of $K_{i,II}$ in K_i by

$$S_i := K_{i,BB} - K_{i,BI}[K_{i,II}]^{-1}K_{i,IB}\,. \tag{2.5}$$

Note, that the application of S_i means actually solving a Dirichlet boundary value problem on the subdomain Ω_i. Since S_i is symmetric positive semidefinite, it defines a seminorm,

$$|v|_{S_i} := \langle S_i\, v,\, v\rangle^{1/2} \qquad \text{for } v \in V^h(\partial\Omega_i), \tag{2.6}$$

that obeys the minimising property

$$|v|_{S_i}^2 \;=\; \min\Big\{ \int_{\Omega_i} \alpha(x)\, |\nabla\widetilde{v}(x)|^2\, dx \;:\; \widetilde{v} \in V^h(\Omega_i)\,,\; \widetilde{v}_{|\partial\Omega_i} = v \Big\}. \tag{2.7}$$

We denote by $\mathcal{H}_{i,\alpha}v$ the function $\widetilde{v} \in V^h(\Omega_i)$ for which the minimum is attained, and we call this function the *discrete α-harmonic extension* of v from $V^h(\partial\Omega_i)$ to $V^h(\Omega_i)$.

The Galerkin projection of (2.1) onto the space $V^h(\Omega)$ (which does not include the essential boundary conditions) leads to the following constrained linear system. Find $\widetilde{u} \in V^h(\Omega)$, $u|_{\partial\Omega} = g_D$ such that (2.1) (substituting $\widetilde{u}$ for u and $\widetilde{v}$ for v) holds for all test functions $\widetilde{v} \in V^h(\Omega)$, $\widetilde{v}|_{\partial\Omega} = 0$, in short

$$\widetilde{K}\,\widetilde{u} \;=\; \widetilde{f}. \tag{2.8}$$

The global stiffness matrix $\widetilde{K}$ and the load vector $\widetilde{f}$ can be assembled from (parts of) the local contributions K_i and f_i, respectively. Non-homogeneous Dirichlet boundary conditions can be treated with standard homogenisation techniques.

FETI methods are special domain decomposition methods to solve system (2.8) in parallel. The common idea of these methods is to decouple the system subdomain-wise and to enforce the continuity of $\widetilde{u}$ across the subdomain interfaces by Lagrange multipliers λ. There are various strategies to eliminate the primal variables and to design parallel preconditioners for the dual system in λ; these are the one-level, dual-primal, and all-floating or total FETI methods, see [10,44], as well as [26,27, 29] for further results developed in the closely related area of boundary elements (BETI).

To simplify the presentation and the proofs, we will follow mainly the all-floating approach where the Dirichlet boundary conditions are incorporated via Lagrange multipliers. However, the results carry over also to the more classical one-level FETI approach, albeit with one additional assumption. We will come back to this below. See [28,31] for details on how some of the proof techniques can also be extended to analyse dual-primal methods.

3 Weighted Poincaré and discrete Sobolev inequalities

The crucial new theoretical tools needed for our analysis below are weighted Poincaré and weighted discrete Sobolev inequalities. As we have seen in [30], the robustness of FETI methods is not affected by variations of the coefficient in the interior of each subdomain Ω_i. This is the reason why in [30] we generalised the Poincaré and discrete Sobolev inequalities to the discrete boundary layer Ω_{i,η_i}. Here, we go one step further and prove certain weighted versions of Poincaré and discrete Sobolev inequalities on the discrete boundary layer Ω_{i,η_i} in order to allow also for some coefficient variation along the interfaces between subdomains.

3.1 Weighted Poincaré inequality – two coefficient regions

Let us consider a single subdomain Ω_i and let us fix $\eta_i > 0$ according to Assumption A2. To state our weighted Poincaré inequality, we further decompose the discrete boundary layer Ω_{i,η_i} into two non-overlapping connected subregions $\Omega^{(1)}_{i,\eta_i}$ and $\Omega^{(2)}_{i,\eta_i}$ such that

$$\overline{\Omega}_{i,\eta_i} \;=\; \overline{\Omega}^{(1)}_{i,\eta_i} \cup \overline{\Omega}^{(2)}_{i,\eta_i}\,, \tag{3.1}$$

see also Fig. 2. Let $\Gamma^{(12)}_{i,\eta_i}$ be the larger of the connected components of the interface $\partial\Omega^{(1)}_{i,\eta_i} \cap \partial\Omega^{(2)}_{i,\eta_i}$. Before we state and prove our weighted Poincaré type inequality, we need some elementary results (see also [30, Appendix]).

Definition 3.1. Let Ω_{i,η_i} be η_i–regular. We say that the partitioning (3.1) is *compatible*, if each of the subregions $\Omega^{(k)}_{i,\eta_i}$ can be partitioned into a union of the patches $\omega_{i,j}$ in Definition 2.6, such that the interface $\Gamma^{(12)}_{i,\eta_i}$ is the union of faces of the patches.

In the following, we assume that Ω_{i,η_i} is η_i-regular and the partitioning (3.1) is *compatible.* For $k \in \{1, 2\}$ we define

$$\Xi^{(k)}_{i,\eta_i} := \{\omega_{i,j} \in \Xi_{i,\eta_i} : \omega_{i,j} \subset \Omega^{(k)}_{i,\eta_i}\},$$

which is a partition of the subregion $\Omega^{(k)}_{i,\eta_i}$ into patches. Patches from $\Xi^{(k)}_{i,\eta_i}$ will be denoted by $\omega^{(k)}_{i,j}$.

Definition 3.2. Let $\omega^{(k)}_{i,j}, \omega^{(k)}_{i,\ell} \in \Xi^{(k)}_{i,\eta_i}$. We call $P_{j\ell}$ a *path* of length $M_{j\ell}$ connecting the patches $\omega^{(k)}_{i,j}$ and $\omega^{(k)}_{i,\ell}$, iff it is a connected union of $M_{j\ell}$ patches from $\Xi^{(k)}_{i,\eta_i}$. Moreover, we denote by $\gamma^{(k)}_{i,j}$ all the faces (resp. edges for $d = 2$) of the patches from $\Xi^{(k)}_{i,\eta_i}$ that are contained in $\partial\Omega_i \cup \Gamma^{(12)}_{i,\eta_i}$.

Lemma 3.3. *Let $\gamma^{(k)}_{i,\ell}$ and $\gamma^{(k)}_{i,j}$ be faces (resp. edges for $d = 2$) of the patches $\omega^{(k)}_{i,j}$, $\omega^{(k)}_{i,\ell} \in \Xi^{(k)}_{i,\eta_i}$ according to Definition 3.2 and let $P_{j\ell}$ be a path of length $M_{j\ell}$ connecting the two patches. Then*

$$\frac{1}{\eta_i^d} \int_{\gamma^{(k)}_{i,j}} \int_{\gamma^{(k)}_{i,\ell}} |u(x) - u(y)|^2 \, ds_x \, ds_y \;\lesssim\; M_{j\ell}\, |u|^2_{H^1(P_{ij})} \qquad \forall u \in H^1(P_{ij}).$$

Proof. The proof follows immediately from [30, Lemma A.2(i)]. □

Lemma 3.4. *For all $u \in H^1(\Omega^{(k)}_{i,\eta_i})$ with $\int_{\Gamma^{(12)}_{i,\eta_i}} u \, ds = 0$, we have*

$$\frac{1}{\eta_i} \|u\|^2_{L^2(\partial\Omega_i^{(k)} \cap \partial\Omega_i)} \;\lesssim\; \Big(\frac{H_i}{\eta_i}\Big)^{\beta} |u|^2_{H^1(\Omega_{i,\eta_i})}\,, \tag{3.2}$$

with $\beta = d$ in general, and $\beta = 2$ if $d = 3$ and $|\Gamma^{(12)}_{i,\eta_i}| \gtrsim H_i\, \eta_i$ (as e. g. in Fig. 2, left).

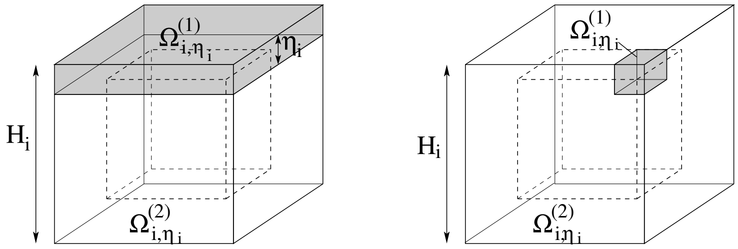

Fig. 2 Different settings of the subregions $\Omega^{(k)}_{i,\eta_i}$ in three dimensions. *Left* $|\Gamma^{(12)}_{i,\eta_i}| \simeq H_i\,\eta_i$, *right* $|\Gamma^{(12)}_{i,\eta_i}| \simeq \eta_i^2$

Proof. Let $\Lambda_i^{(k)} := \partial\Omega_i^{(k)} \cap \partial\Omega_i$. As in [30, Lemma 4.3], we integrate the identity

$$u(x)^2 + u(y)^2 - 2\,u(x)\,u(y) \;=\; \big[u(x) - u(y)\big]^2$$

with respect to x over $\Lambda_i^{(k)}$ and with respect to y over $\Gamma^{(12)}_{i,\eta_i}$. From our assumption that $\int_{\Gamma^{(12)}_{i,\eta_i}} u\,ds = 0$ and by dropping a positive term we obtain

$$|\Gamma^{(12)}_{i,\eta_i}|\,\|u\|^2_{L^2(\Lambda_i^{(k)})} \;\leq\; \int_{\Lambda_i^{(k)}} \int_{\Gamma^{(12)}_{i,\eta_i}} |u(x) - u(y)|^2\,ds_y\,ds_x\,. \tag{3.3}$$

Combining (3.3) and Lemma 3.3 yields

$$\begin{aligned} |\Gamma^{(12)}_{i,\eta_i}|\,\|u\|^2_{L^2(\Lambda_i^{(k)})} \;&\leq\; \sum_{j:\omega^{(k)}_{i,j}\subset\Lambda_i^{(k)}} \;\sum_{\ell:\omega^{(k)}_{i,j}\subset\Gamma^{(12)}_{i,\eta_i}} \int_{\gamma^{(k)}_{i,j}} \int_{\gamma^{(k)}_{i,\ell}} |u(x) - u(y)|^2\,ds_x\,ds_y \\ &\lesssim\; \sum_{j:\omega^{(k)}_{i,j}\subset\Lambda_i^{(k)}} \;\sum_{\ell:\omega^{(k)}_{i,j}\subset\Gamma^{(12)}_{i,\eta_i}} \eta_i^d\,M_{j\ell}\,|u|^2_{H^1(P_{ij})}\,. \end{aligned}$$

Using the regularity of Ω_i and the η_i-regularity of Ω_{i,η_i}, it is easily shown that (i) the first sum contains $\mathcal{O}(|\Lambda_i^{(k)}|/\eta_i^{d-1})$ terms, (ii) the second sum contains $\mathcal{O}(|\Gamma^{(12)}_{i,\eta_i}|/\eta_i^{d-1})$ terms, and (iii) $M_{j\ell} \lesssim H_i/\eta_i$. Therefore, we can conclude that

$$|\Gamma^{(12)}_{i,\eta_i}|\,\|u\|^2_{L^2(\Lambda_i^{(k)})} \;\lesssim\; \frac{|\Lambda_i^{(k)}|}{\eta_i^{d-1}}\,\frac{|\Gamma^{(12)}_{i,\eta_i}|}{\eta_i^{d-1}}\,\eta_i^d\,\frac{H_i}{\eta_i}\,|u|^2_{H^1(\Omega^{(k)}_{i,\eta_i})}$$

and so

$$\frac{1}{\eta_i}\,\|u\|^2_{L^2(\Lambda_i^{(k)})} \;\lesssim\; \frac{|\Lambda_i^{(k)}|H_i}{\eta_i^d}\,|u|^2_{H^1(\Omega^{(k)}_{i,\eta_i})}\,. \tag{3.4}$$

Since $|\Lambda_i^{(k)}| \lesssim H_i^{d-1}$, we obtain inequality (3.2) with $\beta = d$.

Suppose now that $d = 3$ and $|\Gamma^{(12)}_{i,\eta_i}| \gtrsim H_i\,\eta_i$. Using an analogous overlapping argument as in [30, Lemma A.3], one can show that the paths $P_{j\ell}$ connecting the boundary patches with the interface patches can be grouped in such a way that we save one power of H_i/η_i in (3.4), i. e.,

$$\frac{1}{\eta_i}\,\|u\|^2_{L^2(\Lambda_i^{(k)})} \;\lesssim\; \frac{|\Lambda_i^{(k)}|}{\eta_i^2}\,|u|^2_{H^1(\Omega^{(k)}_{i,\eta_i})}\,,$$

from which one easily deduces (3.2). □

To state our weighted Poincaré type inequality, we set on each of the subregions $\Omega^{(k)}_{i,\eta_i}$

$$\underline{\alpha}^{(k)}_{i,\eta_i} := \min_{x\in\Omega^{(k)}_{i,\eta_i}} \alpha(x), \qquad \overline{\alpha}^{(k)}_{i,\eta_i} := \max_{x\in\Omega^{(k)}_{i,\eta_i}} \alpha(x), \tag{3.5}$$

such that $\underline{\alpha}^{(k)}_{i,\eta_i} \leq \alpha(x) \leq \overline{\alpha}^{(k)}_{i,\eta_i}$ for all $x \in \Omega^{(k)}_{i,\eta_i}$.

Lemma 3.5 (weighted Poincaré inequality). *Let $\eta_i > 0$ and let Ω_{i,η_i} be η_i–regular. Suppose that the partitioning* (3.1) *is compatible and $u \in H^1(\Omega_{i,\eta_i})$ with $\int_{\Gamma^{(12)}_{i,\eta_i}} u\, ds = 0$. Then,*

$$\frac{1}{\eta_i^2} \int_{\Omega_{i,\eta_i}} \alpha(x)\, |u(x)|^2\, dx \;\lesssim\; C^*_{i,\eta_i} \int_{\Omega_{i,\eta_i}} \alpha(x)\, |\nabla u(x)|^2\, dx \tag{3.6}$$

with

$$C^*_{i,\eta_i} := \left(\frac{H_i}{\eta_i}\right)^d \max_{k=1,2} \frac{\overline{\alpha}^{(k)}_{i,\eta_i}}{\underline{\alpha}^{(k)}_{i,\eta_i}}\,.$$

If $d = 3$ and $|\Gamma^{(12)}_{i,\eta_i}| \gtrsim H_i\, \eta_i$ (e. g. in Fig. 2, left), then (3.6) *holds with $C^*_{i,\eta_i} = \left(\frac{H_i}{\eta_i}\right)^2 \max_{k=1,2} \frac{\overline{\alpha}^{(k)}_{i,\eta_i}}{\underline{\alpha}^{(k)}_{i,\eta_i}}$.*

Proof. Let $k \in \{1, 2\}$. Since Ω_{i,η_i} is η_i–regular and the partitioning (3.1) is compatible, we can partition $\Omega^{(k)}_{i,\eta_i}$ into a union of patches from the set Ξ_{i,η_i} (cf. Definition 2.6) denoted by $\{\omega^{(k)}_{i,j}\}$, and

$$\frac{1}{\eta_i^2}\, \|u\|^2_{L^2(\Omega^{(k)}_{i,\eta_i})} \;=\; \sum_j \frac{1}{\eta_i^2}\, \|u\|^2_{L^2(\omega^{(k)}_{i,j})}\,.$$

As above let $\Lambda^{(k)}_i := \partial\Omega^{(k)}_i \cap \partial\Omega_i$. Then, by Definition 2.6, $\Lambda^{(k)}_i$ is the union of faces $\gamma^{(k)}_{i,j}$ of the patches $\{\omega^{(k)}_{i,j}\}$. Hence, by a standard Friedrichs type inequality on each patch $\omega^{(k)}_{i,j}$, we have

$$\frac{1}{\eta_i^2}\, \|u\|^2_{L^2(\Omega^{(k)}_{i,\eta_i})} \;\lesssim\; \sum_j \left\{ |u|^2_{H^1(\omega^{(k)}_{i,j})} + \frac{1}{\eta_i}\, \|u\|^2_{L^2(\gamma^{(k)}_{i,j})} \right\} \;=\; |u|^2_{H^1(\Omega^{(k)}_{i,\eta_i})} + \frac{1}{\eta_i}\, \|u\|^2_{L^2(\Lambda^{(k)}_i)}\,. \tag{3.7}$$

Now, on each of the parts $\Lambda^{(k)}_i$ we can apply inequality (3.2) from Lemma 3.4 to bound the L_2–term on the right hand side of (3.7), such that

$$\begin{aligned} \frac{1}{\eta_i^2} \int_{\Omega_{i,\eta_i}} \alpha(x)\, |u(x)|^2\, dx \;&\lesssim\; \sum_{k=1,2} \overline{\alpha}^{(k)}_{i,\eta_i} \left(\frac{H_i}{\eta_i}\right)^\beta |u|^2_{H^1(\Omega^{(k)}_{i,\eta_i})} \\ &\lesssim\; \left(\frac{H_i}{\eta_i}\right)^\beta \max_{k=1,2} \frac{\overline{\alpha}^{(k)}_{i,\eta_i}}{\underline{\alpha}^{(k)}_{i,\eta_i}} \int_{\Omega_{i,\eta_i}} \alpha(x)\, |\nabla u(x)|^2\, dx, \end{aligned}$$

where $\overline{\alpha}^{(k)}_{i,\eta_i}$ and $\underline{\alpha}^{(k)}_{i,\eta_i}$ are defined in (3.5). □

We finish this subsection with a few remarks on Lemma 3.5. First of all note that the Poincaré constant in Lemma 3.5 only depends on the local variation of α on each subregion $\Omega^{(k)}_{i,\eta_i}$ of Ω_{i,η_i}, and is completely independent of the values and the variation of α in the interior of Ω_i and of the contrast between the two regions $\Omega^{(1)}_{i,\eta_i}$ and $\Omega^{(2)}_{i,\eta_i}$. In particular, if $\alpha(x)$ is constant on each of the regions $\Omega^{(k)}_{i,\eta_i}$ then $C^*_{i,\eta_i} = (H_i/\eta_i)^\beta$ (with β depending on the measure of the interface $\Gamma^{(12)}_{i,\eta_i}$).

Note also that there is no restriction on the size of η_i in Lemma 3.5. In particular, if $\eta_i > H_i/2$, then $\Omega_{i,\eta_i} = \Omega_i$ and (3.6) reduces to an inequality on all of Ω_i, i.e.

$$\frac{1}{H_i^2} \int_{\Omega_i} \alpha(x)\, |u(x)|^2\, dx \;\lesssim\; \max_{k=1,2} \frac{\overline{\alpha}^{(k)}_{i,\eta_i}}{\underline{\alpha}^{(k)}_{i,\eta_i}} \int_{\Omega_i} \alpha(x)\, |\nabla u(x)|^2\, dx.$$

Finally, if one of the subregions, say $\Omega^{(2)}_{i,\eta_i}$, is empty, we can set $\Gamma^{(12)}_{i,\eta_i} := \partial\Omega_i$, and (3.6) reduces to

$$\frac{1}{\eta_i^2} \int_{\Omega_{i,\eta_i}} \alpha(x)\, |u(x)|^2\, dx \;\lesssim\; \left(\frac{H_i}{\eta_i}\right)^2 \frac{\overline{\alpha}^{(1)}_{i,\eta_i}}{\underline{\alpha}^{(1)}_{i,\eta_i}} \int_{\Omega_{i,\eta_i}} \alpha(x)\, |\nabla u(x)|^2\, dx.$$

Obviously, Lemma 3.5 can be generalised in a straightforward way to $M_i > 2$ subregions $\Omega^{(k)}_{i,\eta_i}$ by introducing $M_i - 1$ functionals $\int_{\Gamma^{(k\ell)}_{i,\eta_i}} u\, ds$ acting on appropriately chosen interfaces $\Gamma^{k\ell}_{i,\eta_i}$. However, since there is only one degree of freedom per subdomain in our coarse space (see Lemma 5.2 below),

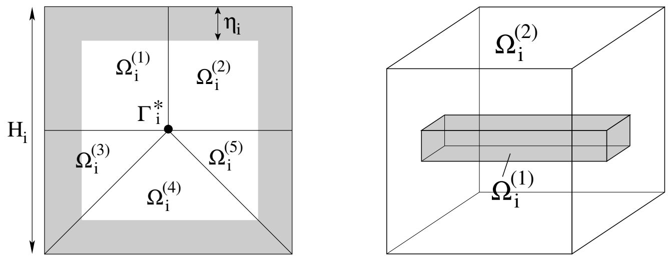

Fig. 3 *Left* multiple coefficient regions, admissible for Lemma 3.6, in grey: $\Omega^{(k)}_{i,\eta_i}$, *right* beam type example

this will be of no use here. Instead, we prove a weighted discrete Sobolev type inequality in the next subsection. Weighted Poincaré inequalities with more than one functional (based on [4]) were used by Xu and Zhu [47] recently to prove coefficient–robustness of geometric multigrid in the case of piecewise constant coefficients that are resolved by the coarse meshes. A very similar weighted Poincare inequality to the one we gave here (with one functional) was also recently proved by Galvis and Efendiev [14] and used in the analysis of two-level overlapping Schwarz.

3.2 Weighted discrete Sobolev inequality – multiple coefficient regions

Let us again consider a single subdomain Ω_i, but now let $\{\Omega_i^{(k)}\}_{k=1,\ldots,M_i}$ be a non-overlapping partition of Ω_i into M_i connected subregions $\Omega_i^{(k)}$, i.e.

$$\overline{\Omega}_i = \overline{\Omega}_i^{(1)} \cup \ldots \cup \overline{\Omega}_i^{(M_i)}, \tag{3.8}$$

with a non-empty intersection

$$\Gamma_i^* := \bigcap_{k=1}^{M_i} \overline{\Omega}_i^{(k)}. \tag{3.9}$$

For technical reasons, we have to assume that each of the subregions $\Omega_i^{(k)}$ is a regular domain in the sense of Definition 2.1, with $\operatorname{diam}(\Omega_i^{(k)}) \simeq H_i$. Note that this implies $M_i = \mathcal{O}(1)$, see also Fig. 3, left.

Now, let $\eta_i > 0$ and let $\Omega^{(k)}_{i,\eta_i} := \Omega_i^{(k)} \cap \Omega_{i,\eta_i}$, i.e. the part of $\Omega_i^{(k)}$ in the boundary layer. Assume that Ω_{i,η_i} is η_i-regular. Analogously to Definition 3.1, we say that the partition (3.8) is *compatible* if each of the subregions $\Omega^{(k)}_{i,\eta_i}$ is a union of patches from Definition 2.6 and the intersection Γ_i^* a union of faces, edges, and vertices of these patches. For each $\Omega^{(k)}_{i,\eta_i}$, let $\underline{\alpha}^{(k)}_{i,\eta_i}$ and $\overline{\alpha}^{(k)}_{i,\eta_i}$ be as defined in (3.5).

Lemma 3.6 (weighted discrete Sobolev inequality)**.** *Let $\eta_i > 0$ and let Ω_{i,η_i} be η_i–regular. Suppose that the partitioning* (3.8) *is compatible and that*

$$\alpha(x) \;\gtrsim\; \underline{\alpha}^{(k)}_{i,\eta_i} \qquad \forall x \in \Omega_i^{(k)} \quad \forall k = 1, \ldots, M_i\,. \tag{3.10}$$

Let $\mathcal{T}_{H_i}(\Omega_i)$ denote the coarse triangulation in Definition 2.1, chosen such that it resolves each of the subregions $\Omega_i^{(k)}$, and assume that $X_i^* \subset \Gamma_i^*$ is a face, an edge, or a vertex of $\mathcal{T}_{H_i}(\Omega_i)$. Then, for all $u \in V^h(\Omega_i)$ with $\int_{X_i^*} u\, ds = 0$ (or with $u(X_i^*) = 0$, if X_i^* is a vertex), we have

$$\frac{1}{\eta_i^2} \int_{\Omega_{i,\eta_i}} \alpha(x)\, |u(x)|^2\, dx \;\lesssim\; C^*_{i,\eta_i} \int_{\Omega_i} \alpha(x)\, |\nabla u(x)|^2\, dx \tag{3.11}$$

with

$$C^*_{i,\eta_i} := \sigma(H_i/h_i)\, \frac{H_i}{\eta_i}\, \max_{k=1}^{M_i} \frac{\overline{\alpha}^{(k)}_{i,\eta_i}}{\underline{\alpha}^{(k)}_{i,\eta_i}}$$

and

$$\sigma(H_i/h_i) := \begin{cases} 1 & \text{if } X_i^* \text{ is a face (resp. edge) of } \mathcal{T}_{H_i}(\Omega_i) \text{ and } d = 3 \text{ (resp. } d = 2\text{)}, \\ (1 + \log(H_i/h_i)) & \text{if } X_i^* \text{ is an edge (resp. vertex) of } \mathcal{T}_{H_i}(\Omega_i) \text{ and } d = 3 \text{ (resp. } d = 2\text{)}, \\ H_i/h_i & \text{if } X_i^* \text{ is a vertex and } d = 3. \end{cases}$$

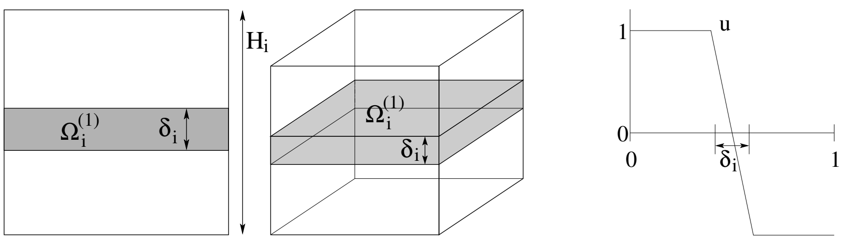

Fig. 4 *Left* Plate type examples in two and three dimensions, *right* counterexample, see Sect. 3.3

Proof. The proof is quite similar to that of Lemma 3.5. However, instead of applying Lemma 3.4 to (3.7) we use the following discrete Sobolev inequality on each of the subregions $\Omega_i^{(k)}$:

$$\|u\|^2_{L^2(\partial\Omega_i^{(k)} \cap \partial\Omega_i)} \lesssim H_i\, \sigma(H_i/h_i)\, |u|^2_{H^1(\Omega_i^{(k)})}\,. \qquad (3.12)$$

This inequality follows from [44, Lemma 4.15, Lemma 4.21, and Sect. 4.6.1]. □

Note that provided the coefficient $\alpha(x)$ fulfils condition (3.10), the constant C^*_{i,η_i} in Lemma 3.6 depends only on the local variation of α within the boundary layer $\Omega^{(k)}_{i,\eta_i}$ of each subregion, but not on the contrast in the value of $\alpha(x)$ between subregions. In particular, if α is constant in each $\Omega^{(k)}_{i,\eta_i}$ then C^*_{i,η_i} is completely independent of variations of α. Note also that we do not require any upper bound on $\alpha(x)$ in the subdomain *interior*. The size of the constant C^*_{i,η_i} is completely independent of the size of $\alpha(x)$ in the interior and so $\alpha(x)$ can be *arbitrarily large* in the interior.

Remark 3.7. The beam type setting in Figure 3, right, cannot be treated with the weighted Poincaré inequality in Lemma 3.5 because the beam is not connected in the boundary layer. However, it can be treated with Lemma 3.6 since the intersection of the two subregions contains a face of $\mathcal{T}_{H_i}(\Omega_i)$.

The statements of both lemmas also hold under weaker assumptions, and we will present a series of such extensions in Section 6. A more general framework for weighted Poincaré inequalities based on a generalisation of the notion of quasi-monotonicity in [12,39] has recently been developed in [34, 33].

3.3 Counterexample

A simple example where neither Lemma 3.5 nor Lemma 3.6 can be applied is the situation depicted in Fig. 4, i. e. a "plate"–type subregion $\Omega_i^{(1)}$ (in grey) separating the remainder of Ω_i into two disconnected regions. If α is chosen, such that $\alpha(x) = \varepsilon \ll 1$, for all $x \in \Omega_i^{(1)}$, and $\alpha(x) = 1$ in the remainder of Ω_i, then neither the assumptions of Lemma 3.5 nor those of Lemma 3.6 are satisfied (for any choice of η_i). Moreover, choosing u as depicted in Fig. 4, right, in the direction perpendicular to the plate (vertical) and constant in the other direction(s), we get

$$\frac{1}{2}\, H_i^d \;\le\; \int_{\Omega_i} \alpha(x)\, |u(x)|^2\, dx \;\le\; H_i^d \qquad \text{and} \qquad \int_{\Omega_i} \alpha(x)\, |\nabla u(x)|^2\, dx \;=\; 4\,\varepsilon\, \frac{H_i^{d-1}}{\delta_i},$$

for any $0 < \delta_i < H/2$, showing that the constant C^*_{i,H_i} in Lemma 3.5 must depend on the size of the jump in $\alpha(x)$ in this example and $C^*_{i,H_i} \ge \varepsilon^{-1}\, \frac{\delta_i}{4\,H_i}$.

Note that for $\varepsilon \gg 1$ the assumptions of Lemmas 3.5 and 3.6 are not fulfilled either, but in contrast, here a robust inequality can be proved (see Section 6).

4 FETI methods for multiscale elliptic PDEs

In this section, we first describe briefly the classical one-level and the all-floating FETI method, paying particular attention to the proposed modifications in the multiscale context. The section ends with the main result of the paper, Theorem 4.1.

4.1 Description of one-level and all-floating FETI

Following [44, Sect. 6.3], we introduce separate unknowns $u_i \in V^h(\Omega_i)$ on the subdomains and denote by $u = [u_1, \ldots, u_N]^\top$ the discontinuous approximation of $\widetilde{u}$ in $\prod_{i=1}^N V^h(\Omega_i)$. The continuity of the solution is enforced by constraints of the form

$$u_i(x^h) - u_j(x^h) \;=\; 0 \qquad \text{for } x^h \in \Gamma_{ij}^h\,. \tag{4.1}$$

Note that at nodes where more than two subdomains meet this introduces redundancies. In this work, we consider only fully redundant constraints, i. e., the full set of possible constraints is used. In the all-floating formulation (cf. [10, 27]), we incorporate the Dirichlet boundary conditions by additional constraints of the form

$$u_i(x^h) - g_D(x^h) \;=\; 0 \qquad \text{for } x^h \in \partial\Omega_i^h \cap \partial\Omega. \tag{4.2}$$

Note that unlike the continuity constraints (4.1), the Dirichlet constraints (4.2) are completely local to each subdomain. Let N_C denote the total number of constraints in (4.1) and (4.2) and set $U := \mathbb{R}^{N_C}$. Then we can write (4.1) and (4.2) compactly as

$$\sum_{i=1}^N B_i\, u_i \;=\; b \;\in\; U, \qquad \text{or equivalently,} \qquad B\, u \;=\; b. \tag{4.3}$$

The operators $B_i : V^h(\Omega_i) \to U$ can be represented by signed Boolean matrices. The full jump operator $B : \prod_{i=1}^N V^h(\Omega_i) \to U$ is defined as $B := [B_1, \ldots, B_N]$. The entries of the vector $b \in U$ that correspond to the constraints in (4.2) contain the values $g_D(x^h)$. All other entries of b are zero. By a simple energy minimisation argument, it follows from the above that solving (2.8) is equivalent to finding $u = [u_1, \ldots, u_N]^\top$ and $\lambda \in U$ satisfying the saddle point system

$$\begin{pmatrix} K_1 & & 0 & B_1^\top \\ & \ddots & & \vdots \\ 0 & & K_N & B_N^\top \\ B_1 & \cdots & B_N & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_N \\ \lambda \end{pmatrix} \;=\; \begin{pmatrix} f_1 \\ \vdots \\ f_N \\ b \end{pmatrix}. \tag{4.4}$$

System (4.4) is uniquely solvable modulo an element in $\{[0, \ldots, 0]^\top\} \times \ker B^\top$ if and only if the block $K := \operatorname{diag}(K_i)$ is SPD on $\ker B$, or equivalently, $\ker K \cap \ker B = \{0\}$. This condition is true whenever the Dirichlet boundary is non-empty. We refer to U as the space of Lagrange multipliers. The more classical one-level FETI formulation leads to a very similar saddle point system. See [30, 44] for details.

Recall that each of the operators K_i is only positive semi-definite and $\ker K_i = \operatorname{span}\{\mathbf{1}_{\Omega_i}\}$. We introduce operators $R_i : \mathbb{R} \to V^h(\Omega_i) : \xi_i \mapsto \xi_i\, \mathbf{1}_{\Omega_i}$ such that $\operatorname{range} R_i = \ker K_i$. Let $K_i^\dagger$ denote some pseudoinverse of K_i. Then, under the compatibility condition

$$f_i - B_i^\top \lambda \;\in\; \operatorname{range} K_i\,,$$

we can eliminate the unknowns u_i from (4.4), i. e.

$$u_i \;=\; K_i^\dagger [f_i - B_i^\top \lambda] + R_i\, \xi_i\,, \tag{4.5}$$

for some $\xi = [\xi_i]_{i=1}^N$. Finally, using that $\operatorname{range} K_i = \ker R_i^\top$ and with the abbreviations

$$\begin{aligned} &K := \operatorname{diag}(K_i), && f := [f_1, \ldots, f_N]^\top, && R := \operatorname{diag}(R_i), && K^\dagger := \operatorname{diag}(K_i^\dagger), \\ &F := B\, K^\dagger\, B^\top, && G := B\, R, && d := B\, K^\dagger f - b, && e := R^\top f, \end{aligned}$$

we arrive at the dual formulation (see [44] for details): Find $(\lambda, \xi) \in U \times \mathbb{R}^N$ such that

$$\begin{pmatrix} F & -G \\ G^\top & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \xi \end{pmatrix} \;=\; \begin{pmatrix} d \\ e \end{pmatrix}. \tag{4.6}$$

In practice, this saddle point system is solved using a projection

$$P : U \to \ker G^\top \subset U \qquad \text{such that} \qquad P := I - Q\,G\,(G^\top Q\,G)^{-1}\,G^\top, \tag{4.7}$$

where a careful choice of the SPD scaling operator $Q : U \to U$ will be crucial to render the method robust to coefficient variation (see (4.14)–(4.15) below). Introducing the subspace

$$V := \{\lambda \in U : \langle B\,z\,, \lambda\rangle = 0 \quad \forall z \in \ker K\} \;=\; \ker G^\top \;=\; \operatorname{range} P \tag{4.8}$$

and using the projection operator the saddle point system (4.6) can be reduced to solving

$$P^\top F\,\widetilde{\lambda} \;=\; P^\top (d - F\,\lambda_0), \tag{4.9}$$

for $\widetilde{\lambda} \in V$, where $\lambda_0 = Q\,G(G^\top Q\,G)^{-1}e$. The original variables λ and ξ can then be recovered from the relations $\lambda = \lambda_0 + \widetilde{\lambda}$ and $\xi = (G^\top Q\,G)^{-1} G^\top Q\,(F\,\lambda - d)$.

Several things are worth mentioning. First note that $G^\top Q\,G$ is the Galerkin projection of $B^\top Q\,B$ onto $\ker K$. Since $\ker K \cap \ker B = \{0\}$, the operator $G^\top Q\,G$ is invertible as long as Q is SPD on $\operatorname{range} G$. Furthermore, since equation (4.9) is SPD on the subspace V modulo $\ker B^\top$, it can be solved using a projected preconditioned conjugate gradient method. The actual solution u can finally be recovered using (4.5). Note that even if λ is only unique modulo an element in $\ker B^\top$, the solution u is always unique (see e. g. [28] for a more detailed discussion). The crucial ingredients that will make the method robust with respect to varying coefficients are the choice of Q and of the preconditioner M^{-1}. In the sequel, we present suitable choices for Q and M^{-1}, generalising the method analysed by Klawonn and Widlund and building on the results in [30, §5.2 & 5.3].

4.2 Choice of Q and M^{-1} for varying coefficients

We follow [30, §5.2]. In order to define M^{-1} we need to introduce scaling operators D_i for the Boolean matrices B_i on the space U of Lagrange multipliers. For each subdomain Ω_i and for each $x^h \in \partial\Omega_i$, we define the pointwise weight

$$\widehat{\alpha}_i(x^h) := \max_{\tau \subset \omega_i(x^h)} \alpha_{|\tau}\,, \tag{4.10}$$

where $\omega_i(x^h) \subset \overline{\Omega}_i$ is the (local) patch of all elements in $\mathcal{T}_i$ that contain node x^h (see [30, Fig. 5]). Furthermore, we define the weighted counting functions

$$\delta_i^\dagger(x^h) := \begin{cases} \widehat{\alpha}_i(x^h)\Big[\sum\limits_{k\in\mathcal{N}(x^h)} \widehat{\alpha}_k(x^h)\Big]^{-1} & \text{for } x^h \in \partial\Omega_i^h\,,\\ 0 & \text{for } x^h \in \Gamma_S^h \setminus \partial\Omega_i^h\,, \end{cases} \tag{4.11}$$

where $\mathcal{N}(x^h) := \{k : x^h \in \partial\Omega_k\}$, the index set of the subdomains sharing node $x^h \in \Gamma^h$. Each function $\delta_i^\dagger(\cdot)$ can be interpreted as a finite element function on the skeleton Γ_S, and the set of all these functions provides a partition of unity on the skeleton, cf. [44, Section 6.2.1]. Now, to define D_i, let $\lambda_{ij}(x^h)$ denote the component of $\lambda \in U$ which corresponds to the constraint (of type (4.1)) at an interface node $x^h \in \Gamma_{ij}^h$ and let $\lambda_{iD}(x^h)$ denote the component of λ corresponding to the constraint (of type (4.2)) at a Dirichlet node $x^h \in \partial\Omega_i^h \cap \partial\Omega$. Let $D_i : U \to U$ be the diagonal matrix such that

$$\begin{cases} (D_i\lambda)_{ij}(x^h) := \delta_j^\dagger(x^h)\lambda_{ij}(x^h) & \text{for } x^h \in \Gamma_{ij}^h\,,\\ (D_i\lambda)_{iD}(x^h) := \lambda_{iD}(x^h) & \text{for } x^h \in \partial\Omega_i^h \cap \partial\Omega. \end{cases} \tag{4.12}$$

Then our FETI preconditioner is chosen to be

$$M^{-1} := \sum_{i=1}^N D_i\,B_i \begin{pmatrix} S_i & 0\\ 0 & 0\end{pmatrix} B_i^\top D_i\,. \tag{4.13}$$

where the S_i are the Schur complements of the stiffness matrices K_i defined in (2.5).

The linear operator Q which appears in the projection P in (4.7) is (usually) also set to be a diagonal matrix with

$$\begin{cases} (Q\,\lambda)_{ij}(x^h) := \min(\widehat{\alpha}_i(x^h),\, \widehat{\alpha}_j(x^h))\, q_i(x^h)\, \lambda_{ij}(x^h) & \text{for } x^h \in \Gamma_{ij}^h\,, \\ (Q\,\lambda)_{iD}(x^h) := \widehat{\alpha}_i(x^h)\, q_i(x^h)\, \lambda_{iD}(x^h) & \text{for } x^h \in \partial\Omega_i^h \cap \partial\Omega, \end{cases} \tag{4.14}$$

where, in three dimensions,

$$q_i(x^h) := \begin{cases} (1+\log(H_i/h_i))\dfrac{h_i^2}{H_i} & \text{if } x^h \text{ lies on a subdomain face,} \\ h_i & \text{if } x^h \text{ lies on a subdomain edge or vertex.} \end{cases} \tag{4.15}$$

In two dimensions, $q_i(x^h) = (1+\log(H_i/h_i))\, h_i/H_i$ for subdomain edges and $q_i(x^h) = 1$ for vertices. Note that $q_i(x^h) \simeq q_j(x^h)$ for neighbouring subdomains since $H_i \simeq H_j$ and $h_i \simeq h_j$. If the coefficient α is piecewise constant with respect to the subdomains, our choices of M^{-1} and Q coincide with the ones given in [26,27] (which builds on [21]).

In each step of the projected preconditioned conjugate gradient method, we have to apply $P^\top F$ and $P\,M^{-1}$. Therefore, the main ingredients of FETI methods are local Dirichlet and regularised Neumann solves on the subdomains Ω_i as well as a coarse solve with the operator $G^\top Q\, G$, which is sparse, see [30,44]. We assume that these types of problems can be handled by direct solvers.

We remark that if we do not impose the Dirichlet boundary conditions by Lagrange multipliers but incorporate them in the local spaces $V(\Omega_i)$, we obtain the standard one-level FETI method, cf. [30,44]. There, some of the operators K_i are regular, thus some of the kernel spanning operators R_i are not needed, and the dimension of the coarse space gets smaller, but otherwise the setup and the choice of Q and M^{-1} is the same.

4.3 Main result

We are now ready to state the main result of this paper and present new condition number bounds for the standard one-level and for the all-floating FETI method. The proofs are postponed to Section 5. Before we give these results, we need one final technical assumption on the local variation of $\alpha(x)$.

Assumption A3. Let $i \in \{1,\ldots,N\}$ and let $\omega_{i,j} \in \Xi_{i,\eta_i}$ be one of the patches covering Ω_{i,η_i} in Definition 2.6. We assume that $\widehat{\alpha}_i(x^h)$ is constant on each face f and on each edge e of $\partial\omega_{i,j} \cap \partial\Omega_i$, i. e. we assume that the coefficients do not vary locally on any of the patch (boundary) faces or edges in any of the subdomain boundary layers.

Recall that faces and edges are always considered to be open, i. e. they do not contain their boundary nodes. Thus, large jumps of $\alpha(x)$ across the subregions $\Omega_i^{(k)}$ are of course still allowed. In Section 6, we will sketch how we can weaken Assumption A3.

Theorem 4.1 (all-floating FETI)**.** *Let $\{\eta_i\}$ and $\alpha(x)$ be such that Assumptions A1, A2 and A3 hold. On each subdomain let C^*_{i,η_i} be the minimum of the values of C^*_{i,η_i} in Lemma 3.5 and in Lemma 3.6 with compatible partitionings* (3.1) *and* (3.8)*, respectively, such that* (3.10) *holds. Then the condition number κ of the preconditioned all-floating FETI system described above satisfies*

$$\kappa \;\lesssim\; \max_{j=1}^{N} \frac{H_j}{\eta_j}\;\max_{i=1}^{N}\Big(C^*_{i,\eta_i}\,(1+\log(H_i/h_i))^2\Big). \tag{4.16}$$

The hidden constant is independent of H_i, η_i, h_i and N, as well as of the contrast in the coefficient α. It does depend on the local variations of α on each patch $\omega_{i,j}$. The constants C^*_{i,η_i} depend on (i) $\max_k \overline{\alpha}^{(k)}_{i,\eta_i}/\underline{\alpha}^{(k)}_{i,\eta_i}$, (ii) a power of H_i/η_i, and possibly (iii) a logarithmic or linear term in H_i/h_i (see Section 3 for details).

Proof. Postponed to Section 5. □

Corollary 4.2 (one-level FETI). *Let the assumptions of Theorem 4.1 hold. In addition, for any non-floating subdomain Ω_i (i. e. for any subdomain that touches the Dirichlet boundary), let $\Omega_{i,\eta_i}^{(k)}$ in* (3.1) *and $\Omega_i^{(k)}$ in* (3.8) *be such that each of them touches the Dirichlet boundary at least in an edge (resp. vertex) of the triangulation $\mathcal{T}_{H_i}(\Omega_i)$ in Definition 2.1 (introduced in Lemma 3.6) in three (resp. two) dimensions. Then the condition number κ of the one-level FETI system also satisfies* (4.16).

We finish this section with a few remarks. First, we would like to illustrate the result of Theorem 4.1 for a special case: If $\eta_i \simeq H_i$, $\alpha(x)$ is constant (or mildly varying) in each of the subregions $\Omega_{i,\eta_i}^{(k)}$, and the interface Γ_i^* from Lemma 3.6 can be chosen to contain at least a coarse edge in three dimensions, then

$$\kappa \;\lesssim\; (1+\log(H_i/h_i))^3$$

at worst. Moreover, if the number of subregions $M_i \le 2$ for all subdomains Ω_i, then Lemma 3.5 applies and the cubic dependence is reduced to a quadratic one. Secondly, we would like to emphasize that if we can apply Lemma 3.5 in each subdomain Ω_i the estimates are completely independent of the values of $\alpha(x)$ in the subdomain interiors $\Omega_i \setminus \Omega_{i,\eta_i}$, see also [30, Remark 3.5].

Statements similar to that of Theorem 4.1 can also be shown under weaker assumptions. We present such extensions in Section 6. There and in Section 7 we will also see that the extra factor of $\max_{j=1}^N H_j/\eta_j$ in (4.16) is an artefact of our proof which we do not observe numerically. However, we are only able to eliminate it in the case that Lemma 3.6 applies in each subdomain (see Theorem 6.3 below), or by making a different choice of Q in the FETI preconditioner (see Theorem 6.4 below).

5 Proof of Theorem 4.1 and Corollary 4.2

As in our previous work [30], we give proofs for the three-dimensional case; the two-dimensional case is analogous. Section 5.1 introduces an abstract framework on the operator level, similar to that of the usual proof, cf. [44, Sect. 6.3]. Section 5.2 introduces some technical tools which we need specifically for the case of varying coefficients. Finally, in Section 5.3, we prove the crucial estimate for the projection operator defined in (5.2) below. In contrast to the proof idea outlined in [30, Sect. 4] our proof below reduces the crucial bounds to standard results on *patches* $\omega_{i,j}$ (cf. Definition 2.6) rather than on subdomains. We concentrate on the all-floating formulation and the proof of Theorem 4.1. In Section 5.4 we sketch how to extend the proof to one-level FETI and thus prove Corollary 4.2.

5.1 Abstract framework

First, we define the spaces

$$W_i := V^h(\partial\Omega_i), \qquad W := \prod_{i=1}^N W_i\,,$$

in which we will carry out the analysis, we write $S_i : W_i \to W_i$, and we define $S : W \to W$ by $S := \operatorname{diag}(S)$. We recall that each S_i induces the seminorm $|w_i|_{S_i} := \langle S_i\, w_i,\, w_i\rangle^{1/2}$, cf. (2.6), and that $|w_i + c|_{S_i} = |w_i|_{S_i}$ for any constant $c \in \mathbb{R}$. On the product space W we define

$$|w|_S := \Big(\sum_{i=1}^N |w_i|_{S_i}^2\Big)^{1/2} \qquad \text{for } w \in W.$$

Furthermore, we define the space

$$V' := \{\mu \in U : \langle B\,z,\, Q\,\mu\rangle = 0 \quad \forall z \in \ker K\} \;=\; \operatorname{range} P^\top, \tag{5.1}$$

which can be shown to be isomorphic to the space V defined in (4.8). In the following, we assume that $U = \operatorname{range} B$, i. e., $\ker B^\top = \{0\}$. The general case, where we have to work in the factor spaces modulo $\ker B^\top$, follows then from this special case; for details see e. g. [28,44]. Finally, we define the operator

$$P_D := \big[B_1^\top D_1\, B_1 \,\big|\, \ldots \,\big|\, B_N^\top D_N B_N\big], \tag{5.2}$$

which can be shown to be a projection fulfilling $B\,P_D = B$. In other words, $I - P_D$ is a projection to the functions that are continuous across the subdomain interfaces and that fulfil the homogeneous Dirichlet conditions on $\partial\Omega$. Analogously to [21] we can show that

$$(P_D\, w)_i(x^h) \;=\; \begin{cases} \sum_{j\in\mathcal{N}(x^h)} \delta_j^\dagger(x^h)\,\big[w_i(x^h) - w_j(x^h)\big] & \text{for } x^h \in \partial\Omega_i^h \cap \Gamma, \\ w_i(x^h) & \text{for } x^h \in \partial\Omega_i^h \cap \partial\Omega. \end{cases} \tag{5.3}$$

In the following, we will regard P_D as an operator mapping W to W, because B acts only on degrees of freedom on the subdomain boundaries. Similarly, we will occasionally regard B as a mapping from W to U, and $B^\top : U \to W$. Note also that $\ker S_i = \operatorname{span}\{\mathbf{1}_{\partial\Omega_i}\}$. As a second important identity we have

$$B^\top M^{-1} B \;=\; P_D^\top S\, P_D\,. \tag{5.4}$$

Using the fact that P_D is a projection, it can be shown that the preconditioner $P\,M^{-1}$ is SPD as a mapping from V' to V as long as Q is SPD on $\operatorname{range} G = B\,(\ker K)$; for details see e. g. [21,28]. Therefore, $P\,M^{-1}$ has a well-defined SPD inverse $M : V \to V'$. To show our bound on κ we show the spectral bounds

$$\langle M\,\lambda,\,\lambda\rangle \;\le\; \langle F\,\lambda,\,\lambda\rangle \;\lesssim\; C^*\,\langle M\,\lambda,\,\lambda\rangle \qquad \forall \lambda \in V, \tag{5.5}$$

with

$$C^* := \Big(\max_{j=1}^{N} \frac{H_j}{\eta_j}\Big)\; \max_{i=1}^{N} \Big(C^*_{i,\eta_i}\,(1 + \log(H_i/h_i))^2\Big).$$

The lower bound in (5.5) can be shown by algebraic arguments following [44, Theorem 6.15], independently of our particular choices of Q and D_i. Using similar algebraic arguments the upper bound can be reduced to an estimate in the space W, which we will give after the following lemma.

Lemma 5.1. *For any $w \in W$, there exists a unique $z_w \in \ker S$ such that $B(w + z_w) \in V'$. Moreover,*

$$\|B\, z_w\|_Q \;\le\; \|B\, w\|_Q\,,$$

where $\|\mu\|_Q := \langle \mu,\, Q\,\mu\rangle^{1/2}$. The unique element z_w is explicitly given by

$$z_w \;=\; \operatorname*{argmin}_{z\in\ker K} \|B(w+z)\|_Q \;=\; -R\,(G^\top Q\, G)^{-1} G^\top Q\, B\, w,$$

which shows that the mapping $w \mapsto z_w$ is linear. Furthermore, $w \mapsto -z_w$ is a projection onto $\ker S$, *i. e., the piecewise constant functions, and this projection is orthogonal with respect to the inner product induced by $B^\top Q\, B$.*

Proof. The proof follows directly from [44, Lemma 6.12]. □

As shown in [44, Sect. 6.3], the crucial estimate

$$|P_D(w + z_w)|_S^2 \;\lesssim\; C^*\, |w|_S^2 \qquad \forall w \in W. \tag{5.6}$$

implies the upper bound in (5.5). In order to apply our weighted Poincaré and Sobolev type inequalities from Lemmas 3.5 and 3.6, we define for each $i = 1, \ldots, N$ the linear functional $g_i : W_i \to \mathbb{R}$ by

$$g_i(w_i) := \begin{cases} \frac{1}{|\Gamma^{(12)}_{i,\eta_i}|} \int_{\Gamma^{(12)}_{i,\eta_i}} \mathcal{H}_{i,\alpha} w_i\, ds, & \text{if Lemma 3.5 is applied,} \\ \frac{1}{|X_i^*|} \int_{X_i^*} \mathcal{H}_{i,\alpha} w_i\, ds, & \text{if Lemma 3.6 is applied and } X_i^* \text{ is a face or edge of } \mathcal{T}_{H_i}(\Omega_i), \\ (\mathcal{H}_{i,\alpha}\, w_i)(X_i^*), & \text{if Lemma 3.6 is applied and } X_i^* \text{ is a vertex,} \end{cases}$$

depending on which lemma we wish to use, and the subspaces

$$W_i^\perp := \big\{w_i \in W_i : g_i(w_i) = 0\big\}, \qquad W^\perp := \prod_{i=1}^{N} W_i^\perp. \tag{5.7}$$

Above, $\mathcal{H}_{i,\alpha} w_i$ denotes the discrete α-harmonic extension of w_i, see equation (2.7).

Lemma 5.2. *Inequality* (5.6) *holds for all* $w \in W$ *if*

$$|P_D(w + z_w)|_S^2 \;\lesssim\; C^* \, |w|_S^2 \qquad \forall w \in W^\perp. \tag{5.8}$$

Proof. First, by Lemma 5.1, $z_y = -y$ for all $y \in \ker S$, and using the linearity, we obtain the invariance

$$w + z_w \;=\; (w + y) + z_{w+y} \qquad \forall w \in W\,,\; y \in \ker S.$$

Secondly, we have the invariance $|w + y|_S = |w|_S$ for all $w \in W$ and $y \in \ker S$. Thus both sides of inequality (5.6) are invariant if we add a y from $\ker S$ to w. Finally, for all $w \in W$ we can find a unique $y_w \in \ker S$ such that $w - y_w \in W^\perp$ by choosing $(y_w)_i := g_i(\mathcal{H}_{i,\alpha} w_i)$. □

5.2 Technical tools

To conclude our proof, we only need to show inequality (5.8). If $\alpha(x)$ is constant on each subdomain Ω_i, this inequality is shown using an additive splitting into terms corresponding to subdomain faces, edges, and vertices, which is motivated from formula (5.3). In our case, however, the functions $\delta_j^\dagger$ are in general no longer constant on such subdomain faces or edges, indeed they can have arbitrary large jumps. Therefore, we need a finer splitting into terms corresponding to faces, edges, and vertices of the patches forming Ω_{i,η_i}, cf. Definition 2.6. We denote by

- $\mathcal{F}_i = \{F\}$ the set of *subdomain* faces,
- $\mathcal{E}_i = \{E\}$ the set of *subdomain* edges,
- $\mathcal{V}_i = \{V\}$ the set of *subdomain* vertices

of Ω_i, and by

- $\mathbb{F}_i = \{f\}$ the set of *patch* faces,
- $\mathbb{E}_i = \{e\}$ the set of *patch* edges,
- $\mathbb{V}_i = \{v\}$ the set *patch* of vertices

with respect to patches from Ξ_{i,η_i} which are part of the subdomain boundary $\partial\Omega_i$. Recall that $\mathcal{F}_i^\Gamma$, $\mathcal{E}_i^\Gamma$, $\mathcal{V}_i^\Gamma$, and $\mathcal{F}_i^D$, $\mathcal{E}_i^D$, $\mathcal{V}_i^D$ denote the subsets of faces, edges, and vertices lying on Γ and $\partial\Omega$, respectively. Correspondingly, we define the subsets $\mathbb{F}_i^\Gamma$, $\mathbb{E}_i^\Gamma$, $\mathbb{V}_i^\Gamma$ and $\mathbb{F}_i^D$, $\mathbb{E}_i^D$, and $\mathbb{V}_i^D$. For convenience we also define $\mathcal{X}_i := \mathcal{F}_i \cup \mathcal{E}_i \cup \mathcal{V}_i$, and $\mathbb{X}_i := \mathbb{F}_i \cup \mathbb{E}_i \cup \mathbb{V}_i$. For each $\mathrm{x} \in \mathbb{X}_i$ we define the union of touching patches $\omega_i(\mathrm{x})$ by

$$\overline{\omega_i(\mathrm{x})} := \bigcup\nolimits_{\mathrm{x} \subset \partial\omega_{i,j}} \overline{\omega_{i,j}}\,. \tag{5.9}$$

Note, that by Definition 2.1, $\omega_i(\mathrm{x})$ is a regular domain. Without loss of generality, we assume that the patches are aligned with the triangulation $\mathcal{T}_i$. If they are not, we can use the Scott-Zhang quasi-interpolation operator [42] as we did in [30].

Similar to [44, Section 4.6], we define the finite element cut-off functions

- $\vartheta_v \in V^h(\Omega_i)$ as being 1 at the vertex v, and zero on all other nodes.
- $\vartheta_e \in V^h(\Omega_i)$ as being 1 at the nodes on the (open) edge e, and zero on all other nodes,
- $\vartheta_f \in V^h(\Omega_i)$ as being 1 at the nodes on the (open) face f, zero on all nodes in $\overline{\Omega_i} \setminus (\omega_i(f) \cup f)$, and discrete harmonic inside of $\omega_i(f)$.
- θ_v, θ_e, and $\theta_f \in V^h(\partial\Omega_i)$ as the traces of ϑ_v, ϑ_e, and ϑ_f, respectively.

To be more exact, we would have to write $\vartheta_{i,v}$, $\theta_{i,v}$, etc., but the domain index i will always be clear from the context and is therefore left out.

Throughout the whole section, we make use of the nodal interpolator I^h onto $V^h(\Omega_i)$ (resp. $V^h(\partial\Omega_i)$) which is continuous in the H^1-seminorm (resp. $H^{1/2}$-seminorm) and in the L^2-norm for quadratic functions. See [44, Lemma 3.9].

By definition the functions θ_x provide a partition of unity on $\partial\Omega_i$ in the sense that

$$\sum_{\mathrm{x} \in \mathbb{X}_i} I^h(\theta_\mathrm{x} u) \;=\; u \qquad \forall u \in W_i\,. \tag{5.10}$$

The following lemma states that cutting a function u by one of the functions ϑ_{x} does not increase the energy considerably.

Lemma 5.3. *For* $\mathrm{x} \in \mathbb{X}_i$ *let* $\omega_{i,j} \in \Xi_{i,\eta_i}$ *be an arbitrary patch such that* $\mathrm{x} \subset \partial\omega_{i,j}$. *Then,*

$$|I^h(\vartheta_{\mathrm{x}} u)|^2_{H^1(\omega_i(\mathrm{x}))} \;\lesssim\; (1+\log(\eta_i/h_i))^2 \Big\{ |u|^2_{H^1(\omega_{i,j})} + \frac{1}{\eta_i^2}\|u\|^2_{L^2(\omega_{i,j})} \Big\}.$$

Proof. For a face f there is only one such patch, i. e., $\omega_i(f) = \omega_{i,j}$. Due to [44, Lemma 4.24],

$$|I^h(\vartheta_f\, u)|^2_{H^1(\omega_i(f))} \;\lesssim\; (1+\log(\eta_i/h_i))^2 \Big\{ |u|^2_{H^1(\omega_{i,j})} + \frac{1}{\eta_i^2}\|u\|^2_{L^2(\omega_{i,j})} \Big\}.$$

For an edge e, [44, Lemma 4.16 and Lemma 4.19] yield

$$|I^h(\vartheta_e u)|^2_{H^1(\omega_i(e))} \;\lesssim\; \|u\|^2_{L^2(e)} \;\lesssim\; (1+\log(\eta_i/h_i)) \Big\{ |u|^2_{H^1(\omega_{i,j})} + \frac{1}{\eta_i^2}\|u\|^2_{L^2(\omega_{i,j})} \Big\}.$$

For a vertex v recall that ϑ_v is the nodal basis function associated with v. We easily obtain

$$|I^h(\vartheta_v u)|^2_{H^1(\omega_i(v))} \;\lesssim\; h_i\,|u(v)|^2 \;\lesssim\; \|u\|^2_{L^2(e)}\,,$$

for some edge $e \in \mathbb{E}_i$ with $v \in \overline{e}$ and $e \subset \overline{\omega_{i,j}}$. From here, we can continue as above. □

The next lemma states that the cut-off functions ϑ_{x} can be used to estimate the energy norm by a decomposition into patches.

Lemma 5.4. *For a function* $u \in W_i$, *we have*

$$|u|^2_{S_i} \;\lesssim\; \sum_{\mathrm{x}\in\mathbb{X}_i} \int_{\omega_i(\mathrm{x})} \alpha(x)\,|\nabla(I^h(\vartheta_{\mathrm{x}}\widetilde{u}))(x)|^2\,dx,$$

where $\widetilde{u}$ *is an arbitrary extension of* u *from* $\partial\Omega_i$ *to* $V^h(\Omega_{i,\eta_i})$. *The hidden constant is independent of the number of patches in* Ω_{i,η_i}.

Proof. First, we convince ourselves that the function

$$v := \sum_{\mathrm{x}\in\mathbb{X}_i} I^h(\vartheta_{\mathrm{x}}\widetilde{u})$$

is an extension of u from $\partial\Omega_i$ to Ω_i with its support in Ω_{i,η_i}, and that $v \in V^h(\Omega_i)$. This is true because each node in $\partial\Omega_i^h$ belongs to a unique $\mathrm{x} \in \mathbb{X}_i$ and ϑ_{x} vanishes on $\partial\Omega_i^h \setminus \mathrm{x}$, and the supports of the ϑ_{x} lie entirely in Ω_{i,η_i}. Using the minimum property (2.7) of the Schur complement, we have

$$|u|^2_{S_i} \;\leq\; \int_{\Omega_{i,\eta_i}} \alpha(x)\,|\nabla v(x)|^2\,dx.$$

Since (i) each ϑ_{x} is only supported in $\omega_i(\mathrm{x})$ which consists only of a finite number of patches, (ii) each patch is contained in finitely many supports $\omega_i(\mathrm{x})$ and (iii) the supports $\omega_i(\mathrm{x})$ have finite overlap, we can conclude that

$$|u|^2_{S_i} \;\lesssim\; \sum_{\mathrm{x}\in\mathbb{X}_i} \int_{\omega_i(\mathrm{x})} \alpha(x)|\nabla v(x)|^2\,dx,$$

which proves the desired statement. □

The last tool is inspired by [20]. For an illustration see Fig. 5.

Lemma 5.5. *Let* $\mathrm{x} \in \mathbb{X}_i \cap \mathbb{X}_j$ *and let* $\omega_{j,\ell}$ *be a patch with* $\mathrm{x} \in \overline{\omega_{j,\ell}}$. *Then there exists an operator* $\Pi_{\mathrm{x}}^{j\to i} : V^h(\omega_{j,\ell}) \to V^h(\omega_i(\mathrm{x}))$ *such that for all* $u \in V^h(\omega_{j,\ell})$,

$$\begin{aligned} (\Pi_{\mathrm{x}}^{j\to i}u)_{|\mathrm{x}} \;&=\; u_{|\mathrm{x}}\,, \\ |\Pi_{\mathrm{x}}^{j\to i}u|^2_{H^1(\omega_i(\mathrm{x}))} + \frac{1}{\eta_i^2}\,\|\Pi_{\mathrm{x}}^{j\to i}u\|^2_{L^2(\omega_i(\mathrm{x}))} \;&\lesssim\; |u|^2_{H^1(\omega_{j,\ell})} + \frac{1}{\eta_i^2}\,\|u\|^2_{L^2(\omega_{j,\ell})}\,, \end{aligned} \tag{5.11}$$

the operator is even continuous in the L^2*-norm and the* H^1*-seminorm, separately.*

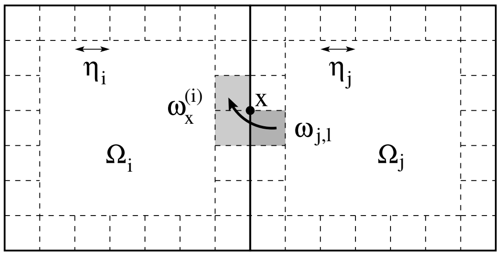

Fig. 5 Illustration of the operator $\Pi_{\rm x}^{j\to i}$ from Lemma 5.5.

Proof. Let $\mathcal{U}$ be the open connected union of $\omega_i(\mathrm{x})$ and $\omega_{j,\ell}$ such that $\operatorname{diam}\mathcal{U} \simeq \eta_i$. In a first step, we construct a continuous extension operator $\mathcal{E} : H^1(\omega_{j,\ell}) \to H^1(\mathcal{U})$ making use of the fact that $\omega_{j,\ell}$ is Lipschitz, and for the time being assuming that $\operatorname{diam}\omega_{j,\ell} = 1$. Following Stein [43] (see also [1, pp. 146]), there exists an extension operator $\widetilde{\mathcal{E}} : H^1(\omega_{j,\ell}) \to H^1(\mathbb{R}^d)$ with $(\widetilde{\mathcal{E}}u)_{|\omega_{j,\ell}} = u$ and

$$\|\widetilde{\mathcal{E}}u\|_{L^2(\mathbb{R}^d)} \lesssim \|u\|_{L^2(\omega_{j,\ell})}, \qquad \|\widetilde{\mathcal{E}}u\|_{H^1(\mathbb{R}^d)} \lesssim \|u\|_{H^1(\omega_{j,\ell})}.$$

Using the mean value $\overline{u} := |\omega_{j,\ell}|^{-1} \int_{\omega_{j,\ell}} u\,dx$ we define

$$\mathcal{E} : H^1(\omega_{j,\ell}) \to H^1(\mathcal{U}) : u \mapsto \big[\widetilde{\mathcal{E}}(u - \overline{u})\big]_{|\mathcal{U}} + \overline{u}.$$

Obviously, $(\mathcal{E}u)_{|\omega_{j,\ell}} = u$. This operator is stable in L^2, as we have

$$\|\mathcal{E}u\|_{L^2(\mathcal{U})} \leq \|\widetilde{\mathcal{E}}(u - \overline{u})\|_{L^2(\mathbb{R}^d)} + \|\overline{u}\|_{L^2(\mathcal{U})} \lesssim \|u - \overline{u}\|_{L^2(\omega_{j,\ell})} + \|\overline{u}\|_{L^2(\mathcal{U})} \lesssim \|u\|_{L^2(\omega_{j,\ell})},$$

where in the last step we used Cauchy's inequality and the fact that $|\mathcal{U}| \simeq |\omega_{j,\ell}|$. By using Poincaré's inequality on $\omega_{i,\ell}$ we can also conclude the stability in the H^1-seminorm,

$$|\mathcal{E}u|_{H^1(\mathcal{U})} \leq \|\widetilde{\mathcal{E}}(u - \overline{u})\|_{H^1(\mathbb{R}^d)} + |\overline{u}|_{H^1(\mathcal{U})} \lesssim \|u - \overline{u}\|_{H^1(\omega_{j,\ell})} \lesssim |u - \overline{u}|_{H^1(\omega_{j,\ell})}.$$

Using a simple dilation argument the two above equations remain also valid when the diameter of $\omega_{i,\ell}$ differs from one, and the constants involved depend only on the shapes of $\omega_{j,\ell}$ and $\mathcal{U}$.

In a second step we use the quasi-interpolation operator $\Pi^h : H^1(\mathcal{U}) \to V^h(\mathcal{U})$ introduced by Scott and Zhang [42] (see also [6] and [20]). This operator is similar to the one by Clément but averages on manifolds instead of patches, which makes it possible to preserve boundary values. In our current setting, we can choose the averaging manifolds such that

$$\begin{aligned} & & (\Pi^h u)_{|\mathrm{x}} &= u_{|\mathrm{x}} && \forall u \in H^1(\mathcal{U}),\ u|_{\omega_{j,\ell}} \in V^h(\omega_{j,\ell}), \\ |\Pi^h u|^2_{H^1(\mathcal{U})} &\lesssim |u|^2_{H^1(\mathcal{U})}, & \|\Pi^h u\|^2_{L^2(\mathcal{U})} &\lesssim \|u\|^2_{L^2(\mathcal{U})} && \forall u \in H^1(\mathcal{U}). \end{aligned}$$

Defining $\Pi_{\rm x}^{j\to i} u := (\Pi^h \mathcal{E}u)_{|\omega_{j,\ell}}$ we meet the requirements of the lemma. □

5.3 The P_D estimates

In this subsection, we show inequality (5.8) by estimating $|P_D\, w|^2_S$ and $|P_D\, z_w|^2_S$ separately. For compact notation, we define for a generic domain D, the α-weighted (semi)norms

$$\|u\|_{L^2(D),\alpha} := \Big(\int_D \alpha(x)\,|u(x)|^2\,dx\Big)^{1/2}, \qquad |u|_{H^1(D),\alpha} := \Big(\int_D \alpha(x)\,|\nabla u(x)|^2\,dx\Big)^{1/2}. \tag{5.12}$$

Lemma 5.6. *Let the assumptions of Theorem 4.1 hold. Then,*

$$|P_D\, w|^2_S \lesssim \max_{j=i}^{N} \Big\{ C^*_{i,\eta_i}\,(1 + \log(\eta_i/h_i))^2 \Big\}\, |w|^2_S \qquad \forall w \in W^\perp.$$

Proof. Due to Assumption A3, the functions $\widehat{\alpha}_i$ defined on $\partial\Omega_i^h$ are piecewise constant with respect to (patch) faces $f \in \mathbb{F}_i^\Gamma$ and edges $e \in \mathbb{E}_i^\Gamma$, and are thus equal to a constant $\widehat{\alpha}_{i\,|\mathrm{x}}$ on each $\mathrm{x} \in \mathbb{X}_i^\Gamma$. As a consequence, the functions $\delta_i^\dagger$ share the same property, and we define $\delta_{i\,|\mathrm{x}}^\dagger$ analogously. Let i now be fixed. Using identity (5.3) and the partition of unity property (5.10) we obtain that

$$|(P_D\, w)_i|_{S_i}^2 \;=\; \Big| \sum_{\mathrm{x}\in\mathbb{X}_i^\Gamma} \sum_{j\in\mathcal{N}_\mathrm{x}} \delta_{j\,|\mathrm{x}}^\dagger\, I^h\big(\theta_\mathrm{x}(w_i - w_j)\big) \;+\; \sum_{\mathrm{x}\in\mathbb{X}_i^D} I^h(\theta_\mathrm{x} w_i)\Big|_{S_i}^2 \tag{5.13}$$

Let Ω_j be one of the neighbours of Ω_i. For a fixed $\mathrm{x} \in \mathbb{X}_j$, large jumps in $\alpha(x)$ can occur in $\omega_i(\mathrm{x})$. We can, however, always find one patch $\omega_{i,k} \in \Xi_{i,\eta_i}$ such that $\omega_{i,k} \subset \omega_i(\mathrm{x})$ and

$$\|\alpha\|_{L^\infty(\omega_i(\mathrm{x}))} \;\leq\; \|\alpha\|_{L^\infty(\omega_{i,k})} \;\leq\; \Big(\sup_{x,y\in\omega_{i,k}} \frac{\alpha(x)}{\alpha(y)}\Big)\, \widehat{\alpha}_{i\,|\mathrm{x}}\,. \tag{5.14}$$

In other words, $\omega_{i,k}$ is the patch in the union $\omega_i(\mathrm{x})$ of patches touching x where the maximum is attained (locally). Obviously, the estimate above depends only on the *local* variation in $\alpha(x)$ but not on the contrast of $\alpha(x)$ between the subregions $\Omega_{i,\eta_i}^{(k)}$. Let the operator $\Pi_\mathrm{x}^{j\to i} : V^h(\omega_{j,\ell}) \to V^h(\omega_i(\mathrm{x}))$ be defined according to Lemma 5.5. With the definitions

$$\begin{aligned} \widetilde{w}_j &:= \mathcal{H}_{j,\alpha} w_j \qquad \text{for } j = 1,\ldots,N,\\ \widetilde{w}_j^\mathrm{x} &:= \Pi_\mathrm{x}^{j\to i}\widetilde{w}_j \qquad \text{for } \mathrm{x} \in \mathbb{X}_i \cap \mathbb{X}_j\,, \end{aligned} \tag{5.15}$$

we have by construction that $(\widetilde{w}_j)_{|\partial\Omega_j} = w_j$ and $(\widetilde{w}_j^\mathrm{x})_{|\mathrm{x}} = (w_j)_{|\mathrm{x}}$ for all j. From the definition (5.13), we see that the function

$$v_i := \sum_{\mathrm{x}\in\mathbb{X}_i^\Gamma} \sum_{j\in\mathcal{N}_\mathrm{x}} \delta_{j\,|\mathrm{x}}^\dagger\, I^h\big(\vartheta_\mathrm{x}(\widetilde{w}_i - \widetilde{w}_j^\mathrm{x})\big) \;+\; \sum_{\mathrm{x}\in\mathbb{X}_i^D} I^h(\vartheta_\mathrm{x}\widetilde{w}_i) \tag{5.16}$$

is an extension of $(P_D\, w)_i$ from W_i to $V^h(\Omega_i)$. By Lemma 5.4 we can conclude that

$$\begin{aligned} |(P_D\, w)_i|_{S_i}^2 \;\lesssim\; & \sum_{\mathrm{x}\in\mathbb{X}_i^\Gamma} \sum_{j\in\mathcal{N}_\mathrm{x}} \underbrace{\big(\delta_{j,\mathrm{x}}^\dagger\big)^2 \int_{\omega_i(\mathrm{x})} \alpha\, \big|\nabla\big[I^h\big(\vartheta_\mathrm{x}(\widetilde{w}_i - \widetilde{w}_j^\mathrm{x})\big)\big]\big|^2\, dx}_{=:\psi_{ij,\mathrm{x}}} \;+\\ & + \sum_{\mathrm{x}\in\mathbb{X}_i^D} \underbrace{\int_{\omega_i(\mathrm{x})} \alpha\, |\nabla(I^h(\vartheta_\mathrm{x}\widetilde{w}_i))|^2\, dx}_{=:\psi_{i,\mathrm{x}}}\,. \end{aligned} \tag{5.17}$$

Using (5.14), we have

$$\psi_{i,\mathrm{x}} \;\leq\; \Big(\sup_{x,y\in\omega_{i,k}} \frac{\alpha(x)}{\alpha(y)}\Big)\, \widehat{\alpha}_{i\,|\mathrm{x}}\, |I^h(\vartheta_\mathrm{x}\widetilde{w}_i)|_{H^1(\omega_i(\mathrm{x}))}^2 \qquad \forall \mathrm{x} \in \mathbb{X}_i^D. \tag{5.18}$$

With the same arguments and the elementary inequality

$$\widehat{\alpha}_i(x^h)\,\big(\delta_j^\dagger(x^h)\big)^2 \;\leq\; \min\big(\widehat{\alpha}_i(x^h), \widehat{\alpha}_j(x^h)\big) \qquad \forall x^h \in \Gamma_{ij}^h\,, \tag{5.19}$$

cf. [44, Sect. 6.2.3], we obtain that for all $\mathrm{x} \in \mathbb{X}_i^\Gamma$ and $j \in \mathcal{N}_\mathrm{x}$,

$$\begin{aligned} \psi_{ij,\mathrm{x}} \;&\lesssim\; \Big(\sup_{x,y\in\omega_{i,k}} \frac{\alpha(x)}{\alpha(y)}\Big)\, \big(\delta_{j\,|\mathrm{x}}^\dagger\big)^2\, \widehat{\alpha}_{i\,\mathrm{x}}\, |I^h(\vartheta_x(\widetilde{w}_i - \widetilde{w}_j^\mathrm{x}))|_{H^1(\omega_i(\mathrm{x}))}^2\\ &\lesssim\; \Big(\sup_{x,y\in\omega_{i,k}} \frac{\alpha(x)}{\alpha(y)}\Big)\, \Big\{\widehat{\alpha}_{i\,|\mathrm{x}}\, |I^h(\vartheta_x\widetilde{w}_i)|_{H^1(\omega_i(\mathrm{x}))}^2 + \widehat{\alpha}_{j\,|\mathrm{x}}\, |I^h(\vartheta_x\widetilde{w}_j^\mathrm{x})|_{H^1(\omega_i(\mathrm{x}))}^2\Big\}. \end{aligned} \tag{5.20}$$

From here on, for a simpler presentation, we will not make the dependency on the local variations $\sup_{x,y\in\omega_{i,k}} \alpha(x)/\alpha(y)$ explicit anymore, but hide them using the $\lesssim$ symbolism. The combination of (5.17), (5.18), and (5.20) then yields

$$|(P_D\, w)_i|_{S_i}^2 \;\lesssim\; \sum_{\mathrm{x}\in\mathbb{X}_i} \widehat{\alpha}_{i\,|\mathrm{x}}\, |I^h(\vartheta_x\widetilde{w}_i)|_{H^1(\omega_i(\mathrm{x}))}^2 + \sum_{\mathrm{x}\in\mathbb{X}_i^\Gamma} \sum_{j\in\mathcal{N}_\mathrm{x}\setminus\{i\}} \widehat{\alpha}_{j\,|\mathrm{x}}\, |I^h(\vartheta_x\widetilde{w}_j^\mathrm{x})|_{H^1(\omega_i(\mathrm{x}))}^2\,. \tag{5.21}$$

For later purposes we introduce the following construction. For a fixed $x \in \mathbb{X}_i \cap \mathbb{X}_j$, we choose patches $\omega_{i,m} \in \Xi_{i,\eta_i}$ and $\omega_{j,\ell} \in \Xi_{j,\eta_j}$ such that

$$
\begin{aligned}
\widehat{\alpha}_{i\,|\mathrm{x}} &\leq \Big(\sup_{x,y \in \omega_{i,m}} \frac{\alpha(x)}{\alpha(y)} \Big)\, \alpha(x) \qquad \forall x \in \omega_{i,m}\,, \\
\widehat{\alpha}_{j\,|\mathrm{x}} &\leq \Big(\sup_{x,y \in \omega_{j,\ell}} \frac{\alpha(x)}{\alpha(y)} \Big)\, \alpha(x) \qquad \forall x \in \omega_{j,\ell}\,,
\end{aligned}
\tag{5.22}
$$

i. e., we choose the patches where $\widehat{\alpha}_{i\,|\mathrm{x}}$ and $\widehat{\alpha}_{j\,|\mathrm{x}}$ are attained. Note that m and ℓ depend on x.

We continue now to further estimate (5.21). The terms in the first sum of this expression are estimated using Lemma 5.3, which yields

$$
\begin{aligned}
\widehat{\alpha}_{i\,|\mathrm{x}}\, |I^h(\vartheta_{\mathrm{x}} \widetilde{w}_i)|^2_{H^1(\omega_i(\mathrm{x}))} &\lesssim \widehat{\alpha}_{i\,|\mathrm{x}}\, (1 + \log(\eta_i/h_i))^2 \Big\{ |\widetilde{w}_i|^2_{H^1(\omega_{i,m})} + \frac{1}{\eta_i^2} \|\widetilde{w}_i\|^2_{L^2(\omega_{i,m})} \Big\} \\
&\lesssim (1 + \log(\eta_i/h_i))^2 \Big\{ |\widetilde{w}_i|^2_{H^1(\omega_{i,m}),\alpha} + \frac{1}{\eta_i^2}\, \|\widetilde{w}_i\|^2_{L^2(\omega_{i,m}),\alpha} \Big\},
\end{aligned}
\tag{5.23}
$$

where in the last line we used (5.22). The second sum in (5.21) is estimated using Lemma 5.3 and Lemma 5.5. For a fixed $\mathrm{x} \in \mathbb{X}_i^\Gamma$ and $j \in \mathcal{N}_{\mathrm{x}} \setminus \{i\}$ we obtain (using that $\widetilde{w}_j^{\mathrm{x}} = \Pi_{\mathrm{x}}^{j \to i} \widetilde{w}_j$)

$$
\begin{aligned}
&\widehat{\alpha}_{j\,|\mathrm{x}}\, |I^h(\vartheta_{\mathrm{x}} \widetilde{w}_j^{\mathrm{x}})|^2_{H^1(\omega_i(\mathrm{x}))} \\
&\quad\lesssim \widehat{\alpha}_{j\,|\mathrm{x}}\, (1 + \log(\eta_i/h_i))^2 \Big\{ |\Pi_{\mathrm{x}}^{j \to i} \widetilde{w}_j|^2_{H^1(\omega_i(\mathrm{x}))} + \frac{1}{\eta_i^2}\, \|\Pi_{\mathrm{x}}^{j \to i} \widetilde{w}_j\|^2_{L^2(\omega_i(\mathrm{x}))} \Big\} \\
&\quad\lesssim \widehat{\alpha}_{j\,|\mathrm{x}}\, (1 + \log(\eta_i/h_i))^2 \Big\{ |\widetilde{w}_j|^2_{H^1(\omega_{j,\ell})} + \frac{1}{\eta_j^2}\, \|\widetilde{w}_j\|^2_{L^2(\omega_{j,\ell})} \Big\} \\
&\quad\lesssim (1 + \log(\eta_i/h_i))^2 \Big\{ |\widetilde{w}_j|^2_{H^1(\omega_{j,\ell}),\alpha} + \frac{1}{\eta_j^2}\, \|\widetilde{w}_j\|^2_{L^2(\omega_{j,\ell}),\alpha} \Big\},
\end{aligned}
\tag{5.24}
$$

where in the last line we used again (5.22). Combining estimates (5.21), (5.23), and (5.24), we obtain with a finite summation argument that

$$
|(P_D\, w)_i|^2_{S_i} \lesssim \sum_{j \in \mathcal{N}_i} (1 + \log(\eta_i/h_i))^2 \Big\{ |\widetilde{w}_j|^2_{H^1(\Omega_{j,\eta_j}),\alpha} + \frac{1}{\eta_i^2}\, \|\widetilde{w}_j\|^2_{L^2(\Omega_{j,\eta_j}),\alpha} \Big\},
$$

where $\mathcal{N}_i = \{j : \overline{\Omega}_i \cap \overline{\Omega}_j \neq \emptyset\}$ is the index set of the neighbours of Ω_i (including i). Since $w_j \in W_j^\perp$, the function $\widetilde{w}_j = \mathcal{H}_{j,\alpha} w_j$ satisfies $g_j(\widetilde{w}_j) = 0$. Hence, Lemma 3.5 and Lemma 3.6 yield

$$
|(P_D\, w)_i|^2_{S_i} \lesssim \sum_{j \in \mathcal{N}_i} C^*_{j,\eta_j}\, (1 + \log(\eta_i/h_i))^2\, |\widetilde{w}_j|^2_{H^1(\Omega_j),\alpha}\,. \tag{5.25}
$$

Finally, using that $h_j \simeq h_i$, $\eta_j \simeq \eta_i$, $|\widetilde{w}_j|_{H^1(\Omega_j),\alpha} = |w_j|_{S_j}$, and that each subdomain has only finitely many neighbours, we obtain the statement of Lemma 5.6. □

Lemma 5.7. *Let the assumptions of Theorem 4.1 hold. In particular, let Q be chosen according to* (4.14)–(4.15). *Then,*

$$
|P_D\, z_w|^2_S \lesssim \Big(\max_{k=1}^N \frac{H_k}{\eta_k} \Big) \max_{j=1}^N \Big\{ C^*_{j,\eta_j}\, (1 + \log(H_k/h_k))^2 \Big\}\, |w|^2_S \qquad \forall w \in W^\perp.
$$

Proof. Recall that $z_w \in \ker S$, i. e., z_w is constant on each subdomain. We denote these constant components by z_i. Let i be fixed. With the same arguments as in Lemma 5.6, the function

$$
v_i := \sum_{x \in \mathbb{X}_i^\Gamma} \sum_{j \in \mathcal{N}_x} \delta^\dagger_{j\,|x}\, I^h(\vartheta_x(z_i - z_j)) + \sum_{x \in \mathbb{X}_i^D} I^h(\vartheta_x z_i)
$$

is an extension of $(P_D\, z_w)_i$ from W_i to $V^h(\Omega_i)$. An application of Lemma 5.4 yields

$$\begin{aligned} |(P_D z_w)_i|^2_{S_i} &\lesssim \sum_{x\in\mathbb{X}_i^\Gamma}\sum_{j\in\mathcal{N}_x} (\delta^\dagger_{j\,|x})^2\, |I^h(\vartheta_x(z_i-z_j))|^2_{H^1(\omega_i(x)),\alpha} + \sum_{x\in\mathbb{X}_i^D} |I^h(\vartheta_x z_i)|^2_{H^1(\omega_i(x)),\alpha} \\ &= \sum_{x\in\mathbb{X}_i}\sum_{j\in\mathcal{N}_x} (\delta^\dagger_{j\,|x})^2\, |\vartheta_x|^2_{H^1(\omega_i(x)),\alpha}\, |z_i-z_j|^2 + \sum_{x\in\mathbb{X}_i^D} |\vartheta_x|^2_{H^1(\omega_i(x)),\alpha}\, |z_i|^2. \end{aligned}$$

As in the proof of Lemma 5.6, we do not make explicit the dependency on the local variations $\sup_{x,y\in\omega_{i,k}} \alpha(x)/\alpha(y)$. Using (5.14), i. e., $\|\alpha\|_{L^\infty(\omega_i(x))} \lesssim \widehat{\alpha}_{i\,|x}$, and due to [44, Lemma 4.16, Lemma 4.25] we have

$$\begin{aligned} |\vartheta_f|^2_{H^1(\omega_i(x)),\alpha} &\lesssim \varphi_{i,f} := \widehat{\alpha}_{i\,|f}\,(1+\log(\eta_i/h_i))\,\eta_i \qquad &&\forall f\in\mathbb{F}_i\,, \\ |\vartheta_e|^2_{H^1(\omega_i(x)),\alpha} &\lesssim \varphi_{i,e} := \widehat{\alpha}_{i\,|e}\,\eta_i &&\forall e\in\mathbb{E}_i\,, \\ |\vartheta_v|^2_{H^1(\omega_i(x)),\alpha} &\lesssim \varphi_{i,v} := \widehat{\alpha}_{i\,|v}\,h_i &&\forall v\in\mathbb{V}_i\,. \end{aligned}$$

Obviously, for each edge e (and for each vertex v) we can find a face f with $e\subset\overline{f}$ (resp. an edge e with $v\in\overline{e}$) such that

$$\varphi_{i,e} \le \varphi_{i,f} \qquad \text{and} \qquad \varphi_{i,v} \le \varphi_{i,e}\,,$$

by choosing the one touching the patch with the effectively largest coefficient. Since each patch face contains only a finite number of patch edges and vertices, and due to the elementary inequality (5.19) we obtain

$$\begin{aligned} |(P_D z_w)_i|^2_{S_i} \lesssim &\sum_{X\in\mathcal{X}_i^\Gamma\cap\mathcal{X}_j^\Gamma}\ \sum_{x\in\mathbb{X}_i,\,x\subset X} \min(\varphi_{i,x},\,\varphi_{j,x})\,|z_i-z_j|^2 + \\ &+ \sum_{X\in\mathcal{X}_i^D}\ \sum_{x\in\mathbb{X}_i,\,x\subset X} \varphi_{i,x}\,|z_i|^2, \end{aligned} \tag{5.26}$$

where the x in the inner sums are of the same kind (face/edge/vertex) as the X in the outer sums. We treat subdomain face, edge, and vertex contributions separately.

- Since each face $f\in\mathbb{F}_i$ contains $\mathcal{O}((\eta_i/h_i)^2)$ nodes, we can bound the subdomain face contributions in (5.26) from above by

$$\begin{aligned} &\sum_{F\in\mathcal{F}_i^\Gamma\cap\mathcal{F}_j^\Gamma}\ \sum_{f\in\mathbb{F}_i,\,f\subset F}\ \sum_{x^h\in f^h} \min(\widehat{\alpha}_{i\,|f},\,\widehat{\alpha}_{j\,|f})\ \underbrace{(1+\log(\eta_i/h_i))\,\frac{h_i^2}{\eta_i}}_{\le\, q_i(x^h)\,H_i/\eta_i}\ |z_i-z_j|^2 \\ &+ \sum_{F\in\mathbb{F}_i^D}\ \sum_{f\in\mathbb{F}_i\, f\subset F}\ \sum_{x^h\in f^h} \widehat{\alpha}_{i\,|f}\ \underbrace{(1+\log(\eta_i/h_i))\,\frac{h_i^2}{\eta_i}}_{\le\, q_i(x^h)\,H_i/\eta_i}\ |z_i|^2, \end{aligned} \tag{5.27}$$

  where f^h is the set of interior nodes on f.
- Since each edge $e\in\mathbb{E}_i$ contains $\mathcal{O}(\eta_i/h_i)$ nodes, we can bound the subdomain edge contributions in (5.26) from above by

$$\begin{aligned} &\sum_{E\in\mathcal{E}_i^\Gamma\cap\mathcal{E}_j^\Gamma}\ \sum_{e\in\mathbb{E}_i,\,e\subset E}\ \sum_{x^h\in e^h} \min(\widehat{\alpha}_{i\,|e},\,\widehat{\alpha}_{j\,|e})\ \underbrace{h_i}_{=\,q_i(x^h)}\ |z_i-z_j|^2 \\ &+ \sum_{E\in\mathcal{E}_i^D}\ \sum_{e\in\mathbb{E}_i,\,e\subset E}\ \sum_{x^h\in e^h} \widehat{\alpha}_{i\,|e}\ \underbrace{h_i}_{=q_i(x^h)}\ |z_i|^2, \end{aligned}$$

  where e^h is the set of interior nodes on e.
- Similarly, the subdomain vertex contributions from (5.26) can be bounded by

$$\sum_{V\in\mathcal{V}_i^\Gamma\cap\mathcal{V}_j^\Gamma} \min(\widehat{\alpha}_i(V),\,\widehat{\alpha}_j(V))\ \underbrace{h_i}_{=q_i(V)}\ |z_i-z_j|^2 + \sum_{V\in\mathcal{V}_i^D} \widehat{\alpha}_i(V)\ \underbrace{h_i}_{=q_i(V)}\ |z_i|^2.$$

According to [21], we observe that the expressions $|z_i - z_j|$ and $|z_i|$ are components of $B\,z_w$. Collecting all terms appropriately and using the definition (4.14) of Q, we can conclude by Lemma 5.1 that

$$|P_D\,z_w|_S^2 \;\lesssim\; \Big(\max_{k=1}^N \frac{H_k}{\eta_k}\Big)\,\|B\,z_w\|_Q^2 \;\lesssim\; \Big(\max_{k=1}^N \frac{H_k}{\eta_k}\Big)\,\|B\,w\|_Q^2\,. \tag{5.28}$$

In the following, we split $\|B\,w\|_Q^2$ into face, edge, and vertex terms,

$$\|B\,w\|_Q^2 \;=\; \sum_{i=1}^N \Big\{ \sum_{\mathsf{x}\in\mathbb{X}_i^\Gamma\cap\mathbb{X}_j^\Gamma} r_{ij}^{\mathsf{x}} + \sum_{\mathsf{x}\in\mathbb{X}_i^D} r_i^{\mathsf{x}} \Big\},$$

with $r_{ij}^{\mathsf{x}} := \min(\widehat{\alpha}_{i\,|\mathsf{x}},\, \widehat{\alpha}_{j\,|\mathsf{x}})\, q_{i\,|\mathsf{x}} \sum_{x^h\in\mathsf{x}^h} \big(w_i(x^h) - w_j(x^h)\big)^2$ and $r_i^{\mathsf{x}} := \widehat{\alpha}_{i\,|\mathsf{x}}\; q_{i\,|\mathsf{x}} \sum_{x^h\in\mathsf{x}^h} w_i(x^h)^2$, and treat patch face, edge, and vertex terms again separately.

– Recall that $q_{i\,|f} = (1+\log(H_i/h_i))\,h_i^2/H_i$. Due to the quasi-uniformity assumption on the mesh on Ω_i we obtain for the patch face terms that

$$r_{ij}^f \;\lesssim\; \min(\widehat{\alpha}_{i\,|f}, \widehat{\alpha}_{j\,|f})\,(1+\log(H_i/h_i))\,\frac{1}{H_i}\;\underbrace{\|w_i - w_j\|_{L^2(f)}^2}_{\lesssim \|w_i\|_{L_2(f)}^2 + \|w_j\|_{L_2(f)}^2}\;,$$

$$r_i^f \;\lesssim\; \widehat{\alpha}_{i\,|f}\,(1+\log(H_i/h_i))\,\frac{1}{H_i}\,\|w_i\|_{L^2(f)}^2\,.$$

In the following, we use construction (5.22) for $\mathsf{x} = f$, i. e., $\widehat{\alpha}_{i\,|f} \lesssim \alpha_{|\omega_{i,m}}$ and $\widehat{\alpha}_{j\,|f} \lesssim \alpha_{|\omega_{j,\ell}}$. A standard Sobolev norm equivalence (cf. [44, Sect. A.4]) yields

$$\frac{1}{\eta_i}\|w_i\|_{L^2(f)}^2 \;\lesssim\; |\mathcal{H}_{i,\alpha} w_i|_{H^1(\omega_{i,m})}^2 + \frac{1}{\eta_i^2}\,\|\mathcal{H}_{i,\alpha} w_i\|_{L^2(\omega_{i,m})}^2\,,$$

and analogously, the L^2-norm of w_j on f is bounded in terms of the scaled H^1-norm on $\omega_{j,\ell}$. Combining these with the estimate before we obtain after summation that

$$\sum_{f\in\mathbb{F}_i^\Gamma\cap\mathbb{F}_j^\Gamma} r_{ij}^f + \sum_{f\in\mathbb{F}_i^D} r_i^f \;\lesssim\; \frac{\eta_i}{H_i}\,\Big(1+\log\Big(\frac{H_i}{h_i}\Big)\Big) \sum_{j\in\mathcal{N}_i} \Big\{ |\mathcal{H}_{j,\alpha} w_j|_{H^1(\Omega_{j,\eta_j}),\alpha}^2 + \frac{1}{\eta_j^2}\,\|\mathcal{H}_{j,\alpha} w_j\|_{L^2(\Omega_{j,\eta_j}),\alpha}^2 \Big\}. \tag{5.29}$$

– For the patch edge terms we obtain by similar arguments that

$$r_{ij}^e \;\lesssim\; \widehat{\alpha}_{i\,|e}\,\|w_i\|_{L^2(e)}^2 + \widehat{\alpha}_{j\,|e}\,\|w_j\|_{L^2(e)}^2\,, \qquad r_i^e \;\lesssim\; \widehat{\alpha}_{i\,|e}\|w_i\|_{L^2(e)}^2\,.$$

Due to [44, Lemma 4.16] and construction (5.22) for $\mathsf{x} = e$, we have

$$\|w_j\|_{L^2(e)}^2 \;\lesssim\; (1+\log(\eta_j/h_j))\,\Big\{ |\mathcal{H}_{j,\alpha} w_j|_{H^1(\omega_{j,\ell})}^2 + \frac{1}{\eta_j^2}\,\|\mathcal{H}_{j,\alpha} w_j\|_{L^2(\omega_{j,\ell})}^2 \Big\},$$

and the analogous estimate for $\|w_i\|_{L^2(e)}^2$. Collecting and summing all terms, we obtain

$$\sum_{e\in\mathbb{E}_i^\Gamma\cap\mathbb{E}_j^\Gamma} r_{ij}^e + \sum_{e\in\mathbb{E}_i^D} r_i^e \;\lesssim\; (1+\log(\eta_i/h_i)) \sum_{j\in\mathcal{N}_i} \Big\{ |\mathcal{H}_{j,\alpha} w_j|_{H^1(\Omega_{j,\eta_j}),\alpha}^2 + \frac{1}{\eta_j^2}\,\|\mathcal{H}_{j,\alpha} w_j|_{L^2(\Omega_{j,\eta_j}),\alpha}^2 \Big\}.$$

– The patch vertex terms can be estimated trivially by patch edge terms.

Combining all the estimates, noticing that

$$|\mathcal{H}_{i,\alpha} w_i|_{H^1(\Omega_{i,\eta_i}),\alpha}^2 \;\le\; |w_i|_{S_i}^2\,, \qquad \eta_i/H_i \;\le\; 1,$$

and using Lemma 3.5 or Lemma 3.6 we obtain

$$|(P_D\,z_w)_i|_{S_i}^2 \;\lesssim\; \Big(\max_{k=1}^N \frac{H_k}{\eta_k}\Big) \sum_{j\in\mathcal{N}_i} C_{j,\eta_j}^*\,(1+\log(H_j/h_j))\,|w_j|_{S_j}^2 \qquad \forall w\in W^\perp,\ i=1,\dots,N,$$

which directly implies the statement of Lemma 5.7. □

Combining Lemma 5.6 and Lemma 5.7, we finally obtain inequality (5.8). This finishes the proof of Theorem 4.1.

5.4 Proof of Corollary 4.2

The proof of Corollary 4.2 requires no new ideas. For non-floating subdomains, since the function w_i vanishes at least on an edge of $\mathcal{T}_{H_i}(\Omega_i)$ in each subregion $\Omega_i^{(k)}$, it is possible to use standard Friedrichs or discrete Poincaré-Friedrichs inequalities. If $M_i = 2$ for all subdomains Ω_i, the quadratic bound $(1+\log(H_i/h_i))^2$ is obtained using the usual trick of introducing an average (over a face of $\mathcal{T}_{H_i}(\Omega_i)$), as it is described e. g. in [21,30].

6 Extensions

In this section, we give a series of extensions of our theory to more general situations, and discuss how some of our technical assumptions may be relaxed.

6.1 Generalisations of Lemma 3.6 – Overlapping subregions

The following two generalisations work with subregions that overlap in order to allow for more general coefficient distributions. Note that they greatly extend the applicability of Lemma 3.6.

Lemma 6.1 (overlapping subregions I)**.** *Let Ω_{i,η_i} be subdivided into M_i non-overlapping subregions $\Omega_{i,\eta_i}^{(k)}$ (as in* (3.1) *for the case $M_i = 2$). For each $k = 1, \ldots, M_i$, let $\Omega_i^{(k)}$ be a regular extension of $\Omega_{i,\eta_i}^{(k)}$ to the interior of Ω_i . Note, however, that in contrast to Lemma 3.6 the regions $\Omega_i^{(k)}$ may now overlap in $\Omega_i\backslash\Omega_{i,\eta_i}$. Assume again that each subregion $\Omega_i^{(k)}$ is resolved by the coarse triangulation $\mathcal{T}_{H_i}(\Omega_i)$ and that $X_i^* \subset \Gamma_i^*$ is a face, edge, or vertex of $\mathcal{T}_{H_i}(\Omega_i)$. Furthermore, suppose that*

$$\alpha(x) \;\gtrsim\; \max_{k:\, x\in\Omega_i^{(k)}} \underline{\alpha}_{i,\eta_i}^{(k)}, \qquad \textit{for all} \quad x \in \Omega_i\,, \tag{6.1}$$

with $\underline{\alpha}_{i,\eta_i}^{(k)}$ as defined in (3.5)*. Then the weighted Sobolev inequality* (3.11) *holds with C^*_{i,η_i} as given in Lemma 3.6 and an extra factor of M_i in the hidden constant.*

Proof. Following the proof of Lemma 3.6 and applying (3.12) in each of the subregions $\Omega_i^{(k)}$, we get

$$\frac{1}{\eta_i^2}\int_{\Omega_{i,\eta_i}} \alpha(x)\,|u(x)|^2\,dx \;\lesssim\; \sum_{k=1}^{M_i} \frac{\overline{\alpha}_{i,\eta_i}^{(k)}}{\underline{\alpha}_{i,\eta_i}^{(k)}}\,\sigma(H_i/h_i)\,\frac{H_i}{\eta_i}\,\underline{\alpha}_{i,\eta_i}^{(k)}\,|u|^2_{H^1(\Omega_i^{(k)})}$$

The result follows directly by using the assumption (6.1). □

Lemma 6.2 (overlapping subregions II)**.** *As in Lemma 3.6, let Ω_i be partitioned into M_i regular subregions $\Omega_i^{(k)}$ according to* (3.8)*, but in contrast to Lemma 3.6 and Lemma 6.1, the regions $\Omega_i^{(k)}$ may overlap (even in the boundary layer). Assume again that each subregion $\Omega_i^{(k)}$ is resolved by the coarse triangulation $\mathcal{T}_{H_i}(\Omega_i)$ and that $X_i^* \subset \Gamma_i^*$ is a face, edge, or vertex of $\mathcal{T}_{H_i}(\Omega_i)$. Suppose that there are constants $\underline{\alpha}_{i,\mathrm{art}}^{(k)}$ and $\overline{\alpha}_{i,\mathrm{art}}^{(k)}$ such that*

$$\begin{array}{llll} (i) & \underline{\alpha}_{i,\mathrm{art}}^{(k)} \;\le\; \alpha(x) & & \forall x \in \Omega_i^{(k)} \setminus \Omega_{i,\eta_i}^{(k)}\,, \\ (ii) & \max_{k:\, x\in\Omega_{i,\eta_i}^{(k)}} \underline{\alpha}_{i,\mathrm{art}}^{(k)} \;\le\; \alpha(x) \;\le\; & \max_{k:\, x\in\Omega_{i,\eta_i}^{(k)}} \overline{\alpha}_{i,\mathrm{art}}^{(k)} & \forall x \in \Omega_{i,\eta_i}\,. \end{array}$$

Then, the weighted Sobolev inequality (3.11) *holds with* $C^*_{i,\eta_i} = M_i\,\sigma(H_i/h_i)\,\dfrac{H_i}{\eta_i}\left(\max_{k=1}^{M_i}\dfrac{\overline{\alpha}_{i,\mathrm{art}}^{(k)}}{\underline{\alpha}_{i,\mathrm{art}}^{(k)}}\right).$

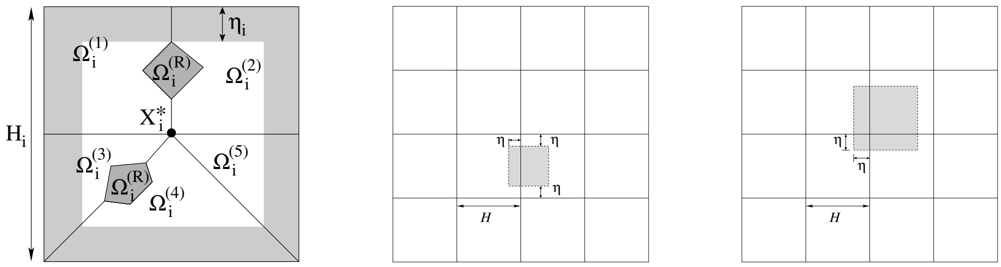

Fig. 6 Different coefficient distributions. *Left:* small inclusions $\Omega_i^{(R)}$, *middle* Example 1 (edge island), *right* Example 2 (crosspoint island)

Proof. We define the sets $\widetilde{\Omega}_{i,\eta_i}^{(k)} := \{x \in \Omega_{i,\eta_i}^{(k)} : k = \text{argmax}_{j:\, x \in \Omega_{i,\eta_i}^{(j)}}\, \overline{\alpha}_{i,\text{art}}^{(j)}\}$ that form a non-overlapping partition of the boundary layer. Using assumption (ii), as well as (3.12), we obtain as in Lemma 6.1

$$\frac{1}{\eta_i^2} \int_{\Omega_{i,\eta_i}} \alpha(x)\, |u(x)|^2\, dx \;\leq\; \sum_{k=1}^{M_i} \frac{\overline{\alpha}_{i,\text{art}}^{(k)}}{\underline{\alpha}_{i,\text{art}}^{(k)}}\, \sigma(H_i/h_i)\, \frac{H_i}{\eta_i} \int_{\Omega_i^{(k)}} \underline{\alpha}_{i,\text{art}}^{(k)}\, |\nabla u(x)|^2\, dx.$$

Since assumptions (i) and (ii) imply that $\underline{\alpha}_{i,\text{art}}^{(k)} \leq \alpha(x)$ for $x \in \Omega_i^{(k)}$, this leads to the desired result. □

An important application of Lemma 6.2 is the example introduced in Section 3.3 (Fig. 4, left) for $\varepsilon \gg 1$, i.e. for a *larger* coefficient in the plate $\Omega_i^{(1)}$ than in the remainder of Ω_i. As in Section 3.3, let $\alpha(x) = \varepsilon$ for all $x \in \Omega_i^{(1)}$ and set $\alpha(x) = 1$ elsewhere in Ω_i with $\varepsilon \gg 1$. Now we can apply Lemma 6.2 with $\Omega_i^{(2)} := \Omega_i$, $\underline{\alpha}_{i,\text{art}}^{(1)} = \overline{\alpha}_{i,\text{art}}^{(1)} = \varepsilon$ and $\underline{\alpha}_{i,\text{art}}^{(2)} = \overline{\alpha}_{i,\text{art}}^{(2)} = 1$. Since the intersection of $\Omega_i^{(1)}$ and $\Omega_i^{(2)}$ is all of $\Omega_i^{(1)}$ we can choose X_i^* to be a coarse face (resp. edge) in three (resp. two) dimensions, and so the weighted Sobolev inequality (3.11) holds with $C_{i,H_i}^* = \mathcal{O}(1)$. This trick does not work when $\varepsilon \ll 1$.

We can further generalise Lemmas 3.6, 6.1 and 6.2 to coefficient distributionss with "soft" (small coefficient) inclusions that are separated from the boundary. In other words, the union of the subregions $\Omega_i^{(k)}$ need not be the whole of Ω_i as long as (i) the entire boundary layer Ω_{i,η_i} remains covered by the subregions, and (ii) for each of the subregions, inequality (3.12) holds. Such an example is illustrated in Figure 6, left: the union of the subregions $\Omega_i^{(k)}$ does not comprise all of Ω_i and excludes the two inclusions $\Omega_i^{(R)}$ located in the interior of Ω_i. We can now freely choose the coefficient in $\Omega_i^{(R)}$, in particular to be *arbitrarily small* (which has not been possible up to now). In contrast to the counterexample in Section 3.3 (the plate example with a smaller coefficient in the plate), the inclusions here do not *separate* regions of larger coefficients from each other.

6.2 Generalisations of Theorem 4.1 and Corollary 4.2

We believe that the extra factor $\max_{j=1}^N \frac{H_j}{\eta_j}$ in Theorem 4.1 and Corollary 4.2 is an artefact of our proof and can be eliminated. Our numerical experiments in Section 7.5 suggest this is possible. We are not able to show an improved bound for the general situation, but are able to do so, (i) if Lemma 3.6 applies on each subdomain and (ii) for two alternative choices of the scaling operator Q in (4.7).

Theorem 6.3. *Under the assumptions of Theorem 4.1, and assuming in addition that* $\frac{H_i}{\eta_i} \simeq \frac{H_j}{\eta_j}$, *for all* $1 \leq i, j \leq N$, *and that Lemma 3.6 applies on each subdomain, we can get the improved bound*

$$\kappa \;\lesssim\; \max_{i=1}^{N} \left(C_{i,\eta_i}^*\, (1 + \log(H_i/h_i))^2 \right) \tag{6.2}$$

for the condition number of the all-floating FETI system with C_{i,η_i}^* *as given in Lemma 3.6.*

Proof. First note that the bound for $|(P_D\, w)|_S^2$ in Lemma 5.6 does not involve the extra factor. On the other hand, we have from (5.28) that

$$|(P_D\, z_w)|_S^2 \;\lesssim\; \big(\max_{k=1}^{N} \frac{H_k}{\eta_k}\big)\, \|B\, w\|_Q^2. \tag{6.3}$$

The contributions to the right hand side of (6.3) from faces can be bounded via (5.29), which involves a factor of η_i/H_i that cancels the factor $\big(\max_k \frac{H_k}{\eta_k}\big)$ in (6.3) (due the additional assumptions on η_i).

In order to improve the bound on the edge (and vertex) contributions, define for each subdomain edge $E \in \mathcal{E}_i$ the set $E^{(k)} := E \cap \partial\Omega_i^{(k)}$, which is either empty, or an edge of the regular subregion $\Omega_i^{(k)}$ and so $\operatorname{diam} E^{(k)} \simeq H_i$. We can now collect the edge contributions from w_i from each non-empty $E^{(k)}$ and apply [44, Lemma 4.16] and inequality (3.12) to obtain

$$\begin{aligned}
\sum_{e \subset E^{(k)}} \widehat{\alpha}_{i\,|e}\, \|w_i\|_{L^2(e)}^2 \;&\le\; \overline{\alpha}_{i,\eta_i}^{(k)}\, \|w_i\|_{L^2(E^{(k)})}^2 \\
&\lesssim\; \overline{\alpha}_{i,\eta_i}^{(k)}\, (1+\log(H_i/h_i)) \Big\{ |\mathcal{H}_{i,\alpha} w_i|_{H^1(\Omega_i^{(k)})}^2 + \frac{1}{H_i}\, \|\mathcal{H}_{i,\alpha} w_i\|_{L^2(\partial\Omega_i^{(k)})}^2 \Big\} \\
&\lesssim\; \underbrace{\frac{\overline{\alpha}_{i,\eta_i}^{(k)}}{\underline{\alpha}_{i,\eta_i}^{(k)}}\, \sigma(H_i/h_i)}_{=\,\frac{\eta_i}{H_i}\, C_{i,\eta_i}^*}\, (1+\log(H_i/h_i))\, \underbrace{\underline{\alpha}_{i,\eta_i}^{(k)}\, |\mathcal{H}_{i,\alpha} w_i|_{H^1(\Omega_i^{(k)})}^2}_{\le\, |w_i|_{S_i}^2}\,.
\end{aligned}$$

The factor η_i/H_i can again be used to cancel $\big(\max_k \frac{H_k}{\eta_k}\big)$ in (6.3). The edge contributions from w_j are bounded analogously. As shown in Section 5, the bound for vertex contributions can be reduced to the above edge estimate. The face, edge and vertex estimates are then combined as in Section 5 to obtain a bound for $|(P_D\, z_w)|_S^2$ that does not involve the extra factor $\big(\max_k \frac{H_k}{\eta_k}\big)$. □

Theorem 6.4. *The improved bound* (6.2) *also holds under the assumptions of Theorem 4.1 (without the additional assumptions in Theorem 6.3), if we make one of the following two alterative choices of the scaling operator Q in* (4.7)*:*

(i) $Q = M^{-1}$,

(ii) diagonal Q as defined in (4.14)*, but with*

$$q_i(x^h) := \begin{cases} (1+\log(H_i/h_i))\,\dfrac{h_i^2}{\eta_i} & \text{if } x^h \text{ lies on a subdomain face,} \\ h_i & \text{if } x^h \text{ lies on a subdomain edge or vertex,} \end{cases} \tag{6.4}$$

in three dimensions (and suitable modifications in two dimensions).

Note that the second choice requires an a priori knowledge of the parameter η_i for each subdomain Ω_i which may not be available in practice.

Proof. The case $Q = M^{-1}$ can be proved as above, using [44, Lemma 6.14] instead of Lemma 5.7.

If Q is chosen according to (6.4), then the factors H_i/η_i and η_i/H_i in (5.27) and (5.29), respectively, disappear and we can avoid the introduction of an extra factor $\max_{k=1}^N H_k/\eta_k$ as well. □

Corollaries to both of these theorems for the one-level FETI method analogous to Corollary 4.2 can of course be obtained again very easily as in the case of Theorem 4.1.

Remark 6.5. Assumption A3 ensures that the functions $\delta_j^\dagger$ in (4.11) are constant on boundary edges and faces of the boundary layer patches $\omega_{i,j}$ in Definition 2.6. We can drop Assumption A3, if we choose

$$\widehat{\alpha}_i(x^h) := \max_{j:\, x^h \in \partial\omega_{i,j}} \|\alpha\|_{L^\infty(\omega_{i,j})}$$

instead of (4.10). However, this choice for $\widehat{\alpha}_i(x^h)$ requires even more a priori knowledge on the coefficient than (6.4) and is not very suitable for implementations.

However, we believe that the statements of Theorem 4.1 and Corollary 4.2 can also be proved for the original choice of $\widehat{\alpha}_i(x^h)$ in (4.10) under a suitable smoothness assumption on $\widehat{\alpha}_i(x^h)$, such as

$$\frac{|\widehat{\alpha}_i(x^h) - \widehat{\alpha}_i(y^h)|}{|x^h - y^h|} \lesssim \frac{1}{\eta_i} \widehat{\alpha}_i(x^h),$$

where x^h and y^h are two neighbouring nodes on a boundary face (resp. edge) of one of the boundary patches $\omega_{i,j}$. This assumption basically excludes rapid oscillations of $\alpha(x)$ along the boundary of any of the patches, but still allows for large jumps between patches.

Remark 6.6. Our proof of Theorem 4.1 in Section 5 uses only arguments on *patches* $\omega_{i,j}$ instead of subdomains. Indeed, the fact that the subdomains are regular (cf. Assumption A1) is only used to fulfill the requirements of Lemma 3.5 and Lemma 3.6. Let us drop Assumption A1 and instead assume a general partitioning of Ω into η_i-regular subdomains Ω_i, where $H_i \simeq H_j$ and $\eta_i \simeq \eta_j$ for neighboring subdomains. Suppose furthermore that each subdomain splits into M_i subregions $\Omega_i^{(k)}$ with common interface Γ_i^* such that there exists a decomposition into regular patches of diameter $\mathcal{O}(\eta_i)$ that resolve the interfaces between the subregions and the subdomains. Then the statement of Lemma 3.6 remains true with constant $C^*_{i,\eta_i} := \sigma_i^*(H_i,\, \eta_i,\, h_i)\, \frac{H_i}{\eta_i} \max_{k=1}^{M_i} \frac{\overline{\alpha}^{(k)}_{i,\eta_i}}{\underline{\alpha}^{(k)}_{i,\eta_i}}$, if the following bound holds on each of the subregions:

$$\|u\|^2_{L^2(\partial\Omega_i^{(k)} \cap \partial\Omega_i)} \leq H_i\, \sigma_i^*(H_i,\, \eta_i,\, h_i)\, |u|^2_{H^1(\Omega_i^{(k)})} \qquad \forall u \in V^h(\Omega_i^{(k)}) \text{ with } \int_{X_i^*} u\, ds = 0, \tag{6.5}$$

where $X_i^* \subset \Gamma_i^*$ is some manifold (not necessarily a straight edge or a flat face) that is resolved by the patch decomposition. Equivalently, the term on the left hand side of (6.5) may be replaced by $\frac{1}{\eta_i}\, \|u\|^2_{L^2(\Omega^{(k)}_{i,\eta_i})}$. In order to get such inequalities with an explicit form of $\sigma_i^*(H_i, \eta_i, h_i)$, one can use the framework of Section 3.1 or in the appendix of [30]. We refer also to the recently developed, more general framework in [34,33].

In order to get the statement of Theorem 4.1 under the above conditions, we need further that each (not necessarily straight) subdomain edge touches $\mathcal{O}(H_i/\eta_i)$ of the patches, and in three dimensions each (not necessarily flat) subdomain face touches $\mathcal{O}((H_i/\eta_i)^2)$ patches.

More details on these generalisations and some further extensions are given in the recent papers [34, 33].

7 Numerical results

In this section, we would like to confirm our new theoretical results. We point out that results for interior “island” coefficients as well as for interface variation are already contained in [30, Section 5] and in [31]. Note also, that our new theory explains the robustness of one-level FETI in case of the nonlinear magnetostatics with large interface variation in [30, Section 5.4], see also [32].

In all our computations, we used PARDISO [41] as sparse direct solver for the subdomain problems. Also, in all the following computations the diagonal choice of Q according to (4.15) has been used, unless indicated otherwise.

7.1 Edge and crosspoint islands

In Example 1 we choose Ω to consist of 16 squares with an island coefficient cutting through a subdomain edge (cf. Fig. 6, middle). In order to rule out symmetries we have shifted the centre of the island to the right of the subdomain interface. In Figure 6 and in what follows, H denotes the subdomain width and η denotes the characteristic geometric scale of the coefficient island. We set the coefficient to 1 outside the island (shaded region) and to a constant α_I inside. We impose Dirichlet boundary conditions on the entire boundary $\partial\Omega$ and choose a piecewise constant right-hand side.

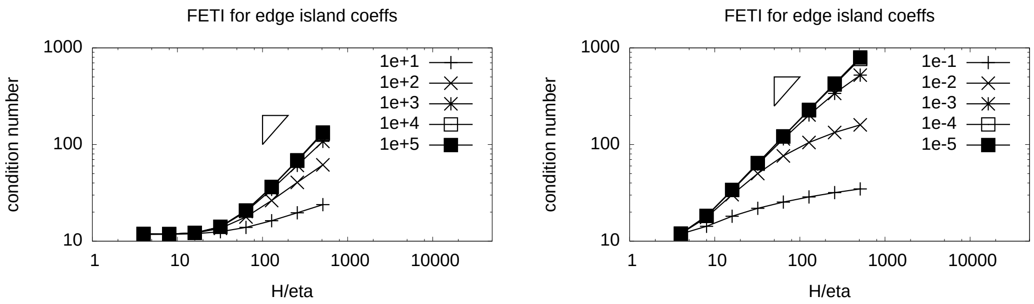

Fig. 7 Example 1: estimated condition numbers; $H/h = 512$, varying ratio H/η and varying magnitude of the jump α_I. *Left* $\alpha_I > 1$, *right* $\alpha_I < 1$

$\frac{H}{\eta}$	$\frac{H}{h} = 64$		128		256		512	
4	17	(19)	17	(19)	20	(22)	22	(23)
8	22	(19)	23	(24)	25	(25)	26	(26)
16	22	(23)	24	(25)	26	(28)	28	(30)
32	23	(25)	25	(27)	27	(29)	29	(31)
64	26	(27)	27	(30)	30	(32)	31	(33)
128	–		33	(31)	36	(36)	37	(37)
256	–		–		43	(39)	46	(42)
512	–		–		–		52	(49)

Table 1 Example 1: number of CG iterations; island coefficient with $\alpha_I = 10^{+5}$, in brackets: $\alpha_I = 10^{-5}$.

$\frac{H}{\eta}$	$\frac{H}{h} = 64$		128		256		512	
4	6.9	(7.0)	6.9	(6.9)	10.1	(10.1)	11.9	(12.0)
8	6.9	(11.0)	8.4	(13.3)	10.1	(15.7)	11.9	(18.3)
16	7.7	(20.8)	9.0	(25.0)	10.5	(29.4)	12.2	(22.5)
32	10.8	(39.0)	11.7	(47.3)	12.8	(55.7)	14.1	(64.3)
64	18.4	(72.0)	19.1	(88.2)	19.9	(104.9)	20.8	(121.7)
128	–		35.0	(163.2)	35.7	(195.8)	36.5	(229.1)
256	–		–		67.9	(363.6)	68.8	(429.0)
512	–		–		–		133.2	(800.5)

Table 2 Example 1: estimated condition numbers; island coefficient with $\alpha_I = 10^{+5}$, in brackets: $\alpha_I = 10^{-5}$.

Figure 7 shows the condition numbers (estimated by Lanczos' method) for a fixed ratio of $H/h = 512$ and for different values of α_I. We see that, as predicted by our theoretical bounds, the method is perfectly robust with respect to the values of α_I. The growth of the condition number with respect to H/η appears to be linear, which suggests that our theoretical bounds may be slightly pessimistic in their predicted dependence of κ on H/η. This is also confirmed in Tables 1 and 2, where we display the number of PCG iterations (to achieve a relative residual reduction of 10^{-8}) and the estimated condition numbers for the cases $\alpha_I = 10^{+5}$ and $\alpha_I = 10^{-5}$. Note, however, that it is possible to come up with examples where the dependence on H/η is quadratic (see Section 7.5).

In Example 2, we choose the coefficient distribution sketched in Figure 6, right. Again, we set α to a constant α_I in the island (the shaded square), and 1 elsewhere. Note that here, the width/height of the island remains of fixed size H. In Figure 8 and in Table 3, we see that the condition number behaves essentially like in Example 1.

7.2 Standard one-level versus all-floating FETI

In Example 3, we consider a coefficient island cutting through a domain that touches the Dirichlet boundary, cf. Fig. 9, left. As before, we impose Dirichlet boundary conditions on the whole of $\partial\Omega$ and choose $\alpha = \alpha_I = \text{const}$ inside the shaded square, and $\alpha = 1$ elsewhere. As one can see in Table 4, the standard one-level FETI method is not robust when $\alpha_I > 1$, whereas the all-floating method remains robust. The reason why the number of PCG iterations stays small in all cases (even when

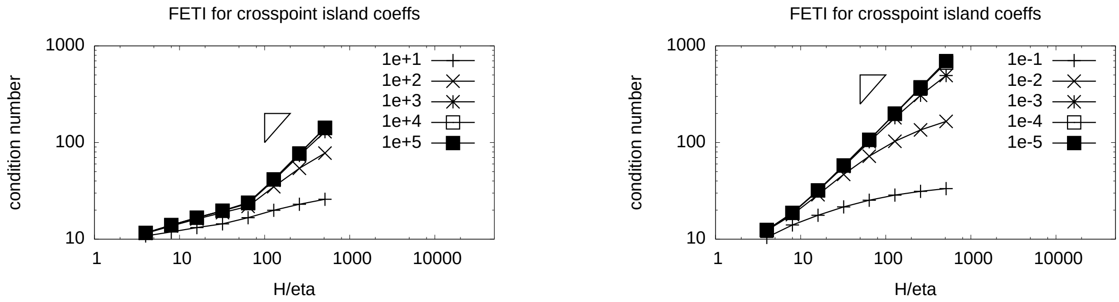

Fig. 8 Example 2: estimated condition numbers; $H/h = 512$, varying ratio H/η and varying magnitude of the jump α_I. *Left* $\alpha_I > 1$, *right* $\alpha_I < 1$

$\frac{H}{\eta}$	$\frac{H}{h} = 64$		128		256		512	
4	7.36	(8.06)	8.92	(9.77)	10.64	(11.63)	12.51	(13.64)
8	8.74	(11.99)	10.45	(14.44)	12.31	(17.03)	14.34	(19.78)
16	10.83	(19.83)	12.62	(23.95)	14.60	(28.19)	16.75	(32.58)
32	15.30	(34.19)	16.85	(41.71)	18.63	(49.38)	20.66	(57.18)
64	26.19	(59.91)	27.45	(74.11)	28.81	(88.67)	30.30	(103.38)
128	–		50.98	(132.61)	52.32	(160.58)	53.60	(189.14)
256	–		–		101.48	(291.42)	103.36	(346.84)
512	–		–		–		221.25	(635.45)

Table 3 Example 2: estimated condition numbers; island coefficient with $\alpha_I = 10^{+5}$, in brackets: $\alpha_I = 10^{-5}$.

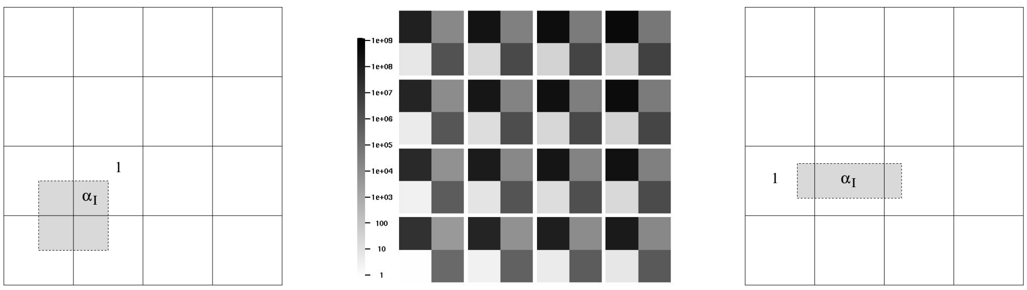

Fig. 9 Three different coefficient distributions and subdomain partitionings. *Left* Example 3, *middle* Example 4, *right* Example 5

	std. one-level		all-floating			std. one-level		all-floating	
α_I	it	cond	it	cond	α_I	it	cond	it	cond
1	17	8.32	19	7.11	1	17	8.32	19	7.11
10^{-1}	21	8.40	22	7.32	10^{+1}	24	$2.59 \cdot 10^1$	22	7.25
10^{-2}	21	8.42	22	7.49	10^{+2}	27	$2.05 \cdot 10^2$	21	7.31
10^{-3}	21	8.43	22	7.51	10^{+3}	31	$2.00 \cdot 10^3$	21	7.32
10^{-4}	21	8.43	22	7.51	10^{+4}	31	$1.99 \cdot 10^4$	21	7.32
10^{-5}	21	8.43	22	7.51	10^{+5}	36	$1.99 \cdot 10^5$	21	7.32
10^{-6}	21	8.43	22	7.51	10^{+6}	38	$1.99 \cdot 10^6$	22	7.32
10^{-7}	21	8.43	22	7.51	10^{+7}	42	$1.99 \cdot 10^7$	21	7.32

Table 4 Example 3: iteration numbers and condition number estimates, standard one-level vs. all-floating FETI, $H/h = 128$

the condition number blows up) is due to the fact that we have only considered one coefficient island. This is related to spectral clustering effects, which are explained e.g. in [15].

The fact that one-level FETI is fully robust for $\alpha_I < 1$ is not contradicting our theory and is perfectly explained by Lemma 6.2, using the overlapping regions $\Omega_i^{(1)} = \Omega_i \setminus \Omega_i^{(I)}$ and $\Omega_i^{(2)} = \Omega_i$, the artificial coefficients $\underline{\alpha}_{i,\mathrm{art}}^{(1)} = \overline{\alpha}_{i,\mathrm{art}}^{(1)} = 1$ and $\underline{\alpha}_{i,\mathrm{art}}^{(2)} = \overline{\alpha}_{i,\mathrm{art}}^{(2)} = \alpha_I$, as well as the manifold $X_i^* := \partial\Omega_i \cap \partial\Omega$

H/h	32	64	128	256	512
std. one-level	25 (28.27)	27 (36.30)	29 (43.94)	30 (50.69)	32 (56.49)
all-floating	26 (20.38)	30 (28.90)	33 (37.07)	34 (44.18)	37 (50.13)

Table 5 Example 4: number of PCG iterations, estimated condition numbers in brackets

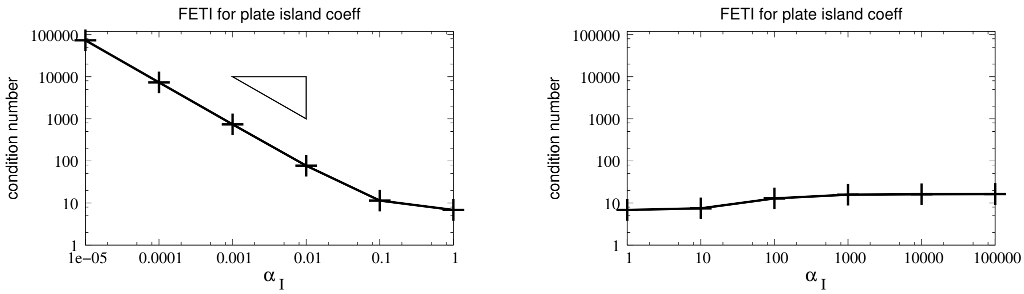

Fig. 10 Example 5: estimated condition numbers; $H/h = 64$, varying magnitude of the jump α_I. *Left:* $\alpha_I < 1$, *right:* $\alpha_I > 1$.

(which is part of the Dirichlet boundary) in all subdomains that contain the region $\Omega_i^{(I)}$ and touch the Dirichlet boundary. Then Lemma 6.2 applies with $\sigma(H_i/h_i) = 1$, and due to Corollary 4.2 we have robustness.

7.3 "Multi-valued" coefficients

In Example 4, we choose 16 subdomains and the coefficient distribution shown in Figure 9, middle. Again, we choose Dirichlet boundary conditions on the whole of $\partial\Omega$. The coefficient takes *four* different values in each subdomain varying in a non-quasi-monotone way, namely in anticlockwise order 1, 10^5, 10^3, and 10^7. To rule out any unforeseen effects caused by a periodic variation, we further multiply the coefficient α at every point $x \in [0,1]^2$ by $(1 + \lfloor 8x_1 \rfloor)(1 + \lfloor 8x_2 \rfloor)$. Table 5 shows the iteration numbers and estimated condition numbers for standard one-level and all-floating FETI. As we expect, the all-floating method is robust in α. We also observe robustness of one-level FETI. Indeed, that can be shown theoretically (as in the previous subsection) by simply choosing overlapping subregions like in Lemma 6.2 that all connect to a part of the Dirichlet boundary.

7.4 Non-robust behaviour

In Example 5, we would like to demonstrate that the condition number can deteriorate considerably if our assumptions on the coefficient distribution fail. Figure 9, right, shows the setup, i. e. a plate that cuts through one of the subdomains as in Figure 4 in Section 3.3. We choose a fixed discretisation of $H/h = 64$ and let α_I vary in the range $[10^{-5}, 10^{+5}]$. The two graphs in Figure 10 show the estimated condition number for all-floating FETI. It clearly depends in a linear way on α_I^{-1} for $\alpha_I < 0$ as outlined in Section 3.3, since the weighted Poincaré / Sobolev inequality fails to hold. Again, due to spectral clustering effects, the method itself is not affected: the number of PCG iterations lies between 14 and 21. For the case $\alpha_I > 0$, the condition number is independent of α_I, which is in perfect accordance to Lemma 6.2.

7.5 Sharpness of the bounds

Finally, in Example 6, which is due to M. Sarkis, we would like to address the sharpness of our condition number bounds with respect to the geometric parameters. The setup is shown in Figure 11, left, with inclusions in two neighbouring subdomains with coefficient values both *larger* and *smaller*

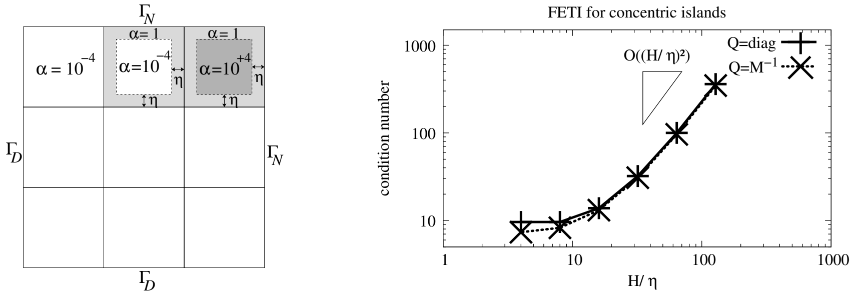

Fig. 11 Example 6: *Left* setup, *right* estimated condition number for fixed discretisation $H/h = 256$, varying parameter η, and two choices of Q (diagonal according to (4.15) as well as $Q = M^{-1}$)

than in the boundary layers. We prescribe homogeneous Dirichlet boundary conditions on Γ_D and homogeneous Neumann boundary conditions on Γ_N. For the choice $Q = M^{-1}$, Theorem 6.4 predicts a condition number bound of $\mathcal{O}((H/\eta)^2\,(1 + \log(H/h))^2)$ in this case, and this quadratic dependence on H/η can be observed numerically: Figure 11, right, shows the estimated condition number of the all-floating FETI method for a fixed discretisation of $H/h = 256$ varying the ratio H/η. There are two graphs in that figure; one corresponding to our default diagonal choice of Q, and another one corresponding to $Q = M^{-1}$, but as one can see the condition numbers are essentially the same. This suggests that it might be possible to eliminate the extra factor H/η also in Theorem 4.1. For the standard one-level FETI methods, the condition numbers are virtually the same.

Sharper results for the case that *all* coefficients in the subdomain interiors are smaller ("inclusion soft type") are sketched in a recent proceedings paper by Dryja and Sarkis, cf. [11].

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