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Convergence of Restricted Additive Schwarz with impedance transmission conditions for discretised Helmholtz problems

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Abstract

The Restricted Additive Schwarz method with impedance transmission conditions, also known as the Optimised Restricted Additive Schwarz (ORAS) method, is a simple overlapping one-level parallel domain decomposition method, which has been successfully used as an iterative solver and as a preconditioner for discretized Helmholtz boundary-value problems. In this paper, we give, for the first time, a convergence analysis for ORAS as an iterative solver – and also as a preconditioner – for nodal finite element Helmholtz systems of any polynomial order. The analysis starts by showing (for general domain decompositions) that ORAS is an unconventional finite element approximation of a classical parallel iterative Schwarz method, formulated at the PDE (non-discrete) level. This non-discrete Schwarz method was recently analysed in [Gong, Gander, Graham, Lafontaine, Spence, arXiv 2106.05218], and the present paper gives a corresponding discrete version of this analysis. In particular, for domain decompositions in strips in 2-d, we show that, when the mesh size is small enough, ORAS inherits the convergence properties of the Schwarz method, independent of polynomial order. The proof relies on characterising the ORAS iteration in terms of discrete 'impedance-to-impedance maps', which we prove (via a novel weighted finite-element error analysis) converge as $h \to 0$ in the operator norm to their non-discrete counterparts.

1 Introduction

The Helmholtz equation, as the time-harmonic form of the wave equation, arises in many scientific and engineering applications, including acoustics, seismic imaging and earthquake modelling. Solutions at high-frequency are often required; for example, in inverse scattering when imaging fine details of a scatterer. However, solving the discretised Helmholtz equation at high frequency is very challenging, first because of the huge number of degrees of freedom (required, in general, to approximate the oscillating solutions) and second because the system matrices are non-Hermitian and highly indefinite. There exist considerable efforts in the literature for finding efficient algorithms for solving discrete Helmholtz problems, e.g., to name two of the most successful, the "shifted Laplace" preconditioner using a multigrid strategy [15] and sweeping algorithms and their variants, e.g., [14, 6, 36], which can be viewed as multiplicative domain decomposition methods. Substantial recent reviews of solvers for discrete Helmholtz problems can be found, for example in [19] and in the introductions to [23] and [36]. However, most practical methods are justified by empirical experiments and there are relatively few rigorous convergence analyses for usable methods.

In this paper, we give a new analysis of the Restricted Additive Schwarz preconditioner with local impedance transmission conditions, also called the ORAS ('Optimised Restricted Additive Schwarz') method, which is arguably the most successful one-level inherently-parallel domain decomposition method for Helmholtz problems. ORAS can be applied on very general geometries, does not require parameter-tuning, and (as the numerical experiments in [20] show) can even be robust to increasing frequency (in the sense that, in some situations, the number of iterations is independent of frequency). More generally, ORAS can be combined with coarse spaces to improve its robustness properties; see, e.g., [3]. ORAS has been used as a stand-alone solver and also in conjunction with a coarse correction (sometimes applied hierarchically) in substantial scientific applications (e.g., [37, 3, 2]). The ORAS preconditioner has quite a large literature, e.g., [35, 10, 25] but, although there are arguments partially explaining its success (e.g., [13], [10, §2.3.2]), there is no rigorous convergence theory for Helmholtz problems. It is worth mentioning that there has been considerable recent interest in convergence theory for non-overlapping domain decomposition methods for Helmholtz problems, e.g., [29, 7, 9]; these algorithms and the corresponding analyses are quite distinct from the method and analysis given here. Here we are motivated to investigate overlapping methods because of the wide practical use of (variants of) Restricted Additive Schwarz methods (e.g., [37]), because of the lack of theoretical understanding of these methods, and also because our previous work showed promising performance of this method in practice (e.g., [2, 22]). Moreover in the recent paper [20] the benefit of overlap was shown rigorously at the non-discrete (PDE) level.

Since the system matrices arising from Helmholtz problems are typically non-Hermitian and indefinite, the Krylov method of choice is GMRES, and a standard approach for predicting its convergence uses the 'Elman estimate', which requires that the norm of the system matrix is bounded above and its field of values is bounded away from the origin. This theory was successfully applied to a simple variant of the ORAS preconditioner (called SORAS), for Helmholtz problems with some absorption [23, 22]. However this theory was unable to predict a convergence rate for the ORAS (or SORAS) preconditioner applied to the Helmholtz problem with no absorption; in fact the field of values of the ORAS preconditioned system matrix, for a particular example, was shown in [22, Figure 2] to include the origin of the complex plane and to have a growing boundary as the frequency increases. Nevertheless, [22] also shows that ORAS still works well as preconditioner for the Helmholtz problem with no absorption and here we develop a novel analysis (avoiding analysing the field of values) that explains this.

1.1 The Helmholtz problem and its discretisation

Although the ORAS method can be applied to very general scattering problems in general geometries, we give its analysis here for the model interior impedance problem defined as follows. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain. Given the source function $f \in L^2(\Omega)$, the impedance data $g \in L^2(\partial\Omega)$, and the frequency $k > 0$, we consider the problem of finding the solution u of

$$\begin{aligned} \Delta u + k^2 u &= -f \quad \text{in} \quad \Omega, \\ \partial_n u - \mathrm{i}ku &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where ∂_n denotes the outward normal derivative on $\partial\Omega$. We write this problem in variational form: find $u \in H^1(\Omega)$ such that

$$a(u, v) = F(v) \text{ for all } v \in H^1(\Omega),$$

where, for all $u, v \in H^1(\Omega)$,

$$a(u, v) := \int_\Omega \nabla u \cdot \overline{\nabla v} - k^2 u\, \overline{v} - \int_{\partial\Omega} \mathrm{i}ku\, \overline{v} \quad \text{and} \quad F(v) := \int_\Omega f\, \overline{v} + \int_{\partial\Omega} g\, \overline{v}. \tag{1.2}$$

We are concerned with solvers for the discrete version of (1.1) in a nodal finite element space $V \subset H^1(\Omega)$ consisting of continuous piecewise polynomials of total degree $\le p$ on a conforming simplicial mesh T^h with mesh diameter h, i.e., we seek the solution $u_h \in V$ to the problem

$$a(u_h, v_h) = F(v_h) \text{ for all } v_h \in V. \tag{1.3}$$

This paper concerns iterative solvers and preconditioners for computing u_h, especially for k large. Let V' be the dual space of V. It is useful to define the discrete operators $\mathcal{A}_h : V \mapsto V'$ and $F_h \in V'$ by

$$(\mathcal{A}_h u_h)(v_h) := a(u_h, v_h) \quad \text{and} \quad F_h(v_h) = F(v_h) \quad \text{for all} \quad u_h, v_h \in V_h, \tag{1.4}$$

and then to write equation (1.3), equivalently, as the equation

$$\mathcal{A}_h u_h = F_h \tag{1.5}$$

to be solved in V' for the solution $u_h \in V$. Under a mesh resolution condition linking h and k, this equation always has a unique solution u_h (see Theorem 1.1 below).

1.2 Related previous work at the PDE level

In the recent paper [20] we studied the particular parallel overlapping Schwarz method (specified in (2.7)-(2.10) below) for the problem (1.1) at the PDE (i.e., non-discrete) level. In (2.7)-(2.10), $\{\Omega_j\}_{j=1}^N$, are a set of N polygonal or Lipschitz polyhedral subdomains forming an overlapping cover of the global domain Ω. (When we come to the finite element counterpart, we assume that each subdomain boundary $\partial\Omega_j$ is resolved by the finite element mesh T^h.) The method (2.7)-(2.10) can be thought of as a generalisation of the classical algorithm of Després [8], [1] to the case of overlapping subdomains.

The convergence theory for (2.7)-(2.10) given in [20] starts by showing that the vector of errors on each subdomain:

$$\boldsymbol{e}^n = (e_1^n, e_2^n, \dots e_N^n)^\top, \quad \text{where} \quad e_j^n = u|_{\Omega_j} - u_j^n, \quad j = 1, \dots, N, \tag{1.6}$$

(where u is the solution of (1.1) and u_j^n is given in (2.7)– (2.9)) satisfies a fixed-point iteration of the form

$$\boldsymbol{e}^{n+1} = \boldsymbol{\mathcal{T}} \boldsymbol{e}^n, \tag{1.7}$$

with $\boldsymbol{\mathcal{T}}$ defined by (2.19)-(2.21). The paper [20] showed that, for general Lipschitz subdomains, the fixed-point operator $\boldsymbol{\mathcal{T}}$ acts in the tensor product of Helmholtz-harmonic functions (i.e., solutions of $(\Delta + k^2)u = 0$) on each subdomain. Using this setting, [20] characterised the powers of $\boldsymbol{\mathcal{T}}$ in terms of (compositions of) "impedance-to-impedance maps" linking pairs of subdomains with non-trivial intersection.

In addition, the paper [20] then gave, for the special case of 2-d rectangular domains covered by overlapping strips, sufficient conditions for $\boldsymbol{\mathcal{T}}^{sN}$ to be a contraction, where N is the number of subdomains and s is a small integer. These conditions were formulated in terms of norms of (compositions of) impedance-to-impedance maps; the conditions were investigated (and verified in certain cases) by both analytical and numerical techniques. To illustrate the theory, experiments with finite element approximations of the parallel Schwarz method were given in [20].

1.3 The main results of this paper

While the results in [20] gave sufficient conditions for the power contractivity of the Schwarz method at the PDE level for domain decompositions in strips, they did not prove the power contractivity of the corresponding method in the practical finite element case (namely, ORAS, defined explicitly in §2.1). The present paper establishes, for the first time, conditions for ORAS to be power contractive. The main results of the present paper (and the technical obstacles that had to be overcome) are as follows.

- For general Lipschitz domains and subdomains (in 2- and 3-d), Theorem 2.5 shows that the ORAS iteration can be interpreted as a finite element approximation of the parallel Schwarz method (2.7)-(2.10); a subtlety of this interpretation is an appropriate discretization of the boundary condition (2.8).

- In this general set-up, ORAS is formulated in (3.5) as a fixed point iteration of the form

$$\boldsymbol{e}_h^{n+1} = \boldsymbol{\mathcal{T}}_h \boldsymbol{e}_h^n,$$

where $\boldsymbol{e}_h^n := (e_{h,1}, \dots, e_{h,N})^\top$ is the discrete analogue of (1.6) and is defined precisely in §3.1. It is shown in §3.2 that $e_{h,\ell}^n$ is discrete Helmholtz-harmonic on Ω_ℓ (i.e., is a solution at the finite-element level of the homogeneous Helmholtz problem on Ω_ℓ). The impedance data of a discrete

Helmholtz-harmonic function on Ω_ℓ is introduced in Lemma 3.4 and this defines a norm (subject to a mesh resolution condition) on the space of discrete Helmholtz-harmonic functions (see Proposition 3.6); this norm is the discrete analogue of the boundary impedance norm (or 'pseudo-energy') introduced in [8] and is used to analyse the power contractivity of $\mathcal{T}_h$.

• Theorem 4.9 shows that, for decompositions in strips in 2-d, the action of (powers of) $\mathcal{T}_h$ can be expressed in terms of (compositions of) discrete impedance-to-impedance maps, analogous to the theory at the continuous level in [20, §4].

• Theorem 5.5 proves that, for these 2-d strip decompositions, the discrete impedance-to-impedance maps converge in the L^2 norm to their continuous counterparts as $h \to 0$ for any fixed k; this result shows that the numerical procedure used in [20] for computing the norms of the impedance-to-impedance maps is reliable. To prove norm convergence of the discrete impedance-to-impedance maps, we adapt the elliptic-projection argument introduced in [16, 17] for Helmholtz finite-element analysis to a weighted setting, using enhanced regularity at interior interfaces.

• The main result of the paper, Theorem 4.17, is that, for these 2-d strip decompositions, if the parallel Schwarz method is power contractive, then so is ORAS for h sufficiently small; furthermore, the power contractivity of ORAS is independent of the polynomial degree of the finite elements (see Corollary 4.18). The proof that ORAS "inherits" the properties of the parallel Schwarz method is not obvious, because the discrete and continuous fixed-point operators, $\mathcal{T}_h$ and $\mathcal{T}$, operate in different spaces (here called $\mathbb{V}_0$ and $\mathbb{U}_0$); nevertheless we show that, for all n, $\|\mathcal{T}_h^n\|_{\mathbb{V}_0} \to \|\mathcal{T}^n\|_{\mathbb{U}_0}$ as $h \to 0$.

• Since the experiments in [20] essentially used the ORAS method to illustrate the parallel Schwarz method, we just summarise these in §6, but we also provide some additional experiments, especially designed to illustrate that the number of ORAS iterations is independent of h and p.

Organisation of paper. §2 introduces the ORAS and the parallel Schwarz methods and explores their connections. §3 obtains the fixed point iteration satisfied by ORAS and examines the spaces in which it operates. §4 restricts attention to decompositions of a rectangular domain into strips, introduces the discrete impedance-to-impedance maps, and proves that, for h small enough, ORAS is power contractive if the parallel Schwarz method has the same property. §4.6 recalls results from [20] that give conditions for the power contractivity of the parallel Schwarz method. §5 shows the norm convergence of the discrete impedance-to-impedance maps. §6 gives numerical experiments.

1.4 Wellposedness of discrete Helmholtz problems

The Helmholtz problem (1.1) at the continuous level is well-posed for all $k > 0$ (see, e.g., [32, §8]). The following result guarantees the existence of a unique solution $u_h \in V$ to the discrete problem (1.5).

Theorem 1.1. *If Ω is convex, then there exists a dimensionless constant C, independent of h, k, and the diameter of Ω (but dependent on p) such that if*

$$h^{2p}k^{2p+1} \leq \frac{C}{\mathrm{diam}(\Omega)}, \tag{1.8}$$

then the problem (1.5) has a unique solution $u_h \in V$.

References for the proof. This result, without the explicit dependence on diam(Ω), was proved in [11, Corollary 6.11]. The result with this explicit dependence (and for the more-general case when the Helmholtz equation has variable coefficients) was proved in [30, Theorem 2.39]; indeed, the condition (1.8) is [30, equation (2.62)], the constants $\mathscr{C}(n)$ and C_{cond} are dimensionless (see their definitions in (2.65) and Table 2.4, respectively), and the constant C_{stab} defined in (2.61) can be written as a non-dimensional quantity multiplied by diam(Ω) (see, e.g., [27, Equation 3.5] and the associated discussion). □

In the rest of the paper we implicitly assume that u_h exists, so that the discussion of iterative methods for finding u_h makes sense. Note that: (i) estimates for the error $u-u_h$ can also be proven under the condition (1.8) (see [11] and, e.g., the overviews in [23, Remark 2.9], [27, §1]), and (ii) if Ω is not convex, then well-posedness of (1.5) can still be proved, but under more restrictive conditions on h and k (since one no longer has H^2 regularity of the solution; see the discussion in the last paragraph of Remark 5.22).

2 Restricted Additive Schwarz with impedance transmission condition (ORAS)

2.1 Definition of the ORAS method

We assume that each subdomain boundary $\partial\Omega_j$ is resolved by the finite element mesh T^h. We denote the diameter of Ω_j by H_j and set $H = \max_j H_j$. We introduce a partition of unity $\{\chi_j\}_{j=1}^N$, with the properties

$$\left.\begin{array}{ll}\text{for each} \quad j, & \text{supp}\chi_j \subset \overline{\Omega_j}, \quad 0 \le \chi_j(\boldsymbol{x}) \le 1 \text{ when } \boldsymbol{x} \in \overline{\Omega_j}, \\ \text{and} & \sum_j \chi_j(\boldsymbol{x}) = 1 \text{ for all } \boldsymbol{x} \in \overline{\Omega}.\end{array}\right\} \tag{2.1}$$

Notation 2.1. *On any curve (in 2-d)/surface (in 3-d) $\Gamma \subset \Omega$, we let $\langle v, w\rangle_\Gamma = \int_\Gamma v\overline{w}$. On any subdomain $D \subset \Omega$, we introduce the local sesquilinear forms*

$$\widetilde{a}_D(v,w) := \int_D \nabla v \cdot \overline{\nabla w} - k^2 v\,\overline{w} \quad \textit{and} \quad a_D(v,w) := \widetilde{a}_D(v,w) - \mathrm{i}k\langle v, w\rangle_{\partial D};$$

note that a_D arises in the variational formulation of a local Helmholtz impedance problem on D. When $D = \Omega$, we denote $\widetilde{a}_\Omega(\cdot,\cdot)$ by $\widetilde{a}(\cdot,\cdot)$ and $a_\Omega(\cdot,\cdot)$ by $a(\cdot,\cdot)$.

Define the local finite element space on $\overline{D}$ by $V(D) := \{v_h|_{\overline{D}} \ : \ v_h \in V\}$ with corresponding nodes $N^h(\overline{D}) \subset N^h(\overline{\Omega})$, where $N^h(\overline{\Omega})$ denotes the nodes of V. For simplicity, we write V_j instead of $V(\Omega_j)$. Then, analogously to (1.4) we define $\mathcal{A}_{h,j} : V_j \to V_j'$ by

$$(\mathcal{A}_{h,j}v_{h,j})(w_{h,j}) := a_{\Omega_j}(v_{h,j}, w_{h,j}) \quad \text{for all } v_{h,j}, w_{h,j} \in V_j.$$

We assume that h is small enough so that each local problem is well-posed:

Assumption 2.2. *For each $j = 1,\ldots,N$, $\mathcal{A}_{h,j} : V_j \to V_j'$ is invertible.*

Remark. If Ω and each Ω_j, $j = 1,\ldots,N$ are convex, and (1.8) holds, then Assumption 2.2 holds, since $(\text{diam}(\Omega))^{-1} \le H_j^{-1}$.

We also need prolongations and restrictions linking local and global problems. For any subset D of $\overline{\Omega}$ (which we assume to be a union of elements of the mesh T^h), we define the prolongation $\mathcal{R}_{h,D}^\top : V(D) \to V$ defined for all $v_{h,D} \in V(D)$ by

$$(\mathcal{R}_{h,D}^\top v_{h,D})(x_i) = \begin{cases} v_{h,D}(x_i) & x_i \in N^h(\overline{D}), \\ 0 & x_i \in N^h(\overline{\Omega})\backslash N^h(\overline{D}). \end{cases} \tag{2.2}$$

Note that the extension $\mathcal{R}_{h,D}^\top v_{h,D}$ is defined *nodewise*: It coincides with $v_{h,D}$ at finite element nodes in $\overline{D}$ and vanishes at nodes in $\overline{\Omega}\backslash\overline{D}$. Thus $\mathcal{R}_{h,D}^\top v_{h,D} \in V \subset H^1(\Omega)$ is a finite element approximation of the zero extension of $v_{h,D}$ to all of Ω. (The zero extension itself – (2.24) below – is not in $H^1(\Omega)$ in general.) For simplicity, we denote $\mathcal{R}_{h,\Omega_j}^\top : V_j \to V$ by $\mathcal{R}_{h,j}^\top$.

Then we can define the restriction operator $\mathcal{R}_{h,j} : V' \to V_j'$ by duality, i.e.,

$$(\mathcal{R}_{h,j}G_h)(v_{h,j}) := G_h(\mathcal{R}_{h,j}^\top v_{h,j}) \quad \text{for all} \ \ v_{h,j} \in V_j, \ G_h \in V',. \tag{2.3}$$

Finally, in restricted additive Schwarz methods, prolongation from local to global is done using a partition of unity, and so we also define a weighted prolongation $\widetilde{\mathcal{R}}_{h,j}^\top : V_j \to V$ by, for all $v_{h,j} \in V_j$,

$$\widetilde{\mathcal{R}}_{h,j}^\top v_{h,j} = \mathcal{R}_{h,j}^\top (\Pi_h(\chi_j v_{h,j})),$$

where Π_h is the nodal interpolation onto V. Note that

$$\sum_\ell \widetilde{\mathcal{R}}_{h,\ell}^\top (w_h|_{\Omega_\ell}) \;=\; w_h, \text{ for all } w_h \in V. \tag{2.4}$$

Then the ORAS preconditioner is the operator $\mathcal{B}_h^{-1} : V' \to V$ defined by

$$\mathcal{B}_h^{-1} \;:=\; \sum_j \widetilde{\mathcal{R}}_{h,j}^\top \mathcal{A}_{h,j}^{-1} \mathcal{R}_{h,j}. \tag{2.5}$$

The corresponding preconditioned Richardson iterative method for (1.5) can be written as

$$u_h^{n+1} = u_h^n + \mathcal{B}_h^{-1}(F_h - \mathcal{A}_h u_h^n); \tag{2.6}$$

the matrix version of this is discussed in [21]. In §2.3, we identify (2.6) as a discrete version of the following classical iterative method for the Helmholtz problem at the PDE level.

2.2 The Schwarz method with impedance transmission condition

Starting with the Helmholtz problem (1.1) and the domain decomposition $\{\Omega_j\}_{j=1}^N$, the parallel Schwarz method is the following: given the nth iterate u^n defined on Ω, solve the local problem on Ω_j for u_j^{n+1} ,

$$(\Delta + k^2) u_j^{n+1} = -f \qquad \text{in } \Omega_j, \tag{2.7}$$

$$\left(\frac{\partial}{\partial n_j} - \mathrm{i}k\right) u_j^{n+1} = \left(\frac{\partial}{\partial n_j} - \mathrm{i}k\right) u^n \qquad \text{on } \Gamma_j := \partial\Omega_j \backslash \partial\Omega, \tag{2.8}$$

$$\left(\frac{\partial}{\partial n_j} - \mathrm{i}k\right) u_j^{n+1} = g \qquad \text{on } \partial\Omega_j \cap \partial\Omega, \tag{2.9}$$

where $\partial/\partial n_j$ denotes the outward normal derivative on $\partial\Omega_j$. The new iterate u^{n+1} is then the weighted sum of the local solutions

$$u^{n+1} := \sum_j \chi_j u_j^{n+1}. \tag{2.10}$$

This method is a generalization of the classical algorithm of Després [8], [1] to the case of overlapping subdomains, and [20] performs a convergence analysis of it. To derive the problem satisfied by the error, note that if u denotes the solution of (1.1), then, $u_j := u|_{\Omega_j}$ satisfies

$$(\Delta + k^2) u_j = -f \qquad \text{in } \Omega_j, \tag{2.11}$$

$$\left(\frac{\partial}{\partial n_j} - \mathrm{i}k\right) u_j = \left(\frac{\partial}{\partial n_j} - \mathrm{i}k\right) u \qquad \text{on } \Gamma_j, \tag{2.12}$$

$$\left(\frac{\partial}{\partial n_j} - \mathrm{i}k\right) u_j = g \qquad \text{on } \partial\Omega_j \cap \partial\Omega, \tag{2.13}$$

Then, using (2.10) and (2.1), the global error $e^n := u - u^n$ can be written as

$$e^n = \sum_\ell \chi_\ell u|_{\Omega_\ell} - \sum_\ell \chi_\ell u_\ell^n = \sum_\ell \chi_\ell e_\ell^n, \quad \text{where} \quad e_\ell^n = u|_{\Omega_\ell} - u_\ell^n.$$

Thus, subtracting (2.7)-(2.9) from (2.11)-(2.13), we obtain

$$(\Delta + k^2)e_j^{n+1} = 0 \quad \text{in} \quad \Omega_j, \tag{2.14}$$

$$\left(\frac{\partial}{\partial n_j} - \mathrm{i}k\right) e_j^{n+1} = \left(\frac{\partial}{\partial n_j} - \mathrm{i}k\right) e^n = \sum_\ell \left(\frac{\partial}{\partial n_j} - \mathrm{i}k\right) \chi_\ell e_\ell^n \quad \text{on} \quad \Gamma_j \tag{2.15}$$

$$\left(\frac{\partial}{\partial n_j} - \mathrm{i}k\right) e_j^{n+1} = 0 \quad \text{on} \quad \partial\Omega_j \cap \partial\Omega. \tag{2.16}$$

In [20], this method was analysed in the following spaces. For D a Lipschitz domain, let

$$U(D) := \left\{u \in H^1(D) : \Delta u + k^2 u \in L^2(D),\ \partial u/\partial n - \mathrm{i}ku \in L^2(\partial D)\right\},$$

and let the corresponding 'Helmholtz harmonic' space be defined by

$$U_0(D) := \left\{u \in H^1(D) : \Delta u + k^2 u = 0 \text{ in } D,\ \partial u/\partial n - \mathrm{i}ku \in L^2(\partial D)\right\} \subset U(D); \tag{2.17}$$

we equip $U_0(D)$ with the 'boundary (pseudo-)energy norm' (see [8], [20, §3.1])

$$\|v\|^2_{U_0(D)} \ := \ \left\|\frac{\partial v}{\partial n} - \mathrm{i}kv\right\|^2_{L^2(\partial D)}. \tag{2.18}$$

The error vector (1.6) thus satisfies the fixed-point iteration (1.7), where the matrix of operators $\boldsymbol{\mathcal{T}} = (\mathcal{T}_{j,\ell})_{j,\ell=1}^N$ is defined as follows. For any $j, \ell \in \{1, \ldots, N\}$ (not necessarily equal), we define, for $v_\ell \in U(\Omega_\ell)$,

$$(\Delta + k^2)(\mathcal{T}_{j,\ell} v_\ell) = 0 \quad \text{in} \quad \Omega_j, \tag{2.19}$$

$$\left(\frac{\partial}{\partial n_j} - \mathrm{i}k\right)(\mathcal{T}_{j,\ell} v_\ell) = \left(\frac{\partial}{\partial n_j} - \mathrm{i}k\right)(\chi_\ell v_\ell) \quad \text{on} \quad \Gamma_j, \tag{2.20}$$

$$\left(\frac{\partial}{\partial n_j} - \mathrm{i}k\right)(\mathcal{T}_{j,\ell} v_\ell) = 0 \quad \text{on} \quad \partial\Omega_j \cap \partial\Omega. \tag{2.21}$$

The analysis in [20] then shows that $\boldsymbol{\mathcal{T}}$ is a bounded operator on the tensor product space

$$\mathbb{U}_0 := \prod_{j=1}^N U_0(\Omega_j), \tag{2.22}$$

Moreover, under certain assumptions, [20] shows that $\boldsymbol{\mathcal{T}}^N$ is a contraction, thus guaranteeing the convergence of the parallel Schwarz method (2.7)-(2.10). This paper proves analogous estimates for ORAS; we begin by connecting ORAS to (2.7)-(2.10).

2.3 Connection between (2.7)-(2.10) and ORAS

In this subsection, we show that a finite element approximation of (2.7)-(2.10) yields (2.6). First note that the variational form of (2.7)-(2.9) is: find $u_j^{n+1} \in H^1(\Omega_j)$ such that

$$a_{\Omega_j}(u_j^{n+1}, v_j) = F(\mathcal{R}_j^\top v_j) + \left\langle \left(\frac{\partial}{\partial n_j} - \mathrm{i}k\right) u^n, v_j \right\rangle_{\Gamma_j} \quad \text{for all } v_j \in H^1(\Omega_j), \tag{2.23}$$

where F as given by (1.2) and $\mathcal{R}_j^\top$ is the zero extension operator with domain $H^1(\Omega_j)$:

$$\mathcal{R}_j^\top v_j = \begin{cases} v_j & \text{in } \overline{\Omega_j}, \\ 0 & \text{in } \overline{\Omega} \backslash \overline{\Omega_j}. \end{cases} \tag{2.24}$$

It is not immediately clear how best to approximate the second term on the right-hand side of (2.23) when u^n is replaced by a finite element approximation. To understand how to deal with this issue, it is useful to introduce the following non-standard sesquilinear form associated with (1.1)

$$\alpha(w,v) = -\int_\Omega ((\Delta + k^2)w)\overline{v} + \left\langle \left(\frac{\partial}{\partial n} - \mathrm{i}k\right) w, v \right\rangle_{\partial\Omega}, \tag{2.25}$$

This is well-defined for $w \in U(\Omega)$ and $v \in L^2(\Omega)$, as long as v has $L^2(\partial\Omega)$ trace. Indeed, by applying Green's identity, we see that

$$\alpha(w,v) = a(w,v) \quad \text{if} \quad w \in U(\Omega) \quad \text{and} \quad v \in H^1(\Omega). \tag{2.26}$$

Another application of Green's identity gives the following formula for the boundary integral term in (2.23).

Proposition 2.3. *If $(w, v_j) \in U(\Omega) \times H^1(\Omega_j)$, then*

$$\begin{aligned} \left\langle \left(\frac{\partial}{\partial n_j} - \mathrm{i}k\right) w, v_j \right\rangle_{\Gamma_j} &= a_{\Omega_j}(w, v_j) - \alpha(w, \mathcal{R}_j^\top v_j) \\ &=: b_j(w, v_j) \end{aligned} \tag{2.27}$$

Even though $\mathcal{R}_j^\top v_j$ is not, in general, in $H^1(\Omega)$, the right-hand side of (2.27) is well-defined. To obtain our finite element analogue of (2.23), we define an analogue of b_j on the product space $H^1(\Omega) \times V_j$ by

$$b_{h,j}(w, v_{h,j}) := a_{\Omega_j}(w, v_{h,j}) - a(w, \mathcal{R}_{h,j}^\top v_{h,j}) \quad \text{for all } (w, v_{h,j}) \in H^1(\Omega) \times V_j. \tag{2.28}$$

To obtain (2.28) from (2.27) we have replaced $\mathcal{R}_j^\top$ (simple extension by zero) by the node-wise finite element extension operator $\mathcal{R}_{h,j}^\top$. Since $\mathcal{R}_{h,j}^\top v_{h,j} \in H^1(\Omega)$ we can then use (2.26) to replace α in (2.27) by a in (2.28).

We now show formally that $b_{h,j}$ in (2.28) approximates the boundary integral b_j in (2.27). (Later we prove a convergence result for this approximation in a specific geometric setting; see Definition 5.3 and Theorem 5.5.) Let $v_{h,j} \in V_j \cap H^1_{0,\Gamma_j}(\Omega_j)$, where Γ_j is defined in (2.8) and

$$H^1_{0,\Gamma_j}(\Omega_j) = \{v \in H^1(\Omega_j) \ : \ v = 0 \text{ on } \Gamma_j\}. \tag{2.29}$$

By (2.2) and (2.24), $\mathcal{R}_{h,j}^\top v_{h,j} = \mathcal{R}_j^\top v_{h,j}$ and so $a(w, \mathcal{R}_{h,j}^\top v_{h,j}) = a_{\Omega_j}(w, v_{h,j})$ and thus (2.28) implies that

$$b_{h,j}(w, v_{h,j}) = 0 \quad \text{for all} \quad v_{h,j} \in V_j \cap H^1_{0,\Gamma_j}(\Omega_j); \tag{2.30}$$

this is the discrete analogue of the fact that b_j is a boundary integral over Γ_j.

We use $b_{h,j}$ to obtain the finite element analogue of (2.23), which we combine with the analogue of (2.10) to obtain the following.

Definition 2.4 (Finite element analogue of parallel Schwarz method)**.** *Given $u_h^n \in V$, let $u_{h,j}^{n+1} \in V_j$ be the solution of*

$$a_{\Omega_j}(u_{h,j}^{n+1}, v_{h,j}) = F(\mathcal{R}_{h,j}^\top v_{h,j}) + b_{h,j}(u_h^n, v_{h,j}), \quad \textit{for all } v_{h,j} \in V_j, \tag{2.31}$$

and then set

$$u_h^{n+1} := \sum_\ell \widetilde{\mathcal{R}}_{h,\ell}^\top u_{h,\ell}^{n+1}, \tag{2.32}$$

The next theorem shows (2.31), (2.32) is equivalent to ORAS.

Theorem 2.5 (Connection between the parallel Schwarz method and ORAS)**.** *With the same starting guess, the iterates produced by (2.31), (2.32) coincide with those produced by (2.6).*

Proof. Combining (2.31) with the definition (2.28) of $b_{h,j}$, and using (2.3) and (1.4), we obtain

$$a_{\Omega_j}(u_{h,j}^{n+1} - u_h^n|_{\Omega_j}, v_{h,j}) = F_h(\mathcal{R}_{h,j}^\top v_{h,j}) - a(u_h^n, \mathcal{R}_{h,j}^\top v_{h,j}) \\ = (\mathcal{R}_{h,j}(F_h - \mathcal{A}_h u_h^n))(v_{h,j}). \tag{2.33}$$

Hence $u_{h,j}^{n+1} - u_h^n|_{\Omega_j} = \mathcal{A}_{h,j}^{-1}\mathcal{R}_{h,j}(F_h - \mathcal{A}_h u_h^n)$ and, by (2.32) and (2.5),

$$u_h^{n+1} = \sum_j \widetilde{\mathcal{R}}_{h,j}^\top u_{h,j}^{n+1} \\ = \sum_j \widetilde{\mathcal{R}}_{h,j}^\top u_h^n|_{\Omega_j} + \mathcal{B}_h^{-1}(F_h - \mathcal{A}_h u_h^n). \tag{2.34}$$

By (2.4), the first term on the right-hand side of (2.34) is u_h^n, so this formula coincides with (2.6). ☐

The following lemma is used frequently in the remainder of the analysis.

Lemma 2.6. *(i) For all* $(w, v_{h,j}) \in H^1(\Omega) \times V_j$*,*

$$b_{h,j}(w, v_{h,j}) = -\widetilde{a}_{\Omega\backslash\Omega_j}(w, \mathcal{R}_{h,j}^\top v_{h,j}) + \mathrm{i}k\langle w, \mathcal{R}_{h,j}^\top v_{h,j}\rangle_{\partial\Omega\backslash\partial\Omega_j} - \mathrm{i}k\langle w, v_{h,j}\rangle_{\Gamma_j}. \tag{2.35}$$

(ii) If either $\ell = j$ *or* $\Omega_\ell \cap \Omega_j = \emptyset$*, then*

$$b_{h,j}(\widetilde{\mathcal{R}}_{h,\ell}^\top w_{h,\ell}, v_{h,j}) = 0 \quad \textit{for all} \ \ w_{h,\ell} \in V_\ell \quad \textit{and} \quad v_{h,j} \in V_j.$$

Proof. To prove part (i) we use the definition (2.28) and recall Notation 2.1, to obtain

$$b_{h,j}(w, v_{h,j}) : = \widetilde{a}_{\Omega_j}(w, v_{h,j}) - \widetilde{a}(w, \mathcal{R}_{h,j}^\top v_{h,j}) - \mathrm{i}k\langle w, v_{h,j}\rangle_{\partial\Omega_j} + \mathrm{i}k\langle w, \mathcal{R}_{h,j}^\top v_{h,j}\rangle_{\partial\Omega}.$$

By (2.2), $\mathcal{R}_{h,j}^\top v_{h,j} = v_{h,j}$ on $\overline{\Omega_j}$, and thus

$$b_{h,j}(w, v_{h,j}) : = -\widetilde{a}_{\Omega\backslash\Omega_j}(w, \mathcal{R}_{h,j}^\top v_{h,j}) - \mathrm{i}k\langle w, v_{h,j}\rangle_{\partial\Omega_j\backslash\partial\Omega} + \mathrm{i}k\langle w, \mathcal{R}_{h,j}^\top v_{h,j}\rangle_{\partial\Omega\backslash\partial\Omega_j}.$$

Part (i) then follows since $\Gamma_j = \partial\Omega_j\backslash\partial\Omega$. Part (ii) follows from Part (i) using the facts that (a) $\widetilde{\mathcal{R}}_{h,\ell}^\top w_{h,\ell}$ vanishes outside $\overline{\Omega_\ell}$ and on Γ_ℓ, and (b) if $\Omega_\ell \cap \Omega_j = \emptyset$ then Ω_j and Ω_ℓ are separated by at least one layer of elements. ☐

3 General properties of the ORAS iteration

3.1 The error propagation operator

We now obtain the discrete analogue of (2.14)-(2.16). Recall that u_h is the finite element approximation of (1.1) (as defined in (1.3)) and Ω_j is the overlapping cover of Ω introduced in §1.2. With $u_{h,j}^n$ and u_h^n as defined in (2.31) and (2.32), we introduce the local and global errors, defined respectively, by

$$e_{h,j}^n := u_h|_{\Omega_j} - u_{h,j}^n \quad \text{and} \quad e_h^n := u_h - u_h^n.$$

Proposition 3.1 (The error equation)**.** *For each* $j = 1, \ldots, N$*,*

$$a_{\Omega_j}(e_{h,j}^{n+1}, v_{h,j}) = b_{h,j}(e_h^n, v_{h,j}) \tag{3.1}$$

and

$$e_h^n = \sum_\ell \widetilde{\mathcal{R}}_{h,\ell}^\top e_{h,\ell}^n. \tag{3.2}$$

Proof. To obtain (3.2), we first observe that

$$\sum_\ell \widetilde{\mathcal{R}}_{h,\ell}^\top e_{h,\ell}^n = \sum_\ell \widetilde{\mathcal{R}}_{h,\ell}^\top u_h|_{\Omega_\ell} - \sum_\ell \widetilde{\mathcal{R}}_{h,\ell}^\top u_{h,\ell}^n = u_h - u_h^n,$$

where we used (2.4) and (2.32). We then combine (2.33) and (1.3) to obtain

$$a_{\Omega_j}(u_{h,j}^{n+1} - u_h^n|_{\Omega_j}, v_{h,j}) = a(u_h - u_h^n, \mathcal{R}_{h,j}^\top v_{h,j}) = a(e_h^n, \mathcal{R}_{h,j}^\top v_{h,j}). \tag{3.3}$$

Since $u_{h,j}^{n+1} - u_h^n|_{\Omega_j} = (u_h - u_h^n)|_{\Omega_j} - (u_h|_{\Omega_j} - u_{h,j}^{n+1}) = e_h^n|_{\Omega_j} - e_{h,j}^{n+1}$, substituting this in (3.3) and rearranging yields (3.1). □

The last result motivates the following definition.

Definition 3.2 (Discrete error propagation operator)**.** *For $j = 1, \dots, N$, and $w_{h,\ell} \in V_\ell$, let $\mathcal{T}_{h;j,\ell} w_{h,\ell} \in V_j$ be defined as the solution of*

$$a_{\Omega_j}(\mathcal{T}_{h;j,\ell} w_{h,\ell}, v_{h,j}) = b_{h,j}(\widetilde{\mathcal{R}}_{h,\ell}^\top w_{h,\ell}, v_{h,j}) \quad \textit{for all } v_{h,j} \in V_j. \tag{3.4}$$

This definition and Proposition 3.1 imply that the error of the ORAS iterative method satisfies the fixed point iteration

$$e_{h,j}^{n+1} = \sum_\ell \mathcal{T}_{h;j,\ell} e_{h,\ell}^n \quad \text{for } j = 1, \dots, N, \quad \text{written compactly as} \quad \boldsymbol{e}_h^{n+1} = \boldsymbol{\mathcal{T}}_h \, \boldsymbol{e}_h^n, \tag{3.5}$$

where $\boldsymbol{e}_h^n := (e_{h,1}^n, e_{h,2}^n, \cdots, e_{h,N}^n)^\top$. In this setting $\boldsymbol{\mathcal{T}}_h$ is a operator acting on the tensor product space $\mathbb{V} := \Pi_{\ell=1}^N V_\ell$.

We see immediately that (3.5) is the discrete analogue of the corresponding fixed point iteration for the parallel Schwarz method given in (1.7). The convergence analysis for the Schwarz method in [20] is carried out in the space $\mathbb{U}_0$ (defined by (2.22)). We now create a finite-element analogue of this space; to do this, we need to address the fact that functions $v \in U_0(\Omega_j)$ satisfy the homogeneous Helmholtz equation $\Delta v + k^2 v = 0$, but the Helmholtz operator is not defined on the finite element space V_j.

3.2 Discrete Helmholtz-harmonic spaces

Recall that Γ_ℓ is defined by (2.8) and $H^1_{0,\Gamma_\ell}(\Omega_\ell)$ is defined by (2.29).

Definition 3.3 (Discrete Helmholtz-harmonic spaces)**.** *Let $V_{0,\ell} \subset V_\ell$ be defined by*

$$V_{0,\ell} \ := \Big\{ w_{h,\ell} \in V_\ell \ : \ a_{\Omega_\ell}(w_{h,\ell}, v_{h,\ell}) = 0 \quad \textit{for all} \ \ v_{h,\ell} \in V_\ell \cap H^1_{0,\Gamma_\ell}(\Omega_\ell) \Big\}, \tag{3.6}$$

and let the corresponding tensor product space be defined by $\mathbb{V}_0 \ := \displaystyle\prod_{\ell=1}^N V_{0,\ell}$.

Observe that $V_{0,\ell}$ consists of finite-element approximations to functions in $U_0(\Omega_\ell)$ (defined in (2.17)) which satisfy $(\Delta + k^2)w = 0$ in Ω_ℓ, $\partial_n w - \mathrm{i}kw = 0$ on $\partial\Omega_\ell \backslash \Gamma_\ell = \partial\Omega_\ell \cap \partial\Omega$. Note that in $V_{0,\ell}$ there is no constraint on Γ_ℓ because of the zero Dirichlet boundary condition on Γ_ℓ in the space of test functions in (3.6). Recalling the equations (2.14)-(2.16) satisfied by the error e_ℓ^n at the PDE level, we see that $V_{0,\ell}$ is the discrete analogue of the space in which the error e_ℓ^n of the parallel Schwarz method lies. To define a discrete analogue of the corresponding norm (2.18), we first prove the following lemma, which extracts the "discrete impedance data" of a function $w_{h,\ell} \in V_{0,\ell}$.

Lemma 3.4 (Discrete impedance data)**.** *Given $w_{h,\ell} \in V_{0,\ell}$, there exists a unique finite element function $\mathrm{imp}^h_{\Gamma_\ell} w_{h,\ell} \in V(\Gamma_\ell)$ (where $V(\Gamma_\ell)$ is the space of finite element functions on Γ_ℓ) such that*

$$\big\langle \mathrm{imp}^h_{\Gamma_\ell} w_{h,\ell}, v_{h,\ell} \big\rangle_{\Gamma_\ell} \ = \ a_{\Omega_\ell}(w_{h,\ell}, v_{h,\ell}) \quad \textit{for all} \ \ v_{h,\ell} \in V_\ell. \tag{3.7}$$

We refer to $\mathrm{imp}^h_{\Gamma_\ell} w_{h,\ell}$ *as the* discrete impedance data *of* $w_{h,\ell}$.

Proof. Let $g_{h,\Gamma_\ell} \in V(\Gamma_\ell)$ be the unique solution of

$$\langle g_{h,\Gamma_\ell}, v_{h,\Gamma_\ell} \rangle_{\Gamma_\ell} \ = \ a_{\Omega_\ell}(w_{h,\ell}, \widehat{v}_{h,\ell}) \quad \text{for all} \ \ v_{h,\Gamma_\ell} \in V(\Gamma_\ell),$$

where $\widehat{v}_{h,\ell}$ denotes the zero nodal extension of v_{h,Γ_ℓ} to V_ℓ (i.e., the finite element function that coincides with v_{h,Γ_ℓ} on Γ_ℓ and has value zero at all other nodes of $\overline{\Omega_\ell}$).

Now, given $v_{h,\ell} \in V_\ell$, let v_{h,Γ_ℓ} be its restriction to Γ_ℓ and define $\widehat{v}_{h,\ell}$ as above. Then $\widetilde{v}_{h,\ell} := v_{h,\ell} - \widehat{v}_{h,\ell} \in V_\ell \cap H^1_{0,\Gamma_\ell}(\Omega_\ell)$. By (3.6), $a_{\Omega_\ell}(w_{h,\ell}, \widetilde{v}_{h,\ell}) = 0$, and hence

$$a_{\Omega_\ell}(w_{h,\ell}, v_{h,\ell}) = a_{\Omega_\ell}(w_{h,\ell}, \widehat{v}_{h,\ell}) = \langle g_{h,\Gamma_\ell}, v_{h,\Gamma_\ell}\rangle_{\Gamma_\ell} = \langle g_{h,\Gamma_\ell}, v_{h,\ell}\rangle_{\Gamma_\ell},$$

which, with $\mathrm{imp}^h_{\Gamma_\ell} := g_{h,\ell}$, is exactly (3.7). □

Observe that if $w_{h,\ell}$ satisfies (3.7), then it is the finite element approximation of the problem $(\Delta + k^2)w = 0$ on $\partial\Omega_\ell$, with w satisfying an impedance condition on $\partial\Omega_\ell$, with impedance data $\mathrm{imp}^h_{\Gamma_\ell} w_{h,\ell}$ on Γ_ℓ and 0 on $\partial\Omega\backslash\Gamma_\ell$. This fact justifies calling $\mathrm{imp}^h_{\Gamma_\ell} w_{h,\ell}$ "the discrete impedance data" of $w_{h,\ell}$ and allows us to introduce the following norm on $V_{0,\ell}$, which is the analogue of the norm on $U_0(\Omega_\ell)$ defined in (2.18).

Definition 3.5 (Discrete Helmholtz-harmonic norm)**.** *For $w_{h,\ell} \in V_{0,\ell}$ and $\boldsymbol{w}_h := (w_{h,1}, \ldots, w_{h,N})^\top \in \mathbb{V}_0$, let*

$$\|w_{h,\ell}\|_{V_{0,\ell}} := \|\mathrm{imp}^h_{\Gamma_\ell} w_{h,\ell}\|_{L^2(\Gamma_\ell)} \quad \textit{and} \quad \|\boldsymbol{w}_h\|_{\mathbb{V}_0} := \Big(\sum_\ell \|w_{h,\ell}\|^2_{V_{0,\ell}}\Big)^{1/2}. \tag{3.8}$$

Proposition 3.6. *The expressions in (3.8) are indeed norms on $V_{0,\ell}$ and $\mathbb{V}_0$.*

Proof. Suppose $\|w_{h,\ell}\|_{V_{0,\ell}} = 0$ for some $w_{h,\ell} \in V_{0,\ell}$. Then $\mathrm{imp}^h_{\Gamma_\ell} w_{h,\ell}$ must vanish on Γ_ℓ. This implies, via (3.7), that $a_{\Omega_\ell}(w_{h,\ell}, v_{h,\ell}) = 0$ for all $v_{h,\ell} \in V_\ell$. Then, by Assumption 2.2, $w_{h,\ell} = 0$. The other norm axioms follow easily from the definitions in (3.8). □

Proposition 3.7. *For each j, ℓ, the operator $\mathcal{T}_{h;j,\ell}$ given in (3.4) is well-defined and maps V_ℓ to $V_{0,j}$. Moreover $\boldsymbol{\mathcal{T}}_h : \mathbb{V} \to \mathbb{V}_0$.*

Proof. By Assumption 2.2, the problem (3.4) has a unique solution and so $\mathcal{T}_{h,j,\ell}$ is well-defined. Moreover, combining (2.30) with (3.4) and (3.6), we see that $\mathcal{T}_{h;j,\ell} v_{h,\ell} \in V_{0,j}$ for all $v_{h,\ell} \in V_\ell$, as required. □

4 Convergence theory for strip domain decompositions

4.1 The error propagation operator

In this subsection we obtain a convergence theory for (2.31)-(2.32) when the domain is partitioned into strips, mirroring the analogous theory in [20, §4] for (2.7)-(2.9). We assume that Ω is a rectangle of height H and the subdomains Ω_ℓ also have height H and are bounded by vertical sides denoted by Γ^-_ℓ and Γ^+_ℓ. We assume the subdomain Ω_ℓ is only overlapped by $\Omega_{\ell-1}$ and $\Omega_{\ell+1}$ (with $\Omega_0 = \emptyset = \Omega_N$). Recalling that $\Gamma_\ell := \partial\Omega_\ell \setminus \partial\Omega$, we have the following simple decomposition

$$\Gamma_\ell = \Gamma^+_\ell \cup \Gamma^-_\ell .$$

The geometry and notation is illustrated in Figure 1.

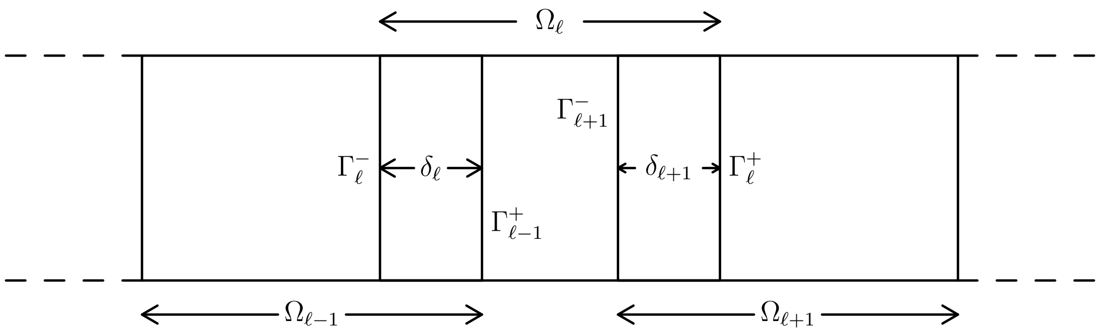

Figure 1: Three overlapping subdomains in 2-d

Using Part (ii) of Lemma 2.6 in the definition of $\mathcal{T}_{h:j,\ell}$ (Definition 3.2), we see that $\boldsymbol{\mathcal{T}}_h$ takes the tridiagonal form

$$\boldsymbol{\mathcal{T}}_h = \begin{pmatrix} 0 & \mathcal{T}_{h;1,2} & & & & \\ \mathcal{T}_{h;2,1} & 0 & \mathcal{T}_{h;2,3} & & & \\ & \mathcal{T}_{h;3,2} & 0 & \mathcal{T}_{h;3,4} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mathcal{T}_{h;N-1,N-2} & 0 & \mathcal{T}_{h;N-1,N} \\ & & & & \mathcal{T}_{h;N,N-1} & 0 \end{pmatrix} =: \boldsymbol{\mathcal{L}}_h + \boldsymbol{\mathcal{U}}_h\,, \tag{4.1}$$

where $\boldsymbol{\mathcal{L}}_h$ and $\boldsymbol{\mathcal{U}}_h$ are the lower and upper triangular components of $\boldsymbol{\mathcal{T}}_h$.

4.2 Decomposition of $V_{0,\ell}$ and discrete impedance data

The following subspaces of $V_{0,\ell}$ and $\mathbb{V}_0$ characterise the range spaces of the operators $\boldsymbol{\mathcal{L}}_h, \boldsymbol{\mathcal{U}}_h$ in (4.1).

Definition 4.1 (Discrete Helmholtz-harmonic spaces with impedance data on $\Gamma_\ell^{\pm}$)**.** *For $s \in \{+,-\}$, let*

$$V_{0,\ell}^s := \Big\{ w_{h,\ell} \in V_\ell \ : \ a_{\Omega_\ell}(w_{h,\ell}, v_{h,\ell}) = 0 \quad \textit{for all} \ \ v_{h,\ell} \in V_\ell \cap H^1_{0,\Gamma_\ell^s}(\Omega_\ell) \Big\}, \tag{4.2}$$

and let the corresponding tensor-product space be defined by

$$\mathbb{V}_0^s = \big\{ \boldsymbol{w}_h \in \mathbb{V} \ : \ w_{h,\ell} \in V_{0,\ell}^s \quad \textit{for all} \ \ \ell = 1,\ldots,N \big\}.$$

By Definition 3.3 and Lemma 3.4, functions in $w_{h,\ell} \in V_{0,\ell}^{\pm}$ satisfy the finite element approximation of the homogeneous interior impedance problem on Ω_ℓ with impedance data supported only on $\Gamma_\ell^{\pm}$. In the next lemma we give a simple formula for the norm on $V_{0,\ell}^s$.

Lemma 4.2. *Suppose $w_{h,\ell} \in V_{0,\ell}^s$ and $s \in \{+,-\}$. Then $\mathrm{imp}^h_{\Gamma_\ell} w_{h,\ell} = 0$ on $\Gamma_\ell \backslash \Gamma_\ell^s$ and*

$$\|w_{h,\ell}\|_{V_{0,\ell}} = \|\mathrm{imp}^h_{\Gamma_\ell^s} w_{h,\ell}\|_{L^2(\Gamma_\ell^s)} \quad \textit{where} \quad \mathrm{imp}^h_{\Gamma_\ell^s} w_{h,\ell} := (\mathrm{imp}^h_{\Gamma_\ell} w_{h,\ell})|_{\Gamma_\ell^s}. \tag{4.3}$$

Proof. Let $w_{h,\ell} \in V_{0,\ell}^s \subset V_{0,\ell}$. By Lemma 3.4, $w_{h,\ell}$ satisfies (3.7) for all $v_{h,\ell} \in V_\ell$. Now choose $v_{h,\ell}$ to be the zero nodal extension of the function $(\mathrm{imp}^h_{\Gamma_\ell} w_{h,\ell})|_{\Gamma_\ell \backslash \Gamma_\ell^s}$. Then $v_{h,\ell} \in V_\ell \cap H^1_{0,\Gamma_\ell^s}$ and, by (4.2), $a_{\Omega_\ell}(w_{h,\ell}, v_{h,\ell}) = 0$. Thus, the definition (3.7) of $\mathrm{imp}^h_{\Gamma_\ell} w_{h,\ell}$ implies that $\|\mathrm{imp}^h_{\Gamma_\ell} w_{h,\ell}\|^2_{L^2(\Gamma_\ell \backslash \Gamma_\ell^s)} = 0$, and the result follows. □

Using this result we can now obtain the following decomposition of $V_{0,\ell}$, which is crucial in the convergence theory in §4.5.

Lemma 4.3 (Decomposition of $V_{0,\ell}$)**.** *For each $\ell = 1,\ldots,N$,*

$$V_{0,\ell} = V_{0,\ell}^- \oplus V_{0,\ell}^+. \tag{4.4}$$

Moreover, if $w_{h,\ell} = w_{h,\ell}^- + w_{h,\ell}^+$ with $w_{h,\ell}^s \in V_{0,\ell}^s$, then

$$\|w_{h,\ell}\|^2_{V_{0,\ell}} \ = \ \|w_{h,\ell}^-\|^2_{V_{0,\ell}^-} + \|w_{h,\ell}^+\|^2_{V_{0,\ell}^+}. \tag{4.5}$$

Proof. Let $w_{h,\ell} \in V_{0,\ell}$; for each $s \in \{+,-\}$, define $\mathrm{imp}^h_{\Gamma_\ell^s} w_{h,\ell}$ as in the second equation in (4.3) and then define $w_{h,\ell}^s \in V_{0,\ell}$ by

$$a_{\Omega_\ell}(w_{h,\ell}^s, v_{h,\ell}) = \big\langle \mathrm{imp}^h_{\Gamma_\ell^s} w_{h,\ell}, v_{h,\ell} \big\rangle_{\Gamma_\ell^s} \quad \text{for all} \ \ v_{h,\ell} \in V_\ell.$$

Then, for all $v_{h,\ell} \in V_\ell$,

$$a_{\Omega_\ell}(w_{h,\ell}^- + w_{h,\ell}^+, v_{h,\ell}) = \big\langle \mathrm{imp}^h_{\Gamma_\ell^-} w_{h,\ell}, v_{h,\ell} \big\rangle_{\Gamma_\ell^-} + \big\langle \mathrm{imp}^h_{\Gamma_\ell^+} w_{h,\ell}, v_{h,\ell} \big\rangle_{\Gamma_\ell^+} = \big\langle \mathrm{imp}^h_{\Gamma_\ell} w_{h,\ell}, v_{h,\ell} \big\rangle_{\Gamma_\ell}.$$

So, by (3.7) and Assumption 2.2, $w_{h,\ell} = w^-_{h,\ell} + w^+_{h,\ell}$.

We now show that the sum in (4.4) is direct. Suppose $w_{h,\ell} \in V^-_{0,\ell} \cap V^+_{0,\ell}$. Then we can write any $v_{h,\ell} \in V_\ell$ as $v_{h,\ell} = v^-_{h,\ell} + v^+_{h,\ell}$ where $v^s_{h,\ell} \in V_\ell \cap H^1_{0,\Gamma^s_\ell}(\Omega_\ell)$, and then, by (4.2),

$$a_{\Omega_\ell}(w_{h,\ell}, v_{h,\ell}) = a_{\Omega_\ell}(w_{h,\ell}, v^-_{h,\ell}) + a_{\Omega_\ell}(w_{h,\ell}, v^+_{h,\ell}) = 0;$$

therefore $w_{h,\ell} = 0$ by Assumption 2.2. The expression (4.5) then follows since, by (4.3),

$$\begin{aligned}\|w_{h,\ell}\|^2_{V_{0,\ell}} = \|\mathrm{imp}^h_{\Gamma_\ell} w_{h,\ell}\|^2_{L^2(\Gamma_\ell)} &= \|\mathrm{imp}^h_{\Gamma^+_\ell} w_{h,\ell}\|^2_{L^2(\Gamma^+_\ell)} + \|\mathrm{imp}^h_{\Gamma^-_\ell} w_{h,\ell}\|^2_{L^2(\Gamma^-_\ell)} \\ &= \|w^-_{h,\ell}\|^2_{V^-_{0,\ell}} + \|w^+_{h,\ell}\|^2_{V^+_{0,\ell}}.\end{aligned}$$

□

Lemma 4.4 (Mapping properties of $\mathcal{T}_{h,j,\ell}$)**.**

$$(i) \quad \mathcal{T}_{h;j,j-1} : V_{j-1} \to V^-_{0,j} \quad and \quad (ii) \quad \mathcal{T}_{h;j,j+1} : V_{j+1} \to V^+_{0,j}. \tag{4.6}$$

Moreover $\boldsymbol{\mathcal{L}}_h : \mathbb{V} \to \mathbb{V}^-_0$ *and* $\boldsymbol{\mathcal{U}}_h : \mathbb{V} \to \mathbb{V}^+_0$.

Proof. We give the proof of (4.6) (i) only; the proof of (ii) is analogous, and the statement about $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{U}}$ follows from (4.6). Let $w_{h,j-1} \in V_{j-1}$ and $v_{h,j} \in V_j \cap H^1_{0,\Gamma^-_j}(\Omega_j)$. Then, by Lemma 2.6,

$$\begin{aligned} b_{h,j}(\widetilde{\mathcal{R}}^\top_{h,j-1} w_{h,j-1}, v_{h,j}) = &- \widetilde{a}_{\Omega\backslash\Omega_j}(\widetilde{\mathcal{R}}^\top_{h,j-1} w_{h,j-1}, \mathcal{R}^\top_{h,j} v_{h,j}) \\ &+ \mathrm{i}k\langle\widetilde{\mathcal{R}}^\top_{h,j-1} w_{h,j-1}, \mathcal{R}^\top_{h,j} v_{h,j}\rangle_{\partial\Omega\backslash\partial\Omega_j} - \mathrm{i}k\langle\widetilde{\mathcal{R}}^\top_{h,j-1} w_{h,j-1}, v_{h,j}\rangle_{\Gamma_j}.\end{aligned}$$

Now $\widetilde{\mathcal{R}}^\top_{h,j-1} w_{h,j-1}$ is supported inside $\overline{\Omega_{j-1}}$ and so vanishes on Γ^+_j. Moreover, $v_{h,j}$ is supported in $\overline{\Omega_j}$ and vanishes on Γ^-_j. Thus $\mathcal{R}^\top_{h,j} v_{h,j}$ also vanishes in $(\Omega\backslash\Omega_j) \cap \Omega_{j-1}$ and on $(\partial\Omega\backslash\partial\Omega_j) \cap \partial\Omega_{j-1}$. Thus

$$b_{h,j}(\widetilde{\mathcal{R}}^\top_{h,j-1} w_{h,j-1}, v_{h,j}) = 0. \tag{4.7}$$

By (3.4), $a_{\Omega_j}(\mathcal{T}_{h,j,j-1} w_{h,j-1}, v_{h,j}) = 0$ for all $v_{h,j} \in V_j \cap H^1_{0,\Gamma^-_j}(\Omega_j)$, and thus $\mathcal{T}_{h,j,j-1} w_{h,j-1} \in V^-_{0,j}$ by (4.2). □

4.3 Discrete impedance-to-impedance maps

In [20], power contractivity of $\boldsymbol{\mathcal{T}}$ is proved by studying the so-called 'impedance-to-impedance maps' defined as follows. Given $g \in L^2(\Gamma^s_\ell)$, where $s \in \{+,-\}$, let $w_\ell \in H^1(\Omega_\ell)$ be the solution of

$$a_{\Omega_\ell}(w_\ell, v_\ell) = \langle g, v_\ell\rangle_{\Gamma^s_\ell} \quad \text{for all } \; v_\ell \in H^1(\Omega_\ell),$$

(i.e., w_ℓ solves the Helmholtz problem on Ω_ℓ with vanishing source and impedance data g on Γ^s_ℓ and zero elsewhere). Then $\mathcal{I}_{\Gamma^s_\ell \to \Gamma^+_{\ell-1}}$ and $\mathcal{I}_{\Gamma^s_\ell \to \Gamma^-_{\ell+1}}$ are the maps that take g to the corresponding impedance data of w_ℓ on $\Gamma^+_{\ell-1}$ and $\Gamma^-_{\ell+1}$, respectively; i.e.,

$$\mathcal{I}_{\Gamma^s_\ell \to \Gamma^+_{\ell-1}} g = \left(\left(\frac{\partial}{\partial n_{\ell-1}} - \mathrm{i}k\right) w_\ell\right)\bigg|_{\Gamma^+_{\ell-1}} \quad \text{and} \quad \mathcal{I}_{\Gamma^s_\ell \to \Gamma^-_{\ell+1}} g = \left(\left(\frac{\partial}{\partial n_{\ell+1}} - \mathrm{i}k\right) w_\ell\right)\bigg|_{\Gamma^-_{\ell+1}}. \tag{4.8}$$

To analyse the power contractivity of $\boldsymbol{\mathcal{T}}_h$, we need to study the discrete versions of these impedance-to-impedance maps; we first introduce some notation.

Notation 4.5 (Extension from and restriction to the boundary)**.** *Let* $s \in \{+,-\}$.
(i) If v_{h,Γ^s_ℓ} *denotes a generic element of* $V(\Gamma^s_\ell)$*, then* $\widehat{v}^s_{h,\ell} \in V_\ell$ *denotes its zero nodal extension to* V_ℓ *(i.e., the finite element function that coincides with* v_{h,Γ^s_ℓ} *on* Γ^s_ℓ *and has value zero at all other nodes of* $\overline{\Omega_\ell}$*).*
(ii) Conversely, if $v_{h,\ell}$ *denotes a generic element of* V_ℓ*, then* v_{h,Γ^s_ℓ} *denotes its restriction to* Γ^s_ℓ *and, as above,* $\widehat{v}^s_{h,\ell} \in V_\ell$ *denotes the zero nodal extension of* v_{h,Γ^s_ℓ} *to* Ω_ℓ*. We also define* $\widetilde{v}^s_{h,\ell} = v_{h,\ell} - \widehat{v}^s_{h,\ell} \in V_\ell$*, so that* $\widetilde{v}^s_{h,\ell} \in V_\ell$ *vanishes on* Γ^s_ℓ*.*

We now define discrete analogues of (4.8). We use the notation introduced in Notation 4.5 and also define

$$\Omega_{\ell,j} := \Omega_\ell \cap \Omega_j.$$

Definition 4.6 (Discrete impedance-to-impedance map). *For $g \in L^2(\Gamma_\ell^s)$ with $s \in \{+,-\}$, let $w_{h,\ell} \in V_\ell^s$ be the unique solution of*

$$a_{\Omega_\ell}(w_{h,\ell}, v_{h,\ell}) = \langle g, v_{h,\ell}\rangle_{\Gamma_\ell^s} \quad \textit{for all} \;\; v_{h,\ell} \in V_\ell. \tag{4.9}$$

Define $\mathcal{I}^h_{\Gamma_\ell^s\to\Gamma_{\ell-1}^+} : L^2(\Gamma_\ell^s) \to V(\Gamma_{\ell-1}^+)$ and $\mathcal{I}^h_{\Gamma_\ell^s\to\Gamma_{\ell+1}^-} : L^2(\Gamma_\ell^s) \to V(\Gamma_{\ell+1}^-)$ as the solutions of

$$\big\langle \mathcal{I}^h_{\Gamma_\ell^s\to\Gamma_{\ell-1}^+} g, v_{h,\Gamma_{\ell-1}^+}\big\rangle_{\Gamma_{\ell-1}^+} = a_{\Omega_{\ell,\ell-1}}(w_{h,\ell}, \widehat{v}^+_{h,\ell-1}) - a_{\Omega_\ell}(w_{h,\ell}, \mathcal{R}^\top_{h,\ell-1}\widehat{v}^+_{h,\ell-1}), \tag{4.10}$$

for any $v_{h,\Gamma_{\ell-1}^+} \in V(\Gamma_{\ell-1}^+)$, and

$$\big\langle \mathcal{I}^h_{\Gamma_\ell^s\to\Gamma_{\ell+1}^-} g, v_{h,\Gamma_{\ell+1}^-}\big\rangle_{\Gamma_{\ell+1}^-} = a_{\Omega_{\ell,\ell+1}}(w_{h,\ell}, \widehat{v}^-_{h,\ell+1}) - a_{\Omega_\ell}(w_{h,\ell}, \mathcal{R}^\top_{h,\ell+1}\widehat{v}^-_{h,\ell+1}), \tag{4.11}$$

for any $v_{h,\Gamma_{\ell+1}^-} \in V(\Gamma_{\ell+1}^-)$.

Observe that (4.10) is the analogue of (2.28) with Ω replaced by Ω_ℓ and Ω_j replaced by $\Omega_{\ell,\ell-1}$, and hence (4.10) and (4.11) can be seen as discrete analogues of (4.8). To estimate impedance-to-impedance maps we always use the L^2 norm on the domain and co-domain; for simplicity we denote this as $\|\cdot\|$. In Section 5, we prove the following result.

Theorem 4.7. *Let $s \in \{+,-\}$. Then*

$$\|\mathcal{I}_{\Gamma_\ell^s\to\Gamma_{\ell-1}^+} - \mathcal{I}^h_{\Gamma_\ell^s\to\Gamma_{\ell-1}^+}\| \to 0 \quad \textit{and} \quad \|\mathcal{I}_{\Gamma_\ell^s\to\Gamma_{\ell+1}^-} - \mathcal{I}^h_{\Gamma_\ell^s\to\Gamma_{\ell+1}^-}\| \to 0 \quad \textit{as} \quad h \to 0, \tag{4.12}$$

where the first limit holds for $\ell = 2, \ldots, N$, and the second limit holds for $\ell = 1, \ldots, N-1$.

4.4 Characterisation of $\mathcal{T}_{h,j,\ell}$ using discrete impedance-to-impedance maps

The following lemma helps connect the impedance maps defined in (4.10) and (4.11) with the action of the operator $\mathcal{T}_{h;j,\ell}$, and is used to prove Theorem 4.9 (the main result of this section).

Lemma 4.8. *Let $s \in \{+,-\}$, $g \in L^2(\Gamma_\ell^s)$, and suppose $w_{h,\ell}$ satisfies (4.9). Then*

$$\big\langle \mathcal{I}^h_{\Gamma_\ell^s\to\Gamma_{\ell-1}^+} g, v_{h,\ell-1}\big\rangle_{\Gamma_{\ell-1}^+} = b_{h,\ell-1}(\widetilde{\mathcal{R}}^\top_{h,\ell} w_{h,\ell}, v_{h,\ell-1}) \quad \textit{for all} \;\; v_{h,\ell-1} \in V_{\ell-1}, \tag{4.13}$$

and

$$\big\langle \mathcal{I}^h_{\Gamma_\ell^s\to\Gamma_{\ell+1}^-} g, v_{h,\ell+1}\big\rangle_{\Gamma_{\ell+1}^-} = b_{h,\ell+1}(\widetilde{\mathcal{R}}^\top_{h,\ell} w_{h,\ell}, v_{h,\ell+1}) \quad \textit{for all} \;\; v_{h,\ell+1} \in V_{\ell+1}. \tag{4.14}$$

Proof. We only prove (4.13); the proof of (4.14) is similar. Let $v_{h,\ell-1} \in V_{\ell-1}$. We now apply the decomposition in Notation 4.5(ii) (on $\Omega_{\ell-1}$ with s equal to $+$), to write $v_{h,\ell-1} = \widehat{v}^+_{h,\ell-1} + \widetilde{v}^+_{h,\ell-1}$, where $\widehat{v}^+_{h,\ell-1}$ coincides with $v_{h,\ell-1}$ on $\Gamma^+_{\ell-1}$ and is zero at other nodes and $\widetilde{v}^+_{h,\ell-1}$ vanishes on $\Gamma^+_{\ell-1}$. Thus,

$$\big\langle \mathcal{I}^h_{\Gamma_\ell^s\to\Gamma_{\ell-1}^+} g, \widetilde{v}^+_{h,\ell-1}\big\rangle_{\Gamma_{\ell-1}^+} = 0 = b_{h,\ell-1}(\widetilde{\mathcal{R}}^\top_{h,\ell} w_{h,\ell}, \widetilde{v}^+_{h,\ell-1}),$$

where the second equality follows by an argument analogous to that used to obtain (4.7).

Therefore, to complete the proof of (4.13), we only need to prove that

$$\big\langle \mathcal{I}^h_{\Gamma_\ell^s\to\Gamma_{\ell-1}^+} g, \widehat{v}^+_{h,\ell-1}\big\rangle_{\Gamma_{\ell-1}^+} = b_{h,\ell-1}(\widetilde{\mathcal{R}}^\top_{h,\ell} w_{h,\ell}, \widehat{v}^+_{h,\ell-1}). \tag{4.15}$$

By (4.10), Notation 4.5 (ii) and Notation 2.1,

$$\begin{aligned}
\big\langle \mathcal{I}^h_{\Gamma^s_\ell \to \Gamma^+_{\ell-1}} g, \widehat{v}^+_{h,\ell-1} \big\rangle_{\Gamma^+_{\ell-1}} &= \big\langle \mathcal{I}^h_{\Gamma^s_\ell \to \Gamma^+_{\ell-1}} g, v_{h,\Gamma^+_{\ell-1}} \big\rangle_{\Gamma^+_{\ell-1}} \\
&= a_{\Omega_{\ell,\ell-1}}(w_{h,\ell}, \widehat{v}^+_{h,\ell-1}) - a_{\Omega_\ell}(w_{h,\ell}, \mathcal{R}^\top_{h,\ell-1}\widehat{v}^+_{h,\ell-1}) \\
&= \widetilde{a}_{\Omega_{\ell,\ell-1}}(w_{h,\ell}, \widehat{v}^+_{h,\ell-1}) - \widetilde{a}_{\Omega_\ell}(w_{h,\ell}, \mathcal{R}^\top_{h,\ell-1}\widehat{v}^+_{h,\ell-1}) \\
&\quad - \mathrm{i}k\langle w_{h,\ell}, \widehat{v}^+_{h,\ell-1}\rangle_{\partial\Omega_{\ell,\ell-1}} + \mathrm{i}k\langle w_{h,\ell}, \mathcal{R}^\top_{h,\ell-1}\widehat{v}^+_{h,\ell-1}\rangle_{\partial\Omega_\ell} \\
&= -\widetilde{a}_{\Omega_\ell\backslash\Omega_{\ell-1}}(w_{h,\ell}, \mathcal{R}^\top_{h,\ell-1}\widehat{v}^+_{h,\ell-1}) - \mathrm{i}k\langle w_{h,\ell}, \widehat{v}^+_{h,\ell-1}\rangle_{\Gamma^+_{\ell-1}} \\
&\quad + \mathrm{i}k\langle w_{h,\ell}, \mathcal{R}^\top_{h,\ell-1}\widehat{v}^+_{h,\ell-1}\rangle_{\partial\Omega_\ell\backslash\partial\Omega_{\ell,\ell-1}},
\end{aligned} \tag{4.16}$$

where the last step used the facts that $\mathcal{R}^\top_{h,\ell-1}\widehat{v}^+_{h,\ell-1} = \widehat{v}^+_{h,\ell-1}$ on $\Omega_{\ell,\ell-1}$ and $\partial\Omega_{\ell,\ell-1} = (\partial\Omega_\ell \cap \partial\Omega_{\ell,\ell-1}) \cup \Gamma^+_{\ell-1}$. Now, to obtain (4.15), we set $w_h = \widetilde{\mathcal{R}}^\top_{h,\ell} w_{h,\ell}$, which is supported in $\overline{\Omega_\ell}$ and satisfies $w_h = w_{h,\ell}$ in $\mathring{\Omega}_\ell := \Omega_\ell \backslash (\Omega_{\ell-1} \cup \Omega_{\ell+1})$. Also $\mathcal{R}^\top_{h,\ell-1}\widehat{v}^+_{h,\ell-1}$ is supported in $\Omega_{\ell-1} \cup \mathring{\Omega}_\ell$, so

$$\widetilde{a}_{\Omega_\ell\backslash\Omega_{\ell-1}}(w_h, \mathcal{R}^\top_{h,\ell-1}\widehat{v}^+_{h,\ell-1}) = \widetilde{a}_{\Omega_\ell\backslash\Omega_{\ell-1}}(w_{h,\ell}, \mathcal{R}^\top_{h,\ell-1}\widehat{v}^+_{h,\ell-1}).$$

Moreover, since $w_h = w_{h,\ell}$ on $\Gamma^+_{\ell-1}$ and $w_h = 0$ on $\Gamma^-_{\ell-1}$,

$$\langle w_{h,\ell}, \widehat{v}^+_{h,\ell-1}\rangle_{\Gamma^+_{\ell-1}} = \langle w_h, \widehat{v}^+_{h,\ell-1}\rangle_{\Gamma_{\ell-1}}.$$

In a similar way, $\langle w_{h,\ell}, \mathcal{R}^\top_{h,\ell-1}\widehat{v}^+_{h,\ell-1}\rangle_{\partial\Omega_\ell\backslash\partial\Omega_{\ell,\ell-1}} = \langle w_h, \mathcal{R}^\top_{h,\ell-1}\widehat{v}^+_{h,\ell-1}\rangle_{\partial\Omega\backslash\partial\Omega_{\ell-1}}$. Therefore, using these last three relations in (4.16), and then (2.35), we obtain

$$\begin{aligned}
\big\langle \mathcal{I}^h_{\Gamma^s_\ell \to \Gamma^+_{\ell-1}} g, \widehat{v}^+_{h,\ell-1} \big\rangle_{\Gamma^+_{\ell-1}} &= -\widetilde{a}_{\Omega\backslash\Omega_{\ell-1}}(w_h, \mathcal{R}^\top_{h,\ell-1}\widehat{v}^+_{h,\ell-1}) - \mathrm{i}k\langle w_h, \widehat{v}^+_{h,\ell-1}\rangle_{\Gamma_{\ell-1}} \\
&\qquad + \mathrm{i}k\langle w_h, \mathcal{R}^\top_{h,\ell-1}\widehat{v}^+_{h,\ell-1}\rangle_{\partial\Omega\backslash\partial\Omega_{\ell-1}} \\
&= b_{h,\ell-1}(w_h, \widehat{v}^+_{h,\ell-1}) = b_{h,\ell-1}(\widetilde{\mathcal{R}}^\top_{h,\ell} w_{h,\ell}, \widehat{v}^+_{h,\ell-1}),
\end{aligned}$$

as required. □

Theorem 4.9 (Connection between $\mathcal{T}_{h,j,\ell}$ and $\mathcal{I}^h_{\Gamma^s_\ell \to \Gamma^t_j}$). *Let $w_{h,\ell} \in V^s_{0,\ell}$ with $s \in \{+,-\}$. Then for any $(j,t) \in \{(\ell-1,+),(\ell+1,-)\}$,*

$$\mathrm{imp}^h_{\Gamma^t_j}(\mathcal{T}_{h;j,\ell} w_{h,\ell}) = \mathcal{I}^h_{\Gamma^s_\ell \to \Gamma^t_j}\left(\mathrm{imp}^h_{\Gamma^s_\ell} w_{h,\ell}\right), \tag{4.17}$$

Proof. Since $w_{h,\ell} \in V^s_{0,\ell}$, by (4.3) and (3.7), with $g_h := \mathrm{imp}^h_{\Gamma^s_\ell} w_{h,\ell}$, we have

$$a_{\Omega_\ell}(w_{h,\ell}, v_{h,\ell}) = \langle g_h, v_{h,\ell}\rangle_{\Gamma^s_\ell} \quad \text{for all} \quad v_{h,\ell} \in V_\ell.$$

Using (4.13)/ (4.14) with $g = g_h$ and then (3.4), we have, for any $v_{h,j} \in V_j$,

$$\big\langle \mathcal{I}^h_{\Gamma^s_\ell \to \Gamma^t_j} g_h, v_{h,j}\big\rangle_{\Gamma^t_j} = b_{h,j}(\widetilde{\mathcal{R}}^\top_{h,\ell} w_{h,\ell}, v_{h,j}) = a_{\Omega_j}(\mathcal{T}_{h;j,\ell} w_{h,\ell}, v_{h,j}),$$

Since Lemma 4.4 implies that $\mathcal{T}_{h;j,\ell} w_{h,\ell} \in V^t_{0,j}$, using (4.3) and (3.7) again, we have

$$\mathcal{I}^h_{\Gamma^s_\ell \to \Gamma^t_j} g_h = \mathrm{imp}^h_{\Gamma^t_j}(\mathcal{T}_{h;j,\ell} w_{h,\ell}).$$

Then (4.17) follows since $g_h = \mathrm{imp}^h_{\Gamma^s_\ell} w_{h,\ell}$. □

The next subsection proves the main result of the paper –Theorem 4.17– which shows that if, for some n, $\boldsymbol{\mathcal{T}}^n$ is a contraction, then so is $\boldsymbol{\mathcal{T}}_h^n$ for sufficiently small h. This result is obtained by proving that $\|\boldsymbol{\mathcal{T}}_h^n\|_{\mathbb{V}_0} \to \|\boldsymbol{\mathcal{T}}^n\|_{\mathbb{U}_0}$.

4.5 Convergence of norms

For $s \in \{+,-\}$, we let $\mathbf{\Gamma}^s := \prod_{\ell=1}^N \Gamma_\ell^s$, and $\mathbb{V}(\mathbf{\Gamma}^s) := \prod_{\ell=1}^N V(\Gamma_\ell^s) \subseteq \prod_{\ell=1}^N L^2(\Gamma_\ell^s) =: L^2(\mathbf{\Gamma}^s)$, equipped with the norm

$$\|\boldsymbol{g}\|^2_{L^2(\mathbf{\Gamma}^s)} := \sum_{\ell=1}^N \|g_\ell\|^2_{L^2(\Gamma_\ell^s)} \quad \text{for } \boldsymbol{g} = (g_1, \cdots, g_N) \in L^2(\mathbf{\Gamma}^s).$$

The maps $\mathbf{imp}_s^h : \mathbb{V}_0 \mapsto \mathbb{V}(\mathbf{\Gamma}^s)$ and $\boldsymbol{\mathcal{I}}^h_{s\to t} : L^2(\mathbf{\Gamma}^s) \mapsto \mathbb{V}(\mathbf{\Gamma}^t)$ are then defined by

$$\mathbf{imp}_s^h := \operatorname{diag}\left(\mathrm{imp}^h_{\Gamma_1^s}, \mathrm{imp}^h_{\Gamma_2^s}, \cdots, \mathrm{imp}^h_{\Gamma_N^s}\right),$$

and

$$\boldsymbol{\mathcal{I}}^h_{s\to -} := \begin{pmatrix} 0 & & & & \\ \mathcal{I}^h_{\Gamma_1^s\to\Gamma_2^-} & 0 & & & \\ & \mathcal{I}^h_{\Gamma_2^s\to\Gamma_3^-} & 0 & & \\ & & \ddots & \ddots & \\ & & & \mathcal{I}^h_{\Gamma_{N-1}^s\to\Gamma_N^-} & 0 \end{pmatrix}, \quad \boldsymbol{\mathcal{I}}^h_{s\to +} := \begin{pmatrix} 0 & \mathcal{I}^h_{\Gamma_2^s\to\Gamma_1^+} & & & \\ & 0 & \mathcal{I}^h_{\Gamma_3^s\to\Gamma_2^+} & & \\ & & \ddots & \ddots & \\ & & & 0 & \mathcal{I}^h_{\Gamma_N^s\to\Gamma_{N-1}^+} \\ & & & & 0 \end{pmatrix},$$

where $\mathrm{imp}^h_{\Gamma_\ell^s}$ is defined in Lemma 4.2. The relation (4.17) implies the following corollary.

Corollary 4.10. *For $\boldsymbol{v}_h \in \mathbb{V}_0$,*

$$\mathbf{imp}_-^h \boldsymbol{\mathcal{L}}_h \boldsymbol{v}_h = \boldsymbol{\mathcal{I}}^h_{-\to-} \mathbf{imp}_-^h \boldsymbol{v}_h + \boldsymbol{\mathcal{I}}^h_{+\to-} \mathbf{imp}_+^h \boldsymbol{v}_h \quad \text{and} \quad \mathbf{imp}_+^h \boldsymbol{\mathcal{L}}_h \boldsymbol{v}_h = 0, \tag{4.18}$$

and

$$\mathbf{imp}_+^h \boldsymbol{\mathcal{U}}_h \boldsymbol{v}_h = \boldsymbol{\mathcal{I}}^h_{-\to+} \mathbf{imp}_-^h \boldsymbol{v}_h + \boldsymbol{\mathcal{I}}^h_{+\to+} \mathbf{imp}_+^h \boldsymbol{v}_h \quad \text{and} \quad \mathbf{imp}_-^h \boldsymbol{\mathcal{U}}_h \boldsymbol{v}_h = 0, \tag{4.19}$$

Proof. We only prove (4.18); the proof of (4.19) is similar. The fact that $\mathbf{imp}_+^h \boldsymbol{\mathcal{L}}_h \boldsymbol{v}_h = 0$ follows from Lemmas 4.4 and 4.2. We now consider the first part of (4.18), the ℓth entry of which is

$$\mathrm{imp}_{\Gamma_\ell^-} \mathcal{T}_{h,\ell,\ell-1} v_{h,\ell-1} = \mathcal{I}^h_{\Gamma_{\ell-1}^-\to\Gamma_\ell^-} \mathrm{imp}^h_{\Gamma_{\ell-1}^-} v_{h,\ell-1} + \mathcal{I}^h_{\Gamma_{\ell-1}^+\to\Gamma_\ell^-} \mathrm{imp}^h_{\Gamma_{\ell-1}^+} v_{h,\ell-1}, \tag{4.20}$$

for any $v_{h,\ell-1} \in V_{0,\ell}$. Now, using (4.4) to write $v_{h,\ell-1} = v^-_{h,\ell-1} + v^+_{h,\ell-1}$ and Lemma 4.2, we obtain (4.20) from Theorem 4.9. □

Next we introduce vectorised notation to write (4.18) and (4.19) in a more compact form. Let $\mathbf{\Gamma} = \mathbf{\Gamma}^- \times \mathbf{\Gamma}^+$ and define $\mathbb{V}(\mathbf{\Gamma}) := V(\mathbf{\Gamma}^-) \times V(\mathbf{\Gamma}^+) \subseteq L^2(\mathbf{\Gamma}) =: L^2(\mathbf{\Gamma}^-) \times L^2(\mathbf{\Gamma}^+)$, with the norm

$$\|\boldsymbol{g}\|^2_{L^2(\mathbf{\Gamma})} := \|\boldsymbol{g}^-\|^2_{L^2(\mathbf{\Gamma}^-)} + \|\boldsymbol{g}^+\|^2_{L^2(\mathbf{\Gamma}^+)} \quad \text{for all } \boldsymbol{g} = (\boldsymbol{g}^-, \boldsymbol{g}^+) \in L^2(\mathbf{\Gamma}).$$

We then define the maps $\mathbf{imp}^h : \mathbb{V}_0 \mapsto \mathbb{V}(\mathbf{\Gamma})$ and $\boldsymbol{\mathcal{I}}^h : L^2(\mathbf{\Gamma}) \mapsto \mathbb{V}(\mathbf{\Gamma})$ by

$$\mathbf{imp}^h = \begin{pmatrix} \mathbf{imp}_-^h \\ \mathbf{imp}_+^h \end{pmatrix} \quad \text{and} \quad \boldsymbol{\mathcal{I}}^h = \begin{pmatrix} \boldsymbol{\mathcal{I}}^h_{-\to-} & \boldsymbol{\mathcal{I}}^h_{+\to-} \\ \boldsymbol{\mathcal{I}}^h_{-\to+} & \boldsymbol{\mathcal{I}}^h_{+\to+} \end{pmatrix}.$$

Then we have the following theorem

Theorem 4.11. *For any $\boldsymbol{v}_h \in \mathbb{V}_0$,*

$$\mathbf{imp}^h \left(\boldsymbol{\mathcal{T}}_h \boldsymbol{v}_h\right) = \boldsymbol{\mathcal{I}}^h \left(\mathbf{imp}^h \boldsymbol{v}_h\right). \tag{4.21}$$

Proof. By (4.1), $\boldsymbol{\mathcal{T}}_h = \boldsymbol{\mathcal{L}}_h + \boldsymbol{\mathcal{U}}_h$. By Lemma 4.4, $\mathbf{imp}_-^h \boldsymbol{\mathcal{U}}_h \boldsymbol{v}_h = \mathbf{0} = \mathbf{imp}_+^h \boldsymbol{\mathcal{L}}_h \boldsymbol{v}_h$, and thus

$$\mathbf{imp}^h \left(\boldsymbol{\mathcal{T}}_h \boldsymbol{v}_h\right) = \begin{pmatrix} \mathbf{imp}_-^h \left(\boldsymbol{\mathcal{T}}_h \boldsymbol{v}_h\right) \\ \mathbf{imp}_+^h \left(\boldsymbol{\mathcal{T}}_h \boldsymbol{v}_h\right) \end{pmatrix} = \begin{pmatrix} \mathbf{imp}_-^h \left(\boldsymbol{\mathcal{L}}_h \boldsymbol{v}_h\right) \\ \mathbf{imp}_+^h \left(\boldsymbol{\mathcal{U}}_h \boldsymbol{v}_h\right); \end{pmatrix} \tag{4.22}$$

the result then follows by combining (4.22) with the first equations in (4.18) and (4.19). □

Using (4.21) n times, we have the following corollary.

Corollary 4.12. *For any integer $n \geq 1$,*

$$\mathbf{imp}^h\left(\boldsymbol{\mathcal{T}}_h^n \boldsymbol{v}_h\right) = \left(\boldsymbol{\mathcal{I}}^h\right)^n \left(\mathbf{imp}^h \boldsymbol{v}_h\right) \quad \textit{for any } \boldsymbol{v}_h \in \mathbb{V}_0. \tag{4.23}$$

The next lemma gives an alternative expression for the norm $\|\cdot\|_{\mathbb{V}_0}$.

Lemma 4.13. *For any $\boldsymbol{w}_h \in \mathbb{V}_0$,*

$$\|\boldsymbol{w}_h\|_{\mathbb{V}_0} = \|\mathbf{imp}^h \boldsymbol{w}_h\|_{L^2(\boldsymbol{\Gamma})}. \tag{4.24}$$

Proof. Let $\boldsymbol{w}_h = (w_{h,1}, \cdots, w_{h,N}) \in \mathbb{V}_0$, and use Lemma 4.3 to write $w_{h,\ell} = w_{h,\ell}^- + w_{h,\ell}^+$. By (3.8),

$$\|\boldsymbol{w}_h\|_{\mathbb{V}_0}^2 = \sum_\ell \|w_{h,\ell}\|_{\mathbb{V}_{0,\ell}}^2 = \sum_\ell \|\mathrm{imp}_{\Gamma_\ell}^h w_{h,\ell}\|_{\mathbb{V}_{0,\ell}}^2.$$

Then, using Lemmas 4.3 and 4.2, we obtain

$$\begin{aligned}\|\boldsymbol{w}_h\|_{\mathbb{V}_0}^2 &= \sum_\ell \|\mathrm{imp}_{\Gamma_\ell^-}^h w_{h,\ell}^-\|_{L^2(\Gamma_\ell^-)}^2 + \sum_\ell \|\mathrm{imp}_{\Gamma_\ell^+}^h w_{h,\ell}^+\|_{L^2(\Gamma_\ell^+)}^2 \\ &= \|\mathrm{imp}_-^h \boldsymbol{w}_h\|_{L^2(\boldsymbol{\Gamma}^-)}^2 + \|\mathrm{imp}_-^h \boldsymbol{w}_h\|_{L^2(\boldsymbol{\Gamma}^+)}^2 = \|\mathrm{imp}^h \boldsymbol{w}_h\|_{L^2(\boldsymbol{\Gamma})}^2.\end{aligned}$$

□

Lemma 4.14. *For any integer $n \geq 1$,*

$$\|\boldsymbol{\mathcal{T}}_h^n\|_{\mathbb{V}_0} = \left\|\left(\boldsymbol{\mathcal{I}}^h\right)^n\right\|_{L^2(\boldsymbol{\Gamma})}. \tag{4.25}$$

Proof. By Lemma 4.13 and Corollary 4.12,

$$\begin{aligned}\|\boldsymbol{\mathcal{T}}_h^n\|_{\mathbb{V}_0} &= \sup_{\boldsymbol{w}_h \in \mathbb{V}_0} \frac{\|\boldsymbol{\mathcal{T}}_h^n \boldsymbol{w}_h\|_{\mathbb{V}_0}}{\|\boldsymbol{w}_h\|_{\mathbb{V}_0}} = \sup_{\boldsymbol{w}_h \in \mathbb{V}_0} \frac{\|\mathbf{imp}^h\left(\boldsymbol{\mathcal{T}}_h^n \boldsymbol{w}_h\right)\|_{L^2(\boldsymbol{\Gamma})}}{\|\mathbf{imp}^h\ \boldsymbol{w}_h\|_{L^2(\boldsymbol{\Gamma})}} \\ &= \sup_{\boldsymbol{w}_h \in \mathbb{V}_0} \frac{\|(\boldsymbol{\mathcal{I}}^h)^n \left(\mathbf{imp}^h\ \boldsymbol{w}_h\right)\|_{L^2(\boldsymbol{\Gamma})}}{\|\mathbf{imp}^h\ \boldsymbol{w}_h\|_{L^2(\boldsymbol{\Gamma})}} \leq \left\|\left(\boldsymbol{\mathcal{I}}^h\right)^n\right\|_{L^2(\boldsymbol{\Gamma})}.\end{aligned}$$

The lemma then follows if we can show that $\|\boldsymbol{\mathcal{T}}_h^n\|_{\mathbb{V}_0} \geq \|(\boldsymbol{\mathcal{I}}^h)^n\|_{L^2(\boldsymbol{\Gamma})}$.

For each ℓ, s, let $P_{\Gamma_\ell^s}^h$ denote the L^2 orthogonal projection from $L^2(\Gamma_\ell^s)$ to $V(\Gamma_\ell^s)$, and let $\boldsymbol{P}^h = \mathrm{diag}(P_{\Gamma_1^-}^h, \ldots, P_{\Gamma_N^-}^h, P_{\Gamma_1^+}^h, \ldots, P_{\Gamma_N^+}^h)$. A simple examination of Definition 4.6 shows that $\mathcal{I}_{\Gamma_\ell^s \to \Gamma_j^t} = \mathcal{I}_{\Gamma_\ell^s \to \Gamma_j^t} P_{\Gamma_\ell^s}^h$, and so

$$\boldsymbol{\mathcal{I}}^h \boldsymbol{g} = \boldsymbol{\mathcal{I}}^h \boldsymbol{P}^h \boldsymbol{g} \quad \text{for all } \ \boldsymbol{g} \in L^2(\boldsymbol{\Gamma}). \tag{4.26}$$

Let $\boldsymbol{g} = (\boldsymbol{g}^-, \boldsymbol{g}^+)$ with $\boldsymbol{g}^s = (g_{\Gamma_1^s}, g_{\Gamma_2^s}, \cdots, g_{\Gamma_N^s}) \in \mathbb{V}(\boldsymbol{\Gamma}^s)$. Let the components of $\boldsymbol{w}_h \in \mathbb{V}_0$ be defined as the solutions of

$$a_\ell(w_{h,\ell}, v_{h,\ell}) = \langle g_{\Gamma_\ell^-}, v_{h,\ell}\rangle_{\Gamma_\ell^-} + \langle g_{\Gamma_\ell^+}, v_{h,\ell}\rangle_{\Gamma_\ell^+} \quad \text{for all } \ v_{h,\ell} \in V_\ell.$$

Then, by Lemma 3.4,

$$\mathbf{imp}^h \boldsymbol{w}_h = \boldsymbol{P}^h \boldsymbol{g}. \tag{4.27}$$

Thus, by using (4.26)-(4.27) and (4.23), (4.24), we have

$$\begin{aligned}\left\|\left(\boldsymbol{\mathcal{I}}^h\right)^n\right\|_{L^2(\boldsymbol{\Gamma})} &= \sup_{\boldsymbol{g} \in L^2(\boldsymbol{\Gamma})} \frac{\|\left(\boldsymbol{\mathcal{I}}^h\right)^n \boldsymbol{g}\|_{L^2(\boldsymbol{\Gamma})}}{\|\boldsymbol{g}\|_{L^2(\boldsymbol{\Gamma})}} \leq \sup_{\boldsymbol{g} \in L^2(\boldsymbol{\Gamma})} \frac{\|\left(\boldsymbol{\mathcal{I}}^h\right)^n \boldsymbol{P}^h \boldsymbol{g}\|_{L^2(\boldsymbol{\Gamma})}}{\|\boldsymbol{P}^h \boldsymbol{g}\|_{L^2(\boldsymbol{\Gamma})}} \\ &\leq \sup_{\boldsymbol{w}_h \in \mathbb{V}_0} \frac{\|\left(\boldsymbol{\mathcal{I}}^h\right)^n \left(\mathbf{imp}^h \boldsymbol{w}_h\right)\|_{L^2(\boldsymbol{\Gamma})}}{\|\mathbf{imp}^h \boldsymbol{w}_h\|_{L^2(\boldsymbol{\Gamma})}} = \sup_{\boldsymbol{w}_h \in \mathbb{V}_0} \frac{\|\mathbf{imp}^h\left(\boldsymbol{\mathcal{T}}_h^n \boldsymbol{w}_h\right)\|_{L^2(\boldsymbol{\Gamma})}}{\|\mathbf{imp}^h \boldsymbol{w}_h\|_{L^2(\boldsymbol{\Gamma})}} = \|\boldsymbol{\mathcal{T}}_h^n\|_{\mathbb{V}_0}.\end{aligned}$$

□

The analogous results to Theorem 4.11, Corollary 4.12 and Lemmas 4.13 and 4.14 can be easily established at the continuous level. Define the continuous vectorised maps $\mathbf{imp}_s : \mathbb{U}_0 \mapsto L^2(\mathbf{\Gamma}^s)$ and $\boldsymbol{\mathcal{I}}_{s\to t} : L^2(\mathbf{\Gamma}^s) \mapsto L^2(\mathbf{\Gamma}^t)$, as follows,

$$\mathbf{imp}_s := \mathrm{diag}\left(\mathrm{imp}_{\Gamma_1^s}, \mathrm{imp}_{\Gamma_2^s}, \cdots, \mathrm{imp}_{\Gamma_N^s}\right),$$

where $\mathrm{imp}_{\Gamma_\ell^s} v_\ell = (\partial/\partial n_\ell - \mathrm{i}k)v_\ell$ on Γ_ℓ^s, and

$$\boldsymbol{\mathcal{I}}_{s\to -} := \begin{pmatrix} 0 & & & & \\ \mathcal{I}_{\Gamma_1^s\to\Gamma_2^-} & 0 & & & \\ & \mathcal{I}_{\Gamma_2^s\to\Gamma_3^-} & 0 & & \\ & & \ddots & \ddots & \\ & & & \mathcal{I}_{\Gamma_{N-1}^s\to\Gamma_N^-} & 0 \end{pmatrix}, \quad \boldsymbol{\mathcal{I}}_{s\to +} := \begin{pmatrix} 0 & \mathcal{I}_{\Gamma_2^s\to\Gamma_1^+} & & & \\ & 0 & \mathcal{I}_{\Gamma_3^s\to\Gamma_2^+} & & \\ & & \ddots & \ddots & \\ & & & 0 & \mathcal{I}_{\Gamma_N^s\to\Gamma_{N-1}^+} \\ & & & & 0 \end{pmatrix}. \tag{4.28}$$

Then define the maps $\mathbf{imp} : \mathbb{U}_0 \mapsto L^2(\mathbf{\Gamma})$ and $\boldsymbol{\mathcal{I}} : L^2(\mathbf{\Gamma}) \mapsto L^2(\mathbf{\Gamma})$ By

$$\mathbf{imp} = \begin{pmatrix} \mathbf{imp}_- \\ \mathbf{imp}_+ \end{pmatrix} \quad \text{and} \quad \boldsymbol{\mathcal{I}} = \begin{pmatrix} \boldsymbol{\mathcal{I}}_{-\to -} & \boldsymbol{\mathcal{I}}_{+\to -} \\ \boldsymbol{\mathcal{I}}_{-\to +} & \boldsymbol{\mathcal{I}}_{+\to +} \end{pmatrix}.$$

Lemma 4.15. *For any integer $n \geq 1$,*

$$\mathbf{imp}\left(\boldsymbol{\mathcal{T}}^n \boldsymbol{v}\right) = \boldsymbol{\mathcal{I}}^n\left(\mathbf{imp}\ \boldsymbol{v}\right) \quad \textit{for any } \boldsymbol{v} \in \mathbb{U}_0,$$

and

$$\|\boldsymbol{\mathcal{T}}^n\|_{\mathbb{U}_0} = \|\boldsymbol{\mathcal{I}}^n\|_{L^2(\mathbf{\Gamma})}. \tag{4.29}$$

Lemma 4.16.

$$\|\boldsymbol{\mathcal{I}}^h - \boldsymbol{\mathcal{I}}\|_{L^2(\mathbf{\Gamma})} \to 0 \quad \textit{as } h \to 0. \tag{4.30}$$

Moreover, there exist a constant $C > 0$ such that, for any $\boldsymbol{g} \in L^2(\mathbf{\Gamma})$ and sufficiently small h,

$$\|\boldsymbol{\mathcal{I}}\boldsymbol{g}\|_{L^2(\mathbf{\Gamma})} \leq C\|\boldsymbol{g}\|_{L^2(\mathbf{\Gamma})} \quad \textit{and} \quad \|\boldsymbol{\mathcal{I}}^h\boldsymbol{g}\|_{L^2(\mathbf{\Gamma})} \leq C\|\boldsymbol{g}\|_{L^2(\mathbf{\Gamma})}. \tag{4.31}$$

Proof. (4.30) is a direct corollary of Theorem 4.7. The boundedness of $\boldsymbol{\mathcal{I}}$ follows from [20, Lemma 3.8]. The boundedness of $\boldsymbol{\mathcal{I}}^h$ follows from the boundedness of $\boldsymbol{\mathcal{I}}$ and (4.30). □

Finally we obtain our main result.

Theorem 4.17. *For any fixed integer $n \geq 1$,*

$$\|\boldsymbol{\mathcal{T}}_h^n\|_{\mathbb{V}_0} \to \|\boldsymbol{\mathcal{T}}^n\|_{\mathbb{U}_0} \quad \textit{as } h \to 0.$$

Proof. By (4.25) and (4.29), and the reverse triangle inequality,

$$\left|\|\boldsymbol{\mathcal{T}}_h^n\|_{\mathbb{V}_0} - \|\boldsymbol{\mathcal{T}}^n\|_{\mathbb{U}_0}\right| = \left|\left\|\left(\boldsymbol{\mathcal{I}}^h\right)^n\right\|_{L^2(\mathbf{\Gamma})} - \|\boldsymbol{\mathcal{I}}^n\|_{L^2(\mathbf{\Gamma})}\right| \leq \left\|\left(\boldsymbol{\mathcal{I}}^h\right)^n - \boldsymbol{\mathcal{I}}^n\right\|_{L^2(\mathbf{\Gamma})}.$$

So it is sufficient to show that $\|(\boldsymbol{\mathcal{I}}^h)^n - \boldsymbol{\mathcal{I}}^n\|_{L^2(\mathbf{\Gamma})} \to 0 \quad$ as $h \to 0$. We now prove by induction that

$$\left\|\left(\boldsymbol{\mathcal{I}}^h\right)^n - \boldsymbol{\mathcal{I}}^n\right\|_{L^2(\mathbf{\Gamma})} \leq nC^{n-1}\left\|\boldsymbol{\mathcal{I}}^h - \boldsymbol{\mathcal{I}}\right\|_{L^2(\mathbf{\Gamma})}, \tag{4.32}$$

where C is as in (4.31); the result then follows by combining (4.32) with (4.30) (for any fixed n). First observe that (4.32) obviously holds for $n = 1$. Assuming it holds $n = m-1$ and using (4.31), we have, for $n = m$,

$$\begin{aligned} \left\|\left(\boldsymbol{\mathcal{I}}^h\right)^m - \boldsymbol{\mathcal{I}}^m\right\|_{L^2(\mathbf{\Gamma})} &= \left\|\left(\left(\boldsymbol{\mathcal{I}}^h\right)^{m-1} - \boldsymbol{\mathcal{I}}^{m-1}\right)\boldsymbol{\mathcal{I}}^h + \boldsymbol{\mathcal{I}}^{m-1}\left(\boldsymbol{\mathcal{I}}^h - \boldsymbol{\mathcal{I}}\right)\right\|_{L^2(\mathbf{\Gamma})} \\ &\leq \left\|\left(\left(\boldsymbol{\mathcal{I}}^h\right)^{m-1} - \boldsymbol{\mathcal{I}}^{m-1}\right)\boldsymbol{\mathcal{I}}^h\right\|_{L^2(\mathbf{\Gamma})} + \left\|\boldsymbol{\mathcal{I}}^{m-1}\left(\boldsymbol{\mathcal{I}}^h - \boldsymbol{\mathcal{I}}\right)\right\|_{L^2(\mathbf{\Gamma})} \\ &\leq (m-1)C^{m-2}\left\|\boldsymbol{\mathcal{I}}^h - \boldsymbol{\mathcal{I}}\right\|_{L^2(\mathbf{\Gamma})} C + C^{m-1}\left\|\boldsymbol{\mathcal{I}}^h - \boldsymbol{\mathcal{I}}\right\|_{L^2(\mathbf{\Gamma})} \\ &= mC^{m-1}\left\|\boldsymbol{\mathcal{I}}^h - \boldsymbol{\mathcal{I}}\right\|_{L^2(\mathbf{\Gamma})}; \end{aligned}$$

thus (4.32) holds for any integer $n \geq 1$ and the proof is complete. □

4.6 Power contractivity of the ORAS iteration

If $\boldsymbol{\mathcal{T}}^n$ is a contraction, then Theorem 4.17 has the following corollary.

Corollary 4.18. *Suppose $\|\boldsymbol{\mathcal{T}}^n\|_{\mathbb{U}_0} < C < 1$ and suppose ORAS is implemented using finite elements of degree p. Then for all $\varepsilon > 0$ there exists $h_0 = h_0(p, \varepsilon) > 0$, such that*

$$\|\boldsymbol{\mathcal{T}}_h^n\|_{\mathbb{V}_0} < C + \varepsilon \quad \textit{for all} \;\; h \le h_0(p, \varepsilon).$$

That is, the power contractive property of ORAS is independent of p and h for h sufficiently small (with a p-dependent threshold).

In the rest of this section, we discuss the results of [20], giving insight into when the condition $\|\boldsymbol{\mathcal{T}}^n\|_{\mathbb{U}_0} < C < 1$ holds. The paper [20] studies the power contractivity of the operator $\boldsymbol{\mathcal{T}}$ by using the decomposition $\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{L}} + \boldsymbol{\mathcal{U}}$ into lower and upper triangular parts (analogous to (4.1)). Analogously to Lemma 4.4, $\boldsymbol{\mathcal{L}}$ and $\boldsymbol{\mathcal{U}}$ are maps on vectors of Helmholtz-harmonic functions and the products $\boldsymbol{\mathcal{L}\mathcal{U}}$, $\boldsymbol{\mathcal{U}\mathcal{L}}$ can be characterised, respectively, in terms of the action of right-to-left and left-to-right impedance-to-impedance maps. Computations in [20] show that impedance-to-impedance maps that switch direction are typically $\ll 1$ in norm, while impedance maps that preserve direction have norm very close to 1. With ρ denoting the maximum of the norms of the impedance-to-impedance maps that switch direction, and γ denoting the maximum of the norms of the maps that preserve direction, it is proved rigorously in [26] that, for a particular model problem, ρ decreases with $\mathcal{O}(\delta^{-2})$, for k sufficiently large where δ is the width of the overlap of neighbouring subdomains. Also, since the norm of any discrete impedance map can be computed (by solving a local eigenvalue problem) and since the discrete map converges in norm to the continuous map, we can also compute ρ, γ, with guaranteed accuracy, for moderate k. Computations in [20] show that typically that $\boldsymbol{\mathcal{T}}^n$ is a contraction when $n \ge N$ and that ρ is decreases rapidly with increasing δ while $\gamma \approx 1$. Thus, upper bounds on $\|\boldsymbol{\mathcal{T}}^n\|_{\mathbb{U}_0}$ in terms of ρ and γ, where the upper bound $\to 0$ as $\rho \to 0$, help justify observations of power contractivity of $\boldsymbol{\mathcal{T}}^n$. A key result in [20] is that, for any $n \ge N$,

$$\|\boldsymbol{\mathcal{T}}^n\|_{\mathbb{U}_0} \;\le\; 2\sum_{j=1}^{n-1} \begin{pmatrix} n-1 \\ j \end{pmatrix} \max_{p \in \mathcal{P}(n,j)} \|p(\mathcal{L}, \mathcal{U})\|_{\mathbb{U}_0}, \tag{4.33}$$

where $\mathcal{P}(n, j)$ is the set of monomials of degree n in 2 variables with the property that $p(\boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{U}})$ contains j switches from $\boldsymbol{\mathcal{L}}$ to $\boldsymbol{\mathcal{U}}$ or $\boldsymbol{\mathcal{U}}$ to $\boldsymbol{\mathcal{L}}$. Hence terms in (4.33) corresponding to $j = 1$ contain one impedance map that switches direction and $N - 1$ maps that preserve direction, leading to the estimate:

$$\|\boldsymbol{\mathcal{T}}^N\|_{\mathbb{U}_0} \;\le\; 4\gamma^{N-1}(N-1)\rho \;+\; \mathcal{O}(\rho^2). \tag{4.34}$$

While (4.34) shows that, for fixed N, $\mathcal{T}^N$ is contracting when ρ is small enough, computations in [20] suggest that the power contractive property is independent of N, for fixed ρ. This property is explained in [20] by estimating terms $\|p(\boldsymbol{\mathcal{L}}, \boldsymbol{\mathcal{U}})\|_{\mathbb{U}_0}$ for $p \in \mathcal{P}(N, 1)$ more carefully in terms of norms of *compositions* of impedance-to-impedance maps; see [20, §4]. The analysis in [20, §4] also obtains refinements of (4.34) that explain practical observations [20, Figure 6.1] that the convergence profile of the algorithm takes a sharp jump downwards after each batch of N iterations. Finally we highlight that [20] also contains extensive experiments on non-strip-type domain decompositions, and these indicate that power contractivity persists in this more-general situation.

5 Convergence of discrete impedance-to-impedance maps

In this section we prove Theorem 4.7, thus establishing the norm convergence of the discrete impedance-to-impedance maps to their continuous counterparts.

To reduce notational technicalities, we note that the proof is the same for all ℓ. Moreover, by symmetry, if we prove Theorem 4.7 for s equal to $-$ then we also have the result for s equal to $+$. Furthermore, for any choice of s, the proofs of the two convergence results in (4.12) are almost identical, so we just prove the first one. Finally by the affine change of coordinates $x = H\tilde{x}$, a

Helmholtz problem on any of the domains Ω_ℓ in Figure 1 can be scaled to a Helmholtz problem on the rectangle $[0,L]\times[0,1]$, for some L, leading only to a multiplicative change in the wavenumber k. Thus in this section, without loss of generality we restrict to the canonical domain $\Omega = [0,L]\times[0,1]$ depicted in Figure 2. We study only the left-to-right impedance map (which we denote $\mathcal{I}$), which takes, as input, left-facing impedance data on Γ^-, solves the Helmholtz problem on Ω and evaluates the right-facing impedance data at the vertical interface Γ_δ, which is situated a distance $\delta \in (0,L)$ from Γ^-. In the case of a physical subdomain Ω_ℓ (see Figure 1), Γ^- corresponds to Γ_ℓ^- and Γ_δ corresponds to $\Gamma_{\ell-1}^+$.

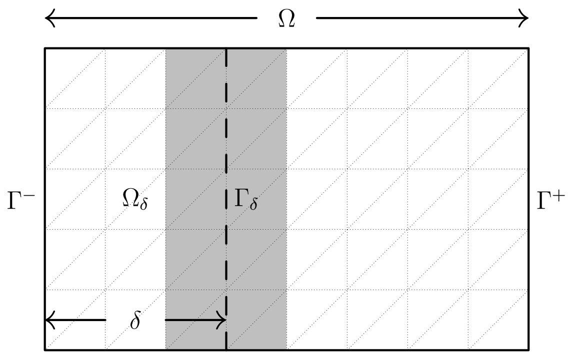

Figure 2: The canonical domain Ω, and the domain Ω_δ bounded by Γ^- and Γ_δ.

For clarity, we now redefine the map $\mathcal{I}$, as well as its finite element counterpart $\mathcal{I}^h$, essentially rewriting the definitions given in (4.8), (4.10) and (4.11), but using the simplified setting of Figure 2. Throughout this section (only) we need the following notations, which are equivalent to notations used previously, but recast in the simpler setting of the canonical domain.

Notation 5.1 (Notation specific to canonical domain)**.** *$V = V(\Omega)$ denotes the finite element space on Ω (with mesh T^h, of diameter h and assumed to resolve the interior interface Γ_δ). $V(\Omega_\delta)$ and $V(\Gamma_\delta)$ denote the restrictions of V to Ω_δ and Γ_δ respectively. For any $v_\delta \in H^1(\Omega_\delta)$, $\mathcal{R}_\delta^\top v_\delta$ denotes its zero extension to all of Ω (analogous to (2.24)). For any $v_{h,\delta} \in V(\Omega_\delta)$, $\mathcal{R}_{h,\delta}^\top v_{h,\delta} \in V$ denotes its zero nodal extension to all of Ω (analogous to (2.2)). For any $v_{h,\Gamma_\delta} \in V(\Gamma_\delta)$, $\widehat{v}_{h,\delta} \in V(\Omega_\delta)$ denotes its zero nodal extension to all of Ω_δ (analogous to Notation (4.5)(i)). We also need to pay special attention to the mesh elements that touch the interface Γ_δ (see the shaded region in Figure 2), and so we set*

$$\Omega'_\delta := \bigcup \{\tau \in T^h : \tau \cap \Gamma_\delta \neq 0\}, \tag{5.1}$$

where the elements τ are assumed closed.

Assumption 5.2 (Mesh assumption)**.** *T^h is shape-regular. In addition, the elements touching the interface Γ_δ are quasiuniform in the sense that there exists a constant $C_0 \geq 1$ such that*

$$h \leq C_0 h_\tau \quad \text{for all} \;\; \tau \subset \Omega'_\delta. \tag{5.2}$$

Definition 5.3 ($\mathcal{I}$ and $\mathcal{I}^h$ defined in canonical domain)**.** *Given $g \in L^2(\Gamma^-)$, let $u \in H^1(\Omega)$ and $u_h \in V$ be the solutions of*

$$a(u,v) = \langle g, v\rangle_{\Gamma^-} \quad \text{for all} \quad v \in H^1(\Omega), \tag{5.3}$$

$$a(u_h,v_h) = \langle g, v_h\rangle_{\Gamma^-} \quad \text{for all} \quad v_h \in V(\Omega). \tag{5.4}$$

The continuous left-to-right impedance map $\mathcal{I} : L^2(\Gamma^-) \to L^2(\Gamma_\delta)$ is defined (analogously to (4.8)) as

$$\mathcal{I}g = \left(\left(\frac{\partial}{\partial x_1} - \mathrm{i}k\right)u\right)\Big|_{\Gamma_\delta} \in L^2(\Gamma_\delta)$$

The discrete counterpart of $\mathcal{I}$ is $\mathcal{I}^h : L^2(\Gamma^-) \to V(\Gamma_\delta)$, defined (analogously to (4.10)) by

$$\left\langle \mathcal{I}^h g, v_{h,\Gamma_\delta}\right\rangle_{\Gamma_\delta} := a_{\Omega_\delta}(u_h, \widehat{v}_{h,\delta}) - a(u_h, \mathcal{R}_{h,\delta}^\top \widehat{v}_{h,\delta}) \quad \text{for all} \quad v_{h,\Gamma_\delta} \in V(\Gamma_\delta), \tag{5.5}$$

The rest of this section is devoted to proving Theorem 5.5 below, which is sufficient to establish Theorem 4.7. From here on our estimates are explicit in the parameters k and h. In this context, we use the following notation.

Notation 5.4. *We write $A \lesssim B$ if $A \leq CB$ where C is independent of h and k. We write $A \sim B$ when $A \lesssim B$ and $B \lesssim A$. Since the entire theory of this paper is underpinned by the condition (1.8) and since we are studying the limit $h \to 0$ we assume, without loss of generality, that*

$$hk \leq 1 \quad \text{and} \quad k \geq k_0 > 0. \tag{5.6}$$

Theorem 5.5 (Norm convergence of discrete imp-to-imp maps)**.** *Suppose Assumption 5.2 holds and $h^{1/2}k$ is sufficiently small. Then,*

$$\|\mathcal{I} - \mathcal{I}^h\| \ \lesssim \ h^{1/2}k + hk^3.$$

5.1 Proof of Theorem 5.5

In the following we use the Helmholtz energy inner product for any subdomain $\Omega' \subset \Omega$,

$$(v, w)_{1,k,\Omega'} := (\nabla v, \nabla w)_{\Omega'} + k^2(v, w)_{\Omega'}, \tag{5.7}$$

and we denote the induced norm by $\|\cdot\|_{1,k,\Omega'}$; when $\Omega' = \Omega$ we just write $(\cdot,\cdot)_{1,k}$ and $\|\cdot\|_{1,k}$.

Theorem 5.5 is a direct corollary of the following two results (by setting $\alpha = 1/2$ in Theorem 5.7).

Lemma 5.6. *If Assumption 5.2 holds, then*

$$\|\mathcal{I}g - \mathcal{I}^h g\|_{L^2(\Gamma_\delta)} \ \lesssim \ h^{1/2}k\|g\|_{L^2(\Gamma^-)} + h^{-1/2}\|u - u_h\|_{1,k,\Omega'_\delta}. \tag{5.8}$$

Theorem 5.7. *Suppose that Assumption 5.2 holds and that, given $\alpha \in [0, 1/2]$, $h^{2-\alpha}k^3$ is sufficiently small. Then*

$$\|u - u_h\|_{1,k,\Omega'_\delta} \lesssim \left(h^{\alpha+1/2}k + h^{3/2}k^3\right)\|g\|_{L^2(\Gamma^-)}.$$

Lemma 5.6 is proved at the end of this subsection. The proof of Theorem 5.7 is more involved, and is postponed until §5.2. Although Theorem 5.7 is stated and proved for Ω given in Figure 2, appropriate analogues of it hold for more general geometries; see Remark 5.22.

First we recall that since $\delta > 0$, the regularity of the solution u of (5.3) with respect to the data g is much better in a neighbourhood of the interface Γ_δ than it is near the part of the boundary Γ^- where the impedance condition involving g is imposed.

Lemma 5.8. *If u is the solution of (5.3), then, given $\beta > 1/2$,*

$$\|u\|_{1,k} \lesssim \|g\|_{L^2(\Gamma^-)} \quad \text{and} \quad \|u\|_{H^{3/2}(\Omega)} \lesssim k^\beta\|g\|_{L^2(\Gamma^-)}. \tag{5.9}$$

If D be a convex polygonal subdomain of Ω with $\mathrm{dist}(D, \Gamma^-) \gtrsim 1$, then

$$\|u\|_{H^2(D)} \lesssim k\|g\|_{L^2(\Gamma^-)}. \tag{5.10}$$

Proof. The first inequality of (5.9) follows from [20, Lemma 2.4] with $f = 0$. The second inequality of (5.9) follows from [20, Theorem 2.9] with $f = 0$ there. To prove (5.10), first observe that if $\mathrm{dist}(D', \partial\Omega) > 0$, then by interior elliptic regularity for the Laplacian (see, e.g., [28, Theorem 4.16]), and $\Delta u = k^2 u$,

$$\|u\|_{H^2(D')} \lesssim \|\Delta u\|_{L^2(D)} + \|u\|_{H^1(D)} \ \lesssim \ k^2\|u\|_{L^2(D)} + \|u\|_{H^1(D)}\,;$$

the bound (5.10) in this case then follows from the first inequality in (5.9).

It is therefore sufficient to prove (5.10) for D supported in a neighbourhood of $\partial\Omega\backslash\Gamma^s$. In this case, by elliptic regularity up to the boundary for the Laplacian (see, e.g., [28, Theorem 4.18]), using the fact that D is convex,

$$\begin{aligned}\|u\|_{H^2(D)} &\lesssim \|\Delta u\|_{L^2(D)} + \|u\|_{H^1(D)} + \|\partial u/\partial n\|_{H^{1/2}(\partial\Omega\cap D)} \\ &= k^2\|u\|_{L^2(D)} + \|u\|_{H^1(D)} + k\|u\|_{H^{1/2}(\partial\Omega\cap D)}\,,\end{aligned}$$

The bound (5.10) then follows from the trace result $\|u\|_{H^{1/2}(\Gamma_2)} \leq \|u\|_{H^{1/2}(\partial\Omega)} \lesssim \|u\|_{H^1(\Omega)}$ and the first inequality in (5.9). □

Lemma 5.9. *Given $g \in L^2(\Gamma^-)$, let u and u_h be the solutions of (5.3) and (5.4) respectively. Then*

$$\|\mathcal{I}g - \mathcal{I}^h g\|_{L^2(\Gamma_\delta)} \leq \inf_{v_{h,\Gamma_\delta} \in V^h(\Gamma_\delta)} \|\mathcal{I}g - v_{h,\Gamma_\delta}\|_{L^2(\Gamma_\delta)} + \sup_{v_{h,\Gamma_\delta} \in V^h(\Gamma_\delta)\setminus\{0\}} \frac{|a_{\Omega_\delta}(u - u_h, \widehat{v}_{h,\delta})|}{\|v_{h,\Gamma_\delta}\|_{L^2(\Gamma_\delta)}}, \quad (5.11)$$

where $\widehat{v}_{h,\delta}$ is defined in Notation 5.1.

Proof. Let $\mathcal{P}^h_{\Gamma_\delta} : L^2(\Gamma_\delta) \mapsto V^h(\Gamma_\delta)$ denote the L^2-orthogonal projection onto $V^h(\Gamma_\delta)$ and set $z_h = \mathcal{P}^h_{\Gamma_\delta}\mathcal{I}g$. Then, since $z_h - \mathcal{I}^h g \in V(\Gamma_\delta)$,

$$\begin{aligned}\|\mathcal{I}g - \mathcal{I}^h g\|_{L^2(\Gamma_\delta)} &\leq \|\mathcal{I}g - z_h\|_{L^2(\Gamma_\delta)} + \|z_h - \mathcal{I}^h g\|_{L^2(\Gamma_\delta)} \\ &= \inf_{v_{h,\Gamma_\delta} \in V^h(\Gamma_\delta)} \|\mathcal{I}g - v_{h,\Gamma_\delta}\|_{L^2(\Gamma_\delta)} + \sup_{v_{h,\Gamma_\delta} \in V(\Gamma_\delta)\setminus\{0\}} \frac{\left|\langle z_h - \mathcal{I}^h g, v_{h,\Gamma_\delta}\rangle_{\Gamma_\delta}\right|}{\|v_{h,\Gamma_\delta}\|_{L^2(\Gamma_\delta)}} \\ &= \inf_{v_{h,\Gamma_\delta} \in V^h(\Gamma_\delta)} \|\mathcal{I}g - v_{h,\Gamma_\delta}\|_{L^2(\Gamma_\delta)} + \sup_{v_{h,\Gamma_\delta} \in V^h(\Gamma_\delta)\setminus\{0\}} \frac{\left|\langle \mathcal{I}g - \mathcal{I}^h g, v_{h,\Gamma_\delta}\rangle_{\Gamma_\delta}\right|}{\|v_{h,\Gamma_\delta}\|_{L^2(\Gamma_\delta)}}.\end{aligned}$$

By (2.27),

$$\langle \mathcal{I}g, v_\delta\rangle_{\Gamma_\delta} = a_{\Omega_\delta}(u, v_\delta) - \alpha(u, \mathcal{R}_\delta^\top v_\delta) \quad \text{for all} \quad v_\delta \in H^1(\Omega_\delta). \quad (5.12)$$

with α defined by (2.25). Now we use (5.12) with $v_\delta = \widehat{v}_{h,\delta}$, and combine it with (5.5) to obtain

$$\left\langle \mathcal{I}g - \mathcal{I}^h g, v_{h,\Gamma_\delta}\right\rangle_{\Gamma_\delta} = a_{\Omega_\delta}(u - u_h, \widehat{v}_{h,\delta}) - \alpha\left(u, \mathcal{R}_\delta^\top \widehat{v}_{h,\delta}\right) + a(u_h, \mathcal{R}_{h,\delta}^\top \widehat{v}_{h,\delta}). \quad (5.13)$$

The proof is completed by showing that the last two terms in (5.13) vanish. For the first term we recall that u is Helmholtz-harmonic, and so, by (2.25),

$$\alpha(u, \mathcal{R}_\delta^\top \widehat{v}_{h,\delta}) = \left\langle \partial u/\partial n - \mathrm{i}ku, \mathcal{R}_\delta^\top \widehat{v}_{h,\delta}\right\rangle_{\partial\Omega};$$

this last expression vanishes because $\partial u/\partial n - \mathrm{i}ku = 0$ on $\partial\Omega\backslash\Gamma^-$ and $\mathcal{R}_\delta^\top \widehat{v}_{h,\delta} = 0$ on Γ^-. Also, by (5.4),

$$a(u_h, \mathcal{R}_{h,\delta}^\top \widehat{v}_{h,\delta}) = \langle g, \mathcal{R}_{h,\delta}^\top \widehat{v}_{h,\delta}\rangle_{\Gamma^-} = 0,$$

thus completing the proof. □

Proof of Lemma 5.6. We bound the two terms on the right-hand side of (5.11). For the first term, by [12, Theorem 6.1] and the standard scaling argument

$$\inf_{v_{h,\Gamma_\delta} \in V^h(\Gamma_\delta)} \|\mathcal{I}g - v_{h,\Gamma_\delta}\|_{L^2(\Gamma_\delta)} \lesssim h^{1/2}\|\mathcal{I}g\|_{H^{1/2}(\Gamma_\delta)}.$$

Then, with u defined by (5.3),

$$\|\mathcal{I}g\|_{H^{1/2}(\Gamma_\delta)} = \left\|\left(\frac{\partial u}{\partial x_1} - \mathrm{i}k\right) u\right\|_{H^{1/2}(\Gamma_\delta)} \lesssim \|u\|_{H^2(\Omega\backslash\overline{\Omega_\delta})} + k\|u\|_{H^1(\Omega)} \lesssim k\|g\|_{L^2(\Gamma^-)},$$

where we used the trace theorem and the bounds (5.9) and (5.10), the latter with $D = \Omega\backslash\overline{\Omega_\delta}$. Combining the last two displayed estimates gives the first term on the right-hand side of (5.8).

For the second term on the right-hand side of (5.11), we recall that $\widehat{v}_{h,\delta}$ is supported in Ω'_δ. Using the boundedness of the sesquilinear form a_{Ω_δ} (see, e.g., [23, Lemma 2.4(i)]) we obtain

$$|a_{\Omega_\delta}(u - u_h, \widehat{v}_{h,\delta})| \lesssim \|u - u_h\|_{1,k,\Omega'_\delta}\|\widehat{v}_{h,\delta}\|_{1,k,\Omega'_\delta}. \quad (5.14)$$

Now, using a standard inverse estimate elementwise on Ω'_δ and the local quasiuniformity (5.2), we obtain

$$
\begin{aligned}
\|\widehat{v}_{h,\delta}\|^2_{1,k,\Omega'_\delta} &= \sum_{\substack{\tau\in T^h\\ \tau\subset\Omega'_\delta}} \left(\int_\tau |\nabla \widehat{v}_{h,\delta}|^2 + k^2|\widehat{v}_{h,\delta}|^2\right) \lesssim (h^{-2}+k^2) \sum_{\substack{\tau\in T^h\\ \tau\subset\Omega'_\delta}} \int_\tau |\widehat{v}_{h,\delta}|^2 \\
&\sim (h^{-2}+k^2) \sum_{\tau\subset\Omega'_\delta} h_\tau^2 \sum_{x_j\in\tau} |(\widehat{v}_{h,\delta})(x_j)|^2 \qquad (x_j \text{ are the nodes of the mesh}) \\
&\sim h(h^{-2}+k^2) \sum_{\tau\subset\Omega'_\delta} \int_{\Gamma_\delta\cap\overline{\tau}} |v_{h,\Gamma_\delta}|^2 \qquad (\text{since } \widehat{v}_{h,\delta} \text{ vanishes at all nodes not on } \Gamma_\delta) \\
&= h^{-1}(1+(hk)^2)\|v_{h,\Gamma_\delta}\|^2_{L^2(\Gamma_\delta)}.
\end{aligned}
$$

Inserting this bound into (5.14) and using the resulting bound in the second term on the right-hand side of (5.11), we obtain the second term on the right-hand side of (5.8); we have therefore proved the result (5.8). □

5.2 Proof of Theorem 5.7 via weighted error analysis

Definition 5.10. *With Ω as shown in Figure 2 and given $\alpha\in[0,1/2]$, let $\omega\in C^\infty(\overline{\Omega})$ be defined by*

$$\omega(\boldsymbol{x}) = \widetilde{\omega}(x_1), \quad \boldsymbol{x}=(x_1,x_2)\in\overline{\Omega}, \tag{5.15}$$

where $\widetilde{\omega}\in C^\infty[0,1]$ satisfies

$$\widetilde{\omega}(x_1) = \begin{cases} h^\alpha, & x_1\in[0,\delta-k^{-1}],\\ 1, & x_1\in[\delta-h,L],\end{cases} \tag{5.16}$$

$0\le\widetilde{\omega}(x_1)\le 1$ *for all $x_1\in[0,L]$, and and*

$$|\widetilde{\omega}^{(r)}(x_1)| \lesssim k^r \quad \text{for all } x_1\in[0,L] \quad \text{and all } r\ge 1. \tag{5.17}$$

Note that if $\alpha=0$, then $\omega\equiv1$.

Proposition 5.11 (Properties of the weight function ω)**.**

$$(i) \quad w\equiv 1 \quad \text{on } \Omega'_\delta \quad \text{and} \quad w\equiv h^\alpha \quad \text{on an } \mathcal{O}(1) \text{ neighbourhood of } \Gamma^-; \tag{5.18}$$

$$(ii) \quad \|\partial^\gamma\omega\|_{L^\infty(\Omega)} \lesssim k^{|\gamma|}; \tag{5.19}$$

$$(iii) \quad \max_{\tau\in T^h}\left(\max_{\boldsymbol{x}\in\tau}\omega(\boldsymbol{x})/\min_{\boldsymbol{x}\in\tau}\omega(\boldsymbol{x})\right) \le 1+h^{1-\alpha}k; \tag{5.20}$$

Proof. Parts (i), (ii) follow from (5.18) and (5.17). Part (iii) follows from the estimate

$$\max_{\tau\in T^h}\left(\sup_{\boldsymbol{x}\in\tau}\omega(\boldsymbol{x})/\inf_{\boldsymbol{x}\in\tau}\omega(\boldsymbol{x})\right) \le \max_{\tau\in T^h}\sup_{\boldsymbol{x}_0\in\tau}\sup_{\boldsymbol{x},\tilde{\boldsymbol{x}}\in\tau}\frac{\omega(\boldsymbol{x}_0)+|\nabla\omega(\tilde{\boldsymbol{x}})||\boldsymbol{x}-\boldsymbol{x}_0|}{\omega(\boldsymbol{x}_0)},$$

which is obtained using Taylor's theorem. □

Let

$$\mathring{\Omega} := \bigcup\{\tau\in T^h : \omega\not\equiv h^\alpha \text{ on } \tau\}, \tag{5.21}$$

and note that $\mathring{\Omega}$ therefore contains $\mathrm{supp}(\omega-h^\alpha)$. With a slight abuse of the notation (5.7), we define the weighted norm, for any $u\in H^1(\Omega)$ and $\beta\in\{\pm1\}$,

$$\|u\|^2_{1,k,\omega^\beta} := k^2\|\omega^\beta u\|^2_{L^2(\Omega)} + \|\omega^\beta\nabla u\|^2_{L^2(\Omega)}. \tag{5.22}$$

Outline of the proof of Theorem 5.7 The proof consists of combining the inequality

$$\|u - u_h\|_{1,k,\Omega'_\delta} \leq \|u - u_h\|_{1,k,\omega}, \tag{5.23}$$

which follows from the definition of $\|\cdot\|_{1,k,\omega}$ and Proposition 5.11, with the following two lemmas.

Lemma 5.12. *Suppose Assumption (5.2) and (5.6) hold. If $h^{2-\alpha}k^3$ is sufficiently small, then*

$$\begin{aligned}\|u - u_h\|_{1,k,\omega} \lesssim\ & \big(1 + h^{1-\alpha}k^2\big)\|u - v_h\|_{1,k,\omega} + k\big(1 + h^{1-\alpha}k\big)^2\|u - v_h\|_{L^2(\mathring{\Omega})} \\ & + hk^2\big(1 + h^{1-\alpha}k\big)^2\|u - v_h\|_{1,k} \quad \text{for all } v_h \in V.\end{aligned} \tag{5.24}$$

Lemma 5.13. *Let $u \in U_0(\Omega)$ be the solution of (5.3), let $\alpha \in [0, 1/2]$, and let I_h be the Scott-Zhang interpolant [34] in the space V. Then, for any $\beta > 1/2$,*

$$\begin{aligned}\|u - I_h u\|_{1,k,\omega} &\lesssim (h^{\alpha+1/2}k^\beta + hk)\|g\|_{L^2(\Gamma^-)}, \\ \|u - I_h u\|_{L^2(\mathring{\Omega})} &\lesssim h^2 k\|g\|_{L^2(\Gamma^-)}, \\ \|u - I_h u\|_{1,k} &\lesssim (h^{1/2}k^\beta + hk)\|g\|_{L^2(\Gamma^-)}.\end{aligned} \tag{5.25}$$

Proof of Theorem 5.7 assuming Lemmas 5.12 and 5.13. Combining (5.23) and (5.24), and then choosing $v_h = I_h u$ and using Lemma 5.13, we obtain

$$\begin{aligned}\|u - u_h\|_{1,k,\Omega'_\delta} &\lesssim \Big[\big(1 + h^{1-\alpha}k^2\big)h^{\alpha+1/2}k + k\big(1 + h^{1-\alpha}k\big)^2 h^2 k \\ &\qquad\qquad + hk^2\big(1 + h^{1-\alpha}k\big)^2 h^{1/2}k\Big]\|g\|_{L^2(\Gamma^-)} \\ &\lesssim \Big[h^{\alpha+1/2}k + h^{3/2}k^3 + \Big(h^2k^2 + h^{3/2}k^3\Big)\left(1 + h^{1-\alpha}k\right)^2\Big]\|g\|_{L^2(\Gamma^-)}.\end{aligned}$$

Since $h^2k^2 \leq h^{3/2}k^3 \leq h^{2-\alpha}k^3$, the result follows if we can show that $h^{1-\alpha}k \lesssim 1$. However, by the hypothesis of the theorem,

$$h \leq Ck^{-3/(2-\alpha)} \quad \text{so that} \quad h^{1-\alpha}k \leq Ck^{1-3(1-\alpha)/(2-\alpha)},$$

which is $\lesssim 1$ since $\alpha \leq 1/2$. It therefore remains to prove Lemmas 5.12 and 5.13.

Proof of Lemma 5.13. We only prove the first inequality in (5.25); the others follow similarly. Using the property of the weight function in (5.16) and applying [34, Theorem 4.1] with $p = 2, m = 1$, $\ell = 3/2$, and then with $\ell = 2$, we have, for all $\beta > 1/2$,

$$\begin{aligned}\|\omega\nabla(u - I_h u)\|_{L^2(\Omega)} &\lesssim \|h^\alpha \nabla(u - I_h u)\|_{L^2(\Omega\backslash\mathring{\Omega})} + \|\nabla(u - I_h u)\|_{L^2(\mathring{\Omega})} \\ &\lesssim h^{\alpha+1/2}\|u\|_{H^{3/2}(\Omega\backslash\mathring{\Omega})} + h\|u\|_{H^2(\mathring{\Omega})} \leq (h^{\alpha+1/2}k^\beta + hk)\|g\|_{L^2(\Gamma^-)},\end{aligned}$$

where we used the bounds (5.9) and (5.10), the latter with $D = \mathring{\Omega}$. Similarly, using [34, Theorem 4.1] with $p = 2, m = 0$, $\ell = 3/2$, and then with $\ell = 2$, we find that, for all $\beta > 1/2$,

$$\|\omega(u - I_h u)\|_{L^2(\Omega)} \lesssim \|h^\alpha(u - I_h u)\|_{L^2(\Omega\backslash\mathring{\Omega})} + \|(u - I_h u)\|_{L^2(\mathring{\Omega})} \leq (h^{\alpha+3/2}k^\beta + h^2k)\|g\|_{L^2(\Gamma^-)}.$$

Combining these last two estimates and using (5.6) yields the first inequality in (5.25).

Proof of Lemma 5.12. This proof requires several auxiliary results. First in Lemma 5.14 we prove a trace inequality in the non-standard weighted setting and use it to obtain the boundedness of the sesquilinear form a in a weighted context (Corollary 5.15). Then, using a certain elliptic-projection operator (Definition 5.16), we prove in Lemma 5.18 that $u - u_h$ has a better estimate in the weighted L^2 norm than in the norm $\|\cdot\|_{1,k,\omega}$ (see Remark 5.20 below for discussion on how this result is related to results obtained using this "elliptic-projection" argument, which was first introduced in the Helmholtz context in [16, 17]). From Lemma 5.18 we then prove Corollary 5.19 and Lemma 5.21, which together prove Lemma 5.12.

Lemma 5.14. *Suppose* (5.6) *holds, and let* $\beta = \pm 1$. *Then, for all* $\epsilon \in (0,1]$,

$$k\|\omega^\beta u\|^2_{L^2(\partial\Omega)} \lesssim \epsilon^{-1}k^2\|\omega^\beta u\|^2_{L^2(\Omega)} + \epsilon\|\omega^\beta \nabla u\|^2_{L^2(\Omega)} \quad \text{for all } \ u \in H^1(\Omega), \tag{5.26}$$

where the omitted constant is independent of ϵ *(but may depend on* k_0*).*

Proof. By density, it is sufficient to prove the estimates for all $u \in C^1(\overline{\Omega})$. In the proof we make use of the standard inequality $2ab \le \epsilon a^2 + \epsilon^{-1}b^2$, valid for all a, b and $\epsilon > 0$.

Noting that ω is constant on $\widetilde{\Omega} := [0, \delta - k_0^{-1}] \times [0,1]$ and using the multiplicative trace inequality, we have

$$\begin{aligned}\|\omega^\beta u\|_{L^2(\Gamma^-)} \lesssim \|\omega^\beta u\|^{1/2}_{L^2(\widetilde{\Omega})}\|\nabla(\omega^\beta u)\|^{1/2}_{L^2(\widetilde{\Omega})} &= \|\omega^\beta u\|^{1/2}_{L^2(\widetilde{\Omega})}\|\omega^\beta \nabla u\|^{1/2}_{L^2(\widetilde{\Omega})}\\ &\le k\epsilon^{-1}\|\omega^\beta u\|_{L^2(\Omega)} + \epsilon k^{-1}\|\omega^\beta\nabla u\|_{L^2(\Omega)}\end{aligned} \tag{5.27}$$

the same estimate holds for $\|\omega^\beta u\|_{L^2(\Gamma^+)}$, by an analogous argument.

Let $\boldsymbol{\mu}(\boldsymbol{x}) = (0, 2x_2 - 1)$ and observe that

$$\nabla\omega \cdot \boldsymbol{\mu} = 0 \text{ on } \Omega, \quad \boldsymbol{\mu}\cdot\boldsymbol{\nu} = 1 \text{ on } \partial\Omega\backslash(\Gamma^-\cup\Gamma^+) \quad \text{and } \boldsymbol{\mu}.\boldsymbol{\nu} = 0 \quad \text{on } \Gamma^-\cup\Gamma^+, \tag{5.28}$$

where $\boldsymbol{\nu}$ is the outward normal on $\partial\Omega$. By the divergence theorem,

$$\|\omega^\beta u\|^2_{L^2(\partial\Omega\backslash(\Gamma^-\cup\Gamma^+))} = \int_{\partial\Omega}\omega^{2\beta}|u|^2\boldsymbol{\mu}\cdot\boldsymbol{\nu} = \int_\Omega \nabla\cdot(\omega^{2\beta}|u|^2\boldsymbol{\mu}). \tag{5.29}$$

Since $u, \omega \in C^1(\overline{\Omega})$, by the first equation in (5.28),

$$\nabla\cdot(\omega^{2\beta}|u|^2\boldsymbol{\mu}) = \omega^{2\beta}|u|^2\nabla\cdot\boldsymbol{\mu} + 2\omega^{2\beta}\Re(\overline{u}\nabla u)\cdot\boldsymbol{\mu};$$

then, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\left|\int_\Omega \nabla\cdot(\omega^{2\beta}|u|^2\boldsymbol{\mu})\right| &\le \|\nabla\cdot\boldsymbol{\mu}\|_{L^\infty(\Omega)}\|\omega^\beta u\|^2_{L^2(\Omega)} + 2\|\boldsymbol{\mu}\|_{L^\infty(\Omega)}\|\omega^\beta u\|_{L^2(\Omega)}\|\omega^\beta\nabla u\|_{L^2(\Omega)}\\ &\le 2(1+\epsilon^{-1}k)\|\omega^\beta u\|^2_{L^2(\Omega)} + \epsilon k^{-1}\|\omega^\beta\nabla u\|^2_{L^2(\Omega)};\end{aligned} \tag{5.30}$$

the result then follows from (5.27), (5.29) and (5.30) since we are assuming $k \ge k_0$. □

Lemma 5.14 allows us to prove the following boundedness estimate for the sesquilinear form $a(\cdot,\cdot)$ in the weighted setting.

Corollary 5.15. *Given* $k_0 > 0$ *there exists a constant* $C_{\rm cont}$ *such that, for all* $k \ge k_0$,

$$|a(u,v)| \le C_{\rm cont}\|u\|_{1,k,\omega}\|v\|_{1,k,\omega^{-1}} \quad \text{for all } \ u, v \in H^1(\Omega).$$

Proof. By Cauchy-Schwarz inequality,

$$\begin{aligned}|a(u,v)| &= \left|(\omega\nabla u, \omega^{-1}\nabla v) - k^2(\omega u, \omega^{-1}v) - \mathrm{i}k\langle\omega u, \omega^{-1}v\rangle_{\partial\Omega}\right|\\ &\lesssim \|\omega\nabla u\|_{L^2(\Omega)}\|\omega^{-1}\nabla v\|_{L^2(\Omega)} + k^2\|\omega u\|_{L^2(\Omega)}\|\omega^{-1}v\|_{L^2(\Omega)} + k\|\omega u\|_{L^2(\partial\Omega)}\|\omega^{-1}v\|_{L^2(\partial\Omega)}\\ &\lesssim \left(\|\omega\nabla u\|^2_{L^2(\Omega)} + k^2\|\omega u\|_{L^2(\Omega)}\right)^{1/2}\left(\|\omega^{-1}\nabla v\|^2_{L^2(\Omega)} + k^2\|\omega^{-1}v\|^2_{L^2(\Omega)}\right)^{1/2},\end{aligned} \tag{5.31}$$

where the last term in (5.31) is estimated using (5.26) with $\epsilon = 1$. □

Define the sesquilinear form $a_\star(\cdot,\cdot)$ by

$$a_\star(u,v) := (\nabla u, \nabla v)_\Omega - \mathrm{i}k\langle u, v\rangle_{\partial\Omega}.$$

Then $a_\star(\cdot,\cdot)$ is continuous and coercive on $H^1(\Omega)$ and $\|v\|_\star := \sqrt{a_\star(v,v)}$ is a norm on $H^1(\Omega)$ with

$$\|v\|_\star \sim \|\nabla v\|_{L^2(\Omega)} + k^{1/2}\|v\|_{L^2(\partial\Omega)}; \tag{5.32}$$

see, e.g., [4, Lemma 5.3], [27, Lemma 7.1].

Definition 5.16 (Elliptic projection $\mathcal{P}_h$)**.** *Given $u \in H^1(\Omega)$, define $\mathcal{P}_h u \in V$ by*

$$a_\star(v_h, \mathcal{P}_h u) = a_\star(v_h, u) \qquad \textit{for all } v_h \in V.$$

By the Lax–Milgram theorem, $\mathcal{P}_h$ is well-defined and we have the Galerkin orthogonality

$$a_\star(v_h, u - \mathcal{P}_h u) = 0 \qquad \text{for all } v_h \in V. \tag{5.33}$$

Lemma 5.17 (Approximation properties of $\mathcal{P}_h$)**.** *For all $u \in H^1(\Omega)$,*

$$\|u - \mathcal{P}_h u\|_\star \lesssim \min_{v_h \in V} \|u - v_h\|_{1,k} \qquad \textit{and} \qquad \|u - \mathcal{P}_h u\|_{L^2(\Omega)} \lesssim h \, \|u - \mathcal{P}_h u\|_\star \, . \tag{5.34}$$

References for the proof. This follows from [4, §5.5] or [27, Lemma 7.4], using the regularity result of [5]. □

The next lemma uses the following regularity result for the adjoint problem with zero impedance data. Let $\phi \in H^1(\Omega)$ be the solution of $a(v, \phi) = (v, f)$ for all $v \in H^1(\Omega)$. Then, by [18, Lemma 2.12],

$$\|\phi\|_{H^2(\Omega)} \lesssim k\|f\|_{L^2(\Omega)}. \tag{5.35}$$

Lemma 5.18. *If $\alpha \in [0, 1/2]$ and $h^{2-\alpha}k^3$ is sufficiently small, then, with ∂^r any partial derivative of order $r \geq 0$,*

$$\|(\partial^r \omega)(u - u_h)\|_{L^2(\Omega)} \;\lesssim\; h^{1-\alpha}k^{r+1}\|u - v_h\|_{1,k,\omega} \qquad \textit{for all } v_h \in V. \tag{5.36}$$

Proof. Set $e := u - u_h$ and let $\phi \in H^1(\Omega)$ be the solution of the adjoint problem

$$a(v, \phi) = \big(v, (\partial^r \omega)^2 e\big) \qquad \text{for all} \;\; v \in H^1(\Omega). \tag{5.37}$$

By (5.35) and (5.19),

$$\|\phi\|_{H^2(\Omega)} \lesssim k\|(\partial^r \omega)^2 e\|_{L^2(\Omega)} \lesssim k^{r+1}\|(\partial^r \omega) e\|_{L^2(\Omega)}. \tag{5.38}$$

The motivation for considering this particular adjoint problem is the following. By (5.37), Galerkin orthogonality for both $a(\cdot,\cdot)$ and $a_\star(\cdot,\cdot)$ (5.33), for any $v_h \in$ V,

$$\|(\partial^r \omega) e\|^2_{L^2(\Omega)} = a(e, \phi) = a(e, \phi - \mathcal{P}_h \phi) = a_\star(u - v_h, \phi - \mathcal{P}_h \phi) - k^2(e, \phi - \mathcal{P}_h \phi). \tag{5.39}$$

We see that to obtain (5.36) from (5.39), we need to bound $\phi - \mathcal{P}_h \phi$. By (5.32), (5.34), (5.6) and (5.38),

$$\begin{aligned} \|\nabla(\phi - \mathcal{P}_h \phi)\|_{L^2(\Omega)} + k^{1/2}\|\phi - \mathcal{P}_h \phi\|_{L^2(\partial\Omega)} &\lesssim \min_{v_h \in V} \|\phi - v_h\|_{1,k} \\ &\lesssim h(1 + hk)\|\phi\|_{H^2(\Omega)} \lesssim hk^{r+1}\|(\partial^r \omega) e\|_{L^2(\Omega)}, \end{aligned} \tag{5.40}$$

Arguing similarly, but starting with the second inequality in (5.34), we find that

$$\|\phi - \mathcal{P}_h \phi\|_{L^2(\Omega)} \lesssim h^2\|\phi\|_{H^2(\Omega)} \lesssim h^2 k^{r+1}\|(\partial^r \omega) e\|_{L^2(\Omega)}. \tag{5.41}$$

For the second term of the right-hand side of (5.39), Definition 5.10 and (5.41) imply that

$$\begin{aligned} \big|k^2(e, \phi - \mathcal{P}_h \phi)\big| = k^2 \left|\big(\omega e, \omega^{-1}(\phi - \mathcal{P}_h \phi)\big)\right| &\leq k^2\|\omega e\|_{L^2(\Omega)}\|\omega^{-1}(\phi - \mathcal{P}_h \phi)\|_{L^2(\Omega)} \\ &\leq k^2\|\omega e\|_{L^2(\Omega)} h^{-\alpha}\|\phi - \mathcal{P}_h \phi\|_{L^2(\Omega)} \\ &\lesssim h^{2-\alpha}k^{r+3}\|\omega e\|_{L^2(\Omega)}\|(\partial^r \omega) e\|_{L^2(\Omega)}. \end{aligned} \tag{5.42}$$

For the first term of the right-hand side of (5.39), we write

$$a_\star(u - v_h, \phi - \mathcal{P}_h \phi) = (\omega \nabla(u - v_h), \omega^{-1}\nabla(\phi - \mathcal{P}_h \phi)) - \mathrm{i}k\langle \omega(u - v_h), \omega^{-1}(\phi - \mathcal{P}_h \phi)\rangle_{\partial\Omega};$$

using the Cauchy-Schwarz inequality (as in the proof of Corollary 5.15) we obtain

$$
\begin{aligned}
|a_\star(u-v_h,\phi-\mathcal{P}_h\phi)| \lesssim \|u-v_h\|_{1,k,\omega}\|\phi-\mathcal{P}_h\phi\|_{1,k,\omega^{-1}} &\leq \|u-v_h\|_{1,k,\omega}h^{-\alpha}\|\phi-\mathcal{P}_h\phi\|_{1,k} \\
&\lesssim h^{1-\alpha}k^{r+1}\|u-v_h\|_{1,k,\omega}\|(\partial^r\omega)e\|_{L^2(\Omega)}, \qquad (5.43)
\end{aligned}
$$

where in the last step we used (5.40), (5.41) and (5.6). Combining (5.39), (5.42) and (5.43), we find that

$$
\|(\partial^r\omega)e\|_{L^2(\Omega)} \lesssim h^{1-\alpha}k^{r+1}\|u-v_h\|_{1,k,\omega} + h^{2-\alpha}k^{r+3}\|\omega e\|_{L^2(\Omega)}. \qquad (5.44)
$$

Since this holds for all $r \geq 0$ we can choose $r=0$ and use the assumption that $h^{2-\alpha}k^3$ is sufficiently small to obtain that

$$
\|\omega e\|_{L^2(\Omega)} \lesssim h^{1-\alpha}k\|u-v_h\|_{1,k,\omega}.
$$

Inserting this into the right-hand side of (5.44) yields

$$
\begin{aligned}
\|(\nabla_r\omega)e\|_{L^2(\Omega)} &\lesssim h^{1-\alpha}k^{r+1}\|u-v_h\|_{1,k,\omega} + h^{2-\alpha}k^{r+3}h^{1-\alpha}k\|u-v_h\|_{1,k,\omega} \\
&= h^{1-\alpha}k^{r+1}(1+h^{2-\alpha}k^3)\|u-v_h\|_{1,k,\omega};
\end{aligned}
$$

thus, the result (5.36) follows from the fact that $h^{2-\alpha}k^3$ is sufficiently small. □

The following corollary is obtained by putting $r=0$ in Lemma 5.18 and bounds the L^2 norm on the left-hand side of (5.24).

Corollary 5.19. *If $h^{2-\alpha}k^3$ is sufficiently small, then*

$$
\|\omega(u-u_h)\|_{L^2(\Omega)} \lesssim h^{1-\alpha}k\|u-v_h\|_{1,k,\omega} \quad \textit{for all } v_h \in V. \qquad (5.45)
$$

Remark 5.20 (Link with other work using the "elliptic-projection argument")**.** *Recall that the "Schatz argument" proves, for the FEM applied to (1.1), under the assumption of H^2 regularity, the Aubin-Nitsche type bound that*

$$
\|u-u_h\|_{L^2(\Omega)} \lesssim hk\|u-u_h\|_{1,k};
$$

see [33, 31]. The "elliptic projection" argument applied to this set up proves the stronger bound that, if h^2k^3 is sufficiently small, then, for all $v_h \in V$,

$$
\|u-u_h\|_{L^2(\Omega)} \lesssim hk\|u-v_h\|_{1,k}; \qquad (5.46)
$$

see, e.g., [4, Theorem 5.2], [27, Equation 5.2], with this argument relying on the regularity result of [5]. This argument was first introduced in the setting of discontinuous Galerkin methods by [16, 17], with the analogue of (5.46) appearing in [16, Lemma 5.2], [17, Lemma 4.3], and [38, Lemma 4.2]; see the literature review in, e.g., [30, §2.3].

Observe that our weighted estimate (5.45) generalises (5.46), and reduces to the latter when $\alpha = 0$ (and hence $\omega \equiv 1$ by (5.16)).

To complete the proof of Lemma 5.18 we need to bound the H^1 semi-norm on the left-hand side of (5.24). This bound is obtained in the following lemma.

Lemma 5.21. *Let $\alpha \in [0,1/2]$. If $h^{2-\alpha}k^3$ is sufficiently small, then for any $v_h \in V$,*

$$
\begin{aligned}
\|\omega\nabla(u-u_h)\|_{L^2(\Omega)} \lesssim& \big(1+h^{1-\alpha}k^2\big)\|u-v_h\|_{1,k,\omega} + k\big(1+h^{1-\alpha}k\big)^2\|u-v_h\|_{L^2(\mathring{\Omega})} \\
&+ hk^2\big(1+h^{1-\alpha}k\big)^2\|u-v_h\|_{1,k}. \qquad (5.47)
\end{aligned}
$$

Proof. Set $e = u-u_h$. By the definition of $a(\cdot,\cdot)$,

$$
\|\omega\nabla e\|^2_{L^2(\Omega)} = a(e,\omega^2 e) - (\nabla e, 2\omega(\nabla\omega)e) + k^2(e,\omega^2 e) + \mathrm{i}k\langle e,\omega^2 e\rangle; \qquad (5.48)
$$

we now bound each of the four terms on the right-hand side.

In what follows, v_h is an arbitrary element of V and $\epsilon \in (0,1]$ is arbitrary. For the second term on the right-hand side of (5.48), using Cauchy-Schwarz inequality and (5.36), we have

$$\begin{aligned} |(\nabla e, 2\omega(\nabla\omega)e)| &\le \epsilon\|\omega\nabla e\|^2_{L^2(\Omega)} + \epsilon^{-1}\|(\nabla\omega)e\|^2_{L^2(\Omega)} \\ &\lesssim \epsilon\|\omega\nabla e\|^2_{L^2(\Omega)} + \epsilon^{-1}(h^{1-\alpha}k^2)^2\|u-v_h\|^2_{1,k,\omega}. \end{aligned} \tag{5.49}$$

For the third term on the right-hand side of (5.48), using (5.36), we have

$$k^2(e,\omega^2 e) \;=\; k^2\|\omega e\|^2_{L^2(\Omega)} \;\lesssim\; (h^{1-\alpha}k^2)^2\|u-v_h\|^2_{1,k,\omega}. \tag{5.50}$$

For the fourth term on the right-hand side of (5.48), using (5.26) and then (5.36), we have,

$$\begin{aligned} k\langle e,\omega^2 e\rangle_{\partial\Omega} \;=\; k\|\omega e\|^2_{L^2(\partial\Omega)} &\lesssim \epsilon\|\omega\nabla e\|^2_{L^2(\Omega)} + \epsilon^{-1}k^2\|\omega e\|^2_{L^2(\Omega)} \\ &\lesssim \epsilon\|\omega\nabla e\|^2_{L^2(\Omega)} + \epsilon^{-1}(h^{1-\alpha}k^2)^2\|u-v_h\|^2_{1,k,\omega} \end{aligned} \tag{5.51}$$

Therefore, combining (5.49), (5.50), (5.51), and (5.48), and then choosing the ϵ parameters in (5.49) and (5.51) to be sufficiently small (independent of k), we obtain

$$\|\omega\nabla e\|^2_{L^2(\Omega)} \lesssim |a(e,\omega^2 e)| + \left(h^{1-\alpha}k^2\right)^2\|u-v_h\|^2_{1,k,\omega}. \tag{5.52}$$

More effort is needed to bound the first term on the right-hand side of (5.48)/(5.52). For any $v_h \in V$, set

$$\psi_h := \omega^2(u_h - v_h). \tag{5.53}$$

Then, by Galerkin orthogonality,

$$\begin{aligned} a(e,\omega^2 e) = a(u-u_h,\omega^2(u-u_h)) &= a(u-u_h,\omega^2(u-v_h)) - a(u-u_h,\omega^2(u_h-v_h)) \\ &= a(u-u_h,\omega^2(u-v_h)) - a(u-u_h,\psi_h-\Pi_h\psi_h), \end{aligned} \tag{5.54}$$

where Π_h denotes the nodal interpolation operator that maps a continuous function on Ω into its interpolant in V.

We now deal with each of the two terms in (5.54) separately. For the first term, by Corollary 5.15 and the definition (5.22) of $\|\cdot\|_{1,k,\omega}$,

$$\begin{aligned} \left|a(u-u_h,\omega^2(u-v_h))\right| &\lesssim \|u-u_h\|_{1,k,\omega}\|\omega^2(u-v_h)\|_{1,k,\omega^{-1}} \\ &\le \epsilon\left(\|\omega\nabla e\|^2_{L^2(\Omega)} + k^2\|\omega e\|^2_{L^2(\Omega)}\right) + \epsilon^{-1}\|\omega^2(u-v_h)\|^2_{1,k,\omega^{-1}}. \end{aligned} \tag{5.55}$$

Then, by (5.19) and (5.21),

$$\begin{aligned} \|\omega^2(u-v_h)\|^2_{1,k,\omega^{-1}} &= \|\omega^{-1}\nabla\left(\omega^2(u-v_h)\right)\|^2_{L^2(\Omega)} + k^2\|\omega^{-1}\omega^2(u-v_h)\|^2_{L^2(\Omega)} \\ &\lesssim \|u-v_h\|^2_{1,k,\omega} + \|2(\nabla\omega)(u-v_h)\|^2_{L^2(\Omega)} \\ &\lesssim \|u-v_h\|^2_{1,k,\omega} + k^2\|u-v_h\|^2_{L^2(\mathring{\Omega})}. \end{aligned} \tag{5.56}$$

Thus, combining (5.55), (5.56), and (5.45), we obtain the following estimate for the first term in (5.54):

$$\left|a(u-u_h,\omega^2(u-v_h))\right| \lesssim \epsilon\|\omega\nabla e\|^2_{L^2(\Omega)} + \left(\epsilon(h^{1-\alpha}k^2)^2 + \epsilon^{-1}\right)\|u-v_h\|^2_{1,k,\omega} + \epsilon^{-1}k^2\|u-v_h\|^2_{L^2(\mathring{\Omega})}. \tag{5.57}$$

For the second term in (5.54), by Corollary 5.15 and (5.45),

$$\begin{aligned} |a(u-u_h,\psi_h-\Pi_h\psi_h)| &\lesssim \|u-u_h\|_{1,k,\omega}\|\psi_h-\Pi_h\psi_h\|_{1,k,\omega^{-1}} \\ &\le \epsilon\left(\|\omega\nabla e\|^2_{L^2(\Omega)} + k^2\|\omega e\|^2_{L^2(\Omega)}\right) + \epsilon^{-1}\|\psi_h-\Pi_h\psi_h\|^2_{1,k,\omega^{-1}} \\ &\le \epsilon\|\omega\nabla e\|^2_{L^2(\Omega)} + \epsilon(h^{1-\alpha}k^2)^2\|u-v_h\|^2_{1,k,\omega} + \epsilon^{-1}\|\psi_h-\Pi_h\psi_h\|^2_{1,k,\omega^{-1}}. \end{aligned} \tag{5.58}$$

Now, with h_τ denoting the diameter of τ, standard element-wise estimates for Π_h yield

$$\begin{aligned}
\|\psi_h - \Pi_h\psi_h\|^2_{1,k,\omega^{-1}} &= \sum_{\tau\in T^h}\int_\tau \omega^{-2}\left(k^2|\psi_h-\Pi_h\psi_h|^2 + |\nabla(\psi_h-\Pi_h\psi_h)|^2\right)\,dx \\
&\le \sum_{\tau\in T^h}\left(\min_{\boldsymbol{x}\in\tau}\omega(\boldsymbol{x})\right)^{-2}\int_\tau \left(k^2|\psi_h-\Pi_h\psi_h|^2 + |\nabla(\psi_h-\Pi_h\psi_h)|^2\right)\,dx \\
&\lesssim \sum_{\tau\in T^h}\left(\min_{\boldsymbol{x}\in\tau}\omega(\boldsymbol{x})\right)^{-2}\left((hk)^2+1\right)h_\tau^{2p}|\psi_h|^2_{H^{p+1}(\tau)}\,dx \\
&\lesssim (1+h^{1-\alpha}k)^2\sum_{\tau\in T^h}\sum_{|\beta|=p+1} h_\tau^{2p}\|\omega^{-1}(\partial^\beta\psi_h)\|^2_{L^2(\tau)}, \qquad (5.59)
\end{aligned}$$

where in the last step we used (5.6) and (5.20).

Now, to estimate (5.59), we recall the definition (5.53) of ψ_h and use the multivariate Leibnitz rule (with β and γ denoting multiindices) to obtain

$$\begin{aligned}
\|\omega^{-1}(\partial^\beta\psi_h)\|_{L^2(\tau)} &\lesssim \sum_{0\le\gamma\le\beta}\|\omega^{-1}\partial^\gamma(\omega^2)\partial^{\beta-\gamma}(u_h-v_h)\|_{L^2(\tau)} \qquad (5.60)\\
&= \sum_{r=1}^{p+1}\sum_{|\gamma|=r}\|\omega^{-1}\partial^\gamma(\omega^2)\partial^{\beta-\gamma}(u_h-v_h)\|_{L^2(\tau)},
\end{aligned}$$

where in the last step we used the fact that $|\beta| = p+1$ and $u_h - v_h$ is a polynomial of degree p on τ, the term with $\gamma = 0$ in (5.60) vanishes. Also, simple induction shows that

$$\|\omega^{-1}\partial^\gamma(\omega^2)\|_{L^\infty(\Omega)} \lesssim \begin{cases} k, & \text{when } |\gamma| = 1,\\ h^{-\alpha}k^{|\gamma|} & \text{when } |\gamma| > 1.\end{cases}$$

Using this and an element-wise inverse estimate for derivatives of polynomials on τ, we obtain (using again (5.6))

$$\begin{aligned}
\|\omega^{-1}(\partial^\beta\psi_h)\|_{L^2(\tau)} &\lesssim \left(h_\tau^{-p}k + h^{-\alpha}\sum_{r=2}^{p+1}h_\tau^{r-p-1}k^r\right)\|u_h-v_h\|_{L^2(\tau)} \\
&\lesssim h_\tau^{-p}k\left(1+h^{1-\alpha}k\right)\|u_h-v_h\|_{L^2(\tau)}. \qquad (5.61)
\end{aligned}$$

Also, since the derivatives of ω vanish outside of $\mathring{\Omega}$, the upper bound (5.61) can be replaced by zero for elements τ that do not intersect $\mathring{\Omega}$. Hence, combining (5.61) and (5.59) yields

$$\|\psi_h-\Pi_h\psi_h\|^2_{1,k,\omega^{-1}} \lesssim \left(1+h^{1-\alpha}k\right)^4 k^2\|u_h-v_h\|^2_{L^2(\mathring{\Omega})},$$

Inserting this into (5.58), we have

$$\begin{aligned}
|a(u-u_h,\psi-\Pi_h\psi)| &\lesssim \epsilon\|\omega\nabla e\|^2_{L^2(\Omega)} + \epsilon(h^{1-\alpha}k^2)^2\|u-v_h\|^2_{1,k,\omega} \\
&\quad + \epsilon^{-1}\left(1+h^{1-\alpha}k\right)^4 k^2\|u_h-v_h\|^2_{L^2(\mathring{\Omega})}. \qquad (5.62)
\end{aligned}$$

Now, by the triangle inequality and (5.45) (with $\alpha = 0$ and hence $\omega \equiv 1$),

$$\|u_h-v_h\|_{L^2(\mathring{\Omega})} \lesssim (hk)\|u-v_h\|_{1,k} + \|u-v_h\|_{L^2(\mathring{\Omega})}. \qquad (5.63)$$

Combining (5.54), (5.57), (5.62), and (5.63), we obtain

$$\begin{aligned}
|a(e,\omega^2 e)| \le& \epsilon\|\omega\nabla e\|^2_{L^2(\Omega)} + \left(\epsilon(h^{1-\alpha}k^2)^2+\epsilon^{-1}\right)\|u-v_h\|^2_{1,k,\omega} \\
&+ \epsilon^{-1}\left(1+h^{1-\alpha}k\right)^4h^2k^4\|u-v_h\|^2_{1,k} + \epsilon^{-1}\left(1+h^{1-\alpha}k\right)^4k^2\|u-v_h\|^2_{L^2(\mathring{\Omega})}.
\end{aligned}$$

The result (5.47) then follows from combining this last inequality with (5.52). □

Remark 5.22 (Generalising Theorem 5.7 to more-general geometries)**.** *Inspecting the proof of Theorem 5.7, we see that the key result to generalise is the trace inequality of Lemma 5.14. The proof of this uses the fact that in the region where ω is non-constant (in a neighbourhood of Γ_δ), it is constant in the normal direction – see the first equation in* (5.28)*.*

Thus an analogous result to Theorem 5.7 holds for any convex polygon and interface Γ_δ, provided a weight function with analogous properties can be constructed. This is the case, in particular, when the interior interface is separated from both *the boundary where the non-zero impedance condition is imposed* and *vertices of the polygon.*

The requirement that the polygon is convex could be removed, but then one would need to use $H^{3/2+\varepsilon}$ regularity instead of H^2 regularity, and this would change the powers of h and k in the final error bound.

6 Numerical experiments

In this section, we give numerical experiments to validate our theoretical results on the convergence of the impedance-to-impedance maps, and on the performance of the ORAS preconditioner. Because there are substantial results in other papers [20, 21, 22], we are brief; in particular, [20] used the ORAS iteration for small h as an illustration of the theory on convergence of the parallel Schwarz iteration. All the experiments were implemented within the FreeFEM++ software [24] and were run on the University of Bath's Balena HPC system. All the experiments concern rectangular 2-d domains, discretised using nodal elements of degree $p = 1, 2, 3, 4$ on uniform triangular meshes.

6.1 Computations of the impedance-to-impedance maps

We illustrate the convergence result Theorem 5.5 by computing impedance-to-impedance maps on the canonical domain in Figure 2. The map that takes left-facing impedance data on Γ^- to left-facing impedance data on Γ_δ we call $\mathcal{I}_{-\to-}$. The map that takes right-facing impedance data on Γ^+ to left-facing impedance data on Γ_δ we call $\mathcal{I}_{+\to-}$. The finite element approximations are $\mathcal{I}^h_{-\to-}, \mathcal{I}^h_{+\to-}$. With $\mathcal{I}^h$ denoting either of these maps, (4.26) implies that $\mathcal{I}^h g = \mathcal{I}^h g_h$ where g_h is the L^2-orthogonal projection of g onto the corresponding finite element space. Therefore $\mathcal{I}^h$ acts only on finite-dimensional spaces, and its norm can be computed by solving an appropriate matrix eigenvalue problem - more details are in [20].

First experiment. This experiment verifies Theorem 5.5. Since the exact norm of any map $\mathcal{I}$ is unknown, we approximate it by computing $\|\mathcal{I}^{h_0}\|$ with $h_0 = 1/(32k)$. Tables 1-2 give the errors $|\|\mathcal{I}^h_{t\to-}\| - \|\mathcal{I}^{h_0}_{t\to-}\||$ for $t = +$ and $t = -$, respectively. Both tables show that the error is decreasing as h decreases and mostly the rate is faster than $\mathcal{O}(h)$. In this experiment, the finite-element degree $p = 1$ and the quantity δ in Figure 2 is chosen to be $1/4$.

$k\backslash h$	$\frac{1}{2k}$	$\frac{1}{4k}$	$\frac{1}{8k}$	$\frac{1}{16k}$
10	1.3e-2	4.9e-3	1.6e-3	5.0e-4
20	6.4e-3	2.2e-3	2.0e-3	8.2e-4
40	2.1e-2	2.3e-3	1.2e-4	6.4e-5
80	3.5e-2	8.9e-3	1.4e-3	1.4e-4

Table 1: Computations of $\|\mathcal{I}^h_{+\to-}\| - \|\mathcal{I}^{h_0}_{+\to-}\||$ with $h_0 = 1/(32k)$.

$k\backslash h$	$\frac{1}{2k}$	$\frac{1}{4k}$	$\frac{1}{8k}$	$\frac{1}{16k}$
10	6.4e-3	2.4e-3	1.1e-3	3.9e-4
20	5.6e-3	2.1e-4	3.2e-5	1.2e-5
40	5.4e-3	1.8e-5	7.4e-6	1.1e-6
80	2.6e-3	7.5e-6	5.2e-4	7.6e-5

Table 2: Computations of $|\|\mathcal{I}^h_{-\to-}\| - \|\mathcal{I}^{h_0}_{-\to-}\||$ with $h_0 = 1/(32k)$.

Second experiment. This experiment studies how the norms of these maps vary with δ. Table 3 gives results for $\|\mathcal{I}^h_{+\to-}\|$ (a right-to left map) and we observe that it decreases as δ increases. (A rigorous proof of a closely related result is described in [20, §4.4.4]. The smallness of this norm is a driver for the power contractivity of $\boldsymbol{\mathcal{T}}_h$ – see §4.6.)

Table 4 gives results for $\|\mathcal{I}^h_{-\to-}\|$ (a left-to-left impedance map). This norm remains very close to 1. In this experiment, we fixed $h = 80^{-5/4}$ and used polynomials of degree $p = 2$.

More computations on the impedance-impedance maps can be found in [20]. A summary of the results is as follows.

1. The norms of right-to-left (or left-to-right) impedance maps can be made controllably small by making both the length of the canonical domain and the overlap long enough (see [20, Experiment 1]).

2. The norm of the composition of such maps is much smaller than the product of the norms of the individual maps in the composition. This is again an important factor determining power contractivity of $\boldsymbol{\mathcal{T}}_h$ (see also [20, Experiment 3-4] and the discussion in §4.6).

$k\backslash\delta$	h	$2h$	$4h$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$1-4h$	$1-2h$	$1-h$
10	0.938	0.889	0.807	0.216	0.116	0.102	0.015	0.007	0.003
20	0.925	0.866	0.771	0.237	0.156	0.097	0.029	0.014	0.007
40	0.906	0.836	0.732	0.279	0.180	0.134	0.070	0.036	0.018
80	0.883	0.804	0.703	0.336	0.220	0.124	0.149	0.087	0.045

Table 3: Values of $\|\mathcal{I}^h_{+\to-}\|$ for different δ and k

$k\backslash\delta$	h	$2h$	$4h$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$1-4h$	$1-2h$	$1-h$
10	1.000	1.000	1.000	0.996	0.980	0.945	0.899	0.898	0.897
20	1.000	1.000	1.000	1.001	1.000	0.999	0.994	0.994	0.994
40	1.000	1.001	1.001	1.003	1.001	1.000	1.000	1.000	1.000
80	1.001	1.002	1.003	1.002	1.002	1.002	1.002	1.000	1.000

Table 4: Values of $\|\mathcal{I}^h_{-\to-}\|$ for different δ and k

6.2 Tests of ORAS as a preconditioner

Substantial numerical experiments on the ORAS preconditioner were already performed in [22, 21, 20]. We first summarise the results of these experiments and then focus on some new experiments that are closely related to the theoretical results in this paper.

When the domain is decomposed into strips, the following holds.

1. If the norm of left-to-right impedance map becomes smaller, which is achievable by making the length of local domain and the overlap size larger, the convergence of the ORAS iterative method becomes faster; (see [20, Experiment 1]).

2. The relative error history of the ORAS iterative method has a sudden reduction of the error after each batch of N steps; (see [20, Experiment 2]). Actually, the Nth power of the ORAS

error propagation matrix is a contraction in the Helmholtz energy norm, while the $(N-1)$th power is not (see [21, Table 1]). Here N is the number of subdomains.

Third experiment. This experiment confirms that the ORAS iterative method has a similar convergence property to the parallel Schwarz method, and thus is independent of h and the polynomial degree p. We decompose a rectangular domain of unit height and of length 16/3 into eight overlapping subdomains overlap width 1/2. In Table 5-6, we list the number of iterations (brackets for GMRES iterations) required to obtain a reduction of 10^{-6} on the residual in Euclidean norm. Table 5 displays the results of experiments using different mesh sizes and using lowest order finite element space. Table 6 displays the results of experiments using different finite-element orders and mesh size fixed at $\frac{2\pi}{10k}$. The results show that the convergence of ORAS as an iterative method and as a preconditioner for GMRES is independent of the mesh size h and the polynomial degree p of the local finite element space and even improves slightly for small h and large p.

$k\backslash h$	$\frac{2\pi}{10k}$	$\frac{2\pi}{10k}\cdot\frac{1}{2}$	$\frac{2\pi}{10k}\cdot\frac{1}{4}$	$\frac{2}{10k}\cdot\frac{1}{8}$
20	14 (14)	14 (13)	14 (12)	14 (11)
40	14 (14)	14 (13)	14 (12)	13 (11)
80	14 (14)	14 (13)	14 (11)	12 (10)
120	14 (14)	14 (12)	13 (11)	12 (10)

Table 5: Iteration counts for the iterative method (GMRES) for different mesh size, 8 strip-type subdomains, $p = 1$

$k\backslash p$	1	2	3	4
20	14 (14)	14 (12)	13 (12)	13 (11)
40	14 (13)	14 (13)	13 (11)	13 (10)
80	14 (13)	13 (13)	13 (10)	12 (9)
120	14 (13)	13 (11)	13 (9)	12(9)

Table 6: Iteration counts for the iterative method (GMRES) for different polynomial degrees, 8 strip-type subdomains, $h = 2\pi/10k$

Finally we consider the case when a square domain is decomposed into uniform square subdomains (a so-called checkerboard decomposition) and each subdomain is then extended to provide an overlapping cover. Previous computations showed the following.

1. The ORAS iterative method appears robust as k increases provided the subdomains are large enough and overlap is generous enough (see [22, Experiment 6.3]).

2. While the boundary of the field of values of the ORAS preconditioned matrix is growing as k increases, and the field of values contains the origin (see [22, Figure 2]), the norm of the power of the error propagation matrix is decreasing as the power increases, and finally it becomes a contraction (see [21, Figure 1]).

Fourth experiment. In this final experiment, we investigate how the mesh size and polynomial degree affect the convergence of the ORAS iterative method in the checkerboard case. We choose a non-overlapping partition with diameter $H \sim k^{-0.4}$ and extend it to overlapping subdomains by adding neighbouring elements with distance smaller than $H/4$. Tables 7-8 list the number of iterations required to obtain 10^{-6} reduction on the initial residual. These experiments indicate that when using a checkerboard domain decomposition, the performance of ORAS as an iterative solver and preconditioner for GMRES is also independent of h and p.

$k\backslash h$	$\frac{2\pi}{10k}$	$\frac{2\pi}{10k}\cdot\frac{1}{2}$	$\frac{2\pi}{10k}\cdot\frac{1}{4}$	$\frac{2}{10k}\cdot\frac{1}{8}$
40	14 (13)	14 (13)	14 (13)	14 (13)
80	18 (17)	18 (17)	18 (16)	16 (15)
120	20 (19)	21 (19)	21 (18)	18 (17)
160	23 (22)	23 (22)	23 (21)	20 (19)

Table 7: Checkerboard decomposition: Iterations for the iterative method (GMRES) as $h \to 0$, checkerboard case, $p = 1$

$k\backslash p$	1	2	3	4
40	15 (14)	14 (13)	14 (13)	13 (12)
80	18 (17)	18 (16)	16 (15)	15 (14)
120	21 (20)	22 (18)	19 (17)	18 (16)
160	23 (22)	22 (21)	20 (19)	19 (17)

Table 8: Checkerboard decomposition: Iterations for the iterative method (GMRES) as p increases , $h = 2\pi/(10k)$

An outline approach to extend the theory (at the PDE level) from the strip decomposition to the checkerboard decomposition is given in [20, §6.3].

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