Passivity analysis and control of nonlinear systems modelled by bond graphs

R. Galindo[a] and R.F. Ngwompo[b]

[a]Faculty of Electrical and Mechanical Engineering, Autonomous University of Nuevo Leon, Av. Universidad, San Nicolas de los Garza, Nuevo Leon, 66450, Mexico;
[b]University of Bath, Department of Mechanical Engineering, Faculty of Engineering and Design, Bath, BA2 7AY, UK

ARTICLE HISTORY
Compiled July 18, 2022

ABSTRACT
Recent results on the passivity analysis and control of physical systems, based on the balance of dissipated and internally generated energy, are generalized to nonlinear systems represented by bond graphs. For linear systems, the internally generated energy associated with modulated sources can be coupled with the dissipative field, so that if external energy sources are excluded, then the system is passive (or dissipative) if the resulting composite multiport field is passive. Such a result for linear systems was previously conveniently expressed in terms of a characteristic matrix being positive semi-definite. Parasitic elements of previous works are no longer required, which allows working on the original bond graph of lower dimension than the augmented bond graph and for no-linear systems avoid inverting the dissipative non-linear constitutive relations. For nonlinear systems, passivity is now considered through the explicit difference between the dissipated and the internally generated energy. If this energy difference is positive the system is passive. For control systems, the current work proposes that the controller is designed to have a structure similar to the plant (linear or nonlinear) and its parameters are chosen to assure that in closed-loop the difference between the dissipated and the internally generated energy is positive. In particular, the control parameters can be chosen to assign a desired dissipated energy or to cancel by feedback the internally generated energy and to add damping, therefore achieving sufficient condition for the passivity of the closed-loop system.

KEYWORDS
Bond Graph; Junction structure; Feedback; Passivity analysis; Physical and Passivity-Based Control; Three-tank system

1. Introduction

The passivity property is of great interest in systems analysis and control as it achieves robust stability (see (Triverio, Grivet-Talocia, Nakhala, Canavero, & Achar, 2007) and (Brogliato, Lozano, Maschke, & Egeland, 2007)) and also ensures a level of safety by guaranteeing that the balance between the energy extracted and energy supplied to the system never exceeds the initial energy at any time. As stated in the work of

[a]Contact R. Galindo. Email: rgalindoro@gmail.com, rene.galindoorz@uanl.edu.mx
[b]Email: R.F.Ngwompo@bath.ac.uk

(J. C. Gil & Sira-Ramirez, 1997), “the determination of passivity using the available energy function can result in a complex process disregarding the case studied. For this reason, an alternate way was thought that allows to determine passivity using a bond graph”. In this early work, the scattering matrix of scalar linear systems was used that is closely related to passivity, *i.e.*, the real positiveness of the transfer function. In Bond Graph (BG) representation, active bonds are used to manipulate only one variable. These active bonds are widely used in mechatronic and control systems for the interconnection of sub-systems through internal modulated sources of energy that are interpreted as power scaling transformers and gyrators (Li & Ngwompo, 2005). Such interconnected sub-systems can lead to an overall non-dissipated system.

There is a wide literature about energy-based analysis and design, energy-based analysis and control design and analysis and control of Port-Hamiltonian systems(see for instance (Meigooni & Mollaioli, 2021), (Guerrero et al., 2019) and (Karimi & Binazadeh, 2019), respectively). The present work focus on BG analysis and control design, due to BG allows to analyze the structure of the system, obtain symbolic equations and analyze the energy interactions.

BG representation has been used to propose a generic framework for the design of controllers in the physical domain, where the controller and the plant were all represented by their BG models (P. J. Gawthrop, 1995). This physical model-based control has found several applications such as in nano-scale positioning (P. J. Gawthrop, Wagg, & Nield, 2007) or substructuring (P. J. Gawthrop, Bhikkaji, & Moheimani, 2010) for example. The present article follows the same idea of controllers design in the physical (bond graph) domain but to achieve passivity and a focus on dynamic feedback rather than the observer-based approach adopted by (P. J. Gawthrop, 1995) and (Gonzalez-A, 2016). Another approach to deal with the problem under consideration is based on port-Hamiltonian systems (see (Ortega & García, 2004) and (Van der Schaft, 2006)). Standard bond graphs generate a class of port-Hamiltonian systems (see (Golo, der Schaft, Breedveld, & Maschke, 2003)). This was exploited for the disturbance decoupling and model matching control problems in the work of (Vink, Ballance, & Gawthrop, 2006).

A method for the passification of mechatronic systems is proposed by (Li & Ngwompo, 2005) using power scaling transformers and gyrators with the condition that those power scaling elements are not part of any causal loops. Their approach does not, therefore, cover many control systems in a closed-loop configuration. For linear systems with internal modulated sources in an open or closed-loop configuration, (Galindo & Ngwompo, 2017) and (Ngwompo & Galindo, 2017) proposed a method to decompose the system into a passive field that consists of storage elements and a multiport composite field that encompasses modulated sources and dissipative elements. Such systems can also be represented by a bond graph that has a passive storage field connected to a pseudo-junction structure consisting of an inner power-conservative junction sub-structure and an outer junction sub-structure that contains power scaling elements connected to the dissipative field augmented with parasitic elements if necessary. The outer power scaling junction sub-structure and dissipative field implicitly implement the net balance between the energy internally generated by the modulated sources and the dissipated energy by the resistive elements. Its associated multiport composite field determines the overall passivity of the system through the semi-positive definiteness of its characteristics matrix.

The present work does not require pseudo-junction structures or parasitic elements as proposed in (Galindo & Ngwompo, 2017) and (Ngwompo & Galindo, 2017). This is an important contribution that allows working on the original BG of lower dimensions than the augmented BG. So, the causality assignment of the dissipative elements does not change and for no-linear systems avoid inverting the dissipative non-linear constitutive relations. First, the problem statement is given in section 2. Passivity analysis and control are realized in sections 3 and 4 based on the balance of the dissipated and internally generated energy. These energies are joined up into a multi-port coupled dissipated field. Cascade and feedback interconnections of subsystems without loading effect, are considered. Assuming that the storage field is passive and only power external sources are applied, then the overall system is passive if the multi-port coupled dissipative field is passive (see the work of (Beaman & Rosenberg, 1988)). Passivity analysis and control of Linear Time-Invariant (LTI) systems are tackled in section 5. The cases of less or more dissipative than storage elements are covered. For LTI systems the dissipated and internally generated energies have quadratic forms. Hence, the overall system is passive if the associated matrix is positive semidefinite, leading to the same results of (Galindo & Ngwompo, 2017) and (Ngwompo & Galindo, 2017), when the dissipative elements are equal to the storage elements or parasitic elements are added.

The results are illustrated through examples and the non-linear results are applied to a three-tank system example in section 4.

Notation.- $\mathcal{I}_p$ and 0_p are the identity and zero matrices of dimensions $p \times p$, respectively; $diag\{a_1, a_2, \dots, a_n\}$ is a diagonal matrix of dimension $n \times n$ whose elements are $a_1, a_2, \dots, a_n$; and a real matrix M is positive semidefinite if and only if the symmetric part $\frac{1}{2}\left(M + M^T\right)$ is positive semidefinite, where M^T is the transpose of M.

2. Problem statement

A Bond Graph (BG) model of a system in integral causality is described by the junction structure of Fig. 1 where C and I are storage elements in integral causality, R is the dissipative field, D_e and D_f are detectors (sensors) of effort and flow. The junction structure $S(0, 1, TF, GY, MS_e, MS_f)$, linking these elements, is an assemblage of $0-$junctions, $1-$junctions, transformers, TF, gyrators, GY, and modulated sources of effort and flow, MS_e and MS_f. Let m, n, p and q be the input, the state, the output and the dissipative space dimensions, respectively. The state vector $x(t) \in \Re^{n\times 1}$ is associated with the energy variables of the storage elements in integral causality; $z(t) \in \Re^{n\times 1}$ is the co-energy vector; $D_o(t) \in \Re^{q\times 1}$ and $D_i(t) \in \Re^{q\times 1}$ are vectors of variables associated with R; $u(t) \in \Re^{m\times 1}$ is the system input; $y(t) \in \Re^{m\times 1}$ is the conjugated system output, *i.e.*, $y^T(t)u(t)$ is power; and $y_m(t) \in \Re^{p\times 1}$ is the measured system output. In general $y_m(t) \neq y(t)$ in control or mechatronics applications.

The system energy at time t, $E(t)$, is,

$$E(t) = E(t_0) + E_{su} + E_g - E_{st} - E_d \tag{1}$$

where $E(t_0)$ is the initial energy, $E_{su} := \int y^T(t)u(t)dt$ is the supply energy, $E_g := \int y_g^T(t)u_g(t)dt$ is the internally generated energy, $E_{st} := \int \dot{x}^T(t)z(t)dt$ is the stored energy and $E_d := \int D_i^T(t)D_o(t)dt$ is the dissipated energy, with $y_g(t)$ and $u_g(t)$ being the conjugated output and input of the internal sources MS_e and MS_f, respectively.

A dissipative or passive system satisfies,

$$E_{su} + E_g \leq E_{st} + E_d \tag{2}$$

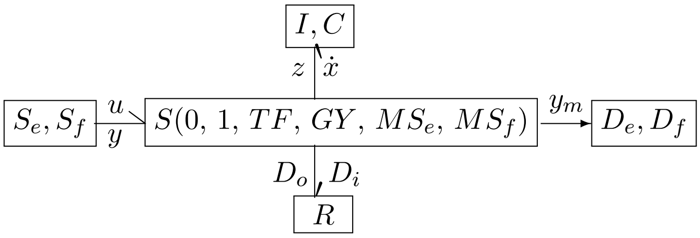

Figure 1. Junction structure associated with a bond graph in integral causality

that from Eq. (1) implies $E(t) \leq E(t_0)$, $\forall t$. BG that use active bonds does not necessarily satisfy Eq. (2). These non-dissipative systems may arise in the system interconnection of subsystems or in systems that include internal modulated sources. This is the case of BG including power-scaling elements (Li & Ngwompo, 2005).

In the work of (Galindo & Ngwompo, 2017) and (Ngwompo & Galindo, 2017), the internal sources are combined with the dissipative field while the storage field, assumed to be passive, remains unchanged. This process is equivalent to creating a composite field that implements the difference between E_g and E_d resulting in either a net dissipative or net source of the energy field. From the formal definition of passivity in terms of the energy that can be extracted from a system being finite, it is clear that excluding the external supply of energy, a system is passive if each element of the system is passive (Beaman & Rosenberg, 1988). Therefore, if all R, C and I elements are passive in a multiport bond graph model containing internal modulated sources, the overall passivity of the system can be investigated through the above mentioned composite field or the net balance between the energy internally generated E_g and the dissipated energy E_d and passivity is guaranteed if,

$$0 \leq E_d - E_g \tag{3}$$

The condition (3) was used as an alternative way to express and to check that the characteristic matrix of the composite field was positive semidefinite (see Appendix 2 of (Ngwompo & Galindo, 2017)). This approach is generalized here for a wider class of systems. In particular, a conservative condition is $E_g = 0$ that assures passivity if $0 \leq E_d$. Also, a well-known technique to achieve passivity is to add damping (see for instance the works of (Garcia, Rimaux, & Delgado, 2006) and (P. Gawthrop, Neild, & Wagg, 2015)), *i.e.*, to increase E_d such that inequality (3) is satisfied.

In order to simplify passivity analysis and control the following assumptions are made:

Assumption 1. $y_g(t) = f\left(z(t)\right)$ and $u_g(t) = g\left(z(t)\right)$, and

Assumption 2. $D_i(t)$ is neither a function of $D_o(t)$ or $u(t)$.

Assumption 1 implies that the internal modulated sources are related through the junction structure to the storage elements and Assumption 2 states that there are no causal loops between R-elements or direct causal paths linking sources to R-elements. A way to assure that this assumption is satisfied is to add parasitic elements as in (Galindo & Ngwompo, 2017) and (Ngwompo & Galindo, 2017). Under Assumption 2 the relationships for the junction structure $S(0,\, 1,\, TF,\, GY,\, MS_e,\, MS_f)$ are given by:

$$\begin{bmatrix} \dot{x}(t) \\ D_i(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & 0_q & 0_{q\times m} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{bmatrix} z(t) \\ D_o(t) \\ u(t) \end{bmatrix} \tag{4}$$

where the block partition of $S(0,\, 1,\, TF,\, GY,\, MS_e,\, MS_f)$ is according to the dimensions of $z(t)$, $D_o(t)$ and $u(t)$.

Under Assumptions 1 and 2, the problems under consideration are the following:
Problem 1: Passivity analysis of systems with internal modulated sources, that is, to determine if such systems are passive using inequality (3).
Problem 2: Passivity-Based Control (PBC) of systems, that is, given a system represented by its bond graph model, to design a control such that the whole system satisfies inequality (3).

3. Non-linear systems

A solution to Problem 1, *i.e.*, the proposed approach for the passivity analysis is summarized in the following result,

Lemma 3.1. *Let $z(t) \in \Re^{n\times 1}$ and $D_i(t) \in \Re^{q\times 1}$. Under Assumptions 1 and 2, the non-linear system modelled by bond graph is passive if,*

$$0 \leq \int_0^t \left[D_i^T(\tau)\phi(D_i(\tau)) - f^T\left(z(\tau)\right) g\left(z(\tau)\right) \right] d\tau \tag{5}$$

where all the elements of $D_i(t)$ are directly related to the ones of $z(t)$ for $q \leq n$, and all the elements of $z(t)$ are directly related to the elements of $D_i(t)$ for $q > n$.

Proof. Since $D_o(t) = \phi\left(D_i(t)\right)$ then $E_d = \int_0^t D_i^T(\tau)\phi(D_i(\tau))d\tau$. Under Assumption 1, $E_g = \int_0^t f^T\left(z(\tau)\right) g\left(z(\tau)\right) d\tau$. Assumption 2 assures that $D_i(t) = S_{21}z(t)$ and hence the result follows from inequality (3), applying the result of (Beaman & Rosenberg, 1988). □

The case $q = n$ can be obtained by adding parasitic elements as in (Galindo & Ngwompo, 2017) and (Ngwompo & Galindo, 2017). However, for non-linear systems it is difficult in general to invert the non-linear constitutive relation of the dissipative field, so, the parasitic elements must not change the causality assignment of the dissipative field. The parasitic elements increase the complexity of the augmented BG. Instead, passivity can be analysed through inequality (5), including the case $q = n$.

Since $E_d > 0$, a solution to Problem 2 is to design a control such that $E_g = 0$. It is proposed that the controller has a structure similar to the plant, as realized for linear systems by (Galindo & Ngwompo, 2017). In the following result, the control parameters are selected such that $E_g = 0$ and E_d becomes a desired positive semidefinite quadratic function. So, a solution of Problem 2 is proposed by,

Lemma 3.2. *Suppose that the non-linear sub-systems modelled by bond graph are interconnected in feedback with no loading effect as shown in Fig. 2, and under Assumptions 1 and 2. Let $z_a(t) \in \Re^{n_a\times 1}$ and $z_b(t) \in \Re^{n_b\times 1}$ be the coenergy vectors of the controller and the plant, respectively, and $D_i^a(t) \in \Re^{q_a\times 1}$ and $D_i^b(t) \in \Re^{q_b\times 1}$ be the*

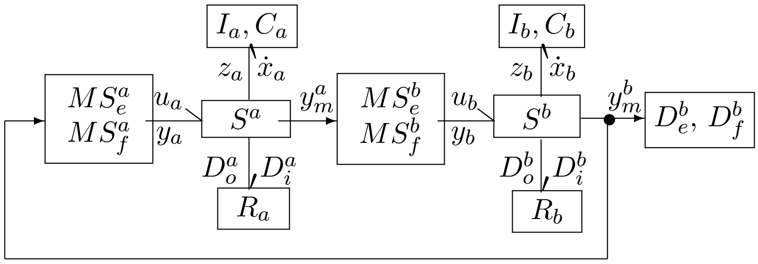

Figure 2. Feedback interconnection of bond graph models

inputs of the dissipative field of the controller and the plant, respectively. If the control parameters are selected such that,

$$\begin{aligned} f_b^T(z_b(t))\,u_b(t) &= -f_a^T(z_a(t))\,u_a(t) \\ \left(D_i^b(t)\right)^T \phi(D_i^b(t)) &= \left(D_i^d(t)\right)^T L D_i^d(t) - \left(D_i^a\right)^T(t)\phi(D_i^a(t)) \end{aligned} \tag{6}$$

where $D_i^d(t) \in \Re^{q_d \times 1}$ is the input of the desired dissipative field of the feedback system and $L = L^T \geq 0$. Then, the feedback system modelled by BG is passive.

Proof. From Eq. (6), $E_g = \int_0^t \left[f_a^T(z_a(\tau))\,u_a(t) + f_b^T(z_b(\tau))\,u_b(t)\right] d\tau = 0$ and $E_d = \int_0^t \left[\left(D_i^a(\tau)\right)^T \phi(D_i^a(\tau)) + \left(D_i^b(\tau)\right)^T \phi(D_i^b(\tau))\right] d\tau = \int_0^t \left(D_i^d(\tau)\right)^T L D_i^d(\tau) d\tau$. Hence, since $L = L^T \geq 0$ then $E_d \geq 0$ and from Lemma 3.1, the feedback system is passive. □

A common case is when the constitutive relations of the plant sensor and actuator are affine functions of the coenergy vector of the plant and the controller output. In the following result, the control parameters are selected such that the plant and controller inputs be linear functions of the coenergy vectors of the controller and of the plant, respectively,

Lemma 3.3. *Suppose that the sub-systems modelled by bond graph are interconnected in feedback with no loading effect as shown in Fig. 2 and under Assumptions 1 and 2. Let m_a and p_a be the input and output dimensions of the controller, respectively, m_b and p_b be the input and output dimensions of the plant. The plant output and input $y_m^b(t) \in \Re^{p_b \times 1}$ and $u_b(t) \in \Re^{m_b \times 1}$, respectively, satisfy,*

$$\begin{aligned} y_m^b(t) &= K_1 z_b(t) + K_2 \text{ and} \\ u_b(t) &= K_3 y_m^a(t) + K_4 \end{aligned} \tag{7}$$

where $z_b(t) \in Re^{n_b \times 1}$ and $y_m^a(t) \in \Re^{p_a \times 1}$ are the coenergy vector of the plant and the controller output, respectively, K_1 to K_4 are real matrices of appropriate dimensions, and K_3 is a non-singular matrix. The control parameters are selected such that,

$$\begin{aligned} y_m^a(t) &= K_1 z_a(t) - K_3^{-1} K_4 \\ u_a(t) &= K_3\left(-y_m^b(t) + K_2\right) \end{aligned} \tag{8}$$

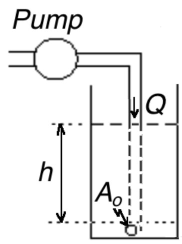

Figure 3. A tank system.

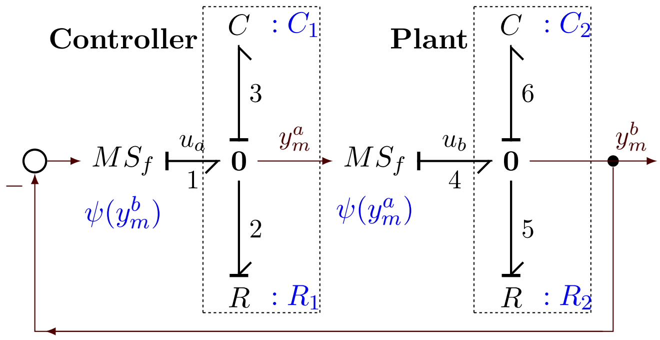

Figure 4. BG of two $R - C$ subsystems interconnected in feedback.

Then,

$$\begin{aligned} u_b(t) &= K_3 K_1 z_a(t), \\ u_a(t) &= -K_3 K_1 z_b(t) \text{ and} \\ E_g &= \int_0^t \left[-f_a^T (z_a(\tau)) K_3 K_1 z_b(t) + f_b^T (z_b(\tau)) K_3 K_1 z_a(t) \right] d\tau \end{aligned} \tag{9}$$

Proof. The result follows directly from equations (7) and (8) substituting $y_m^a(t)$ and $y_m^b(t)$ into $u_a(t)$ and $u_b(t)$. □

The proposed control strategy and the results of the above Lemmas are illustrated by the following example, where the design of a controller for a non-linear system is considered so that the overall system is passive. In this example, $y_m^a(t) = z_a(t)$ and $y_m^b(t) = z_b(t)$ and from Lemma 3.3, the controller can be designed such that E_g has a quadratic form, *i.e.*, $E_g = \int_0^t z^T(\tau) G z(\tau) d\tau$, where $G \in \Re^{n \times n}$ is a skew-symmetric matrix, then E_g vanishes and inequality (5) is accomplished. Also, it is possible to add damping increasing E_d or to assign a desired dissipated energy.

Example 3.4. The example presented in (Galindo & Ngwompo, 2016) is considered. The level of liquid in a tank is to be controlled and for this, a controller having the same structure as the plant is chosen. The tank system is shown in Fig. 3 where $Q(t)$ is the supply flow rate in m^3/s, $h(t)$ is the liquid level in meters and A_o the cross-section of the flow leak in m^2. Let A be the cross-section of the tank in m^2, $\gamma = \rho g$ be the specific weight, g be the earth gravity, ρ be the flow density, and $\theta_1, \ldots, \theta_4$ be real constant sensor parameters. This tank is modelled by the $R_2 - C_2$ system in Fig. 4 where the liquid level $h(t)$, the plant control signal $u_b(t) = f_4 = \psi\left(y_m^a(t)\right)$ and the constitutive relationship of the R-element $D_o^b(t) = \phi_b\left(D_i^b(t)\right)$ are given by,

$$\begin{array}{l} h(t) = \frac{e_6}{\gamma} = -\theta_3 y_m^b(t) + \theta_4, \\ f_4 = Q(t) = \theta_3 A\left(\theta_1 \alpha_2 y_m^a(t) + \theta_2\right) \text{and} \\ f_5 = Q_0(t) = \alpha_0 A_o \sqrt{\frac{2e_6}{\rho}} \end{array} \tag{10}$$

respectively, with $y_m^b(t)$ being the output in volts of a liquid level meter, $y_m^a(t)$ being the input in volts of the pump that modulates the flow source in Fig. 4, α_2 being the gain of the modulated source of flow, $Q_0(t)$ being the liquid leakage flow and $0 \leq \alpha_0 \leq 1$ the proportion of leakage. So, the plant output $y_m^b(t)$ and input $u_b(t) = f_4$ satisfy Eq. (7) where $z_b(t) = e_6$, $u_b(t) = f_4$,

$$\begin{array}{l} K_1 = \frac{-1}{\theta_3 \gamma},\ K_2 = \frac{\theta_4}{\theta_3}, \\ K_3 = \theta_3 A \theta_1 \alpha_2 \text{ and } K_4 = \theta_3 A \theta_2 \end{array} \tag{11}$$

Let $R_1 - C_1$ in Fig. 4 be a proposed model for the non-linear controller that has the same structure as the plant. Applying Lemma 3.3, the control parameters are selected from Eq. (8),

$$\begin{array}{l} y_m^a(t) = \frac{-1}{\theta_3 \gamma} e_3 - \frac{\theta_2}{\theta_1 \alpha_2}, \\ f_1 = \theta_1 A \alpha_2 \left(-\theta_3 y_m^b(t) + \theta_4\right) \text{ and} \\ f_2 = \alpha_1 A_o \sqrt{\frac{2e_3}{\rho}} \end{array} \tag{12}$$

where $z_a(t) = e_3$, $u_a(t) = f_1$, α_1 is a control parameter, $y_m^a(t)$ is in volts and negative feedback is selected of the measured $y_m^b(t)$ signal. These two $R - C$ systems have a feedback interconnection as shown in Fig. 4. Then, from Lemma 3.3 the non-linear controller leads to $u_b(t)$ and $u_a(t)$ be linear functions of $e_3(t)$ and $e_6(t)$, respectively, *i.e.*,

$$\begin{array}{l} f_4 = \frac{-\theta_1 A \alpha_2}{\gamma} e_3 \text{ and} \\ f_1 = \frac{\theta_1 A \alpha_2}{\gamma} e_6 \end{array} \tag{13}$$

Also, since $y_m^a(t) = z_a(t)$ and $y_m^b(t) = z_b(t)$, then $E_g = 0$ and has a quadratic form. In the BG of Fig. 4, $q = n$ and from the third rows of equations (10) and (12) and inequality (5) the system is passive if

$$E_d = A_o \sqrt{\frac{2}{\rho}} \int_0^t \left(\alpha_1 e_2 \sqrt{e_2} + \alpha_0 e_5 \sqrt{e_5}\right) d\tau \geq 0 \tag{14}$$

The above result is the same as in (Galindo & Ngwompo, 2016) but without using junction or pseudo-junction structures. The left-hand term of inequality (14) is the total dissipated energy of the controller and the plant. With α_0 being a fixed parameter of the plant (proportion of leakage), the closed-loop system is passive for all α_1 so that $0 \leq \alpha_1 \leq 1$ and the designer can tune α_1 to change the energy dissipation rate to achieve the desired performance, with $\alpha_1 = 0$ being analogous to a completely closed and $\alpha_1 = 1$ to a completely open leak valve, respectively. Alternatively, applying Lemma 3.2 the control parameters can be selected such that E_d be a desired positive semidefinite quadratic function.

Table 1. Simulation parameters.

Earth gravity	$g = 9.81\ m/s^2$
Fluid density	$\rho = 1000\ kg/m^3$
Tank cross-section area	$A = 0.0154\ m^2$
Leakage cross-section area	$A_0 = 0.05 \times 10^{-3}\ m^2$
Proportion of leakage	$\alpha_0 = 0.5$
Level sensor parameters	$\theta_1 = 0.0103\ s^{-1}$
	$\theta_2 = 0.1022\ V/s$
	$\theta_3 = 0.0338143\ m/V$
	$\theta_4 = 0.3115872\ m$
Initial height of liquid	$h = 0.4\ m$

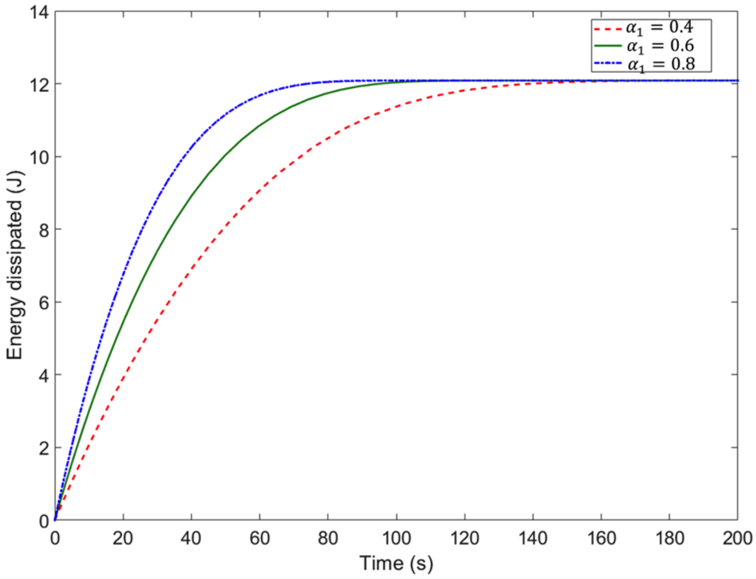

Figure 5. Energy dissipated for proportion of leakage $\alpha_1 = 0.4$, 0.6 and 0.8

For the physical interpretation of the above results, a simulation of the tank system in Fig. 3 in closed-loop configuration is done to assess the overall system passivity with the parameters in Table 1. The model in Fig. 4 has two modulated sources with the controller parameters chosen so that the net energy generated by both sources is zero, *i.e.*, $E_g = 0$, that is, the energy consumed by the plant modulated source is compensated by the energy supplied by the controller modulated source, and the net energy is zero at all instants. The system is passive according to Eq. (3). The condition $E_g = 0$ is sufficient but not necessary for the system to be passive. However, if this condition is satisfied, only the dissipative R-elements in Fig. 4 and not the modulated sources control the rate at which the initial energy in the system (stored in C-elements) is dissipated. The initial potential energy, $E(t_0)$ in the tank is $Ah^2\rho g/2 = 12.1J$. Since $E_g = 0$ and the energy is dissipated, then E_{st} and $E(t)$ tend to zero and from Eq. (1), E_d tends to $E(t_0)$ as shown in Fig. 5. Also, Fig 5 shows that as the parameter α_1 of the controller R_1-element (*i.e.* the proportion of leakage) increases, the initial energy stored in the tank is dissipated at a faster rate. □

Now the cascade and feedback interconnections of power-conservative systems modelled by BG and satisfying Assumptions 1 and 2, are considered. These configurations are common in control and mechatronics systems. Besides, for Linear Time-Invariant (LTI) systems, using the parametrisation of all stabilizing controllers the closed-loop

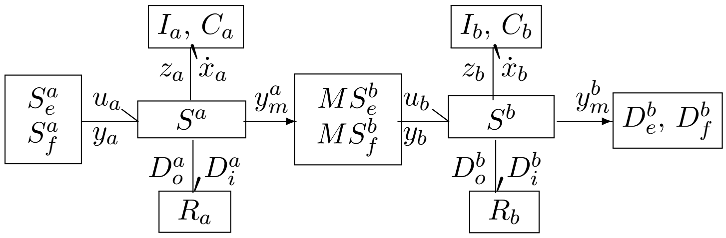

Figure 6. Bond graph model of systems interconnected in cascade (with no loading effect)

transfer functions are affine functions of the free control parameters (see (Vidyasagar, 1985)). In particular, in a one-degree of freedom feedback configuration the transfer function from the output reference, $y_d(t)$ to the plant output $y(t)$ is $T_o(s) = N(s)\tilde{N}_{\mathcal{K}}(s)$ (see (Vidyasagar, 1985)), where $N(s)$ and $\tilde{N}_{\mathcal{K}}(s)$ are the numerators of the coprime factorizations of the plant and the controller, respectively, while in a two-degrees of freedom feedback configuration, $T_o(s) = N(s)Q(s)$ (see (Vidyasagar, 1985)), where $Q(s)$ is a free control parameter. So, in both cases, the feedback system can be regarded as a cascade interconnection of $N(s)$ and $\tilde{N}_{\mathcal{K}}(s)$ or of $N(s)$ and $Q(s)$, respectively.

The junction structures associated with the BG of the controller $\mathcal{K}$ and the BG of the plant P are denoted by S^a and S^b, respectively, and are shown in the cascade and feedback configurations of figures 2 and 6. It is assumed that the sub-systems are interconnected with no loading effect, that is, these sub-systems are interconnected through active (signal) bonds that modulate sources of effort or flow. Due to this connection, the overall system may not conserve energy.

When the constitutive relations of the plant sensor and actuator are affine functions of the coenergy vector of the plant and the controller output, applying Lemma 3.3, $u_a(t)$ and $u_b(t)$ become linear functions of $z_b(t)$ and $z_a(t)$, respectively, *i.e.*, the controller is designed such that these linear relations are satisfied. If $y_m^a(t) = z_a(t)$ and $y_m^b(t) = z_b(t)$, then Problem 2 is solved by,

Theorem 3.5. *Suppose that the sub-systems modelled by bond graph are interconnected in cascade and feedback with no loading effect as shown in figures 2 and 6, respectively, and under Assumptions* 1 *and* 2. *Let m_a, n_a and p_a be the input, state and output space dimensions of the controller, respectively, m_b, n_b and p_b be the input, state and output space dimensions of the plant,*

$$\begin{array}{l} u_b(t) = K_b y_m^a(t) \quad \textit{and} \\ u_a(t) = -K_a y_m^b(t) \end{array} \tag{15}$$

where $K_b \in \Re^{m_b \times p_a}$ and $K_a \in \Re^{m_a \times p_b}$ are non-singular matrices composed of the control gains, and two junction structures $S^a(0,\, 1,\, TF,\, GY)$ and $S^b(0,\, 1,\, TF,\, GY)$ of bond graphs modelling the controller and the plant, respectively,

$$\begin{bmatrix} \dot{x}_a(t) \\ D_i^a(t) \\ y_m^a(t) \end{bmatrix} = \begin{bmatrix} S_{11}^a & S_{12}^a & S_{13}^a \\ S_{21}^a & 0_{q_a} & 0_{q_a \times m_a} \\ S_{31}^a & S_{32}^a & S_{33}^a \end{bmatrix} \begin{bmatrix} z_a(t) \\ D_o^a(t) \\ u_a(t) \end{bmatrix}, \tag{16}$$

and

$$\begin{bmatrix} \dot{x}_b(t) \\ D_i^b(t) \\ y_m^b(t) \end{bmatrix} = \begin{bmatrix} S_{11}^b & S_{12}^b & S_{13}^b \\ S_{21}^b & 0_{q_b} & 0_{q_b \times m_b} \\ S_{31}^b & S_{32}^b & S_{33}^b \end{bmatrix} \begin{bmatrix} z_b(t) \\ D_o^b(t) \\ u_b(t) \end{bmatrix} \tag{17}$$

where $x_a(t) \in \Re^{n_a \times 1}$, $z_a(t) \in \Re^{n_a \times 1}$, $D_i^a(t) \in \Re^{q_a \times 1}$, $D_o^a(t) = \phi_a\left(D_i^a(t)\right) \in \Re^{q_a \times 1}$, $y_a(t) \in \Re^{p_a \times 1}$, $u_a(t) \in \Re^{m_a \times 1}$, $x_b(t) \in \Re^{n_b \times 1}$, $z_b(t) \in \Re^{n_b \times 1}$, $D_i^b(t) \in \Re^{q_b \times 1}$, $D_o^b(t) = \phi_b\left(D_i^b(t)\right) \in \Re^{q_b \times 1}$, $y_b(t) \in \Re^{p_b \times 1}$ *and* $u_b(t) \in \Re^{m_b \times 1}$.

Then, the cascade interconnection is passive if

$$0 \leq E_d - \int_0^t f_b^T\left(z_b(\tau)\right) S_{13}^b K_b \eta_a(\tau) d\tau \tag{18}$$

where

$$\begin{array}{l} E_d := \int_0^t \left[\left(D_i^a(\tau)\right)^T \phi_a(D_i^a(\tau)) + \left(D_i^b(\tau)\right)^T \phi_b(D_i^b(\tau))\right] d\tau \text{ and} \\ \eta_a(t) := S_{31}^a z_a(t) + S_{32}^a \phi_a\left(D_i^a(t)\right) \end{array} \tag{19}$$

and the feedback interconnection is passive if,

$$\begin{array}{l} 0 \leq E_d + \int_0^t f_a^T\left(z_a(\tau)\right) S_{13}^a \bar{\Delta}^{-1} K_a \eta_c(\tau) d\tau \\ - \int_0^t f_b^T\left(z_b(\tau)\right) S_{13}^b \Delta^{-1} K_b \eta_d(\tau) d\tau \end{array} \tag{20}$$

where

$$\begin{array}{l} \eta_b(t) := S_{31}^b z_b(t) + S_{32}^b \phi_b\left(D_i^b(t)\right), \\ \eta_c(t) := \eta_b(t) + S_{33}^b K_b \eta_a(t) \text{ and} \\ \eta_d(t) := \eta_a(t) - S_{33}^a K_a \eta_b(t) \end{array} \tag{21}$$

and the Schur complements are,

$$\begin{array}{l} \Delta := \mathcal{I}_{m_b} + K_b S_{33}^a K_a S_{33}^b \\ \bar{\Delta} := \mathcal{I}_{m_a} + K_a S_{33}^b K_b S_{33}^a \end{array} \tag{22}$$

Proof. Under Assumption 1, $\left(y_m^a(t)\right)^T u_a(t)$ is power, and then, as stated in the work of (Beaman & Rosenberg, 1988), whether each element of a model is passive then the system is passive. Hence, $u_a(t) = 0$ is considered for the passivity analysis. Then, from the cascade interconnection of Fig. 6,

$$u_b(t) = K_b \eta_a(t) \tag{23}$$

Since the effective input to the system is $S_{13}^b u_b(t)$, under Assumption 2, $E_g = \int_0^t f_b^T\left(z_b(\tau)\right) S_{13}^b u_b(\tau) d\tau$, then, the result of inequality (18) follows applying Lemma 3.1. From the feedback interconnection of Fig. 2,

$$\begin{bmatrix} \mathcal{I}_{m_a} & K_a S_{33}^b \\ -K_b S_{33}^a & \mathcal{I}_{m_b} \end{bmatrix} \begin{bmatrix} u_a \\ u_b \end{bmatrix} = \begin{bmatrix} -K_a \eta_b(t) \\ K_b \eta_a(t) \end{bmatrix} \tag{24}$$

Let $\bar{\Delta} := \mathcal{I}_{m_a} - A_{12}A_{21}$, using (see (Zhou, Doyle, & Glover, 1996)), $\begin{bmatrix} \mathcal{I} & A_{12} \\ A_{21} & \mathcal{I} \end{bmatrix}^{-1} = \begin{bmatrix} \bar{\Delta}^{-1} & -\bar{\Delta}^{-1}A_{12} \\ -A_{21}\bar{\Delta}^{-1} & \mathcal{I} + A_{21}\bar{\Delta}^{-1}A_{12} \end{bmatrix}$, then $\bar{\Delta}$ is given by Eq. (22) and,

$$\begin{aligned} \mathcal{I} + A_{21}\bar{\Delta}^{-1}A_{12} &= \mathcal{I}_{m_b} - K_b S_{33}^a \bar{\Delta}^{-1} K_a S_{33}^b \\ &= \Delta^{-1} \end{aligned} \tag{25}$$

where Δ is given by Eq. (22). Hence,

$$\begin{bmatrix} u_a^T & u_b^T \end{bmatrix}^T = \begin{bmatrix} \bar{\Delta}^{-1} & -\bar{\Delta}^{-1} K_a S_{33}^b \\ K_b S_{33}^a \bar{\Delta}^{-1} & \Delta^{-1} \end{bmatrix} \begin{bmatrix} -K_a \eta_b(t) \\ K_b \eta_a(t) \end{bmatrix} \tag{26}$$

Since the effective inputs to the system are $S_{13}^a u_a(t)$ and $S_{13}^b u_b(t)$, under Assumption 2,

$$E_g = \int_0^t \left[f_a^T (z_a(\tau)) S_{13}^a u_a(\tau) + f_b^T (z_b(\tau)) S_{13}^b u_b(\tau) \right] d\tau \tag{27}$$

So, the result of inequality (20) follows applying Lemma 3.1. □

Remark 1. If $f_a(z_a(t)) = z_a(t)$, $f_b(z_b(t)) = z_b(t)$ and the outputs are directly related to the storage elements, *i.e.*, $S_{33}^a = S_{33}^b = 0$ and $S_{32}^a = S_{32}^b = 0$, then from Theorem 3.5, $\Delta = \bar{\Delta} = \mathcal{I}$, $\eta_d = \eta_a = S_{31}^a z_a$ and $\eta_c = \eta_b = S_{31}^b z_b$. Hence, the feedback interconnection is passive if,

$$\begin{aligned} 0 \leq E_d + \int_0^t z_a^T(\tau) S_{13}^a K_a S_{31}^b z_b(\tau) d\tau \\ - \int_0^t z_b^T(\tau) S_{13}^b K_b S_{31}^a z_a(\tau) d\tau \end{aligned} \tag{28}$$

Moreover, if the controller and the plant have junction structures that satisfy $S_{13}^b = \left(S_{31}^b\right)^T$ and $S_{13}^a = \left(S_{31}^a\right)^T$ that are widely assumed in Port-Hamiltonian systems (see (Castaños, Ortega, Van der Schaft, & Astolfi, 2009)), then the feedback interconnection is passive if $K_a = K_b^T$. Moreover, since $z(t) = \frac{\partial H}{\partial x}$ where H is the Hamiltonian function and if $\phi\left(D_i(t)\right) = \phi\left(S_{21} z(t)\right) = \psi\left(x(t)\right) z(t)$, then, from equations (16) and (17), the plant and the controller have the structure of port-Hamiltonian systems,

$$\begin{aligned} \dot{x}(t) &= \left[S_{11} + \psi\left(x(t)\right)\right] \frac{\partial H}{\partial x} \\ y(t) &= S_{31} \frac{\partial H}{\partial x} \end{aligned} \tag{29}$$

where the dissipated function $\psi\left(x(t)\right) \geq 0$ and their interconnection will be passive (see (Brogliato et al., 2007)). □

In Example 3.4, the junction structures for the controller and the plant are,

$$\begin{bmatrix} f_3 \\ e_2 \\ y_m^a(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_3 \\ f_2 \\ u_a(t) \end{bmatrix}, \tag{30}$$

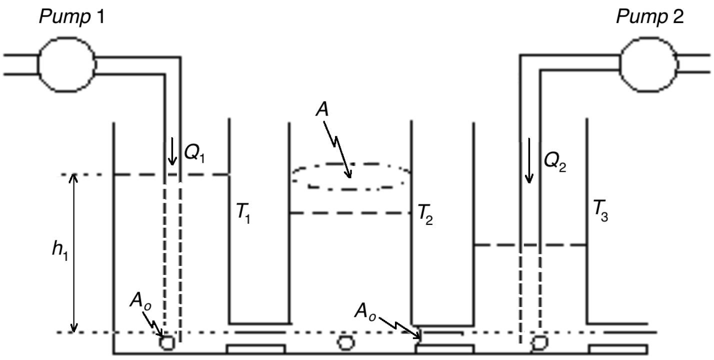

Figure 7. A tank system.

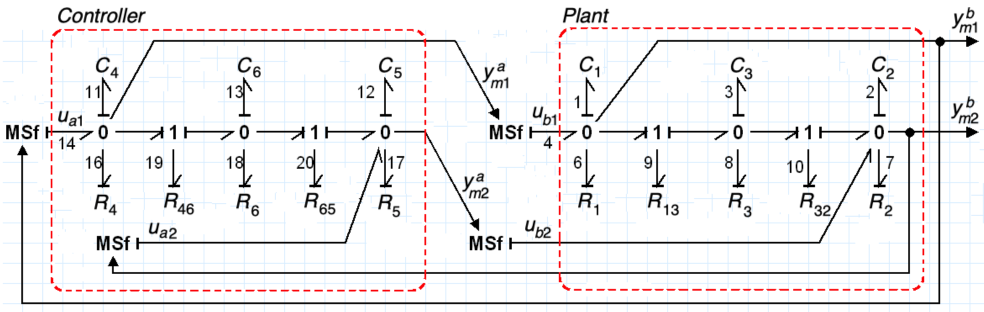

Figure 8. BG of a controller and three tank systems interconnected in feedback.

and

$$\begin{bmatrix} f_6 \\ e_5 \\ y_m^b(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_6 \\ f_5 \\ u_b(t) \end{bmatrix} \tag{31}$$

respectively. So, from Theorem 3.5 and from Eq. (13), $K_b = \frac{-A\theta_1\alpha_2}{\gamma}$, $K_a = \frac{-\theta_1 A\alpha_2}{\gamma}$, $\eta_a(t) = \eta_d(t) = z_a(t)$, $\eta_b(t) = \eta_c(t) = z_b(t)$ and $\Delta = \bar{\Delta} = \mathcal{I}$. The dissipated energy E_d is given in Example 3.4 and from inequality (20) the system is passive, that is, $0 \leq E_d$.

Theorem 3.5 is applied to a three-tank system example in the next section.

4. Three-tank system

Example 4.1. A Three-tank system scheme is shown in Fig. 7 and its modelled by bond graph in Fig. 8. The following is a non-linear model of a Three-Tank-System,

$$\begin{array}{ll} \frac{dh(t)}{dt} & = \frac{1}{A}\left[\begin{array}{c} Q_1(t) - Q_{13}(t) - Q_{10}(t) \\ Q_2(t) + Q_{32}(t) - Q_{20}(t) \\ Q_{13}(t) - Q_{32}(t) - Q_{30}(t) \end{array}\right] \\ y_m^b(t) & = \left[\begin{array}{c} e_1 \\ e_2 \end{array}\right] = \left[\begin{array}{c} h_1(t) \\ h_2(t) \end{array}\right] \\ Q_{13}(t) & = \alpha_{13} A_o sign\left(h_{13}(t)\right)\sqrt{2g\left|h_{13}(t)\right|} \\ Q_{32}(t) & = \alpha_{32} A_o sign\left(h_{32}(t)\right)\sqrt{2g\left|h_{32}(t)\right|} \\ Q_{i0}(t) & = \alpha_{i0} A_o \sqrt{2gh_i(t)}, \quad i = 1, \ldots, 3 \end{array} \tag{32}$$

where $h_{13}(t) := h_1(t) - h_3(t)$, $h_{32}(t) := h_3(t) - h_2(t)$, the states $h_i(t)$, $i = 1, \ldots, 3$ are the liquid levels in meters, $Q_i(t)$, $i = 1, 2$ are the supplying flow rates in $m^3/\sec.$, $Q_{i0}(t)$ being the liquid leakage flow and $0 \leq \alpha_{i0} \leq 1$ the proportion of leakage, A is the cross section of tank in m^2, α_{13} and α_{32} are the outflow coefficients, A_o is the cross-section of flow leak and connection pipes in m^2, $\gamma = \rho g$ is the specific weight, g is the earth gravity and ρ be the flow density. The liquid leakage is modelled in Fig. 8 by R_1 to R_3 that have the non-linear constitutive relations, $D_o^b(t) = \phi_b\left(D_i^b(t)\right)$, given by Eq. (32). In the work of (Galindo, 2005) an identification procedure is used to get the relations for the pumps and sensors, that is, $y_m^b(t) = \left[\begin{array}{cc} e_6 & e_7 \end{array}\right]^T$, $u_b(t) = \left[\begin{array}{cc} Q_1 & Q_2 \end{array}\right]^T = \left[\begin{array}{cc} f_4 & f_5 \end{array}\right]^T = \psi\left(y_m^a(t)\right)$ and $D_o^b(t) = \phi_b\left(D_i^b(t)\right)$ are,

$$\begin{array}{l} Q_1(t) = f_4 = c_{13}A\left(c_{11}\alpha_{b1}y_{m1}^a(t) + c_{12}\right) \\ Q_2(t) = f_5 = c_{23}A\left(c_{21}\alpha_{b2}y_{m2}^a(t) + c_{22}\right) \\ h_i(t) = \frac{e_i}{\gamma} = -c_{i3}y_{mi}^b(t) + c_{i4}, \quad i = 1, \ldots, 3 \end{array} \tag{33}$$

where $y_{mi}^b(t)$, $i = 1, \ldots, 3$ are the liquid levels in volts, $\alpha_{bi}y_{mi}^a(t)$, $i = 1, 2$ are the supplying flow rates in volts of the pumps that modulates the flow sources in Fig. 8, α_{b1} and α_{b2} are control gains, and c_{ij}, $i = 1, \ldots, 3$, $j = 1, \ldots, 4$ are constant parameters. Also, from Fig. 8, $D_i^b = \left[\begin{array}{ccccc} e_6 & e_7 & e_8 & e_9 & e_{10} \end{array}\right]^T$, $D_i^a = \left[\begin{array}{ccccc} e_{16} & e_{17} & e_{18} & e_{19} & e_{20} \end{array}\right]^T$, $z_b = \left[\begin{array}{ccc} e_1 & e_2 & e_3 \end{array}\right]^T$ and $z_a = \left[\begin{array}{ccc} e_{11} & e_{12} & e_{13} \end{array}\right]^T$. So, since $e_1 = e_6$, $e_2 = e_7$, $e_3 = e_8$, $e_{11} = e_{16}$, $e_{12} = e_{17}$ and $e_{13} = e_{18}$ then, the elements of z_b and z_a are directly related to the ones of D_i^b and D_i^a, respectively, that is, $q_a > n_a$, $q_b > n_b$ and $S_{22}^b = S_{23}^b = 0$. So, $E_d = \int_0^t \left(D_i^a(\tau)D_o^a(\tau) + D_i^b(\tau)D_o^b(\tau)\right)d\tau$ is,

$$\begin{array}{l} E_d = A_o\sqrt{\frac{2}{\rho}}\int_0^t\left[\sum_{i=6}^{8}\left(\alpha_i e_i\sqrt{e_i} + \alpha_{1i}e_{1i}\sqrt{e_{1i}}\right) + \right. \\ \left. \alpha_{13}e_9\sqrt{|e_9|}sign\left(\frac{e_9}{\gamma}\right) + \alpha_{32}e_{10}\sqrt{|e_{10}|}sign\left(\frac{e_{10}}{\gamma}\right)\right]d\tau + \\ +A_o\sqrt{\frac{2}{\rho}}\int_0^t\left[\alpha_{46}e_{19}\sqrt{|e_{19}|}sign\left(\frac{e_{19}}{\gamma}\right) + \right. \\ \left. \alpha_{65}e_{20}\sqrt{|e_{20}|}sign\left(\frac{e_{20}}{\gamma}\right)\right]d\tau \end{array} \tag{34}$$

So, the plant output $y_m^b(t) = \left[\begin{array}{cc} e_1 & e_2 \end{array}\right]^T$ and input $u_b(t) = \left[\begin{array}{cc} f_4 & f_5 \end{array}\right]^T$ satisfy Eq.

(7) where,

$$K_1 = \begin{bmatrix} \frac{-1}{c_{13}\gamma} & 0 & 0 \\ 0 & \frac{-1}{c_{23}\gamma} & 0 \end{bmatrix}, \; K_2 = \begin{bmatrix} \frac{c_{14}}{c_{13}} \\ \frac{c_{24}}{c_{23}} \end{bmatrix},$$
$$K_3 = \begin{bmatrix} c_{13}Ac_{11}\alpha_{b1} & 0 \\ 0 & c_{23}Ac_{21}\alpha_{b2} \end{bmatrix} \text{ and } K_4 = \begin{bmatrix} c_{13}Ac_{12} \\ c_{23}Ac_{22} \end{bmatrix} \tag{35}$$

An analogous structure to the plant is proposed for the controller where, $y_m^a(t) = \begin{bmatrix} e_{11} & e_{12} \end{bmatrix}^T$, $u_a(t) = \begin{bmatrix} f_{14} & f_{15} \end{bmatrix}^T$ and $D_o^a(t) = \phi_a\left(D_i^a(t)\right)$. Applying Lemma 3.3, the control parameters are selected from Eq. (8),

$$\begin{array}{l} y_{m1}^a(t) = \frac{-1}{c_{13}\gamma}e_{11} - \frac{c_{12}}{c_{11}\alpha_{b1}}, \\ y_{m2}^a(t) = \frac{-1}{c_{23}\gamma}e_{12} - \frac{c_{22}}{c_{21}\alpha_{b2}}, \\ f_{14} = c_{11}A\alpha_{b1}\left(-c_{13}y_{m1}^b(t) + c_{14}\right), \\ f_{15} = c_{21}A\alpha_{b2}\left(-c_{23}y_{m2}^b(t) + c_{24}\right) \text{ and} \\ f_{1i} = \alpha_{1i}A_o\sqrt{\frac{2e_{1i}}{\rho}}, \quad i = 6, \ldots, 8 \end{array} \tag{36}$$

where α_{1i}, $i = 6, \ldots, 8$ are control parameters, and $u_{ai}(t)$ and $y_{mi}^a(t)$, $i = 1, 2$ are in volts. Negative feedback is considered and the interconnection of the plant and the controller is shown in Fig. 8. Then, the non-linear controller leads to $u_a(t) = -K_a y_m^b(t)$ and $u_b(t) = K_b y_m^a(t)$ be linear functions of $y_m^b(t)$ and $y_m^a(t)$, respectively, where,

$$\begin{array}{l} K_b = \frac{-A}{\gamma}\begin{bmatrix} c_{11}\alpha_{b1} & 0 \\ 0 & c_{21}\alpha_{b2} \end{bmatrix} \text{ and} \\ K_a = \frac{-A}{\gamma}\begin{bmatrix} c_{11}\alpha_{b1} & 0 \\ 0 & c_{21}\alpha_{b2} \end{bmatrix} \end{array} \tag{37}$$

Since, $S_{33}^a = S_{33}^b = S_{32}^a = S_{32}^b = 0$ then from Theorem 3.5, $\Delta = \bar{\Delta} = \mathcal{I}$, $\eta_c(t) = \eta_b(t)$, $\eta_d(t) = \eta_a(t)$,

$$\eta_a(t) = S_{31}^a z_a(t) = \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix}, \; \eta_b(t) = S_{31}^b z_b(t) = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{38}$$

where $S_{31}^a = S_{31}^b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Also, $f\left(z_a(t)\right) = z_a(t)$ and $f\left(z_b(t)\right) = z_b(t)$, so,

$$\begin{array}{l} f_a^T\left(z_a(t)\right)S_{13}^a = \begin{bmatrix} e_{11} & e_{12} \end{bmatrix}, \\ f_b^T\left(z_b(t)\right)S_{13}^b = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \end{array} \tag{39}$$

where $S_{13}^a = S_{13}^b = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Hence, applying Theorem 3.5, from inequality (20), the overall system is passive, that is, $0 \leq E_d$, since E_g is a quadratic form with a skew symmetric matrix, *i.e.*, $E_g = 0$. So, the feedback system is passive for all $0 \leq \alpha_{1i} \leq 1$, $i = 6, \ldots, 8$, the designer can tune α_{1i} adding damping and achieving the desired performance. □

Theorem 3.5 is applied to linear time-invariant systems in the following section.

5. Linear time-invariant systems

For Linear Time-Invariant (LTI) systems E_g can be described by a quadratic form of $z(t)$ or $D_i(t)$, that is,

$$E_g = \int_0^t z^T(\tau) G z(\tau) d\tau \text{ or } E_g = \int_0^t (D_i(\tau))^T G D_i(\tau) d\tau \tag{40}$$

where $G \in \Re^{n\times n}$. Moreover, if G is a skew-symmetric matrix or the controller assures that, then E_g vanish. Also, $E_g \geq 0 \iff G \geq 0$. For LTI systems, three cases are considered in the following result,

Lemma 5.1. *Let $z(t) \in \Re^{n\times 1}$ and $D_i(t) \in \Re^{q\times 1}$. Under Assumptions 1 and 2 an LTI system modelled by bond graph is passive if,*

(1) when $E_g = \int_0^t z^T(\tau) G z(\tau) d\tau$,

$$\begin{array}{l} 0 \leq diag\{L,\, 0_{n-q}\} - G, \;\; q < n, \\ 0 \leq L - diag\{G,\, 0_{q-n}\}, \;\; q > n \text{ and} \\ 0 \leq L - G, \;\; q = n \end{array} \tag{41}$$

(2) when $E_g = \int_0^t (D_i(\tau))^T G D_i(\tau) d\tau$

$$0 \leq L - G, \;\; q \geq n \tag{42}$$

□

Proof. Under Assumption 2 the relationships for the junction structure $S(0,\, 1,\, TF,\, GY,\, MS_e,\, MS_f)$ are given by Eq. (4). Then,

(1) For $q < n$, all the elements of $D_i(t)$ are directly related to the ones of $z(t)$ and let these relationships be the first q rows of $z(t)$ then $S_{12} = \left[\begin{array}{cc} -\mathcal{I}_q & 0_{q\times(n-q)} \end{array}\right]^T$ and $S_{21} = \left[\begin{array}{cc} \mathcal{I}_q & 0_{q\times(n-q)} \end{array}\right]$,

(2) For $q > n$, all the elements of $z(t)$ are directly related to the ones of $D_i(t)$ and let these relationships be the first q rows of $D_i(t)$ then $S_{12} = \left[\begin{array}{cc} -\mathcal{I}_n & 0_{n\times(q-n)} \end{array}\right]$ and $S_{21} = \left[\begin{array}{cc} \mathcal{I}_n & 0_{n\times(q-n)} \end{array}\right]^T$, and

(3) For $q = n$, all the elements of $D_i(t)$ are directly related to all the ones of $z(t)$, then $S_{12} = -\mathcal{I}_n$ and $S_{21} = \mathcal{I}_n$.

The constitutive relations of the dissipative field of LTI systems modelled by bond graphs is $D_o(t) = L D_i(t)$ where $L \in \Re^{q\times q}$ is a matrix composed of $1/R$, R and multiport R elements. Hence, from inequality (3) and Eq. (40), the system is passive if,

$$\begin{array}{l} 0 \leq \int_0^t \left[z^T(t) \left(diag\{L,\, 0_{n-q}\} - G\right) z(t)\right] d\tau, \; q < n, \\ 0 \leq \int_0^t \left[D_i^T(t) \left(L - diag\{G,\, 0_{q-n}\}\right) D_i(t)\right] d\tau, \; q > n \text{ and} \\ 0 \leq \int_0^t \left[z^T(t) \left(L - G\right) z(t)\right] d\tau, \; q = n \end{array} \tag{43}$$

arriving at the result. □

Clearly, in Lemma 5.1 if $q < n$ then all the elements of $z(t)$ are directly related

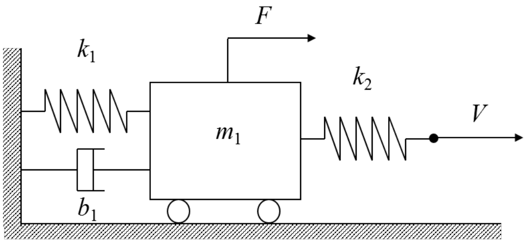

Figure 9. A two-port mechanical system

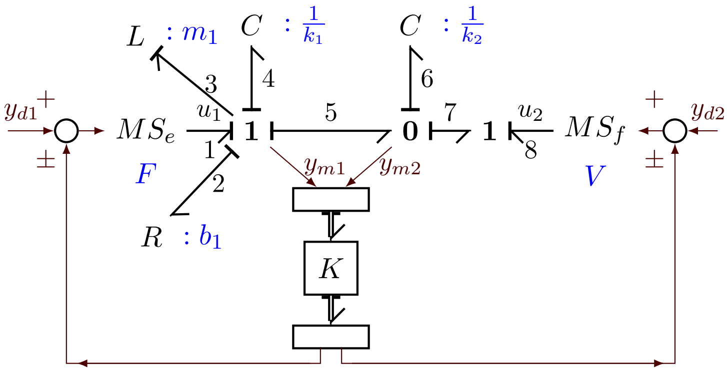

Figure 10. Closed loop configuration of the mechanical system.

to the ones of $D_i(t)$ and E_d is a quadratic form of $z(t)$, while if $q > n$ then all the elements of $D_i(t)$ are directly related to the ones of $z(t)$ and E_d is a quadratic form of $D_i(t)$. Hence, in both cases, $E_d - E_g$ is described by a quadratic form.

The results of Lemma 5.1 are applied in the following example, to the PBC of a mechanical system.

Example 5.2.

The example of the work of (Ngwompo & Galindo, 2017) of the mechanical system shown in Fig. 9 and modelled by BG in Fig. 10, is considered, where m_1, b_1, and k_i, $i = 1, 2$, are the mass, the damping coefficient and the stiffness parameters, respectively. Force $e_1(t)$ and velocity $f_8(t)$ are inputs applied to the system and the outputs are the velocity $y_1(t)$ of the mass and the spring force $y_2(t)$ as indicated. This mechanical system is considered in the closed-loop configuration of Fig. 10, so that $u = K(y_d \pm y)$ with the modulating gain matrix given by,

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \tag{44}$$

Different from the work of (Ngwompo & Galindo, 2017), an augmented bond graph model is not required. From Fig. 10, $D_i(t) = f_2 = f_3$ and f_3 is an element of $z(t) = \begin{bmatrix} f_3 & e_4 & e_6 \end{bmatrix}^T$, $D_o(t) = e_2$, so, $L = b_1$. Under Assumption 1, $y^T(t)u(t) = y_d^T(t)u_d(t)$ is power, hence $y_d(t) = 0$ is considered for the passivity analysis. So, from Fig. 10,

$E_g = \int_0^t (e_1 f_1 + e_8 f_8)\, d\tau$ is the quadratic form,

$$E_g = \int_0^t \left[(\pm K_{11} f_3 \pm K_{12} e_6)\, f_3 - e_6\, (\pm K_{21} f_3 \pm K_{22} e_6) \right] d\tau \tag{45}$$

Applying Lemma 5.1, since $q < n$, then, the closed loop system is passive if $0 \leq L_{cl} = diag\{L,\, 0\} - G$ where,

$$L_{cl} := \begin{bmatrix} b_1 \mp K_{11} & 0 & \mp K_{12} \\ 0 & 0 & 0 \\ \pm K_{21} & 0 & \pm K_{22} \end{bmatrix} \tag{46}$$

The positive semi-definiteness of L_{cl} determines the passivity of the closed-loop system. For this, Sylvester's criterion is applied to the symmetric part of this matrix, *i.e.*,

$$\tfrac{1}{2}(L_{cl} + L_{cl}^T) = \begin{bmatrix} b_1 \mp K_{11} & 0 & \frac{1}{2}(\mp K_{12} \pm K_{21}) \\ 0 & 0 & 0 \\ \frac{1}{2}(\mp K_{12} \pm K_{21}) & 0 & \pm K_{22} \end{bmatrix} \tag{47}$$

that is the same result as the work of (Ngwompo & Galindo, 2017), without the skew-symmetric matrix S_{11}, and as the parasitic elements R_2 and R_3 tend to infinity. So, for positive feedback, the passivity conditions are $K_{11} \leq b_1$, $|K_{21} - K_{12}| \leq 2\sqrt{(b_1 - K_{11})K_{22}}$ and $K_{22} \geq 0$, while for negative feedback, the passivity conditions are any $K_{11} \geq 0$, $K_{21} = K_{12}$ and $K_{22} = 0$. □

The following Corollary applies Theorem 3.5 to the cascade and feedback interconnections of power-conservative LTI systems satisfying Assumptions 1 and 2. It is assumed that the sub-systems are interconnected with no loading effect, that is, these sub-systems are interconnected through active (signal) bonds that modulate sources of effort or flow. Due to this connection, the overall system may not conserve energy.

Corollary 5.3. *Suppose that the junction structures associated to the bond graph of the LTI controller $\mathcal{K}$ and LTI plant P are given by equations (16) and (17), respectively, and are interconnected in cascade and feedback with no loading effect as shown in figures 6 and 2, respectively, and under Assumptions* 1 *and* 2*, $f_a\,(z_a(t)) = z_a(t)$ and $f_b\,(z_b(t)) = z_b(t)$. Then, the cascade interconnection is passive if,*

$$\begin{array}{l} 0 \leq diag\{L,\, 0_{n_a - q_a}\} - G_{c1}, \;\; q_a < n_a \\ 0 \leq L - diag\{G_{c2},\, 0_{q_a - n_a}\}, \;\; q_a > n_a \text{ and} \\ 0 \leq L - G_{c3}, \;\; q_a = n_a \end{array} \tag{48}$$

where

$$\begin{array}{l} G_{c1} := \begin{bmatrix} 0_{n_a} & 0_{n_a \times n_b} \\ S_{13}^b K_b \Theta_a & 0_{n_b} \end{bmatrix}, \;\; L := diag\{L_a,\, L_b\}, \\ G_{c2} := \begin{bmatrix} 0_{n_a} & 0_{n_a \times n_b} \\ S_{13}^b K_b \Gamma_a & 0_{n_b} \end{bmatrix}, \;\; G_{c3} := \begin{bmatrix} 0_{n_a} & 0_{n_a \times n_b} \\ S_{13}^b K_b \Upsilon_a & 0_{n_b} \end{bmatrix} \end{array} \tag{49}$$

being $\Theta_a := S_{31}^a + \begin{bmatrix} S_{32}^a L_a & 0 \end{bmatrix}$, $\Gamma_a := S_{31}^a + \begin{bmatrix} S_{32}^a L_a \\ 0 \end{bmatrix}$, $\Upsilon_a := S_{31}^a + S_{32}^a L_a$, $L_a \in \Re^{q_a \times q_a}$ *and* $L_b \in \Re^{q_b \times q_b}$ *matrices composed of* $1/R$, R *and multiport* R *elements of the* *constitutive relations of the dissipative field of $\mathcal{K}$ and P modelled by bond graphs, that is, $D_o^a(t) = L_a D_i^a(t)$ and $D_o^b(t) = L_b D_i^b(t)$, respectively. The feedback interconnection is passive if,*

$$\begin{array}{l} 0 \leq diag\{L,\, 0_{n_a-q_a+n_b-q_b}\} - G_{f1},\ q_a < n_a,\ q_b < n_b, \\ 0 \leq L - diag\{G_{f2},\, 0_{q_a-n_a+q_b-n_b}\},\ q_a > n_a,\ q_b > n_b \text{ and} \\ 0 \leq L - G_{f3},\ q_a = n_a,\ q_b = n_b \end{array} \tag{50}$$

where

$$\begin{array}{l} G_{f1} := \\ \begin{bmatrix} -S_{13}^a \bar{\Delta}^{-1} K_a S_{33}^b K_b \Theta_a & -S_{13}^a \bar{\Delta}^{-1} K_a \Theta_b \\ S_{13}^b \Delta^{-1} K_b \Theta_a & -S_{13}^b \Delta^{-1} K_b S_{33}^a K_a \Theta_b \end{bmatrix}, \\ G_{f2} := \\ \begin{bmatrix} -S_{13}^a \bar{\Delta}^{-1} K_a S_{33}^b K_b \Gamma_a & -S_{13}^a \bar{\Delta}^{-1} K_a \Gamma_b \\ S_{13}^b \Delta^{-1} K_b \Gamma_a & -S_{13}^b \Delta^{-1} K_b S_{33}^a K_a \Gamma_b \end{bmatrix} \text{ and} \\ G_{f3} := \\ \begin{bmatrix} -S_{13}^a \bar{\Delta}^{-1} K_a S_{33}^b K_b \Upsilon_a & -S_{13}^a \bar{\Delta}^{-1} K_a \Upsilon_b \\ S_{13}^b \Delta^{-1} K_b \Upsilon_a & -S_{13}^b \Delta^{-1} K_b S_{33}^a K_a \Upsilon_b \end{bmatrix} \end{array} \tag{51}$$

being $\Upsilon_b := S_{31}^b + S_{32}^b L_b$, $\Psi := \left(\mathcal{I} + K_a S_{33}^b K_b\right)^{-1}$, $\Phi := \mathcal{I} - S_{33}^a \Psi K_a S_{33}^b K_b$, $\Theta_b := S_{31}^b + \begin{bmatrix} S_{32}^b L_b & 0 \end{bmatrix}$, $\Gamma_b := S_{31}^b + \begin{bmatrix} S_{32}^b L_b \\ 0 \end{bmatrix}$, $\Delta := \mathcal{I}_{m_b} + K_b S_{33}^a K_a S_{33}^b$ *and* $\bar{\Delta} := \mathcal{I}_{m_a} + K_a S_{33}^b K_b S_{33}^a$.

Proof. Using $D_o^a(t) = L_a D_i^a(t)$ and $D_o^b(t) = L_b D_i^b(t)$, equations (19) and (21) become,

$$\begin{array}{l} \eta_a(t) = S_{31}^a z_a(t) + S_{32}^a L_a D_i^a(t) \text{ and} \\ \eta_b(t) = S_{31}^b z_b(t) + S_{32}^b L_b D_i^b(t) \end{array} \tag{52}$$

Then, since $f_b(z_b(t)) = z_b(t)$, from Theorem 3.5 the cascade interconnection is passive if,

$$0 \leq E_d - \int_0^t z_b^T(\tau) S_{13}^b K_b \eta_a(\tau) d\tau \tag{53}$$

where,

$$E_d := \int_0^t \left[\left(D_i^a(\tau)\right)^T L_a D_i^a(\tau) + \left(D_i^b(\tau)\right)^T L_b D_i^b(\tau)\right] d\tau \tag{54}$$

For $q_a \leq n_a$, all the elements of $D_i^a(t)$ are directly related to the ones of $z_a(t)$ and let these relationships be the first q_a rows of $z_a(t)$ then $S_{12}^a = \begin{bmatrix} -\mathcal{I}_{q_a} & 0_{q_a \times (n_a - q_a)} \end{bmatrix}^T$ and $S_{21}^a = \begin{bmatrix} \mathcal{I}_{q_a} & 0_{q_a \times (n_a - q_a)} \end{bmatrix}$. For $q_a > n_a$, all the elements of $z_a(t)$ are directly related to the ones of $D_i^a(t)$ and let these relationships be the first q_a rows of $D_i^a(t)$ then $S_{12}^a = \begin{bmatrix} -\mathcal{I}_{n_a} & 0_{n_a \times (q_a - n_a)} \end{bmatrix}$ and $S_{21}^a = \begin{bmatrix} \mathcal{I}_{n_a} & 0_{n_a \times (q_a - n_a)} \end{bmatrix}^T$. Hence, since $D_i^a(t) = S_{21}^a z_a(t)$, from inequality (18),

$$\begin{array}{l} 0 \leq E_d - \int_0^t z_b^T(\tau) S_{13}^b K_b \Theta_a z_a(\tau) d\tau,\ q_a \leq n_a, \\ 0 \leq E_d - \int_0^t z_b^T(\tau) S_{13}^b K_b \Gamma_a z_a(\tau) d\tau,\ q_a > n_a \end{array} \tag{55}$$

So, the result of inequalities (48) follows applying Lemma 5.1 with $z(t) := \begin{bmatrix} z_a^T(t) & z_b^T(t) \end{bmatrix}^T$ and $D_i(t) := \begin{bmatrix} (D_i^a(\tau))^T & \left(D_i^b(\tau)\right)^T \end{bmatrix}^T$. Also, from Eq. (21), for $q_a \leq n_a$ and $q_b \leq n_b$

$$\begin{array}{l} \eta_c(t) := \Theta_b z_b(t) + S_{33}^b K_b \Theta_a z_a(t) \text{ and} \\ \eta_d(t) := \Theta_a z_a(t) - S_{33}^b K_b \Theta_b z_b(t) \end{array} \tag{56}$$

and for $q_a > n_a$ and $q_b > n_b$,

$$\begin{array}{l} \eta_c(t) = \Gamma_b z_b(t) + S_{33}^b K_b \Gamma_a z_a(t) \text{ and} \\ \eta_d(t) = \Gamma_a z_a(t) - S_{33}^b K_b \Gamma_b z_b(t) \end{array} \tag{57}$$

Hence, applying Theorem 3.5, since $f_a\left(z_a(t)\right) = z_a(t)$ and $f_b\left(z_b(t)\right) = z_b(t)$, for $q_a \leq n_a$ and $q_b \leq n_b$, the feedback interconnection is passive if,

$$\begin{array}{rl} 0 \leq & E_d + \int_0^t z_a^T(\tau) S_{13}^a \bar{\Delta}^{-1} K_a \left[\Theta_b z_b(\tau) + S_{33}^b K_b \Theta_a z_a(\tau)\right] d\tau + \\ & - \int_0^t z_b^T(\tau) S_{13}^b \Delta^{-1} K_b \left[\Theta_a z_a(\tau) - S_{33}^b K_b \Theta_b z_b(\tau)\right] d\tau \end{array} \tag{58}$$

and for $q_a > n_a$ and $q_b > n_b$,

$$\begin{array}{rl} 0 \leq & E_d + \int_0^t z_a^T(\tau) S_{13}^a \Psi K_a \left[\Gamma_b D_i^b(\tau) + S_{33}^b K_b \Gamma_a D_i^a(\tau)\right] d\tau + \\ & - \int_0^t z_b^T(\tau) S_{13}^b K_b \left[\Phi \Gamma_a D_i^a(\tau) - S_{33}^a \Psi K_a \Gamma_b D_i^b(\tau)\right] d\tau \end{array} \tag{59}$$

Hence, the result of inequalities (50) follows applying Lemma 5.1. □

Remark 2. *If $q_a = n_a$ and $q_b = n_b$, removing the skew-symmetric matrices S_{11}^a and S_{11}^b, Corollary* 1 *includes the result of Theorem 3.5 about the cascade interconnection of the work of* (Ngwompo & Galindo, 2017)*. Also, if the plant and the controller are described by state-space realizations, a useful result is the one given by the work of* (Gonzalez & Galindo, 2009) *with the change of sign of the storage field proposed by* (Galindo & Ngwompo, 2017)*, that is, if* $\begin{bmatrix} \dot{x}_a(t) \\ D_i^a(t) \\ y_a(t) \end{bmatrix} = \begin{bmatrix} 0 & -\mathcal{I} & B_a \\ \mathcal{I} & 0 & 0 \\ C_a F_a^{-1} & 0 & D_a \end{bmatrix} \begin{bmatrix} z_a(t) \\ D_o^a(t) \\ u_a(t) \end{bmatrix}$, $\begin{bmatrix} \dot{x}_b(t) \\ D_i^b(t) \\ y_b(t) \end{bmatrix} = \begin{bmatrix} 0 & -\mathcal{I} & B_b \\ \mathcal{I} & 0 & 0 \\ C_b F_b^{-1} & 0 & D_b \end{bmatrix} \begin{bmatrix} z_b(t) \\ D_o^b(t) \\ u_b(t) \end{bmatrix}$, $L_a = -A_a F_a^{-1}$ *and* $L_b = -A_b F_b^{-1}$ *then,*

$$G_c := \begin{bmatrix} 0 & 0 \\ B_b K_b C_a F_a^{-1} & 0 \end{bmatrix} \tag{60}$$

and $L = diag\{-A_a F_a^{-1},\ -A_b F_b^{-1}\}$*, that is the result of the cascade interconnection of* (Galindo & Ngwompo, 2017)*. Using Eq. (25), G_{f3} can be rewritten as,*

$$\begin{array}{rl} G_{f3} := & \begin{bmatrix} -S_{13}^a \\ -S_{13}^b K_b S_{33}^a \end{bmatrix} \bar{\Delta}^{-1} K_a \begin{bmatrix} S_{33}^b K_b \Upsilon_a & \Upsilon_b \end{bmatrix} + \\ & \begin{bmatrix} 0 & 0 \\ S_{13}^b K_b \Upsilon_a & 0 \end{bmatrix} \end{array} \tag{61}$$

Then, Corollary 1 *also includes the result of Theorem* 4.1 *about the feedback interconnection of the work of* (Galindo & Ngwompo, 2017)*.* □

If $p = n$, in the work of (Galindo & Ngwompo, 2017) it is proposed an approximation of the derivative to control position outputs when the controller is designed for speed control. So, the tracking control problem is solved. Also, the problem of pole placement for a particular class of systems is considered in the work of (Galindo & Ngwompo, 2017).

6. Conclusions

The proposed approach based on the dissipative and internally generated energies, and on a controller with a similar structure to the plant, greatly simplify the passivity analysis and control. The control parameters of the feedback system are selected such that the difference between the dissipated and the internally generated energies be positive. This can be accomplished by adding damping or assigning a desired dissipated energy. Moreover, the dissipative analysis is simplified by the design of a controller such that the internally generated energy vanishes. So, the control parameters and variables have physical meaning and the implementation of the controller is facilitated. The method provides a guide for the selection of the controller structure and the assignment of relevant parameters. Further, for linear time-invariant systems with the same dissipative and storage dimensions, the results reduce to results published in the literature. Hence, the system is passive if the matrix associated with the quadratic form of the energies is positive semidefinite. The results have potential applications in control and mechatronics for analysis, design and optimization.

References

Beaman, J. J., & Rosenberg, R. C. (1988, December). Constitutive and modulation structure in bond graph modeling. *Journal of Dynamic Systems, Measurement and Control*, *110*, 395-402.

Brogliato, B., Lozano, R., Maschke, B., & Egeland, O. (2007). *Dissipative systems analysis and control, theory and applications.* Springer-Verlag.

Castaños, F., Ortega, R., Van der Schaft, A., & Astolfi, A. (2009). Asymptotic stabilization via control by interconnection of port-hamiltonian systems. *Automatica*, *45*(7), 1611-1618.

Galindo, R. (2005). Mixed sensitivity $\mathcal{H}_\infty$ control of a three-tank-system. In (p. 1764-1769). American Control Conference.

Galindo, R., & Ngwompo, R. F. (2016). A bond graph pseudo-junction structure for non-linear non-conservative systems. In *Ieee 11th ukacc international conference on control.*

Galindo, R., & Ngwompo, R. F. (2017). Passivity-based control of linear time-invariant systems modelled by bond graph. *Int. J. of Control*, *91*(2), 420-436.

Garcia, J., Rimaux, S., & Delgado, M. (2006, December). Bond graphs in the design of adaptive passivity-based controllers for dc/dc power converters. In (p. 132-137). Mumbai: IEEE International Conference on Industrial Technology.

Gawthrop, P., Neild, S., & Wagg, D. (2015). Dynamically dual vibration absorbers: a bond graph approach to vibration control. *Systems Science & Control Engineering*.

Gawthrop, P. J. (1995). Physical model-based control: A bond graph approach. *Journal of the Franklin Institute - Engineering and Applied Mathematics*, *332*(3), 285-305.

Gawthrop, P. J., Bhikkaji, B., & Moheimani, S. O. R. (2010). Physical-model-based control of a piezoelectric tube for nano-scale positioning applications. *Mechatronics*, *20*(1), 74-84.

Gawthrop, P. J., Wagg, D., & Nield, S. (2007). Bond graph based control and substructuring. *Simulation Modelling Practice and Theory*, *17*(1), 211-227.

Golo, G., der Schaft, A. V., Breedveld, P., & Maschke, B. (2003). *Hamiltonian formulation of bond graphs, in r. johansson and a. rantzer, editors, nonlinear and hybrid systems in automotive control, pp. 351 – 372.* Springer-Verlag, London.

Gonzalez, G., & Galindo, R. (2009). Removing the algebraic loops of a bond graph model. *Proc. of the Institution of Mechanical Engineers Part I: J. of Systems and Control Engineering*, *222*(I6), 543-556.

Gonzalez-A, G. (2016). A bond graph model of a singularly perturbed lti mimo system with a slow state estimated feedback. *Proc. of the Institution of Mechanical Engineers Part I: J. of Systems and Control Engineering*.

Guerrero, M., Hernández, O., Lozano, R., García, C., Valencia, G., & López, F. (2019). Energy-based control and lmi-based control for a quadrotor transporting a payload. *Mathematics*, *7*, 1090.

J. C. Gil, M. D., A. Pedraza, & Sira-Ramirez, H. (1997). Flatness and passivity from a bond graph. In (Vol. 2, p. 1516-1521). IEEE Int. Conf. on Systems, Man, and Cybernetics. Computational Cybernetics and Simulation.

Karimi, M., & Binazadeh, T. (2019). Energy-based hamiltonian approach in $\mathcal{H}_\infty$ controller design for n-degree of freedom mechanical systems. *Systems Science & Control Engineering*, *7*(1).

Li, P. Y., & Ngwompo, R. F. (2005). Power scaling bond graph approach to the passification of mechatronic systems-with application to electrohydraulic valves. *ASME J. of Dynamic Systems, Measurement and Control*, *127*(4), 633-641.

Meigooni, F. S., & Mollaioli, F. (2021). Simulation of seismic collapse of simple structures with energy-based procedures. *Soil Dynamics and Earthquake Engineering*, *145*, 106733.

Ngwompo, R. F., & Galindo, R. (2017). Passivity analysis of linear physical systems with internal energy sources modelled by bond graphs. *Proc. of the Institution of Mechanical Engineers Part I: J. of Systems and Control Engineering*, *231*(1), 14-28.

Ortega, R., & García, E. (2004). Interconnection and damping assignment passivity-based control: A survey. *European Journal of Control*, *10*, 432-450.

Triverio, P., Grivet-Talocia, S., Nakhala, M. S., Canavero, C., & Achar, R. (2007). Stability, causality, and passivity in electrical interconnect models. *IEEE Trans. on Advanced Packaging*, *30*(4), 795-808.

Van der Schaft, A. (2006). Port-hamiltonian systems: an introductory survey. Proc. of the Int. Congress of Mathematicians.

Vidyasagar, M. (1985). *Control system synthesis: A factorization approach.* The MIT Press Cambridge.

Vink, D., Ballance, D., & Gawthrop, P. (2006). Bond graphs in model matching control. *Mathematical and Computer Modelling of Dynamical Systems: Methods, Tools and Applications in Engineering and Related Sciences*, *12*(2-3), 249-261.

Zhou, K., Doyle, J. C., & Glover, K. (1996). *Robust and optimal control.* Prentice Hall.