Time-averaging under fast periodic forcing of parabolic partial differential equations: Exponential estimates

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Abstract

The phases of a large class of parabolic partial differential equations with rapid time-periodic forcing can separated up to exponential small errors. The originally nonautonomous equation is transformed such that the nonautonomous terms are exponentially small in the period h of the forcing. This is a counterpart for partial differential equations of the theorem by Neishtadt [Nei84] for ordinary differential equations. In our case the exponential rate depends on time t and the estimates have the form $h\exp(-c(t)h^{-\frac{1}{3}})$.

1 Introduction

Consider first an ordinary differential equation

$$\begin{aligned} \dot{x} &= f(x), \\ x(0) &= x_0 \in \mathbf{R}^m. \end{aligned}$$

We want to compare the flow $\Phi(t,u)$ with the evolution defined by a nonautonomous fast time-periodic perturbation

$$\begin{aligned} \dot{x} &= f(x) + h^p g(x, \frac{t}{h}, h), \\ x(0) &= x_0 \in \mathbf{R}^m. \end{aligned}$$

Our main intention is not to give best estimates to compare these perturbations with the original flow. Instead we want to compare them to some modified equation $\dot{x} = \tilde{f}(x)$, with $\tilde{f}$ inside the same class as f. In the case of periodic forcing this will lead to averaging methods. For such rapidly forced equations Neishtadt [Nei84] proved:

Theorem (Neishtadt,1984) *Let f and g be real analytic and bounded on a complex extension of some domain D. Then there exists a time-periodic, real analytic near-identity coordinate change*

$$x = y + hU(y, \frac{t}{h}, h),$$

such that the transformed equation has only exponentially small nonautonomous remainder terms

$$\dot{y} = F(y) + \alpha(y, \frac{t}{h}, h)$$

with $\|\alpha\| < c_2 \exp(-c_1/h)$ *and* $\|F - f\| < c_3 h^p$ *uniformly on a complex extension of* D.

So when choosing the right coordinates, the influence of the fast phase is nearly neglible. The fast and the slow phase can be separated.

When considering special solutions we can also explain, why there has to be still some error $\|x_n - \tilde{x}(nh)\|$ for any $\tilde{f}$ for most forcing terms g. One example is the behavior near homoclinic orbits, where a solution $\Gamma(t)$ converges to a hyperbolic steady state $\bar{x}$ for $t \to \pm\infty$. There are generically effects which belong genuinely to nonautonomous dynamical systems [FS96]. In these systems homoclinic orbits are usually transversal and consist out of discrete points instead of a continuous arc. The stable and unstable manifold intersect transversally at some homoclinic point and give rise to shift type dynamics. Several quantities describing these effects were shown by Fiedler and Scheurle [FS96] to be exponentially small, but generically to be positive.

Averaging up to exponentially small remainder terms is until now completely unknown in the literature of infinite dimensional dynamical system or partial differential equations. We consider semilinear parabolic partial differential equations. The general setting for these equations is given for example by Henry [Hen83]. Some general results on averaging of such rapidly forced partial differential equations are given for example in Henry [Hen83, theorem 3.4.9] and Ilyin [Il98]. In these articles the main subject is continuous dependence on h, hence averaging in the 0^{th} order in h. Our aim is averaging beyond any finite order for some concrete examples, where is enough regularity in space and time for our analysis. We will consider a system of reaction-diffusion equations with periodic boundary conditions as a model. We discus possible generalizations including the Navier-Stokes equation and other nonlinearities in section 5.

We will prove a result like the Neishtadt theorem for a rapidly forced system of reaction-diffusion equations with $U = (u_1, \ldots, u_n)$, $D = \text{diag}(d_1, \ldots, d_n)$ with $d_i > 0$, $F = (f_1, \ldots, f_n)$ and $G = (g_1, \ldots, g_n)$:

$$\frac{\partial}{\partial t} U(x,t) - D\Delta U(x,t) = F(U(x,t)) + hG(U(x,t), \frac{t}{h}, h) \tag{1.1}$$

with periodic boundary conditions on $\Omega = [0, l]^d, d = 1, 2, 3$ and initial conditions $U(0) = U_0$ in the Sobolev space $H^s_{per}(\Omega, \mathbf{R}^n) \subset C^0(\Omega, \mathbf{R}^n)$, i.e. $s > \frac{d}{2}$, for a definition see e.g. [Tem88, p.48].

Theorem 1 *Suppose that the nonlinearities* $F(.), G(., \tau, h) : \mathbf{R}^n \to \mathbf{R}^n$ *are entire functions. Let* G *be continuous with respect to* t *and* h *and periodic in* $\tau = \frac{t}{h}$ *with period* 1. *Furthermore assume that there exist global solutions for all initial conditions.*

Then the equation can be transformed on bounded sets by a real analytic and time-periodic change of coordinates for $0 < h < h_1$

$$U = V + hW(V, \frac{t}{h}, h) \tag{1.2}$$

with W *bounded on any ball of radius* R *in* H^s_{per}. *The transformed nonautonomous terms* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *are exponentially small after a short transient, but the equation may contain nonlocal terms* $\bar{F} = (\bar{f}_1, \ldots, \bar{f}_n)$:

$$\frac{\partial}{\partial t} V(x,t) - D\Delta V(x,t) = F(V(x,t)) + \bar{F}(V(t), h)(x) + \alpha(V(t), \frac{t}{h}, h)(x), \tag{1.3}$$

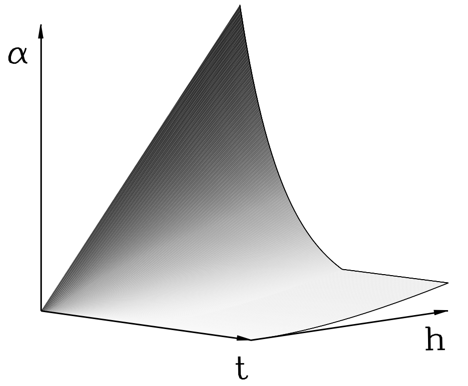

Figure 1: The effect of nonautonomous term $|\alpha| = -h\exp(-\min(c,t)h^{-\frac{1}{3}})$ is exponentially small after a transient.

with $V(0) = U_0$*,* $|V(0)|_{H^s_{per}} < R$ *and* $t \in (0,T)$

$$\begin{aligned}\sup_{|V(0)|_{H^s_{per}}<R} |\alpha(V(t))|_{H^s_{per}} &\leq c_2 h\exp(-\min(t,c_1)h^{-\frac{1}{3}}),\\ \sup_{|V(0)|_{H^s_{per}}<R} |\bar{F}(V(t))|_{H^s_{per}} &\leq c_3 h + c_4\exp(-\min(t,c_5)h^{-\frac{1}{3}}) \qquad (1.4)\end{aligned}$$

where $c_1, c_2, c_3, c_4, c_5, h_1$ *do not depend on* U_0*.*

So after a transient we can separate the fast and the slow phase. Some remarks discussing this theorem are appropriate:

1. The transformed equation leaves the class of local superposition-type nonlinearities. The transformed nonlinearity $\bar{F}$ and α can only be evaluated nonlocally in space like $\bar{F}(V(t),h)(x)$, whereas F and G can be computed by point evaluation $F(V(x,t))$. A different version of theorem 1, where the assumptions on F and G are formulated in terms of operators on certain Banach spaces, can be found in theorem 2 in section 3. There the transformed equation remains in the same class as the original one.

2. Our construction is discontinuous in h. But if G is smooth in h, then $\bar{F}$ and α can be chosen to depend smoothly on h for $h > 0$. Smoothness may break down for $h \to 0$ in general. An exception is analyzed in remark 9.

3. For fixed $h > 0$, we transform the equation only in a finite dimensional subspace: Its image $W(H^s_{per})$ is finite dimensional for fixed h. Furthermore the transformation is the identity at multiple of the period: $W(.,k,.) = 0, k \in \mathbf{Z}$.

4. The transformed nonlinearities consist out of two parts. First there is a finite dimensional part, where a finite-dimensional averaging result is applied. Secondly we have to consider some further correction terms caused by this approximation of the infinite-dimensional problem. The formulas for the correction terms are given in (3.19,3.20). Some of the additional infinite-dimensional terms give rise to the time dependent error

terms $\exp(-\min(t,T^*)h^{-\frac{1}{3}})$. These errors are damped due to the parabolic regularisation. A computation of $\bar{F}(V,h)$ at least in the leading order of h is given in (5.12).

5. The estimates on $\bar{F},\alpha$ can be improved when we estimate the L^2 norm. Then the correction term $\bar{F}$ converges to 0 for $h\to 0$:

$$\sup_{|V(0)|_{H^s_{per}}<M_1} |\bar{F}(V(t))|_{L^2} \le c_3 h + \tilde{c}_4 h^{\frac{s}{3}} \exp(-\min(t,T^*)h^{-\frac{1}{3}}). \tag{1.5}$$

   For α we get

$$\begin{aligned}\sup_{|V(0)|_{H^s_{per}}<M_1} |\alpha(V(t),\frac{t}{h},h)|_{L^2} \quad &\le \quad c_2 h \exp(-c_1 h^{-\frac{1}{3}}) \\ &\qquad + \tilde{c}_5 h^{1+\frac{s}{3}} \exp(-\min(t,T^*)h^{-\frac{1}{3}}).\end{aligned} \tag{1.6}$$

   A proof is given in remark 8.

6. Usually one wants to compare the autonomous part of (1.3)

$$\frac{\partial}{\partial t}V(x,t) - D\Delta V(x,t) = F(V(x,t)) + \bar{F}(V(t),h)(x) \tag{1.7}$$

   with the full equation. If we start at $t=0$ we get an error estimate of order $h^{\frac{s+4}{3}}$, when using the above L^2 estimates and the variation of constants formula. But if we start dropping the nonautonomous part after some transient time, we get an exponential estimate: $h\exp(-ch^{-\frac{1}{3}})$, see lemma 10.

7. We have not to assume global solutions. It suffices to assume, that the forward orbits of points in the ball of radius R in H^s_{per} stay inside a larger ball of radius M_0 for the finite time T. The constants will then not depend on T, but only on M_0.

As an application to an improved geometric theory we analyze the influence of rapid forcing to hyperbolic equilibria and to homoclinic orbits in one-parameter families of equations. We compare the period h map of the semi-evolution $\tilde{S}^h(h,0,.)$ defined by the forced equation (1.1) with the time-h map of the semiflow defined by the averaged equation (1.7). Then we can show several effects to be exponentially small.

When the original unforced equation (1.1 with $h=0$) possesses a hyperbolic equilibrium, then (1.1) possesses a hyperbolic periodic orbit $\tilde{u}_h$ of period h and thus $\tilde{S}^h(h,0,.)$ a hyperbolic fixed point. We will show, that $\tilde{u}_h$ is exponentially close in h to a hyperbolic equilibrium of the averaged equation (1.7) and that their phase portraits are also exponentially close, when being close enough to these points. Then the splitting of homoclinic orbits can only happen in exponentially small wedges in parameter space. We can estimate the size of the splitting at least at some points of a possibly discrete homoclinic orbit.

The rest of the paper is organized as follows. In the next section we prove very high (Gevrey) regularity of solutions of (1.1) and for equations with quite general nonlinearities, which is the essentiell tool in the proof of theorem 1. In section 3 we show theorem 1 by proving a theorem 2 in a wider class of nonlinearities. In section 4 the local averaging theorem 1 is applied to describe the qualitative behavior near equilibria and homoclinic orbits. A discussion of the assumptions, possible generalizations and related results can be found in section 5.

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2 Gevrey Regularity

In this section we discuss the regularity of solutions of equations like (1.1) for $U = (u_1, \dots, u_n)$:

$$\frac{\partial}{\partial t}U(x,t) - D\Delta U(x,t) = F(U(x,t)) + G(U(x,t),t,h) \quad (2.1)$$

$$U(t_0) = U_0 \quad (2.2)$$

We extend results of Ferrari and Titi [FT98] to show that the solutions lie in some Gevrey class under appropriate conditions on the nonlinearities $F = (f_1, \dots, f_n)$ and $G = (g_1, \dots, g_n)$. We will state the conditions on the nonlinearities F and G as general operators. First we collect some preliminaries about Gevrey classes. Then we proceed to give a uniqueness and existence result for regular solutions using Galerkin approximations. Solutions will be shown to be in Gevrey classes. This extremely high regularity of the solutions and the exponential decay of the Fourier coefficients will be a key ingredient for the proof of theorem 1.

2.1 Properties of Gevrey classes

We consider Gevrey classes $G_\sigma^{\frac{s}{2}}(\Omega, \mathbf{R}^n)$, which contains functions, for which the Fourier modes decay exponentially fast. A norm is given by

$$|v|_{G_\sigma^{\frac{s}{2}}} = \left(\sum_{k=1}^{n} \sum_{j \in \mathbf{Z}^d} v_k^j v_k^j (1 + d_k \|j\|^2)^s \exp(2\sigma\|j\|) \right)^{\frac{1}{2}}. \quad (2.3)$$

where v_k^j is the k^{th} component of the Fourier coefficient of the Fourier expansion into $\exp(i\frac{2\pi}{l} j \cdot x)$ of the periodic functions v. Alternatively we can define Gevrey classes as

$$G_\sigma^{\frac{s}{2}}(\Omega, \mathbf{R}^n) = D\left((-D\Delta)^{\frac{s}{2}} \exp(\sigma(-\Delta)^{\frac{1}{2}}) \right). \quad (2.4)$$

Note that $G_0^{\frac{s}{2}}(\Omega, \mathbf{R}^n) = H_{per}^s(\Omega, \mathbf{R}^n)$. For $\sigma > 0$ the Fourier coefficients V_j of $V \in G_\sigma^{\frac{s}{2}}(\Omega, \mathbf{R}^n)$ decay like $\|j\|^{-s} \exp(-\sigma\|j\|)$.

We list some properties of $G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$. Most of the proofs are adapted from [FT98]. The Gevrey class $G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$ is a Hilbert space with scalar product

$$(v,w)_{G_\sigma^{\frac{s}{2}}} = \sum_{k=1}^{n} \sum_{j \in \mathbf{Z}^d} v_k^j w_k^j (1 + d_k \|j\|^2)^s \exp(2\sigma\|j\|) \quad (2.5)$$

where v^j and w^j are the Fourier coefficients in $\mathbf{R}^n$ of the periodic functions v and w.

For $s > \frac{d}{2}$ and $\sigma > 0$ all functions $U : \Omega \to \mathbf{R}^n, U \in G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$ are real analytic in the spatial variable $x \in \Omega$. The Gevrey classes are Banach algebras under point-wise multiplication for $s > \frac{d}{2}$. The constants are even independent of the exponent σ:

Lemma 1 *If $u, v \in G_\sigma^{s/2}$ with $s > \frac{d}{2}$ then w defined by some point-wise multiplication is in $G_\sigma^{s/2}$: Let $w(x) = \begin{pmatrix} u^{l_1}(x) \cdot v^{m_1}(x) \\ \vdots \\ u^{l_n}(x) \cdot v^{m_n}(x) \end{pmatrix}$ for arbitrary components $l_k, m_k \in 1, \dots, n$ and there exists a constant C_s not depending on u, v and σ, such that*

$$|w|_{G_\sigma^{s/2}} \le C_s |u|_{G_\sigma^{s/2}} |v|_{G_\sigma^{s/2}}. \tag{2.6}$$

Proof: This result is a direct extension of the scalar case ($n = 1$) of [FT98, lemma 1]. This lemma yields for each component $w^k = u^{l_k} v^{m_k} \in D\left((-\Delta)^{\frac{s}{2}} \exp(\sigma(-\Delta)^{\frac{1}{2}})\right)$ with $|w^k|_{G_\sigma^{s/2}} \le C_s |u^{l_k}|_{G_\sigma^{s/2}} |v^{m_k}|_{G_\sigma^{s/2}}$. Summing these inequalities leads to (2.6). □

Using this lemma we can show, that a big class of superposition-type nonlinearities map $G_\sigma^{s/2}$ into itself. The function F is required to be real analytic on some ball $B_{\mathbf{R}^n}(R, 0)$ and that it is given by power series which is convergent on whole $B_{\mathbf{R}^n}(R, 0)$, i.e. $F(U) = \sum_{|j|=0}^\infty \beta_j U^j$ with multi-indices $j \in \mathbf{N}_0^n$ and $\beta_j \in \mathbf{R}^n$. Then we can explicitly choose a majorising function a given by a power series $\sum_{l=0}^\infty a_l s^l$ with $a_l = \sum_{|j|=l} \|\beta_j\|$, such that

$$\|F(U)\| \le a(\|U\|) \text{ for all } U \in B_{\mathbf{R}^n}(R, 0). \tag{2.7}$$

Lemma 2 *Let $F : B_{\mathbf{R}^n}(R, 0) \to \mathbf{R}^n$ be real analytic and given by a power series, which is convergent on whole $B_{\mathbf{R}^n}(R, 0)$ with a majorising function as in (2.7). If $u \in G_\sigma^{s/2}$ with $|u|_{G_\sigma^{s/2}} \le R_0$ and $s > \frac{d}{2}$, such that $|u|_{BC} \le \frac{R}{C_s}$, then*

$$|F(U)|_{G_\sigma^{s/2}} \le (1 + C_s^{-1}) a(C_s |U|_{G_\sigma^{s/2}}). \tag{2.8}$$

Furthermore $F : B_{G_\sigma^{s/2}}(R_0, 0) \to G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$ is differentiable and $F'(U)$ is a uniformly bounded linear mapping in $L^2(\Omega, \mathbf{R}^n)$ for $|U|_{H_{per}^s} \le R_0$.

Proof: The first part is an easy generalization of the scalar case in [FT98]. We use the Taylor series expansion of F and estimate partial sums:

$$\begin{aligned} \left| \sum_{|k|=0}^N \beta_k U^k \right|_{G_\sigma^{s/2}} &\le \sum_{|k|=0}^N \|\beta_k\| |U^k|_{G_\sigma^{s/2}} \le \|\beta_0\| + \sum_{|k|=1}^N \|\beta_k\| C_s^{|k|-1} |U|_{G_\sigma^{s/2}}^k \\ &\le (1 + C_s^{-1}) \sum_{|k|=0}^N \|\beta_k\| C_s^{|k|} |U|_{G_\sigma^{s/2}}^k \le (1 + C_s^{-1}) a(C_s |U|_{G_\sigma^{s/2}}). \end{aligned}$$

As $\left|\sum_{|k|=M}^N \beta_k U^k\right|_{G_\sigma^{s/2}} \le (1 + C_s^{-1}) \sum_{|k|=M}^N \|\beta_k\| C_s^{|k|} |U|_{G_\sigma^{s/2}}^k \to 0$ holds for $N, M \to \infty$, when passing to the limit for $N \to \infty$ we get $\sum_{|k|=0}^N \beta_k U^k \to \tilde{F}(.) \in G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$ by the completeness of $G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$. The norm of $G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$ is stronger than the uniform convergence norm, since even $G_0^{s/2}(\Omega, \mathbf{R}^n) = H_{per}^s(\Omega, \mathbf{R}^n)$ embeds continuously into $BC(\Omega, \mathbf{R}^n)$. Hence the analyticity of $F(u)$ in u and the uniform boundedness of U in Ω imply $\sum_{|k|=0}^N \beta_k U^k \to F(U(.))$. Therefore $\tilde{F}(x) = F(U(x))$.

To show, that F is differentiable as a map from $B_{G_\sigma^{s/2}}(R, 0)$ to $G_\sigma^{s/2}$, we show

$$|F(U(.) + hV(.)) - F(U(.)) - F'(U(.))hV(.)|_{G_\sigma^{s/2}} = o(h)$$

for $U,V\in G_\sigma^{s/2}$. $F:B_{\mathbf{R}^n}(R,0)\to\mathbf{R}^n$ is differentiable, since F is analytic. Furthermore each component of F', given by the partial derivative $\frac{\partial f_k}{\partial u_l}$, is majorised by a power series $\tilde{a}$. Hence applying the first part of the lemma gives, that $\frac{\partial f_k}{\partial u_l}(U)\in G_\sigma^{s/2}$ and thus $F'(U(.))hV(.)\in G_\sigma^{s/2}$. Taking a standard representation of the remainder r of the Taylor expansion $r(U(.),hV(.))=F(U(.)+hV(.))-F(U(.))-F'(U(.))hV(.)$ gives

$$r(U(.),hV(.))=\int_0^1 F''(U(.)+thV(.))(hV(.),hV(.))dt.$$

As F'' is given by a convergent power series, each $F''(U(.)+thV(.))(V(.),V(.))\in G_\sigma^{s/2}$ for $U,V\in G_\sigma^{s/2}$ and thus $|r(U,hV)|_{G_\sigma^{s/2}}\le h^2C$. Hence $F:B_{G_\sigma^{s/2}}(R,0)\to G_\sigma^{s/2}$ is differentiable and its derivative is given by $F'(U)$.

Then we can also estimate $|F'(U)V|_{L^2}=|F'(U(.))V(.)|_{L^2}$. As $H_{per}^s=G_0^{s/2}$ embeds continuously into $BC(\Omega,\mathbf{R}^n)$ and $F'(U(x))$ is bounded for $U(x)\in B_{\mathbf{R}^n}(R,0)$, we obtain $|F'(U(.))V(.)|_{L^2}\le M|V(.)|_{L^2}$ and the lemma is proved. □

2.2 Existence and uniqueness of regular solutions

The setting of Nemitski operators $F:G_\sigma^{s/2}\to G_\sigma^{s/2}$ defined by superposition-type nonlinearities as in lemma 2 leads to the general setting of differentiable operators $G_\sigma^{s/2}\to G_\sigma^{s/2}$. We prove for this kind of nonlinearities an existence and uniqueness result for regular solutions. So we will first define, what we mean by a solution of the initial value problem (2.1) and (2.2).We rewrite (2.1) with general nonlinear operators

$$\begin{aligned}\frac{\partial}{\partial t}U+AU &= F(U)+G(U,t) \\ U(0) &= U_0\end{aligned}\tag{2.9}$$

with $A=-\text{diag}(d_1,\dots,d_n)\Delta$ on $\Omega=[0,l]^d$ with periodic boundary conditions. Let $d_{min}=\min(d_1,\dots,d_n)>0$.

Definition 3 *Let $U_0\in H_{per}^s(\Omega,\mathbf{R}^n)$ with $s>\frac{d}{2}$. A function*

$$U\in C([0,t_1],H_{per}^s(\Omega,\mathbf{R}^n))\cap L^2([0,t_1],D(A))$$

is called a regular solution of (2.1) and (2.2) on $[t_0,t_1]$, if $\frac{dU}{dt}\in L^2([t_0,t_1],L^2(\Omega,\mathbf{R}^n))$ and

$$(\frac{dU}{dt},\phi)_{L^2}+(A^{\frac{1}{2}}U,A^{\frac{1}{2}}\phi)_{L^2}-(F(U),\phi)_{L^2}-(G(U,\frac{t}{h},h),\phi)_{L^2}=0\tag{2.10}$$

for every $\phi\in H_{per}^1(\Omega,\mathbf{R}^n)$ and almost all $t\in[0,t_1]$.

The initial time $t_0=0$ is used just for notational simplicity. For U_0,F,G we use the following assumptions

$$\begin{array}{l} U_0\in H_{per}^s \text{ with } s>\frac{d}{2} \\ F(.),G(.,t):B_{G_\sigma^{s/2}}(R,0)\to G_\sigma^{s/2} \text{ differentiable in } U \\ F'(U),G'(U)\in L(L^2(\Omega,\mathbf{R}^n),L^2(\Omega,\mathbf{R}^n)) \text{ for } U \text{ bounded in } H_{per}^s \\ G \text{ continuous in } t,h \\ F(.),G(.) \text{ bounded with constants not depending on } \sigma,t: \\ \quad |F(U)|_{G_\sigma^{s/2}}\le C_s a(C_s|U|_{G_\sigma^{s/2}}) \\ \quad |G(U,t)|_{G_\sigma^{s/2}}\le C_s b(C_s|U|_{G_\sigma^{s/2}}) \\ \text{with } a,b \text{ monotone increasing on } [0,\frac{R}{C_s}) \end{array}\tag{H.1}$$

This hypothesis is fulfilled by all nonlinearities fulfilling the assumptions of lemma 2. But we might have to decrease the radius R.

Proposition 4 *Let the hypothesis (H.1) be satisfied. Also assume* $|U|_{H^s_{per}} \leq M_0 < \frac{R}{C_{emb}}$ *for some* $M_0 > 0$. *Then there exists a constant* $T^*(M_0)$ *such that equation (2.9) has a unique regular solution on* $[0, T^*)$ *with initial value* U_0. *Furthermore* $U(.,t) \in G_t^{s/2}$ *and* $|U(.,t)|_{G_t^{s/2}} \leq 2C_0|U(0)|_{H^s} + 1$ *for* $t \in [0, T^*)$.

Proof:

Using results in Henry [Hen83] we could prove local uniqueness and existence of solutions for a different definition of a solution. Both coincide when F and G are highly regular, such that we have classical solutions, but we will follow [FT98]. Their existence proof will lead directly to the higher space regularity. We start by proving uniqueness.

Assume, that U is a regular solution, then $AU \in L^2(\Omega, \mathbf{R}^n)$. From definition 3 we have

$$\frac{d}{dt}(U, \phi)_{L^2} = (-AU + F(U) + G(U,t), \phi)_{L^2}$$

for almost all $t \in [0,T)$ and all $\phi \in H^1_{per}(\Omega, \mathbf{R}^n)$. By regularity of U and (H.1) we have

$$-AU(t) + F(U(t)) + G(U(t),t) \in L^2([0,T), L^2(\Omega, \mathbf{R}^n)) \subset L^1([0,T), L^2(\Omega, \mathbf{R}^n)).$$

Hence we get e.g. by [Tem77, lemma 1.1, ch.3], that

$$\frac{d}{dt}U(t) + AU(t) = F(U(t)) + G(U(t),t) \tag{2.11}$$

holds in $L^2(\Omega, \mathbf{R}^n)$ for almost every $t \in [0,T]$.

Letting U, V be any two regular solutions on $[0,T]$ with initial value U_0. By equation (2.11) we have

$$\frac{d}{dt}(U-V)(t) + A(U(t) - V(t)) = F(U(t)) - F(V(t)) + G(U(t),t) - G(V(t),t)$$

Since $(U-V)(t) \in H^s_{per}(\Omega, \mathbf{R}^n) \subset L^2(\Omega, \mathbf{R}^n)$ we can take the scalar product with $(U-V)(t)$ to get

$$\begin{aligned} & (\frac{d}{dt}(U-V)(t), (U-V)(t))_{L^2} + (A(U(t)-V(t)), U-V)_{L^2} \\ = \; & (F(U(t)) - F(V(t)), (U-V)(t))_{L^2} + (G(U(t),t) - G(V(t),t), (U-V)(t))_{L^2} \end{aligned}$$

As $\frac{d}{dt}(U-V)(t) \in L^2([0,T), L^2(\Omega, \mathbf{R}^n))$ we have that the first term on the left hand side equals $\frac{1}{2}\frac{d}{dt}|(U-V)(t)|_{L^2}$. Using the mean value theorem and the boundedness of $F'(U)$ and $G'(U,t)$ as a linear mapping from $L^2(\Omega, \mathbf{R}^n)$ to $L^2(\Omega, \mathbf{R}^n)$ we get

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}|(U-V)(t)|^2_{L^2} + |A^{\frac{1}{2}}(U-V)(t)|^2_{L^2} \\ \leq \; & (|F(U(t)) - F(V(t))|_{L^2} + |G(U(t),t) - G(V(t),t)|_{L^2})\,|(U-V)(t)|_{L^2} \\ \leq \; & \left(\sup_{|W|_{H^s_{per}} \leq K} |F'(W)|_{L(L^2,L^2)} |(U-V)(t)|_{L^2} \right. \\ & \left. + \sup_{|W|_{H^s_{per}} \leq K} |G'(W,t)|_{L(L^2,L^2)} |(U-V)(t)|_{L^2} \right) |(U-V)(t)|_{L^2} \end{aligned}$$

with $K \leq 2\max(\sup|U(t)|_{H^s_{per}}, \sup|V(t)|_{H^s_{per}})$, hence we obtain

$$\frac{1}{2}\frac{d}{dt}|(U-V)(t)|^2_{L^2} + \frac{1}{2}|A^{\frac{1}{2}}(U-V)(t)|^2_{L^2} \leq C_K|(U-V)(t)|^2_{L^2}$$

and by the Gronwall inequality

$$|U(t)-V(t)|^2_{L^2} \leq |U(0)-V(0)|^2_{L^2}e^{C_K t} = 0.$$

Thus U and V coincide and the uniqueness is proven.

To show existence we use a Galerkin approximation as an analytic tool. So

$$U_N(x,t) = \sum_{\|j\|\leq N} \begin{pmatrix} \alpha_N^{1,j}(t) \\ \vdots \\ \alpha_N^{n,j}(t) \end{pmatrix} \exp(i\frac{2\pi}{l}jx).$$

with multi-index $j \in \mathbf{Z}^d$ defines a sequence of approximative solutions and fulfills the ordinary differential equation

$$\begin{aligned} \dot{U}_N + AU_N &= P_N F(U_N) + P_N G(U_N, t) \\ U_N(0) &= P_N U_0 \end{aligned} \tag{2.12}$$

This is a well-posed ordinary differential equation with a unique solution for every N for some interval $[0, T_N]$, because the vector field in (2.12) is locally Lipschitz in U_N and continuous in time t. To show convergence for the approximating solutions in $G^{s/2}_\sigma(\Omega, \mathbf{R}^n)$ we derive a priori estimates for the U_N, not depending on N. Note that $U_N(t) \in G^{s/2}_\sigma(\Omega, \mathbf{R}^n)$ for all $t \in [0, T_N]$ and $\sigma \geq 0$.

We apply $A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}$ to the differential equation (2.12) and take the L^2 scalar product with $A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t)$ to get

$$\begin{aligned} &\left(A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}\frac{d}{dt}U_N(t) + A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}AU_N(t), A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t)\right)_{L^2} \\ = &\left(A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}P_N F(U_N(t)) + A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}P_N G(U_N(t),t), A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t)\right)_{L^2}. \end{aligned}$$

Using the equality

$$\begin{aligned} &\left(A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}\frac{d}{dt}U_N(t), A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t)\right)_{L^2} \\ = &\frac{1}{2}\frac{d}{dt}\left|A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t)\right|^2_{L^2} - \left(A^{\frac{s}{2}}(-\Delta)^{\frac{1}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t), A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t)\right)_{L^2} \end{aligned}$$

we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left|A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t)\right|^2_{L^2} + \left(A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}AU_N(t), A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t)\right)_{L^2} \\ = &\left(A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}P_N F(U_N(t)), A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t)\right)_{L^2} \\ &+\left(A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}P_N G(U_N(t),t), A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t)\right)_{L^2} \\ &+\left(A^{\frac{s}{2}}(-\Delta)^{\frac{1}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t), A^{\frac{s}{2}}e^{t(-\Delta)^{\frac{1}{2}}}U_N(t)\right)_{L^2} \end{aligned}$$

Then we use the symmetry of $A^{\frac{1}{2}}$ and for the last term on the right hand side the Cauchy-Schwarz inequality and the Young inequality $ab < \epsilon a^2 + C_\epsilon b^2$ with $C_\epsilon = \frac{1}{4\epsilon}$. Thus we estimate

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}|U_N(t)|^2_{G_\sigma^{s/2}} + |U_N(t)|^2_{G_\sigma^{(s+1)/2}} \\
\leq\ & \epsilon |A^{\frac{s}{2}}(-\Delta)^{\frac{1}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t)|^2_{L^2} + C_\epsilon |U_N(t)|^2_{G_t^{s/2}} \\
& + |(A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} P_N F(U_N(t)), A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t))_{L^2}| \\
& + |(A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} P_N G(U_N(t),t), A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t))_{L^2}|
\end{aligned}
$$

As $|\Delta V|_{L^2} \leq \frac{1}{d_{min}}|AV|_{L^2}$ with $d_{min} = \min(d_1, \dots, d_n)$ we have

$$
|A^{\frac{s}{2}}(-\Delta)^{\frac{1}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t)|^2_{L^2} \leq \frac{1}{d_{min}} |U_N(t)|^2_{G_t^{(s+1)/2}}
$$

and then

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}|U_N(t)|^2_{G_t^{s/2}} + (1 - \frac{\epsilon}{d_{min}})|U_N(t)|^2_{G_t^{(s+1)/2}} \\
\leq\ & C_\epsilon |U_N(t)|^2_{G_t^{s/2}} + |(A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} P_N F(U_N(t)), A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t))_{L^2}| \\
+\ & |(A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} P_N G(U_N(t),t), A^{\frac{s}{2}} e^{t(-\Delta)^{\frac{1}{2}}} U_N(t))_{L^2}| \qquad (2.13)
\end{aligned}
$$

Using the Cauchy-Schwarz inequality and dividing by $|U_N(t)|_{G_t^{s/2}}$ in inequality (2.13), choosing $\epsilon = \frac{d_{min}}{2}$ and using

$$
\begin{aligned}
|P_N F(U_N)|_{G_\sigma^{s/2}} &\leq |F(U_N)|_{G_\sigma^{s/2}} \text{ and} \\
|P_N G(U_N,t)|_{G_\sigma^{s/2}} &\leq |G(U_N,t)|_{G_\sigma^{s/2}}
\end{aligned}
$$

we get

$$
\frac{d}{dt}|U_N(t)|_{G_t^{s/2}} + \frac{1}{2}|U_N(t)|_{G_t^{s+1/2}} \leq \frac{1}{d_{min}} C|U_N(t)|_{G_t^{s/2}} + |F(U_N)|_{G_t^{s/2}} + |G(U_N,t)|_{G_t^{s/2}}.
$$

The nonlinearities can be estimated using (H.1) and by integrating time

$$
\begin{aligned}
& |U_N(t)|_{G_t^{s/2}} + \frac{1}{2}\int_0^t |U_N(\theta)|_{G_\theta^{(s+1)/2}} d\theta \\
\leq\ & |P_N U_0|_{G_0^{s/2}} + \int_0^t \frac{1}{d_{min}} C|U_N(\theta)|_{G_\theta^{s/2}} d\theta + \int_0^t C_s a(C_s|U_N(\theta)|_{G_\theta^{s/2}}) d\theta \\
+\ & \int_0^t C_s b(C_s|U_N(\theta)|_{G_\theta^{s/2}}) d\theta \qquad (2.14)
\end{aligned}
$$

Now we try to bound the first term on the left hand side. As $|P_N U_0|_{G_0^{s/2}} \leq |U_0|_{G_0^{s/2}} \leq C_0|U_0|_{H^s_{per}}$ and as $|U_N(t)|_{G_t^{s/2}}$ is continuous in t, the norm of the solution can be bounded for some time $T_N > 0$:

$$
|U_N(t)|_{G_t^{s/2}} \leq 2|U_0|_{G_0^{s/2}} + 1 \leq 2C_0|U_0|_{H^s_{per}} + 1 = \tilde{M}_1
$$

As the majorising functions are increasing in $\mathbf{R}_+$, we have for $t \in [0, T_N]$ by (2.14):

$$\begin{aligned} & |U_N(t)|_{G_t^{s/2}} \\ \leq \quad & C_0|U_0|_{H^s_{per}} + t\left[\frac{1}{d_{min}} C\tilde{M}_1 + C_s a(C_s\tilde{M}_1) + C_s b(C_s\tilde{M}_1)\right] \end{aligned} \tag{2.15}$$

So we have a bound on $|U_N(t)|_{G_t^{s/2}}$ not depending on N, but we still have to show that T_N is bounded away from 0 for all N. The inequality (2.15) gives also some estimate. When considering $|U_N(t)|_{G_t^{s/2}} \leq 2C_0|U_0|_{H^s_{per}} + 1 = \tilde{M}_1$ we get

$$T^*(|U_0|_{H^s_{per}}) = \frac{C_0|U_0|_{H^s_{per}} + 1}{\frac{1}{d_{min}} C\tilde{M}_1 + C_s a(C_s\tilde{M}_1) + C_s b(C_s\tilde{M}_1)} \tag{2.16}$$

and thus $0 < T^* \leq T_N$ for all N. Hence we have now for $t \in [0, T^*]$

$$\sup_{0 \leq t \leq T^*} |U_N(t)|_{G_t^{s/2}} \leq C. \tag{2.17}$$

with C only depending on $|U_0|_{H^s_{per}}$. Proceeding as in [FT98] there exists a subsequence of U_N that converges to some limit function U strongly in $L^2((0,T^*), H^s_{per})$ and weakly in $L^2((0,T^*), H^{s+1}_{per})$. As in [FT98] one shows that U satisfies (2.9). Using further estimates and arguments as in [Tem88, ch III, sec 1.1] the solution is continuous in time, even $U \in C([0,T^*], H^s_{per}(\Omega, \mathbf{R}^n)) \cap L^2([0,T^*], D(A)$. Hence U satisfies definition 3 and is the unique solution.

It remains to show, that $U(t) \in G_t^{s/2}(\Omega, \mathbf{R}^n)$ is bounded for $t \in [0, T^*]$. We will prove this by contradiction using the estimate (2.17) and the uniqueness of the solution. So suppose $U(t) \in G_t^{s/2}(\Omega, \mathbf{R}^n)$ is not bounded, then there exists an interval $[t_0, t_1]$ and m such that $|P_m U(t)|_{G_t^{s/2}} > C + \epsilon$ but $|U_N(t)|_{G_t^{s/2}} < C$ for all N and $t \in [t_0, t_1]$. Thus we have

$$\begin{aligned} C(t_1 - t_0) \quad & \geq \quad \int_{t_0}^{t_1} |U_N(t)|_{G_t^{s/2}} dt \geq \int_{t_0}^{t_1} |P_m U_N(t) - P_m U(t) + P_m U(t)|_{G_t^{s/2}} dt \\ & \geq \quad \int_{t_0}^{t_1} |P_m U(t)|_{G_t^{s/2}} dt - \int_{t_0}^{t_1} |P_m U_N(t) - P_m U(t)|_{G_t^{s/2}} dt \\ & > \quad (C+\epsilon)(t_1 - t_0) - \int_{t_0}^{t_1} \exp(2t_1\sqrt{1+m^2}) |P_m U_N(t) - P_m U(t)|_{H^s_{per}} dt \\ & > \quad (C + \frac{\epsilon}{2})(t_1 - t_0) \end{aligned}$$

for N large enough as $U_N \to U$ in $L^2([t_0,t_1], H^s_{per})$ and hence in $L^1([t_0,t_1], H^s_{per})$, thus there is a contradiction. Therefore the solution U is bounded in Gevrey classes by $|U(t)|_{G_t^{s/2}} < C = 2C_0|U_0|_{H^s_{per}} + 1$. □

For large time intervals we get the following easy corollary as in [FT98].

Corollary 5 *If $|U(t)|_{H^s_{per}} \leq M_0$ for $t \in [0,T]$, then $U(t) \in G_t^{s/2}(\Omega, \mathbf{R}^n)$ for $t \in [0, T^*]$ and $U(t) \in G_{T^*}^{s/2}(\Omega, \mathbf{R}^n)$ for $t \in [T^*, T]$ with T^* given explicitly by (2.16).*

Proof : We show $U(T^* + t) \in G_{T^*}^{s/2}$ for $0 \leq t \leq \min(T^*, T - T^*)$, the corollary follows by induction. We define $V(\theta) = U(\theta + t)$ for $0 \leq t, \theta \leq T^*$. Then as $|V(0)|_{H^s_{per}} \leq M_0$ we can apply proposition 4 to V gives $V(\theta) \in G_\theta^{s/2}(\Omega, \mathbf{R}^n)$ and hence $U(T^* + t) = V(T^*) \in G_{T^*}^{s/2}(\Omega, \mathbf{R}^n)$. □

Thus we have high regularity in space, as U is real analytic in the spatial variable $x \in \Omega$. After a transient the spatial Fourier coefficients of the solutions decay like $\|j\|^{-s} \exp(-\sigma\|j\|)$. We could also prove high regularity in time following Promislow [Pro91]. If the nonlinearities F, G are analytic in U and t, then U is analytic in t. Time can be extended to a complex strip, but we will not use this high regularity in this paper.

3 Averaging

We prove theorem 1 using Neishtadt's methods [Nei84] and using the high regularity in space, that we proved in proposition 4. Let us describe the idea of the proof first. We split the phase space into a finite dimensional part, where we use a Neishtadt-like coordinate change, and into an infinite dimensional part of exponential decay. The finite dimensional part will be the Galerkin approximation space H_N, where N is chosen depending on h with $N \to \infty$ for $h \to 0$. Using the Neishtadt result, the forcing term can be removed up to exponentially small terms with constants depending on N. Due to Gevrey regularity the influence of the higher modes is also exponentially small with $N \to \infty$. We will prove the theorem by combining both exponential smallness results together for an appropriate coupling of N and h.

In the proof we do not use directly, that the nonlinearities are of superposition-type. In fact we prove a version of theorem 1 in a larger class of nonlinearities F and G. We consider nonlinearities F and G, which are operators on Gevrey classes fulfilling some analyticity conditions.

We will first describe these general assumptions. Then we state theorem 2 which is the version of the main result for the larger class of nonlinearities. Before proving theorem 2 we show that theorem 2 implies theorem 1. We end this section with two remarks on L^2 estimates and smooth dependence on the parameter h.

We consider an equation like (2.1) but with small rapid forcing.

$$\begin{aligned} \frac{\partial}{\partial t}U - D\Delta U &= F(U) + h^p G(U, \frac{t}{h}, h) \\ U(t_0) &= U_0 \end{aligned} \tag{3.1}$$

In a small generalization to (1.1) the amplitude of the rapid forcing has the form some power h^p with $p > 0$. This does not complicate the proof and is needed in other work of the author. In addition to the assumption (H.1) given in section 2 we will need further assumptions on the smoothness of F and G. They are defined on the ball $B_{G_\sigma^{s/2}}(R, 0)$. Let F and G be real analytic, when restricted to the finite dimensional approximation spaces H_N with

$$H_N = \{ \sum_{j \in \mathbf{Z}^d, \|j\| \le N} \begin{pmatrix} U^{1,j} \\ \vdots \\ U^{n,j} \end{pmatrix} \exp(i\frac{2\pi}{l} jx) | U^{l,j} \in \mathbf{C}, U^{l,j} = \overline{U^{l,-j}} \}.$$

Then the restricted and projected nonlinearities

$$P_N F, P_N G : D_N = H_N \cap B_{G_\sigma^{s/2}}(R, 0) \to H^N$$

are real analytic taking $H_N \cong \mathbf{R}^k$ for $k = dim_{\mathbf{R}}(H_N)$. Moreover we assume that the analytic continuation to a complex δ-neighborhood (see figure 2)

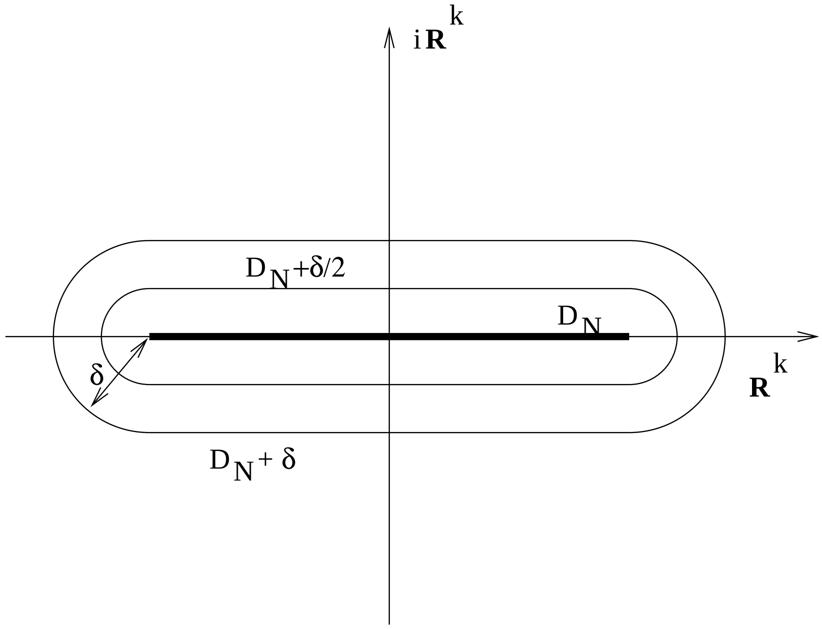

Figure 2: The domain D_N and its complex extensions.

$$D_N + \delta := \{U = \sum_{j \in Z^d, \|j\| \leq N} U^j \exp(i\frac{2\pi}{l}jx) | U^j \in \mathbf{C}^n, \inf_{V \in D_N} |U - V|_{G_\sigma^{s/2}(\Omega, \mathbf{C}^n)} < \delta\}$$

is uniformly bounded in σ, N, t and h. To summarise

$$\begin{array}{l} P_N F, P_N G : D_N \to H_N \text{ is real analytic} \\ \sup_{U_N \in D_N + \delta} |P_N F(U_N)|_{G_\sigma^{s/2}} \leq B_1 \\ \sup_{U_N \in D_N + \delta, t \in \mathbf{R}} |P_N G(U_N, t, h)|_{G_\sigma^{s/2}} \leq B_2 \end{array} \quad \text{(H.2)}$$

Theorem 2 *Assume that the hypotheses (H.1) and (H.2) hold. Let G be periodic in $\tau = \frac{t}{h}$ with period 1 and fix the perturbation order $p > 0$. Suppose for initial values U_0 with $|U_0|_{H^s} \leq M_1$, that the forward orbit remains bounded, i.e. $|S(U_0, t)|_{H^s} \leq M_0 < R$ for $t \in [0, T)$. Then there exists a real analytic and time-periodic change of coordinates for $0 < h < h_1$*

$$U = V + hW(V, \frac{t}{h}, h) \quad (3.2)$$

with the following properties: W is bounded on H^s_{per}. Its image $W(H^s_{per}, \frac{t}{h}, h)$ is finite dimensional for fixed h and $W(., 0, .) = 0$. The transformed nonautonomous terms are exponentially small after a transient, but the equation may contain additional nonlocal terms $\bar{F} = (\bar{f}_1, \ldots, \bar{f}_n)$:

$$\frac{\partial}{\partial t} U(x,t) - D\Delta U(x,t) = F(U(t))(x) + \bar{F}(U(t), h)(x) + \alpha(U(t), \frac{t}{h}, h)(x), \quad (3.3)$$

with

$$\sup_{|V(0)|_{H^s_{per}} < M_1} |\bar{F}(V(t))|_{H^s_{per}} \leq c_3 h^p + c_4 \exp(-\min(t, T^*) h^{-\frac{1}{3}}), \quad (3.4)$$

$$\sup_{|V(0)|_{H^s_{per}}<M_1} |\alpha(V(t))|_{H^s_{per}} \quad \leq \quad c_2 h^p \exp(-c_1 h^{-\frac{1}{3}}) + c_5 h^p \exp(-\min(t,T^*) h^{-\frac{1}{3}}).$$

Here the constants c_1, c_2, c_3, h_1 only depend on B_1 and B_2. The maximal Gevrey exponent T^*, given by proposition 5, and the constants c_4, c_5 depend on the majorising functions a and b in $(H.1)$ and M_0.

Remark 6 *Theorem 2 implies theorem 1.*

Proof : First we have to check that the nonlinearities F and G described in theorem 1 fulfill hypotheses (H.1) and (H.2).

F and G satisfy the assumptions of lemma 2. A majorising power series can be chosen as in equation (2.7) for F and for G uniformly in t. Hence F and G fulfill hypothesis (H.1) with a changed radius for the domain $B_{G^{s/2}_\sigma}(R, 0)$.

When we restrict and project F and G to H_N, they are power series of the Fourier-coefficients $\alpha^N_j = (\alpha^N_{j,1}, \dots, \alpha^N_{j,n})^T \in \mathbf{R}^n, j \in \mathbf{Z}^d, \|j\| \leq N$ of the functions

$$U_N = \sum_{\|j\| \leq N} \alpha^N_j \exp(i\frac{2\pi}{l} jx).$$

This can be seen, when we insert U_N into F:

$$F(U_N) = \sum_{k \in \mathbf{N}^n_0} \beta_k U^k_N$$

Taking a single monomial $\beta_k U^k_N$ first, we derive

$$\beta_k U^k_N = \beta_k \sum_{j \in \mathbf{Z}^d} (\sum_{\substack{j_1, \dots, j_n \in \mathbf{Z}^d \\ \|j_1\|, \dots, \|j_n\| \leq N \\ k_1 j_1 + \dots + k_n j_n = j}} (\alpha^N_{j_1,1})^{k_1} \cdot \dots \cdot (\alpha^N_{j_n,n})^{k_1} \exp(i\frac{2\pi}{l} jx))$$

with $|U^k_N|_{G^{s/2}_\sigma} \leq C^{|k|}_s |U_N|^{|k|}_{G^{s/2}_\sigma}$ by lemma 1. When considering the projection to a single Fourier coefficient in the range we get

$$\begin{aligned} &< F(U_N), \exp(-i\frac{2\pi}{l} jx) > \\ = &\sum_{k \in \mathbf{N}^n_0} \beta^k \left(\sum_{j_1, \dots, j_n \in \mathbf{Z}^d, \|j_1\|, \dots, \|j_n\| \leq N, k_1 j_1 + \dots + k_n j_n = j} (\alpha^N_{j_1,1})^{k_1} \cdot \dots \cdot (\alpha^N_{j_n,n})^{k_n} \right) \end{aligned}$$

This power series converges for $|\alpha^N_{j,l}| \leq R$ by lemma 2. Hence $P_N F$ is real analytic when restricted to $H_N \cap B_{G^{s/2}_\sigma}(R, 0)$. In the same manner this can be shown for G.

As $F, G : B_{\mathbf{R}^n}(R, 0) \to \mathbf{R}^n$ are supposed to be entire on $\mathbf{R}^n$, they are bounded on any bounded set in $\mathbf{C}^n$. Hence we can extend the domain of the α^N_j to a complex neighborhood of $D_N = B_{G^{s/2}_\sigma}(R, 0) \cap H_N$. For the bounds of F and G in $G^{s/2}_\sigma$, we just have to take $G^{s/2}_\sigma$-norms in the domain using lemma 2. The inequality $|F(U)|_{G^{s/2}_\sigma} \leq C_s a(C_s |U|_{G^{s/2}_\sigma})$ also holds for complex extended U. Then $|F(U_N)|_{G^{s/2}_\sigma}$ is bounded on $D_N + \delta$ by $C_s a(C_s(|U|_{G^{s/2}_\sigma} + \delta))$. Similarly we get a bound for G. Thus $(H.2)$ also holds.

The existence of global solutions and the regularisation assures, that the forward orbit of any ball in H^s_{per} remains bounded for any finite time. The assertions of theorem 2 imply those of theorem 1. □

3.1 Proof of Theorem 2

The proof can be divided into four steps. In the first two steps we analyze a finite dimensional problem by reducing to the Galerkin approximation

$$\begin{aligned} \dot{U}_N + AU_N &= P_N F(U_N) + h^p P_N G(U_N, \frac{t}{h}, h) \\ U_N(0) &= P_N U_0 \in H_N, \end{aligned}$$

where $A = -P_N D\Delta$ and where we choose the parameter N depending on h. When λ_N is the largest eigenvalue of $-Id\Delta$ restricted to the Galerkin approximation space H_N, we couple N and h by $\lambda_N^{1+\beta} h \leq 1$, where $\beta > 0$ is to be chosen later. Step 1 and step 2 are adapted from the Neishtadt theorem for the finite dimensional case. In step 1 we make successive formal coordinate changes, such that the nonautonomous terms are formally of higher order in h in the transformed equation

$$\dot{V}_N + AV_N = P_N F(V_N) + \bar{F}(V_N, h) + \alpha(V_N, \frac{t}{h}, h). \tag{3.5}$$

Step 2 will give estimates uniform in h and $N(h)$. We will perform $r \sim \frac{1}{h^\gamma}$ successive coordinate changes, where γ is chosen in the proof. In the transformed equation the nonautonomous terms are exponentially small in h. In step 3 and step 4 we will consider again the full infinite dimensional problem. We do the formal coordinate change in step 3. In step 4 we will finally prove the estimates and hence justify the use of the finite dimensional approximation.

Step 1: Formal coordinate changes

We describe the formal coordinate changes needed to remove nonautonomous terms. For a moment we suppress the dependence of U on N. The situation after j coordinate changes is given by

$$\dot{U} + AU = P_N F(U) + \bar{F}_j(U, h) + \alpha_j(U, \frac{t}{h}, h) \tag{3.6}$$

with $< \alpha_j > (U, h) = \int_0^h \alpha_j(U, \frac{\theta}{h}, h) d\theta = 0$ and U in a complex extended domain D_N^j with $D_N + \frac{\delta}{2} \subset D_N^j \subset D_N + \delta$, see figure 2. Before performing the first coordinate change, we have $\bar{F}_0 = h^p < G >$ and $\alpha_0 = h^p G - h^p < G >$.

Starting with (3.6) the next coordinate change is given by

$$U = V + hW_j(V, \tau, h) \tag{3.7}$$

with W periodic in $\tau = \frac{t}{h}$ with period 1. Substitution into (3.6) yields to

$$\begin{aligned} & \dot{V} + h\frac{\partial}{\partial V} W_j(V, \tau, h)\dot{V} + \frac{\partial}{\partial \tau} W_j(V, \tau, h) + A(V + hW_j(V, \tau, h)) \\ = \; & P_N F(V + hW_j(V, \tau, h)) + \bar{F}_j(V + hW_j(V, \tau, h), h) + \alpha_j(V + hW_j(V, \tau, h), \tau, h) \end{aligned}$$

A formal Taylor expansion in V gives

$$\begin{aligned} \dot{V} \; = \; & \left(I + h\frac{\partial}{\partial V} W_j(V, \tau, h)\right)^{-1} \Big\{ P_N F(V) + \frac{\partial P_N F(V)}{\partial V} hW_j(V, \tau, h) + \bar{F}_j(V, h) \\ & + \frac{\partial \bar{F}_j(V, h)}{\partial V} hW_j(V, \tau, h) - AV - hAW_j(V, \tau, h) \\ & + \alpha_j(V, \tau, h) + \frac{\partial \alpha_j(V, \tau, h)}{\partial V} hW_j(V, \tau, h) + \text{h.o.t.} - \frac{\partial}{\partial \tau} W_j(V, \tau, h) \Big\} \\ =: \; & -AV + P_N F(V) + \bar{F}_j(V) + a \end{aligned} \tag{3.8}$$

The term of lowest order in h, which are time-dependent, is $\alpha_j(V,\tau,h)$. To remove this term we choose

$$W_j(V,\tau,h) = \int_0^\tau \alpha_j(V,\theta,h)d\theta.$$

Then we choose

$$\begin{aligned} \bar{F}_{j+1}(V) &= \bar{F}_j(V) + < a >, \\ \alpha_{j+1}(V,\tau,h) &= a - < a > . \end{aligned} \quad (3.9)$$

Step 2: Estimates for the finite dimensional system

We give rigorous estimates for the formal procedure of step 1. Suppose r substitutions are made altogether for a fixed h. The domain D_N^j after j substitutions is given by $D_N^j = D_N^0 - jK(h)h$ where $K(h)$ is chosen later and $D_N^0 = D_N + \delta$.

We use the notation $\|f\|_{G_\sigma^{s/2},D+\gamma} = \sup_{U\in D+\gamma}|f(U)|_{G_\sigma^{s/2}}$ for some complex extension $D+\gamma$ of $D \subset G_\sigma^{s/2}(\Omega,\mathbf{R})$. Then by construction we have

$$\begin{aligned} \|\alpha_0\|_{G_\sigma^{s/2},D_N+\delta} &\leq 2B_2h^p \\ \|\bar{F}_0\|_{G_\sigma^{s/2},D_N+\delta} &\leq B_2h^p \end{aligned}$$

uniformly in σ. We will show inductively for $1 \leq j \leq r$:

$$\|\bar{F}_j\|_{G_\sigma^{s/2},D_N+\delta/2} \leq B_1 \quad (3.10)$$

$$\|\alpha_j\|_{G_\sigma^{s/2},D_N+\delta/2} \leq M_j \text{ with } M_j = 2^{-j}B_2h^p. \quad (3.11)$$

We will choose h_1 and K such that r substitutions are defined for $0 < h < h_1$, $V \in D_N^{r+1} = D_N^r - Kh \neq \emptyset$ and the inductive assumptions (3.10) and (3.11) are fulfilled.

For the estimates we need the lemma about Cauchy estimates with arbitrary norms:

Lemma 7 *Let $f : \Omega \subset \mathbf{C}^k \to \mathbf{C}^k$ be analytic and $\|f\|_\Omega = \sup_{u\in\Omega}\|f(u)\|$. Then $\|\frac{\partial f}{\partial x}\|_{\Omega-\delta} \leq \frac{\|f\|_\Omega}{\delta}$ for any norm $\|.\|$ on $\mathbf{C}^k$.*

The lemma follows directly from the usual one-dimensional Cauchy formula. For any $u \in \Omega - \delta$ we take a circle in the complex plane defined by $y - u = zx, z \in \mathbf{C}, |z| = \delta$, letting without restriction $\|x\| = 1$. Then we have

$$\begin{aligned} f(u) &= \frac{1}{2\pi i}\oint_{y-u=zx,z\in\mathbf{C},|z|=\delta} \frac{f(y)}{u-y}dy \\ \frac{\partial f}{\partial x}(u) &= \frac{1}{2\pi i}\oint_{y-u=zx,z\in\mathbf{C},|z|=\delta} f(y)\frac{\partial}{\partial x}\left(\frac{1}{u-y}\right)dy. \end{aligned}$$

Thus

$$\|\frac{\partial f}{\partial x}\|_{\Omega-\delta} \leq \frac{1}{2\pi}2\pi\delta\frac{\|f\|_\Omega}{\|y-u\|^2} = \frac{M}{\delta}$$

and the lemma is proved. □

To continue the proof of the theorem, assume for induction that (3.10) and (3.11) hold for j. To simplify notation we suppress the arguments of W and the dependence on time and parameters N,h in the functions. All norms are $G_\sigma^{s/2}$ norms.

Then for the j^{th} coordinate change we obtain

$$\|hW_j\|_{D_N^j} \leq M_jh, \quad (3.12)$$

which yields by lemma 7 to

$$\|h\frac{\partial W_j}{\partial u}\|_{D_N^j-Kh} \le \frac{M_j}{K}. \tag{3.13}$$

We estimate the higher order term a in (3.7).

$$\begin{aligned}
\|a\|_{D_N^j-Kh} &\le \| \left[I + h\frac{\partial}{\partial V}W_j\right]^{-1} \left\{P_N F(V+hW_j) + \bar{F}_j(V+hW_j)\right. \\
&\quad \left. +\alpha_j(V+hW_j) - \alpha_j(V) - A(V+hW_j)\right\} \\
&\quad - \left[-AV + P_N F(V) + \bar{F}_j(V)\right] \|_{D_N^j-Kh} \\
&\le \| \left[I + h\frac{\partial}{\partial V}W_j\right]^{-1} \{-hAW_j + P_N F(V+hW_j) - P_N F(V) \\
&\quad +\bar{F}_j(V+hW_j) - \bar{F}_j(V) + \alpha_j(V+hW_j) - \alpha_j(V) \\
&\quad -h\frac{\partial}{\partial V}W_j \left[-AV + P_N F(V) + \bar{F}_j(V)\right]\} \|_{D_N^j-Kh} \\
&\quad \text{Neumann series and mean value theorem give} \\
&\le \sum_{k=0}^{\infty} \|h\frac{\partial}{\partial V}W_j\|^k_{D_N^j-Kh} \left\{ \|hW_j\|_{D_N^j} \left[\lambda_N + \|\frac{\partial}{\partial V}P_N F\|_{D_N^j-Kh}\right.\right. \\
&\quad \left. +\|\frac{\partial}{\partial V}\bar{F}_j\|_{D_N^j-Kh} + \|\frac{\partial}{\partial V}\alpha_j\|_{D_N^j-Kh}\right] \\
&\quad \left. +\|h\frac{\partial}{\partial V}W_j\|_{D_N^j-Kh} \left[\lambda_N\|V\|_{D_N^j} + \|P_N F\|_{D_N^j} + \|\bar{F}_j\|_{D_N^j}\right]\right\} \\
&\quad \text{lemma 7 gives} \\
&\le 2hM_j\left[\lambda_N + \frac{B_1}{Kh} + \frac{2B_1}{Kh}\right] + \frac{M_j}{K}\left[\lambda_N R + B_1 + 2B_1\right]
\end{aligned}$$

Using $\lambda_N^{1+\beta}h \le 1$, which is equivalent to $\lambda_N \le h^{\frac{-1}{1+\beta}}$, and setting

$$K(h) = \tilde{K}h^{\frac{-1}{1+\beta}} \tag{3.14}$$

we get

$$\|a\|_{D_N^j-Kh} \le M_j\left[2hh^{\frac{-1}{1+\beta}} + 6\frac{B_1}{\tilde{K}h^{\frac{-1}{1+\beta}}} + \frac{R}{\tilde{K}} + 3\frac{B_1}{\tilde{K}h^{\frac{-1}{1+\beta}}}\right] \le \frac{1}{4}M_j$$

for $\tilde{K}$ large enough. Therefore

$$\|\alpha_{j+1}\|_{D_N^{j+1}} < \frac{M_j}{2} = M_{j+1}$$

and

$$\|\bar{F}_{j+1} - \bar{F}_j\|_{D_N^{j+1}} < \frac{M_j}{4}.$$

Hence

$$\begin{aligned}
\|\bar{F}_{j+1}\|_{D_N^{j+1}} &\le \sum_{k=0}^{j}\|\bar{F}_{k+1} - \bar{F}_k\|_{D_N^{j+1}} + \|\bar{F}_0\|_{D_N^{j+1}} \\
&\le \frac{1}{4}\sum_{k=0}^{j} M_k + B_2h^p \le \frac{1}{2}B_2h^p + B_2h^p \le B_1
\end{aligned} \tag{3.15}$$

for h_1 small enough. Thus the inductive statements (3.10) and (3.11) are satisfied for $j+1$ for such a h_1 and the above choice of K . So we can carry out the coordinate changes as long as $D_N^r \neq \emptyset$, i.e.

$$r = \frac{\delta}{2K(h)h} = \frac{\delta}{2\tilde{K}h^{\frac{-1}{1+\beta}}h} = \frac{\delta}{2\tilde{K}h^{\frac{\beta}{1+\beta}}}$$

Letting $\alpha_* = \alpha_{r(h)}(V, \frac{t}{h}, h, N(h))$ and $\bar{F}_* = \bar{F}_{r(h)}(V, h, N(h))$ we get estimates for $G_\sigma^{s/2}$-norms uniform in $h \to 0, N(h) \to \infty$:

$$\|\alpha_*\|_{D_N+\delta/2, G_\sigma^{s/2}} < 2^{-r} B_2 h^p < c_2 h^p \exp(-c_1 h^{-\frac{\beta}{1+\beta}}), \tag{3.16}$$

$$\|\bar{F}_*\|_{D_N+\delta/2, G_\sigma^{s/2}} < Ch^p. \tag{3.17}$$

Step 3: Transformation of the complete system

Next we deal with the complete infinite dimensional system. We linearly extend the coordinate change to other Galerkin modes by

$$U = V + hW(P_N V, \frac{t}{h}, h, N)$$

where $W(P_N V, \frac{t}{h}, h, N) \in H_N$ and $N = N(h)$ is chosen maximally such that $\lambda_N^{1+\beta} h \leq 1$ holds. Thus in the new coordinates we get, when suppressing arguments $\frac{t}{h}$ and h of W

$$\begin{aligned} & \dot{V} + h\frac{\partial}{\partial P_N V} W(P_N V)\dot{V} + \frac{\partial}{\partial \tau} W(P_N V) + A(V + hW(P_N V)) \\ = \;& F(V + hW(P_N V)) + h^p G(V + hW(P_N V), \frac{t}{h}, h). \end{aligned}$$

Solving for $\dot{V}$ gives

$$\begin{aligned} \dot{V} \;=\;& \left[I + h\frac{\partial}{\partial P_N V} W(P_N V) \right]^{-1} \{-A(V + hW(P_N V)) + F(V + hW(P_N V)) \\ & + h^p G(V + hW(P_N V), \frac{t}{h}, h) - \frac{\partial}{\partial \tau} W(P_N V, \frac{t}{h}, h)\}. \end{aligned}$$

Splitting $G_\sigma^{s/2}(\Omega, \mathbf{R}^n)$ into H_N and

$$H_N^\perp = \{U \in G_\sigma^{s/2} | U = \sum_{\|j\|>N} U^j \exp\left(i\frac{2\pi}{l} j.x\right)\}$$

and using

$$\left[I + h\frac{\partial}{\partial P_N V} W(P_N V) \right]^{-1}_{|H_N^\perp} = I_{|H_N^\perp}$$

we obtain

$$\begin{aligned} & \dot{V} + A(I - P_N)V \\ = \;& (I - P_N) \left\{ F(V + hW(P_N V)) + h^p G(V + hW(P_N V), \frac{t}{h}, h) \right\} \\ & + P_N \left[I + h\frac{\partial}{\partial P_N V} P_N W(P_N V) \right]^{-1} P_N\{-A(V + hW(P_N V)) \end{aligned}$$

$$
\begin{aligned}
&+F(V+hW(P_NV))+h^pG(V+hW(P_NV),\frac{t}{h},h)-\frac{\partial}{\partial\tau}P_NW(V,\frac{t}{h},h)\} \\
= \quad & (I-P_N)\left\{F(V+hW(P_NV))+h^pG(V+hW(P_NV),\frac{t}{h},h)\right\} \\
&+P_N\left\{-AV+F(V)+\bar F_*(V,h,N)+\alpha_*(V,\frac{t}{h},h,N)\right\} \\
&+P_N\left[I+h\frac{\partial}{\partial P_NV}W(P_NV)\right]^{-1}P_N\{F(V+hW(P_NV)) \\
&\quad +h^pG(V+hW(P_NV),\frac{t}{h},h)-F(P_NV+hW(P_NV)) \\
&\quad -h^pG(P_NV+hW(P_NV),\frac{t}{h},h)\}.
\end{aligned}
$$

Thus we have an equation as in (1.3)

$$
\dot V+AV=F(V)+\bar F(V,h)+\alpha(V,\frac{t}{h},h) \tag{3.18}
$$

with

$$
\bar F(V,h) = \bar F_*(V,h,N(h))+<b_1+b_2> \tag{3.19}
$$

$$
\alpha(V,\frac{t}{h},h) = \alpha_*(V,\frac{t}{h},h,N(h))+b_1+b_2-<b_1+b_2>, \tag{3.20}
$$

where we get additional correction terms. First there is an error in the higher Galerkin modes ($> N$), as they were neglected in the coordinate transformation

$$
b_1=(I-P_N)\left\{F(V+hW(P_NV))-F(V)+h^pG(V+hP_NW(V),\frac{t}{h},h)\right\}. \tag{3.21}
$$

Secondly there is an error in the lower modes ($\le N$) due to the neglected influence of the higher Galerkin modes to the lower ones

$$
\begin{aligned}
b_2 \quad = \quad & \left[I+h\frac{\partial}{\partial P_NV}W(P_NV)\right]^{-1}P_N\left\{F(V+hW(P_NV))-F(P_NV+hW(P_NV))\right. \\
&+h^pG(V+hW(P_NV),\frac{t}{h},h)-h^pG(P_NV+hW(P_NV),\frac{t}{h},h)\}.
\end{aligned} \tag{3.22}
$$

Step 4: Estimating the complete system

To prove the estimate (3.4) and (3.5) we will show, that (3.18) has Gevrey regular solutions. Thus we have to check the assumptions (H.1) of proposition 4 for the nonlinearities $\bar F(V,h,N)+\alpha(V,\frac{t}{h},h,N)+b_1+b_2$.

The nonlinearities $\bar F_*(V,h,N)+\alpha_*(V,\frac{t}{h},h,N)$, which were defined by the finite dimensional part, fulfill these estimates on $B_{G_\sigma^{\frac{s}{2}}}(R,0)$ by (3.16) and (3.17).

It remains to show, that

$$
F(.+hW(P_N.)),G(.+hP_NW(.),\frac{t}{h},h):B_{G_\sigma^{\frac{s}{2}}}(R,0)\to G_\sigma^{\frac{s}{2}}(\Omega,\mathbf{R}^n)
$$

is continuously differentiable and bounded. $V+hW(V)$ is given by the successive coordinate changes $U=V+hW_j(V)$. Using (3.12) we get that $V+hW(V)$ is a near-identity and differentiable coordinate change uniformly for every $G_\sigma^{\frac{s}{2}}$ norm. The linear mapping

$$
\left[I+h\frac{\partial}{\partial P_NV}P_NW(V)\right]^{-1}P_N
$$

in b_2 is near to the identity by (3.13). Thus the estimates on F and G hold for b_1 and b_2 too.

Hence (3.18) has Gevrey regular solutions and we can estimate $|\alpha(V(t),\frac{t}{h},h)|_{H^s_{per}}$ and $|\bar{F}(V(t),h)|_{H^s_{per}}$. We start with $\bar{F}(V,h) = \bar{F}_*(V,h,N) + < b_1 + b_2 >$. For the finite dimensional part we have $|\bar{F}_*(V,h,N)|_{H^s_{per}} \le c_3 h^p$ by (3.17). Estimating $< b_1 >$ we obtain for $V \in H^s_{per}(\Omega, \mathbf{R}^n)$:

$$
\begin{aligned}
& |< b_1 >|_{H^s_{per}} \\
\le \quad & |(I-P_N)\left\{F(V+hW(P_NV)) - F(V) + h^pG(V+hP_NW(V),\frac{t}{h},h)\right\}|_{H^s_{per}} \\
\le \quad & \sup_{U=V+rW(P_NV),r\in[0,h]} |F'(U)|_{L(H^s_{per},H^s_{per})}|hW(P_NV)|_{H^s_{per}} + h^pB_1 \le Ch^p
\end{aligned}
$$

because $|hW(P_NV)| \le Ch^{p+1}$. Next we deal with the term $< b_2 >$ created by the Galerkin approximation in the finite dimensional part. Letting $V = V(t)$ and $K = \{rV + (1-r)P_NV(t) + hW(P_NV(t)), r \in [0,1]\}$, we get

$$
\begin{aligned}
& |< b_2 >|_{H^s_{per}} \\
\le \quad & |\left[I + h\frac{\partial}{\partial P_NV}P_NW(V)\right]^{-1} P_N \left\{F(V+hW(V)) + h^pG(V+hP_NW(V),\frac{t}{h},h)\right. \\
& \left. -F(P_NV+hW(P_NV)) - h^pG(P_NV+hW(P_NV),\frac{t}{h},h)\right\}|_{H^s_{per}} \\
\le \quad & 2(\sup_K |F'(U)|_{L(H^s_{per},H^s_{per})}|V(t)-P_NV(t)|_{H^s_{per}} \\
& +\sup_K |G'(U(t))|_{L(H^s_{per},H^s_{per})}|V(t)-P_NV(t)|_{H^s_{per}}) \\
\le \quad & C|V(t)-P_NV(t)|_{H^s_{per}} \le C\exp\left(-\min(t,T^*)\sqrt{\lambda_{N+1}}\right),
\end{aligned}
$$

where we used the boundedness of $V(t)$ in the Gevrey norm $G^{s/2}_{\min(t,T^*)}$ and thus if we subtract the projection to H_N

$$
|V(t)-P_NV(t)|_{H^s_{per}} \le |V(t)|_{G^{s/2}_{\min(t,T^*)}} \exp\left(-\min(t,T^*)\sqrt{\lambda_{N+1}}\right). \tag{3.23}
$$

Next we need a more careful analysis for the nonautonomous part:

$$
|\alpha(V(t),\frac{t}{h},h)|_{H^s_{per}} = |\alpha_*(V,\frac{t}{h},h,N) + b_1 + b_2 - < b_1 + b_2 >|_{H^s_{per}}.
$$

For the first part we have by (3.16):

$$
|\alpha_*(V,\frac{t}{h},h,N)|_{H^s_{per}} \le c_2h^p\exp(-c_1h^{-\frac{\beta}{1+\beta}})
$$

For $b_1 - < b_1 >$ we use the same analysis as for $< b_1 >$ above, except we note that $V(t)$ is bounded in the Gevrey norm $G^{s/2}_{\min(t,T^*)}$. Hence $b_1(V(t)) \in G^{s/2}_{\min(t,T^*)}(\Omega,\mathbf{R}^n)$ and thus with $K = \{V + rW(P_NV), r \in [0,h]\}$ and $\sigma(t) = \min(t,T^*)$

$$
\begin{aligned}
|b_1 - < b_1 >|_{H^s_{per}} \quad \le \quad & 2|(I-P_N)\{F(V(t)+hW(P_NV(t))) \\
& -F(V(t)) + h^pG(V(t)+hW(P_NV(t)),\frac{t}{h},h)\}|_{H^s_{per}}
\end{aligned}
$$

$$\begin{aligned}
&\le \left(\sup_{U\in K} |F'(U)|_{L(G^{s/2}_{\sigma(t)}, G^{s/2}_{\sigma(t)})} |hW(P_NV)|_{G^{s/2}_{\sigma(t)}} + h^p B_1 \right) \\
&\quad \cdot \exp\left(-\sigma(t)\sqrt{\lambda_{N+1}}\right) \\
&\le Ch^p \exp\left(-\sigma(t)\sqrt{\lambda_{N+1}}\right) = Ch^p \exp\left(-\min(t,T^*)\sqrt{\lambda_{N+1}}\right).
\end{aligned}$$

For $b_2(V(t))- < b_2(V(t)) >$ we get

$$\begin{aligned}
&|b_2 - < b_2 > |_{H^s_{per}} \\
\le\ & | \left[I + h\frac{\partial}{\partial P_N V} W(P_NV, \frac{t}{h}, h)\right]^{-1} \\
&\cdot P_N \left\{ F(V + hW(P_NV)) + h^p G(V + hW(P_NV), \frac{t}{h}, h) \right. \\
&\left. -F(P_NV + hW(P_NV)) - h^p G(P_NV + hW(P_NV), \frac{t}{h}, h) \right\} - \int_0^1 b_2(V,\theta) d\theta |_{H^s_{per}} \\
&\text{adding and subtracting a partially averaged term gives} \\
=\ & | \left[I + h\frac{\partial}{\partial P_N V} W(P_NV, \frac{t}{h}, h)\right]^{-1} \\
&\cdot P_N \left\{ F(V + hW(P_NV)) + h^p G(V + hW(P_NV), \frac{t}{h}, h) \right. \\
&\quad \left. -F(P_NV + hW(P_NV)) - h^p G(P_NV + hW(P_NV), \frac{t}{h}, h) \right\} \\
&- \int_0^1 \left[I + h\frac{\partial}{\partial P_N V} W(P_NV, \theta, h)\right]^{-1} d\theta \\
&\cdot P_N \left\{ F(V + hW(P_NV)) + h^p G(V + hW(P_NV), \frac{t}{h}, h) \right. \\
&\quad \left. -F(P_NV + hW(P_NV)) - h^p G(P_NV + hW(P_NV), \frac{t}{h}, h) \right\} \\
&+ \int_0^1 \left[I + h\frac{\partial}{\partial P_N V} W(P_NV, \theta, h)\right]^{-1} d\theta \\
&\cdot P_N \left\{ F(V + hW(P_NV)) + h^p G(V + hW(P_NV), \frac{t}{h}, h) \right. \\
&\quad \left. -F(P_NV + hW(P_NV)) - h^p G(P_NV + hW(P_NV), \frac{t}{h}, h) \right\} \\
&- \int_0^1 \left[I + h\frac{\partial}{\partial P_N V} W(P_NV, \theta, h)\right]^{-1} \\
&\cdot P_N \left\{ F(V + hW(P_NV)) + h^p G(V + hW(P_NV), \theta, h) \right. \\
&\quad \left. -F(P_NV + hW(P_NV)) - h^p G(P_NV + hW(P_NV), \theta, h) \right\} d\theta |_{H^s_{per}} \\
\le\ & \left| \left[I + h\frac{\partial}{\partial P_N V} W(P_NV, \frac{t}{h}, h)\right]^{-1} - \int_0^1 \left[I + h\frac{\partial}{\partial P_N V} W(P_NV(t), \theta, h)\right]^{-1} d\theta \right|_{H^s_{per}} \\
&\cdot \left| P_N \left\{ F(V + hW(P_NV)) + h^p G(V + hW(P_NV), \frac{t}{h}, h) \right. \right.
\end{aligned}$$

$$
\begin{aligned}
& -F(P_N V + hW(P_N V)) - h^p G(P_N V + hW(P_N V), \frac{t}{h}, h)\Big\}\Big|_{H^s_{per}} \\
& +\Big|\int_0^1 \left[I + h\frac{\partial}{\partial P_N V} W(P_N V, \theta, h)\right]^{-1} \\
& \cdot P_N \Big\{ F(V + hW(P_N V, \frac{t}{h}, h)) + h^p G(V + hW(P_N V, \frac{t}{h}, h), \frac{t}{h}, h) \\
& \quad -F(P_N V + hW(P_N V, \frac{t}{h}, h)) - h^p G(P_N V + hW(P_N V, \frac{t}{h}, h), \frac{t}{h}, h) \\
& \quad -F(V + hW(P_N V, \theta, h)) - h^p G(V + hW(P_N V, \theta, h), \theta, h) \\
& \quad +F(P_N V + hW(P_N V, \theta, h)) + h^p G(P_N V + hW(P_N V, \theta, h), \theta, h)\Big\}\, d\theta\Big|_{H^s_{per}} \\
\leq\ & C h^p |b_2|_{H^s_{per}} \\
& +\Big|\int_0^1 \left[I + h\frac{\partial}{\partial P_N V} W(P_N V, \theta, h)\right]^{-1} P_N\{ h^p G(V + hW(P_N V), \frac{t}{h}, h) \\
& -h^p G(P_N V + hW(P_N V), \frac{t}{h}, h) - h^p G(V + hW(P_N V), \theta, h) \\
& +h^p G(P_N V + hW(P_N V), \theta, h) \\
& +(F(V + hW(P_N V, \frac{t}{h}, h)) - F(V + hW(P_N V, \theta, h))) \\
& -(F(P_N V + hW(P_N V, \frac{t}{h}, h)) - F(P_N V + hW(P_N V, \theta, h)))\} d\theta\Big|_{H^s_{per}}
\end{aligned}
$$

the last two lines are both of order h^p and the difference of these lines can still be estimated as in the analysis of $\bar{F}$

$$
\leq\ C h^p \exp\left(-\min(t, T^*)\sqrt{\lambda_{N+1}}\right) = C h^p \exp\left(-\min(t, T^*) h^{\frac{-1}{2(1+\beta)}}\right),
$$

because $\sqrt{\lambda_{N+1}} \geq h^{\frac{-1}{2(1+\beta)}}$. When we balance $\exp(-c_1 h^{-\frac{\beta}{1+\beta}})$ and $\exp(-\min(t, T^*) h^{\frac{-1}{2(1+\beta)}})$ we are choosing $\beta = \frac{1}{2}$. Then the estimates are of order $\exp(-ch^{-\frac{1}{3}})$ and $N(h) = \sqrt{\lambda_{N(h)}} = \left[h^{-\frac{1}{3}}\right]$. This gives the estimates (3.4,3.5). □

Remark 8 *L^2 estimates of $\bar{F}$ and α.*

The above estimates on b_1 and b_2 can be improved, when we consider the L^2 norm instead of H^s norms. We use a variant of (3.23):

$$
\begin{aligned}
|V(t) - P_N V(t)|_{L^2} &\leq C\lambda_{N+1}^{-\frac{s}{2}} |V(t) - P_N V(t)|_{H^s_{per}} \\
&\leq C|V(t)|_{G^{s/2}_{\min(t,T^*)}} \lambda_{N+1}^{-\frac{s}{2}} \exp\left(-\min(t, T^*)\sqrt{\lambda_{N+1}}\right) \\
&= C|V(t)|_{G^{s/2}_{\min(t,T^*)}} h^{\frac{s}{3}} \exp\left(-\min(t, T^*) h^{-\frac{1}{3}}\right),
\end{aligned}
$$

where we also used the choice $\lambda_{N+1}^{-\frac{1}{2}} \leq h^{\frac{1}{3}}$. This proves (1.5,1.7).□

Remark 9 *If $V \in G^{s/2}_\sigma$ and if G is smooth in h, then $\bar{F}(V, h)$ and $\alpha(V, \frac{t}{h}, h)$ can be chosen to be smooth in h.*

The coordinate changes in the above proof are not continuous in h, even in the finite dimensional part $\bar{F}_*$ and α_*. The number of averaging steps $[r]$ with $r = \frac{\delta}{2K} h^{-\frac{1}{1+\beta}}$ increases discontinuously for $h \to 0$, as [.] denotes the discontinuous integer part of r. This problem can be solved, when the additional terms are introduced using smooth cut-off functions

$$\chi_k(\tau) = \begin{cases} 0 & \text{for } \tau \le k \\ 1 & \text{for } \tau \ge k+1. \end{cases}$$

When we need $[r]$ averaging steps, then we compute an additional coordinate change $Id + \chi_{[r(h)]}(r(h))W_{[r(h)]+1}$, which only slightly affects the estimates. Then our coordinate change and hence the finite dimensional parts of $\bar{F}, \alpha$ are smooth in h for $h > 0$. Similarly we introduce the increasing number of Galerkin modes using smooth cut-off functions to get smoothness for $h > 0$ in the correction terms b_1, b_2 (3.21,3.22) too.

Next we consider the regularity at $h = 0$ for the finite dimensional part. We briefly sketch, how one can estimate $\left(\frac{d}{dh}\right)^m \bar{F}$. One uses similar estimates on $\left(\frac{d}{dh}\right)^m a$ as in step 2 for a to uniformly bound $\left(\frac{d}{dh}\right)^m \bar{F}_j$ and $\left(\frac{d}{dh}\right)^m \alpha_j$ in each step.

As an example we sketch the first derivative. First we change the estimates on $\|W_j\|_{D_N^j} \le M_j = 2^{-j}B_2h^p$. By performing one more standard averaging step, we get $\|W_j\|_{D_N^j} \le \tilde{M}_j = 2^{-j}\tilde{B}_2h^{p+1}$ maybe on a smaller domain. Then we will prove inductively

$$\begin{aligned} \|\frac{d}{dh}\bar{F}_j\|_{D_N+\frac{\delta}{2}} &\le B_1 \\ \|\frac{d}{dh}\alpha_j\|_{D_N+\frac{\delta}{2}} &\le \tilde{M}_j = 2^{-j}\tilde{B}_2h^p \end{aligned}$$

just as in step 2 of the proof

$$\begin{aligned} \|\frac{d}{dh}a\|_{D_N^j-Kh} \le\ & \|-\left[I + h\frac{\partial}{\partial V}W_j\right]^{-2}\left(\frac{\partial}{\partial V}W_j + h\frac{\partial}{\partial h}\frac{\partial}{\partial V}W_j\right) \\ & \cdot\left\{P_NF(V+hW_j) + \bar{F}_j(V+hW_j) + \alpha_j(V+hW_j)\right. \\ & \quad \left. -P_NF(V) - \bar{F}_j(V) - \alpha_j(V) - hAW_j\right\} \\ & + \left[I + h\frac{\partial}{\partial V}W_j\right]^{-1}\{D_V[P_NF(V+hW_j) + \bar{F}_j(V+hW_j) \\ & \quad +\alpha_j(V+hW_j) - A]\cdot\left(W_j + h\frac{\partial}{\partial h}W_j\right) \\ & + \frac{\partial}{\partial h}\left(\bar{F}_j(V+hW_j) + \alpha_j(V+hW_j)\right) - \frac{\partial}{\partial h}\left(\bar{F}_j(V) + \alpha_j(V)\right)\} \\ & + \left(\frac{\partial}{\partial V}W_j + h\frac{\partial}{\partial h}\frac{\partial}{\partial V}W_j\right)\left[-AV + P_NF(V) + \bar{F}_j(V)\right] \\ & + h\frac{\partial}{\partial V}W_j\frac{\partial}{\partial h}\bar{F}_j(V)\|_{D_N^j-Kh} \\ \le\ & 2\left(\frac{\tilde{M}_jh}{Kh} + \frac{\tilde{M}_jh}{Kh}\right)h^2\tilde{M}_j\left[\lambda_N + \frac{B_1}{Kh} + \frac{2B_1}{Kh}\right] \\ & + 2\{\left(\frac{4B_1}{Kh}\right)\left(2h\tilde{M}_j\right) + h^2\tilde{M}_j\frac{4B_1}{Kh}\} \\ & + 2\frac{\tilde{M}_j}{K}\left[\lambda_NR + B_1 + 2B_1\right] + h\tilde{M}_jB_1 \\ \le\ & \frac{\tilde{M}_j}{2} \end{aligned}$$

for K large enough. Then summing up as in (3.15) will bound the h-derivatives. This can be done for all derivatives $\left(\frac{d}{dh}\right)^m \bar{F}$. So all derivatives are bounded for $h \to 0$ and thus we can smoothly extend down to $h = 0$. Hence we have smoothness in h for the finite dimensional part.

The smoothness of the complete terms $\bar{F}(V, h), \alpha(V, \frac{t}{h}, h)$ for $h = 0$ can easily by shown, if we assume $V \in G_\sigma^{s/2}$. This can be done, if e.g. we guarantee backward existence as on unstable manifolds. For fixed N, b_1 and b_2 can be chosen to be smooth in h, because W and G are smooth in h even for $h \to 0$ and $N(h) \to \infty$. It remains to analyze the influence of $P_{N(h)}$ for $h \to 0$. We have to bound $\frac{d}{dh} P_{N(h)} V$, where we introduce additional Galerkin modes by smooth cut-off functions. If $V \in G_\sigma^{s/2}$, then with $N(h) = h^{-\frac{1}{3}}$

$$|\frac{d}{dh} P_{N(h)} V|_{H^s} = h^{-\frac{4}{3}} \exp(-\sqrt{\lambda_{N(h)}}\sigma) = h^{-\frac{4}{3}} \exp(-h^{-\frac{1}{3}}\sigma) \to 0$$

for $h \to 0$ and similarly for the higher derivatives. Hence we can smoothly extend b_1 and b_2 down to $h = 0$, with $\left(\frac{d}{dh}\right)^n_{|h=0} b_1 = \left(\frac{d}{dh}\right)^n_{|h=0} b_2 = 0$. This proves the remark. □

4 Global Errors

The aim of this section is to get some improved geometric theory by applying the averaging results. We describe the dynamics near forced equilibria and homoclinic orbits. The dynamics of the period-map $\tilde{S}^h(h, 0, .)$ given by (1.3) and the time-h-map $\bar{S}^h(h, .)$ given by the semiflow of (1.7) are compared. By standard theory we get persistence of phase-portraits. For example the distance between the original equilibrium and the one of the forced system is of order $O(h)$. We want to describe the geometry up to exponential small errors, by comparing the dynamical systems given by the rapidly forced equation and the averaged equation. Hence we want to show that the dynamical effects of the rapid forcing are exponentially small in h.

When applying theorem 1 we have to circumvent the transient. The basic idea is given in the following lemma. We get exponential estimates when we can assure backward existence for the forced equation.

Lemma 10 *Let $\tilde{U}(t) \in H^s_{per}, t \in [-T^*, 0]$ be a solution of (1.3). Then the difference between $\bar{U}(t) = \bar{S}^h(t, U_0)$ and $\tilde{U}(t) = \tilde{S}^h(t, 0, U_0)$ with $U_0 = \tilde{U}(0)$ is exponentially small for finite times $[0, T]$: When the solutions remain in some ball $B_{H^s_{per}}(M_0, 0)$ then*

$$|\tilde{S}^h(t, 0, U_0) - \bar{S}^h(t, U_0)|_{H^s_{per}} \leq C \exp(-ch^{-\frac{1}{3}}). \tag{4.1}$$

Proof : We apply the variation of constant formula and estimate the nonautonomous part α of (1.3) using theorem 1. Suppose the solution $\tilde{U}(\theta)$ starts at some negative time $-T^*$. We get the best estimates, when T^* is given as in theorem 1 as the maximal Gevrey exponent. Then we have

$$|\alpha(\tilde{U}(\theta), \frac{\theta}{h}, h)| \leq C \exp(-ch^{-\frac{1}{3}})$$

for $\theta \geq 0$. Thus

$$\begin{aligned} |\tilde{U}(t) - \bar{U}(t)|_{H^s_{per}} &= |\tilde{S}^h(t, 0, U_0) - \bar{S}^h(t, U_0)|_{H^s_{per}} \\ &= |\int_0^t \exp(-A(t-\theta))\{F(\tilde{U}(\theta)) + \bar{F}(\tilde{U}(\theta)) + \alpha(\tilde{U}(\theta)) \end{aligned}$$

$$
\begin{aligned}
& -(F(\bar{U}(\theta)) + \bar{F}(\bar{U}(\theta)))\}d\theta|_{H^s_{per}} \\
\leq\ & L \int_0^t |\tilde{U}(\theta) - \bar{U}(\theta))|_{H^s_{per}} d\theta + tCh \exp(-ch^{-\frac{1}{3}}),
\end{aligned}
$$

if $\tilde{U}(t), \bar{U}(t) \in B_{H^s_{per}}(M_0, 0)$ for $t \in [0, T]$. By the Gronwall inequality we get then the estimate (4.1) for such finite times $[0, T]$. □

Next we describe the phase portrait near hyperbolic equilibria following [Stu95] and [AD91]. The rate of convergence is shown to be exponential. In subsection 4.2 we analyze the persistence of homoclinic orbits in one-parameter families of semiflows and give exponential estimates of the splitting due to the forcing.

4.1 Equilibria

Hyperbolic equilibria and the phase portrait near them persist under small perturbations of the semiflow, as shown in [Stu95] and [AD91] using quite general semigroup methods. We cannot apply these methods directly to get exponential estimates. Especially there is the problem of the transient. In ordinary differential equations the existence of phase portraits near equilibria is closely related to the Grobman-Hartman theorem. For specific examples, where a Grobman-Hartman theorem can be proved, see [BL94],[Lu91].

A point $u_0 \in H^s_{per}$ is called an equilibrium, when $S(t, u_0) = u_0, \forall t \geq 0$. It is a hyperbolic equilibrium, if for any fixed t the spectrum σ of the linearisation $DS(t, u_0)$ at u_0 splits into two parts by the unit circle: there exists $\alpha \in (0, 1)$ such that $\sigma = \sigma_s \cup \sigma_u$

$$\sigma_s = \{\lambda \in \sigma | |\lambda| < \alpha\} \text{ and } \sigma_u = \{\lambda \in \sigma | |\lambda| > \alpha^{-1}\}.$$

The stable and unstable eigenspaces of the linearisation at a hyperbolic equilibrium u_0 are denoted by E^s and E^u. In our setting E^u will always be finite dimensional. We use bounded projections P^s and P^u on E^s and E^u with the additional property $P^s(E^u) = 0$ and $P^u(E^s) = 0$. We will set without restriction $u_0 = 0$. The local stable and unstable manifolds are then given by the graphs of

$$
\begin{aligned}
p_s : E^s \cap B_{H^s_{per}}(\rho, 0) &\rightarrow E^u \text{ for the stable and} \\
p_u : E^u \cap B_{H^s_{per}}(\rho, 0) &\rightarrow E^s \text{ for the unstable manifold.}
\end{aligned}
$$

Proposition 11 *Suppose the original equation (1.1 with $h = 0$) possesses a hyperbolic equilibrium at u_0. Then the forced equation (1.1) has a hyperbolic periodic orbit of period h: $\tilde{S}^h(h, 0, \tilde{u}_h) = \tilde{u}_h$, which is exponentially close to a hyperbolic equilibrium $\bar{u}_h$ of the averaged equation (1.7) with:*

$$
\begin{aligned}
|\tilde{u}_h - \bar{u}_h|_{H^s_{per}} &\leq C \exp(-ch^{-\frac{1}{3}}) && (4.2) \\
|\bar{u}_h - u_0|_{H^s_{per}} &\leq Ch. && (4.3)
\end{aligned}
$$

All further results compare the discrete semigroup $\tilde{S}^h(j \cdot h, 0, .), j \geq 0$ with $\bar{S}(t, .)$. There exists $\rho > 0$ such that the local stable and unstable manifolds of $\tilde{u}_h, \bar{u}_h$ are given by graphs of functions

$$
\begin{aligned}
\tilde{p}^h_s(.), \bar{p}^h_s(.) : E^s \cap B_{H^s_{per}}(\rho, 0) &\rightarrow E^u \\
\tilde{p}^h_u(.), \bar{p}^h_u(.) : E^u \cap B_{H^s_{per}}(\rho, 0) &\rightarrow E^s
\end{aligned}
$$

in the following way with $u_s \in E^s \cap B_{H^s_{per}}(\rho, 0)$:

$$W^{s,loc}(\tilde{u}_h) = \tilde{u}_h + u_s + \tilde{p}^h_s(u_s)$$

and in the same way for the other manifolds too. Moreover the local unstable manifolds are exponentially close on $B_{H^s_{per}}(\rho,0)$:

$$\sup_{U\in P^u(B_{H^s_{per}}(\rho,0))} |(\tilde{p}^h_u(U)+\tilde{u}_h)-(\bar{p}^h_u(U)+\bar{u}_h)|_{H^s_{per}} \le C\exp(-ch^{-\frac{1}{3}}) \tag{4.4}$$

and the local stable manifolds are exponentially close on $B_{H^s_{per}}(\rho,0)$ intersected with the image of $W^{s,loc}(\tilde{u}_h)$ under the regularising map $\tilde{S}^h(T^*,0,.)$:

$$\sup_{U\in P^s\left(B_{H^s_{per}}(\rho,0)\cap \tilde{S}^h(T^*,0,W^{s,loc}(\tilde{u}_h))\right)} |\tilde{p}^h_s(U)+\tilde{u}_h-(\bar{p}^h_s(U)+\bar{u}_h)|_{H^s_{per}} \le C\exp(-ch^{-\frac{1}{3}}). \tag{4.5}$$

On $B_{H^s_{per}}(\rho,0)\cap\tilde{S}^h(T^*,0,B_{H^s_{per}}(\rho,0))$ the orbits of $\tilde{S}^h(t,0,.)$ are exponentially followed by orbits of $\bar{S}^h(t,.)$ in the following sense, taking all times including T^* as multiples of h: For all $U\in B_{H^s_{per}}(\rho,0)\cap\tilde{S}^h(T^*,0,B_{H^s_{per}}(\rho,0))$ there exists $V\in B_{H^s_{per}}(\rho,0)\cap\bar{S}^h(T^*,B_{H^s_{per}}(\rho,0))$, such that as long $\bar{S}^h(t,U)\in B_{H^s_{per}}(\rho,0)$ we have that

$$|\tilde{S}^h(t,0,U)-\bar{S}(t,V)|_{H^s_{per}} \le C\exp(-ch^{-\frac{1}{3}}). \tag{4.6}$$

Remark 12 *It is also possible to compare $\tilde{S}^h(j\cdot h+\theta,\theta,.)$ with $\bar{S}^h(t,.)$, in the same way. The fixed point of $\tilde{S}^h(j\cdot h+\theta,\theta,.)$ is then $\tilde{u}_h(\theta)=\tilde{S}^h(\theta,0,\tilde{u}_h)$. All estimates above hold uniformly in θ. When considering the continuous time $\tilde{S}^h(t,0,.)$, then the stable and unstable manifolds will depend on time. The functions $\tilde{p}^h_s(\theta,.)$ and $\tilde{p}^h_u(\theta,.)$ are periodic in θ, but the estimates in (4.4) and (4.5) still hold.*

Remark 13 *The orbits of $\tilde{S}^h(t,0,.)$ also follow the orbits of $\bar{S}^h(t,.)$, but the author could not easily prove exponential closeness.*

Proof :
Step 1: Equilibrium
The averaged equation has a unique hyperbolic equilibrium $\bar{u}_h$, which is dependent on h near the original equilibrium u_0. This follows from [Stu95, theorem 4.3] or [AD91, theorem 2.2], since the semiflows $\bar{S}^h(t,u)$ given by (1.7) and $S(t,u)$ defined by (1.1 with $h=0$) are close in $C^1(B_{H^s_{per}}(\rho,0),H^s_{per})$. The C^0 distance is given by

$$\begin{aligned}|\bar{S}^h(t,u)-S(t,u)|_{H^s_{per}} &\le |\bar{S}^h(t,u)-\tilde{S}^h(t,u)|_{H^s_{per}}+|\tilde{S}^h(t,u)-S(t,u)|_{H^s_{per}}\\ &\le Ch,\end{aligned}$$

where we used that the coordinate change in theorem 1 is $Id+O(h)$. Then we can estimate the distance $\bar{u}_h-u_0$ following [AD91]. This is based on the following idea: $-Id+S(t,.)$ is invertible for fixed t near 0 and its inverse is Lipschitz, thus

$$\begin{aligned}|\bar{u}_h-u_0|_{H^s_{per}} &\le |\mathrm{Lip}(-Id+S(t,.))^{-1}(-\bar{u}_h+S(t,\bar{u}_h)-(-u_0+S(t,u_0)))|_{H^s_{per}}\\ &= \mathrm{Lip}(-Id+S(t,.))^{-1}|-\bar{u}_h+S(t,\bar{u}_h)|_{H^s_{per}}\\ &= \mathrm{Lip}(-Id+S(t,.))^{-1}|-\bar{S}^h(t,\bar{u}_h)+S(t,\bar{u}_h)|_{H^s_{per}}\\ &\le Ch.\end{aligned}$$

Starting from hyperbolic equilibria of the averaged equation we are now going to analyze 'equilibria' of the forced equation, i.e. we consider periodic orbits of period h. For this we use similar methods as above. The existence of a periodic orbit with period h is given by [Stu95, theorem 4.4] with $\tilde{S}^h(h,0,\tilde{u}_h) = \tilde{u}_h$. Hyperbolicity follows by C^1 closeness of $\tilde{S}^h(h,0,.)$ and $\bar{S}^h(h,.)$. To estimate the difference we use again the same trick as above. $-Id + \bar{S}^h(t,.)$ is invertible for $t > 0$ near $\bar{u}^h$ and its inverse is Lipschitz, then taking t as multiple of h

$$\begin{aligned} |\tilde{u}_h - \bar{u}_h|_{H^s_{per}} &\leq |\text{Lip}(-Id + \bar{S}^h(t,.))^{-1}(-\tilde{u}_h + \bar{S}^h(t,\tilde{u}_h) - (-\bar{u}_h + \bar{S}^h(t,\bar{u}_h)))|_{H^s_{per}} \\ &= \text{Lip}(-Id + \bar{S}^h(t,.))^{-1}|\tilde{S}^h(t,\tilde{u}_h) - \bar{S}^h(t,\tilde{u}_h)|_{H^s_{per}} \end{aligned}$$

As we can guarantee backward existence for $\tilde{u}_h$, we can apply lemma 10 and get exponential closeness:

$$|\tilde{u}^h - \bar{u}_h|_{H^s_{per}} \leq C\exp(-ch^{-\frac{1}{3}}).$$

Step 2: Stable and unstable manifold

Next we deal with the local stable and unstable manifolds. Inside some neighborhood $B_{H^s_{per}}(\rho,0)$ of $u_0 = 0$ the existence and convergence of the local stable and unstable manifolds are essentially given by [Stu95, theorem 4.19, corollary 5.6] and [AD91, theorem 2.2]. We transform the forced and the averaged equations such that the equilibrium is at 0, i.e. $\bar{S}^h_0(t,0,.) = \bar{S}^h(t,0,.+\bar{u}^h) - \bar{u}^h$ and $\tilde{S}^h_0(t,0,.) = \tilde{S}^h(t,0,.+\tilde{u}_h) - \tilde{u}_h$, where $t = j\cdot h, j \in \mathbf{N}_0$. Then the local stable and unstable manifolds are given as graphs of the following functions:

$$\begin{aligned} \tilde{p}^h_s(.), \bar{p}^h_s(.) : E^s \cap B_{H^s_{per}}(\rho,0) &\rightarrow E^u \\ \tilde{p}^h_u(.), \bar{p}^h_u(.) : E^u \cap B_{H^s_{per}}(\rho,0) &\rightarrow E^s. \end{aligned}$$

The estimates can again be improved using lemma 10. So we can estimate $\tilde{p}^h_s(.) - \bar{p}^h_s(.)$ and $\tilde{p}^h_u(.) - \bar{p}^h_u(.)$ following e.g. [AD91]. We define

$$\begin{aligned} C_0(H^s_{per}) &= \{(\gamma(j))_{j\in\mathbf{N}_0} | \gamma(j) \in H^s_{per}, \gamma(j) \rightarrow 0\} \\ C_{0,\rho}(H^s_{per}) &= \{(\gamma(j))_{j\in\mathbf{N}_0} \in C_0(H^s_{per}) | \|\gamma\|_\infty = \sup_{j\geq 0} |\gamma(j)|_{H^s_{per}} < \rho\}. \end{aligned}$$

We start with the unstable manifold. In [AD91]

$$\begin{aligned} T^u_h : C_{0,\rho}(H^s_{per}) &\rightarrow E^u \times C_0(H^s_{per}) \\ T^u_h((\gamma(j))_{j\in\mathbf{N}_0}) &= (\gamma(0)_u; \gamma(j) - \bar{S}^h(t,\gamma(j+1)), j \geq 0) \end{aligned}$$

is shown to be be invertible with Lipschitz continuous inverse for fixed t. Hence taking $u_u \in E^u \cap B_{H^s_{per}}(\rho,0)$ and

$$\begin{aligned} \gamma_1(j) &= \tilde{S}^h_0(-jt,0,u_u + \tilde{p}^h_u(u_u)) \\ \gamma_2(j) &= \bar{S}^h_0(-jt,0,u_u + \bar{p}^h_u(u_u)) \end{aligned}$$

gives

$$\begin{aligned} T^u_h((\gamma_2(j))) &= (u_u, 0) \text{ and} \\ T^u_h((\gamma_1(j))) &= \Big(u_u, \tilde{S}^h_0\left(-jt,0,u_u + \tilde{p}^h_u(u_u)\right) \\ &\quad -\bar{S}^h_0\left(t, \tilde{S}^h_0(-(j+1)t,0,u_u + \tilde{p}^h_u(u_u))\right), j \geq 0\Big). \end{aligned}$$

Then we get

$$
\begin{aligned}
& |\tilde{p}_u^h(u_u) - \bar{p}_u^h(u_u)|_{H^s_{per}} \\
\leq \; & \|\gamma_1 - \gamma_2\|_\infty \leq 2\sigma \|T_h^u((\gamma_1(j))) - T_h^u((\gamma_2(j)))\|_\infty \\
\leq \; & 2\sigma \|T_h^u((\gamma_1(j)))\|_\infty \\
= \; & 2\sigma \|\tilde{S}_0^h(t, 0, \tilde{S}_0^h(-(j+1)t, 0, u_u + \tilde{p}_u^h(u_u))) \\
& -\bar{S}_0^h(t, \tilde{S}_0^h(-(j+1)t, 0, u_u + \tilde{p}_u^h(u_u))), j \geq 0\|_\infty.
\end{aligned}
$$

Each point $\tilde{S}_0^h(-(j+1)t, 0, u_u + \tilde{p}_u^h(u_u)) = \tilde{S}^h(-(j+1)t, 0, \tilde{u}_h + u_u + \tilde{p}_u^h(u_u))$ has a backward orbit in H^s_{per} under (1.3). Thus $\tilde{S}^h(-(j+1)t, 0, \tilde{u}_h + u_u + \tilde{p}_u^h(u_u)) = \tilde{S}^h(T^*, 0, u_j)$ for some $u_j \in B_{H^s_{per}}(\rho, 0)$. Hence for T^* a multiple of h

$$
\begin{aligned}
& \sup_{U \in P^u(B_{H^s_{per}}(\rho,0))} |\tilde{p}_u^h(U) + \tilde{u}_h - (\bar{p}_u^h(U) + \bar{u}_h))|_{H^s_{per}} \\
\leq \; & C \exp(-ch^{-\frac{1}{3}}) + 2\sigma \sup_{j \geq 0} |\tilde{S}_0^h(t, 0, \tilde{S}_0^h(T^*, 0, u_j)) - \bar{S}_0^h(t, \tilde{S}_0^h(T^*, 0, u_j))|_{H^s_{per}} \\
\leq \; & C \exp(-ch^{-\frac{1}{3}}),
\end{aligned}
$$

which shows (4.4). Next we deal with the stable manifolds, where the reasoning is similar. We use with t again a multiple of h

$$
\begin{aligned}
T_h^s : C_{0,\rho}(H^s_{per}) & \rightarrow E^s \times C_0(H^s_{per}) \\
T_h^s((\gamma(j))_{j \in \mathbf{N}}) & = (\gamma(0)_s; \gamma(j+1) - \bar{S}_0^h(t, \gamma(j)), j \geq 0)
\end{aligned}
$$

and with $u_s \in E^s$ we set

$$
\begin{aligned}
\gamma_1(j) & = \tilde{S}_0^h(jt, 0, u_s + \tilde{p}_s^h(u_s)) \\
\gamma_2(j) & = \bar{S}_0^h(jt, u_s + \bar{p}_s^h(u_s)).
\end{aligned}
$$

Again we have

$$
\begin{aligned}
& \sup_{U \in B_{H^s_{per}}(\rho,0) \cap E^s} |\tilde{p}_s^h(U) + \tilde{u}_h - (\bar{p}_s^h(U) + \bar{u}_h)|_{H^s_{per}} \\
\leq \; & C \exp(-ch^{-\frac{1}{3}}) + 2\sigma \|T_h^s(\gamma_1) - T_h^s(\gamma_2)\|_\infty \\
\leq \; & C \exp(-ch^{-\frac{1}{3}}) + 2\sigma \sup_{j \geq 0} |\tilde{S}_0^h(t, \tilde{S}_0^h(jt, 0, u_s + \tilde{p}_s^h(u_s))) \\
& -\bar{S}_0^h(t, \tilde{S}_0^h(jt, 0, u_s + \tilde{p}_s^h(u_s)))|_{H^s_{per}} \\
\leq \; & 2C \exp(-ch^{-\frac{1}{3}}) + 2\sigma \sup_{j \geq 0} |\tilde{S}^h(t, \tilde{S}^h(jt, 0, \tilde{u}_h + u_s + \tilde{p}_s^h(u_s))) \\
& -\bar{S}^h(t, \tilde{S}^h(jt, 0, \tilde{u}_h + u_s + \tilde{p}_s^h(u_s)))|_{H^s_{per}}.
\end{aligned}
$$

The last term is exponentially small, if $u_s + \tilde{p}_s^h(u_s)$ has a backward orbit. This can be guaranteed, if $u_s \in P^s(B_{H^s_{per}}(\rho, 0) \cap \tilde{S}^h(T^*, 0, W^{s,loc}(\tilde{u}_h))$. This proves the estimate (4.5).

Step 3: Phase Portrait

The convergence of phase portraits can be obtained from [Stu95, theorem 4.18] and [AD91, theorem 2.2]. It is proved by solving a discrete boundary value problem. For initial value U_0 we prescribe the projection to the stable eigenspace $P^s U_0$ and for the last value U_J the projection to the unstable eigenspace $P^u U_J$. In [AD91] the mapping

$$
\begin{aligned}
H : B_{H^s_{per}}(\rho, 0)^{J+1} & \rightarrow E^s \times (H^s_{per})^J \times E^u \\
\gamma & \mapsto (P^s(\gamma(0)); \gamma(j+1) - \bar{S}_0^h(t, \gamma(j)), 0 \leq j \leq J-1; P^u(\gamma(J))
\end{aligned}
$$

is shown to be invertible with Lipschitz inverse. For all $u_s \in E^s \cap B_{H^s_{per}}(\rho_0, 0)$ and $u_u \in E^u \cap B_{H^s_{per}}(\rho_0, 0)$ for some smaller $0 < \rho_0 < \rho$ there exists $\gamma \in B_{H^s_{per}}(\rho, 0)^{J+1}$, such that $H(\gamma) = (u_s, 0, u_u)$. Here we now again assume all times to be multiples of the period h.

For the improved rates on $B_{H^s_{per}}(\rho, 0) \cap \tilde{S}^h(T^*, 0, B_{H^s_{per}}(\rho, 0))$ we use again a similar argument as above: Let $\gamma_1 = \tilde{S}^h(j \cdot h, 0, U)$ given with $U \in B_{H^s_{per}}(\rho, 0) \cap \tilde{S}^h(T^*, 0, B_{H^s_{per}}(\rho, 0))$. Take J maximal such that all $\gamma_1(j) \in B_{H^s_{per}}(\rho, 0)$ for all $j \leq J$. Then by [AD91] there exists $\gamma_2 \in B_{H^s_{per}}(\rho, 0)^{J+1}$, which solves the boundary value problem for $\bar{S}^h$, i.e. with $H(\gamma_2) = (P^s U, 0, P^u \gamma_1(J))$. Choosing $V = P^s U + P^u \gamma_2(0))$ we get

$$
\begin{aligned}
& |\tilde{S}^h(t, 0, U) - \bar{S}^h(t, V)|_{H^s_{per}} \leq \sup_{j \geq 0} |\gamma_1(j) - \gamma_2(j)|_{H^s_{per}} \\
\leq \quad & C \sup_{j \geq 0} |H(\gamma_1)(j) - H(\gamma_2)(j)|_{H^s_{per}} = C \sup_{j \geq 0} |\gamma_1(j+1) - \bar{S}^h(h, \gamma_1(j))|_{H^s_{per}} \\
= \quad & C \sup_{j \geq 0} |\tilde{S}^h((j+1) \cdot h, 0, U) - \bar{S}^h(h, \tilde{S}^h(j \cdot h, 0, U))|_{H^s_{per}} \leq C \exp(-ch^{-\frac{1}{3}}),
\end{aligned}
$$

using lemma 10, as U has a backward orbit. This proves (4.6). □

4.2 Homoclinic orbits

Now we analyze the influence of rapid forcing on homoclinic orbits in semiflows defined by a parameter-dependent version of (1.1 with $h = 0$):

$$
\frac{\partial}{\partial t} U(x, t) - D \Delta U(x, t) = F(U(x, t), \lambda) + hG(U(x, t), \frac{t}{h}, h, \lambda). \tag{4.7}
$$

The nonlinearities $F = (f_1, \ldots, f_n)$ and $G = (g_1, \ldots, g_n)$ are supposed to depend smoothly on λ. All other assumptions of theorem 2 are also assumed. Homoclinic orbits $\Gamma(t)$ are solutions, which exist for positive and negative times and converge to the same equilibrium u_0 for $t \to \infty$ and $t \to -\infty$:

$$
\lim_{t \to \pm\infty} \Gamma(t) = u_0.
$$

Hence $\Gamma(t) \subset W^s(u_0) \cap W^u(u_0)$. Homoclinic orbits of the autonomous part (4.7 with $h = 0$) can be generically encountered in one-parameter families of equations only, since the stable and the unstable manifold intersect nontransversally because of the direction of the homoclinic orbit.

The existence of homoclinic orbits can be deduced, for example, from the existence of Takens-Bogdanov bifurcation points in equations depending on two parameters (for a concrete example of reaction-diffusion equations see [Fie86]). In nonautonomous equations the stable and unstable manifold may intersect transversally at points of the homoclinic orbit. The stable and unstable manifolds split and the homoclinic orbit then consists out of discrete points instead of a continuous arc. We will show in the next proposition, that this effect only occurs in an exponentially small wedge in parameter space (λ, h). The splitting distance is also shown to be exponentially small in h. $T_V W^u(u_0, \lambda)$ is the tangent space of the unstable manifold at a point $V = (U, \lambda)$. The transversality condition (4.8) states, that the stable and unstable manifold intersect transversally at the homoclinic orbit in the extended phase space (phase space times parameter space), when we also vary the parameter. Whereas at a transverse homoclinic orbit, the stable and unstable manifold intersect transversally in phase space, which is not possible for nonautonomous systems due to the direction of the flow.

Proposition 14 *Suppose the unforced equation (4.7 with $h = 0$) possesses a homoclinic orbit $\Gamma(t)$ for $\lambda_0 \in \Lambda \subset \mathbf{R}$, which is biasymptotic to the hyperbolic equilibrium u_0 with $dim(W^u(u_0)) = k < \infty$. Also assume that the forward orbit of the unstable manifold exists for all times. Assume that the homoclinic orbit is non-degenerate, i.e.*

$$(\lambda, T_V W^u(u_0, \lambda)) \overline{\pitchfork}_{\lambda_0, \Gamma(t)} (\lambda, T_V W^s(u_0, \lambda)). \tag{4.8}$$

Then the averaged equation (1.7) possesses a homoclinic orbit $\Gamma^h(t)$ for some λ_h with $|\lambda_h - \lambda_0| \leq Ch^{\frac{s+1}{3}}$. The nonautonomous equations (4.7) possess homoclinic orbits near $\Gamma^h(t)$, these may be transverse. They only exist for an exponentially small wedge $|\lambda - \lambda_h| \leq C\exp(-ch^{-\frac{1}{3}})$.

While there exists a transverse homoclinic orbit, the distance of the stable and the unstable manifolds is exponentially small when measuring the distance in points in $\bar{S}^h(T^*, W^{s,loc}(\tilde{u}_h))$.

Proof :

We will first describe the geometric idea of the proof. The existence of the homoclinic orbit for the averaged system, which is a small perturbation of the original equation, is due to the non-degeneracy and transversality in the extended phase space $H^s_{per} \times \Lambda$. The result on exponential small splittings follows with proposition 11. The local stable and unstable manifolds are exponentially close to those of the averaged system, where the manifold do not split at all. When the local unstable manifold evolves forward in time, it only needs some finite time to reach the local stable manifold near the homoclinic orbit. In this finite time the forward orbit of the local unstable manifolds of $\bar{u}_h$ and $\tilde{u}_h$ remain exponentially close. So the stable and unstable manifolds can only intersect, when the distance between the local stable and the forward orbit of the local unstable manifold of the averaged system is exponentially small. This holds only in an exponentially small wedge in parameter space due to transverse unfolding. The rigorous proof will be given in three steps.

Step 1: Local coordinate transformation

We transform the semigroups in such a way, that the hyperbolic equilibrium of the averaged equation is at the origin 0 and that the local stable manifold coincides with E^s in $B_{H^s_{per}}(\frac{\rho}{2}, 0)$. Hence we first transform for all h and λ

$$\bar{S}^h_0(t, u) = \bar{S}^h(t, u + \bar{u}_h) - \bar{u}_h$$

In $B_{H^s_{per}}(\rho, 0)$ we use a smooth cut-off function χ on $\mathbf{R}^+_0$ defined by:

$$\chi(\tau) = \begin{cases} 1 \text{ for } 0 \leq \tau \leq \frac{1}{2} \\ 0 \text{ for } \tau \geq 1. \end{cases}$$

Then we let for $u = u_s + u_u$ with $u_s \in E^s$ and $u_u \in E^u$

$$\bar{S}^h_*(t, u_s + u_u) = \bar{S}^h_0(t, u_s + u_u + \chi(\frac{|u|_{H^s_{per}}}{\rho})\bar{p}^h_s(u_s)) \tag{4.9}$$

The representation of the local unstable manifold is also transformed by this local coordinate transform, but we will still denote the function representing the unstable manifold as a graph by $\bar{p}^h_u$. We use for the forced system the same transformation. We denote the transformed semi-evolution by $\tilde{S}^h_*(t, 0, .)$ and the transformed periodic orbit of period h of the forced system by U_h.

Step 2: Persistence of homoclinic orbits

We will apply the transversal isotopy theorem [AR67, theorem20.2] and the openness of transversal intersection to prove the persistence. The theorems in Abraham and Robbin state:

Theorem *Let $\mathcal{A}, X$ and Y be C^{r+1} manifolds ($r \geq 1$), $\phi : \mathcal{A} \times X \to Y$ a C^{r+1} representation of mappings, $W \subset Y$ a submanifold and $a_0 \in \mathcal{A}$ a point. For $a \in \mathcal{A}$ let $W_a = \phi_a^{-1}(W)$. Assume that*

1. *W is closed in Y,*
2. *X compact and C^{r+3},*
3. *$\phi_{a_0} \pitchfork W$.*

Then there is an open neighborhood N of a_0 in $\mathcal{A}$, such that for $a \in N$ we have that $\phi_a \pitchfork W$ and W_a is C^r-isotopic to W_{a_0}. This implies, that there is a C^r-diffeomorphism $F_a : X \to X$ with $F_a(W_a) = W_{a_0}$ for $a \in N$.

We will choose $\mathcal{A} = [-h_0, h_0]$, $a_0 = 0$ and

$$\begin{aligned} X &= \Lambda \times \overline{(E^u \cap B_{H^s_{per}}(\rho, 0))} \\ Y &= \Lambda \times H^s_{per} \\ W &= \overline{E^s \cap B_{H^s_{per}}(\rho, 0)} \end{aligned}$$

which fulfill all above assumptions. The smoothness assumptions are fulfilled trivially as all manifolds are linear spaces intersected with balls. X is compact, because it is finite dimensional by assumption, bounded and closed.

ϕ describes the forward orbit of the transformed local unstable manifold:

$$\begin{aligned} \phi : [-h_0, h_0] \times \left(\Lambda \times \overline{(E^u \cap B_{H^s_{per}}(\rho, 0))}\right) &\to H^s_{per} \\ \phi_h(\lambda, u_u) &= \bar{S}^h_*(T, u_u + \bar{p}^h_u(u_u), \lambda), \end{aligned} \tag{4.10}$$

where T is chosen large enough, such that we can guarantee $\phi_0(X) \cap W \neq \emptyset$ for $h = 0, \lambda = \lambda_0$. This representation of mappings $\phi_h : X \to Y$ is C^2 in (λ, u_u). As the manifolds are smooth, the transformation in step 1 and $u_u + \bar{p}^h_u(u_u)$ are smooth. $F, \bar{F}$ depend smoothly on λ and u. We obtain by smooth dependence on parameters and initial values (see e.g. [Hen83, Corollary3.4.6]), that $\bar{S}^h_*(T, ., .)$ is smooth in (λ, u_u). The semiflow depends on W^u smoothly on h down to $h = 0$, when using remark 9. The last assumption of the theorem $\phi_0 \pitchfork W$ holds by (4.8).

Hence by the theorem there exists an interval $(-h_*, h_*)$ such that for all $h \in (-h_*, h_*)$: $\phi_h^{-1}(W)$ is diffeomorphic to $\phi_0^{-1}(W)$. Thus there is $\lambda_h \in \Lambda$ such that $W^s(0, \lambda_h, h) \cap W^u(0, \lambda_h, h)$ contains a one-dimensional arc, i.e. for this λ_h there exists a homoclinic orbit.

Next we consider the persistence of homoclinic orbits for the nonautonomous equation. The equilibrium persists as periodic orbit of period h. We fix a single point $\tilde{u}_h$ on the periodic orbit and take only multiples of h as times and apply the same reasoning as above to

$$\begin{aligned} \tilde{\phi} : [-h_0, h_0] \times \left(\Lambda \times \overline{(E^u \cap B_{H^s_{per}}(\rho, 0))}\right) &\to H^s_{per} \\ \tilde{\phi}_h(\lambda, u_u) &= \tilde{S}^h_*(T, u_u + \tilde{p}^h_u(u_u), \lambda), \end{aligned} \tag{4.11}$$

where $\tilde{S}$ denotes the semievolution and $\tilde{p}_u$ describes unstable manifold of $\tilde{u}_h$. Then by the theorem there exists an interval $(-h_*, h_*)$ such that for all $h \in (-h_*, h_*)$: $\tilde{\phi}_h^{-1}(W)$

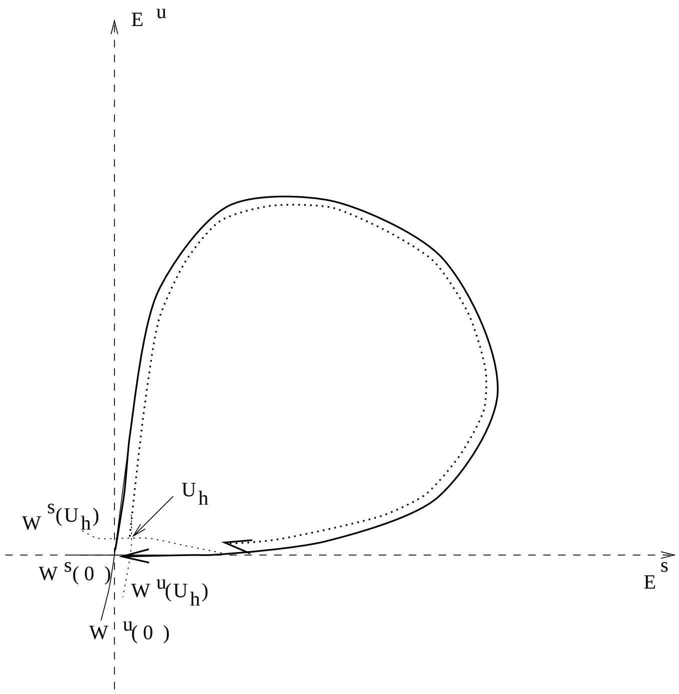

Figure 3: The homoclinic orbit to 0 in the averaged system is closely followed by the unstable manifold (dotted) of U_h, which is a point on the periodic orbit of period h of the forced system.

is diffeomorphic to $\tilde{\phi}_0^{-1}(W)$. Then there is $\lambda_h \in \Lambda$ such that $W^s(0, \lambda_h, h) \cap W^u(0, \lambda_h, h)$ contains homoclinic points, which one expects to be transverse.

Step 3: Exponential smallness

Now we analyze the behavior of the forced system. By proposition 11 the unstable manifolds of the forced and the averaged system are exponentially close in h. In the original coordinates they are given by

$$\begin{aligned} W_h^{u,loc}(\bar{u}_h) &= \{\bar{u}_h + u_u + \bar{p}_u^h(u_u), u_u \in E^u \cap B_{H^s_{per}}(\rho, 0)\} \\ W_h^{u,loc}(\tilde{u}_h(\theta)) &= \{\tilde{u}_h(\theta) + u_u + \tilde{p}_u^h(\theta, u_u), u_u \in E^u \cap B_{H^s_{per}}(\rho, 0)\} \end{aligned}$$

The exponential closeness does not change, when transforming the semigroups as in step 1, where $\bar{u}_h$ is transformed to the origin and $\tilde{u}_h$ to U_h, see figure 3.

Next we consider the semiflow applied to the local unstable manifold. There is a time T as in (4.10), such that the forward orbit of the local unstable manifold intersects with the local stable manifold:

$$\bar{S}_*^h(T, W_h^{u,loc}(0)) \cap W_h^{s,loc}(0) \neq \emptyset$$

After an additional time T^* we have

$$\bar{S}_*^h(T + T^*, W_h^{u,loc}(0)) \cap W_h^{s,loc}(0) \cap \bar{S}_*^h(T^*, W_h^{s,loc}(0)) \neq \emptyset,$$

where the local stable manifolds of 0 for the averaged system and of U_h for the periodically forced system are exponentially close.

Furthermore if the semiflows of the averaged system and the forced system evolve forward for the finite time $T+T^*$, they preserve the closeness of the unstable manifolds. Let $u_1 \in W_h^{u,loc}(U_h(\theta))$ and $u_2 \in W_h^{u,loc}(0)$ with $|u_1-u_2| \le C\exp(-ch^{-\frac{1}{3}})$. Then applying lemma 10 and using that the linear mapping $D\bar{S}_*^h(T+T^*,u)$ is bounded for $u \in K = \{ru_1+(1-r)u_2 | r \in [0,1]\}$ we get

$$\begin{aligned}
& |\tilde{S}_*^h(\theta+T+T^*,\theta,u_1) - \bar{S}_*^h(T+T^*,u_2)|_{H^s_{per}} \\
\le\ & |\tilde{S}_*^h(T+T^*+\theta,\theta,u_1) - \bar{S}_*^h(T+T^*,u_1)|_{H^s_{per}} \\
& +|\bar{S}_*^h(T+T^*,u_1) - \bar{S}_*^h(T+T^*,u_2)|_{H^s_{per}} \\
\le\ & C\exp(-ch^{-\frac{1}{3}}) + \sup_{u\in K} |D\bar{S}_*^h(T+T^*,u)||u_1-u_2|_{H^s_{per}} \\
\le\ & C\exp(-ch^{-\frac{1}{3}})
\end{aligned}$$

The transversality condition (4.8) also holds for $h \neq 0$ by step 2. Thus we can choose a special local coordinate system near homoclinic points P, which are in the forward orbit of the local stable manifold, i.e. $P \in \bar{S}_*^h(T+T^*,W_h^{u,loc}(0)) \cap \bar{S}_*^h(T^*,W_h^{s,loc}(0))$. Then (4.8)

$$(\lambda, T_V W^u(u_0,\lambda)) \pitchfork_{\lambda_h,\Gamma_h(t)} (\lambda, T_U W^s(u_0,\lambda))$$

implies that

$$T_{U,\lambda}\bar{S}_*^h(T+T^*,W^u(0),\lambda) + E^s = H^s_{per},$$

where the tangential of the finite dimensional manifold $\bar{S}_*^h(T+T^*,W^u(U_h),\lambda)$ is taken at $\lambda_h, U \in \bar{S}_*^h(T+T^*,W_h^{u,loc}(0)) \cap W_h^{s,loc}(0)$. Then a coordinate system is given by

$$T_\lambda \bar{S}_*^h(T+T^*,W^u(0),\lambda) \oplus (T_U\bar{S}_*^h(T+T^*,W^u(0),\lambda) + E^s)$$

The stable eigenspace E^s is a subspace of codimension k. The tangent space $T_U\bar{S}_*^h(T+T^*,W^u(0),\lambda)$ is k dimensional, but has a non-zero intersection with E^s, hence $E^\perp = T_\lambda\bar{S}_*^h(T+T^*,W^u(0),\lambda)$ is not zero. Thus by the Taylor theorem $\bar{S}_*^h(T+T^*,W^u(0),\lambda_h+\epsilon)$ has an $E^\perp$ component of order ϵ. Then $\bar{S}_*^h(T+T^*,W^u(U_h),\lambda_h+\epsilon)$ has an $E^\perp$ component of order $\epsilon+\exp(-ch^{-\frac{1}{3}})$. Furthermore at points in $B_{H^s_{per}}(\rho,0) \cap \tilde{S}_*^h(T^*,0,B_{H^s_{per}}(\rho,0))$ the local stable manifold of the forced system $W_h^{s,loc}(U_h)$ is exponentially close of order $\exp(-ch^{-\frac{1}{3}})$ to $W_h^{s,loc}(0)$ by proposition 11. Thus the $E^\perp$ component is at most $\exp(-ch^{-\frac{1}{3}})$. Hence for $|\epsilon| > C\exp(-ch^{-\frac{1}{3}})$ there cannot be an intersection of $\tilde{S}_*^h(T+T^*,W^u(U_h),\lambda_h+\epsilon)$ with $W_h^{s,loc}(U_h)$ and therefore there is no transversal homoclinic orbit near $\Gamma_h(t)$.

Step 4: Size of splitting

The exponential closeness of stable and unstable manifold of U_h in $B_{H^s_{per}}(\rho,0)$ intersected with $\bar{S}_*^h(T^*,B_{H^s_{per}}(\rho,0))$ can be shown in the following way. Let U be a homoclinic point of the forced equation, which is on the forward orbit of the local stable manifold

$$U \in W^{s,loc}(U_h) \cap \tilde{S}_*^h(T^*,0,B_{H^s_{per}}(\rho,0)) \cap \tilde{S}_*^h(T+T^*,W^{u,loc}(U_h)).$$

$W^{s,loc}(U_h)$ is exponentially close to E^s on $\tilde{S}_*^h(T^*,0,B_{H^s_{per}}(\rho,0))$ and $\{\tilde{S}_*^h(t,U), t\in(0,h)\}$ is the part of the unstable manifold that is between two consecutive homoclinic points. Hence it remains to show that $\tilde{S}_*^h(t,U)$ stays exponentially close to E^s, i.e. $P^u\tilde{S}_*^h(t,U)$ is exponentially small. This can be shown using lemma 10, the boundedness of $D\bar{S}^h(t,U)$ and that $E^s \cap B_{H^s_{per}}(\rho,0)$ is invariant under $\bar{S}_*^h(t,.)$

$$|P^u\tilde{S}_*^h(t,U)|_{H^s_{per}}$$

$$
\begin{aligned}
&\leq |\tilde{S}^h_*(t,0,U) - \bar{S}^h_*(t,P^sU)|_{H^s_{per}}\\
&\leq |\tilde{S}^h_*(t,0,U) - \bar{S}^h_*(t,U)|_{H^s_{per}} + |\bar{S}^h_*(t,U) - \bar{S}^h_*(t,P^sU)|_{H^s_{per}}\\
&\leq C\exp(-ch^{-\frac{1}{3}}) + C|U - P^sU|_{H^s_{per}}\\
&\leq C\exp(-ch^{-\frac{1}{3}}).
\end{aligned}
$$

This proves the exponential smallness of the splitting at points of the discrete transversal homoclinic orbit, which are on the forward orbit of the local stable manifold. □

5 Discussion

This last chapter contains remarks outlining generalisations, variants, implications and limitations of the results above.

Computation of the correction terms

$\bar{F}(V,h)$ is computed using a h dependent number of iterative steps. Anyway, we can describe the terms of leading order in h, when neglecting terms, that are damped after the transient. Then:

$$
\bar{F}(V,h) = \text{average}\left(hP_{N(h)}G(P_{N(h)}V, \frac{t}{h}, h) + h(I - P_{N(h)})G(V, \frac{t}{h}, h)\right) + \text{h.o.t.} \tag{5.12}
$$

where $N(h) = [h^{-\frac{1}{3}}]$ was chosen in the proof.

Different domains

Our particular choice of reaction-diffusion systems with periodic boundary conditions was made, because we need Gevrey regularity in our proofs. Other examples of equations with this extreme regularisation can be analyzed in the same way. For scalar reaction-diffusion equations on the Sphere S^2 Cao, Rammaha and Titi [CRT99] show regularisation to Gevrey classes for initial values $u_0 \in H^s_{per}$ with $s \geq \frac{d}{2} + 1$. Ferrari and Titi [FT98] and Promislow [Pro91] also claim that their results hold on $\mathbf{R}^d$ too. Then it should be possible to extend our averaging to reaction-diffusions on these domains.

Gradient dependent nonlinearities

Gevrey regularity results are also possible for gradient dependent nonlinearities $F(u, \nabla u)$, see [FT98, theorem 3]. In our setting they will lead to Nemitski operators $F : G^{s/2}_\sigma \to G^{(s-1)/2}_\sigma$, using a lemma similar to lemma 2. To extend theorem 1 to this setting one needs some changes. In the estimates of α we will need a different coupling $\lambda_N^{1+\beta}h$. We can estimate $|\alpha|_{G^{(s-1)/2}_\sigma}$, because $F(U_N) \in G^{(s-1)/2}_\sigma$. But we still have to estimate terms like $|\alpha|_{G^{s/2}_\sigma}$. When using $|P_N\alpha|_{G^{s/2}_\sigma} \leq \sqrt{\lambda_N}|P_N\alpha|_{G^{(s-1)/2}_\sigma}$ and compensating the additional unbounded factor $\sqrt{\lambda_N}$ by $K(h) = \tilde{K}h^{-\frac{3}{2+2\beta}}$ instead of $\tilde{K}h^{-\frac{1}{1+\beta}}$ as in (3.14), then estimates with exponential terms like $\exp(-ch^{-\frac{1}{4}})$ can be shown.

Navier-Stokes equation

The same method of proof is also applicable to Navier-Stokes equations with periodic boundary conditions. In [FT89] Foias and Temam showed Gevrey regularity for this setting. Actually the Navier-Stokes equation was the first example of regularisation to Gevrey classes for semilinear parabolic equations.

More geometric theory

Equilibria and homoclinic orbits are only two examples of special solutions, where the averaging improves the understanding of nonautonomous effects. Future work should analyze, e.g. as in [Il98], the influence on attractors and other invariant objects.

Approximation of homoclinic orbits
The ideas in proposition 14 can be used to prove the persistence of homoclinic orbits under perturbations of parameter dependent semigroups in a much more general framework. The persistence of phase portrait, local stable and unstable manifolds of hyperbolic equilibria is shown for many examples [Stu95, ch.3]. Then the forward orbit of the local unstable manifolds of the original and the perturbed semigroup will stay close. Under transversality assumptions like (4.8) one should be able to show the existence of a homoclinic orbit even for higher dimensional unstable manifolds.

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