Homogenisation of Exponential Order for Elliptic Systems in Infinite Cylinders

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Abstract

We consider systems of semilinear elliptic equations on infinite cylinders with a nonlinear rapid periodic inhomogeneity in the unbounded direction. We transform the equation, such that the inhomogeneous term is exponentially small in the period of the inhomogeneity for bounded solutions. The results can be used to show that equilibrium solutions persist as periodic solutions with exponentially small modulation. The analytic tools of the paper include the dynamical systems approach to elliptic equations, averaging of exponential order for ordinary differential equations and extreme regularity (Gevrey classes).

1 Introduction and Main Results

Homogenisation theory is a major part of the analysis of partial differential equations with rapidly oscillating coefficients in time and/or space, which aims at eliminating these dependencies. Often effective or homogenized equations are derived which describe the model up to a certain finite order. When looking for higher order expansions as in e.g. [BaPa89] for linear elliptic problems, we find that the correctors must satisfy partial differential equations of higher and higher order, which are often not well-posed. As a general references for results and methods of homogenisation theory see e.g. [BPS99, BJP99, JKO94]. In this paper we follow a different approach. The homogenisation problem is transformed such that the terms in the equations, which are still rapidly oscillating, are beyond any finite order and are in fact exponentially small.

As a particular example we investigate periodic homogenisation of systems of semilinear elliptic equations, which are inhomogeneous in an unbounded direction. Given

$$\partial_t^2 u + D\Delta_x u \quad = \quad f(u, u_t, \nabla_x u) + \varepsilon^p g(u, u_t, \nabla_x u, \frac{t}{\varepsilon}, \varepsilon) \tag{1.1}$$

for some order $p \geq 0$, $u \in \mathbb{R}^n$ in a unbounded cylinder $(t, x) \in \mathbb{R} \times \Omega$ with cross-section $\Omega = [0, L]^d$ and periodic boundary conditions in x. The nonlinearity g is periodic in $\theta = \frac{t}{\varepsilon}$. These equations arise, for example, when analysing the influence of rapid inhomogeneities on travelling waves of reaction-diffusion equations

$$\partial_\tau u = \Delta_{t,x} u + f(u), \tag{1.2}$$

for a survey on travelling waves and their applications see [VVV94].

The equation (1.1) is analysed by combining four main steps. First we use the spatial-dynamics approach to elliptic with an unbounded direction. The unbounded direction is considered as 'time' and an (ill-posed) evolution problem is then studied. Then we use special averaging techniques for this evolution problem, which were first developped in [Mat01] for parabolic equations with rapid time-dependence. As a second step we modify averaging results of exponential order for bounded ordinary differential equations by Neishtadt [Nei84], such that they can be applied to a finite dimensional Galerkin approximation of evolution equation, where the norm of the vector field becomes unbounded with the dimension of the Galerkin approximation k. This will give a homogenised equation, where the inhomogeneous remainder is exponentially small in some algebraic expression of ε. The third step deals with the infinite-dimensional part, which is not considered in the Galerkin approximation. This will then be estimated using the extreme regularity of the solutions. The remaining Galerkin modes decay exponentially in the dimension k of the our approximation space. Balancing both exponential estimates together for an appropriate coupling of ε and k gives an exponential error estimate. For solutions uniformly bounded on $\mathbb{R} \times \Omega$, we will show that the nonautonomous part is of exponential order $O(\exp(-\frac{c}{\sqrt{\varepsilon}}))$ after the transformation.

To formulate our main results, we are rewriting the elliptic system (1.1) of n second-order equations as a system of $2n$ first-order equations. By letting

$$U = \begin{pmatrix} u \\ u_t \end{pmatrix} \tag{1.3}$$

we have

$$U_t = AU + F(U) + \varepsilon^p G(U, \frac{t}{\varepsilon}, \varepsilon) \tag{1.4}$$

with

$$\begin{aligned} AU &= \begin{pmatrix} 0 & I \\ -D\Delta_x & 0 \end{pmatrix} \begin{pmatrix} u \\ u_t \end{pmatrix} \\ F(U) &= \begin{pmatrix} 0 \\ f(u, u_t, \nabla_x u) \end{pmatrix} \\ G(U, \frac{t}{\varepsilon}, \varepsilon) &= \begin{pmatrix} 0 \\ g(u, u_t, \nabla_x u, \frac{t}{\varepsilon}, \varepsilon) \end{pmatrix}. \end{aligned}$$

Then it can be considered as a dynamical system. Dynamical systems methods have shown to be very powerful in understanding elliptic equations in infinite cylinders, see e.g. [Kir82, Mie94, VZ96, FSV99, VZ99]. The unbounded direction t is then called 'time'. We choose then as phase-space some Sobolev space on the cross-section Ω . We let $U \in X = H^s_{per}(\Omega, \mathbb{R}^n) \times H^{s-1}_{per}(\Omega, \mathbb{R}^n)$, for some $s > 0$. All theorems hold, if we fix $s > \frac{d}{2} + 1$.

We consider solutions of (1.4) resp. (1.1) which are uniformly bounded on $\mathbb{R} \times \Omega$ in the following sense:

$$\begin{array}{lcl} u & \in & BC^0(\mathbb{R}, H^s_{per}(\Omega, \mathbb{R}^n)) \cap C^0(\mathbb{R}, H^2_{per}(\Omega, \mathbb{R}^n)) \\ u_t & \in & BC^0(\mathbb{R}, H^{s-1}_{per}(\Omega, \mathbb{R}^n)) \cap C^0(\mathbb{R}, L^2(\Omega, \mathbb{R}^n)) \\ u_{tt} & \in & BC^0(\mathbb{R}, L^2(\Omega, \mathbb{R}^n)), \end{array} \tag{H.1}$$

and (1.1) holds weakly in the cross section for all t. Here

$$\begin{aligned} BC(\mathbb{R}, Y) &= \{u : \mathbb{R} \to Y | u \text{ continuous}, \sup_{t \in \mathbb{R}} |u(t)|_Y < \infty\}, \\ \|u\|_{BC(\mathbb{R}, Y)} &= \sup_{t \in \mathbb{R}} |u(t)|_Y, \end{aligned}$$

we try to stick to the convention that $||.||$ denote a sup-norm, while $|.|$ is the norm in a general Banach space. As we assume smoothness of f, it follows that every weak solution is strong. The choice of the differentiability order s can be relaxed depending on the form of the nonlinearities. The set of bounded solutions, satisfying (H.1) is often also called the (trajectory) attractor $\mathcal{A}$ of the equation. The dynamics are by translation of the the trajectories along the 'time' direction, for a more detailed explanation see e.g. [FSV99]. In particular we consider solutions, which are bounded by a constant R.

$$\mathcal{A}_R := \{u : \mathbb{R} \to X, u \text{ solves (1.4) and fulfills (H.1)}, ||u||_{BC(\mathbb{R},X)} \leq R\}$$

The corresponding attractors of transformed equations will be denoted by $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}_R$.

We make the following assumptions on f and g

$$g(u, \theta, \varepsilon) \text{ is periodic in } \theta \text{ of period 1 and continuous with respect to } \theta \text{ and } \varepsilon \tag{H.2}$$

When f, g do not depend on derivatives, we assume

$$f(.), g(., \theta, \varepsilon) : \mathbb{R}^n \to \mathbb{R}^n \text{ are entire for all } \theta, \varepsilon \text{ and } s > \tfrac{d}{2}. \tag{H.3a}$$

If the nonlinearities involve first order derivatives with respect to t or x, i.e. they have the general form $f(u, u_t, \nabla_x u)$ and $g(u, u_t, \nabla_x u, \theta, \epsilon)$, we assume instead

$$f(., ., .), g(., ., ., \theta, \varepsilon) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \cdot d} \to \mathbb{R}^n \text{ are entire for all } \theta, \varepsilon \text{ and } s > \tfrac{d}{2} + 1. \tag{H.3b}$$

Our main result is the following.

Theorem 1 *Assume the hypothesis (H.2) and depending on the form of the nonlinearities (H.3a) or (H.3b) for the equation (1.4). Fix a perturbation order* $p \geq 0$ *and some bound* $R > 0$.

Then there exist $\varepsilon_0 > 0$, $C_1, C_2, C_3 > 0$ *and a t-periodic transformation of (1.4) on a ball* $B_X(R)$ *for* $0 < \varepsilon < \varepsilon_0$ *to*

$$V_t = AV + F(V) + \bar{F}(V, \varepsilon) + \alpha(V, \frac{t}{\varepsilon}, \varepsilon), \tag{1.5}$$

where $\bar{F}, \alpha$ *are differentiable for* $V \in B_X(R)$, *nonlinear and nonlocal in the cross-section, but local in t. When considering bounded solutions* $V(.) \in BC(\mathbb{R}, X)$ *of the original equation (1.4) - i.e.* $V(.)$ *is in the attractor* $\mathcal{A}_R$ *- or bounded solutions of (1.5) satisfying (H.1) - i.e.* $V(.)$ *is in an attractor* $\tilde{\mathcal{A}}_R$ *- then the influence of the fast scale on* $V(.)$ *is exponentially small, uniformly on balls in* $BC(\mathbb{R}, X)$:

$$\sup_{V(.) \in \mathcal{A}_R \cup \tilde{\mathcal{A}}_R} ||\alpha(V(.), ./\varepsilon, \varepsilon)||_{BC(\mathbb{R},X)} \leq C_1 \exp(-C_2 \varepsilon^{-\frac{1}{2}}) \tag{1.6}$$

The correction term can be estimated on $B_X(R)$ *by*

$$\sup_{V(.) \in B_X(R)} ||\bar{F}(V(.), \varepsilon)||_{BC(\mathbb{R},X)} \leq C_3 \varepsilon^p. \tag{1.7}$$

Remark 1.1 *The estimate (1.6), possibly with changed constants* C_1, C_2, *also hold for* $V(.) \in BC(\mathbb{R}, X)$, *which are bounded solutions of any system of elliptic equations of type (1.4), which fulfills the regularity proposition 2.1.*

This transformation can be used to understand the influence of the rapid forcing on solutions. Our main interest will be here on 'equilibrium' solutions, which do not depend on t for $\epsilon = 0$, see section 4. We will compare the solutions of the truncated equation

$$V_t = AV + F(V) + \bar{F}(V, \varepsilon) \tag{1.8}$$

with the solutions of (1.4) and (1.5). We will see in that 'equilibrium' solutions persist under the forcing as solutions periodic in t, which are exponentially close to equilibria of (1.8). In a forth-coming paper the influence on localised solutions and their relation to pinning effects in (1.2) will be also analysed.

The rest of paper is organised as follows. In section 2 we introduce Gevrey classes as a tool for describing extreme regularity of the solutions of (1.4), (1.5) and (1.8). The bulk part of the proof of theorem 1 is given in section 3, where a version with more general assumptions (H.4) and (H.5) is proved. The influence of rapid forcing on equilibrium solutions is analysed in section 4.

2 Gevrey classes

An important tool is the extremely high regularity of bounded solutions of (1.1). They are real analytic in the cross-section and are in special Gevrey classes. To define these spaces, we collect some properties of the differential operator A. As above, A is defined by

$$AU = \begin{pmatrix} 0 & I \\ -D\Delta_x & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

with $D = \mathrm{diag}(d_1, \ldots, d_n)$. For comparison we also consider the differential operator $\bar{A}$, where all diagonal entries in D are equal:

$$\bar{A}U = \begin{pmatrix} 0 & I \\ -d_{min} I \Delta_x & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{2.1}$$

where $d_{min} = \min(d_1, \ldots, d_n)$.

The operator A has both infinitely many positive and negative eigenvalues. We describe here the situation for periodic boundary conditions. There are strictly positive eigenvalues $\lambda_{k,l} = |k|\sqrt{d_l}\frac{2\pi}{L}$ for $k \in \mathbb{Z}^d \backslash \{0\}, l \in \{1, \ldots, n\}$ with eigenfunctions

$$v_{k,l}(x) = \begin{pmatrix} e_l \exp\left(i\frac{2\pi}{L}k \cdot x\right) \\ |k|\sqrt{d_l}\frac{2\pi}{L} e_l \exp\left(i\frac{2\pi}{L}k \cdot x\right) \end{pmatrix}, \tag{2.2}$$

where e_l is the l^{th} unit vector in $\mathbb{R}^n$. The negative eigenvalues are $\mu_{k,l} = -|k|\sqrt{d_l}\frac{2\pi}{L}$ for $k \in \mathbb{Z}^d, l \in \{1, \ldots, d\}$ with eigenfunctions

$$w_{k,l}(y) = \begin{pmatrix} e_l \exp\left(i\frac{2\pi}{L}k \cdot x\right) \\ -|k|\sqrt{d_l}\frac{2\pi}{L} e_l \exp\left(i\frac{2\pi}{L}k \cdot x\right) \end{pmatrix}. \tag{2.3}$$

We denote by A_+ the part of A with negative eigenvalues, which generates an analytic semigroup for *positive* times:

$$\begin{aligned} A_+ v_{k,l} &= 0 \\ A_+ w_{k,l} &= A w_{k,l} = \mu_{k,l} w_{k,l}. \end{aligned}$$

In the same way, A_- is the part of A, which generates an analytic semigroup for *negative* times

$$\begin{aligned} A_- v_{k,l} &= A v_{k,l} = \lambda_{k,l} w_{k,l} 0 \\ A_- w_{k,l} &= 0. \end{aligned}$$

We will also need the projections P_+ and P_-:

$$\begin{aligned} P_+ v_{k,l} &= 0 \qquad & P_+ w_{k,l} &= w_{k,l} \\ P_- v_{k,l} &= v_{k,l} \qquad & P_- w_{k,l} &= 0, \end{aligned}$$

with $A_+ = AP_+$ and $A_- = AP_-$. Finally define $|A| = -A_+ + A_-$. The same analysis holds of course for the special case of a differential operator like $\bar{A}$, with the same coefficient d_{min} in all n components. The spaces, which we will use, are defined as the domain of the exponential of the operator $|\bar{A}|$. The Gevrey spaces are defined by

$$\mathcal{G}_\sigma^s = D(|A|^s \exp(\sigma |\bar{A}|)).$$

The graph norm can be expressed using the eigenfunction expansion. Letting

$$U = \sum_{k \in \mathbb{Z}^d, l=1,\ldots,n} a_{k,l} v_{k,l} + b_{k,l} w_{k,l}$$

then

$$|U|^2_{\mathcal{G}_\sigma^s} = \sum_{k \in \mathbb{Z}^d, l=1,\ldots,n} (|a_{k,l}|^2 + |b_{k,l}|^2)(1 + \sqrt{d_l}\frac{2\pi}{L}|k|)^{2s} \exp\left(2\sigma |k| \sqrt{d_{min}} \frac{2\pi}{L}\right). \tag{2.4}$$

Note that $\mathcal{G}_0^s = D(|A|^s) = X$.

As we have to deal with nonlocal operators later in the proof anyway, we will reformulate the assumptions (H.3a) and (H.3b) to include a wider class of nonlinearities. Denoting the ball $B_Y(M) = \{u \in Y | |u|_Y \le M\}$ for some Banach space Y, we will assume

For some fixed $R > 0, \sigma_0 > 0, \varepsilon_0 > 0$ the following assumptions hold uniformly in $0 \le \varepsilon \le \varepsilon_0$ and $0 \le \sigma \le \sigma_0$

$F(.), G(., \theta, \varepsilon) : B_{\mathcal{G}_\sigma^s}(2R+1) \to \mathcal{G}_\sigma^s$ are differentiable
$G(., ., \varepsilon) : B_{\mathcal{G}_\sigma^s}(2R+1) \times \mathbb{R} \to \mathcal{G}_\sigma^s$ is continuous

There exist non-decreasing functions $a_s, b_s : [0, 2R+1] \to \mathbb{R}$, (H.4)
independent of ε, θ and $0 \le \sigma \le \sigma_0$:

$$\begin{aligned} |F(U)|_{\mathcal{G}_\sigma^s} &\le a_s(|U|_{\mathcal{G}_\sigma^s}) \\ |G(U,\theta,\varepsilon)|_{\mathcal{G}_\sigma^s} &\le b_s(|U|_{\mathcal{G}_\sigma^s}) \end{aligned}$$

There exists a non-decreasing function $c_{s,\sigma} : [0, 2R+1] \to \mathbb{R}$ independent of ε, θ
$|D_U F(U)|_{L(\mathcal{G}_\sigma^s, \mathcal{G}_\sigma^s)} + |D_U G(U,\theta,\varepsilon)|_{L(\mathcal{G}_\sigma^s, \mathcal{G}_\sigma^s)} \le c_{s,\sigma}(|U|_{\mathcal{G}_\sigma^s})$

The hypothesis (H.4) holds for all functions fulfilling (H.2) and one of (H.3a) or (H.3b). But also functions nonlocal in the cross-section like $F(u)(x) = \int_\Omega u \, dx$ and spatially nonhomogenuous terms like $h(x)$, $h(.) \in \mathcal{G}_{\sigma_0}^s$ are included.

The main regularity result is the following.

Proposition 2.1 *Consider bounded solution of (1.4) fulfilling (H.1) and (H.4) (or for local nonlinearities (H.1), (H.2) and depending on the particular form (H.3a) or (H.3b)). Then for all $R > 0$, there exists $\sigma_R > 0$, such that for all $U(.)$ with $\|U(.)\|_{BC(\mathbb{R},X)} \leq R$,*

$$\|U\|_{BC(\mathbb{R},\mathcal{G}^s_{\sigma_R})} = \sup_{t\in\mathbb{R}} |U(t)|_{\mathcal{G}^s_{\sigma_R}} \leq 2R+1 \tag{2.5}$$

holds.

This proposition and its proof use ideas from similar regularity results for parabolic problems, which were analysed in [FoTe89, Pro91, TBDHT96, FeTi98]. We will show the more general part of proposition 2.1 using (H.4). Finally we will show that the hypotheses (H.2) and (H.3a/b) imply (H.4).

Proof: To establish the regularity result, we will use a Galerkin approximation by ordinary differential equations. We will derive a priori estimates for solutions fulfilling (H.1). The main idea here is to split the phase space in P_+X and P_-X, where A generates regularising semigroups for different time directions. These two subspaces are treated separately and we get estimates for the Galerkin approximation depending on the time direction. To derive estimates at 'time' T^0, we start at some earlier time $T^0 - T$ in the space P_+X and use the regularising of $\exp(A_+t)$ for positive times t, whereas we start in P_-X at some time $T^0 + T$ and then go backwards with the regularising $\exp(A_-t)$. These estimates do not depend on T_0 and thus we get uniform estimates on $\|U\|_{BC(\mathbb{R},\mathcal{G}^s_\sigma)}$.

We will use a Galerkin approximation in forward time for P_+X. In forward time A_+ generates an analytic semigroup on P_+X. We restrict the Galerkin approximation in backward time to P_-X, where A_- generates an analytic semigroup in backward time. So with P^N denoting the projection to H_N, where

$$H_N = \left\{ \mathrm{span}\{v_{k,l}, w_{k,l}\} \,\middle|\, k \in \mathbb{Z}^d, l = 1,\ldots,n, \text{ with } |k|\sqrt{d_{min}}\frac{2\pi}{L} \leq N \right\}. \tag{2.6}$$

We obtain

$$\begin{aligned} \frac{d}{dt}U_N &= AU_N + P^N F(U_N) + P^N G(U_N, \frac{t}{\varepsilon}, \varepsilon) && (2.7)\\ U_N(0) &= P^N U_0. && (2.8) \end{aligned}$$

Considering the P_+ part only, we get

$$\frac{d}{dt}U_N^+ = AU_N^+ + P_+P^N F(U_N) + P_+P^N G(U_N, \frac{t}{\varepsilon}, \varepsilon),$$

where the nonlinearities also depend on the P_-X part. Multiplying both sides with $|A|^s \exp(t|\bar{A}|)$ and taking the $L^2(\Omega, \mathbb{R}^{2n})$-scalar product with $|A|^s \exp(t|\bar{A}|)U_N^+(t)$ gives on P_+X, where $|A| = |A_+|$:

$$\begin{aligned} &\left(|A_+|^s \exp(t|\bar{A}_+|)\frac{d}{dt}U_N^+, |A_+|^s \exp(t|\bar{A}_+|)U_N^+\right)\\ = \;& \left(|A_+|^s \exp(t|\bar{A}_+|)AU_N^+, |A_+|^s \exp(t|\bar{A}_+|)U_N^+\right)\\ &+ \left(|A_+|^s \exp(t|\bar{A}_+|)(P_+P^N F(U_N) + P_+P^N G(U_N, \frac{t}{\varepsilon}, \varepsilon)), |A_+|^s \exp(t|\bar{A}_+|)U_N^+\right), \end{aligned}$$

where we dropped the explicit dependence of U_N on t. Thus

$$\frac{1}{2}\frac{d}{dt}\left||A_+|^s \exp(t|\bar{A}_+|)U_N^+\right|^2_{L^2}$$

$$
\begin{aligned}
& -(|A_+|^s \exp(t|\bar{A}_+|)|\bar{A}_+|U_N^+, |A_+|^s \exp(t|\bar{A}_+|)U_N^+) \\
= \quad & -(|A_+|^{s+1} \exp(t|\bar{A}_+|)U_N^+, |A_+|^s \exp(t|\bar{A}_+|)U_N^+) \\
& +(|A_+|^s \exp(t|\bar{A}_+|)(P_+P^N F(U_N) + P_+P^N G(U_N, \frac{t}{\varepsilon}, \varepsilon)), |A_+|^s \exp(t|\bar{A}_+|)U_N^+)
\end{aligned}
$$

The first term on the right hand side minus the last term on the left hand side is positive and can thus be dropped in an inequality: As $|A_+| - |\bar{A}_+| = |A_+ - \bar{A}_+|$ is symmetric and positive, we obtain

$$
\begin{aligned}
& (|A_+|^{s+1} \exp(t|\bar{A}_+|)U_N, |A_+|^s \exp(t|\bar{A}_+|)U_N^+) \\
& -(|A_+|^s \exp(t|\bar{A}_+|)|\bar{A}_+|U_N, |A_+|^s \exp(t|\bar{A}_+|)U_N^+) \\
= \quad & \left| |A_+ - \bar{A}_+|^{\frac{1}{2}} |A_+|^s \exp(t|\bar{A}_+|)U_N \right|_{L^2}^2 \geq 0
\end{aligned}
$$

This yields

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt} \left| |A_+|^s \exp(t|\bar{A}_+|)U_N^+ \right|_{L^2}^2 \\
\leq \quad & (|A_+|^s \exp(t|\bar{A}_+|)(P_+P^N F(U_N) + P_+P^N G(U_N, \frac{t}{\varepsilon}, \varepsilon)), |A_+|^s \exp(t|\bar{A}_+|)U_N^+).
\end{aligned} \tag{2.9}
$$

The left hand side $|U(t)|_{\mathcal{G}_t^s}$ depends on t both in U and in the norm. As

$$
\frac{1}{2}\frac{d}{dt} \left| |A_+|^s \exp(t|\bar{A}_+|)U_N^+(t) \right|_{L^2}^2 = \frac{d}{dt}\left(\left|U_N^+(t)\right|_{\mathcal{G}_t^s} \right) \left|U_N^+(t)\right|_{\mathcal{G}_t^s}
$$

and by using the Cauchy-Schwarz inequality on the right hand side of (2.9), we can conclude

$$
\begin{aligned}
\frac{d}{dt}\left|U_N^+(t)\right|_{\mathcal{G}_t^s} \quad & \leq \quad |P_+P^N F(U_N) + P_+P^N G(U_N, \frac{t}{\varepsilon}, \varepsilon)|_{\mathcal{G}_t^s} \\
& \leq \quad a_s(|U_N|_{\mathcal{G}_t^s}) + b_s(|U_N|_{\mathcal{G}_t^s})
\end{aligned} \tag{2.10}
$$

Thus we have a bound in Gevrey norms on $U_N^+(t)$ for $t \geq 0$ with increasing exponent, if we also know, that U_N^- is bounded:

$$
|U_N^+(t)|_{\mathcal{G}_t^s} \leq |U_N^+(0)|_{\mathcal{G}_0^s} + \int_0^t a_s(|U_N|_{\mathcal{G}_\tau^s}) + b_s(|U_N|_{\mathcal{G}_\tau^s}) d\tau. \tag{2.11}
$$

Due to continuous dependence on t for the ordinary differential equation, we can assume, that $|U_N(t)|_{\mathcal{G}_t^s}$ is bounded on some interval $[0, T_N]$, i.e. $|U_N(t)|_{\mathcal{G}_t^s} \leq M = 2|U_0|_X + 2$ with $|U_N(T_N)|_{\mathcal{G}_{T_N}^s} = M$. Thus using (2.11) we have an estimate for $t \in [0, T_N]$

$$
|U_N^+(t)|_{\mathcal{G}_t^s} \leq |U_N^+(0)|_X + t(a_s(M) + b_s(M)), \tag{2.12}
$$

with $M = 2|U_0|_X + 2 \leq 2R + 1$. Assuming $|U_N^+(t)|_{\mathcal{G}_t^s} \leq M/2$ we obtain a lower estimate for T_N from

$$
M/2 \leq |U_N^+(T_N)|_{\mathcal{G}_{T_N}^s} \leq |U_0|_X + T_N(a_s(M) + b_s(M)).
$$

Hence

$$
T_N \geq T^* = \frac{1}{a_s(2R+1) + b_s(2R+1)} \tag{2.13}
$$

as long $|U_N^-(t)|_{\mathcal{G}_t^s}$ also remains bounded for $t \in [0, T_N]$.

When considering negative times we get the same estimates for A_-, only the time-direction has to be changed. For $t \in [-T_N, 0]$ we obtain

$$|U_N^-(t)|_{\mathcal{G}^s_{-t}} \leq |U_N^-(0)|_X + |t|(a_s(M) + b_s(M)), \tag{2.14}$$

where T_N is as in (2.13) as long as $|U_N^+(t)|_{\mathcal{G}^s_{|t|}}$ also remains bounded for $t \in [-T_N, 0]$.

We are aiming at uniform estimates on $|U_+^N(t) + U_-^N(t)|_{\mathcal{G}^s_\sigma}$ for some $\sigma > 0$. So consider some fixed T_0, without restriction $T_0 = 0$. Then we start with the P_+X part at some negative time $-T < 0$ and evolve forward, whereas we start with P_-X at some positive time $T > 0$ and evolve backwards. Both evolutions are regularising. Taking (2.12) and (2.14) together gives

$$|U_N(0)|_{\mathcal{G}^s_T} \leq 2(\|U\|_{BC^0(\mathbb{R},X)} + T(a_s(M) + b_s(M))) \tag{2.15}$$

as long $|U_N^-(t)|_{\mathcal{G}^s_t} \leq M/2$ for $t \in [0,T]$ and $|U_N^+(t)|_{\mathcal{G}^s_{|t|}} \leq M/2$ for $t \in [-T, 0]$. But for these we obtain the same estimate, as

$$\begin{aligned} |U_N^-(t)|_{\mathcal{G}^s_t} &\leq |U_N(t)|_{\mathcal{G}^s_T} \text{ for } t \in [0,T] \\ |U_N^+(t)|_{\mathcal{G}^s_{|t|}} &\leq |U_N(t)|_{\mathcal{G}^s_T} \text{ for } t \in [-T,0]. \end{aligned}$$

with $T \leq \frac{1}{a_s(2R+1)+b_s(2R+1)}$. Hence we get uniform estimates with $M = (2\|U\|_{BC^0(\mathbb{R},X)} + 2)$. Thus $|P^N U_0|_{\mathcal{G}^s_T} = |U^N(0)|_{\mathcal{G}^s_T} \leq C$, and $P^N U_0$ converges in $\mathcal{G}^s_\sigma$ for $\sigma < T$, as $\mathcal{G}^s_\sigma$ embeds compactly in $\mathcal{G}^s_T$, therefore $U_0 \in \mathcal{G}^s_\sigma$. Hence if we know, that there is a bounded solution with $U(T_0) = U_0$, then $U_0 \in \mathcal{G}^s_\sigma$. Taking these estimates at all times uniformly gives (2.5)

$$\|U\|_{L^\infty(\mathbb{R},\mathcal{G}^s_{\sigma_R})} \leq 2(\|U\|_{BC^0(\mathbb{R},X)} + \sigma_R\,(a_s(2R+1) + b_s(2R+1))) \leq 2R+1$$

where we let

$$\sigma_R = \frac{1}{2(a_s(2R+1) + b_s(2R+1))} \leq \frac{T^*}{2}.$$

From (2.10) we get similar estimates on $\frac{d}{dt}U(0) \in L^2_{loc}(\mathbb{R}, \mathcal{G}^s_\sigma)$ and by [Tem77, ch.3, lemma1.2] U is continuous in t with values in $\mathcal{G}^s_\sigma$, which proves proposition 2.1 for (H.4).$\Box$

To complete the proof of proposition 2.1, the main key is the next lemma, which will close the remaining gap that the hypotheses (H.2) and (H.3a/b) imply (H.4). It also selects our particular choice of Gevrey classes. The nonlinearities in (1.4) map these Gevrey classes into themselves.

Lemma 2.2 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be entire. Let $U = \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{G}^s_\sigma$ with $s > \frac{d}{2}$, then $F(U) = \begin{pmatrix} 0 \\ f(u) \end{pmatrix} \in \mathcal{G}^s_\sigma$ and*

$$|F(U)|_{\mathcal{G}^s_\sigma} \leq (1 + C_s^{-1})a(C_s|U|_{\mathcal{G}^s_\sigma})$$

for some function $a : \mathbb{R}^+ \to \mathbb{R}$ independent of σ.

Proof: Gevrey spaces for scalar problems are Banach algebras, see e.g. [FeTi98, lemma 1]. For our vector valued Gevrey classes, we consider the $2n$-components separately. We define the scalar Gevrey spaces $\hat{\mathcal{G}}^s_\sigma$ for scalar functions $u(.)$ with $u(x) = \sum_{k\in \mathbb{Z}^d} u_k \exp\left(\frac{2\pi i}{L} k\cdot x\right)$ by

$$
\begin{aligned}
\hat{\mathcal{G}}^s_\sigma &= \{u \in L^2(\Omega, \mathbb{R}) \big| |u|_{\hat{\mathcal{G}}^s_\sigma} < \infty\}, \quad \text{with} \\
|u|^2_{\hat{\mathcal{G}}^s_\sigma} &= \sum_{k\in \mathbb{Z}^d, l=1,\dots,n} (a^2_{k,l} + b^2_{k,l})(1+|k|)^{2s} \exp\left(2\sigma|k|\sqrt{d_{min}}\frac{2\pi}{L}\right).
\end{aligned}
$$

This gives $\mathcal{G}^s_\sigma = (\hat{\mathcal{G}}^s_\sigma)^n \times (\hat{\mathcal{G}}^{s-1}_\sigma)^n$ and $U = (u_1, \dots, u_n, v_1, \dots, v_n)^T \in \mathcal{G}^s_\sigma$ with

$$
\begin{aligned}
u_l &\in \hat{\mathcal{G}}^s_\sigma \text{ for } l = 1, \dots, n \\
v_l &\in \hat{\mathcal{G}}^{s-1}_\sigma \text{ for } l = 1, \dots, n
\end{aligned}
$$

For the component functions $u_{l_1}, u_{l_2} \in \hat{\mathcal{G}}^s_\sigma$ we have by [FeTi98, lemma 1] for $s > \frac{d}{2}$, independent of σ:

$$
|u_{l_1} \cdot u_{l_2}|_{\hat{\mathcal{G}}^s_\sigma} \le C_s |u_{l_1}|_{\hat{\mathcal{G}}^s_\sigma} |u_{l_2}|_{\hat{\mathcal{G}}^s_\sigma}. \tag{2.16}
$$

Then as $f : \mathbb{R}^n \to \mathbb{R}^n$ is entire, there exists a power series $f(u) = \sum_{j\in \mathbb{N}^n_0} a_j u^j$, with $a_j \in \mathbb{R}^n$ and $u^j = u_1^{j_1} \cdot \ldots \cdot u_n^{j_N}$, converging for all $u = (u_1, \dots, u_n) \in \mathbb{R}^n$. Furthermore $a(r) = \sum_{k\in\mathbb{N}_0} \bar{a}_k r^k$ with $\bar{a}_k = \sum_{j\in\mathbb{N}^n_0, |j|=k} |a_j|_{\mathbb{R}^n}$ is convergent for all $r \in \mathbb{R}$ with $|f(u)| \le a(|u|)$.

To show $F(U) \in \mathcal{G}^s_\sigma$, we estimate a power series expansion of $F = (0, f)^T$ and pass to the limit. In fact we even estimate $(f, 0)^T$ by using (2.16)

$$
\begin{aligned}
\left| \sum_{|j|=0}^N (a_j u^j, 0)^T \right|_{\mathcal{G}^s_\sigma} &\le \sum_{|j|=0}^N |a_j|_{\mathbb{R}^n} \left|u^j\right|_{\hat{\mathcal{G}}^s_\sigma} \\
&\le |a_0|_{\mathbb{R}^n} + \sum_{|j|=1}^N |a_j|_{\mathbb{R}^n} C_s^{|j|-1} |U|^{|j|}_{\mathcal{G}^s_\sigma} \\
&\le (1 + C_s^{-1}) \sum_{|j|=0}^N |a_j|_{\mathbb{R}^n} C_s^{|j|} |U|^{|j|}_{\mathcal{G}^s_\sigma} \\
&\le (1 + C_s^{-1}) a(C_s |U|_{\mathcal{G}^s_\sigma})
\end{aligned}
$$

To pass to the limit we see

$$
\left| \sum_{|j|=M}^N (a_j u^j, 0) \right|_{\mathcal{G}^s_\sigma} \le (1 + C_s^{-1}) \sum_{|j|=0}^N |a_j| C_s^{|j|} |U|^{|j|}_{\mathcal{G}^s_\sigma} \to 0
$$

for $M, N \to \infty$. Hence $\sum_{|j|=0}^N (a_j u^j, 0) = \tilde{F}(.)$ exists in $\mathcal{G}^s_\sigma$ by the completeness of $\mathcal{G}^s_\sigma$. The first part u of $U = (u, u_t)$ is uniformly bounded for $\sigma \ge 0$ as $u \in H^s_{per}(\Omega, \mathbb{R}^n) \hookrightarrow BC^0(\Omega, \mathbb{R}^n)$ for $s > \frac{d}{2}$. Furthermore the $\mathcal{G}^s_\sigma$ norm is in the first n component stronger than the uniform convergence norm. Hence the analyticity of $f(.)$ implies $\sum_{|j|=0}^N a_j u^j(.) \to f(u(.))$. Hence $(f(u(.)), 0)^T \in \mathcal{G}^s_\sigma$ and

$$
|F|_{\mathcal{G}^s_\sigma} = |\begin{pmatrix} 0 \\ f \end{pmatrix}|_{\mathcal{G}^s_\sigma} \le |\begin{pmatrix} f \\ 0 \end{pmatrix}|_{\mathcal{G}^s_\sigma} \le (1 + C_s^{-1}) a(C_s |U|_{\mathcal{G}^s_\sigma}) \tag{2.17}
$$

The same argument holds for the time dependent g first for fixed ε, θ. By taking the maximum, we have a uniform majorising function b for all ε in some compact interval and for all times θ, using the continuity with respect to ε and θ in (H.2). Then the lemma holds also in the time-dependent case.□

A similar lemma holds for nonlinearities involving first derivatives, where $u_2 = \partial_t u_1$.

Lemma 2.3 *Let $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n \cdot d} \to \mathbb{R}^n$ be entire. Let $U = (u_1, u_2) \in \mathcal{G}_\sigma^s$ with $s > \frac{d}{2} + 1$, then*

$$F(U) = \begin{pmatrix} 0 \\ f(u_1, u_2, \nabla_x u_1) \end{pmatrix} \in \mathcal{G}_\sigma^s$$

with

$$|F(U)|_{\mathcal{G}_\sigma^s} \leq c_s a(C_s |U|_{\mathcal{G}_\sigma^s})$$

for $a : \mathbb{R}^+ \to \mathbb{R}$ independent of σ and s.

Proof: We first observe that $(u_2, 0)^T \in \mathcal{G}_\sigma^{s-1}$ directly from the definition (2.4) with $|(u_2, 0)^T|_{\mathcal{G}_\sigma^{s-1}} \leq |U|_{\mathcal{G}_\sigma^s}$. Spatial derivatives of u_1 with respect to x give in the eigenfunction expansion $\sum a_{k,l} v_{k,l} + b_{k,l} w_{k,l}$ at most a factor $|k|$, hence $|(\nabla_x u_1, 0)^T|_{\mathcal{G}_\sigma^{s-1}} \leq |U|_{\mathcal{G}_\sigma^s}$. Using the proof of lemma 2.2, especially the last inequality in (2.17) with $s-1$ instead of s, then gives

$$(f(u_1, u_2, \nabla_x u_1), 0)^T \in \mathcal{G}_\sigma^{s-1}.$$

Hence as

$$A(0, f(u_1, u_2, \nabla_x u_1))^T = (f(u_1, u_2, \nabla_x u_1), 0)^T \in \mathcal{G}_\sigma^{s-1}$$

holds, then $(0, f(u_1, u_2, \nabla_x u_1))^T \in \mathcal{G}_\sigma^s$. By (2.17) we then have

$$\begin{aligned} |F(U)|_{\mathcal{G}_\sigma^s} &\leq (1 + C_s^{-1}) a(C_s |U|_{\mathcal{G}_\sigma^s}, C_s |u_2|_{\mathcal{G}_\sigma^{s-1}}, C_s |\nabla_y u_1|_{\mathcal{G}_\sigma^{s-1}}) \\ &\leq c_s a(C_s |U|_{\mathcal{G}_\sigma^s}) \end{aligned}$$

This proves the result □.

Corollary 2.4 *The assumption (H.2) together with one of the assumptions (H.3a) or (H.3b) implies assumption (H.4).*

Proof: For the assumption (H.3a) with $s > \frac{d}{2}$ we can use lemma 2.2 to derive all assumptions on $F = (0, f)^T$ and $G = (0, g)^T$ except the differentiability from $\mathcal{G}_\sigma^s \to \mathcal{G}_\sigma^s$ and the boundedness of $D_U F, D_U G$ on $\mathcal{G}_0^s$. As the nonlinearities are given by power series, they are differentiable. Then $D_U F, D_U G$ can be similarly estimated as in lemma 2.2 and $D_U F(U), D_U G(U, \theta, \varepsilon)$ are bounded linear operators for bounded U, see also [Mat01, lemma 2]. In the same way (H.3b) implies (H.4) with $s > \frac{d}{2} + 1$ by applying lemma 2.3. □

This also completes the proof of proposition 2.1.

3 Averaging

In this section we are going to give the proof of theorem 1. Before going to the technical details, we outline the proof. There are three main steps:

- First we modify averaging results of exponential order for bounded ordinary differential equations by Neishtadt [Nei84], such that they can be applied to a finite dimensional Galerkin approximation of equation (1.4), where the norm of the vector field becomes unbounded with the dimension of the Galerkin approximation k. This will give a homogenised equation, where the inhomogeneous remainder is exponentially small in some algebraic expression of ε.

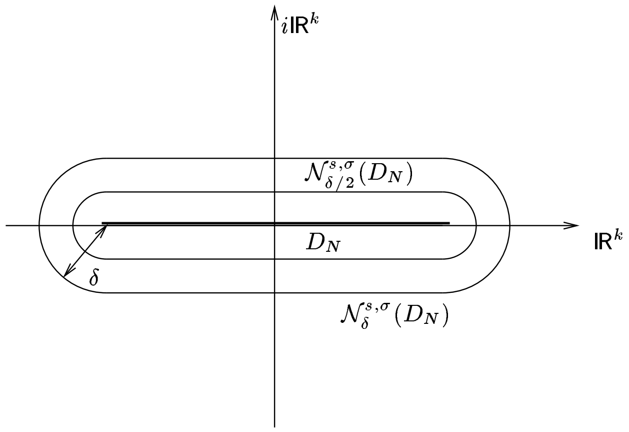

Figure 1: The domain D_N and its complex extensions.

- The second step deals with the infinite-dimensional part, which is not considered in the Galerkin approximation. This will then be estimated using the extreme regularity of bounded solutions. The remaining Galerkin modes decay exponentially in the dimension k of the our approximation space.

- Taking both exponential estimates together for an appropriate coupling of ε and k gives error estimate of order $\exp(-c\varepsilon^{-\frac{1}{2}})$.

Due to the construction of the homogeneous equations by the Galerkin approximation, which is nonlocal in the cross-section and does not preserve the second order structure, there will be nonlocal corrections, which also destroy the original second order structure of (1.1).

Thus we have to deal with nonlocal operators anyway, hence we use the reformulated assumption (H.4) instead of (H.2), (H.3a) and (H.3b). Furthermore we need to assume analyticity on the Galerkin approximation space H_N, which is the span of those eigenfunctions of $\bar{A}$, which have eigenvalues of modulus $\leq N$:

$$H_N = \left\{ \text{span}\{v_{k,l}, w_{k,l}\} \middle| k \in \mathbb{Z}^d, l = 1, \ldots, n, \text{ with } |k|\sqrt{d_{min}}\frac{2\pi}{L} \leq N \right\} \tag{3.1}$$

Then the restricted and projected nonlinearities

$$P^N F, P^N G : D_N := H_N \cap B_{\mathcal{G}^s_\sigma}(M) \to H_N \tag{3.2}$$

are assumed to be real analytic taking $H_N \cong \mathbb{R}^k$ for $k = \dim_{\mathbb{R}}(H_N)$, in accordance with (H.4) we consider a ball of radius $M = 2R + 1$. Moreover we assume that the analytic continuation of the nonlinearities to a complex δ-neighbourhood (see figure 1)

$$\mathcal{N}^{s,\sigma}_{\delta}(D_N) := \{U = \sum_{j \in \mathbb{Z}^d, |j|\frac{\sqrt{d_{min}}2\pi}{L} \leq N} U^j \exp(i\frac{2\pi}{L}jx) | U^j \in \mathbb{C}^n, \inf_{V \in D_N} |U - V|_{\mathcal{G}^s_\sigma(\Omega, \mathbb{C}^n)} < \delta\}$$

is uniformly bounded in σ, N, t and ε, again the case $\sigma = 0$ corresponds to the Sobolev norm of the phase space X. To summarise

$$
\begin{array}{l}
P^N F, P^N G : D_N \to H_N \text{ is real analytic} \\
\sup_{U_N \in \mathcal{N}_\delta^{s,\sigma}(D_N)} |P_N F(U_N)|_{\mathcal{G}_\sigma^s} \le B_1 \\
\sup_{U_N \in \mathcal{N}_\delta^{s,\sigma}(D_N), t \in \mathbb{R}} |P_N G(U_N, t, h)|_{\mathcal{G}_\sigma^s} \le B_2
\end{array}
\tag{H.5}
$$

We again only consider solutions $V(.) \in \mathcal{A}_R \subset BC(\mathbb{R}, X)$, which have a uniform bound R.

Theorem 2 *Assume (H.4) and (H.5) for the nonlinearities of (1.4). Fix the perturbation order $p \ge 0$ and the bound $R > 0$. Then there exists a t-periodic transformation of the phase space $B_X(R)$ for $0 < \varepsilon < \varepsilon_0$, given by*

$$
U = V + \varepsilon W(V, \frac{t}{\varepsilon}, \varepsilon) \tag{3.3}
$$

with the following properties. Both

$$
\begin{array}{rcl}
W(., \theta, \varepsilon) : B_X(R) & \to & X \\
W(., \theta, \varepsilon) : B_{\mathcal{G}_\sigma^s}(2R+1) & \to & \mathcal{G}_\sigma^s
\end{array}
$$

are analytic. Its image $W(X, \frac{t}{\varepsilon}, \varepsilon)$ is finite dimensional for fixed ε and $W(., 0, .) = 0$. The transformed system has the form

$$
\frac{\partial}{\partial t} V(t,x) = AV(t,x) + F(V(t,x)) + \bar{F}(V(t), \varepsilon)(x) + \alpha(V(t), \frac{t}{\varepsilon}, \varepsilon)(x), \tag{3.4}
$$

with $\alpha, \bar{F}$ bounded for $V \in B_X(R)$, but nonlocal in the cross-section.

For all bounded solutions $V(.) \in \mathcal{A}_R \subset BC(\mathbb{R}, X)$ of (1.4) and for all solutions $V(.) \in BC(\mathbb{R}, X)$ of (3.4) fulfilling (H.1) with $\|V(.)\|_{BC(\mathbb{R},X)} \le R$ – i.e. $V(.)$ is in the attractor $\tilde{\mathcal{A}}_R$ –, the following estimates on the transformed terms and their derivatives hold

$$
\begin{array}{rcll}
\sup\limits_{V(.) \in \mathcal{A}_R \cup \tilde{\mathcal{A}}_R} \|\alpha(V(.), ., \varepsilon)\|_{BC(\mathbb{R},X)} & \le & C_2 \exp(-C_1 \varepsilon^{-\frac{1}{2}}). & (3.5) \\
\sup\limits_{V(.) \in B_X(R)} \|\bar{F}(V(.), \varepsilon)\|_{BC(\mathbb{R},X)} & \le & C_3 \varepsilon^p & (3.6) \\
\sup\limits_{V \in \mathcal{A}_R \cup \tilde{\mathcal{A}}_R} \|D_V \alpha(V(.), ., \varepsilon)\|_{L(BC(\mathbb{R},X), BC(\mathbb{R},X))} & \to & 0 \text{ for } \varepsilon \to 0 & (3.7) \\
\sup\limits_{V \in B_X(R)} \|D_V \bar{F}(V(.), \varepsilon)\|_{L(BC(\mathbb{R},X), BC(\mathbb{R},X))} & \to & 0 \text{ for } \varepsilon \to 0 & (3.8)
\end{array}
$$

The exponent C_1 is given by $\min(c_1, \sigma_R)$, where σ_R is the Gevrey exponent in (2.5). The constants $c_1, C_2, C_3, \varepsilon_0$ depend on the majorising functions a_s and b_s in (H.4), on the constants B_1 and B_2 in (H.5) and R.

Remark 3.1 *The estimates (3.5,3.7) hold similarly for all bounded $V(.) \in BC(\mathbb{R}, \mathcal{G}_\sigma^s)$ for some $\sigma > 0$. The property $V(.) \in \mathcal{A}_R \cup \tilde{\mathcal{A}}_R$ is sufficient for this by proposition 2.1.*

Proof:

The proof can be divided into three steps. In the first step we analyse a finite dimensional problem by reducing to the Galerkin approximation

$$
\begin{array}{rcl}
\dot{U}_N & = & AU_N + P^N F(U_N) + \varepsilon^p P^N G(U_N, \frac{t}{\varepsilon}, \varepsilon) \\
U_N(0) & = & P^N U_0 \in H_N,
\end{array}
$$

where $AU_N = P^N AU_N$ and where we choose N depending on ε. The parameter N is by definition an upper bound on the largest eigenvalue of $|\bar{A}|$ restricted to the Galerkin approximation space H_N, see (2.1,3.1). We will couple N and ε by

$$N^{1+\beta}\varepsilon \leq 1, \tag{3.9}$$

where $\beta > 0$ is to be chosen later. Step 1 is adapted from the proof of Neishtadt's theorem [Nei84] about averaging of exponential order for finite dimensional ODE. First we make successive formal coordinate changes, such that the nonautonomous terms are formally of higher order in ε in the transformed equation

$$\dot{V}_N = AV_N + P^N F(V_N) + \bar{F}(V_N, \varepsilon) + \alpha(V_N, \frac{t}{\varepsilon}, \varepsilon). \tag{3.10}$$

Then we will give estimates uniformly in ε and $N(\varepsilon)$ using the Gevrey norms in $\mathcal{G}^s_\sigma$, which will give with $\sigma = 0$ the required estimates in X. We will perform $r \sim 1/\varepsilon^\gamma$ successive coordinate changes, where γ is chosen in the proof. In the transformed equation the nonautonomous terms are exponentially small in ε. In step 2 and step 3 we will consider again the full infinite dimensional problem. We perform the formal coordinate change and prove error estimates for the finite dimensional approximation in step 2. In step 3 we will finally combine both estimates to derive the exponential estimates.

Step 1: Finite dimensional transformation

a) Formal coordinate changes

We describe the formal coordinate changes needed to remove nonautonomous terms. For a moment we suppress the dependence of U on N. The situation after j coordinate changes is given by

$$\dot{U} = AU + P^N F(U) + \bar{F}_j(U, \varepsilon) + \alpha_j(U, \frac{t}{\varepsilon}, \varepsilon) \tag{3.11}$$

with average $\langle\alpha_j\rangle(U, \varepsilon) = \int_0^1 \alpha_j(U, \theta, \varepsilon) d\theta = 0$ and U in a complex extended domain D^j_N with $\mathcal{N}_{\delta/2}(D_N) \subset D^j_N \subset \mathcal{N}_\delta(D_N)$, see figure 1. Before performing the first coordinate change, we have $\bar{F}_0 = \varepsilon^p\langle G\rangle$ and $\alpha_0 = \varepsilon^p(G - \langle G\rangle)$.

Starting with (3.11) the next coordinate change is written as

$$U = V + \varepsilon W_j(V, \tau, \varepsilon) \tag{3.12}$$

with W periodic in $\tau = \frac{t}{\varepsilon}$ with period 1. Substitution into (3.11) yields to

$$\begin{aligned} & \dot{V} + \varepsilon\frac{\partial}{\partial V} W_j(V, \tau, \varepsilon)\dot{V} + \frac{\partial}{\partial \tau} W_j(V, \tau, \varepsilon) \\ = \; & A(V + \varepsilon W_j(V, \tau, \varepsilon)) + P^N F(V + \varepsilon W_j(V, \tau, \varepsilon)) \\ & + \bar{F}_j(V + \varepsilon W_j(V, \tau, \varepsilon), \varepsilon) + \alpha_j(V + \varepsilon W_j(V, \tau, \varepsilon), \tau, \varepsilon) \end{aligned}$$

A formal Taylor expansion in V gives

$$\begin{aligned} \dot{V} \; = \; & \left(I + \varepsilon\frac{\partial}{\partial V} W_j(V, \tau, \varepsilon)\right)^{-1} \Big\{ AV + P^N F(V) + \bar{F}_j(V, \varepsilon) \\ & + \varepsilon A W_j(V, \tau, \varepsilon) + \frac{\partial P^N F(V)}{\partial V}\varepsilon W_j(V, \tau, \varepsilon) + \frac{\partial \bar{F}_j(V, \varepsilon)}{\partial V}\varepsilon W_j(V, \tau, \varepsilon) \\ & + \alpha_j(V, \tau, \varepsilon) + \frac{\partial \alpha_j(V, \tau, \varepsilon)}{\partial V}\varepsilon W_j(V, \tau, \varepsilon) + \text{h.o.t.} - \frac{\partial}{\partial \tau} W_j(V, \tau, \varepsilon)\Big\} \\ =: \; & AV + P^N F(V) + \bar{F}_j(V, \varepsilon) + a(V, \tau, \varepsilon), \end{aligned} \tag{3.13}$$

where the last equality defines a. The term of formal lowest order in ε, which are time-dependent, is $\alpha_j(V,\tau,\varepsilon)$. To remove this term we let

$$W_j(V,\tau,\varepsilon) = \int_0^\tau \alpha_j(V,\theta,\varepsilon)d\theta.$$

Then we choose

$$\bar{F}_{j+1}(V,\varepsilon) = \bar{F}_j(V) + \langle a(V,.,\varepsilon)\rangle, \quad \alpha_{j+1}(V,\tau,h) = a(V,\tau,\varepsilon) - \langle a(V,.,\varepsilon)\rangle. \tag{3.14}$$

b) Estimates for the finite dimensional system

We give rigorous estimates for the formal procedure of part a). Suppose r substitutions are made altogether for a fixed ε. The domain D_N^j after j substitutions is given by $D_N^j = \mathcal{N}^{s,\sigma}_{\delta - jK(\varepsilon)}(D_N)$ where the function $K(\varepsilon)$ is chosen later.

We use again the notation $\|f\|_{\mathcal{G}^s_\sigma, \mathcal{N}^{s,\sigma}_\gamma(D)} = \sup_{U\in\mathcal{N}^{s,\sigma}_\gamma(D)} |f(U)|_{\mathcal{G}^s_\sigma}$ for some complex extension $\mathcal{N}^{s,\sigma}_\gamma(D)$ of $D \subset H_N \cap \mathcal{G}^s_\sigma(\Omega, \mathbb{R})$. Then by construction we have

$$\begin{aligned} \|\alpha_0\|_{\mathcal{G}^s_\sigma, \mathcal{N}^{s,\sigma}_\delta(D_N)} &\le 2B_2\varepsilon^p \\ \|\bar{F}_0\|_{\mathcal{G}^s_\sigma, \mathcal{N}^{s,\sigma}_\delta(D_N)} &\le B_2\varepsilon^p \end{aligned}$$

uniformly in σ. We will show inductively for $1 \le j \le r$:

$$\|\bar{F}_j\|_{\mathcal{G}^s_\sigma, D_N^j} \le B_1 \tag{3.15}$$

$$\|\alpha_j\|_{\mathcal{G}^s_\sigma, D_N^j} \le M_j \text{ with } M_j = 2^{-j}B_2\varepsilon^p. \tag{3.16}$$

We will choose ε_0 and $K(\varepsilon)$ such that r substitutions are defined for $0<\varepsilon<\varepsilon_0$, $V \in D_N^{r+1} = \mathcal{N}^{s,\sigma}_{\delta-(r+1)K(\varepsilon)}(D_N) \neq \emptyset$ and the inductive assumptions (3.15) and (3.16) are fulfilled.

For the estimates we need the following version of the Cauchy estimates, where the vector norm $|.|$ in $\mathbb{C}^k$ is arbitrary. We will apply it on the Galerkin approximation with norms, which are induced by the Gevrey norms.

Lemma 3.2 (Cauchy estimate) *Let $f : \Omega \subset \mathbb{C}^k \to \mathbb{C}^k$ be analytic and $\|f\|_\Omega = \sup_{u\in\Omega}|f(u)|$. Then $|\frac{\partial f}{\partial x}(x)| \le \frac{1}{\eta}\|f\|_\Omega$ for $x \in \Omega$ and $\mathrm{dist}(x, \delta\Omega) \ge \eta$.*

This is a simple application of the Cauchy formula (see [Mat01, lemma7]).

For induction we assume that (3.15) and (3.16) hold for j. To simplify notation we suppress the arguments of W and the dependence on time and parameters N,ε in the functions. We use the notation $\|f\|_\gamma := \|f\|_{\mathcal{G}^s_\sigma, \mathcal{N}^{s,\sigma}_\gamma(D)}$.

Then we obtain for the j^{th} coordinate change

$$\|\varepsilon W_j\|_{\delta_j} \le M_j\varepsilon, \tag{3.17}$$

which yields by lemma 3.2 to

$$\|\varepsilon\frac{\partial W_j}{\partial u}\|_{\delta_j - K(\varepsilon)} \le \frac{M_j\varepsilon}{K(\varepsilon)}. \tag{3.18}$$

We estimate the higher order term a in (3.12).

$$\|a\|_{\delta_j-K(\varepsilon)} \le \| \left[I + \varepsilon\frac{\partial}{\partial V}W_j\right]^{-1} \{P^N F(V+\varepsilon W_j) + \bar{F}_j(V+\varepsilon W_j)$$

$$
\begin{aligned}
&+\alpha_j(V+\varepsilon W_j)-\alpha_j(V)+A(V+\varepsilon W_j)\} \\
&-\left[AV+P^N F(V)+\bar{F}_j(V)\right]\|_{\delta_j-K(\varepsilon)} \\
\leq\ & \|\left[I+\varepsilon\frac{\partial}{\partial V}W_j\right]^{-1}\{\varepsilon AW_j+P^N F(V+\varepsilon W_j)-P^N F(V) \\
&+\bar{F}_j(V+\varepsilon W_j)-\bar{F}_j(V)+\alpha_j(V+\varepsilon W_j)-\alpha_j(V) \\
&-\varepsilon\frac{\partial}{\partial V}W_j\ \left[AV+P^N F(V)+\bar{F}_j(V)\right]\}\|_{\delta_j-K(\varepsilon)}
\end{aligned}
$$

We can estimate $\|AV\|$ by $cN\|V\|$, because $A=B\bar{A}$ with some bounded operator B and $\|\bar{A}_{|H_N}\|\leq N$. Using Neumann series and the mean value theorem we have

$$
\begin{aligned}
\|a\|_{\delta_j-K(\varepsilon)} \quad \leq \quad & \sum_{k=0}^{\infty}\|\varepsilon\frac{\partial}{\partial V}W_j\|^k_{\delta_j-K(\varepsilon)}\left\{\|\varepsilon W_j\|_{\delta_j}\left[cN+\|\frac{\partial}{\partial V}P^N F\|_{\delta_j-K(\varepsilon)}\right.\right. \\
&\left.+\|\frac{\partial}{\partial V}\bar{F}_j\|_{\delta_j-K(\varepsilon)}+\|\frac{\partial}{\partial V}\alpha_j\|_{\delta_j-K(\varepsilon)}\right] \\
&\left.+\|\varepsilon\frac{\partial}{\partial V}W_j\|_{\delta_j-K(\varepsilon)}\left[cN\|V\|_{\delta_j}+\|P^N F\|_{\delta_j}+\|\bar{F}_j\|_{\delta_j}\right]\right\}
\end{aligned}
$$

Applying the Cauchy lemma gives

$$
\|a\|_{\delta_j-K(\varepsilon)} \quad \leq \quad 2\left\{\varepsilon M_j\left[cN+\frac{B_1}{K(\varepsilon)}+\frac{2B_1}{K(\varepsilon)}\right]+\frac{M_j\varepsilon}{K(\varepsilon)}\left[cNM+B_1+2B_1\right]\right\},
$$

where M is the radius of D_N, see 3.2. Using $N^{1+\beta}\varepsilon\leq 1$, which is equivalent to $N\leq\varepsilon^{\frac{-1}{1+\beta}}$, and setting

$$
K(\varepsilon)=\tilde{K}\varepsilon^{\frac{\beta}{1+\beta}} \tag{3.19}
$$

we obtain

$$
\|a\|_{\delta_j-K(\varepsilon)}\leq M_j\left[2\varepsilon\varepsilon^{\frac{-1}{1+\beta}}+6\frac{B_1}{\tilde{K}\varepsilon^{\frac{-1}{1+\beta}}}+2\frac{M}{\tilde{K}}+6\frac{B_1}{\tilde{K}\varepsilon^{\frac{-1}{1+\beta}}}\right]\leq\frac{1}{4}M_j
$$

for $\tilde{K}$ large enough. Therefore

$$
\|\alpha_{j+1}\|_{\delta_{j+1}}<\frac{M_j}{2}=M_{j+1}
$$

and

$$
\|\bar{F}_{j+1}-\bar{F}_j\|_{\delta_{j+1}}<\frac{M_j}{4}.
$$

Hence

$$
\begin{aligned}
\|\bar{F}_{j+1}\|_{\delta_{j+1}} \quad \leq \quad & \|\bar{F}_0\|_{\delta_{j+1}}+\sum_{k=0}^{j}\|\bar{F}_{k+1}-\bar{F}_k\|_{\delta_{j+1}} \\
\leq \quad & \frac{1}{4}\sum_{k=0}^{j}M_k+B_2\varepsilon^p\leq\frac{1}{2}B_2\varepsilon^p+B_2\varepsilon^p\leq B_1
\end{aligned} \tag{3.20}
$$

for ε_0 small enough. Thus the inductive statements (3.15) and (3.16) are satisfied for $j+1$ for such a ε_0 and the above choice of $K(\varepsilon)$. So we can carry out the coordinate changes as long as $\mathcal{N}_{\delta_r}(D_N)\neq\emptyset$. More precisely, for $\delta_r=\delta-rK(\varepsilon)\geq\delta/2$ we need

$$
r=\frac{\delta}{2K(\varepsilon)}=\frac{\delta}{2\tilde{K}\varepsilon^{\frac{\beta}{1+\beta}}}=\frac{\delta}{2\tilde{K}\varepsilon^{\frac{\beta}{1+\beta}}}
$$

Letting $\alpha_* = \alpha_{r(\varepsilon)}(V, \frac{t}{\varepsilon}, \varepsilon, N(\varepsilon))$ and $\bar{F}_* = \bar{F}_{r(\varepsilon)}(V, \varepsilon, N(\varepsilon))$ we get estimates for $\mathcal{G}^s_\sigma$-norms uniform in $\varepsilon \to 0, N(\varepsilon) \to \infty$, as $\mathcal{N}^{s,\sigma}_{\delta/2}(D_N) \subset D_N^{r(\varepsilon)}$:

$$\begin{aligned} \|\alpha_*\|_{\mathcal{G}^s_\sigma, \mathcal{N}^{s,\sigma}_{\delta/2}(D_N)} &< 2^{-r} B_2 \varepsilon^p < c_2 \varepsilon^p \exp(-c_1 \varepsilon^{-\frac{\beta}{1+\beta}}), &(3.21)\\ \|\bar{F}_*\|_{\mathcal{G}^s_\sigma, \mathcal{N}^{s,\sigma}_{\delta/2}(D_N)} &< C\varepsilon^p. &(3.22)\end{aligned}$$

Applying the Cauchy estimate again, also gives estimates on the derivatives:

$$\begin{aligned} \|D_U \alpha_*\|_{\mathcal{G}^s_\sigma, \mathcal{N}^{s,\sigma}_{\delta/4}(D_N)} &< C \exp(-c_1 \varepsilon^{-\frac{\beta}{1+\beta}}), &(3.23)\\ \|D_U \bar{F}_*\|_{\mathcal{G}^s_\sigma, \mathcal{N}^{s,\sigma}_{\delta/2}(D_N)} &< C\varepsilon^p. &(3.24)\end{aligned}$$

Step 2: Transformation of the full system
a) Formal Transformation
Next we deal with the full infinite dimensional system. We let

$$U = V + \varepsilon W(P^N V, \frac{t}{\varepsilon}, \varepsilon, N),$$

which means $(I - P_N)U = (I - P_N)V$ as $W \in H_N$, i.e. the other modes stay unchanged. Here $N = N(\varepsilon)$ is chosen maximally such that $N^{1+\beta}\varepsilon \leq 1$ holds. Thus in the new coordinates we get, when suppressing the arguments $\frac{t}{\varepsilon}$ and ε of W

$$\begin{aligned} &\dot{V} + \varepsilon \frac{\partial}{\partial P^N V} W(P^N V)\dot{V} + \frac{\partial}{\partial \tau} W(P^N V)\\ =\ & A(V + \varepsilon W(P^N V)) + F(V + \varepsilon W(P^N V)) + \varepsilon^p G(V + \varepsilon W(P^N V), \frac{t}{\varepsilon}, \varepsilon).\end{aligned}$$

Solving for $\dot{V}$ gives

$$\begin{aligned} \dot{V} &= \left[I + \varepsilon \frac{\partial}{\partial P^N V} W(P^N V)\right]^{-1} \Big\{A(V + \varepsilon W(P^N V)) + F(V + \varepsilon W(P^N V))\\ &\quad + \varepsilon^p G(V + \varepsilon W(P^N V), \frac{t}{\varepsilon}, \varepsilon) - \frac{\partial}{\partial \tau} W(P^N V, \frac{t}{\varepsilon}, \varepsilon)\Big\}.\end{aligned}$$

We split $\mathcal{G}^s_\sigma$ into the Galerkin approximation space H_N and its $\mathcal{G}^s_\sigma$ orthogonal complement $H_N^\perp$. By using

$$\left[I + \varepsilon \frac{\partial}{\partial P^N V} W(P^N V)\right]^{-1}_{|H_N^\perp} = I_{|H_N^\perp}$$

we obtain

$$\begin{aligned} \dot{V} &= (I - P^N)\Big\{AV + F(V + \varepsilon W(P^N V)) + \varepsilon^p G(V + \varepsilon W(P^N V), \frac{t}{\varepsilon}, \varepsilon)\Big\}\\ &\quad + P^N \left[I + \varepsilon \frac{\partial}{\partial P^N V} P^N W(P^N V)\right]^{-1} P^N \Big\{A(V + \varepsilon W(P^N V))\\ &\qquad + F(V + \varepsilon W(P^N V)) + \varepsilon^p G(V + \varepsilon W(P^N V), \frac{t}{\varepsilon}, \varepsilon) - \frac{\partial}{\partial \tau} P^N W(V, \frac{t}{\varepsilon}, \varepsilon)\Big\}\\ &= (I - P^N)\Big\{AV + F(V + \varepsilon W(P^N V)) + \varepsilon^p G(V + \varepsilon W(P^N V), \frac{t}{\varepsilon}, \varepsilon)\Big\}\end{aligned}$$

$$
\begin{aligned}
&+P^N\left\{AV + F(V) + \bar{F}_*(V,\varepsilon,N) + \alpha_*(V,\frac{t}{\varepsilon},\varepsilon,N)\right\} \\
&+P^N\left[I+\varepsilon\frac{\partial}{\partial P^N V}W(P^N V)\right]^{-1} P^N\Big\{F(V+\varepsilon W(P^N V)) \\
&\quad +\varepsilon^p G(V+\varepsilon W(P^N V),\frac{t}{\varepsilon},\varepsilon) - F(P^N V+\varepsilon W(P^N V)) \\
&\quad -\varepsilon^p G(P^N V+\varepsilon W(P^N V),\frac{t}{\varepsilon},\varepsilon)\Big\}.
\end{aligned}
$$

Thus we have an equation as in (1.5)

$$
\dot{V} = AV + F(V) + \bar{F}(V,\varepsilon) + \alpha(V,\frac{t}{\varepsilon},\varepsilon) \tag{3.25}
$$

where we group the terms in the following way

$$
\begin{aligned}
\bar{F}(V,\varepsilon) &= \bar{F}_*(V,\varepsilon) + \langle b_1(V,.,\varepsilon,N(\varepsilon))\rangle && (3.26)\\
\alpha(V,\frac{t}{\varepsilon},\varepsilon) &= \alpha_*(V,\frac{t}{\varepsilon},\varepsilon,N(\varepsilon)) + b(V,\frac{t}{\varepsilon},\varepsilon,N(\varepsilon)) - \langle b_1(V,.,\varepsilon,N(\varepsilon))\rangle, && (3.27)
\end{aligned}
$$

with an additional correction term $b = b_1 + b_2$. The infinite-dimensional correction b_1 of the error in the higher Galerkin modes (where $\bar{A}$ has eigenvalues of modulus greater than N) occurs as these modes were neglected in the coordinate transformation. We have

$$
\begin{aligned}
& b_1(V,\frac{t}{\varepsilon},\varepsilon,N(\varepsilon)) \\
=\;& (I-P^N)\left\{F(V+\varepsilon W(P^N V)) - F(V) + \varepsilon^p G(V+\varepsilon P^N W(V),\frac{t}{\varepsilon},\varepsilon)\right\}. \qquad (3.28)
\end{aligned}
$$

We get also a correction b_2 in the finite dimensional Galerkin space, as there is an error in the lower modes due to the neglected influence of the higher Galerkin modes to the lower ones. It is given by

$$
\begin{aligned}
& b_2(V,\frac{t}{\varepsilon},\varepsilon,N(\varepsilon)) \\
=\;& \left[I+\varepsilon\frac{\partial}{\partial V}W(P^N V)\right]^{-1} P^N\left\{F(V+\varepsilon W(P^N V)) - F(P^N V+\varepsilon W(P^N V))\right. \\
& \left. +\varepsilon^p G(V+\varepsilon W(P^N V),\frac{t}{\varepsilon},\varepsilon) - \varepsilon^p G(P^N V+\varepsilon W(P^N V),\frac{t}{\varepsilon},\varepsilon)\right\}. \qquad (3.29)
\end{aligned}
$$

b) Estimating the full system

To prove the estimates (3.5, 3.6, 3.7, 3.8) we will show first, that bounded solutions of the transformed equation (3.25) are Gevrey regular. Then the additional terms can be estimated very easily using the following observation:

We use only that $V(t)$ is uniformly bounded in the Gevrey norm $\mathcal{G}^s_\sigma$, this is the reasons, why remark 3.1 holds. When we measure the difference between V and its projection to H_N then we get an exponential estimate:

$$
|V(t) - P^N V(t)|_X \le |V(t)|_{\mathcal{G}^s_\sigma} \exp(-\sigma N). \tag{3.30}
$$

For solutions $V(.)$ in the attractor of (1.4), this holds directly by proposition 2.1. As pointed out in remark 1.1, we do not need to assume that $V(.)$ solves the particular equation, we are transforming, solutions of any other equation, which fulfills the assumptions of proposition 2.1 gives the needed uniform boundedness of the Gevrey norms.

To show it for the transformed equation, we first have to check the assumptions of the regularity proposition 2.1 for the transformed nonlinearity $\bar{F}(.)+\alpha(.,\frac{t}{\varepsilon},\varepsilon,N)+b_1(.,\frac{t}{\varepsilon},\varepsilon,N)+b_2(.,\frac{t}{\varepsilon},\varepsilon,N)$ for fixed N,ε. Then we can use (3.30) to estimate b_1 and b_2. The property (H.1) for $V(.)$ holds by assumption. It remains to check (H.4).

Consider the nonlinearity $\bar{F}_*(.,\varepsilon,N)+\alpha_*(.,\frac{t}{\varepsilon},\varepsilon,N)$, which was defined by the finite dimensional part on $D_N = H_N \cap B_{\mathcal{G}^s_\sigma}(M)$, a ball of radius $M = 2R+1$. By (3.21) and (3.22) we have the existence of majorating functions a_s, b_s independent of σ. The existence of $c_{s,\sigma}$ follows from (3.23,3.24). The differentiability and continuity comes from the fact, that they hold for the original equation and the transformation is analytic in U and continuous in θ. It remains to show the properties for

$$F(.+\varepsilon W(P^N.)), G(.+\varepsilon P^N W(.),\frac{t}{\varepsilon},\varepsilon) : B_{\mathcal{G}^s_\sigma}(2R+1) \to \mathcal{G}^s_\sigma.$$

The transformation $V+\varepsilon W(V)$ is given by the successive coordinate changes $V_j = V_{j+1} + \varepsilon W_{j+1}(V_{j+1})$. Using (3.17, 3.18), $V+\varepsilon W(V)$ is a near-identity and differentiable coordinate change uniformly for every $\mathcal{G}^s_\sigma$ norm. The mapping

$$\left[I + \varepsilon \frac{\partial}{\partial V} P^N W(V)\right]^{-1} P^N$$

occurring in b_2 is near to the identity by (3.18). Thus the estimates on F and G and their derivatives hold in a straight forward way for b_1 and b_2 and their derivatives too. The differentiability and continuity are preserved by the transformation. Thus the assumption (H.4) of proposition 2.1 holds too, i.e. proposition 2.1 is applicable to the solution V of (3.25).

Hence (3.25) has Gevrey regular solutions. We can estimate $|\alpha(V(t),\frac{t}{\varepsilon},\varepsilon)|_X$ and $|\bar{F}(V(t),\varepsilon)|_X$. We start with $\bar{F}(.,\varepsilon) = \bar{F}_*(.,\varepsilon,N) + \langle b_1(.,.,\varepsilon,N) + b_2(.,.,\varepsilon,N)\rangle$. For the finite dimensional part we have $|\bar{F}_*(V,\varepsilon,N)|_X \le c_3\varepsilon^p$ by (3.22) for $V \in X$ *bounded*. Estimating $\langle b_1\rangle$ we obtain for $V \in X = \mathcal{G}^s_0$ bounded:

$$\begin{aligned}
&|\langle b_1(V,.,\varepsilon,N)\rangle|_X \\
\le\ & |(I-P^N)\left\{F(V+\varepsilon W(P^N V)) - F(V) + \varepsilon^p G(V+\varepsilon P^N W(V),\frac{t}{\varepsilon},\varepsilon)\right\}|_X \\
\le\ & \sup_{U=V+rW(P^N V), r\in[0,\varepsilon]} |F'(U)|_{L(X,X)} |\varepsilon W(P^N V)|_X + \varepsilon^p B_1 \le C\varepsilon^p \qquad (3.31)
\end{aligned}$$

because $|\varepsilon W(P^N V)|_X \le C\varepsilon^{p+1}$.

Furthermore we have $\|D_U(\langle b_1\rangle)\|_{L(X,X)} \le C\varepsilon^p$ when $\varepsilon \to 0$ for $U \in X$ bounded, as F and G are C^1 in U. This gives together with (3.24) the estimate (3.8).

Next we need a slightly more careful analysis for the nonautonomous part:

$$|\alpha|_X = |\alpha_* + b_1 + b_2 - \langle b_1\rangle|_X.$$

For the first part we have by (3.21):

$$|\alpha_*(V,\frac{t}{\varepsilon},\varepsilon,N)|_X \le c_2\varepsilon^p \exp(-c_1\varepsilon^{-\frac{\beta}{1+\beta}})$$

For $b_1 - \langle b_1\rangle$ we use the same analysis as for $\langle b_1\rangle$ above, except we also note that V is bounded in the Gevrey norm $\mathcal{G}^s_\sigma$. Hence $b_1(V) \in \mathcal{G}^s_\sigma$ and thus with $K = \{V + r\varepsilon W(P^N V), r \in [0,1]\}$

$$|b_1(V,\frac{t}{\varepsilon},\varepsilon,N) - \langle b_1(V,.,\varepsilon,N)\rangle|_X \quad \le \quad 2|(I-P^N)\{F(V+\varepsilon W(P^N V))$$

$$
\begin{aligned}
& -F(V(t)) + \varepsilon^p G(V + \varepsilon W(P^N V), \frac{t}{\varepsilon}, \varepsilon)\}|_X \\
\leq\ & \left(\sup_{U \in K} |F'(U)|_{L(\mathcal{G}^s_\sigma, \mathcal{G}^s_\sigma)} |\varepsilon W(P^N V)|_{\mathcal{G}^s_\sigma} + \varepsilon^p B_1\right) \\
& \cdot \exp(-\sigma|\lambda_{N+1}|) \\
\leq\ & C\varepsilon^p \exp(-\sigma N) .
\end{aligned}
\tag{3.32}
$$

For $b_2(V)$, which was created by the Galerkin approximation in the finite dimensional part, we obtain an exponential estimate for $V \in \mathcal{G}^s_\sigma$. For $V \in \mathcal{G}^s_\sigma$ define the line segment $K_V = \{rV + (1-r)P^N V + \varepsilon W(P^N V) | r \in [0,1]\}$, we then obtain

$$
\begin{aligned}
& |b_2(V, ., \varepsilon, N)|_X \\
\leq\ & |\left[I + \varepsilon \frac{\partial}{\partial P^N V} W(P^N V)\right]^{-1} P^N \{F(V + \varepsilon W(P^N V)) - F(P^N V + \varepsilon W(P^N V)) \\
& \quad + \varepsilon^p G(V + \varepsilon W(P^N V), \frac{t}{\varepsilon}, \varepsilon) - \varepsilon^p G(P^N V + \varepsilon W(P^N V), \frac{t}{\varepsilon}, \varepsilon)\Big\} |_X \\
\leq\ & 2(\sup_{U \in K_V} |F'(U)|_{L(X,X)} |V(t) - P^N V(t)|_X \\
& \quad + \sup_{U \in K_V} |G'(U(t))|_{L(X,X)} |V - P^N V|_X) \\
\leq\ & C|V - P^N V|_X \leq C \exp(-\sigma N) ,
\end{aligned}
\tag{3.33}
$$

where we used (3.30). As above we also obtain $\|D_U(b_1 - \langle b_1 \rangle + b_2)(V)|_{L(X,X)} \to 0$ for $\varepsilon \to 0$ and $V \in \mathcal{G}^s_\sigma$ for $\sigma > 0$, as F, G are C^1. This gives, together with (3.23), the estimate (3.7).

Step 3: Combining the exponential estimates

We now balance the exponential estimates of Step 1 and Step 2 using the freedom we still have in choosing β in (3.9). Using that N is maximal such that $N \leq \varepsilon^{\frac{-1}{(1+\beta)}}$, the estimates (3.32, 3.33) can be expressed in ε. Then we can balance the two exponential estimates of the different parts of α, the exponent σ is then σ_R given by proposition 2.1.

$$
\begin{array}{ll}
\exp(-c_1 \varepsilon^{-\frac{\beta}{1+\beta}}) & \text{from (3.21)} \\
\exp(-\sigma_R \varepsilon^{-\frac{1}{1+\beta}}) & \text{from (3.32, 3.33)}
\end{array}
$$

The best exponential estimate is obtained when choosing $\beta = 1$. Then the estimates are of order $\exp(-c\varepsilon^{-\frac{1}{2}})$ and $N(\varepsilon) = O(\varepsilon^{-\frac{1}{2}})$. This gives the estimates (3.5,3.6). It just remains to check the boundedness of $\bar{F}$ and α. The transformed nonlinearities $\bar{F}$ and α are bounded on X, since all estimates hold for $\mathcal{G}^s_\sigma$, $\sigma \geq 0$ and thus also for $X = \mathcal{G}^s_0$. The estimates in (3.31,3.32, 3.33) do not yield exponential smallness for the case $\sigma = 0$, but they give then still boundedness. The finite dimensional part is still exponentially small.

Differentiability, in fact analyticity of the transformed nonlinearities and of W with respect to V in some ball of X resp. $\mathcal{G}^s_\sigma$ follows with the Cauchy estimate as W, $\bar{F}$ and α are bounded by the above analysis on a complex extension of such balls in X resp. $\mathcal{G}^s_\sigma$. □

Remark 3.3 *If G is smooth in ε, then the construction can be modified such that $\bar{F}(V, \varepsilon)$ and $\alpha(V, \frac{t}{\varepsilon}, \varepsilon)$ are smooth in ε. This can be done in the same way as in [Mat01, remark9].*

Proof of theorem 1: It remains to check (H.4) and (H.5), such that theorem 2 then implies theorem 1. By corollary 2.4, the assumptions (H.3a)/(H.3b) imply (H.4). To check (H.5), we restrict and project F and G to H_N. Then F, G are given by a convergent power series in the coefficients of the eigenfunction expansion, see [Mat01, remark 6]. This power series converges for the whole H_N. The estimates on the complex extension follow from lemma 2.2 and lemma 2.3 respectively.

Thus the assumptions for theorem 1 imply those of theorem 2 and the proof is complete. $\square$

4 Equilibrium solutions

We now apply theorem 1 and 2 to describe quantitatively the effect of the nonautonomous forcing on special solutions. Our main interest here is on equilibrium solutions of the original – unforced – equation.

Other special solutions, to be analysed in a forth-coming paper, which can be understood as fronts of velocity 0 for parabolic equations as in (1.2)

$$u_\tau = \Delta_{(t,x)} u + f(u, \lambda)$$

on a space-domain $(t, x) \in \mathbb{R} \times \Omega$ with time τ and a parameter $\lambda \in \mathbb{R}$. Typically those pinned fronts exist in the homogenouos case only for special parameter values λ_0, whereas adding nonhomogenouos perturbation $\varepsilon g(u, t)$ will pin fronts in a whole interval I of parameters. Using theorem 1 and theorem 2 it is possible to show, that the length $|I|$ of the pinning interval is exponentially small for rapid forcing terms $\varepsilon g(u, t/\varepsilon)$.

Here we study equilibria, equilibria are solutions $u(t, x) = w(x)$, which do not depend on t, i.e. $\frac{d}{dt} u(t, x) \equiv 0$, i.e. $D\Delta_x w + f(w, 0, \nabla_x w) = 0$. We will consider hyperbolic equilibria. As in dynamical systems the equilibrium $w(x)$ is called hyperbolic, if the linearised operator

$$L = A + DF(w, \nabla_x w, 0)$$

has no spectrum in a strip near the imaginary axis:

$$\mathrm{spec}(L) \cap \{z | |Re(z)| < \delta\} = \emptyset.$$

For another definition of hyperbolic equilibria of elliptic equations on cylinders, see [FSV99, sec. 5]. Note that when we impose periodic boundary conditions, that there is a translation symmetry in x, creating zero eigenvalues. Adding a slow spatial inhomogeneity h in an appropriate Gevrey space to break this symmetry is allowed by (H.4), such that we consider

$$\partial_t^2 u + D\Delta_x u \quad = \quad f(u, u_t, \nabla_x u) + \varepsilon^p g(u, u_t, \nabla_x u, \frac{t}{\varepsilon}, \varepsilon) + h(x), \tag{4.1}$$

and the whole theory still applies. It is standard, that hyperbolic equilibria persist under small autonomous perturbations and that they also persist as solutions, which are ε-periodic in t, under rapid nonautonomous periodic forcing. The distance between original and the perturbation is in the order of the perturbation.

The new result is, that the periodic solutions, which are created by rapid forcing of the hyperbolic equilibria are in fact exponentially close to hyperbolic equilibria of the homogenized equation. Thus the modulation is only exponentially small in the transformed phase-space.

Proposition 4.1 *Let the assumptions of theorem 1 hold. Fix some perturbation order $p > 0$. Furthermore assume that the original unforced equation (1.4 with $\varepsilon = 0$) has a hyperbolic equilibrium p_0. Then the forced solution has a bounded solution $p_\varepsilon(t, x)$, which is periodic in*

t with period ε. The truncated equation (1.8) has a hyperbolic equilibrium solution $\bar{p}_\varepsilon$. For these the following holds:

$$|p_0 - \bar{p}_\varepsilon|_X \leq C\varepsilon^p \tag{4.2}$$

$$|p_\varepsilon(t) - \bar{p}_\varepsilon|_X \leq C\exp(-C_1\varepsilon^{-\frac{1}{2}}), \text{ for all } t \in \mathbb{R} \tag{4.3}$$

i.e. the periodic solution created by the rapid inhomogeneity is exponentially close to the equilibrium of the homogenised equation.

Proof: To prove the persistence of equilibria and the existence of periodic solutions we apply some implicit functions argument, similar to Lord et al. in [LPSS00]. The following version of Banach's fixed point theorem is used, see [LPSS00].

Lemma 4.2 *Suppose Y is a Banach space and $H : Y \to \tilde{Y}$ is C^1. Assume, that there exists a bounded linear and invertible operator $L : Y \to \tilde{Y}$, an element $U_0 \in Y$ and numbers $\eta > 0$ and $0 < \kappa < 1$ such that*

1. $|I - L^{-1}DH(U)|_{L(Y,Y)} \leq \kappa$ *for all* $U \in B_\eta(U_0)$
2. $|L^{-1}H(U_0)|_Y \leq (1-\kappa)\eta$

There exists then a unique point $U_* \in B_\eta(U_0)$ with $H(U_*) = 0$ and

$$|U_0 - U_*|_Y \leq (1-\kappa)^{-1}|L^{-1}H(U_0)|_Y$$

First we show the persistence of hyperbolic equilibria $\bar{p}_\varepsilon$. Consider

$$\begin{aligned} & (A + D_UF(p_0))^{-1}(A(p_0+U) + F(p_0+U) + \bar{F}(p_0+U,\varepsilon)) \\ = \; & U + (A + D_UF(p_0))^{-1}(F(p_0+U) - F(p_0) - D_UF(p_0)U + \bar{F}(p_0+U,\varepsilon)) \\ =: \; & H_\varepsilon(U) \end{aligned}$$

To find equilibria, we look for zeros near $U = 0$. The map H is smooth in U as a map $B_X(R) \to X$ and by (3.6)

$$|H_\varepsilon(0)| \leq |(A + D_UF(p_0))^{-1}\bar{F}(p_0,\varepsilon)| \leq C\varepsilon^p.$$

Furthermore we have

$$\begin{aligned} & \|D_UH_\varepsilon(U) - I\| \\ = \; & \|(A + D_UF(p_0))^{-1}(D_UF(p_0+U) - D_UF(p_0) + D_U\bar{F}(p_0+U,\varepsilon))\| \leq \frac{1}{2} \end{aligned}$$

for all U in a small enough ball in X of radius η, uniformly for $0 \leq \varepsilon \leq \varepsilon_0$ by (3.8). Hence there is by lemma 4.2 a unique zero $\bar{p}_\varepsilon \in B_\eta(U_0) \subset X$ of H_ε with

$$|\bar{p}_\varepsilon - p_0| \leq C\varepsilon^p$$

The linearisation $A + D_UF(\bar{p}_\varepsilon) + D_U\bar{F}(\bar{p}_\varepsilon,\varepsilon)$ is hyperbolic for ε small enough, since F is C^1 and $D_U\bar{F}$ is small. Furthermore $\bar{p}_\varepsilon$ is as smooth in ε as $\bar{F}(U,\varepsilon)$ is.

To find periodic solutions consider

$$H(U) = U_t - \left(AU + F(U) + \bar{F}(U,\varepsilon) + \alpha(U, \frac{t}{\varepsilon}, \varepsilon)\right) \tag{4.4}$$

as an operator on some subspace of $C^0_{per}([0,T],X)$, the space of periodic continuous functions on the finite interval $(0,T)$ with values in X. Using an implicit function argument, we will show, that there exists a unique solution in this subspace near the equilibrium p_0, which will be the desired ε-periodic solution, when letting $T = k\varepsilon$, $k \in \mathbb{N}$.

Let $\tilde{X} = H^r(\Omega, \mathbb{R}^n) \times H^{r-1}(\Omega, \mathbb{R}^n)$ with $r \in \mathbb{R}$ and $r+1 > s > r \geq 0$ and with the convention $H^0 = L^2$. Then on $\tilde{X}$ the domain of A is given by $D(A_{\tilde{X}}) = H^{r+1}(\Omega, \mathbb{R}^n) \times H^r(\Omega, \mathbb{R}^n)$. The Sobolev exponents $s > r$ are chosen such that $F : \tilde{X} \to \tilde{X}$ is differentiable. This is possible for our the local nonlinearity in (4.1) by lemma 2.3 and the proof of corollary 2.4. We consider the following linear operator L_0 related to (4.4). We will show, that L_0 is invertible and which will imply by Fredholm arguments, that $DH(p_0)$ is also invertible. Let P_0 denote the projection to the n-dimensional kernel of A, spanned by $v_{0,l}$ (see (2.2)).

$$\begin{aligned} L_0(U) &= U_t - (A + P_0)U \\ L_0 &: Y \to \tilde{Y} \end{aligned}$$

Where we use the following function spaces $\tilde{Y} = C^{0;\gamma}_{per}([0,T], \tilde{X})$, the space of periodic Hölder continuous functions with norm

$$\|U\|_{\tilde{Y}} = \sup_{t \in [0,T]} |U(t)|_{\tilde{X}} + \sup_{\tau, t \in [0,T]} \frac{|U(t) - U(\tau)|_{\tilde{X}}}{|t - \tau|^\gamma}$$

and $Y = C^1_{per}([0,T], \tilde{X}) \cap C^0_{per}([0,T], D(A)) \cap \{U | U_t - (A + P_0)U \in \tilde{Y}\}$ with norm

$$\|U\|_Y = \sup_{t \in [0,T]} (|U(t)|_{\tilde{X}} + |U'(t)|_{L(\tilde{X},\tilde{X})}) + \sup_{t \in [0,T]} |U(t)|_{D(A)} + \|U_t - (A + P_0)U\|_{\tilde{Y}}.$$

Then L_0 is invertible. As mentioned in the beginning of section 2, A_- generates an analytic semigroup on P_-X for negative times and A_+ generates an analytic semigroup on P_+X for positive times. The latter is not changed by the finite dimensional perturbation to $A_+ + P_0$, which still leaves P_+X invariant.

For $f \in \tilde{Y}$, the inverse is given by

$$\begin{aligned} U(t) &= \int_0^t \exp\big((A_+ + P_0)(t-\tau)\big) P_+ f(\tau) d\tau + \exp\big((A_+ + P_0)t\big) P_+ U(0) \\ &\quad + \int_t^T \exp\big(A_-(\tau - T)\big) P_- f(\tau) d\tau + \exp\big(A_-(t - T)\big) P_- U(0), \end{aligned}$$

where the $U(0)$ and $U(T)$ are chosen to fulfil the periodic boundary conditions $U(0) = U(T)$ on $(0,T)$:

$$\begin{aligned} P_- U(0) &= \Big(I - \exp\big(A_-(-T)\big)\Big)^{-1} \int_0^T \exp\big(A_-(\tau - T)\big) P_- f(\tau) d\tau \\ P_+ U(0) &= \Big(I - \exp\big((A_+ + P_0)T\big)\Big)^{-1} \int_0^T \exp\big((A_+ + P_0)(T - \tau)\big) P_+ f(\tau) d\tau. \end{aligned}$$

From [Hen83, lemma 3.5.1] we then have, that U is even differentiable with values in some fractional power space, which can be identified with the original X, if the Hölder exponent γ is large enough, i.e. for $\gamma > s - r$, then $U \in C^1((0,T), X)$. Next we consider

$$\begin{aligned} L_1 : Y &\to \tilde{Y} \\ L_1(U) &= U_t - AU - D_U F(p_0) U \end{aligned}$$

Again, as $U \in Y$ implies $U \in C^1_{per}([0,T],X)$, we have then $D_U F(p_0)U \in C^1_{per}([0,T],X)$ and thus $L_1(U) \in \tilde{Y}$. As $C^1_{per}([0,T],X)$ embeds compactly into $\tilde{Y} = C^{0,\gamma}([0,T],\tilde{X})$, we have that $D_U F(p_0) : Y \to \tilde{Y}$ is a compact operator. Then $L_1 = L_0 - P_0 + D_U F(p_0)$ is Fredholm of index 0: As L_0 is invertible, it is Fredholm of index 0 and Fredholm properties persist under the compact perturbation $-P_0 + D_U F(p_0)$. Thus L_1 is Fredholm of index 0. Due to the hyperbolicity of the equilibrium p_0, the kernel of L_1 is trivial, thus L_1 is invertible.

Next we apply lemma 4.2 to H given in (4.4) and L given by

$$LU = L_1 U = U_t - AU - D_U F(p_0)U.$$

Then

$$|I - L^{-1}DH(U)|_{L(Y,Y)} = |L^{-1}(D_U F(U) - D_U F(p_0) + D_U(\bar{F}(U,\varepsilon) + \alpha(U,.,\varepsilon))| \leq \frac{1}{2} \quad (4.5)$$

holds for $U \in B_\eta(\bar{p}_\varepsilon) \subset B_{2\eta}(p_0)$ for some fixed small η and for $0 < \varepsilon < \varepsilon_0$, as F is C^1 and $|D_U \bar{F}(U,\varepsilon)|_{L(X,X)} \in o(1)$ by (3.8). For $|D_U \alpha(U,\varepsilon)|$ we need a more careful analysis: By (3.27)

$$\alpha(U,\frac{t}{\varepsilon},\varepsilon) = \alpha_*(U,\frac{t}{\varepsilon},\varepsilon,N(\varepsilon)) + b_1(U,\frac{t}{\varepsilon},\varepsilon,N(\varepsilon)) - \langle b_1(U,.,\varepsilon,N(\varepsilon))\rangle + b_2(U,\frac{t}{\varepsilon},\varepsilon,N(\varepsilon))$$

By the analysis in section 3,

$$|D_U(\alpha_*(V,\frac{t}{\varepsilon},\varepsilon,N(\varepsilon)) + b_1(V,\frac{t}{\varepsilon},\varepsilon,N(\varepsilon)) - \langle b_1(V,.,\varepsilon,N(\varepsilon))\rangle)|_{L(X,X)} \leq C\varepsilon^p$$

for $U \in B_X(R)$. Now consider $D_U b_2$, where in general $|D_U b_2(U,\frac{t}{\varepsilon},\varepsilon,N(\varepsilon))|_{L(X,X)}$ is of order $O(1)$ for $U \in B_X(R)$. We will use here the regularising property of L^{-1}. As the Sobolev exponent r in the definition of $\tilde{X}$ is chosen such that $F : \tilde{X} \to \tilde{X}$ is differentiable we can estimate as follows, suppressing some arguments for notational convenience.

$$\begin{aligned}
& |D_U b_2(U)|_{L(X,\tilde{X})} \\
= \; & |D_U\{\left[I + \varepsilon\frac{\partial}{\partial P^N U}W(P^N U)\right]^{-1} P^N \left\{F(U + \varepsilon W(P^N U)) - F(P^N U + \varepsilon W(P^N U))\right. \\
& \left. +\varepsilon^p G(U + \varepsilon W(P^N U),\frac{t}{\varepsilon},\varepsilon) - \varepsilon^p G(P^N U + \varepsilon W(P^N U),\frac{t}{\varepsilon},\varepsilon)\right\}\}|_{L(X,\tilde{X})} \\
\leq \; & C|DF(U + \varepsilon W(P^N U))(id + \varepsilon\frac{\partial}{\partial P^N U}W(P^N U)) \\
& -DF(P^N U + \varepsilon W(P^N U))(P^N + \varepsilon\frac{\partial}{\partial P^N U}W(P^N U)|_{L(X,\tilde{X})} + C\varepsilon^p \\
\leq \; & C\varepsilon^p + C|DF(U + \varepsilon W(P^N U))(I - P_N)|_{L(X,\tilde{X})} \\
& +|[DF(U + \varepsilon W(P^N U)) - DF(P^N U + \varepsilon W(P^N U))][P_N + \varepsilon\frac{\partial}{\partial P^N U}W(P^N U)]|_{L(X,\tilde{X})} \\
\leq \; & C\varepsilon^p + CN^{r-s} + C\eta,
\end{aligned}$$

as $|(I - P_N)|_{L(X,\tilde{X})} \leq CN^{r-s}$ and $|P^N U - U|_X \leq C\eta$ uniformly in $B_\eta(\bar{p}_\varepsilon)$ for $N(\varepsilon) \geq N_0$ large enough. Choosing η small enough and using $L^{-1} : \tilde{Y} \to Y$ and $D(A_{\tilde{X}}) \subset X$ completes the estimate (4.5).

Furthermore

$$|L^{-1}H(\bar{p}_\varepsilon)|_Y = |L^{-1}\alpha(\bar{p}_\varepsilon,.,\varepsilon))|_Y \leq C\exp\left(-\frac{c}{\sqrt{\varepsilon}}\right),$$

by (1.6). Even if $\bar{p}_\varepsilon$ is in general not in the attractor of the equation (1.5), the estimates as in (1.6) still hold, as $\bar{p}_\varepsilon(t)$ is Gevrey regular, see remark 3.1. Thus by lemma 4.2, there exists a unique solution $\tilde{U}_\varepsilon$ of $H(U) = 0$. All this is uniform in T, so when we are changing T slightly, such that $T = k\varepsilon$ for some $k \in \mathbb{N}$, then the estimates do not change. Then $\tilde{U}_\varepsilon$ is in fact periodic of period ε. Because otherwise the shifted solution $\tilde{U}(.+\varepsilon)$ would be another solution of H near p_0, which contradicts uniqueness. Thus $\tilde{U}_\varepsilon$ can be identified with a periodic solution p_ε with

$$\|\bar{p}_\varepsilon - p_\varepsilon\|_{BC(\mathbb{R},X)} \leq C \exp\left(-\frac{c}{\sqrt{\varepsilon}}\right)$$

This completes the proof of the proposition. □

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