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EXPONENTIALLY SMALL SPLITTING OF HOMOCLINIC ORBITS OF PARABOLIC DIFFERENTIAL EQUATIONS UNDER PERIODIC FORCING

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Abstract. Homoclinic orbits of semilinear parabolic partial differential equations can split under time-periodic forcing as for ordinary differential equations. The stable and unstable manifold may intersect transverse at persisting homoclinic points. The size of the splitting is estimated to be exponentially small of order $\exp(-c/\epsilon)$ in the period ϵ of the forcing with $\epsilon \to 0$.

1. **Introduction.** Generally speaking, there are two major classes of dynamical systems: those with continuous time and those with discrete time. A natural question is then, what are the qualitative differences between them and how can we quantify these? Using suspension, one can interpret invertible dynamical systems in discrete time as Poincaré maps of a dynamical system in continuous time. But this does change the phase space, since the dimension increases. For near-identity diffeomorphisms there is another variant, we can interpret those as the time-ϵ-map of a non-autonomous dynamical system. This non-autonomous dynamical system can be defined by a differential equation with a time-periodic right hand side of period ϵ, see e.g. [8, ch 2].

So a variant of the above question is: What are the differences between autonomous and non-autonomous ordinary differential equations? Two effects are known, which explain a different behaviour. First there are resonances on closed invariant curves. Whereas one has a single periodic orbit in the autonomous case, one has to expect several periodic orbits with heteroclinic connections between them on an invariant curve of a non-autonomous system. The other effect are transverse homoclinic orbits. Homoclinic orbits are orbits, that are biasymptotic for $t \to \pm\infty$ to the same steady u_0. In autonomous differential equations, they consist of a complete arc and without additional assumptions they will not persist under small autonomous perturbations of the vector field. In non-autonomous differential equations, homoclinic orbits can consist out of discrete points, at which stable and unstable manifold may intersect transversely, i.e. the manifolds split. Then the homoclinic orbit persists under small perturbations.

Neishtadt [18] gave for ordinary differential equations with analytic right hand side a general answer how large the differences can be when the frequency of the

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forcing gets larger and its amplitude gets smaller. We consider the ordinary differential equation on a bounded domain $D \subset \mathbb{R}^n$

$$\dot{x} = f(x) + \epsilon g(x, \frac{t}{\epsilon}, \epsilon) \tag{1.1}$$

with f, g analytic and $g(., \tau, .)$ periodic in τ with period 1. So g is periodic in t with period ϵ, so we call this rapid forcing. This can be transformed by a periodic near-identity coordinate change to

$$\dot{y} = F(y) + \epsilon r(y, \frac{t}{\epsilon}, \epsilon) \tag{1.2}$$

with $F - f$ small and the remainder $|r| \leq C\epsilon \exp\left(-c\epsilon^{-1}\right)$ uniform on D. Then it is easy to show, that the non-autonomous effects –including the splitting of homoclinic orbits– are exponentially small in the analytic category. Discussing homoclinic orbits alone, a more careful analysis is possible, several quantities describe the transversality of the unstable and stable manifolds at homoclinic points. Early results for particular examples including lower estimates can be found in [14, 22]. General upper estimates are given by Fontich and Simo [9, 10] for planar near-identity diffeomorphisms, Fontich [11] for rapid forcing of planar vector fields and Fiedler and Scheurle [8] for analytic vector fields. The constants c and C are related to the extension of the homoclinic orbit to *complex* time. The exponent is given by $c = 2\pi\eta$, where η is the width of the complex time strip, to which the homoclinic orbit can be extended. Sharp lower estimates are proved in [5] and [12] for several concrete equations. For the relationship of splitting of separatrices to Arnold diffusion see e.g. [1].

Of course the same questions can also be posed for infinite dimensional dynamical systems: The differences between autonomous and non-autonomous partial differential equations are extensively studied. We will focus on semilinear parabolic equations. For a general reference see Henry [13]. In this class, there are e.g. nonlinear heat, reaction-diffusion and Navier-Stokes equations. For rapid forcing, there are results on averaging by Henry [13, theorem 3.4.9] and by Ilyin on attractors [15]. In [16], there is a counterpart of the result by Neishtadt for a class of reaction diffusion equations, where one can prove high space regularity (Gevrey regularity). Estimates on the non-autonomous remainder are of the form

$$C\epsilon \exp\left(-c(t)\epsilon^{-\frac{1}{3}}\right), \tag{1.3}$$

where the exponent depends on time t: $c(t) = \min(c, t)$, which will give exponential estimates for the initial value problem after any short transient. This implies in [16] then some exponential estimates in the form $C \exp(-c\epsilon^{-\frac{1}{3}})$ on the size of the splitting.

Here we will extend these results by methods similar to those in [8]. Adaptions are needed in the functional analytic formulation as well as in the geometric description of the splitting. We consider general semilinear parabolic equations and will achieve improved estimates of the form $C \exp(-c\epsilon^{-1})$. A more careful geometric description is also possible.

The rest of the paper is organised as follows. In the next section we set up the framework of general parabolic equations, define several quantities describing the splitting of homoclinic orbits and formulate the main theorem. In section 3 we give the proof of the theorem. The paper concludes with a short discussion in section 4.

2. **Main Result.** We consider periodically forced parabolic partial differential equation

$$\begin{aligned} u_t &= Au + f(u;\lambda) + \epsilon^p g(u, \frac{t}{\epsilon}; \lambda, \epsilon) \\ u(0) &= U \in X^\gamma \end{aligned} \tag{2.1}$$

with some unbounded operator A

$$\begin{aligned} &A && \text{has a compact resolvent} \\ &-A && \text{is sectorial on } D(A) \text{ contained in the Hilbert space } X \end{aligned} \tag{H.1}$$

for example $A = \mathrm{diag}(d_1, \ldots, d_n)\Delta$ on some bounded domain Ω with suitable boundary conditions, e.g. for Dirichlet boundary conditions: $X = L^2(\Omega, \mathbb{R}^n)$ and $D(A) = H^2(\Omega, \mathbb{R}^n) \cap H_0^1(\Omega, \mathbb{R}^n)$. The forcing g is time-periodic of period ϵ, i.e. $g(., \tau; ., .) = g(., \tau + 1; ., .)$ is periodic in τ with period 1. To make sense of an extension to complex time, we also have to consider the complexified Hilbert spaces $X_{\mathbb{C}} = X \times X$ with $|(u_1, u_2)|^2_{X_{\mathbb{C}}} = |u_1|^2_X + |u_2|^2_X$ and

$$\begin{aligned} \overline{(u_1, u_2)} &= (u_1, -u_2) \\ (a + bi)(u_1, u_2) &= (au_1 - bu_2, bu_1 + au_2) \\ L(u_1, u_2) &= (Lu_1, Lu_2) \text{ for } L \in L(X, X) \\ L(u_1, u_2) &= Lu_1 + iLu_2 \text{ for } L \in L(X, \mathbb{R}) \end{aligned}$$

The phase space is then denoted by $X^\gamma_{\mathbb{C}}$ which is the fractional interpolation between $X_{\mathbb{C}}$ and $D_{\mathbb{C}}(A)$, the domain of A in $X_{\mathbb{C}}$.

We assume on the nonlinearities

$$\begin{aligned} & f(.;.) : X^\gamma_{\mathbb{C}} \times \Lambda \to X_{\mathbb{C}} \\ & \quad \text{is analytic, see e.g. [13][p.11]} \\ & f(.;.) : X^{\gamma+\gamma_1}_{\mathbb{C}} \times \Lambda \to X^{\gamma_1}_{\mathbb{C}} \\ & \quad \text{is } C^1 \text{ for some } \gamma_1 > 0 \text{ with } \gamma + \gamma_1 < 1 \\ & g(., .; ., .) : X^\gamma_{\mathbb{C}} \times \mathbb{R} \times \Lambda \times (0, \epsilon_0) \to X_{\mathbb{C}} \\ & \quad \text{is analytic and 1-periodic in the second variable} \\ & g(., .; ., .) : X^{\gamma+\gamma_1}_{\mathbb{C}} \times \mathbb{R} \times \Lambda \times (0, \epsilon_0) \to X^{\gamma_1}_{\mathbb{C}} \\ & \quad \text{is } C^1 \text{ for some } \gamma_1 > 0 \text{ with } \gamma + \gamma_1 < 1 \end{aligned} \tag{H.2}$$

Assume for $\epsilon = 0, \lambda = \lambda_0 \in \Lambda \subset \mathbb{R}$, that there exists a homoclinic solution $\Gamma(t)$ to the equilibrium u_0, i.e.

$$\begin{aligned} & \Gamma(t) \to u_0 \text{ for } t \to \pm\infty \text{ is a solution of equation (2.1)} \\ & \Gamma(.) : \Sigma_\eta \to X^\gamma \text{ is analytic in the complex strip } \Sigma_\eta = \{t \in \mathbb{C} \,||\mathrm{Im} t| \leq \eta\} \end{aligned} \tag{H.3}$$

The equilibrium u_0 is assumed to be hyperbolic with a finite dimensional unstable manifold

$$\begin{aligned} Au_0 + f(u_0; \lambda_0) &= 0 \\ |Re\,(\sigma(A + D_U f(u_0; \lambda_0)))| &\geq d > 0 \\ \dim(W^u(u_0)) &= k < \infty \end{aligned} \tag{H.4}$$

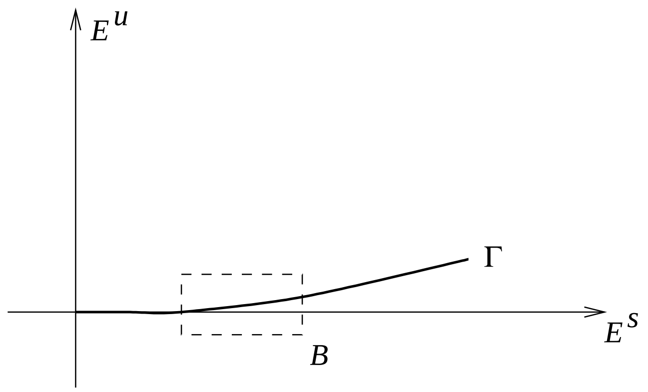

FIGURE 1. Flow-box B

Furthermore we will use non-degeneracy conditions on the linearisation $\mathcal{L}$ along the homoclinic orbit

$$\begin{aligned}\mathcal{L}u(t) &= u_t - Au - D_u f(\Gamma(t);\lambda_0)u\\ \mathcal{L}: E &\to F\\ & \quad F = BC^{0,\nu}(\mathrm{IR}, X), 0 \le \gamma < \nu < 1\\ & \quad E = BC^{1,\nu}(\mathrm{IR}, X) \cap BC^{0,\nu}(\mathrm{IR}, D(A))\end{aligned}$$

As norms we use

$$\begin{aligned}|u|_F &= \sup_{s\in\mathrm{IR}} |u(s)|_X + \sup_{s,t\in\mathrm{IR}} \frac{|u(s)-u(t)|_X}{|s-t|^\nu}\\ |u|_E &= \sup_{s\in\mathrm{IR}} |u(s)|_X + \sup_{s\in\mathrm{IR}} |u'(s)|_X + \sup_{s,t\in\mathrm{IR}} \frac{|u'(s)-u'(t)|_X}{|s-t|^\nu}\\ &\quad + \sup_{s\in\mathrm{IR}} |Au(s)|_X + \sup_{s,t\in\mathrm{IR}} \frac{|Au(s)-Au(t)|_X}{|s-t|^\nu}\end{aligned}$$

Indeed, $\mathcal{L}u \in F$ for $u \in E$. The first two terms $u_t - Au$ of $\mathcal{L}u$ are in F by definition. Then u fulfills the differential equation

$$u_t = Au + h, \text{ for some } h \in F.$$

In fact, this equation is equivalent to $u \in E$ by [3, lemma4.a.6]. As $D_u f(\Gamma(.);\lambda_0) \in BC^1(\mathrm{IR}, L(X^\gamma, X))$ and $u \in BC^{0,\nu}(\mathrm{IR}, D(A)) \subset BC^{0,\nu}(\mathrm{IR}, X^\gamma)$, thus we obtain $D_u f(\Gamma(.);\lambda_0)u(.) \in F$ and $\mathcal{L} \in L(E, F)$.

We will suppose, that the homoclinic orbit splits generically under changes of the parameter:

$$F = BC^{0,\nu}(\mathrm{IR}, X) = \mathrm{range}(\mathcal{L}) \oplus \mathrm{span} f_\lambda(\lambda_0, \Gamma(.)) \tag{H.5}$$

We fix a box B near the local stable manifold of u_0 using the saddle point property [13][theorem5.2.1], see figure 1. Near the saddle point u_0, the local stable manifold of u_0 is a graph over the infinite dimensional stable eigenspace E^s of the linearisation at u_0. It is of codimension k. We denote by W^u_B and W^s_B the parts of the stable and unstable manifolds intersected with B, which contain parts of Γ. They intersect only along Γ by (H.5). The unstable and stable manifold can be parametrised over the tangent space T_u, T_s along Γ. But there is a component $u_\perp$, which is orthogonal to T_u, T_s. This is one-dimensional by the non-degeneracy condition (H.5).

The box B is a tubular neighbourhood of Γ by the following construction. We choose the time $\vartheta \in [0,\Theta]$, which will be used as the coordinate on Γ, such that $\vartheta = 0$ is at beginning of the box and $\vartheta = \Theta$ is at the end. Consider the following infinite dimensional bundle over Γ.

$$V = \{(\vartheta, u) \in \Gamma \times X^\gamma | u \in \dot{\Gamma}^\perp(\vartheta)\}$$

The map

$$\begin{array}{rcl} b : V & \to & X^\gamma \\ (\vartheta, u) & \mapsto & \Gamma(\vartheta) + u \end{array}$$

will give the tubular neighbourhood. $Tb_{(\vartheta,0)}$ is invertible, thus by the inverse function theorem b is a diffeomorphism in a neighbourhood of $(\vartheta, 0)$, see e.g. [4, ch VII]. By compactness of the interval $[0,\Theta]$, there is a uniform neighbourhood of Γ in E, which we take as a new coordinate system of B.

So we take $(\vartheta, u_u, u_s, u_\perp)$ as coordinates in B with the additional assumption $(u_s, \dot{\Gamma}(\vartheta))_{X^\gamma} = 0$, $(u_u, \dot{\Gamma}(\vartheta))_{X^\gamma} = 0$ and $(u_\perp, \dot{\Gamma}(\vartheta))_{X^\gamma} = 0$ for uniqueness. The stable and unstable manifolds persist under perturbations. The perturbed unstable manifold is given by

$$\begin{array}{rcl} u_s & = & u_s^u(\epsilon, \lambda, \vartheta, u_u) \text{ with } (u_s, \dot{\Gamma}(\vartheta))_{X^\gamma} = 0 \\ u_\perp & = & u_\perp^u(\epsilon, \lambda, \vartheta, u_u) \text{ with } u_u \in T_u\Gamma(\vartheta), (u_u, \dot{\Gamma}(\vartheta))_{X^\gamma} = 0 \end{array}$$

and the stable manifold is given by

$$\begin{array}{rcl} u_u & = & u_u^s(\epsilon, \lambda, \vartheta, u_s) \text{ with } (u_u, \dot{\Gamma}(\vartheta))_{X^\gamma} = 0 \\ u_\perp & = & u_\perp^s(\epsilon, \lambda, \vartheta, u_s) \text{ with } u_s \in T_s\Gamma(\vartheta), (u_s, \dot{\Gamma}(t+\vartheta))_{X^\gamma} = 0. \end{array}$$

For $\epsilon = 0, \lambda = \lambda_0$ we have $u_s^u = 0, u_\perp^u = 0$ for all ϑ, u^u and $u_u^s = 0, u_\perp^s = 0$ for all ϑ, u^s.

We define the following splitting quantities $l, d, \omega, \bar{\omega}$. The set $I(\epsilon)$ contains those $\lambda \in \Lambda$, for which

$$W_B^u(\epsilon, \lambda) \cap W_B^s(\epsilon, \lambda) \neq \emptyset$$

This yields the existence of homoclinic points for some ϑ

$$\begin{array}{rcl} u_s & = & u_s^u(\epsilon, \lambda, \vartheta, u_u) \\ u_u & = & u_u^s(\epsilon, \lambda, \vartheta, u_s) \\ 0 & = & u_\perp^u(\epsilon, \lambda, \vartheta, u_u) - u_\perp^s(\epsilon, \lambda, \vartheta, u_s) \end{array}$$

The length of $I(\epsilon)$, which turns out to be an interval, is $l(\epsilon)$. The splitting distance $d(\epsilon, \lambda, \vartheta)$ for fixed ϑ is given by: $d(\epsilon, \lambda, \vartheta) := u_\perp^u(\epsilon, \lambda, \vartheta, u_u) - u_\perp^s(\epsilon, \lambda, \vartheta, u_s)$. Then

$$d(\lambda, \vartheta) = \max_{0 \le \vartheta \le \Theta} d(\epsilon, \lambda, \vartheta) \tag{2.2}$$

is called the splitting distance. It measures the distance between $W_B^u(\epsilon, \lambda)$ and $W_B^s(\epsilon, \lambda)$. The splitting angle $\omega(\epsilon, \lambda)$ can be defined as the slope $d_\vartheta = \frac{\partial}{\partial_\vartheta} d$ of $\vartheta \mapsto d(\lambda, \vartheta)$ at homoclinic points

$$\omega(\epsilon, \lambda) := \max_{\vartheta : d(\epsilon, \lambda, \vartheta) = 0} |d_\vartheta(\epsilon, \lambda, \vartheta)| \tag{2.3}$$

the splitting slope is defined by the maximum of d_ϑ over all ϑ.

$$\bar{\omega}(\epsilon,\lambda) := \max_{0\le\vartheta\le\Theta} |d_\vartheta(\epsilon,\lambda,\vartheta)|. \tag{2.4}$$

Theorem 2.1. *Let assumptions (H.1) to (H.5) hold. Consider the splitting length $l(\epsilon)$, the splitting distance $d(\epsilon,\lambda)$, the splitting angle $\omega(\epsilon,\lambda)$ and the splitting slope $\bar{\omega}(\epsilon,\lambda)$ defined in (2.2,2.3,2.4) in a fixed box B near the local stable manifold of u_0.*

Then there are $\epsilon_0 > 0$ and $C > 0$ such that for all $0 \le \epsilon \le \epsilon_0$, $\lambda \in I(\epsilon)$ the following holds: $I(\epsilon)$ is an interval,

$$0 \le \quad l(\epsilon) \quad \le C\exp\left(-\frac{2\pi\eta}{\epsilon}\right) \tag{2.5}$$

$$0 \le \quad d(\epsilon,\lambda) \quad \le C\exp\left(-\frac{2\pi\eta}{\epsilon}\right) \tag{2.6}$$

$$0 \le \quad \omega(\epsilon,\lambda) \quad \le \bar{\omega}(\epsilon,\lambda) \le C\exp\left(-\frac{2\pi\eta}{\epsilon}\right) \tag{2.7}$$

Furthermore there is a continuous curve $\lambda = \lambda(\epsilon), 0 \le \epsilon \le \epsilon_0$ real analytic for positive ϵ, such that $\lambda(\epsilon) \in I(\epsilon)$ for all ϵ and $|\lambda(\epsilon) - \lambda_0| \le C\epsilon^p$.

3. **Exponentially small splitting.** We adapt the method of proof of [8] for parabolic differential equations. Changes to [8] are especially needed in the functional analytic set-up (lemma 3.1) and when extending equations to complex time-sectors (lemma 3.3). We will set up a Ljapunov-Schmidt reduction (lemma 3.2) and use analyticity and periodicity of the reduced equation to show exponential smallness (lemma 3.4). Then we relate this back to the geometric properties (lemma 3.5).

We rewrite the partial differential equation (2.1) such that $v(t) = 0$ corresponds to the homoclinic orbit with $B(t) = D_u f(\Gamma(t), \lambda_0)$

$$v_t \quad = \quad Av + B(t)v + [f(\Gamma(t)+v;\lambda) - f(\Gamma(t);\lambda_0) - B(t)v + \epsilon^p g(\Gamma(t)+v, \frac{t}{\epsilon}; \lambda, \epsilon)].$$

We will solve

$$(\mathcal{L}(\alpha)v)(t) - \mathcal{F}(\epsilon,\lambda,\alpha,\beta,v(.))(t) \quad = \quad 0 \tag{3.8}$$

$$\int_{-\infty}^{\infty} (v(t), \dot{\Gamma}(t+\alpha))_{X_{\mathbb{C}}}\, dt \quad = \quad 0 \tag{3.9}$$

with

$$\begin{aligned}
(\mathcal{L}(\alpha)v)(t) &= v_t(t) - (Av + B(t+\alpha))v \\
\mathcal{F}(\epsilon,\lambda,\alpha,\beta,v(.))(t) &= f(\Gamma(t+\alpha)+v(t);\lambda) - f(\Gamma(t+\alpha);\lambda_0) \\
&\quad -B(t+\alpha)v(t) + \epsilon^p g(\Gamma(t+\alpha)+v(t), \frac{t}{\epsilon}+\beta; \epsilon, \lambda)]
\end{aligned}$$

where time t and parameter β are real, whereas the parameter α is in the complex strip Σ_η. Then the values will be complex in $X^\gamma_{\mathbb{C}}$, due to the analyticity of f and g in the complexified space $X^\gamma_{\mathbb{C}}$, this is well-defined. Then we define the complex Banach spaces $E_{\mathbb{C}}$ and $F_{\mathbb{C}}$:

$$\begin{aligned}
F_{\mathbb{C}} &= BC^{0,\nu}(\mathbb{R}, X_{\mathbb{C}}), 0 \le \gamma < \nu < 1 \\
E_{\mathbb{C}} &= BC^{1,\nu}(\mathbb{R}, X_{\mathbb{C}}) \cap BC^{0,\nu}(\mathbb{R}, D_{\mathbb{C}}(A)).
\end{aligned}$$

Lemma 3.1. *$\mathcal{L}(\alpha)$ is a bounded linear Fredholm operator of index 0 and analytic as a map*

$$\alpha \mapsto \mathcal{L}(\alpha) \in L(E_{\mathbb{C}}, F_{\mathbb{C}})$$

for all $\alpha \in \Sigma_\eta$. The kernel of $\mathcal{L}(\alpha)$ is spanned by

$$(t \mapsto \dot{\Gamma}(t+\alpha)) \in E_{\mathbb{C}}$$

The $L^2(\mathbb{R}, X_{\mathbb{C}})$ orthogonal complement to the range of $\mathcal{L}(\alpha)$ is spanned by

$$(t \mapsto \Psi(t+\alpha)) \in F_{\mathbb{C}}$$

where $\bar{\Psi}(.+\alpha)$ is the analytic extension to Σ_η of the unique (up to scalar multiples) nontrivial bounded solution of the adjoint equation

$$\dot{\bar{\Psi}}(t) = -(A+B(t))^* \bar{\Psi}(t), t \in \mathbb{R}, \tag{3.10}$$

where * is the real adjoint in X.

Proof: Again the autonomous part $u_t - Au$ is bounded by the definition of $E_{\mathbb{C}}$. So to show that $\mathcal{L}(\alpha)$ is bounded linear for $\alpha \in \Sigma_\eta$, it is enough to show that $B(.+\alpha) = D_x f(\lambda_0, \Gamma(.))$ defines a bounded linear operator $E_{\mathbb{C}} \to F_{\mathbb{C}}$ uniformly in $\alpha \in \Sigma_\eta$. By the definition of E, we have $u(.) \in BC^{0,\nu}(\mathbb{R}, X^\gamma)$. $B(.)$ is differentiable on Σ_η, since $\Gamma(.)$ is analytic. $B(t) \in L(X^\gamma, X)$ is uniformly bounded in t, for t in compact subsets of Σ_η and on the real line. Furthermore

$$\lim_{t\to\pm\infty} \Gamma(t+\alpha) = u_0, \lim_{t\to\pm\infty} \dot{\Gamma}(t+\alpha) = 0 \tag{3.11}$$

holds for all $\alpha \in \Sigma_\eta$, since we can use the flow properties for complex times, as the differential equation $\dot{\Gamma}(t) = A\Gamma(t) + f(\Gamma(t); \lambda_0)$ also holds by analyticity for complex times. Indeed, the solution semi-group $S(t, U; \epsilon)$ of (2.1) is analytic in time and can be extended to complex times. Then $S(t_0 + i\tau, .; 0) : X^\gamma \to X^\gamma$ is continuous for some t_0 and all $|\tau| \le t^*$. By concatenation $S(T_0 + i\mathrm{Im}(\alpha), .; 0)$ is continuous at least in a neighbourhood of the equilibrium u_0, then

$$\begin{aligned} \lim_{t\to\pm\infty} S(t+\alpha, \Gamma(0); 0) &= S(T_0 + i\mathrm{Im}(\alpha), \lim_{t\to\pm\infty} S(t - T_0 + \mathrm{Re}(\alpha), \Gamma(0); 0); 0) \\ &= S(T_0 + i\mathrm{Im}(\alpha), u_0; 0) = u_0. \end{aligned}$$

Thus $B(.+\alpha) \in BC^1(\mathbb{R}, L(X^\gamma, X))$ and $B(.+\alpha)u(.) \in F$. Hence $\mathcal{L}(\alpha) \in L(E_{\mathbb{C}}, F_{\mathbb{C}})$ holds.

Analyticity in α holds, as $\alpha \mapsto B(.+\alpha)$ is analytic as a map $\Sigma_\eta \to L(E_{\mathbb{C}}, F_{\mathbb{C}})$.

The Fredholm properties for $\alpha = 0$ can be found directly in [20] using exponential dichotomies in a more general setting or more implicit in [21]. The results were first proved by Palmer [19] for ordinary differential equations. Results on Fredholm properties and exponential dichotomies for parabolic equations with partially incomplete proofs can be found in [2] and [23]. For the sake of completeness, we give here a direct proof for our special case. We compare $\mathcal{L}(\alpha)$ with the constant coefficient operator

$$\bar{\mathcal{L}}u = u_t - (A + D_u f(u_0; \lambda_0))u$$

The linear operator $A + D_u f(u_0; \lambda_0)$ is hyperbolic by (H.4). Then

$$u_t - (A + D_u f(u_0; \lambda_0))u = h(t), h \in F$$

has a unique bounded strong solution by [13, sec.7.6, example 1, theorem 7.6.3]. Thus $\bar{\mathcal{L}}$ is invertible and hence it is a Fredholm operator of index 0. We show by perturbation arguments, that $\mathcal{L}(\alpha)$ is a Fredholm operator too. Consider the difference $B(.+\alpha) - D_u f(u_0;\lambda_0) = D_u f(\Gamma(.+\alpha);\lambda_0) - D_u f(u_0;\lambda_0))$. By (3.11), we can split this difference

$$B(.+\alpha) - D_u f(u_0;\lambda_0) = K(t) + S(t),$$

where $\|S(.)\|_{L(E_{\mathbb{C}},F_{\mathbb{C}})}$ is arbitrary small and $K(t) = 0$ for $|t| > T^*$. Due to assumption $(H.2)$, we obtain

$$K(.) + S(.) \in BC^1(\mathbb{R}, L(X_{\mathbb{C}}^{\gamma}, X_{\mathbb{C}})) \cap BC^0(\mathbb{R}, L(X_{\mathbb{C}}^{\gamma+\gamma_1}, X_{\mathbb{C}}^{\gamma_1})). \tag{3.12}$$

Furthermore $u \in E_{\mathbb{C}}$ implies $u_t = Au + h$ for some $h \in F_{\mathbb{C}}$. Then u satisfies the variation of constant formula

$$u(t) = \exp(A(t-t_0))u(t_0) + \int_{t_0}^{t} \exp(A(s-t_0))h(s)ds.$$

This implies $u \in BC^1(\mathbb{R}, X_{\mathbb{C}}^{\gamma})$ using [13, lemma3.5.1] and $\gamma < \nu$. Together with $u \in E_{\mathbb{C}}$ this gives

$$u \in BC^1(\mathbb{R}, X_{\mathbb{C}}^{\gamma}) \cap BC^0(\mathbb{R}, X_{\mathbb{C}}^{\gamma+\gamma_1}) \tag{3.13}$$

Taking (3.12,3.13) together, we can choose $K(.)$, such that

$$K(.)u(.) \in BC^1((-(T^*+1), T^*+1), X_{\mathbb{C}}) \cap BC^0((-(T^*+1), T^*+1), X_{\mathbb{C}}^{\gamma_1})$$

for all $u \in E_{\mathbb{C}}$. Furthermore $X_{\mathbb{C}}^{\gamma_1}$ embeds compactly into $X_{\mathbb{C}}$, since A has a compact resolvent. Investigating standard proofs of Ascoli's theorem; see [6]; we obtain that $BC^1((-(T^*+1), T^*+1), X_{\mathbb{C}}) \cap BC^0((-(T^*+1), T^*+1), X_{\mathbb{C}}^{\gamma_1})$ embeds compactly into $BC^{0,\nu}((-(T^*+1), T^*+1), X_{\mathbb{C}})$. Therefore it also embeds compactly into $F_{\mathbb{C}}$. Thus the map $K : E_{\mathbb{C}} \to F_{\mathbb{C}}$ is compact. As Fredholm properties persist under small and under compact perturbations, $\mathcal{L}(\alpha)$ is Fredholm of index 0.

From Fredholm index 0, we get dim $\mathrm{Ker}(\mathcal{L}(\alpha))$ = codim $\mathrm{R}(\mathcal{L}(\alpha))$ and clearly $\dot{\Gamma}(.+\alpha) \in \mathrm{Ker}(\mathcal{L}(\alpha))$ holds. As codim $\mathrm{R}(\mathcal{L}(\alpha)) = 1$ by (H.5), dim $\mathrm{Ker}(\mathcal{L}(\alpha)) = 1$ and $\dot{\Gamma}(.+\alpha)$ spans $\mathrm{Ker}(\mathcal{L}(\alpha))$.

To show the statements about Ψ, we start with $\alpha = 0$. We consider the adjoint equation and the corresponding constant coefficient part

$$\begin{aligned} \mathcal{L}^* u &= u_t + (A^* + D_u f(\Gamma(t);\lambda_0)^*)u \\ \bar{\mathcal{L}}^* u &= u_t + (A^* + D_u f(u_0;\lambda_0)^*)u \end{aligned}$$

As A^* is sectorial and as $A^* + D_u f(u_0;\lambda_0)^*$ is hyperbolic, see [13, sec 7.3], [21, sec 2], the above arguments apply within the Hilbert space $X_{\mathbb{C}}^* = X_{\mathbb{C}}$. Then $\mathcal{L}^*$ is a Fredholm map of index 0. It is an easy calculation to show $\int_{-\infty}^{\infty}(\Psi, \mathcal{L}(0))_X = 0$ for all bounded solutions Ψ of

$$\Psi_t = -(A^* + D_u f(\Gamma(t);\lambda_0)^*)\Psi. \tag{3.14}$$

Thus $\mathrm{Ker}(\mathcal{L}^*) \subset \mathrm{R}(\mathcal{L}(0))^{\perp}$. Similarly $\int_{-\infty}^{\infty}(\phi, \mathcal{L}^*(0))_{X^*} = 0$ for all bounded solutions ϕ of

$$\phi_t = (A + B(t))\phi$$

and $\mathrm{Ker}(\mathcal{L}(0)) \subset \mathrm{R}(\mathcal{L}^*)^{\perp}$. Hence in

$$\text{codim } \mathrm{R}(\mathcal{L}(0)) \geq \dim \mathrm{Ker}(\mathcal{L}^*) = \text{codim } \mathrm{R}(\mathcal{L}^*) \geq \dim \mathrm{Ker}(\mathcal{L}(0)) = \text{codim } \mathrm{R}(\mathcal{L}(0))$$

equality holds and in particular dim $\mathrm{Ker}(\mathcal{L}^*) = 1$ and (3.10) has a – up to a real factor – unique bounded strong solution.

The statements about Ψ for $\alpha \neq 0$ follow by analytic continuation of (3.10) from real to complex time. $\bar{\Psi}$ is complex differentiable, because it fulfils the analytic differential equation (3.10). As we have for all $\alpha \in \Sigma_\eta$

$$\langle \Psi(.+\alpha), \mathcal{L}(\alpha)v \rangle = \int_{-\infty}^{\infty} (\bar{\Psi}(t+\alpha), (\dot{v}(t) - (A + B(t+\alpha))v(t)))_{X_{\mathbb{C}}}\, dt$$

by partial integration for all $v \in E$. The right hand side is zero for all real α, it is analytic in $\alpha \in \Sigma_\eta$, hence complex analytic and the right hand side vanishes for all $\alpha \in \Sigma_\eta$. This completes the proof of the lemma. □

Using this result, we can proceed as in [8] with a standard Ljapunov-Schmidt reduction. P_α is the projection onto the co-kernel of $\mathcal{L}(\alpha)$, i.e.

$$P_\alpha v(.) := \langle v(.), \Psi(.+\alpha)\rangle \Psi(.+\alpha),$$

where $\langle .,. \rangle$ denotes the $L^2(\mathbb{R}, X_{\mathbb{C}})$ scalar product. We will apply the Ljapunov-Schmidt reduction to

$$\begin{aligned} (I - P_\alpha)(\mathcal{L}(\alpha)v - \mathcal{F}(\epsilon,\lambda,\alpha,\beta,v)) &= 0; \langle v(.), \dot{\Gamma}(.+\alpha)\rangle = 0 && (3.15)\\ P_\alpha(\mathcal{L}(\alpha)v - \mathcal{F}(\epsilon,\lambda,\alpha,\beta,v)) &= -P_\alpha \mathcal{F}(\epsilon,\lambda,\alpha,\beta,v) = 0 && (3.16) \end{aligned}$$

Lemma 3.2. *The equation (3.15) can be solved uniquely for v near $\epsilon = 0, \lambda = \lambda_0, v = 0$ by the implicit function theorem. There exist $\epsilon_0 > 0$, $\delta_0 > 0$, $C >$ and a unique $v(.) = v(\epsilon,\lambda,\alpha,\beta,.) \in E_{\mathbb{C}}$ such that*

(i) *v is defined for all real $0 \leq \epsilon \leq \epsilon_0$, $0 \leq \beta \leq 2$, $|\lambda - \lambda_0| \leq \epsilon_0$ and all $\alpha \in \Sigma_\eta$ with $0 \leq$ Re $\alpha \leq 2$.*
(ii) *down to $\epsilon = 0$, v and all its derivatives with respect to λ, α, β are continuous and analytic with respect to λ, α.*
(iii) $|v(\epsilon,\lambda,\alpha,\beta,.) - v(0,\lambda,\alpha,\beta,.)|_{E_{\mathbb{C}}} \leq C\epsilon^p$.
(iv) *v has period 1 in β.*
(v) *$v(\epsilon,\lambda,\alpha,\beta; t+t') = v(\epsilon,\lambda,\alpha+t',\beta+\frac{t'}{\epsilon},t)$ for all $\epsilon,\lambda,\alpha,\beta$ as in (i).*

Proof:

The proof follows along the lines of [8]. We apply the implicit function theorem. A solution of the system (3.15) is given by $v = 0$ for $\epsilon = 0, \lambda = \lambda_0$ and all shift parameters α, β. The linearisation of equation (3.15) with respect to v at $v = 0, \epsilon = 0, \lambda = \lambda_0$ is given by the bounded linear operator

$$\begin{aligned} \hat{\mathcal{L}} : E_{\mathbb{C}} &\to (Id - P_\alpha)F_{\mathbb{C}} \times \mathbb{C} && (3.17)\\ \xi &\mapsto ((Id - P_\alpha)\mathcal{L}(\alpha)\xi, \langle \xi, \dot{\Gamma}(.+\alpha)\rangle) \end{aligned}$$

We have to show that $\hat{\mathcal{L}}$ is invertible. As $\mathcal{L}(\alpha)$ is Fredholm of index 0 and as $Id - P_\alpha$ is Fredholm of index 1, the first part of the system (3.17) $(Id - P_\alpha)\mathcal{L}(\alpha)$ is Fredholm of index 1. Hence the complete system $\hat{\mathcal{L}}$ is Fredholm of index 0. So it remains to show, that $\ker(\hat{\mathcal{L}}) = \{0\}$. By the definition of P_α, we have $P_\alpha\mathcal{L}(\alpha) = 0$ and the kernels of $(Id - P_\alpha)\mathcal{L}(\alpha)$ and $\mathcal{L}(\alpha)$ coincide. By our non-degeneracy assumption (H.5) they are spanned by $\dot{\Gamma}(.+\alpha)$. But the second component $\langle \xi, \dot{\Gamma}(.+\alpha)\rangle \neq 0$ for $\xi = \dot{\Gamma}(.+\alpha)$. Thus $\hat{\mathcal{L}}$ has a trivial kernel and hence is invertible. So we can apply the implicit function theorem to get (i).

We get the regularity properties in (ii) from those of $\mathcal{L}, \dot{\Gamma}$ and $\mathcal{F}$.

To get the estimates in (iii), we use the uniform boundedness of $\mathcal{F}$ and of the $\epsilon^p g$ term and the invertibility of $\hat{\mathcal{L}}$ in the relevant domain.

The periodicity (iv) of v in β holds, because it holds for $\mathcal{F}$ and g.

Property (v) describes the effect of time shifts in equations (3.8), (3.9) and (3.15) by t'. This is equivalent to solving the equation with $\alpha + t'$and $\beta + \frac{t'}{\epsilon}$ as the normalisation $\langle v(. + t'), \dot\Gamma(. + \alpha + t')\rangle$ is translation invariant. □

Until now we do not have any regularity with respect to ϵ. We rescale v and will gain even analyticity for positive ϵ. Let

$$\tilde v(\epsilon, \lambda, \alpha, \beta; \tau) = v(\epsilon, \lambda, \alpha, \beta; \epsilon\tau), \tag{3.18}$$

for which we rewrite lemma 3.2.

Lemma 3.3. (i) *The domains of definition of v and $\tilde v$ coincide.*

(ii) *for $\epsilon > 0$, $\tilde v$ is C^∞ in all variables. $\tilde v$ and all its derivatives are analytic with respect to $\lambda, \alpha, \epsilon$.*

(iii) $|\tilde v(\epsilon, \lambda, \alpha, \beta; .) - \tilde v(0, \lambda, \alpha, \beta; .)|_{E_{\mathbb{C}}} \le C\epsilon^p$.

(iv) *$\tilde v$ has period 1 in β.*

(v) *$\tilde v(\epsilon, \lambda, \alpha, \beta; t + t') = \tilde v(\epsilon, \lambda, \alpha + \epsilon t', \beta + t'; t)$ for all $\epsilon, \lambda, \alpha, \beta$ as in (i).*

Proof: $\tilde v$ fulfils the rescaled version of (3.8,3.9):

$$\begin{aligned}
\frac{d}{d\tau}\tilde v &= \epsilon\,[(A + B(\epsilon\tau + \alpha))\tilde v + f(\Gamma(\epsilon\tau + \alpha) + \tilde v, \lambda) - f(\Gamma(\epsilon\tau + \alpha), \lambda_0) \\
&\quad - B(\epsilon\tau + \alpha)\tilde v + \epsilon^p g(\Gamma(\epsilon\tau + \alpha), \tau + \beta; \lambda, \epsilon) + \tilde v)] &(3.19)\\
0 &= \int_{-\infty}^{\infty} (\tilde v(\tau), \dot\Gamma(\epsilon\tau + \alpha))_{X_{\mathbb{C}}}\, d\tau &(3.20)
\end{aligned}$$

In this operator equation all terms depend analyticly on ϵ, thus $\tilde v$ is analytic in ϵ for real $\epsilon > 0$. To extend ϵ to a complex domain, we have to consider the time dependence of

$$\Gamma(t), t\dot\Gamma(t), t\dot B(t). \tag{3.21}$$

We show, that these tend to u_0 resp. zero exponentially for $|\mathrm{Re} t| \to \infty$ and t in a set

$$\Sigma_{\eta,\delta} := \Sigma_\eta \cup \{t \in \mathbb{C}; |\mathrm{Im}\ t| \le \delta|\mathrm{Re}\ t|\}$$

All these quantities decay exponentially for real $t \to \pm\infty$. As $A + D_u f(u_0, \lambda_0)$ is sectorial, it defines an analytic semi-group for all whole complex time-sector $S_{\delta^*} = \{t \in \mathbb{C}|\arg(t) \le \delta^*\}$. In this complex sectors the terms in (3.21) may grow at most exponentially of uniform rate, as long $\Gamma(t)$ stays in small neighbourhood of the equilibrium u_0. By splitting time into (positive or negative) real part plus some part in S_{δ^*} we achieve exponential convergence of (3.21) in $\Sigma_{\eta,\delta}$ for some $\delta < \delta^*$. By real analyticity of $g(u, \tau; \lambda, \epsilon)$ for $0 \le \epsilon \le \epsilon_0$, it can be extended to such a complex ϵ-sector. This implies complex differentiability and hence analyticity of the operators with respect to ϵ in a sector

$$|\mathrm{Im}\ \epsilon| \le \delta \mathrm{Re}\ \epsilon.$$

The rest of the lemma is just a reformulation of lemma 3.2. □

Now we have solved equation (3.15). It remains to solve the other equation of system (3.15,3.16):

$$P_\alpha(\mathcal{L}(\alpha)v - \mathcal{F}(\epsilon, \lambda, \alpha, \beta, v)) = 0$$

with $v = v(\epsilon, \lambda, \alpha, \beta; .)$. This is equivalent to

$$b(\epsilon, \lambda, \alpha, \beta) := \int_{-\infty}^{\infty} (\bar\Psi(t + \alpha), \mathcal{F}(\epsilon, \lambda, \alpha, \beta, v(\epsilon, \lambda, \alpha, \beta; .))(t))_{X_{\mathbb{C}}}\, dt = 0 \tag{3.22}$$

This bifurcation function has the following properties including the crucial exponential estimate in (iv):

Lemma 3.4. (i) *b has period ϵ in α and period 1 in β. Thus b can be extended to $\beta \in \mathbb{R}$, $0 \le \epsilon \le \epsilon_0$, $|\lambda - \lambda_0| \le \epsilon_0$ and all $\alpha \in \Sigma_\eta$.*

(ii) *For $\epsilon > 0$, b is analytic in all variables. Down to $\epsilon = 0$, v and all its derivatives with respect to λ, α, β are continuous and analytic with respect to λ, α, β.*

(iii) *$b(\epsilon, \lambda, \alpha, \beta) = b(\epsilon, \lambda, \alpha - \epsilon\beta, 0)$.*

(iv) *The Fourier expansion*

$$b(\epsilon, \lambda, \alpha, \beta) = \sum_{k \in \mathbb{Z}} b^k(\epsilon, \lambda) \exp\left(2\pi i k (\frac{\alpha}{\epsilon} - \beta)\right)$$

converges uniformly and absolutely. b^k is as regular with respect to (ϵ, λ) as b. Furthermore for some $\delta > 0$

$$|b^k(\epsilon, \lambda)| \le C_\delta \exp\left(-2\pi(\eta + \delta)\frac{|k|}{\epsilon}\right). \tag{3.23}$$

(v)

$$\begin{aligned} b(0, \lambda, \alpha, \beta) &= b^0(0, \lambda) \\ b^0(0, \lambda) &= 0 \ \mathit{at}\ \lambda = \lambda_0 \\ b^0_\lambda(0, \lambda) &\ne 0 \ \mathit{at}\ \lambda = \lambda_0 \end{aligned}$$

There exists a $C > 0$ such that

$$|b^0(\epsilon, \lambda) - b(0, \lambda)| \le C\epsilon^p.$$

Proof: We start by showing (iii). Consider the time shift

$$S_{t'} v(\epsilon, \alpha, \beta; .) := v(\epsilon, \alpha, \beta; . + t')$$

Then $S_{t'}\mathcal{F}(\epsilon, \alpha, \beta, v) = \mathcal{F}(\epsilon, \alpha + t', \beta + \frac{t'}{\epsilon}, S_{t'}v)$ for real t' by the definition of $\mathcal{F}$. By lemma 3.2, we also have

$$S_{t'} v(\epsilon, \alpha, \beta; .) = v(\epsilon, \alpha + t', \beta + \frac{t'}{\epsilon}; .)$$

This implies

$$\begin{aligned} b(\epsilon, \lambda, \alpha, \beta) &= \langle \Psi(\alpha + .), \mathcal{F}(\epsilon, \alpha, \beta, v(\epsilon, \alpha, \beta, .)) \rangle \\ &= \langle S_{t'}\Psi(\alpha + .), S_{t'}\mathcal{F}(\epsilon, \alpha, \beta, v(\epsilon, \alpha, \beta, .)) \rangle \\ &= \langle \Psi(\alpha + t' + .), \mathcal{F}(\epsilon, \alpha + t', \beta + \frac{t'}{\epsilon}, v(\epsilon, \alpha + t', \beta + \frac{t'}{\epsilon}, .)) \rangle \\ &= b(\epsilon, \lambda, \alpha + t', \beta + \frac{t'}{\epsilon}) \end{aligned}$$

for real t'. Putting $t' = -\epsilon\beta$ proves (iii) for $\epsilon > 0$. For $\epsilon = 0$ it holds by continuity of b.

This implies (i), because $\mathcal{F}$ and v are 1-periodic in β.

Regarding the regularity of b, lemma 3.2, the analyticity of $\mathcal{F}$ with respect to v and (iii) imply that b and all its derivatives with respect to λ, α, β are continuous and analytic in λ, α, β, down to $\epsilon = 0$. To show analyticity including ϵ we use the rescaled function and lemma 3.3. Rescaling the integration variable and suppressing λ

$$\begin{aligned} b(\epsilon, \alpha, \beta) = \ & \epsilon \int_{-\infty}^{\infty} (\bar{\Psi}(\epsilon\tau + \alpha), [f(\Gamma(\epsilon\tau + \alpha) + \tilde{v}(\tau)) - f(\Gamma(\epsilon\tau + \alpha)) \\ & -B(\epsilon\tau + \alpha)\tilde{v}(\tau) + \epsilon^p g(\tau + \beta, \Gamma(\epsilon\tau + \alpha) + \tilde{v}(\tau); \epsilon)])_{X_{\mathbb{C}}}\, d\tau \end{aligned}$$

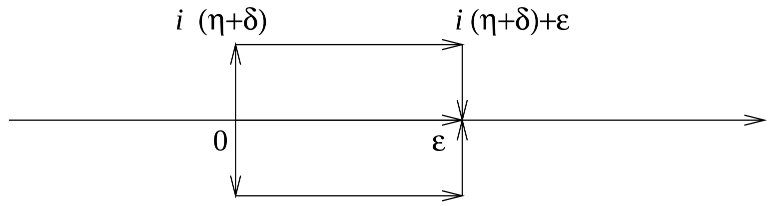

FIGURE 2. Different paths to compute $b^k(\epsilon)$

with $\tilde{v}(\tau) = \tilde{v}(\epsilon, \alpha, \beta; \tau)$. Now analyticity in $\epsilon > 0$ follows as in lemma 3.3, recalling from lemma 3.1 that $\bar{\Psi}$ is complex differentiable. This proves (ii).

Now we prove (iv). By (iii), b has period ϵ in α, as b has period 1 in β. Since Γ is also analytic in the slightly larger closed strip $\Sigma_{\eta+\delta}$, b is also complex analytic in $\alpha \in \Sigma_{\eta+\delta}$. Therefore we can expand $b(\epsilon, \alpha, 0)$ into a Laurent series with respect to $q = \exp(2\pi i \frac{\alpha}{\epsilon})$ for

$$\exp(-2\pi \frac{\eta + \delta}{\epsilon}) < |q| < \exp(2\pi \frac{\eta + \delta}{\epsilon}).$$

This gives the Fourier series of b with the desired regularity properties of the coefficients with respect to ϵ, λ. We need to estimate the Fourier coefficients

$$b^k(\epsilon) = \frac{1}{\epsilon} \int_0^\epsilon \exp(-2\pi i k \frac{\alpha}{\epsilon}) b(\epsilon, \alpha, 0) d\alpha \tag{3.24}$$

As $b(\epsilon, \alpha, 0)$ is complex analytic in $\alpha \in \Sigma_{\eta+\delta}$, path integrals over closed paths vanish. Furthermore b has real period ϵ in α, thus we can choose other integration paths, see figure 2. Integrating from $\pm i(\eta + \delta)$ to $\epsilon \pm i(\eta + \delta)$ yields as b is bounded on $\Sigma_{\eta+\delta}$ by C_δ

$$|b^k(\epsilon)| \leq C_\delta \exp\left(-2\pi |k| \frac{\eta + \delta}{\epsilon}\right)$$

for all ϵ, k and λ. This shows the uniform convergence and hence (iv).

Finally we show (v). As b and b^k are continuous in ϵ, using the estimate in (iv), we see that $b(0, \lambda, \alpha, \beta) = b^0(0, \lambda)$ is independent of α, β. Considering again λ in (3.22) and letting $v(0, \lambda, 0, 0; .) = v(\lambda; .)$ we obtain

$$\begin{aligned} b^0(0, \lambda) &= b(0, \lambda, \alpha = 0, \beta = 0) \\ &= \int_{-\infty}^{\infty} (\bar{\Psi}(t), \mathcal{F}(0, \lambda, 0, 0, v^0(\lambda; t)))_X dt \\ &= \int_{-\infty}^{\infty} (\bar{\Psi}(t), (f(\Gamma(t) + v^0(t), \lambda) - f(\Gamma(t)) - B(t) v^0(\lambda, t)))_X dt \end{aligned} \tag{3.25}$$

As $v^0(\lambda_0; .) \equiv 0$, we get $b^0(0, \lambda_0) = 0$. Next we differentiate with respect to λ at $\epsilon = 0, \lambda = \lambda_0$. Using $B(t) = D_u f(\Gamma(t), \lambda_0)$ we get

$$\begin{aligned} b^0_\lambda(0, \lambda_0) &= \int_{-\infty}^{\infty} (\bar{\Psi}(t), (f_\lambda(\Gamma(t), \lambda_0) + D_u f v^0_\lambda(t) - B(t) v^0_\lambda(\lambda_0, t)))_X dt \\ &= \int_{-\infty}^{\infty} (\bar{\Psi}(t), f_\lambda(\Gamma(t), \lambda_0))_X dt \neq 0 \end{aligned}$$

as $f_\lambda(\Gamma(t),\lambda_0) \notin \mathrm{range}(\mathcal{L}) = \mathrm{span}(\Psi)^\perp$ by the non-degeneracy assumption (H.5). Finally to get the last estimate, consider

$$\begin{aligned} b^0_\lambda(\epsilon,\lambda_0) &= \frac{1}{\epsilon}\int_0^\epsilon b(\epsilon,\alpha,0)d\alpha \\ &= \frac{1}{\epsilon}\int_0^\epsilon \int_{-\infty}^{\infty} (\bar\Psi(t+\alpha),[f(\Gamma(t+\alpha)+v(\epsilon,\alpha,\beta;t),\lambda) - f(\Gamma(t+\alpha);\lambda_0) \\ &\quad -B(t+\alpha)v(\epsilon,\lambda,\alpha,\beta;t) + \epsilon^p g(\Gamma(t+\alpha)+v(t),\frac{t}{\epsilon}+\beta;\lambda,\epsilon)])_X dt d\alpha \end{aligned}$$

The parts in the brackets [.] is only a $O(\epsilon^p)$ perturbation of the respective part in (3.25). So the inner integration with the exponential decaying $\bar\psi(t)$ will also only give an $O(\epsilon^p)$ perturbation of $b(0,\lambda)$ for all α. This also holds for the averaging of α on $[0,\epsilon]$, hence

$$|b^0(\epsilon,\lambda) - b(0,\lambda)| \le C\epsilon^p$$

and the lemma is proved. □

In this lemma we used the essential trick for the exponential smallness: the integration over complex time of a real periodic function.

Up to now we reduced the problem of finding homoclinic orbits of equation (2.1) near Γ for real λ,α:

$$u(t) = \Gamma(t+\alpha) + v(t),$$

with the normalisation $\langle v(.), \dot\Gamma(.+\alpha)\rangle = 0$, is a homoclinic orbit of (2.1), if and only if $v(t) = v(\epsilon,\lambda,\alpha,\beta=0;t)$ and ϵ,λ satisfy the reduced equation

$$b(\epsilon,\lambda,\alpha,\beta=0) = 0 \tag{3.26}$$

We also consider a modified bifurcation function b^*, which is more related to the geometric quantities describing the splitting of homoclinic orbits.

For b^* we use the same decomposition as in (3.15,3.16), only the normalisation $\langle v(.), \dot\Gamma(.+\alpha)\rangle = 0$ is replaced by an orthogonality condition at the beginning of the box B at time $t^* = 0$.

$$(v(0), \dot\Gamma(\alpha))_{X^\gamma_{\mathbb{C}}} = 0 \tag{3.27}$$

Both conditions imply

$$v(.) \notin \mathrm{span}(\frac{d}{d\alpha}\Gamma(.+\alpha)) = \mathrm{Ker}\mathcal{L}(\alpha)$$

Hence we can solve (3.15) with the other normalisation condition locally for v

$$v = v^*(\epsilon,\lambda,\alpha,\beta=0;.)$$

Then we define the modified bifurcation function

$$b^*(\epsilon,\lambda,\alpha,\beta) = \int_{\mathbb{R}} (\bar\Psi(t+\alpha), \mathcal{F}(\epsilon,\lambda,\alpha,\beta,v^*(\epsilon,\lambda,\alpha,\beta;.))(t))_{X_{\mathbb{C}}}\,dt \tag{3.28}$$

This b^* will be related to the geometric splitting properties by the next lemma. Using the (ϑ,u) coordinates in the box B with $\vartheta = 0, u = (u_s,u_u,u_\perp) = 0$ at $\Gamma(t^*=0)$, we get that

$$d(\epsilon,\lambda,\vartheta) = 0 \tag{3.29}$$

holds if and only if

$$b^*(\epsilon,\lambda,\alpha,\beta=0) = 0, \tag{3.30}$$

because both equations are equivalent to the existence of a homoclinic orbit in the $\vartheta = \text{const}$ section at time $t = t^* = 0$.

Lemma 3.5. *Equations (3.26), (3.29) and (3.30) can be solved for λ:*

$$b(\epsilon,\lambda,\alpha,\beta=0) \quad \Leftrightarrow \quad \lambda=\lambda(\epsilon,\alpha) \tag{3.31}$$

$$b^*(\epsilon,\lambda,\alpha,\beta=0) \quad \Leftrightarrow \quad \lambda=\lambda^*(\epsilon,\alpha) \tag{3.32}$$

$$d(\epsilon,\lambda,\vartheta) \quad \Leftrightarrow \quad \lambda=\hat{\lambda}(\epsilon,\vartheta) \tag{3.33}$$

for $0\le \epsilon,\alpha,\vartheta\le\epsilon_0$ with ϵ_0 small enough. λ,λ^* and $\hat{\lambda}$ with their derivatives with respect to α (resp. ϑ) are continuous and analytic in α (resp. ϑ) down to $\epsilon=0$. For $\epsilon>0$ they are also analytic in ϵ. For real $\epsilon,\alpha,\vartheta$ the functions $\lambda,\lambda^*,\hat{\lambda}$ are related by

$$\begin{aligned}\lambda(\epsilon,\alpha) &= \lambda^*(\epsilon,\vartheta(\epsilon,\alpha))\\ \lambda^*(\epsilon,\vartheta) &= \hat{\lambda}(\epsilon,\vartheta),\end{aligned}$$

where $\vartheta(.,.)$ and all its α-derivatives are continuous down to $\epsilon=0$ and $\vartheta(.,.)$ is smooth for $\epsilon>0$. In the ϑ,u coordinates in B, $\vartheta(.,.)$ is given by

$$\vartheta=\alpha+(v(\epsilon,\lambda(\epsilon,\alpha),\alpha,\beta=0;t=0),\dot{\Gamma}(\alpha))_{X^\gamma}. \tag{3.34}$$

Proof: We can solve (3.26), (3.29) and (3.30) for λ, because locally $b_\lambda,b^*_\lambda,d_\lambda$ are non-zero. By lemma 3.4 we have $b(0,\lambda,\alpha,\beta)=b^0(0,\lambda)$ with $b^0_\lambda(0,\lambda_0)\neq 0$. By the same reasoning applied to the modified bifurcation function $b^*_\lambda(0,\lambda)\neq 0$, as the argument in part (v) of lemma 3.4 is independent of the specific normalisation. The non-degeneracy assumption (H.5) implies that the stable and unstable manifolds split with non-zero speed as λ increases through λ_0 in the autonomous case $\epsilon=0$. Hence we get $d_\lambda\neq 0$.

The regularity properties of λ,λ^* can be obtained from those of b and b^* similar to lemma 3.4. As $d(\epsilon,\lambda,\vartheta)=0$ if and only if $b^*(\epsilon,\lambda,\vartheta,\beta=0)=0$, λ^* and $\hat{\lambda}$ are the same functions. Next we prove $\lambda(\epsilon,\alpha)=\lambda^*(\epsilon,\vartheta(\epsilon,\alpha))$, where ϑ is given by (3.34). $b(\epsilon,\lambda(\epsilon,\alpha),\alpha,\beta=0)=0$ implies that

$$u(t)=\Gamma(t+\alpha)+v(\epsilon,\lambda(\epsilon,\alpha),\alpha,\beta=0;t)$$

is a homoclinic orbit of (2.1) with the $L^2(\mathbb{R},X_{\mathbb{C}})$ normalisation for v. Letting $v^*(0):=v(0)-(v(0),\dot{\Gamma}(\alpha))_{X^\gamma}\dot{\Gamma}(\alpha)$, we find

$$\begin{aligned}u(t^*=0) &= \Gamma(\alpha)+v(0)=\alpha\dot{\Gamma}(0)+v(0)\\ &= (\vartheta-(v(0),\dot{\Gamma}(0))_{X^\gamma})\dot{\Gamma}(0)+v(0)\\ &= \Gamma(\vartheta)+v(0)-(v(0),\dot{\Gamma}(\alpha))_{X^\gamma}\dot{\Gamma}(\alpha)\\ &= \Gamma(\vartheta)+v^*(0)\end{aligned}$$

since $\dot{\Gamma}$ is in (ϑ,u) coordinates the unit vector in ϑ direction. Thus we obtain

$$\Big(v^*(0),\dot{\Gamma}(0)\Big)_{X^\gamma}=\Big(v(0)-(v(0),\dot{\Gamma}(0))_{X^\gamma}\dot{\Gamma}(0),\dot{\Gamma}(0)\Big)_{X^\gamma}=0,$$

hence $b^*(\epsilon,\lambda(\epsilon,\alpha),\vartheta,\beta=0)=0$ and therefore $\lambda(\epsilon,\alpha)=\lambda^*(\epsilon,\vartheta(\epsilon,\alpha))$. The smoothness properties of $\vartheta=\vartheta(\epsilon,\alpha)$ given in (3.34) are obtained from those of $v,\tilde{v}$ and $\lambda(.,.)$ in the (ϑ,u) coordinates inside B similar to lemma 3.2 and lemma 3.3. For $\epsilon=0$ we set

$$\vartheta(0,\alpha)=\alpha \tag{3.35}$$

in $\lambda(0,\alpha)=\lambda_0$ and $v=v^*=0$, hence (3.34) defines locally a real diffeomorphism $\alpha\mapsto\vartheta$ and we can use α as a new coordinate instead of ϑ in B.□

Now we are in the position to complete the proof of the main theorem.

Proof of Theorem 2.1

We start by proving the exponential estimates (2.5,2.6,2.7). Then we define the curve $\lambda = \lambda(\epsilon)$ and finally prove the order ϵ^p estimate. Consider first the splitting set $I(\epsilon)$. $\lambda(\epsilon, \alpha)$ solves

$$b(\epsilon, \lambda(\epsilon, \alpha), \alpha, \beta = 0) = 0. \tag{3.36}$$

Hence λ has period ϵ in α as b has by lemma 3.4. Then the splitting set is given by

$$I(\epsilon) = \{\lambda(\epsilon, \alpha) | 0 \le \alpha \le \epsilon\},$$

and is an interval as $\lambda(\epsilon, .)$ is continuous. Then the estimate on $l(\epsilon)$ can be obtained, if we show

$$|\lambda_\alpha| \le C \exp(-\frac{2\pi\eta}{\epsilon}) \tag{3.37}$$

for some constant C. Implicit differentiation of (3.36) yields to

$$\lambda_\alpha = -\frac{b_\alpha}{b_\lambda}$$

As by lemma 3.4 (v), b_λ is bounded away from zero, we only have to show, that b_α can be estimated by an exponential upper bound. This holds, because the Fourier series in lemma 3.4 is uniformly convergent and the estimate on the Fourier coefficients is independent of α. Thus the estimate (3.37) holds and $l(\epsilon)$ is exponentially small.

Now we estimate $d(\epsilon, \lambda, \vartheta)$ and $d_\vartheta(\epsilon, \lambda, \vartheta)$ for $\lambda \in I(\epsilon)$, this will give the estimates on $d(\epsilon, \lambda)$, $\omega(\epsilon, \lambda)$ and $\bar{\omega}(\epsilon, \lambda)$. Since $l(\epsilon)$ is exponentially small and d, d_ϑ have uniform bounded derivatives with respect to λ, see (2.2), we only have to show exponential smallness of $d(\epsilon, \lambda, \vartheta)$ and $d_\vartheta(\epsilon, \lambda, \vartheta)$ at special points $\lambda \in I(\epsilon)$. At $\lambda = \hat{\lambda}(\epsilon, \vartheta)$ we have by lemma 3.5:

$$d(\epsilon, \hat{\lambda}(\epsilon, \vartheta), \vartheta) = 0, \tag{3.38}$$

which proves the exponential smallness of $|d(\epsilon, \lambda, \vartheta)|$ and $|d(\epsilon, \lambda)|$. For $d_\vartheta(\epsilon, \lambda, \vartheta)$ we get at $\lambda = \hat{\lambda}(\epsilon, \vartheta)$ by implicit differentiating in lemma 3.5 and (3.38), when suppressing some arguments

$$\begin{aligned} d_\vartheta(\epsilon, \hat{\lambda}(\epsilon, \vartheta), \vartheta) &= -d_\lambda \hat{\lambda}_\vartheta = -d_\lambda \lambda^*_\vartheta = -(d_\lambda/\vartheta_\alpha) \cdot \lambda_\alpha \\ &= -(d_\lambda(\epsilon, \hat{\lambda}(\epsilon, \vartheta), \vartheta)/\vartheta_\alpha(\epsilon, \alpha)) \cdot \lambda_\alpha(\epsilon, \alpha), \end{aligned}$$

where $\vartheta(\epsilon, \alpha)$ is given by (3.34). By (3.35), $\vartheta_\alpha(\epsilon, \alpha)$ is uniformly bounded away from zero. Then the exponential smallness of λ_α implies the exponential smallness of $d_\vartheta, \bar{\omega}$ and ω. This completes the proof of the exponential estimates (2.5,2.6,2.7).

Finally we show that there is a curve $\epsilon \mapsto \lambda(\epsilon) \in I(\epsilon)$, such that $I(\epsilon)$ is non-empty. $\lambda(\epsilon)$ is defined by solving the equation $b^0(\epsilon, \lambda) = 0$ for the constant coefficient in the Fourier expansion of b, see lemma 3.4:

$$b^0(\epsilon, \lambda(\epsilon)) = 0$$

As $b^0_\lambda \neq 0$, $\lambda(\epsilon)$ exists. It is continuous down to $\epsilon = 0$, and is analytic for $\epsilon > 0$, because b^0 has these regularity properties. To show $\lambda(\epsilon) \in I(\epsilon)$, consider

$$0 = b^0(\epsilon, \lambda(\epsilon)) = \frac{1}{\epsilon} \int_0^\epsilon b(\epsilon, \lambda(\epsilon), \alpha, \beta) d\alpha.$$

b is continuous in α and real for real α. Thus there exists a real α_0, which yields $b(\epsilon, \lambda(\epsilon), \alpha_0, \beta) = 0$, because the above integrand cannot be completely less or greater than 0. Thus $\lambda(\epsilon) = \lambda(\epsilon, \alpha_0) \in I(\epsilon)$.

To show the ϵ^p estimate we use lemma 3.4 (v) to get

$$|[b^0(0,\lambda(0))=0]-b^0(0,\lambda(\epsilon))| = |[b^0(\epsilon,\lambda(\epsilon))=0]-b^0(0,\lambda(\epsilon))| \leq C'\epsilon^p,$$

then as b^0_λ is uniformly bounded away from zero, this implies

$$|\lambda(\epsilon)-\lambda(0)| \leq C\epsilon^p$$

This completes the proof of theorem 1.□

4. **Discussion.** Homoclinic orbits are not as well studied in semilinear parabolic equations as in ordinary differential equations. Perhaps, because homoclinic orbits cannot exist in the easiest examples of parabolic equations like scalar on an interval $[a, b]$

$$u_t = u_{xx} + f(u)$$

with Neumann boundary conditions. There is a Ljapunov functional

$$L(u) = \int_a^b \frac{1}{2}(u_x(x))^2 - F(u(x))dx, \qquad \text{with } F' = f,$$

for which $\partial_t L(u(t)) < 0$ unless u is an equilibrium. Thus homoclinic orbits with

$$\lim_{t\to\pm\infty} L(u(t)) = L(u_0)$$

cannot exist.

But homoclinic orbits do already appear in systems of two reaction-diffusion equations. A one-parameter family of homoclinic orbits are created for example in two-parameter systems near Takens-Bogdanov bifurcation points. An example of such a bifurcation point in a reaction-diffusion system are spatially extended porous catalyst equations

$$\begin{aligned} u_t &= \Delta u - \lambda_1(u,v) \\ \lambda_2 v_t &= \Delta v + \beta\lambda_1(u,v) \end{aligned}$$

with Dirichlet boundary conditions $u = v = 1$, see [7]. Periodic forcing, created by periodic changes of light in light sensitive reactions creates equations as (2.1).

The existence of transverse homoclinic orbit creates complicated dynamics (shift-dynamics) in partial differential equations too, see [2, 20, 21]. In this paper we have shown, that transverse homoclinic orbits can only exist in an exponentially small parameter wedge, but the region, in which the complicated dynamics persist, can be bigger, but can be expected to be very small too.

Theorem 1 improves the estimates on the splitting of homoclinic orbits in [16] in two ways: First it holds for a much larger class of equations. We are working in the general frame work of [13] and only the analyticity assumptions on the complexified phase space are not directly obvious. Whereas in [16] we analysed equations

$$\begin{aligned} u_t &= D\Delta u + f(u;\lambda) + \epsilon^p g(u, \frac{t}{\epsilon}; \lambda, \epsilon) \qquad (4.1) \\ u(0) &= U \in H^s_{per}([0,L]^d, \mathbb{R}^n) \end{aligned}$$

with periodic boundary conditions, for which the existence of extreme regular (Gevrey regular) solutions are shown. The theorems in [16] gave a general estimate on the influence of rapidly oscillating term for general initial data. It can be removed by a transformation in the phase space up to a remainder of order

$$C\epsilon \exp(-c(t)\epsilon^{-1/3}). \qquad (4.2)$$

The estimate on the splitting quantities in [16] are a corollary of the general estimate. As we concentrated in this paper on homoclinic orbits, we could secondly improve the estimate by getting a better exponent in the exponential estimate. We achieve exactly the same estimate as for ordinary differential equations

$$C\exp(-2\pi\eta/\epsilon),$$

which is shown to be sharp in certain examples. Hence although in general the quantitative difference between autonomous and non-autonomous parabolic partial differential equations as in (4.1) can be only estimated by weaker exponential estimates than for ordinary differential equations, the splitting effect induced by non-autonomous forcing could be shown to be of the same size. The functional analytic description of the splitting effects, used in [8], proved to be very efficient.

In the moment, it is still open, if the estimates like (4.2) are sharp in the context of averaging of parabolic partial differential equations. But in another framework for averaging of Hamiltonian partial differential equations, a lower estimate on the non-autonomous remainder larger than for ordinary differential equations could be provided, see [17]. This shows, that there is a genuine difference between ordinary and partial differential equations – as far as averaging is concerned. In the light of this result, it is surprising, that there are effects, which can be analysed by averaging and where is the size of effect is of the same order for ordinary and partial differential equations.

However, there are still differences between the situation of ordinary and parabolic partial differential equations. In the finite dimensional case, the geometry of the splitting near any arbitrary point on the homoclinic orbit can be analysed. For this, one has to choose a flow-box. But general flow-box theorems do not exist for equations like (2.1). So we have to restrict our attention to a part of the local stable manifold near the equilibrium, where we can define a kind of flow-box.

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