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Robust Hedging of Double Touch Barrier Options*

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Abstract. We consider robust pricing of digital options, which pay out if the underlying asset has crossed both upper and lower barriers. We make only weak assumptions about the underlying process (typically continuity), but assume that the initial prices of call options with the same maturity and all strikes are known. In such circumstances, we are able to give upper and lower bounds on the arbitrage-free prices of the relevant options and show that these bounds are tight. Moreover, pathwise inequalities are derived, which provide the trading strategies with which we are able to realize any potential arbitrages. These super- and subhedging strategies have a simple quasi-static structure, their associated hedging error is bounded below, and in practice they carry low transaction costs. We show that, depending on the risk aversion of the investor, they can outperform significantly the standard delta/vega-hedging in presence of market frictions and/or model misspecification. We make use of embeddings techniques; in particular, we develop two new solutions to the (optimal) Skorokhod embedding problem.

Key words. double barrier option, robust hedging, no-arbitrage pricing, Skorokhod embedding, risk neutral distribution, superhedging, subhedging

AMS subject classifications. Primary, 91B28, 91B70; Secondary, 60G40, 60G44

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1. Introduction.

In the standard approach to pricing and hedging, one postulates a model for the underlying asset, calibrates it to the market prices of liquidly traded vanilla options, and then uses the model to derive prices and associated hedges for exotic over-the-counter products. Prices and hedges will be correct only if the model describes the real world perfectly, which is unlikely. The robust (model-independent) approach uses market data to deduce bounds on the prices consistent with *no-arbitrage* and the associated super- and subreplicating strategies, which are robust to model misspecification. More precisely, we start with quoted prices of some liquid options and assume that this market input is consistent with *no-arbitrage*. Then we want to answer two questions. First, for a given exotic option, what is the range of prices that we can charge for it without introducing a model-independent arbitrage? Second, if we see a price outside this range, how do we exploit it to make a riskless profit? In this paper we adopt such an approach to pricing and hedging for digital double barrier options.

The general methodology, which we now outline, is based on solving the *Skorokhod embedding problem* (SEP). We assume *no-arbitrage* and suppose that we know the market prices of calls and puts for all strikes at one maturity T. We are interested in pricing an exotic option

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with payoff given by a path-dependent functional $O(S)_T$. The example we consider here is a *digital double touch* barrier option struck at $(\underline{b}, \overline{b})$ which pays 1 if the stock price reaches both $\underline{b}$ and $\overline{b}$ before maturity T. Our aim is to construct a robust superreplicating strategy of the form

$$O(S)_T \leq F(S_T) + N_T, \tag{1.1}$$

where $F(S_T)$ is the payoff of a finite portfolio of European puts and calls quoted at time zero and N_T are gains from a self-financing trading strategy (typically forward transactions). Furthermore, we want (1.1) to be tight in the sense that we can construct a market model which matches the market prices of calls and puts and in which we have equality in (1.1). The initial price of the portfolio $F(S_T)$ is then the least upper bound on the price of the exotic $O(S)_T$, and the right-hand side (RHS) of (1.1) gives a simple superreplicating strategy at that cost. There is an analogous argument for the lower bound and an analogous subreplicating strategy. We stress that the RHS in (1.1) makes sense as a model-independent superhedge. It requires an initial capital, the price of $F(S_T)$, which is uniquely specified by the prices quoted in the market, and the rest is carried out in a self-financing way. Typically, for any specific payoff $O(S)_T$, one will be able to come up with a variety of random variables X which satisfy $O(S)_T \leq X \leq F(S_T) + N_T$, and hence, in some market models, X may be cheaper than $F(S_T)$. However, such X has no interpretation as a model-independent superreplicating strategy. Indeed, if X is a valid model-independent superreplicating strategy, it has a uniquely specified price at time $t = 0$ from the market quoted prices. This price is independent of the market model and hence, since we required (1.1) to be tight, is equal to the price of $F(S_T)$, as in the extreme model both are equal to the price of $O(S)_T$.

In fact, in order to construct (1.1), we first construct the market model which induces the upper bound on the price of $O(S)_T$ and hence will attain equality in (1.1). For this construction we rely on the theory of Skorokhod embeddings (cf. Obłój [34]). We assume *no-arbitrage* and consider a market model in the risk-neutral measure so that the forward price process $(S_t : t \leq T)$ is a martingale.[1] It follows from the work of Monroe [32] that $S_t = B_{\rho_t}$, for a Brownian motion (B_t) with $B_0 = S_0$ and some increasing sequence of stopping times $\{\rho_t : t \leq T\}$ (possibly relative to an enlarged filtration). Let us further assume that the payoff of the exotic option is invariant under the time change: $O(S)_T = O(B)_{\rho_T}$ a.s. Knowing the market prices of calls and puts for all strikes at maturity T is equivalent to knowing the distribution μ of S_T (cf. Breeden and Litzenberger [6]). Thus, we can see the stopping time $\rho = \rho_T$ as a solution to the SEP for μ. Conversely, let τ be a solution to the SEP for μ; i.e., $B_\tau \sim \mu$ and $(B_{t\wedge\tau} : t \geq 0)$ is a uniformly integrable martingale. Then the process $\tilde{S}_t := B_{\tau \wedge \frac{t}{T-t}}$ is a model for the stock-price process consistent with the observed prices of calls and puts at maturity T. In this way, we obtain a correspondence which allows us to identify market models with solutions to the SEP and vice versa. In consequence, to estimate the fair price of the exotic option $\mathbb{E}O(S)_T$, it suffices to bound $\mathbb{E}O(B)_\tau$ among all solutions τ to the

[1] Equivalently, under a simplifying assumption of zero interest rates, S_t is simply the stock price process. See section 3.2 for further discussion.

SEP. More precisely, we have

$$\inf_{\tau:B_\tau\sim\mu}\mathbb{E}O(B)_\tau \leq \mathbb{E}O(S)_T \leq \sup_{\tau:B_\tau\sim\mu}\mathbb{E}O(B)_\tau, \tag{1.2}$$

where all stopping times τ are such that $(B_{t\wedge\tau})_{t\geq 0}$ is uniformly integrable. Once we compute the above bounds and the stopping times which achieve them, we usually have a good intuition into how to construct the super- (and sub-)replicating strategies (1.1).

A more detailed description of the SEP-driven methodology outlined above can be found in Hobson [27] or in Obłój [35]. The idea of *no-arbitrage* bounds on prices goes back to Merton [31], and a recent survey of the literature can be found in Cox [12]. The methods for robust pricing and hedging of options sketched above go back to the works of Hobson [28] (lookback option) and Brown, Hobson, and Rogers [8] (single barrier options). More recently, Dupire [20] investigated volatility derivatives using the SEP, and Cox, Hobson, and Obłój [13] designed pathwise inequalities to derive price range and robust superreplicating strategies for derivatives paying a convex function of the local time.

Unlike in previous works, e.g., [8], we don't find a unique inequality (1.1) for a given barrier option. Instead we find that, depending on the market input (i.e., prices of calls and puts) and the pair of barriers, different strategies may be optimal. We characterize all of them and give precise conditions for deciding which one should be used. This new difficulty is coming from the dependence of the payoff on both the running maximum and minimum of the process. Solutions to the SEP which maximize or minimize $\mathbb{P}(\sup_{u\leq\tau} B_u \geq \overline{b}, \inf_{u\leq\tau} B_u \leq \underline{b})$ have not been developed previously and are introduced in this paper. These are new probabilistic results of independent interest which we derive here as tools to study our financial problem. As one might suspect, our new solutions are considerably more involved than those by Perkins [36] or Azéma and Yor [2] exploited by Brown, Hobson, and Rogers [8].

From a practical point of view, the no-arbitrage price bounds which we obtain are too wide to be used for pricing. However, our super- or subhedging strategies can still be used. Specifically, suppose an agent sells a double touch barrier option $O(S)_T$ for a premium p. She can then set up our superhedge (1.1) for an initial premium $\overline{p} > p$. At maturity T she holds $H = -O(S)_T + F(S)_T + N_T + p - \overline{p}$, which on average is worth zero, $\mathbb{E}H = 0$, but is also bounded below: $H \geq p - \overline{p}$. In reality, in the presence of model uncertainty and market frictions, this can be an appealing alternative to the standard delta/vega-hedging. Indeed, our numerical simulations in section 3.3 suggest that in the presence of transaction costs a risk averse agent will generally prefer the hedging strategy we construct to a (monitored daily) delta/vega-hedge.

The paper is structured as follows. First we present the setup: our assumptions and terminology and the types of double barriers considered in this paper. Then in section 2 we consider digital double touch barrier options introduced above. We first present super- and subreplicating strategies and then prove in section 2.3 that they induce tight robust bounds on the admissible prices of the double touch options. In section 3 we reconsider our assumptions and investigate some applications. Specifically, in section 3.1 we consider the case when calls and puts with only a finite number of strikes are observed, and in section 3.2 we discuss discontinuities in the price process (S_t), nonzero interest rates, and further additions to the set of available market quoted prices. In section 3.3 we present a numerical investigation of the performance of our super- and subhedging strategies. Section 4 contains the proofs of main theorems. In particular, it contains new solutions to the SEP which are necessary to prove results in section 2.3.

1.1. Setup. In what follows, $(S_t)_{t\geq 0}$ is the forward price process. Equivalently, we can think of the underlying asset with zero interest rates, or an asset with zero cost of carry. In particular, our results can be directly applied in Foreign Exchange markets for currency pairs from economies with equal interest rates. Moving to the spot market with nonzero interest rates is not immediate as our barriers become time-dependent; see section 3.2.

We assume that $(S_t)_{t\geq 0}$ has continuous paths. We comment in section 3.2 that this assumption can be removed or weakened to a requirement that given barriers are crossed continuously. We fix a maturity $T > 0$ and assume that we observe the initial spot price S_0 and the market prices of European calls for all strikes $K > 0$ and maturity T,

$$\Big(C(K) : K \geq 0\Big), \tag{1.3}$$

which we call the *market input*. For simplicity we assume that $C(K)$ is twice differentiable and strictly convex on $(0,\infty)$. Further, we assume that we can enter a forward transaction at no cost. More precisely, let ρ be a stopping time relative to the natural filtration of $(S_t)_{t\leq T}$ such that $S_\rho = \overline{b}$. Then the portfolio corresponding to selling a forward at time ρ has final payoff $(\overline{b} - S_T)\mathbf{1}_{\rho\leq T}$, and we assume its initial price is zero. The initial price of a portfolio with a constant payoff K is K. We denote by $\mathcal{X}$ the set of all calls, forward transactions, and constants, and $\mathrm{Lin}(\mathcal{X})$ is the space of their finite linear combinations, which is precisely the set of portfolios with given initial market prices. For convenience we introduce a pricing operator $\mathcal{P}$ which, to a portfolio with payoff X at maturity T, associates its initial (time zero) price, e.g., $\mathcal{P}K = K$, $\mathcal{P}(S_T - K)^+ = C(K)$, and $\mathcal{P}(\overline{b} - S_T)\mathbf{1}_{\rho\leq T} = 0$. We also assume that $\mathcal{P}$ is linear, whenever defined. Initially, $\mathcal{P}$ is given only on $\mathrm{Lin}(\mathcal{X})$. One of the aims of the paper is to understand extensions of $\mathcal{P}$ which do not introduce arbitrage to $\mathrm{Lin}(\mathcal{X}\cup\{Y\})$, for double touch barrier derivatives Y. Note that linearity of $\mathcal{P}$ on $\mathrm{Lin}(\mathcal{X})$ implies that call-put parity holds, and in consequence we also know the market prices of all European put options with maturity T:

$$P(K) := \mathcal{P}(K - S_T)^+ = K - S_0 + C(K).$$

Finally, we assume that the market admits no *model-independent arbitrage* in the sense that any portfolio of initially traded assets with a nonnegative payoff has a nonnegative price:

$$\forall X \in \mathrm{Lin}(\mathcal{X}) : X \geq 0 \Longrightarrow \mathcal{P}X \geq 0. \tag{1.4}$$

As we do not yet have any probability measure, by $X \geq 0$ we mean that the payoff is nonnegative for any continuous nonnegative stock price path $(S_t)_{t\leq T}$.

By a *market model* we mean a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with a continuous $\mathbb{P}$-martingale (S_t) which matches the market input (1.3). Note that we consider the model under the risk-neutral measure, and the pricing operator is then just the expectation $\mathcal{P} = \mathbb{E}$. Saying that (S_t) matches the market input is equivalent to saying that it starts in the initial spot S_0 a.s. and that $\mathbb{E}(S_T - K)^+ = C(K)$, $K > 0$. This in turn is equivalent to knowing the distribution of S_T (cf. [6, 8]). We denote this distribution by μ and often refer to it as the *law of S_T implied by the call prices.* Our regularity assumptions on $C(K)$ imply that

$$\mu(\mathrm{d}K) = C''(K)\mathrm{d}K, \quad K > 0, \tag{1.5}$$

so that μ has a positive density on $(0, \infty)$. We could relax this assumption and take the support of μ to be any interval $[a, b]$. Introducing atoms would complicate our formulae (essentially without introducing new difficulties).

The running maximum and minimum of the price process are denoted respectively by $\overline{S}_t = \sup_{u \le t} S_u$ and $\underline{S}_t = \inf_{u \le t} S_u$. We are interested in this paper in derivatives whose payoff depends on both $\overline{S}_T$ and $\underline{S}_T$. It is often convenient to express events involving the running maximum and minimum in terms of the first hitting times $H_x = \inf\{t : S_t = x\}$, $x \ge 0$. As an example, note that $\mathbf{1}_{\overline{S}_T \ge \overline{b},\, \underline{S}_T \le \underline{b}} = \mathbf{1}_{H_{\overline{b}} \vee H_{\underline{b}} \le T}$.

We use the notation $a \ll b$ to indicate that a is *much smaller* than b. This is only used to give intuition and is not formal. The minimum and maximum of two numbers are denoted $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$, respectively, and the positive part is denoted by $a^+ = a \vee 0$.

1.2. Connections to other barrier options. The barrier options considered in this paper are fairly specific: we are interested in a double touch option which pays out $\mathbf{1}_{\overline{S}_T \ge \overline{b},\, \underline{S}_T \le \underline{b}}$ at maturity T. It is natural to ask how the problem we consider is connected to similar problems for related barrier options, and also whether the results can be generalized to a wider class of options. One question is: can our results on double barrier options be expressed in terms of the results for single barrier options due to Brown, Hobson, and Rogers [8]? The answer is negative: we are inspired by their paper and we use similar methodology but to solve a different problem, and we cannot apply their results in our setting. More specifically, in [8], the authors develop arbitrage-free bounds on the price of a one-touch digital option (that is, an option which pays out 1 if a given level is crossed before maturity). At first sight one might want to price our double touch option as a sort of compound option which, upon hitting the first barrier, pays out a one-touch option struck at the second barrier. This intuition would work in a model-specific framework, but it breaks down entirely in the model-independent framework that we consider. Specifically, the bounds given in [8] depend on knowing the call prices at the time the option is issued. In our setting, however, we know the call prices initially but make no assumption about how they behave (or even if they are quoted) at intermediate times. In particular we have no a priori information about future call prices at the time the first barrier is hit and so cannot use the bounds derived in [8] for the options we study. On the other hand, we will recover results from [8] as limiting cases of double touch options when one of the barriers degenerates to the spot S_0.

An alternative question that may be asked is: can one use our results to say something about different types of digital barrier options? In this work, we are interested in the option with payoff $\mathbf{1}_{\overline{S}_T \ge \overline{b},\, \underline{S}_T \le \underline{b}}$, but the identity $\mathbf{1}_{\overline{S}_T \ge \overline{b},\, \underline{S}_T \le \underline{b}} = 1 - \mathbf{1}_{\overline{S}_T < \overline{b} \text{ or } \underline{S}_T > \underline{b}}$ immediately allows us to convert a superhedge of $\mathbf{1}_{\overline{S}_T \ge \overline{b},\, \underline{S}_T \le \underline{b}}$ into a subhedge of $\mathbf{1}_{\overline{S}_T < \overline{b} \text{ or } \underline{S}_T > \underline{b}}$, and consequently we can convert an upper bound on the price of $\mathbf{1}_{\overline{S}_T \ge \overline{b},\, \underline{S}_T \le \underline{b}}$ to a lower bound on the price of $\mathbf{1}_{\overline{S}_T < \overline{b} \text{ or } \underline{S}_T > \underline{b}}$. There are also identities which connect the double touch to other double barrier options, for example, $\mathbf{1}_{\overline{S}_T \ge \overline{b},\, \underline{S}_T \le \underline{b}} = \mathbf{1}_{\overline{S}_T \ge \overline{b}} - \mathbf{1}_{\overline{S}_T \ge \overline{b},\, \underline{S}_T > \underline{b}}$. A natural conjecture would then be that an upper bound on the price of $\mathbf{1}_{\overline{S}_T\geq\overline{b},\,\underline{S}_T\leq\underline{b}}$ might translate into a lower bound on the price of $\mathbf{1}_{\overline{S}_T\geq\overline{b},\,\underline{S}_T>\underline{b}}$. However, in the setup we consider this is not the case since the price of the one-touch option, $\mathbf{1}_{\overline{S}_T\geq\overline{b}}$, is not specified under our assumptions, and the lowest possible price of $\mathbf{1}_{\overline{S}_T\geq\overline{b},\,\underline{S}_T>\underline{b}}$ will typically not occur at the same time as the price of the one-touch option is maximized. This situation would alter if we assumed that the one-touch option was traded at a given initial price, in which case the lower bound on the price of $\mathbf{1}_{\overline{S}_T\geq\overline{b},\,\underline{S}_T\leq\underline{b}}$ would correspond to an upper bound on $\mathbf{1}_{\overline{S}_T\geq\overline{b},\,\underline{S}_T>\underline{b}}$. However, then the additional information given in the price of the one-touch option changes the setup of the initial problem and would, in all likelihood, change the bounds we derive in this paper. See section 3.2 for a further discussion of this point. As a consequence, the results in this paper will not extend to other double barrier options beyond the bounds given by the identity $\mathbf{1}_{\overline{S}_T\geq\overline{b},\,\underline{S}_T\leq\underline{b}} = 1 - \mathbf{1}_{\overline{S}_T<\overline{b} \text{ or } \underline{S}_T>\underline{b}}$.

The question of bounds for the double no-touch option $\mathbf{1}_{\overline{S}_T\leq\overline{b},\,\underline{S}_T\geq\underline{b}}$ is considered in Cox and Obłój [14]. In this case, the analysis of the hedges and bounds is relatively straightforward, but the paper focuses much more on subtleties concerning different classes of arbitrage with which we do not concern ourselves in this paper.

1.3. Probabilistic interpretation. The bounds on prices of double touch options developed in Theorems 2.2 and 2.4 correspond, in probabilistic terms, to computing

$$\sup_M \mathbb{P}\left(\sup_t M_t \geq \overline{b},\ \inf_t M_t \leq \underline{b}\right) \quad \text{and} \quad \inf_M \mathbb{P}\left(\sup_t M_t \geq \overline{b},\ \inf_t M_t \leq \underline{b}\right) \tag{1.6}$$

over all uniformly integrable continuous martingales $M = (M_t : 0 \leq t \leq \infty)$ with $M_0 = S_0$ and M_∞ distributed according to μ. To the best of our knowledge, such bounds have not been studied before and hence are of independent interest. As mentioned above, in order to compute these we develop new solutions to the Skorokhod embedding problem. The bounds that we obtain depend in a complex way on μ and $(\underline{b}, \overline{b})$ and are considerably more involved than in the single-sided case $\underline{b} = S_0$, which goes back to Blackwell and Dubins [4] and which was exploited in [8]. More precisely, $\sup_M \mathbb{P}(\sup_t M_t \geq \overline{b})$ is attained by the Azéma–Yor martingale (see Azéma and Yor [3]) simultaneously for all $\overline{b} \geq S_0$. In contrast, the supremum in (1.6) is attained by a different martingale for each pair $(\overline{b}, \underline{b})$. The bounds in (1.6) can be seen as a first step toward studying admissible laws for the triplet $(M_\infty, \sup_t M_t, \inf_t M_t)$, in a similar way as the single-sided case led to studies of admissible laws for $(M_\infty, \sup_t M_t)$ in Rogers [37] and Vallois [40].

2. Robust pricing and hedging.

We now investigate robust pricing and hedging of a double touch option which pays 1 if and only if the stock price goes above $\overline{b}$ and below $\underline{b}$ before maturity: $\mathbf{1}_{\overline{S}_T\geq\overline{b},\,\underline{S}_T\leq\underline{b}}$. We present simple quasi-static super- and subreplicating strategies which prove to be optimal (i.e., replicating) in some market model.[2] Sometimes, by a slight abuse, we refer to these robust strategies as *model-independent*. This emphasises that they work universally under our setup outlined above and do not depend on specific modeling assumptions.

[2] At first how the strategies below were derived may appear rather mysterious. In fact, as explained in the introduction, we first identified these extreme models and analyzed hedging strategies within these models. This way the super- and subreplicating strategies arise quite naturally.

It seems to us that the hedges are most easily expressed by considering four special cases. Each case will provide a super- or subhedge. We will see, however, that, depending on the values of $\underline{b}, \overline{b}$ relative to S_0, a different one will be the smallest superhedge/largest subhedge. In sections 2.1 and 2.2 we will outline the super- and subhedges, and in section 2.3 we will give criteria that allow us to determine exactly into which case a given set of parameter values falls.

The fact that we have four different hedges is rather intuitive. Imagine a trader who has a long position in a digital double touch barrier option and needs to hedge it in a robust way.[3] Then he is likely to think differently about the option depending on where the barriers are relative to the spot. If one of the barriers is very close to the spot, then he can effectively approximate the double touch with a simple one-touch struck at the other barrier. We will see that for some parameter values this is indeed the case, and the double touch has robust prices and superhedges that are identical to the one-touch. These cases are given below as $\overline{H}^I$ and $\overline{H}^{II}$. When barriers are approximately symmetric around the spot, our rough estimation above becomes too costly, and the trader hedges a genuine double touch option. When the barriers are close to the spot—relative to trader's belief about the volatility of the market (which is here inferred from the quoted call prices)—then it is reasonable to build the hedging strategy around the assumption that at least one of them will be hit. The optimal strategy then is described in our hedge $\overline{H}^{IV}$. On the other hand, if the barriers are far away, there will also be situations when neither barrier is struck, and the strategy has to account for that. This is done in $\overline{H}^{III}$. Finally, an analogous story holds for subhedging strategies.

We note also that there are strong similarities between some of the cases that we separate, to the extent that it is natural to ask, for example: can we express $\overline{H}^{IV}$ as a special case of $\overline{H}^{III}$? To some degree, the answer to this is that we can, with a suitable interpretation of some of the parameter values. However, it does not appear to us that making such a change would simplify the analysis in any way, since the special cases would need to be treated separately in any subsequent analysis anyway; rather, we have chosen to express the different super- and subhedges in the manner that appears to convey the most intuitive picture of the differing possible behavior.

2.1. Superhedging. We present here four superreplicating strategies. All our strategies have the same simple structure: we buy an initial portfolio of calls and puts, and when the stock price reaches $\underline{b}$ or $\overline{b}$ we buy or sell forward contracts. Naturally our goal is not only to write a superreplicating strategy but to write *the smallest superreplicating strategy*, and to do so we have to choose the parameters judiciously. As we will see in section 2.3, for a given pair of barriers $\underline{b}, \overline{b}$ exactly one of the superreplicating strategies will induce a tight bound on the derivative's price. We will provide an explicit criterion determining which strategy to use.

$\overline{H}^I$: *superhedge for* $\underline{b} \ll S_0 < \overline{b}$. We buy α puts with strike $K \in (\underline{b}, \infty)$, and when the stock price reaches $\underline{b}$ we buy β forward contracts; see Figure 1. The values of α, β are chosen so that the final payoff on $(0, K)$, provided that the stock price has reached $\underline{b}$, is constant and equal to 1. One easily computes that $\alpha = \beta = (K - \underline{b})^{-1}$. Formally, the superreplication

[3]This can be due to, e.g., high uncertainty about a market model, an illiquid market, or high transaction costs; see section 3.3 for a detailed discussion.

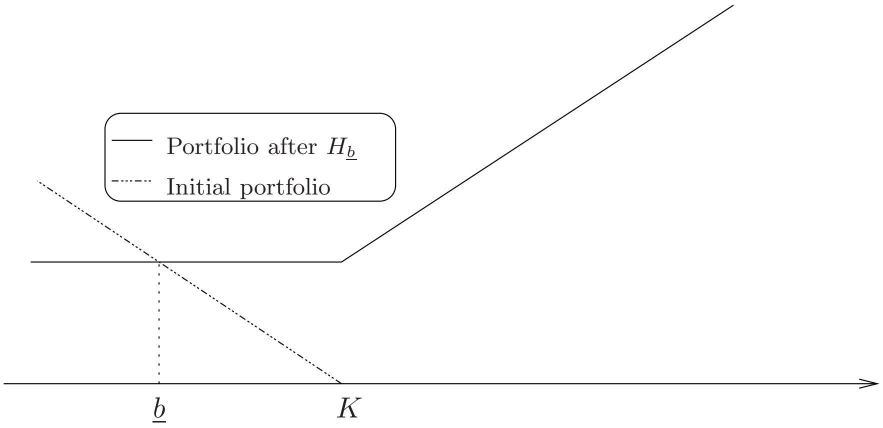


Figure 1. *Superhedge* $\overline{H}^I$.

follows from the following inequality:

$$\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} \leq \mathbf{1}_{\underline{S}_T \leq \underline{b}} \leq \frac{(K - S_T)^+}{K - \underline{b}} + \frac{S_T - \underline{b}}{K - \underline{b}} \mathbf{1}_{\underline{S}_T \leq \underline{b}} =: \overline{H}^I(K), \tag{2.1}$$

where the last term corresponds to a forward contract entered into, at no cost, when $S_t = \underline{b}$. Note that $\mathbf{1}_{\underline{S}_T \leq \underline{b}} = \mathbf{1}_{H_{\underline{b}} \leq T}$.

$\overline{H}^{II}$: *superhedge for* $\underline{b} < S_0 \ll \overline{b}$. This is a mirror image of $\overline{H}^I$: we buy α calls with strike $K \in (0, \overline{b})$, and when the stock price reaches $\overline{b}$ we sell β forward contracts. The values of α, β are chosen so that the final payoff on (K, ∞), provided that the stock price reaches $\overline{b}$, is constant and equal to 1. One easily computes that $\alpha = \beta = (\overline{b} - K)^{-1}$. Formally, the superreplication follows from the following inequality:

$$\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} \leq \mathbf{1}_{\overline{S}_T \geq \overline{b}} \leq \frac{(S_T - K)^+}{\overline{b} - K} + \frac{\overline{b} - S_T}{\overline{b} - K} \mathbf{1}_{\overline{S}_T \geq \overline{b}} =: \overline{H}^{II}(K). \tag{2.2}$$

$\overline{H}^{III}$: *superhedge for* $\underline{b} \ll S_0 \ll \overline{b}$. This superhedge involves a static portfolio of four calls and puts and at most four dynamic trades. The choice of parameters is judicious, which makes the strategy the most complex to describe. Choose

$$0 < K_4 < \underline{b} < K_3 < K_2 < \overline{b} < K_1 \tag{2.3}$$

and buy α_i calls with strike K_i, $i = 1, 2$, and α_j puts with strike K_j, $j = 3, 4$. If the stock price reaches $\overline{b}$ without having hit $\underline{b}$ before, that is when $H_{\overline{b}} < H_{\underline{b}} \wedge T$, sell β_1 forward. If $H_{\underline{b}} < H_{\overline{b}} \wedge T$, at $H_{\underline{b}}$ buy β_2 forwards. When the stock price, having hit $\overline{b}$, first reaches $\underline{b}$, that is, at $H_{\underline{b}} \in (H_{\overline{b}}, T]$, buy $\beta_3 = \alpha_3 + \beta_1$ forwards. Finally, at $H_{\overline{b}} \in (H_{\underline{b}}, T]$ sell $\beta_4 = \alpha_2 + \beta_2$ forwards. The choice of β_3 and β_4 is such that the final payoff after hitting $\overline{b}$ and then $\underline{b}$ (resp., $\underline{b}$ and then $\overline{b}$) is constant and equal to 1 on $[K_4, K_3]$ (resp., $[K_2, K_1]$). We now proceed to impose conditions which determine other parameters. A pictorial representation of the superhedge is given in Figure 2.

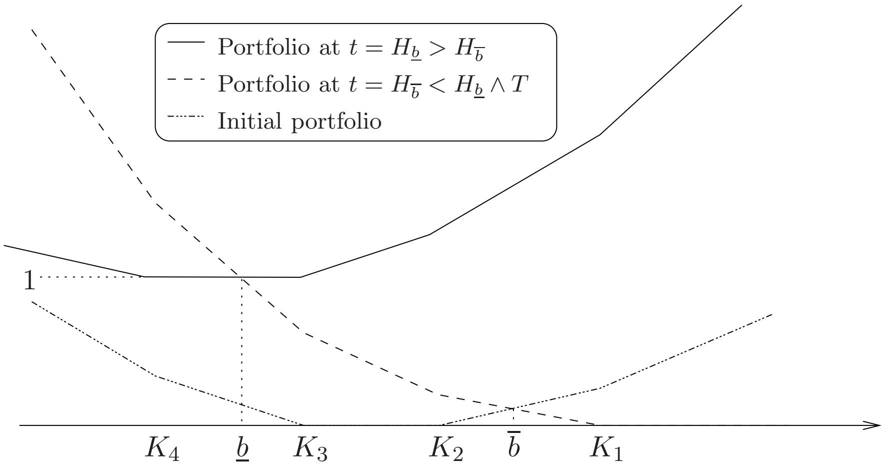


Figure 2. *Superhedge* $\overline{H}^{III}$.

Note that the initial payoff on $[K_3, K_2]$ is zero. After hitting $\overline{b}$ and before hitting $\underline{b}$, the payoff should be zero on $[K_1, \infty)$ and equal to 1 at $\underline{b}$. Likewise, after hitting $\underline{b}$ and before hitting $\overline{b}$, the payoff should be zero on $[0, K_4]$ and equal to 1 at $\overline{b}$. This yields six equations:

$$\left\{\begin{array}{l}\alpha_1 + \alpha_2 - \beta_1 = 0,\\ \alpha_2(K_1 - K_2) - \beta_1(K_1 - \overline{b}) = 0,\\ \alpha_3(K_3 - \underline{b}) - \beta_1(\underline{b} - \overline{b}) = 1,\end{array}\right. \qquad \left\{\begin{array}{l}\alpha_3 + \alpha_4 - \beta_2 = 0,\\ \alpha_3(K_3 - K_4) + \beta_2(K_4 - \underline{b}) = 0,\\ \alpha_2(\overline{b} - K_2) + \beta_2(\overline{b} - \underline{b}) = 1.\end{array}\right. \tag{2.4}$$

The superhedging strategy corresponds to an almost-sure inequality

$$\begin{aligned}\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} \leq\ & \alpha_1(S_T - K_1)^+ + \alpha_2(S_T - K_2)^+ + \alpha_3(K_3 - S_T)^+ + \alpha_4(K_4 - S_T)^+ \\ & - \beta_1(S_T - \overline{b})\mathbf{1}_{H_{\overline{b}} < H_{\underline{b}} \wedge T} + \beta_2(S_T - \underline{b})\mathbf{1}_{H_{\underline{b}} < H_{\overline{b}} \wedge T} \\ & + \beta_3(S_T - \underline{b})\mathbf{1}_{H_{\overline{b}} < H_{\underline{b}} \leq T} - \beta_4(S_T - \overline{b})\mathbf{1}_{H_{\underline{b}} < H_{\overline{b}} \leq T} \\ & =: \overline{H}^{III}(K_1, K_2, K_3, K_4),\end{aligned} \tag{2.5}$$

where the parameters, after solving (2.4), are given by

$$\left\{\begin{array}{l}\alpha_3 = \frac{(K_1-K_2)(\underline{b}-K_4)(\overline{b}-\underline{b})-(K_1-\overline{b})(\overline{b}-K_2)(\underline{b}-K_4)}{(K_1-K_2)(K_3-K_4)(\overline{b}-\underline{b})^2-(K_3-\underline{b})(K_1-\overline{b})(\overline{b}-K_2)(\underline{b}-K_4)},\\ \alpha_1 = \left(1 - \alpha_3\frac{K_3-K_4}{\underline{b}-K_4}(\overline{b} - \underline{b})\right)(K_1 - \overline{b})^{-1},\\ \alpha_2 = \left(1 - \alpha_3\frac{K_3-K_4}{\underline{b}-K_4}(\overline{b} - \underline{b})\right)(\overline{b} - K_2)^{-1},\\ \alpha_4 = \frac{K_3-\underline{b}}{\underline{b}-K_4}\alpha_3,\end{array}\right. \qquad \left\{\begin{array}{l}\beta_1 = \alpha_1 + \alpha_2,\\ \beta_2 = \alpha_3 + \alpha_4.\end{array}\right. \tag{2.6}$$

Using (2.3), one can verify that α_3 and α_1 are nonnegative and thus also α_2 and α_4 and all $\beta_1, \dots, \beta_4$.

$\overline{H}^{IV}$: *superhedge for* $\underline{b} < S_0 < \overline{b}$. Choose $0 < K_2 < \underline{b} < S_0 < \overline{b} < K_1$. The initial portfolio is composed of α_1 calls with strike K_1, α_2 puts with strike K_2, α_3 forward contracts, and α_4 in cash. If we hit $\overline{b}$ before hitting $\underline{b}$, we sell β_1 forwards, and if we hit $\underline{b}$ before hitting $\overline{b}$, we

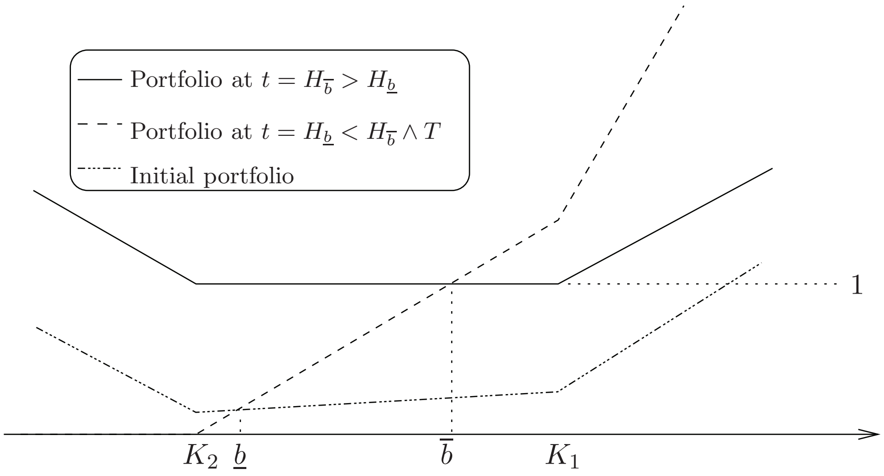


Figure 3. *Superhedge* $\overline{H}^{IV}$.

buy β_2 forwards. The payoff of the portfolio should be zero on $[K_1, \infty)$ (resp., $[0, K_2]$) and equal to 1 at $\underline{b}$ (resp., $\overline{b}$) in the first (resp., second) case. Finally, when we first hit $\underline{b}$ after having hit $\overline{b}$, we buy β_3 forwards, and when we first hit $\overline{b}$ having previously hit $\underline{b}$, we sell β_4 forwards. In both cases the final payoff should then be equal to 1 on $[K_2, K_1]$; see Figure 3. The superhedging strategy corresponds to the following almost-sure inequality:

$$
\begin{aligned}
\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} \leq\ & \alpha_1(S_T - K_1)^+ + \alpha_2(K_2 - S_T)^+ + \alpha_3(S_T - S_0) + \alpha_4 \\
& - \beta_1(S_T - \overline{b})\mathbf{1}_{H_{\overline{b}} < H_{\underline{b}} \wedge T} + \beta_2(S_T - \underline{b})\mathbf{1}_{H_{\underline{b}} < H_{\overline{b}} \wedge T} \\
& + \beta_3(S_T - \underline{b})\mathbf{1}_{H_{\overline{b}} < H_{\underline{b}} \leq T} - \beta_4(S_T - \overline{b})\mathbf{1}_{H_{\underline{b}} < H_{\overline{b}} \leq T} \\
& =: \overline{H}^{IV}(K_1, K_2),
\end{aligned} \tag{2.7}
$$

where, working out the conditions on α_i, β_i, the parameters are

$$
\begin{cases}
\alpha_1 = 1/(K_1 - \underline{b}), \\
\alpha_2 = 1/(\overline{b} - K_2), \\
\alpha_3 = \frac{(K_1 - \underline{b}) - (\overline{b} - K_2)}{(K_1 - \underline{b})(\overline{b} - K_2)}, \\
\alpha_4 = \frac{\underline{b}\overline{b} - K_1 K_2}{(K_1 - \underline{b})(\overline{b} - K_2)} + \alpha_3 S_0,
\end{cases}
\qquad
\begin{cases}
\beta_1 = \alpha_1 + \alpha_3 = 1/(\overline{b} - K_2), \\
\beta_2 = \alpha_2 - \alpha_3 = 1/(K_1 - \underline{b}), \\
\beta_3 = \alpha_1 = 1/(K_1 - \underline{b}), \\
\beta_4 = \alpha_2 = 1/(\overline{b} - K_2).
\end{cases} \tag{2.8}
$$

As we highlighted at the beginning of the section, all our superhedges have the same general structure: they consist of an initial portfolio of cash, puts, and calls and then involve some forward transactions. We presented above four distinct strategies of this type, and one could ask: it is possible to unify them into one general parametric strategy? It is not too difficult to see that the inequalities (2.1), (2.2), and (2.7) can, with the correct modifications of the parameters, be rewritten in the form (2.5); however, in general the relationships in (2.4) and (2.3) will not hold (for (2.7), one needs to take $K_3 = K_1 > \overline{b}$ and $K_2 = K_4 < \underline{b}$).

However, if we suppose simply that

$$K_4 < \underline{b}, \quad K_3 > \underline{b}, \quad K_2 < \overline{b}, \quad K_1 > \overline{b},$$

one can derive the following conditions on the parameters in (2.5), which preserve the inequality: we require $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2 \geq 0$ and

$$\left\{\begin{array}{l} \alpha_1 + \alpha_2 - \beta_1 \geq 0, \\ \alpha_2(K_1 - K_2) - \beta_1(K_1 - \overline{b}) + \alpha_3(K_3 - K_1)^+ \geq 0, \\ \alpha_3(K_3 - \underline{b}) - \beta_1(\underline{b} - \overline{b}) + +\alpha_2(\underline{b} - K_2)^+ \geq 1, \\ \alpha_3 + \alpha_4 - \beta_2 \geq 0, \\ \alpha_3(K_3 - K_4) + \beta_2(K_4 - \underline{b}) + \alpha_2(K_4 - K_2)^+ \geq 0, \\ \alpha_2(\overline{b} - K_2) + \beta_2(\overline{b} - \underline{b}) + \alpha_3(K_3 - \overline{b})^+ \geq 1. \end{array}\right. \tag{2.9}$$

It can then be verified that the four different cases are each specific solutions of this system where the inequalities are tight in differing manners. Of course, checking that these are the only interesting such solutions is nontrivial and will be the content of Theorem 2.2. We also believe that, while perhaps this is presentationally more concise, the unified presentation hides the true nature of the superhedging strategies.

2.2. Subhedging. We present now three constructions of robust subhedges. Depending on the relative distance of barriers from the spot, one of them will turn out to be the most expensive (model-independent) subhedge. We note, however, that there is also a fourth (trivial) subhedge, which has payoff zero and corresponds to an empty portfolio. In fact this will be the most expensive subhedge when $\underline{b} \ll S_0 \ll \overline{b}$, and we can construct a market model in which both barriers are never hit. Details will be given in Theorem 2.4.

$\underline{H}_I$*: subhedge for* $\underline{b} < S_0 < \overline{b}$. Choose $0 < K_2 < \underline{b} < S_0 < \overline{b} < K_1$. The initial portfolio will contain a cash amount, a forward, and calls with five different strikes and will also include two digital options, which pay 1 provided that S_T is above a specified level. Figure 4 demonstrates graphically the hedging strategy, and we note that the effect of the digital options is to provide a jump in the payoff at the points $\underline{b}$, $\overline{b}$.

As in the previous cases, the optimality of the construction will follow from an almost-sure inequality. The relevant inequality is now

$$\begin{aligned} \mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}} \geq\ & \alpha_0 + \alpha_1(S_T - S_0) - \alpha_2(S_T - K_2)^+ + \alpha_3(S_T - \underline{b})^+ - \alpha_3(S_T - K_3)^+ \\ & + \alpha_3(S_T - \overline{b})^+ - (\alpha_3 - \alpha_2)(S_T - K_1)^+ - \gamma_1 \mathbf{1}_{\{S_T > \underline{b}\}} + \gamma_2 \mathbf{1}_{\{S_T \geq \overline{b}\}} \\ & + (\alpha_2 - \alpha_1)(S_T - \underline{b})\mathbf{1}_{\{H_{\underline{b}} < H_{\overline{b}} \wedge T\}} - \alpha_2(S_T - \overline{b})\mathbf{1}_{\{H_{\underline{b}} < H_{\overline{b}} < T\}} \\ & - (\alpha_3 - \alpha_2 + \alpha_1)(S_T - \overline{b})\mathbf{1}_{\{H_{\overline{b}} < H_{\underline{b}} \wedge T\}} + (\alpha_3 - \alpha_2)(S_T - \underline{b})\mathbf{1}_{\{H_{\overline{b}} < H_{\underline{b}} < T\}}. \end{aligned} \tag{2.10}$$

Specifically, we can see that the hedging strategy consists of a portfolio which contains cash α_0, has α_1 forwards, is short α_2 calls at strike K_2, etc. The novel terms here are the digital options; we note further that the digital options can be considered also as the limit of portfolios of calls (see, for example, Bowie and Carr [5]). In our context, we can use their limiting argument to deduce $\mathcal{P}\mathbf{1}_{\{S_T \geq \overline{b}\}} = -C'(K)$.

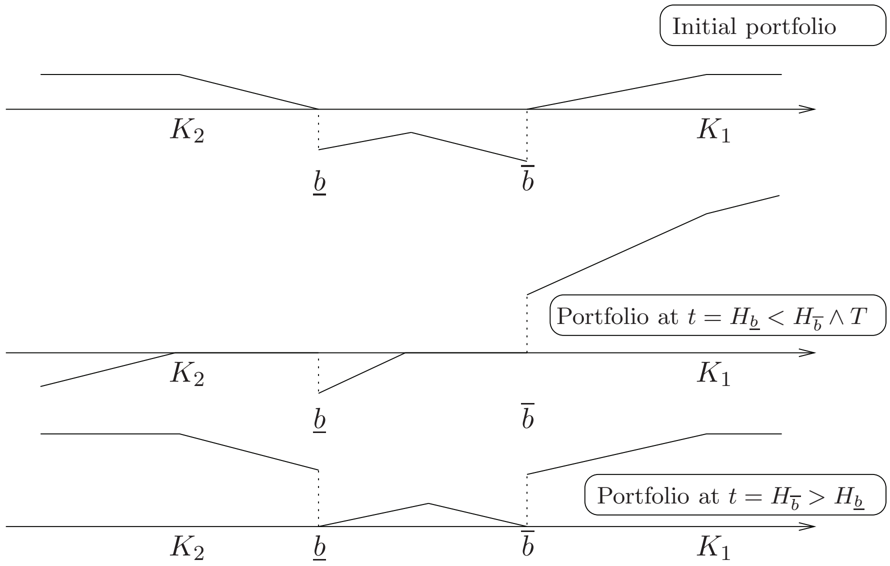


Figure 4. *Subhedge* $\underline{H}_I$.

The strategy to be employed is then as follows: initially, run to either $\underline{b}$ or $\overline{b}$; supposing that $\underline{b}$ is hit first, we buy $(\alpha_2 - \alpha_1)$ forwards, then if we later hit $\overline{b}$, we sell α_2 forwards. A similar strategy is followed if $\overline{b}$ is hit first. As previously, the structure imposes some constraints on the parameters. The relevant constraints are

$$0 = \alpha_0 + \alpha_1(\underline{b} - S_0) - \alpha_2(\underline{b} - K_2), \tag{2.11}$$

$$0 = \alpha_0 + \alpha_1(\overline{b} - S_0) - \alpha_2(\overline{b} - K_2) + \alpha_3(K_3 - \underline{b}) - \gamma_1 + \gamma_2, \tag{2.12}$$

$$1 = \alpha_0 + \alpha_1(K_2 - S_0) + (\alpha_2 - \alpha_1)(K_2 - \underline{b}) - \alpha_2(K_2 - \overline{b}), \tag{2.13}$$

$$1 = \alpha_0 + \alpha_1(K_2 - S_0) - (\alpha_3 - \alpha_2 + \alpha_1)(K_2 - \overline{b}) + (\alpha_3 - \alpha_2)(K_2 - \underline{b}), \tag{2.14}$$

$$\gamma_1 = (K_3 - \underline{b})\alpha_3, \tag{2.15}$$

$$\gamma_2 = (\overline{b} - K_3)\alpha_3, \tag{2.16}$$

$$\frac{K_3 - \underline{b}}{K_1 - \underline{b}} = \frac{\overline{b} - K_3}{\overline{b} - K_2}. \tag{2.17}$$

Equations (2.11) and (2.12) arise from the constraint that initially the payoff is zero at $\underline{b}$, $\overline{b}$; constraints (2.13) and (2.14) come from the constraint that the final payoff is 1 at K_2 when both barriers are hit (in either order); (2.15) and (2.16) represent the fact that, in the intermediate step, at K_3 the gap at $\underline{b}$ (resp., $\overline{b}$) is the size of the respective digital option. The final constraint, (2.17), follows from noting that K_3 is the intersection point of the lines from $(\underline{b}, 0)$ to $(K_1, 1)$ and from $(K_2, 1)$ to $(\overline{b}, 0)$. Note that it follows that the initial payoffs on $(0, K_1)$ and (K_2, ∞) are colinear and that the final payoff in K_1 is 1 when both barriers are hit.

The given equations can be solved to deduce

$$\text{(2.18)}\quad \begin{cases} \alpha_0 = \frac{S_0(K_1+K_2-\overline{b}-\underline{b})+\overline{b}\underline{b}-K_1K_2}{(\overline{b}-K_2)(K_1-\underline{b})}, \\ \alpha_1 = \frac{K_1+K_2-\overline{b}-\underline{b}}{(\overline{b}-K_2)(K_1-\underline{b})}, \\ \alpha_2 = \frac{1}{\overline{b}-K_2}, \\ \alpha_3 = \frac{\overline{b}-K_2+K_1-\underline{b}}{(\overline{b}-K_2)(K_1-\underline{b})}, \end{cases} \qquad \begin{cases} K_3 = \frac{\overline{b}K_1-\underline{b}K_2}{\overline{b}-K_2-\underline{b}+K_1}, \\ \gamma_1 = \frac{\overline{b}-\underline{b}}{\overline{b}-K_2}, \\ \gamma_2 = \frac{\overline{b}-\underline{b}}{K_1-\underline{b}}. \end{cases}$$

We note from the above that $\alpha_2, \alpha_3, \gamma_1$, and γ_2 are all (strictly) positive; further, it can be checked that the quantities $(\alpha_3 - \alpha_2), (\alpha_2 - \alpha_1), (\alpha_3 - \alpha_2 + \alpha_1)$ are all positive. It follows that the construction holds for all choices of K_1, K_2 with $K_2 < \underline{b}$ and $K_1 > \overline{b}$.

For future reference, we define $\underline{H}_I(K_1, K_2)$ to be the random variable given by the RHS of (2.10), where the coefficients are given by the solutions of (2.11)–(2.17).

$\underline{H}_{II}$*: subhedge for* $\underline{b} < S_0 \ll \overline{b}$. While the above hedge can be considered to be the "typical" subhedge for the option, there are two further cases that need to be considered when the initial stock price, S_0, is much closer to one of the barriers than the other. The resulting subhedge will share many of the features of the previous construction; however, the main difference concerns the behavior in the tails: we now have the hedge taking the value 1 in the tails under only one of the possible ways of knocking in (specifically, in the case where $\underline{b} < S_0 \ll \overline{b}$, we get equality in the tails only when $\underline{b}$ is hit first).

A graphical representation of the construction is given in Figure 5. In this case, rather than specifying only K_1 and K_2, we also need to specify $K_3 \in (\underline{b}, \overline{b})$ satisfying

$$\frac{\overline{b} - K_3}{\overline{b} - K_2} \geq \frac{K_3 - \underline{b}}{K_1 - \underline{b}}.$$

This identity implies that, for the initial portfolio, the value of the function just below $\overline{b}$ is strictly smaller than the value of the function just above $\underline{b}$. This can be rearranged to get

$$K_3 \leq \overline{b}\frac{K_1 - \underline{b}}{(K_1 - \underline{b}) + (\overline{b} - K_2)} + \underline{b}\frac{\overline{b} - K_2}{(K_1 - \underline{b}) + (\overline{b} - K_2)}.$$

The actual inequality we use remains the same as in the previous case (2.10), as do some of the constraints:

$$\text{(2.19)}\quad 0 = \alpha_0 + \alpha_1(\underline{b} - S_0) - \alpha_2(\underline{b} - K_2),$$

$$\text{(2.20)}\quad 0 = \alpha_0 + \alpha_1(\overline{b} - S_0) - \alpha_2(\overline{b} - K_2) + \alpha_3(K_3 - \underline{b}) - \gamma_1 + \gamma_2,$$

$$\text{(2.21)}\quad 1 = \alpha_0 + \alpha_1(K_2 - S_0) + (\alpha_2 - \alpha_1)(K_2 - \underline{b}) - \alpha_2(K_2 - \overline{b}),$$

$$\text{(2.22)}\quad 1 = \alpha_0 + \alpha_1(\underline{b} - S_0) + \alpha_2(K_2 - K_1 + \overline{b} - \underline{b}) + \alpha_3(K_3 + K_1 - \underline{b} - \overline{b}) - \gamma_1 + \gamma_2,$$

$$\text{(2.23)}\quad \gamma_1 = (K_3 - \underline{b})\alpha_3,$$

$$\text{(2.24)}\quad \gamma_2 = (\overline{b} - K_3)\alpha_3;$$

(2.19) and (2.20) refer still to having an initial payoff of 0 at $\overline{b}$ and $\underline{b}$, and (2.23) and (2.24) also still relate the size of the digital options to the slopes. The change is in the constraints

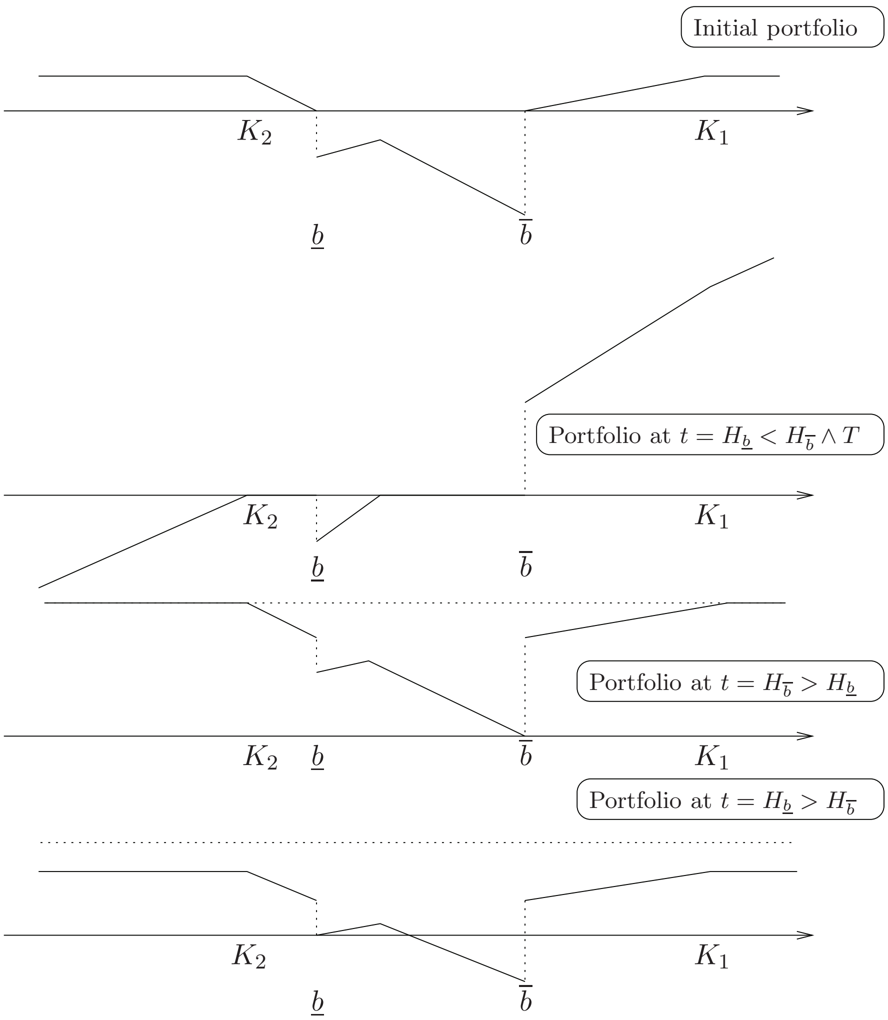


Figure 5. *Subhedge* $\underline{H}_{II}$.

(2.21) and (2.22), which now ensure that the function at K_1 and K_2, after hitting first $\underline{b}$ and then $\overline{b}$, takes the value 1. We note that, in the previous example, where (2.17) held, these are in fact equivalent to (2.13) and (2.14); the fact that (2.17) no longer holds means that we need to be more specific about the constraints.

The solutions to the above are now

(2.25)

$$\begin{cases} \alpha_0 = \frac{(\overline{b}\underline{b}+S_0K_3)(K_1-K_2)-(\underline{b}K_1+S_0\overline{b})(K_3-K_2)-(\overline{b}K_2+S_0\underline{b})(K_1-K_3)}{(\overline{b}-\underline{b})(K_1-K_3)(\overline{b}-K_2)}, \\ \alpha_1 = \frac{K_3(K_1-K_2)-\underline{b}(K_1-K_3)-\overline{b}(K_3-K_2)}{(\overline{b}-\underline{b})(K_1-K_3)(\overline{b}-K_2)}, \\ \alpha_2 = \frac{1}{\overline{b}-K_2}, \\ \alpha_3 = \frac{K_1-K_2}{(K_1-K_3)(\overline{b}-K_2)}, \end{cases} \qquad \begin{cases} \gamma_1 = \frac{(K_3-\underline{b})(K_1-K_2)}{(\overline{b}-K_2)(K_1-K_3)}, \\ \gamma_2 = \frac{(\overline{b}-K_3)(K_1-K_2)}{(\overline{b}-K_2)(K_1-K_3)}. \end{cases}$$

As before, we write $\underline{H}_{II}(K_1, K_2, K_3)$ for the random variable on the RHS of (2.10), where the constants are now specified by (2.25).

$\underline{H}_{III}$*: subhedge for* $\underline{b} \ll S_0 < \overline{b}$*.* The third case here is the corresponding version of the above where we have a large value of K_3, specifically,

$$K_3 \geq \overline{b}\frac{K_1 - \underline{b}}{(K_1 - \underline{b}) + (\overline{b} - K_2)} + \underline{b}\frac{\overline{b} - K_2}{(K_1 - \underline{b}) + (\overline{b} - K_2)},$$

and we need to modify (2.21) and (2.22) appropriately:

$$1 = \alpha_0 + \alpha_1(\overline{b} - S_0) + (\alpha_3 - \alpha_2)(\overline{b} - \underline{b}),$$
$$1 = \alpha_0 + \alpha_1(\overline{b} - S_0) + \alpha_2(K_2 - K_1 + \underline{b} - \overline{b}) + \alpha_3(K_3 + K_1 - 2\underline{b}) - \gamma_1 + \gamma_2.$$

The solutions are now

(2.26)

$$\begin{cases} \alpha_0 = \frac{(\overline{b}\underline{b}+S_0K_3)(K_1-K_2)-(\underline{b}K_1+S_0\overline{b})(K_3-K_2)-(\overline{b}K_2+S_0\underline{b})(K_1-K_3)}{(\overline{b}-\underline{b})(K_3-K_2)(K_1-\underline{b})}, \\ \alpha_1 = \frac{K_3(K_1-K_2)-\underline{b}(K_1-K_3)-\overline{b}(K_3-K_2)}{(\overline{b}-\underline{b})(K_3-K_2)(K_1-\underline{b})}, \\ \alpha_2 = \frac{K_1-K_3}{(K_3-K_2)(K_1-\underline{b})}, \\ \alpha_3 = \frac{K_1-K_2}{(K_3-K_2)(K_1-\underline{b})}, \end{cases} \qquad \begin{cases} \gamma_1 = \frac{(K_3-\underline{b})(K_1-K_2)}{(K_3-K_2)(K_1-\underline{b})}, \\ \gamma_2 = \frac{(\overline{b}-K_3)(K_1-K_2)}{(K_3-K_2)(K_1-\underline{b})}. \end{cases}$$

As before, we write $\underline{H}_{III}(K_1, K_2, K_3)$ for the random variable on the RHS of (2.10), where the constants are now specified by (2.26).

We note that all three subhedges have very similar structure, and it was convenient to represent them using a common inequality (2.10). We could further combine them into a "general" lower bound consisting of (2.10) together with a set of inequalities constraining the parameter choice, out of which one finds three differing possible extremal sets of inequalities. However, similarly to the superhedge, we do not think this would offer any new insights or simplify the presentation.

2.3. Pricing. Consider the double touch digital barrier option with the payoff $\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}}$. As an immediate consequence of the superhedging strategies described in section 2.1 we get an upper bound on the price of this derivative, as follows.

Proposition 2.1. *Given the market input* (1.3)*, no-arbitrage* (1.4) *in the class of portfolios* $\mathrm{Lin}(\mathcal{X} \cup \{\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}}\})$ *implies the following inequality between the prices:*

$$\text{(2.27)} \quad \mathcal{P}\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}} \leq \inf\left\{\mathcal{P}\overline{H}^I(K), \mathcal{P}\overline{H}^{II}(K'), \mathcal{P}\overline{H}^{III}(K_1, K_2, K_3, K_4), \mathcal{P}\overline{H}^{IV}(K_1, K_4)\right\},$$

where the infimum is taken over $K > \underline{b}$*,* $K' < \overline{b}$*, and* $0 < K_4 < \underline{b} < K_3 < K_2 < \overline{b} < K_1$ *and where* $\overline{H}^I$*,* $\overline{H}^{II}$*,* $\overline{H}^{III}$*, and* $\overline{H}^{IV}$ *are given by* (2.1)*,* (2.2)*,* (2.5)*, and* (2.7)*, respectively.*

The purpose of this section is to show that, given the law of S_T and the pair of barriers $\underline{b}, \overline{b}$, we can determine explicitly which superhedges and with what strikes the infimum on the RHS of (2.27) is achieved. We present formal criteria, but we also use labels, e.g., $\underline{b} \ll S_0 < \overline{b}$, which provide an intuitive classification. Furthermore, we will show that we can always construct a market model in which the infimum in (2.27) is the actual price of the double barrier option and therefore exhibit the model-independent least upper bound for the price of the derivative. Subsequently, an analogous reasoning for subhedging and the lower bound is presented.

Let μ be the market implied law of S_T given by (1.5). The barycenter of μ associates with a nonempty Borel set $\Gamma \subset \mathbb{R}$ the mean of μ over Γ via

$$\mu_B(\Gamma) = \frac{\int_\Gamma u\mu(\mathrm{d}u)}{\int_\Gamma \mu(\mathrm{d}u)}. \tag{2.28}$$

For $w < \underline{b}$ and $z > \overline{b}$ let $\rho_-(w) > \underline{b}$ and $\rho_+(z) < \overline{b}$ be the unique points such that the intervals $[w, \rho_-(w)]$ and $[\rho_+(z), z]$ are centered respectively around $\underline{b}$ and $\overline{b}$, that is,

$$\begin{cases} \rho_- : [0, \underline{b}] \to [\underline{b}, \infty) & \text{defined via } \mu_B([w, \rho_-(w)]) = \underline{b}, \\ \rho_+ : [\overline{b}, \infty) \to [0, \overline{b}] & \text{defined via } \mu_B([\rho_+(z), z]) = \overline{b}. \end{cases} \tag{2.29}$$

Note that $\rho_\pm$ are decreasing and well defined as $\mu_B([0,\infty)) = S_0 \in (\underline{b}, \overline{b})$. We need to define two more functions,

$$\begin{cases} \gamma_+(w) \geq \overline{b} & \text{defined via } \mu_B\big([0,w] \cup [\rho_+(\gamma_+(w)), \gamma_+(w)]\big) = \underline{b},\ w \leq \underline{b}, \\ \gamma_-(z) \leq \underline{b} & \text{defined via } \mu_B\big([\gamma_-(z), \rho_-(\gamma_-(z))] \cup [z, \infty)\big) = \overline{b},\ z \geq \overline{b}, \end{cases} \tag{2.30}$$

so that $\gamma_+(\cdot)$ is increasing, $\gamma_-(\cdot)$ is decreasing, and

$$\gamma_+(w) \downarrow \overline{b} \quad \text{as } w \downarrow 0, \qquad \gamma_-(z) \uparrow \underline{b} \quad \text{as } z \uparrow \infty.$$

Note that γ_+ is defined on $[0, w_0]$, where $w_0 = \underline{b} \wedge \sup\{w < \underline{b} : \gamma_+(w) < \infty\}$, and similarly γ_- is defined on $[z_0, \infty]$. We are now ready to state our main theorem.

Theorem 2.2. *Let μ be the law of S_T inferred from the prices of vanillas via (1.5), and consider the double barrier derivative paying $\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}}$ for a fixed pair of barriers $\underline{b} < S_0 < \overline{b}$. Then exactly one of the following is true:*

I *"$\underline{b} \ll S_0 < \overline{b}$": There exists $z_0 > \overline{b}$ such that*[4]

$$\gamma_-(z) \downarrow 0 \ \textit{as}\ z \downarrow z_0 \quad \textit{and} \quad \rho_-(0) \leq \overline{b}. \tag{2.31}$$

Then there is a market model in which $\mathbb{E}\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} = \mathbb{E}\overline{H}^I(\rho_-(0)) = \frac{P(\rho_-(0))}{\rho_-(0) - \underline{b}}$.

II *"$\underline{b} < S_0 \ll \overline{b}$": There exists $w_0 < \underline{b}$ such that*

$$\gamma_+(w) \uparrow \infty \ \textit{as}\ w \uparrow w_0 \quad \textit{and} \quad \rho_+(\infty) \geq \underline{b}.$$

Then there is a market model in which $\mathbb{E}\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}} = \mathbb{E}\overline{H}^{II}(\rho_+(\infty)) = \frac{C(\rho_+(\infty))}{\overline{b} - \rho_+(\infty)}$.

[4]Note that here and subsequently, we use $\uparrow \infty$ and $\downarrow 0$ as meaning only the case where the increasing/decreasing sequence is itself finite/strictly positive, so that in (2.31) we strictly mean $\gamma_-(z) \to 0$ as $z \downarrow z_0$ and $\gamma_-(z) > 0$ for $z > z_0$.

III "$\underline{b} \ll S_0 \ll \overline{b}$": *There exists* $0 \le w_0 \le \underline{b}$ *such that* $\gamma_-(\gamma_+(w_0)) = w_0$ *and* $\rho_-(w_0) \le \rho_+(\gamma_+(w_0))$. *Then there is a market model in which*

$$\begin{aligned}\mathbb{E}\mathbf{1}_{\overline{S}_T \ge \overline{b},\, \underline{S}_T \le \underline{b}} &= \mathbb{E}\overline{H}^{III}\big(\gamma_+(w_0), \rho_+(\gamma_+(w_0)), \rho_-(w_0), w_0\big) \\ &= \alpha_1 C\big(\gamma_+(w_0)\big) + \alpha_2 C\big(\rho_+(\gamma_+(w_0))\big) + \alpha_3 P\big(\rho_-(w_0)\big) + \alpha_4 P\big(w_0\big),\end{aligned} \tag{2.32}$$

where α_i *are given in* (2.6).

IV "$\underline{b} < S_0 < \overline{b}$": *We have* $\overline{b} < \rho_-(0)$, $\underline{b} > \rho_+(\infty)$, *and* $\rho_+(\rho_-(0)) < \rho_-(\rho_+(\infty))$. *Then there is a market model in which*

$$\begin{aligned}\mathbb{E}\mathbf{1}_{\overline{S}_T \ge \overline{b},\, \underline{S}_T \le \underline{b}} &= \mathbb{E}\overline{H}^{IV}\big(\rho_-(0), \rho_+(\infty)\big) \\ &= \alpha_1 C\big(\rho_-(0)\big) + \alpha_2 P\big(\rho_+(\infty)\big) + \alpha_4,\end{aligned} \tag{2.33}$$

where α_i *are given in* (2.8).

We present now the analogues of Proposition 2.1 and Theorem 2.2 for the subhedging case. Whereas above we find an upper bound on the price of the derivative, in this case we will construct a lower bound.

Proposition 2.3. *Given the market input* (1.3), *no-arbitrage* (1.4) *in the class of portfolios* $\mathrm{Lin}(\mathcal{X} \cup \{\mathbf{1}_{\overline{S}_T \ge \overline{b},\, \underline{S}_T \le \underline{b}}, \mathbf{1}_{\{S_T > \underline{b}\}}, \mathbf{1}_{\{S_T \ge \overline{b}\}}\})$ *implies the following inequality between the prices:*

$$\mathcal{P}\mathbf{1}_{\overline{S}_T \ge \overline{b},\, \underline{S}_T \le \underline{b}} \ge \sup\{\mathcal{P}\underline{H}_I(K_1, K_2), \mathcal{P}\underline{H}_{II}(K_1, K_2, K_3), \mathcal{P}\underline{H}_{III}(K_1, K_2, K_3), 0\}, \tag{2.34}$$

where the supremum is taken over $0 < K_2 < \underline{b} < K_3 < \overline{b} < K_1$ *and* $\underline{H}_I, \underline{H}_{II}, \underline{H}_{III}$ *are given by* (2.10) *and the solutions to the relevant set of equations:* (2.18), (2.25), *and* (2.26).

Again, an important aspect of (2.34) is that we can in fact show that the bound is tight—that is, given a set of call prices, there exists a market model under which equality is attained. Recall that under no-arbitrage the prices of digital calls are essentially specified by our market input via $\mathcal{P}\mathbf{1}_{\{S_T \ge \overline{b}\}} = -C'(K)$.

In order to classify the different states, we make the following definitions. Let μ be the law of S_T implied by the call prices. Fix $\underline{b} < S_0 < \overline{b}$, and, given $v \in [\underline{b}, \overline{b}]$, define

$$\begin{aligned}\psi(v) = \inf\Bigg\{ z \in [0, \underline{b}] : \int_{(z,\underline{b}) \cup (v,\overline{b})} u\,\mu(\mathrm{d}u) + \overline{b}\left(\frac{\overline{b} - S_0}{\overline{b} - \underline{b}} - \mu((z,\underline{b}) \cup (v,\overline{b}))\right) \\ = \underline{b}\frac{\overline{b} - S_0}{\overline{b} - \underline{b}} \text{ and } \mu((z,\underline{b}) \cup (v,\overline{b})) \le \frac{\overline{b} - S_0}{\overline{b} - \underline{b}} \Bigg\},\end{aligned} \tag{2.35}$$

$$\begin{aligned}\theta(v) = \sup\Bigg\{ z \ge \overline{b} : \int_{(\underline{b},v) \cup (\overline{b},z)} u\,\mu(\mathrm{d}u) + \underline{b}\left(\frac{S_0 - \underline{b}}{\overline{b} - \underline{b}} - \mu((\underline{b},v) \cup (\overline{b},z))\right) \\ = \overline{b}\frac{S_0 - \underline{b}}{\overline{b} - \underline{b}} \text{ and } \mu((\underline{b},v) \cup (\overline{b},z)) \le \frac{S_0 - \underline{b}}{\overline{b} - \underline{b}} \Bigg\},\end{aligned} \tag{2.36}$$

where we use the convention $\sup\{\emptyset\} = -\infty$, $\inf\{\emptyset\} = \infty$. The functions ψ and θ have a natural interpretation in terms of embedding properties used in the proofs in section 4. For example, the definition of ψ ensures that, on the set where $\psi(v) \ne \infty$, we can diffuse all the mass initially from S_0 to $\{\underline{b}, \overline{b}\}$ and then embed from $\underline{b}$ to $(\psi(v), \underline{b}) \cup (v, \overline{b})$ and a compensating atom at $\overline{b}$ with the remaining mass.

The functions ψ and θ are both decreasing on the sets $\{v \in [\underline{b}, \overline{b}] : \psi(v) < \infty\}$ and $\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\}$, which are both closed intervals. Specifically, we will be interested in the region where both the functions allow for a suitable embedding; define

$$\begin{aligned} \overline{v} &= \min\left\{\sup\{v \in [\underline{b}, \overline{b}] : \psi(v) < \infty\}, \sup\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\}\right\}, \\ \underline{v} &= \max\left\{\inf\{v \in [\underline{b}, \overline{b}] : \psi(v) < \infty\}, \inf\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\}\right\}, \\ \kappa(v) &= \overline{b}\frac{\theta(v) - \underline{b}}{\theta(v) - \underline{b} + \overline{b} - \psi(v)} + \underline{b}\frac{\overline{b} - \psi(v)}{\theta(v) - \underline{b} + \overline{b} - \psi(v)}, \end{aligned} \tag{2.37}$$

where $\sup\{\emptyset\} = -\infty$ and $\inf\{\emptyset\} = \infty$.

Theorem 2.4. *Let μ be the law of S_T inferred from the prices of vanillas via (1.5), consider the double barrier derivative paying $\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}}$ for a fixed pair of barriers $\underline{b} < S_0 < \overline{b}$, and recall (2.35)–(2.37). Then exactly one of the following is true:*

I *"$\underline{b} < S_0 < \overline{b}$": We have $\overline{v} \geq \underline{v}$, and there exists $v_0 \in [\underline{v}, \overline{v}]$ such that $\kappa(v_0) = v_0$. Then there exists a market model in which*

$$\begin{aligned} \mathbb{E}\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}} &= \mathbb{E}\underline{H}_I(\theta(v_0), \psi(v_0)) \\ &= \alpha_0 + \alpha_2(C(\theta(v_0)) - C(\psi(v_0))) + \gamma_2 D(\overline{b}) - \gamma_1 D(\underline{b}) \\ &\quad + \alpha_3\left[C(\underline{b}) + C(\overline{b}) - C(v_0) - C(\theta(v_0))\right], \end{aligned} \tag{2.38}$$

where $D(x)$ is the price of a digital option with payoff $\mathbf{1}_{\{S_T \geq x\}}$ and the values of $\alpha_0, \alpha_2, \alpha_3, \gamma_1, \gamma_2$ are given by (2.18).

II *"$\underline{b} < S_0 \ll \overline{b}$": We have $\overline{v} \geq \underline{v}$ and $\overline{v} < \kappa(\overline{v})$. Then there exists a market model in which*

$$\begin{aligned} \mathbb{E}\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}} &= \mathbb{E}\underline{H}_{II}(\theta(\overline{v}), \psi(\overline{v}), \overline{v}) \\ &= \alpha_0 + \alpha_2(C(\theta(\overline{v})) - C(\psi(\overline{v}))) + \gamma_2 D(\overline{b}) - \gamma_1 D(\underline{b}) \\ &\quad + \alpha_3\left[C(\underline{b}) + C(\overline{b}) - C(\overline{v}) - C(\theta(\overline{v}))\right], \end{aligned} \tag{2.39}$$

where $D(x)$ is the price of a digital option with payoff $\mathbf{1}_{\{S_T \geq x\}}$ and the values of $\alpha_0, \alpha_2, \alpha_3, \gamma_1, \gamma_2$ are given by (2.25).

III *"$\underline{b} \ll S_0 < \overline{b}$": We have $\overline{v} \geq \underline{v}$ and $\underline{v} > \kappa(\underline{v})$. Then there exists a market model in which*

$$\begin{aligned} \mathbb{E}\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}} &= \mathbb{E}\underline{H}_{III}(\theta(\underline{v}), \psi(\underline{v}), \underline{v}) \\ &= \alpha_0 + \alpha_2(C(\theta(\underline{v})) - C(\psi(\underline{v}))) + \gamma_2 D(\overline{b}) - \gamma_1 D(\underline{b}) \\ &\quad + \alpha_3\left[C(\underline{b}) + C(\overline{b}) - C(\underline{v}) - C(\theta(\underline{v}))\right], \end{aligned}$$

where $D(x)$ is the price of a digital option with payoff $\mathbf{1}_{\{S_T \geq x\}}$ and the values of $\alpha_0, \alpha_2, \alpha_3, \gamma_1, \gamma_2$ are given by (2.26).

$\boxed{\text{IV}}$ *"$\underline{b} \ll S_0 \ll \overline{b}$": We have $\overline{v} < \underline{v}$. Then there exists a market model in which $\mathbb{E}\mathbf{1}_{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}} = 0$.*

Furthermore, in cases $\boxed{\text{I}}$–$\boxed{\text{III}}$ we have $\underline{v} = \inf\{v \in [\underline{b}, \overline{b}] : \psi(v) < \infty\} \leq \sup\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\} = \overline{v}$.

3. Applications and practical considerations.

3.1. Finitely many strikes. One important practical aspect where reality differs from the theoretical situation described above concerns the availability of calls with arbitrary strikes. Generally, calls will only trade at a finite set of strikes, $0 = x_0 \leq x_1 \leq \cdots \leq x_N$ (with $x_0 = 0$ corresponding to the asset itself). It is then natural to ask: how does this affect the hedging strategies introduced above? In full generality, this question results in a rather large number of "special" cases that need to be considered separately (for example, the case where no strikes are traded above $\overline{b}$, or the case where there are no strikes traded with $\underline{b} < K < \overline{b}$). In addition, there are differing cases, dependent on whether the digital options at $\underline{b}$ and $\overline{b}$ are traded. Consequently, we will not attempt to give a complete answer to this question, but will consider only the cases where there are "comparatively many" traded strikes and will assume that digital calls are not available to trade. Furthermore, we will apply the theorems of section 2 to measures with atoms. It should be clear how to do this, but a formal treatment would be rather lengthy and tedious, with some extra care needed when the atoms are at the barriers. In addition, as noted in Cox and Obłój [14], there are some rather technical issues relating to forms of arbitrage that need to be carefully considered for some boundary cases. For that reason we state the results of this section only informally.

Mathematically, the presence of atoms in the measure μ means that the call prices are no longer twice differentiable. The function is still convex, but we now have possibly differing left and right derivatives for the function. The implication for the call prices is the following:

$$\mu([x,\infty)) = -C'_-(K) \quad \text{and} \quad \mu((x,\infty)) = -C'_+(K).$$

In particular, atoms of μ will correspond to "kinks" in the call prices.

The first remark to make in the finite-strike case is that, if we replace the supremum/infimum over strikes that appear in expressions such as (2.27) and (2.34) by the maximum/minimum over traded strikes, then the arguments that conclude that these are lower/upper bounds on the price are still valid. The argument breaks down only when we wish to show that these are the best possible bounds. To try to replace the latter, we now need to consider which models might be possible under the given call prices. Our approach will be based on the following type of argument:

(i) suppose that, using only calls and puts with traded strikes, we may construct $\tilde{H}^i$ for $i \in \{I, \dots, IV\}$ such that $\tilde{H}^i \geq \overline{H}^i$ as a function of S_T;

(ii) suppose further that we can find an admissible call price function $C(K)$, $K > 0$, which agrees with the traded prices and such that in the market model $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}^*)$ associated by Theorem 2.2 with the upper bound (2.27) we have $\tilde{H}^i(S_T) = \overline{H}^i(S_T)$, $\mathbb{P}^*$-a.s.;

then the smallest upper bound on the price of a digital double touch barrier option is the cost of the cheapest portfolio $\tilde{H}^i$. This is fairly easy to see: clearly the price is an upper

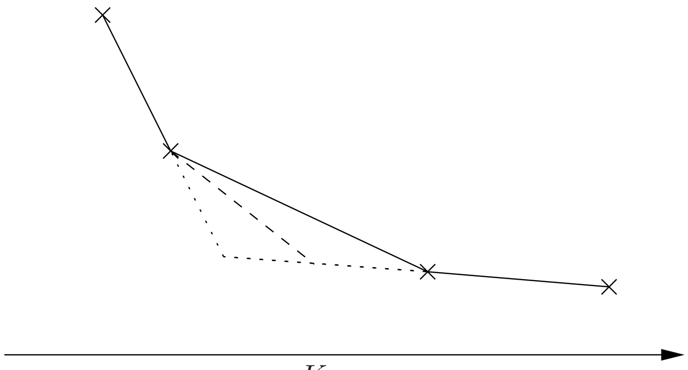


Figure 6. *Possible call price surfaces as a function of the strike. The crosses indicate the prices of traded calls, and the solid line corresponds to the upper bound, which corresponds to placing all possible mass at traded strikes. The lower bound on the interval (x_j, x_{j+1}) is indicated by the dotted line, and the dashed line indicates the surface we will choose when we wish to minimize the call price at K. In this case, we note that there will be mass at K and x_j but not at x_{j+1}. There is a second case where K is below the kink in the dotted line, when the resulting surface would place mass at K and x_{j+1} but not at x_j .*

bound on the price of the option, since $\tilde{H}^i$ superhedges, and under $\mathbb{P}^*$ this upper bound is attained. Indeed, by assumption on $\tilde{H}^i$, in the market model associated with $\mathbb{P}^*$ we have $\mathcal{P}\tilde{H}^i = \mathbb{E}^*\tilde{H}^i = \mathbb{E}^*\overline{H}^i = \mathcal{P}\overline{H}^i$. Consequently, by Theorem 2.2, the price of the traded portfolio $\tilde{H}^i$ and the price of the digital double touch barrier option are equal under the market model $\mathbb{P}^*$. Note that in (ii) above it is in fact enough to have $\tilde{H}^i(S_T) = \overline{H}^i(S_T)$ just for the $\overline{H}^i$ which attains equality in (2.27).

We now wish to understand the possible models that might correspond to a given set of call prices $\{C(x_i); 0 \le i \le N\}$. Simple arbitrage constraints (see, e.g., Carr and Madan [11] or Theorem 3.1 in Davis and Hobson [16][5]) require that the call prices at other strikes (if traded) are such that the function $C(K)$ is convex and decreasing. This allows us to deduce that, for K such that $x_j < K < x_{j+1}$ for some j, we must have

$$C(K) \le C(x_j)\frac{x_{j+1} - K}{x_{j+1} - x_j} + C(x_{j+1})\frac{K - x_j}{x_{j+1} - x_j}, \tag{3.1}$$

$$C(K) \ge C(x_j) + \frac{C(x_j) - C(x_{j-1})}{x_j - x_{j-1}}(K - x_j), \tag{3.2}$$

$$C(K) \ge C(x_{j+1}) - \frac{C(x_{j+2}) - C(x_{j+1})}{x_{j+2} - x_{j+1}}(x_{j+1} - K); \tag{3.3}$$

see Figure 6 for a graphical representation. These inequalities therefore provide upper and lower bounds on the call price at strike K, and it can be seen that the upper bound and lower bound are tight by choosing suitable models: in the upper bound, the corresponding

[5]We suppose also that our call prices do not exhibit what is termed here as weak arbitrage.

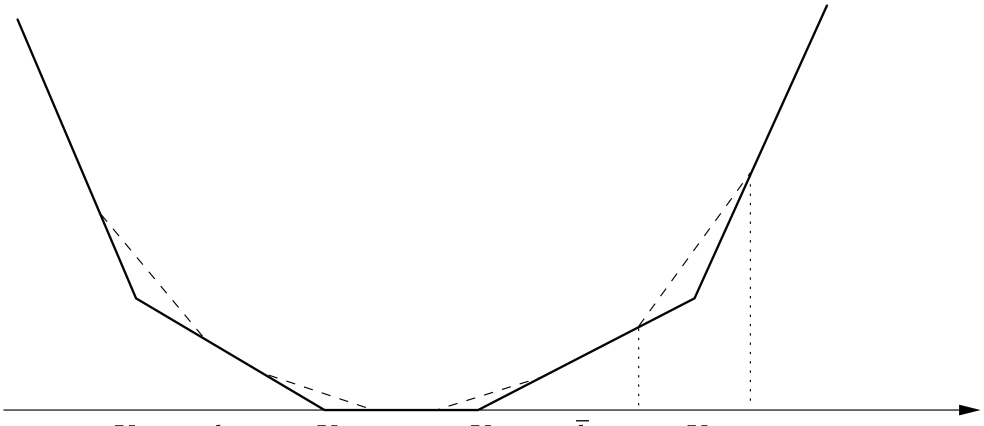


Figure 7. *An optimal superhedge* $\overline{H}^{III}$ *in the case where only finitely many strikes are traded. The lower (solid) payoff denotes the optimal construction under the chosen extension of call prices to all strikes, and the upper (dashed) payoff denotes the payoff actually constructed. Note that, for example,* x_j *is the largest traded strike below* K_1*, and* x_{j+1} *is the smallest traded strike greater than* K_1*.*

model places all mass of the law of S_T at the strikes x_i; in the lower bounds, the larger of the two possible terms can be attained with a law that places mass at K, and at other x_i's except, in (3.2), at x_j, and, in (3.3), at x_{j+1}. Moreover, provided that there is at least one traded call between two strikes K, K', we can (for example) choose a law that attains the maximum possible call price at K and the minimum possible price at K', while we need at least two traded strikes between K and K' should we wish these both to be able to attain their minimum possible price. In general, two intermediate strikes will be sufficient for all constructions, and we will assume that this property holds for all "relevant" points in what follows.

Consider first the case where we wish to superhedge the double touch option. We will consider the model $\mathbb{P}^*$ which corresponds (through Theorem 2.2) to the call prices obtained by linearly interpolating the prices at $x_0, \dots, x_n$—in particular, $C(K)$ is the maximum possible value call prices may take under the assumption of no arbitrage—and assume we are in case $\boxed{\text{III}}$ of Theorem 2.2 so that (2.32) holds. In particular, $\overline{H}^{III}$ would be a perfect hedge if all call options were traded. The key idea is to consider the portfolio depicted in Figure 7, where the smaller payoff is the static part of $\overline{H}^{III}$ and the upper payoff is one that may be constructed using only strikes that are actually traded. Then, although the upper payoff is strictly larger between x_j and x_{j+1}, say, where $x_j < K_1 < x_{j+1}$, the points at which this occurs are not attained by S_T since, under $\mathbb{P}^*$, the law of S_T is supported only by the points x_i. Consequently, both the upper and lower payoff have a.s. the same payoff under $\mathbb{P}^*$ and therefore the same expectation and price. Define $\tilde{H}^{III}$ to be $\overline{H}^{III}$ with its static part replaced with the upper payoff in Figure 7. Then $\tilde{H}^{III}$ is a superhedge and a perfect hedge under $\mathbb{P}^*$ and hence is the least expensive superhedge. Analogous arguments (with the same choice of $C(K)$) hold for the other hedges $\overline{H}^{I}$, $\overline{H}^{II}$, and $\overline{H}^{IV}$.

Consider now the lower bound. To keep things simple we begin by altering slightly the problem: rather than the payoff $\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}}$, we consider a subhedge of the option with payoff $\mathbf{1}_{\overline{S}_T > \overline{b},\, \underline{S}_T < \underline{b}}$.[6] Then (2.10) still holds[7] with the new term $\mathbf{1}_{\overline{S}_T > \overline{b},\, \underline{S}_T < \underline{b}}$ on the left-hand side (LHS), provided that we modify the digital options on the RHS to $\mathbf{1}_{\{S_T \geq \underline{b}\}}$ and $\mathbf{1}_{\{S_T > \overline{b}\}}$. So we may still consider the optimal portfolio (in all three cases) as being short a collection of calls at strikes K_1, K_2, and K_3, long calls at $\underline{b}$ and $\overline{b}$, and holding digital options at each of these points. Intuitively, we should look for a model which will maximize the cost of the calls at K_1, K_2, and K_3 and minimize the cost at $\overline{b}$ and $\underline{b}$, as well as maximizing the cost of a digital call at $\underline{b}$ and minimizing the cost of the digital call at $\overline{b}$. The former conditions correspond to choosing the call prices which give the upper bound (3.1), so we choose the call price which linearly interpolates $C(x_i)$ except when $x_i \leq \underline{b} \leq x_{i+1}$ and $x_i \leq \overline{b} \leq x_{i+1}$. In the latter cases, we wish to minimize the call price, so we choose the prices corresponding to the appropriate lower bound (3.2) or (3.3), which have a kink at $\underline{b}$ and at (exactly) one of its two adjacent traded strikes, and likewise for $\overline{b}$. We note that the prices of the digital calls (which are either minus the left gradient or minus the right gradient of the call prices at the barrier) will also now be optimized when they trade in exactly the forms specified above (that is, the digital call at $\underline{b}$ pays out only if the asset is greater than or equal to $\underline{b}$, while the digital call at $\overline{b}$ will pay out if the asset is strictly larger than $\overline{b}$ at maturity).

The above procedure specifies uniquely a complete set of call prices $C(K)$, which match the market input and which are our candidate for the largest lower bound among the possible models. Then we note that a construction similar to that given in Figure 8 will work—the main difference from the superhedge case is that, at the discontinuity, there are two possible cases that need to be considered, and the optimal subhedge will depend on behavior of $C(K)$ at the strikes adjacent to $\underline{b}$ and $\overline{b}$. More precisely, the portfolio given by the dotted line in Figure 8 corresponds to the case when $C(K)$ has no kinks at the traded strikes to the immediate right of both barriers (i.e., S_T has no atoms at these strikes). The other three possibilities are straightforward modifications. The argument then proceeds as above: in the model given by Theorem 2.4, subhedge $\underline{H}_i$ achieves equality in (2.34) (modified to account for the changes from inequalities to strict inequalities and vice versa), and the portfolio constructed in Figure 8 is a.s. equal[8] to $\underline{H}_i$, so that they must have the same price. The resulting subhedge, constructed using only calls and puts with traded strikes, is therefore a hedge under the chosen model and therefore the optimal lower bound.

[6]Under the optimal model (see the proofs in section 4), the distribution of S_T has atoms at the barriers which are embedded by mass which has already hit the other barrier. With the modified payoff, $\mathbf{1}_{\overline{S}_T > \overline{b},\, \underline{S}_T < \underline{b}}$, this will still give equality in the subhedge, but this would not have been the case with the original payoff. One could now consider the option $\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}}$ by approximating it with a sequence of payoffs of the form $\mathbf{1}_{\overline{S}_T > \overline{b} - \varepsilon,\, \underline{S}_T < \underline{b} + \varepsilon}$, which would (in the optimal embedding) place atoms "just inside the barriers"—it is this behavior of the "optimal" construction that required us to consider the modified payoff in this case. A more careful analysis of the behavior in the limit is nontrivial, but the consequences for a similar example can be seen in Cox and Obłój [14].

[7]Technically, for (2.10) to still hold, we actually need to modify the hitting times so that we consider the entrance times of, e.g., $(0, \underline{b})$ rather than the hitting time of $\underline{b}$, and be able to trade forward with strike $\underline{b}$ at this stopping time. In practice, this will not be crucial, since, for example, continuity of the asset price means that this entrance time may be suitably approximated, and it will commonly be the case that the two stopping times are in fact equal.

[8]Assuming that the optimal K_3 chosen is separated from $\underline{b}$ and $\overline{b}$ by a traded call price.

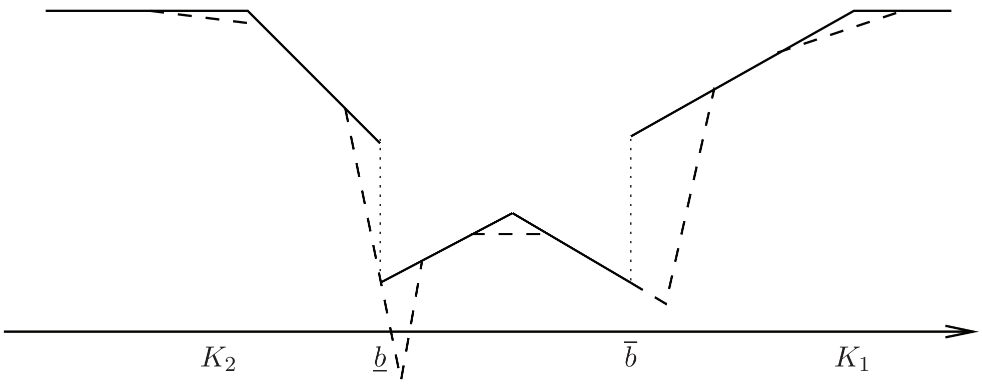


Figure 8. *An optimal subhedge in the case where only finitely many strikes are traded. The upper (solid) payoff denotes the optimal construction under the chosen extension of call prices to all strikes, and the lower (dashed) payoff denotes the payoff actually constructed. Observe that there are two possible constructions at the discontinuity—which type of construction is optimal will depend on whether the optimal lower bound on the call price at $\underline{b}$ or $\overline{b}$ corresponds to the bound* (3.2) *or* (3.3).

3.2. Toward relaxation of our market assumptions. So far, we have made a number of idealized assumptions about the setup that will not necessarily be true in an application. In the previous section, we described how having only finitely many traded strikes may affect our results; in this section we briefly describe how several other aspects that are not captured immediately by our assumptions might alter the results presented.

Continuity of paths. Throughout the paper, we have assumed that $(S_t)_{t\geq 0}$ has continuous paths. In fact we can relax this assumption considerably. First, it is relatively simple to see that if we assume only that barriers $\underline{b}, \overline{b}$ are crossed in a continuous manner, then all of our results remain true. Second, if we make no continuity assumptions,[9] then all our superhedges still work—jumping over the barrier only makes the appropriate forward transaction more profitable. In contrast, our subhedges do not work, and lower bounds on prices are trivially zero and are attained in the model where $S_t = S_0$ for all $t < T$, and then it jumps to the final position S_T.

Zero interest rates. The assumption that we are working with a forward price (or, more specifically, that interest rates are zero) is an important assumption for the methods we have used. Without assuming that the cost-of-carry is zero, the position of the barriers, after discounting, will alter over the lifetime of the option, and our methods are not easily adapted to this setting. There are perhaps two possible resolutions: the first is to look at the "worst-case" barrier values and compute the upper and lower bounds using these values, resulting in upper and lower estimates on the value of the option. Note, however, that there may not be models that correspond to these worst-case scenarios. Second, a more practical response may be to carry out a model-specific adjustment to the model-independent bounds, which may involve something along the following lines: choose an "average" interest adjusted value for each of the barriers and solve the robust problem for these barriers (and discounted stock price process).

[9]We always suppose that the processes have right-continuous paths with left limits.

The idea is to use this as a basic hedge, and, in general, one should expect that the performance of the hedge may be fairly close to a super- or subhedge. Now choose your favorite model, look at the paths where the super- or subhedge fails, and add in a model-specific adjustment. If the choice of model which one makes is moderately good, one would expect an adjustment that is of relatively small transactional size compared to the model-independent component, but which performed relatively well across most models, thus maintaining the relatively robust nature of the initial hedge.

Additional traded options. In practice, in most markets where digital double barrier options are traded, there will be other options traded—for example, one-touch barrier options or call options with earlier maturities. In both of these cases, one would expect these options to include further information about the price of the double-barrier option which, one may hope, would impact the resulting price bounds and hedges. In theory, we believe this to be the case; however, it would seem that a complete analysis along the lines carried out in this work that also included these extra options would be much more involved—this additional complexity can be seen, for example, in the work of Brown, Hobson, and Rogers [7], where including calls traded at a single intermediate time considerably complicates the picture established in Hobson [28]. In this case, the most natural question to ask would appear to be: how might the bounds and hedges alter if, in addition to the call prices with the same maturity, one could also trade in one-touch barrier options with the same maturity? From a theoretical point of view, experience with Skorokhod embeddings suggests this may be a very hard problem, but it is an interesting direction for further research.

3.3. Hedging comparisons. In practice, one would expect that the prices we derive as upper and lower bounds using the techniques of this paper will be rather far apart and well outside typical bid/ask spreads. Consequently, the use of these techniques as a method for pricing is unlikely to be successful, although it will give a good indication of the size of the model-risk associated with a given model-based price. However, our techniques also provide superhedging and subhedging strategies that may be helpful. Consider a trader who has sold a double barrier option (at a price determined by some model perhaps) and who wishes to hedge the resulting risk. In a Black–Scholes world, the trader could remove the risk from his position by delta-hedging the short position. However, there are a number of practical considerations that would interfere with such an approach:

Discrete hedging. A notable source of errors in the hedge will be the fact that the hedging portfolio cannot be continuously adjusted; rather the delta of the position might be adjusted on a periodic basis, resulting in an inexact hedge of the position. While including a gamma hedge could improve this, the hedge will never be perfect. In addition, there is an organizational cost (and risk) to setting up such a hedging operation that might be important.

Transaction costs. A second consideration is that each trade will incur a certain level of transaction costs. These might be tiny for delta-hedging, but this is no longer the case for vega-hedging. To minimize the total transaction costs, the trader would like to be able to trade as infrequently as possible. Of course, this means that there will be a necessary trade-off with the discretization errors incurred above.

Model risk. The final concern for the trader would be: am I hedging with the correct model? Using an incorrect model will of course result in systematic hedging errors due to, e.g., incorrectly estimated volatility, but could also lead to large losses should the model fail to incorporate structural effects such as jumps. A delta-hedge would typically be improved with a vega-hedge, which in turn raises the issue of transaction costs as mentioned above.

We claim that the constructions developed in section 2 address some of these issues: there is no need for regular recalculation of the Greeks of the position, although the breaching of the barrier still requires monitoring; since there are only a small number of transactions, it seems likely that the transaction costs may be reduced; our hedge has been derived using robust techniques, so that we will still be hedged even if the market does not behave according to our initial model, and behavior such as jumps (at least for the upper bound of the double touch) will not affect this.

A further consideration that will be of importance to a hedger is the distribution of the returns, even before the transaction costs are deducted. Under the hedging strategy suggested above, if the trader has sold the option using the "correct" price (plus a small profit) and set up the robust superhedging strategy suggested at a higher price, on average the trader will come out even, as he will if he delta hedges. His comparison between the approaches would then come down to the respective risk involved in the different hedges. For a delta/vega-hedge this is typically symmetric about zero; however, the robust hedge will be very asymmetric since it is bounded below. If the trader is particularly worried about the possible tail of his trading losses as a measure of risk, this strict cut-off could be very advantageous. The delta/vega-hedge, on the other hand, has the appeal of having a lower variance of hedging errors.

Of course a variety of such strategies (typically known as static, semistatic, or robust) have been suggested in the literature, under a variety of more or less restrictive assumptions on the price process, and mostly for single barrier options and variants such as knock-out calls. We have already mentioned the paper by Brown, Hobson, and Rogers [8], which makes very limited restrictions on the underlying price process. More restrictive is the work of Bowie and Carr [5], and subsequent papers of Carr and Chou [9] and Carr, Ellis, and Gupta [10]. Here the authors assume that the volatility satisfies a symmetry assumption, and as a consequence one can, for example, hedge a knock-out call with the barrier above the strike by holding the vanilla call and being short a call at a certain strike above the barrier. By the assumption on the volatility, whenever the underlying asset hits the barrier, both calls have the same value, and the position may be closed out for zero value. A related technique is due to Derman, Ergener, and Kani [18], followed up by Andersen, Andreasen, and Eliezer [1] and Fink [23]. The idea here is to use other traded options to make the value of the hedging portfolio equal to zero along the barrier when liquidated. In the simplest form, a portfolio of calls above the barrier at different strikes and/or maturities is purchased so that the portfolio value at selected times before maturity is zero. Extensions allow this idea to be used for stochastic volatility, and even to cover jumps, at the expense of needing possibly a very large portfolio of options. More recently, work of Nalholm and Poulsen [33] unifies both these approaches and allows a fairly general set of asset dynamics, as does work by Giese and Maruhn [25], where the authors find an optimal portfolio by setting up an optimization problem. Note, however, that all these strategies assume a known model for the underlying asset and also that the hedging assets will be liquid enough for the portfolio to be liquidated at the price specified under the model. In addition, since some of the hedging portfolios can involve a large number of options, it is not clear that the static hedges here will be efficient at resolving the issues of transaction costs (and operational simplicity) or model risk. A numerical investigation of the performance of a range of these hedges in practice has been conducted in Engelmann et al. [21], and similar investigations for a different class of static hedges appear in Davis, Schachermayer, and Tompkins [17] and Tompkins [38].

There are also more classical, theoretical approaches to the problem of hedging where the problem is considered in an incomplete market (without which, of course, a perfect hedge would be possible). In this situation (see, for example, Föllmer and Schweizer [24]), one wants to solve an optimal control problem where the aim is to minimize the "risk" of the hedging error, where risk is interpreted suitably (perhaps with regards to a utility function or a risk measure). More recently, in a combination of the static and dynamic approaches, Ilhan, Jonsson, and Sircar [30] have considered the problem of risk minimization over an initial static portfolio and a dynamic trading strategy in the underlying asset.

The most notable difference between the studies described above and the ideas of the previous sections is that we make very few modeling assumptions on the underlying asset, whereas most of the approaches listed require a single model to be specified, with respect to which the results will then be optimal. In particular, these techniques are unable to say anything about hedging losses should the assumed model actually be incorrect.

Of course, the criteria under which we have constructed our hedges—that they are the smallest robust superhedging strategy, or the greatest robust subhedging strategy—do not necessarily mean that the behavior of the hedges in "normal" circumstances will be particularly suitable: we would expect that the hedge would perform best in extreme market conditions; however, in order for it to be suitable as a hedge against model risk, one would also want the performance of the hedge to generally be reasonable. To see how this strategy compares, we will now consider some Monte Carlo–based comparisons with the standard delta/vega-hedging techniques. The comparisons will take the following form:

(i) We choose the Heston model for the "true" underlying asset and compute the time-0 call prices under this model at a range of strike prices, and the time-0 price of a double barrier option.

(ii) We compute the optimal super- and subhedges for the digital double touch barrier option based on the observed call prices, and suppose that the hedger purchases these portfolios using the cash received from the buyer (and borrowing/investing the difference between the portfolios).

(iii) For comparison purposes, we also hedge the option using a suitable delta/vega-hedge with daily updating. For the robust hedges, we will consider both the case where hitting of the barrier is monitored daily and the case where it is monitored exactly.

In the numerical examples, we assume that the underlying process is the Heston stochastic volatility model (Heston [26])

$$\begin{cases} dS_t = \sqrt{v_t} S_t dW_t^1, & S_0 = S_0,\ v_0 = \sigma_0 \\ dv_t = \kappa(\theta - v_t)dt + \xi\sqrt{v_t}dW_t^2, & d\langle W^1, W^2\rangle_t = \rho dt, \end{cases} \tag{3.4}$$

with parameters

(3.5) $S_0 = 1.4990,\ \sigma_0^2 = 0.0110,\ \kappa = 3.8626,\ \theta = 0.0169,\ \xi = 0.5004,\ \text{and}\ \rho = -0.1850.$

These parameters, taken from Ulmer [39], resulted from the calibration of a Heston model to market prices of European options with different strikes and maturities (a total of 176 options) on the EUR/USD foreign exchange spot rate on January 14, 2010. This gives us realistic dynamics for our hypothetical forward market, which in fact is not far from the real spot market, given that 1-year interest rates were then similar for EUR and USD (deposit rates reported by Bloomberg were respectively 1.1% and 0.905%).

Transactions in S_t carry a 0.15% transaction cost, and buying or selling call/put options carries a 1% transaction cost.[10] The delta/vega-hedge is constructed using the Black–Scholes delta of the option, but using the at-the-money implied volatility assuming that the call prices are correct (i.e., they follow the Heston model). While not perfect as a hedge, empirical evidence as in Dumas, Fleming, and Whaley [19] or Engelmann, Fengler, and Schwendner [22] suggests that the hedge is reasonable even without the vega component, although it is also the case that more sophisticated methods should result in an improvement of this benchmark.

We consider a short and a long position in a digital double touch barrier option with payoff $\mathbf{1}_{\overline{S}_T \geq \overline{b},\, \underline{S}_T \leq \underline{b}}$ for each combination of upper and lower barriers $\overline{b} = 1.47, 1.52, 1.57$ and $\underline{b} = 1.35, 1.39, 1.43$. We then compare hedging performance of our robust super- and subhedges and the standard delta/vega-hedges by running 20,000 Monte Carlo simulations for each of the nine combinations of upper and lower barrier, and analyzing the resulting loss/gains from trading.[11] We note that whether the barrier is monitored exactly or daily (so that the trades in the robust hedges may not occur exactly at the barrier) will have a noticeable difference on the cumulative distribution function. To highlight this difference, the daily hedge will be type (A) and the exact hedge type (B).

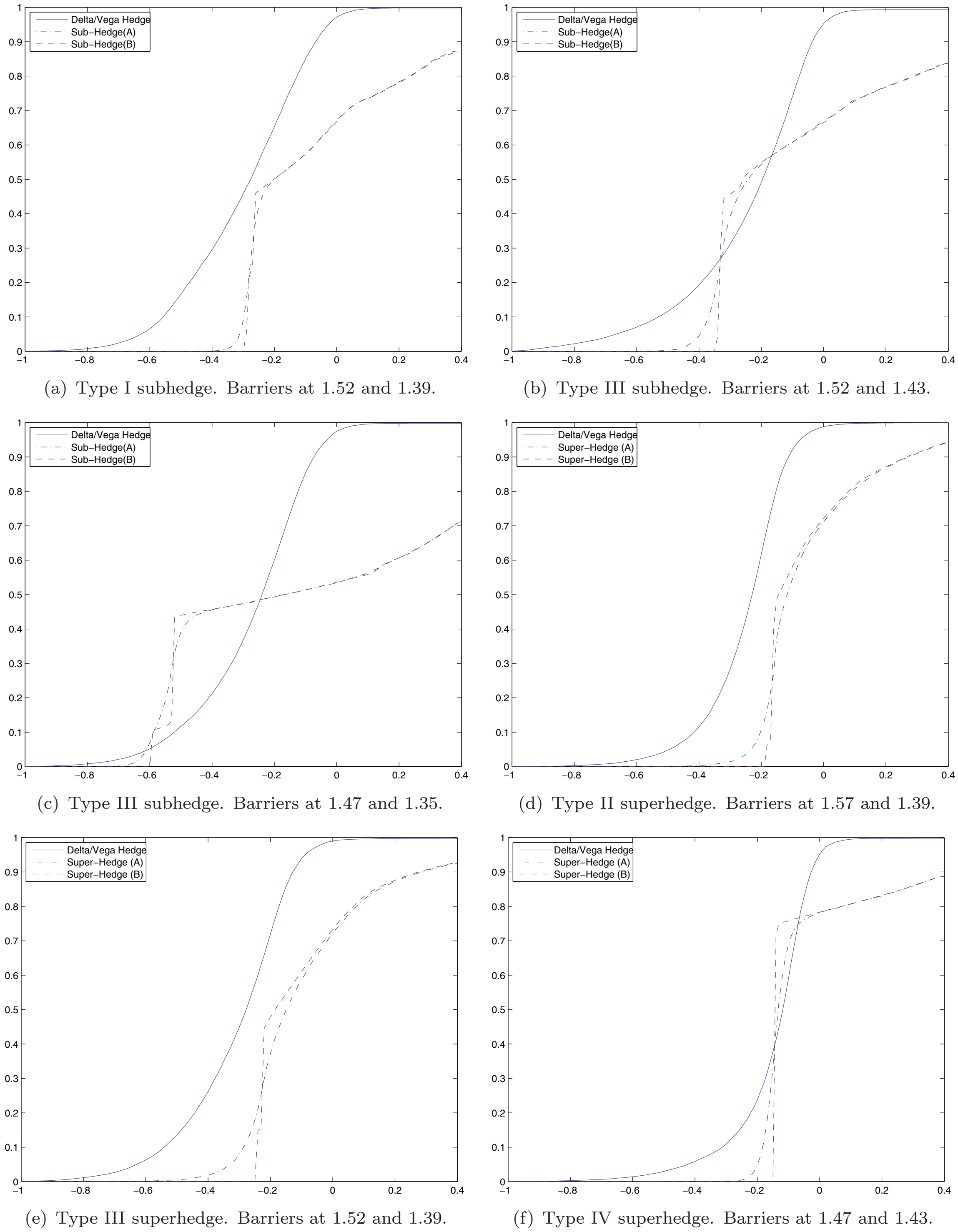
The cumulative distributions of hedging errors for a selection of barrier pairs are given in the graphs in Figure 9. Figure 9 shows that the robust super- and subhedges introduced in this paper can incur losses in a manner similar to the delta/vega strategies. However, the comparatively large losses are less frequent for the robust strategies in comparison to the delta/vega-hedging strategy. Indeed, in Table 1 we show that an agent with an exponential utility[12] $U(x) = 1-\exp(-x)$ would systematically, strongly prefer the error distribution of our hedges to that of the delta/vega-hedge. Note also that if we allow our hedges to monitor the barrier crossings exactly, the corresponding cumulative distributions of hedging errors have losses which are bounded below. It is interesting, however, to note that in terms of utility of hedging errors (cf. Table 1) this doesn't really change the performance of our hedges.

Finally, we consider how the performance of our hedges compares if we vary the barrier levels. The hedge for the short position fairly consistently outperforms delta/vega-hedging. In the long position, on the other hand, the hedge appears to perform worse as the barriers

[10] The latter has a predominant effect in our simulations, and the results below remain essentially the same even when transactions in S_t carry no transaction costs.

[11] The resulting hedging errors were mean-adjusted as in Tompkins [38] for consistency. The adjustments are of order 0.01 and have no qualitative influence on our results.

[12] We take the parameter α in the utility function $U(x) = 1-\exp(-\alpha x)$ to be 1, but a sensible parametrization may be to take $\alpha = 10^{-6}$ and to consider a contract with notional value £10^6.



(a) Type I subhedge. Barriers at 1.52 and 1.39.

(b) Type III subhedge. Barriers at 1.52 and 1.43.

(c) Type III subhedge. Barriers at 1.47 and 1.35.

(d) Type II superhedge. Barriers at 1.57 and 1.39.

(e) Type III superhedge. Barriers at 1.52 and 1.39.

(f) Type IV superhedge. Barriers at 1.47 and 1.43.

Figure 9. *Cumulative distributions of hedging errors under different scenarios of a long position* (a)–(c) *and a short position* (d)–(f) *in a double touch option with barriers at different levels under the Heston model* (3.4)–(3.5). *The robust hedge* (A) *monitors barrier crossing daily, while the robust hedge* (B) *is allowed to monitor exactly the moments of barrier crossings.*

Table 1

Comparison of exponential utilities of hedging errors of positions (Pos) in double touch options under Heston model (3.4)–(3.5) *resulting from delta/vega* ($\Delta/\mathcal{V}$) *hedging and our robust (Rob) super- or subhedging strategies. The robust hedge* (A) *monitors barrier crossing daily, while the robust hedge* (B) *is allowed to monitor exactly the moments of barrier crossings. Type of the robust hedge and the strikes used in its construction are reported.*

Barriers	Pos	$\Delta/\mathcal{V}$	Rob (A)	Rob (B)	Type	K_1	K_2	K_3	K_4
1.47–1.35	Short	−0.3258	−0.0690	−0.0674	IV	1.5017	1.1611		
1.47–1.35	Long	−0.3278	−0.1774	−0.1767	III	1.7421	1.2546	1.416	
1.47–1.39	Short	−0.3183	−0.0605	−0.0589	IV	1.5818	1.1611		
1.47–1.39	Long	−0.3180	−0.1139	−0.1117	III	1.6753	1.2947	1.4416	
1.47–1.43	Short	−0.1666	−0.0414	−0.0406	IV	1.7487	1.1611		
1.47–1.43	Long	−0.1698	−0.1550	−0.1495	I	1.5751	1.3214	1.4549	
1.52–1.35	Short	−0.3272	−0.0501	−0.0483	III	1.5551	1.4883	1.4015	1.2880
1.52–1.35	Long	−0.3263	−0.0609	−0.0623	III	1.9558	1.0275	1.4549	
1.52–1.39	Short	−0.3750	−0.0824	−0.0786	III	1.5885	1.4616	1.4416	1.3214
1.52–1.39	Long	−0.3799	−0.0779	−0.0795	I	1.7287	1.1477	1.4549	
1.52–1.43	Short	−0.3121	−0.0668	−0.0654	IV	1.7487	1.3414		
1.52–1.43	Long	−0.3169	−0.1107	−0.1082	II	1.6018	1.2078	1.4549	
1.57–1.35	Short	−0.2363	−0.0313	−0.0303	III	1.6152	1.5351	1.3748	1.3214
1.57–1.35	Long	−0.2348	−0.0421	−0.0445	IV				
1.57–1.39	Short	−0.2850	−0.0441	−0.0423	III	1.6486	1.5150	1.4149	1.3614
1.57–1.39	Long	−0.2875	−0.0603	−0.0617	II	1.9758	0.9341	1.4349	
1.57–1.43	Short	−0.2702	−0.0660	−0.0636	III	1.7755	1.4683	1.4416	1.4149
1.57–1.43	Long	−0.2795	−0.0841	−0.0838	II	1.6619	1.1277	1.4349	

get closer together. However, for the parameters given, the robust strategy appears to consistently outperform the delta/vega strategy. Naturally when the barriers vary, the types of super-
and subhedges which we use change. Table 1 also shows which types are optimal depending on the values of the barriers, and indicates the values of the corresponding strikes. This provides an illustration of the intuitive labels we gave to each case in section 2.

The results presented in this section clearly show that the hedges we advocate are in many circumstances an improvement on the classical hedges. We stress that this was meant to be an indicative enquiry and not a comprehensive numerical study of performance of our robust hedging methods. Naturally, there is a large literature (e.g., Hodges and Neuberger [29], Cvitanić and Karatzas [15], Whalley and Wilmott [41]) on more sophisticated techniques that might offer a considerable improvement over the classical hedge we have implemented. In our case the delta/vega-hedge was severely impacted by high transaction costs associated with trading options, and it is possible that such improvements would reverse the relative performance of the hedges. Further, it would be interesting to compare the performance of our hedges against various static and semistatic hedges cited above. Such study could feature different market scenarios and levels of model misspecification as well as more involved performance measures than just expected utility for one value of risk aversion. We believe this is an interesting avenue for further research.

However, we hope that the numerical evidence presented above does at least convince the reader that the robust hedges are competitive with dynamic hedging, and that their differing nature means that they might well prove a more suitable approach in situations where market conditions are dramatically different from the idealized Black–Scholes world—e.g., large transaction costs, illiquidity, or parameter uncertainty. There also seems scope for a more sophisticated approach based on the robust hedges but allowing for some model-based trading: for example, a hybrid of a robust portfolio and some dynamic trading could be used to reduce some of the overhedge in the simple robust hedge. Finally, our simulations include only a very simple form of model misspecification. We would expect that if a trader believes in a model which is significantly different from the real world model, in particular if real world dynamics change dramatically within the time horizon $[0,T]$, then our robust hedging strategies would outperform hedging "using the wrong model."

4. Proofs.

Let $(B_t)_{t\geq 0}$ be a standard real valued Brownian motion starting from B_0, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual hypothesis. We recall that for any probability measure ν on $\mathbb{R}_+$ with $\nu_B(\mathbb{R}_+) = B_0$ we can find a stopping time τ such that $B_\tau \sim \nu$ and $(B_{t\wedge\tau})$ is a uniformly integrable martingale. Such stopping is simply a solution to the Skorokhod embedding problem (SEP), and a number of different explicit solutions are known; see Obłój [34] for an overview of the domain. Note also that when $\nu([a,b]) = 1$ then $(B_{t\wedge\tau})$ is a uniformly integrable martingale if and only if $B_t \in [a,b]$, $t \leq \tau$, a.s. In the remainder, when speaking about embedding a measure we implicitly mean embedding it in a uniformly integrable (UI) manner in (B_t).

Recall that if $B_0 = S_0$ and τ is an embedding of μ, then $S_t := B_{\tau\wedge\frac{t}{T-t}}$, $t \leq T$, is a market model which matches the market input (1.3). In what follows we will be constructing embeddings τ of μ such that the associated market model attains equality in our super- or subhedging inequalities. Stopping times τ will often be compositions of other stopping times embedding (rescaled) restrictions of μ or some other intermediary measures. Unless specified otherwise, the choice of particular intermediary stopping times has no importance, and we do not specify it—one's favorite solution to the SEP can be used.

Proof of Theorem 2.2. We start with some preliminary lemmas and then prove Theorem 2.2. In the body of the proof, cases $\boxed{\text{I}}$–$\boxed{\text{IV}}$ refer to the cases stated in Theorem 2.2. We note that, by considering $z_0 = \gamma_+(w_0)$, $\boxed{\text{III}}$ is equivalent to

$$\text{There exists } z_0 \geq \overline{b} \text{ such that } \gamma_+(\gamma_-(z_0)) = z_0 \text{ and } \rho_+(z_0) \geq \rho_-(\gamma_-(z_0)).$$

We recall, without proof, straightforward properties of the barycenter function (2.28) which will be useful in what follows.

Lemma 4.1. *The barycenter function defined in* (2.28) *satisfies*

(i) $\mu_B(\Gamma) \geq a$, $\Gamma_1 \subset (0,a) \Longrightarrow \mu_B(\Gamma \setminus \Gamma_1) \geq a$,

(ii) $a \leq \mu_B(\Gamma) \leq \mu_B(\Gamma_1) \leq b$ *and* $\mu(\Gamma \cap \Gamma_1) = 0 \Longrightarrow a \leq \mu_B(\Gamma \cup \Gamma_1) \leq b$.

We separate the proof into two steps. In the first step we prove that exactly one of $\boxed{\text{I}}$–$\boxed{\text{IV}}$ holds. In the second step we construct the appropriate embeddings and market models which achieve the upper bounds on the prices.

Step 1. This step is divided into four possible cases, (A)–(D), corresponding to four regions in which $\underline{b}, \overline{b}$ may lie. For each region, we shall show that exactly one of the instances $\boxed{\text{I}}$–$\boxed{\text{IV}}$ will hold, and therefore in general exactly one them can hold. We start with technical lemmas which are proved after the cases are considered.

Lemma 4.2. *If* $\rho_+(\gamma_+(w_0)) < \rho_-(w_0)$ *for some* w_0*, then* $\rho_+(\gamma_+(w)) < \rho_-(w)$ *for all* $w \le w_0$*. Similarly, if* $\rho_-(\gamma_-(z_0)) > \rho_+(z_0)$ *for some* z_0*, then* $\rho_-(\gamma_-(z)) > \rho_+(z)$ *for all* $z \ge z_0$*.*

Lemma 4.3. *If* $\overline{b} \ge \rho_-(0)$ *and* $\underline{b} \le \rho_+(\infty)$*, then at least of one of the functions* $\gamma_\pm$ *is bounded on its domain. In particular at most one of* $\boxed{\text{I}}$ *and* $\boxed{\text{II}}$ *may be true.*

Lemma 4.4. $\boxed{\text{IV}}$ *implies not* $\boxed{\text{III}}$*.* $\boxed{\text{III}}$ *implies not (*$\boxed{\text{I}}$ *or* $\boxed{\text{II}}$*).*

Case (a): $\overline{b} \ge \rho_-(0)$, $\underline{b} \le \rho_+(\infty)$. We note first that the case $\boxed{\text{IV}}$ is not possible, and that the second half of the conditions for $\boxed{\text{I}}$ and $\boxed{\text{II}}$ are trivially true. Suppose that neither $\boxed{\text{I}}$ nor $\boxed{\text{II}}$ holds. If we have $\gamma_-(\overline{b}) = 0$, then $\boxed{\text{III}}$ holds with $w_0 = 0$, and if $\gamma_+(\underline{b}) = \infty$, then $\boxed{\text{III}}$ holds with $w_0 = \underline{b}$. We may thus assume that $\gamma_+(\cdot)$ is bounded above and $\gamma_-(\cdot)$ is bounded away from zero, which in turn implies

$$\gamma_-(\gamma_+(\underline{b})) < \underline{b} \quad \text{and} \quad \gamma_-(\gamma_+(0)) > 0.$$

The function $\gamma_-(\gamma_+(\cdot))$ is continuous and increasing on $(0, \underline{b}]$, and thus we must have w_0 such that $\gamma_-(\gamma_+(w_0)) = w_0$. Finally, suppose that for such a w_0 we in fact have $\rho_-(w_0) > \rho_+(\gamma_+(w_0))$; then Lemma 4.2 implies $\rho_-(0) > \rho_+(\gamma_+(0)) = \overline{b}$, contradicting our assumptions. That only one of the cases $\boxed{\text{I}}$–$\boxed{\text{III}}$ holds now follows from Lemmas 4.3 and 4.4.

Case (b): $\overline{b} \ge \rho_-(0)$, $\underline{b} > \rho_+(\infty)$. It follows that neither $\boxed{\text{II}}$ nor $\boxed{\text{IV}}$ is possible. Observe that Lemma 4.2 implies that $\rho_-(w) \le \rho_+(\gamma_+(w))$ for all $w \le \underline{b}$—if this were not true, then $\overline{b} = \rho_+(\gamma_+(0)) < \rho_-(0)$. Suppose further that $\boxed{\text{I}}$ does not hold. If $\gamma_-(\overline{b}) = 0$, then $\boxed{\text{III}}$ holds with $w_0 = 0$.

So assume instead that $\gamma_-(\cdot)$ is bounded away from zero, and therefore that $w < \gamma_-(\gamma_+(w))$ for w close to zero. If we show also that $\gamma_-(\gamma_+(\rho_+(\infty))) \le \rho_+(\infty)$, then by continuity of $\gamma_-(\gamma_+(\cdot))$ there exists a suitable w_0 for which $\boxed{\text{III}}$ holds. Let $\Gamma_1 = (\rho_+(\infty), \infty)$ and $\Gamma_2 = (\rho_+(\gamma_+(\rho_+(\infty))), \gamma_+(\rho_+(\infty)))$. We have by definition $\mu_B(\Gamma_1) = \overline{b} = \mu_B(\Gamma_2)$ so that $\mu_B(\Gamma_1 \setminus \Gamma_2) = \overline{b}$, since $\Gamma_2 \subset \Gamma_1$. Let $\Gamma = (\rho_+(\infty), \rho_-(\rho_+(\infty))) \cup (\gamma_+(\rho_+(\infty)), \infty)$ and note that $\mu_B(\Gamma) = \overline{b}$ is equivalent to $\gamma_-(\gamma_+(\rho_+(\infty))) = \rho_+(\infty)$. Noting that $\rho_-(w) \le \rho_+(\gamma_+(w))$ for all $w \le \underline{b}$ implies $\rho_-(\rho_+(\infty)) \le \rho_+(\gamma_+(\rho_+(\infty)))$, and using Lemma 4.1, we have

$$\mu_B(\Gamma) = \mu_B\Big((\Gamma_1 \setminus \Gamma_2) \setminus \big(\rho_-(\rho_+(\infty)), \rho_+(\gamma_+(\rho_+(\infty)))\big)\Big) \ge \overline{b},$$

which implies that $\gamma_-(\gamma_+(\rho_+(\infty))) \le \rho_+(\infty)$. As previously, it remains to note that Lemma 4.4 implies exclusivity of $\boxed{\text{III}}$ and $\boxed{\text{I}}$.

Case (c): $\overline{b} < \rho_-(0)$, $\underline{b} \le \rho_+(\infty)$. This case is essentially identical to Case (b) above.

Case (d): $\overline{b} < \rho_-(0)$, $\underline{b} > \rho_+(\infty)$. Note that we now cannot have either of $\boxed{\text{I}}$ or $\boxed{\text{II}}$. Suppose further that $\boxed{\text{IV}}$ does not hold—or rather, the weaker:

$$\rho_+(\rho_-(0)) > \underline{b},$$
$$\rho_-(\rho_+(\infty)) < \overline{b}.$$

Let $\Gamma_w = (w, \rho_-(w)) \cup (\overline{b}, \infty)$, and observe that $\mu_B(\Gamma_w)$ decreases as w decreases, provided that $\rho_-(w) < \overline{b}$. We have

$$\mu_B(\Gamma_{\rho_+(\infty)}) \ge \mu_B((\rho_+(\infty), \infty)) = \overline{b}.$$

Our assumption $\rho_-(\rho_+(\infty)) < \overline{b} < \rho_-(0)$ implies that $\rho_-^{-1}(\overline{b}) \in (0, \rho_+(\infty))$ so that $\Gamma_{\rho_-^{-1}(\overline{b})} \supsetneq (\rho_+(\infty), \infty)$ and in consequence $\mu_B(\Gamma_{\rho_-^{-1}(\overline{b})}) < \overline{b}$. Using continuity of $w \to \mu_B(\Gamma_w)$, we conclude that there exists a $w_1 \in (\rho_-^{-1}(\overline{b}), \rho_+(\infty)]$ with $\mu_B(\Gamma_{w_1}) = \overline{b}$, or equivalently $\gamma_-(\overline{b}) = w_1$. A symmetric argument implies that $\gamma_+(\underline{b}) > 0$. We conclude, as in Case (A), that there exists a w_0 such that $\gamma_-(\gamma_+(w_0)) = w_0$.

It remains to show that if $\boxed{\text{IV}}$ does not hold, and the first half of the condition for $\boxed{\text{III}}$ holds, then so too does the second condition. Suppose that w_0 is a point satisfying $\gamma_-(\gamma_+(w_0)) = w_0$, and suppose for a contradiction that $\rho_-(w_0) > \rho_+(\gamma_+(w_0))$. Since the sets $(\rho_+(\gamma_+(w_0)), \gamma_+(w_0))$ and $(w_0, \rho_-(w_0)) \cup (\gamma_+(w_0), \infty)$ are both centered at $\overline{b}$ and overlap, it follows that $\mu_B([w_0, \infty)) > \overline{b}$ and in consequence $\rho_+(\infty) < w_0$. Symmetric arguments imply that $\rho_-(0) > \gamma_+(w_0)$. Applying $\rho_-(\cdot), \rho_+(\cdot)$ to these inequalities, we further deduce that

$$\rho_-(w_0) < \rho_-(\rho_+(\infty)),$$
$$\rho_+(\rho_-(0)) < \rho_+(\gamma_+(w_0)),$$

which, together with the assumption that $\rho_-(w_0) > \rho_+(\gamma_+(w_0))$, implies

$$\rho_+(\rho_-(0)) < \rho_+(\gamma_+(w_0)) < \rho_-(w_0) < \rho_-(\rho_+(\infty)),$$

contradicting $\boxed{\text{IV}}$ not holding. Hence Step 1 is complete once we prove the lemmas stated at the start, as follows.

Proof of Lemma 4.2. Consider $w < w_0$ with $\rho_+(\gamma_+(w_0)) < \rho_-(w_0)$. The latter implies $\mu_B((w_0, \rho_+(\gamma_+(w_0)))) < \underline{b}$, so that $\mu_B((0, \gamma_+(w_0))) < \underline{b}$. Suppose now that $\rho_+(\gamma_+(w)) \geq \rho_-(w)$. As ρ_- is decreasing and γ_+ is increasing, we have $\rho_-(w_0) < \rho_-(w) \leq \rho_+(\gamma_+(w))$ and $\overline{b} < \gamma_+(w) < \gamma_+(w_0)$. We then have

$$\begin{aligned} \underline{b} &= \mu_B\big((0, \rho_-(w)) \cup (\rho_+(\gamma_+(w)), \gamma_+(w))\big) \\ &= \mu_B\Big((0, \gamma_+(w_0)) \setminus \big[(\rho_-(w_0), \rho_+(\gamma_+(\omega))) \cup \big((\gamma_+(w), \gamma_+(w_0))\big)\big]\Big) \\ &\leq \mu_B\big((0, \gamma_+(w_0))\big) < \underline{b}, \end{aligned}$$

which gives the desired contradiction. ■

Proof of Lemma 4.3. Define $\Gamma = (0, \underline{b}) \cup (\overline{b}, \infty)$ and consider $\mu_B(\Gamma)$. If both $\boxed{\text{I}}$ and $\boxed{\text{II}}$ hold, or more generally if $\gamma_-(z_0) = 0$ and $\gamma_+(w_0) = \infty$ for some $z_0 \geq \overline{b}$, $w_0 \leq \underline{b}$, we have

$$\mu_B((0, \rho_-(0)) \cup (\overline{b}, \infty)) \geq \mu_B((0, \rho_-(0)) \cup (z_0, \infty)) = \overline{b} \tag{4.1}$$

and

$$\mu_B((0, \underline{b}) \cup (\rho_+(\infty), \infty)) \leq \mu_B((0, w_0) \cup (\rho_+(\infty), \infty)) = \underline{b}. \tag{4.2}$$

Now suppose $\mu_B(\Gamma) < \overline{b}$. Then

$$\mu_B(\Gamma \cup (\underline{b}, \rho_-(0))) < \overline{b},$$

contradicting (4.1), and similarly if $\mu_B(\Gamma) > \underline{b}$,

$$\mu_B(\Gamma \cup (\rho_+(\infty), \overline{b})) > \underline{b}$$

contradicts (4.2). ■

Proof of Lemma 4.4. $\boxed{\text{IV}} \Longrightarrow$ not $\boxed{\text{III}}$: Assume that both $\boxed{\text{IV}}$ and $\boxed{\text{III}}$ hold. From the definition of $\gamma_+(w_0)$ and $\rho_-(w_0)$ we have that $\mu_B(\Gamma_+) = \underline{b}$ for $\Gamma_+ = (0, \rho_-(w_0)) \cup (\rho_+(\gamma_+(w_0)), \gamma_+(w_0))$. This implies that $\gamma_+(w_0) \geq \rho_-(0)$ since otherwise

$$\mu_B(\Gamma_+) = \mu_B\Big((0, \rho_-(0)) \setminus \big\{ \big(\rho_-(w_0), \rho_+(\gamma_+(w_0))\big) \cup \big(\gamma_+(w_0), \rho_-(0)\big) \big\} \Big) > \underline{b},$$

where we also used the assumption $\rho_-(w_0) \leq \rho_+(\gamma_+(w_0))$. Likewise, using $\gamma_-(\gamma_+(w_0)) = w_0$, we see that $w_0 \leq \rho_+(\infty)$. Applying ρ_- to the last inequality and using our assumptions, we obtain

$$\rho_+(\rho_-(0)) < \rho_-(\rho_+(\infty)) \leq \rho_-(w_0) \leq \rho_+(\gamma_+(w_0)).$$

In consequence, $\gamma_+(w_0) < \rho_-(0)$, which gives the desired contradiction.

$\boxed{\text{III}} \Longrightarrow$ not ($\boxed{\text{I}}$ or $\boxed{\text{II}}$): Suppose that $\boxed{\text{III}}$ and $\boxed{\text{II}}$ hold together. Let $w_1 < \underline{b}$ be the point given by $\boxed{\text{II}}$ such that $\gamma_+(w_1) = \infty$, and w_0 the point in $\boxed{\text{III}}$ such that $\gamma_-(\gamma_+(w_0)) = w_0$. Naturally, as $\gamma_-(\gamma_+(w_1)) = \gamma_-(\infty) = \underline{b} > w_1$ we have that $w_0 < w_1$. Observe also that $\mu_B((0, \rho_-(w_1)) \cup (\rho_+(\infty), \infty)) = \underline{b}$ and $\mu_B(\mathbb{R}) = S_0 \in (\underline{b}, \overline{b})$, which readily imply $\underline{b} < \rho_-(w_1) < \rho_+(\infty) < \overline{b}$. Let us further denote $r = \min\{\rho_-(w_0), \rho_+(\infty)\}$ and $R = \max\{\rho_-(w_0), \rho_+(\infty)\}$ so that finally, using our assumptions,

$$w_0 < w_1 < \underline{b} < \rho_-(w_1) < r \leq R \leq \rho_+(\gamma_+(w_0)) \leq \overline{b} \leq \gamma_+(w_0). \tag{4.3}$$

By definition we have

$$\int_{\rho_+(\infty)}^{\infty} (\overline{b} - u)\mu(\mathrm{d}u) = 0 = \int_{w_0}^{\rho_-(w_0)} (\overline{b} - u)\mu(\mathrm{d}u) + \int_{\rho_+(\gamma_+(w_0))}^{\infty} (\overline{b} - u)\mu(\mathrm{d}u).$$

Subtracting these two quantities, we arrive at

$$\int_{R}^{\rho_+(\gamma_+(w_0))} (\overline{b} - u)\mu(\mathrm{d}u) = \int_{w_0}^{r} (\overline{b} - u)\mu(\mathrm{d}u), \quad \text{and using (4.3), we deduce}$$

$$\mu\Big(\big(R, \rho_+(\gamma_+(w_0))\big) \Big) > \mu\Big((w_0, r) \Big). \tag{4.4}$$

Using the properties of our functions again, we have

$$\int_{-\infty}^{w_1} (u - \underline{b})\mu(\mathrm{d}u) + \int_{\rho_+(\infty)}^{\infty} (u - \underline{b})\mu(\mathrm{d}u) = 0 = \int_{-\infty}^{w_0} (u - \underline{b})\mu(\mathrm{d}u) + \int_{\rho_+(\gamma_+(w_0))}^{\gamma_+(w_0)} (u - \underline{b})\mu(\mathrm{d}u),$$

which after subtracting, using $\int_{w_0}^{w_1} (u - \underline{b})\mu(\mathrm{d}u) = -\int_{\rho_-(w_1)}^{\rho_-(w_0)} (u - \underline{b})\mu(\mathrm{d}u)$, yields

$$\int_{\rho_+(\infty)}^{\rho_+(\gamma_+(w_0))} (u - \underline{b})\mu(\mathrm{d}u) - \int_{\rho_-(w_1)}^{\rho_-(w_0)} (u - \underline{b})\mu(\mathrm{d}u) + \int_{\gamma_+(w_0)}^{\infty} (u - \underline{b})\mu(\mathrm{d}u) = 0. \tag{4.5}$$

The last term in (4.5) is positive, and for the first two terms, using (4.4), we have

$$\begin{aligned} \int_{R}^{\rho_+(\gamma_+(w_0))} (u - \underline{b})\mu(\mathrm{d}u) &\geq (R - \underline{b})\mu\Big(\big(R, \rho_+(\gamma_+(w_0))\big) \Big) \\ &> (R - \underline{b})\mu\Big((\rho_-(w_1), r) \Big) \geq \int_{\rho_-(w_1)}^{r} (u - \underline{b})\mu(\mathrm{d}u). \end{aligned} \tag{4.6}$$

This readily implies that the LHS of (4.5) is strictly positive, leading to the desired contradiction.

The case when $\boxed{\text{III}}$ and $\boxed{\text{I}}$ hold together is similar. ■

Step 2. Construction of relevant embeddings. Our strategy is now as follows. For each of the four exclusive cases $\boxed{\text{I}}$–$\boxed{\text{IV}}$ we construct a stopping time τ which solves the SEP for μ and such that for the price process $S_t := B_{\frac{t}{T-t}\wedge\tau}$ the appropriate superhedge $\overline{H}^I - \overline{H}^{IV}$ is in fact a perfect hedge. The stopping time τ will be a composition of stopping times, each of which is a solution to an embedding problem for a (rescaled) restriction of μ to appropriate intervals.

Suppose that $\boxed{\text{II}}$ holds. This embedding is closely related to the classical embedding of Azéma and Yor [2] used in the work of Brown, Hobson, and Rogers [8] on one-sided barrier options. There, it is used to achieve equality in the second inequality in (2.2). However, in order to also achieve equality in the first inequality we need to modify the embedding slightly. Let τ_1 be a UI embedding, in $(B_t)_{t\geq 0}$ with $B_0 = S_0$, of

$$\nu^1 = \mu|_{(w_0,\rho_+(\infty))} + p\delta_{\underline{b}}, \quad \text{where } p = 1 - \mu((w_0,\rho_+(\infty))),$$

which is centered in S_0. Let $\nu^2 = \frac{1}{p}\mu|_{\mathbb{R}_+\setminus(w_0,\rho_+(\infty))}$, which is a probability measure with $\nu^2_B(\mathbb{R}_+) = \underline{b}$, and let τ_2 be the Azéma–Yor embedding (cf. section 5 in [34]) of ν^2, i.e.,

$$\tau_2 = \inf\left\{t > 0 : \overline{B}_t \geq \nu^2_B([B_t,\infty))\right\},$$

which is a UI embedding of ν^2 when $B_0 = \underline{b}$. Note that $\nu^2_B([x,\infty)) = \mu_B([x,\infty))$ for $x \geq \rho_+(\infty)$ and that

$$\{\overline{B}_{\tau_2} \geq \overline{b}\} = \{B_{\tau_2} \geq \rho_+(\infty)\}, \quad \text{since } \nu^2_B((\rho_+(\infty),\infty)) = \overline{b}.$$

We define our final embedding as follows: we first embed ν^1, and then the atom in $\underline{b}$ is diffused into ν^2 using the Azéma–Yor procedure, i.e.,

$$\tau := \tau_1 \mathbf{1}_{B_{\tau_1}\neq\underline{b}} + \tau_2\circ\tau_1 \mathbf{1}_{B_{\tau_1}=\underline{b}}, \tag{4.7}$$

where $B_0 = S_0$. Clearly, τ is a UI embedding of μ, and $S_t := B_{\frac{t}{T-t}\wedge\tau}$ defines a model for the stock price which matches the given prices of calls and puts; i.e., $S_T \sim \mu$. Furthermore, $\{\overline{S}_T \geq \overline{b}\} = \{\overline{S}_T \geq \overline{b}, \underline{S}_T \leq \underline{b}\} = \{S_T \geq \rho_+(\infty)\}$, since $\rho_+(\infty) < \overline{b}$ implies $\{\tau = \tau_1\} \subseteq \{\overline{S}_T < \overline{b}\}$ and $\{\tau \neq \tau_1\} \subseteq \{\underline{S}_T \leq \underline{b}\}$. It follows that

$$\mathbf{1}_{\overline{S}_T\geq\overline{b},\, \underline{S}_T\leq\underline{b}} = \overline{H}^{II}(\rho_+(\infty)).$$

Suppose that $\boxed{\text{I}}$ holds. This is a mirror image of $\boxed{\text{II}}$. We first embed $\nu^1 = \mu|_{(\rho_-(0),z_0)} + p\delta_{\overline{b}}$, with $p = 1-\mu((\rho_-(0),z_0))$. Then the atom in $\overline{b}$ is diffused into $\mu|_{\mathbb{R}_+\setminus(\rho_-(0),z_0)}$ using the reversed Azéma–Yor stopping time (cf. section 5.3 in [34]). The resulting stopping time τ and the stock price model $S_t := B_{\frac{t}{T-t}\wedge\tau}$ satisfy $\mathbf{1}_{\overline{S}_T\geq\overline{b},\, \underline{S}_T\leq\underline{b}} = \overline{H}^I(\rho_-(0))$.

Suppose that $\boxed{\text{III}}$ holds. We describe the embedding in words before writing it formally. We first embed μ on $(\rho_-(w_0), \rho_+(\gamma_+(w_0)))$, or we stop when we hit $\overline{b}$ or $\underline{b}$. If we hit $\overline{b}$, then we embed μ on $(\gamma_+(w_0),\infty)$ or we run until we hit $\underline{b}$. Likewise, if we first hit $\underline{b}$, then we embed μ on $(0,w_0)$ or we run till we hit $\overline{b}$. Finally, from $\underline{b}$ and $\overline{b}$ we embed the remaining bits of μ.

We now formalize these ideas. Let

$$\nu^1 = p\delta_{\underline{b}} + \mu|_{(\rho_-(w_0),\rho_+(\gamma_+(w_0)))} + \big(1-p-\mu(\rho_-(w_0),\rho_+(\gamma_+(w_0)))\big)\delta_{\overline{b}}, \tag{4.8}$$

where p is chosen so that $\nu^1_B(\mathbb{R}_+) = S_0$. Define two more measures,

$$\begin{aligned} \nu^2 &= \mu\big([w_0,\rho_-(w_0)]\big)\delta_{\underline{b}} + \mu|_{(\gamma_+(w_0),\infty)},\\ \nu^3 &= \mu|_{(0,w_0)} + \mu\big([\rho_+(\gamma_+(w_0)),\gamma_+(w_0)]\big)\delta_{\overline{b}}, \end{aligned} \tag{4.9}$$

and note that by definition $\nu^2_B(\mathbb{R}_+) = \overline{b}$ and $\nu^3_B(\mathbb{R}_+) = \underline{b}$. Furthermore, as the barycenter of

$$\nu^3 + \mu|_{((\rho_-(w_0),\rho_+(\gamma_+(w_0)))} + \nu^2$$

is equal to the barycenter of μ, and from the uniqueness of p in (4.8), we deduce that

$$\nu^3(\mathbb{R}_+) = p \quad \text{and} \quad \nu^2(\mathbb{R}_+) = q = \big(1-p-\mu(\rho_-(w_0),\rho_+(\gamma_+(w_0)))\big).$$

Let τ_1 be a UI embedding of ν^1 (for $B_0 = S_0$), τ_2 be a UI embedding of $\frac{1}{q}\nu^2$ (for $B_0 = \overline{b}$), and τ_3 be a UI embedding of $\frac{1}{p}\nu^3$ (for $B_0 = \underline{b}$). Further, let τ_4 and τ_5 be UI embeddings respectively of

$$\frac{1}{\mu\big((w_0,\rho_-(w_0))\big)}\mu|_{(w_0,\rho_-(w_0))} \quad \text{and} \quad \frac{1}{\mu\big((\rho_+(\gamma_+(w_0)),\gamma_+(w_0))\big)}\mu|_{(\rho_+(\gamma_+(w_0)),\gamma_+(w_0))},$$

where the starting points are respectively $B_0 = \underline{b}$ and $B_0 = \overline{b}$. We are ready to define our stopping time. Let $B_0 = S_0$, and write $H_z = \inf\{t : B_t = z\}$. We put

$$\begin{aligned} \tau := \tau_1 &\mathbf{1}_{\tau_1 < H_{\underline{b}}\wedge H_{\overline{b}}}\\ &+ \tau_2\circ\tau_1 \mathbf{1}_{H_{\overline{b}}=\tau_1}\mathbf{1}_{\tau_2\circ\tau_1 < H_{\underline{b}}}\\ &+ \tau_4\circ\tau_2\circ\tau_1 \mathbf{1}_{H_{\overline{b}}=\tau_1}\mathbf{1}_{H_{\underline{b}}=\tau_2\circ\tau_1}\\ &+ \tau_3\circ\tau_1 \mathbf{1}_{H_{\underline{b}}=\tau_1}\mathbf{1}_{\tau_3\circ\tau_1 < H_{\overline{b}}}\\ &+ \tau_5\circ\tau_3\circ\tau_1 \mathbf{1}_{H_{\underline{b}}=\tau_1}\mathbf{1}_{H_{\overline{b}}=\tau_3\circ\tau_1}, \end{aligned} \tag{4.10}$$

and it is immediate from the properties of our measures that $B_\tau \sim \mu$ and $(B_{t\wedge\tau})$ is a UI martingale. Furthermore, with $S_t := B_{\frac{t}{T-t}\wedge\tau}$, we see that

$$\mathbf{1}_{\overline{S}_T\geq\overline{b},\,\underline{S}_T\leq\underline{b}} = \overline{H}^{III}\big(\gamma_+(w_0),\rho_+(\gamma_+(w_0)),\rho_-(w_0),w_0\big), \quad \text{a.s.}$$

Finally, suppose that $\boxed{\text{IV}}$ holds. In this case, we initially run to $\{\underline{b},\overline{b}\}$ without stopping any mass. Then, from $\overline{b}$, we either run to $\underline{b}$ or embed μ on $(\rho_-(0),\infty)$. The mass which is at $\underline{b}$ after the first step is either run to $\overline{b}$ or used to embed μ on $(0,\rho_+(\infty))$. The mass which remains at $\underline{b}$ and $\overline{b}$ is then used to embed the remaining part of μ on $(\rho_+(\infty),\rho_-(0))$.

To begin with, we define the measures

$$\nu^1 = \left[\frac{S_0 - \underline{b}}{\overline{b} - \underline{b}} - \mu((\rho_-(0), \infty))\right] \delta_{\underline{b}} + \mu|_{(\rho_-(0),\infty)},$$
$$\nu^2 = \left[\frac{\overline{b} - S_0}{\overline{b} - \underline{b}} - \mu((0, \rho_+(\infty))\right] \delta_{\overline{b}} + \mu|_{(0,\rho_+(\infty))}.$$

Then ν^1 is a measure, since $\mu((\rho_-(0), \infty)) < \frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}$: noting that $\overline{b} < \rho_-(0)$, we get

$$\begin{aligned} 0 &= \int_0^{\rho_-(0)} (u - S_0)\mu(\mathrm{d}u) + \int_{\rho_-(0)}^{\infty} (u - S_0)\mu(\mathrm{d}u) \\ &\geq (\underline{b} - S_0)\mu((0, \rho_-(0))) + (\overline{b} - S_0)\mu((\rho_-(0), \infty)), \end{aligned}$$

and the statement follows. Moreover, we can see that $\nu^1_B(\mathbb{R}_+) = \overline{b}$:

$$\begin{aligned} &\int_{\rho_-(0)}^{\infty} (u - \overline{b})\mu(\mathrm{d}u) + \frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}(\underline{b} - \overline{b}) - \mu((\rho_-(0), \infty))(\underline{b} - \overline{b}) \\ &\quad = \int_{\rho_-(0)}^{\infty} (u - \underline{b})\mu(\mathrm{d}u) + (\underline{b} - S_0) \\ &\quad = (S_0 - \underline{b}) - \int_0^{\rho_-(0)} (u - \underline{b})\mu(\mathrm{d}u) + (\underline{b} - S_0) = 0. \end{aligned}$$

Similar results hold for ν^2.

Consequently, we can construct the first stages of the embedding. The final stage is to run from $\underline{b}$ and $\overline{b}$ to embed the remaining mass. Of course, it does not matter exactly how we do this from the optimality point of view, since these paths have already struck both barriers, but we do need to check that the embedding is possible. It is clear that the means and probabilities match, but unlike the previously considered cases, we now have initial mass in two places, and the existence of a suitable embedding is not trivial. To resolve this, we note the following: suppose we can find a point $z^* \in (\rho_+(\rho_-(0)), \rho_-(\rho_+(\infty)))$ such that $\nu^1(\{\underline{b}\}) = \mu((\rho_+(\infty), z^*))$. Then because $\mu_B((\rho_+(\infty), \rho_-(\rho_+(\infty)))) = \underline{b}$, we can find $z_1 \in (\rho_+(\infty), z^*)$ such that the measure

$$\nu^3 = \mu|_{(\rho_+(\infty), z_1)} + (\nu^1(\{\underline{b}\}) - \mu((\rho_+(\infty), z_1)))\delta_{z^*}$$

has barycenter $\underline{b}$. There is a similar construction for ν^4, a point z_2 which will embed mass from ν^2 at $\overline{b}$ to μ on $(z_2, \rho_-(0))$, and an atom at z^*. In the final stage, we can then embed the mass from z^* to (z_1, z_2).

It remains to show that we can find such a point z^*. To do this, we check that there is sufficient mass being stopped at $\underline{b}$ at the end of the second step (i.e., paths which have first hit $\overline{b}$ and then $\underline{b}$). Specifically, we need to show that

$$\frac{S_0 - \underline{b}}{\overline{b} - \underline{b}} - \mu((\rho_-(0), \infty)) \geq \mu((\rho_+(\infty), \rho_+(\rho_-(0)))).$$

Rearranging and using the definitions of the functions ρ_+ and ρ_-, this is equivalent to

$$\begin{aligned}(S_0 - \underline{b}) &\geq \int_{\rho_+(\infty)}^{\infty} (u - \underline{b})\,\mu(\mathrm{d}u) - \int_{\rho_+(\rho_-(0))}^{\rho_-(0)} (u - \underline{b})\mu(\mathrm{d}u)\\ &\geq \int_{(\rho_+(\infty),\rho_+(\rho_-(0)))\cup(\rho_-(0),\infty)} (u - \underline{b})\,\mu(\mathrm{d}u)\\ &\geq (S_0 - \underline{b}) - \int_{(0,\rho_+(\infty))\cup(\rho_+(\rho_-(0)),\rho_-(0))} (u - \underline{b})\,\mu(\mathrm{d}u).\end{aligned}$$

Using the definitions of the appropriate functions, this can be seen to be equivalent to

$$0 \leq \int_{\rho_+(\rho_-(0))}^{\rho_-(\rho_+(\infty))} (u - \underline{b})\,\mu(dy),$$

which follows since $\rho_+(\rho_-(0)) > \underline{b}$. The construction of the appropriate stopping time, and its optimality, follow as previously. This ends the proof of Theorem 2.2. ■

In order to prove Theorem 2.4 we start with an auxiliary lemma.

Lemma 4.5. *Either we may construct an embedding of μ under which the process never hits both $\overline{b}$ and $\underline{b}$, or*

$$\inf\{v \in [\underline{b},\overline{b}] : \psi(v) < \infty\} \geq \inf\{v \in [\underline{b},\overline{b}] : \theta(v) > -\infty\}, \tag{4.11}$$

$$\sup\{v \in [\underline{b},\overline{b}] : \psi(v) < \infty\} \geq \sup\{v \in [\underline{b},\overline{b}] : \theta(v) > -\infty\}, \tag{4.12}$$

and we may then write

$$\underline{v} = \inf\{v \in [\underline{b},\overline{b}] : \psi(v) < \infty\} \leq \sup\{v \in [\underline{b},\overline{b}] : \theta(v) > -\infty\} = \overline{v}, \tag{4.13}$$

where $\underline{v}, \overline{v}$ are given in (2.37).

Proof. We begin by showing that if $\theta(v) = -\infty$ for all $v \in [\underline{b},\overline{b}]$, then there exists an embedding of μ which does not hit both $\underline{b}$ and $\overline{b}$.

For $w \geq \overline{b}$, define $\alpha_*(w)$ to be the mass that must be placed at $\underline{b}$ in order for the barycenter of this mass plus μ on $(\overline{b}, w)$ to be $\overline{b}$, so that $\alpha_*(w)$ satisfies

$$\alpha_*(w)\underline{b} + \int_{\overline{b}}^{w} u\,\mu(\mathrm{d}u) = \overline{b}\left(\alpha_*(w) + \mu((\overline{b}, w))\right).$$

It follows that $\alpha_*(w)$ exists, although there is no guarantee that it is less than $1 - \mu((\overline{b}, w))$. In addition, define $\beta^*(w)$ to be

$$\beta^*(w) = \inf\left\{\beta \in [\underline{b},\overline{b}] : \int_{(\underline{b},\beta)\cup(\overline{b},w)} u\,\mu(\mathrm{d}u) = \overline{b}\int_{(\underline{b},\beta)\cup(\overline{b},w)} \mu(\mathrm{d}u)\right\}.$$

If this is finite, then it is the point at which $\mu_B((\underline{b}, \beta^*(w)) \cup (\overline{b}, w)) = \overline{b}$. Note also that $\beta^*(w)$ is increasing as a function of w and is continuous when $\beta^*(w) < \infty$. Define

$$\begin{aligned}p_*(w) &= \mu((\overline{b}, w)) + \alpha_*(w),\\ p^*(w) &= \mu((\underline{b}, \beta^*(w)) \cup (\overline{b}, w)).\end{aligned}$$

Suppose initially that $\beta^*(w) \le \overline{b}$ for all $w \ge \overline{b}$. Then we may assign the following interpretations to these quantities: $p_*(w)$ is the smallest amount of mass that we can start at $\overline{b}$ and run to embed μ on $(\overline{b}, w)$, $(\underline{b}, \cdot)$ and an atom at $\underline{b}$, and $p^*(w)$ is the largest amount of mass that we may do this with; the smallest amount is attained by running all the mass below $\overline{b}$ to $\underline{b}$, while the largest probability is attained by running all this mass to $(\underline{b}, \beta^*(w))$. The assumption that $\beta^*(w) \le \overline{b}$ implies that this upper bound does not run out of mass to embed. Moreover, by adjusting the size of the atom at $\underline{b}$, we can embed an atom of any size between $p_*(w)$ and $p^*(w)$ from $\overline{b}$ in this way. Recalling the definition of $\theta(v)$, we conclude that there exists v such that $\theta(v) = w$ if and only if $p_*(w) \le \frac{S_0 - \underline{b}}{\overline{b} - \underline{b}} \le p^*(w)$. Finally, note that the functions $p_*(w)$ and $p^*(w)$ are both increasing in w, and further that $p_*(\overline{b}) = 0$. Consequently, if there is no v such that $\theta(v) > -\infty$, and if $\beta^*(w) \le \overline{b}$ for all $w \ge \overline{b}$, we must have $p^*(\infty) := \lim_{w \to \infty} p^*(w) < \frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}$.

So suppose $p^*(\infty) < \frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}$. We now construct an embedding as follows: from S_0, we initially run to either $\overline{b}$ or

$$b_* = \frac{S_0 - p^*(\infty)\overline{b}}{1 - p^*(\infty)}.$$

Since $p^*(\infty) < \frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}$, then $b_* \in (\underline{b}, S_0)$, and the probability that we hit $\overline{b}$ before b_* is $p^*(\infty)$. In addition, by the definition of $\beta^*(w)$, we deduce that the set $(\underline{b}, \beta^*(\infty)] \cup [\overline{b}, \infty)$ is given mass $p^*(\infty)$ by μ, and that the barycenter of μ on this set is $\overline{b}$. We may therefore embed the paths from $\overline{b}$ to this set, and the paths from b_* to the remaining intervals, $[0, \underline{b}] \cup (\beta^*(\infty), \overline{b})$, and we note that no paths will hit both $\overline{b}$ and $\underline{b}$.

Suppose instead that $\beta^*(w_0) = \overline{b}$ for some w_0, with $p^*(w_0) \le \frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}$. (If the latter condition does not hold, then using the fact that $\beta^*(w)$ is left-continuous and increasing, we can find a w such that $\beta^*(w) < \overline{b}$ and $p^*(w) = \frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}$, and therefore, by the arguments above, there exists v with $\theta(v) > -\infty$.) We may then continue to construct measures with barycenter $\overline{b}$, which are equal to μ on $(\underline{b}, w)$ for $w > w_0$ and have a compensating atom at $\underline{b}$. As we increase w, eventually either w reaches ∞ or the mass of the measure reaches $\frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}$. In the latter case, we know $\theta(\overline{b}) = w$, contradicting $\theta(v) = -\infty$ for all $v \in [\underline{b}, \overline{b}]$. So consider the former case: we obtained that the measure which is μ on $(\underline{b}, \infty)$ with a further atom at $\underline{b}$ to give barycenter $\overline{b}$ has total mass (p, say) less than $\frac{S_0 - \underline{b}}{\overline{b} - \underline{b}}$. We show that this is impossible: divide μ into its restriction to $(0, z)$ and $[z, \infty)$, where z is chosen so that $\mu([z, \infty)) = p$. Then $z < \underline{b}$, and the barycenter of the restriction to $[z, \infty)$ is strictly smaller than the barycenter of the measure with the mass on $[z, \underline{b})$ placed at $\underline{b}$, which is the measure described above and which has barycenter $\overline{b}$. Additionally, the barycenter of the lower restriction of μ must be strictly smaller than $\underline{b}$. Moreover, we may calculate the barycenter of μ by considering the barycenter of the two restrictions; since μ has mean (and therefore barycenter) S_0, we must have

$$S_0 = (1-p)\mu_B((0, z)) + p\mu_B([z, \infty)) < (1-p)\underline{b} + p\overline{b} < \underline{b}\frac{\overline{b} - S_0}{\overline{b} - \underline{b}} + \overline{b}\frac{S_0 - \underline{b}}{\overline{b} - \underline{b}} = S_0,$$

which is a contradiction.

We conclude that, if $\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\}$ is empty, there is an embedding of μ which does not hit both $\overline{b}$ and $\underline{b}$. A similar result follows for $\psi(v)$. In particular, if we assume that there is no such embedding, then there exists v such that $\psi(v) < \infty$, and (not necessarily the same) v such that $\theta(v) > -\infty$. We now wish to show that (4.11) holds. Suppose not. Then

$$v_* := \inf\{v \in [\underline{b}, \overline{b}] : \psi(v) < \infty\} < \inf\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\} =: v^*.$$

Moreover, we can deduce from the definition of $\theta(v)$ that since $v^* > \underline{b}$, we must have $\theta(v^*) = \infty$. Now consider the barycenter of the measure which is taken by running from S_0 to $\underline{b}$ and $\overline{b}$, and then from $\underline{b}$ to $(\psi(v_*), \underline{b}) \cup (v_*, \overline{b})$, with a compensating mass at $\overline{b}$, so that the measure has barycenter $\underline{b}$, and from $\overline{b}$ to $(\underline{b}, v^*) \cup (\overline{b}, \theta(v^*)) = (\underline{b}, v^*) \cup (\overline{b}, \infty)$, with a compensating mass at $\underline{b}$, so that the measure has barycenter $\overline{b}$. Then the whole law of the resulting process must have mean S_0, since this can be done in a uniformly integrable way, but the resulting distribution is at least μ on $(\psi(v_*), \infty)$ (it is twice μ on (v_*, v^*), has atoms at $\underline{b}$ and $\overline{b}$, and is μ elsewhere), and zero on $(0, \psi(v_*))$, so must have mean greater than S_0, which is a contradiction. A similar argument shows (4.12). Hence, we may conclude (still under the assumption that there is no embedding which never hits both $\underline{b}$ and $\overline{b}$) that the equalities in (4.13) hold. It remains to show the inequality when $\underline{v}, \overline{v} \in (\underline{b}, \overline{b})$. However, this is now almost immediate: the forms of $\overline{v}, \underline{v}$ imply that μ gives mass $\frac{\overline{b}-S_0}{\overline{b}-\underline{b}}$ to the set $(\psi(\underline{v}), \underline{b}) \cup (\underline{v}, \overline{b})$ and gives mass $\frac{S_0-\underline{b}}{\overline{b}-\underline{b}}$ to the set $(\underline{b}, \overline{v}) \cup (\overline{b}, \theta(\overline{v}))$. If $\overline{v} < \underline{v}$, this implies that μ gives mass 1 to the set $(\psi(\underline{v}), \overline{v}) \cup (\underline{v}, \theta(\overline{v})) \subsetneq [0, \infty)$, contradicting the positivity of μ. ■

Proof of Theorem 2.4. From Lemma 4.5 case IV and the last statement of the theorem follow. Assume from now on that $\underline{v} \leq \overline{v}$. We note first that $\psi(v)$ and $\theta(v)$ are both continuous and decreasing on $[\underline{v}, \overline{v}]$, and consequently $\kappa(v)$ is also continuous and decreasing as a function of v on $[\underline{v}, \overline{v}]$. It follows that the three cases $\kappa(\underline{v}) < \underline{v}$, $\kappa(\overline{v}) > \overline{v}$, and the existence of $v_0 \in [\underline{v}, \overline{v}]$ such that $\kappa(v_0) = v_0$ are exclusive and exhaustive. We consider each case separately:

I Suppose that there exists $v_0 \in [\underline{v}, \overline{v}]$ such that $\kappa(v_0) = v_0$. By the definition of $\psi(v)$, we can run all the mass initially from S_0 to $\{\underline{b}, \overline{b}\}$ and then embed (in a UI way) from $\underline{b}$ to $(\psi(v_0), \underline{b}) \cup (v_0, \overline{b})$ and a compensating atom at $\overline{b}$ with the remaining mass, and similarly from $\overline{b}$ to $(\underline{b}, v_0) \cup (\overline{b}, \theta(v_0))$ with an atom at $\underline{b}$. The mass now at $\underline{b}$ and $\overline{b}$ can then be embedded in the remaining tails in a suitable way—the means and masses must agree, since the initial stages were embedded in a UI manner, and the remaining mass all lies outside $[\underline{b}, \overline{b}]$. We denote by τ the stopping time which achieves the embedding.

Now we compare both sides of the inequality in (2.10), where we choose $K_1 = \theta(v_0), K_2 = \psi(v_0)$, and therefore, as a consequence of the definition of $\kappa(v)$, we also have $K_3 = v_0$. The key observation is now that the mass is stopped only at points where the inequality is an equality: mass which hits $\overline{b}$ initially either stops in the interval $(\underline{b}, K_3) \cup (\overline{b}, K_1)$, when there is equality in (2.10), or it goes on to hit $\underline{b}$, and from this point also continues to the tails $(0, K_2) \cup (K_1, \infty)$, where there is again equality between both sides of (2.10). Taking expectations on the RHS of (2.10), we get the terms on the RHS of (2.38), and we conclude that (2.38) holds in the market model $S_t := B_{\tau \wedge \frac{t}{T-t}}$.

II Suppose now that $\kappa(\overline{v}) > \overline{v}$. Then we must have $\overline{v} = \sup\{v \in [\underline{b}, \overline{b}] : \theta(v) > -\infty\}$ by Lemma 4.5, and then $\overline{v} < \kappa(\overline{v}) \leq \overline{b}$. So, by the definition of $\theta(v)$, since $\overline{v}$ exists and is less than $\overline{b}$, we must have

$$\int_{(\underline{b},\overline{v})\cup(\overline{b},\theta(\overline{v}))} u\,\mu(\mathrm{d}u) = \overline{b}\frac{S_0-\underline{b}}{\overline{b}-\underline{b}} \quad \text{and} \quad \mu((\underline{b},\overline{v})\cup(\overline{b},\theta(\overline{v}))) = \frac{S_0-\underline{b}}{\overline{b}-\underline{b}},$$

and we can embed from $\overline{b}$ (having initially run to $\{\underline{b},\overline{b}\}$) to $(\underline{b},\overline{v})\cup(\overline{b},\theta(\overline{v}))$ *without* leaving an atom at $\underline{b}$. Similarly, we can also run from $\underline{b}$ to $(\psi(\overline{v}),\underline{b})\cup(\overline{v},\overline{b})$ with an atom at $\overline{b}$. The atom can then be embedded in the tails $(0,\psi(\overline{v}))\cup(\theta(\overline{v}),\infty)$ in a UI manner. We now need to show that when we take $K_3=\overline{v}, K_1=\theta(\overline{v})$, and $K_2=\psi(\overline{v})$ we get the required equality in (2.39). The main difference from the above case occurs in the case where we hit $\overline{b}$ initially and then hit $\underline{b}$: we no longer need equality in (2.10), since this no longer occurs in our optimal construction; however, what remains to be checked is that the inequality does hold on this set. Specifically, we need to show that

$$1 \geq \alpha_0+\alpha_1(K_2-S_0)-(\alpha_3-\alpha_3+\alpha_1)(K_2-\overline{b})+(\alpha_3-\alpha_2)(K_2-\underline{b}).$$

Using (2.20) and (2.25), we see that this occurs when

$$\frac{(K_3-K_2)(K_1-\underline{b})}{(\overline{b}-K_2)(K_1-K_3)} \leq 1,$$

which rearranges to give

$$K_3 \leq \overline{b}\frac{K_1-\underline{b}}{(K_1-\underline{b})+(\overline{b}-K_2)}+\underline{b}\frac{\overline{b}-K_2}{(K_1-\underline{b})+(\overline{b}-K_2)}.$$

This is satisfied by our choice of $\overline{v}$ as K_3, and $K_1=\theta(\overline{v}), K_2=\psi(\overline{v})$.

III This is symmetric to case II. ■

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