The radiating part of circular sources

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Abstract

An analysis is developed linking the form of the sound field from a circular source to the radial structure of the source, without recourse to far-field or other approximations. It is found that the information radiated into the field is limited, with the limit fixed by the wavenumber of source multiplied by the source radius (Helmholtz number). The acoustic field is found in terms of the elementary fields generated by a set of line sources whose form is given by Chebyshev polynomials of the second kind, and whose amplitude is found to be given by weighted integrals of the radial source term. The analysis is developed for tonal sources, such as rotors, and, for Helmholtz number less than two, for random disk sources. In this case, the analysis yields the cross-spectrum between two points in the acoustic field. The analysis is applied to the problems of tonal radiation, random source radiation as a model problem for jet noise, and to noise cancellation, as in active control of noise from rotors. It is found that the approach gives an accurate model for the radiation problem and explicitly identifies those parts of a source which radiate.

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I. INTRODUCTION

A problem in many applications is that of sound generated by circular sources. These include rotors of various types such as aircraft propellers and fans, wind turbines and cooling fans; vibrating systems such as loudspeakers; ducts such as aircraft engines, ventilation systems and exhausts; and distributed sources with axial symmetry such as jets. There have been numerous studies of the noise generation and radiation process in each of these areas extending over many decades. These studies can be divided into those which examine the relationship between the acoustic source and the physical processes which give rise to it, for example the work of Lighthill[1] and of Ffowcs Williams and Hawkings[2] which relate aerodynamic quantities to acoustic sources, and those which examine the radiated field generated by a given source distribution, such as methods for prediction of the field radiated by pistons and loudspeakers[3–6] or from a known rotating source distribution[7–10].

There are a number of areas where these issues, those of generation and radiation, overlap. One is the general area of source identification. There have been many attempts to develop methods which use acoustic measurements to infer, in greater or lesser detail, the source distribution responsible for the acoustic field. In the case of rotating sources, some examples include cooling fans[11–13] and propellers[14–17], while a number of groups have developed methods for the inverse problem for radiation from a duct termination[18–22]. Such studies can have a number of motivations. The first is to use near-field data, for example those taken in wind-tunnel tests, to predict the far acoustic field. In this case, the requirement is to extract information about source strength and directivity, but there is no need to know which processes generate the source. A second motivation, however, is the identification of the noisiest parts of the source with a view to reduction of noise at source, for example the identification of "hot spots" caused by unsteady loading on a cooling fan[11,12]. In this case,

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the link between the aerodynamics and the source is an essential part of the solution of the problem.

In each of the applications of source identification listed, the authors have recognized that the problem is (very) ill-conditioned. This can be attributed to physical causes, and is not merely an artifact of the methods used. Recent analysis[23–25] has given a framework for the study of this ill-conditioning by quantifying the source information which is radiated into the acoustic near and far fields. As described below, it has been found that the source can be decomposed into orthogonal modes based on Chebyshev polynomials, only a limited number of which radiate a detectable acoustic field, with the limit being fixed by the source frequency.

A second area where the issues of generation and radiation overlap is that of jet noise. Lighthill's acoustic analogy[1] is accepted as an exact theory for noise generation by turbulence and there is solid evidence for the validity of his source term, as demonstrated by high quality numerical simulation[26]. This knowledge, however, is not sufficient to explain certain features of jet noise, in particular the low radiation efficiency of subsonic jets and the low order structure of the acoustic field. It is known that subsonic jets radiate only a small fraction of the source energy, a view given support by the very small changes in the flow which suffice to give large reductions in noise, when control is applied[27]. It is also known that the acoustic far field of a jet is significantly simpler than the flow field. In a recent study[28], modal decomposition of the far-field noise and of the flow field of a Mach 0.9 jet showed that 24 modes were sufficient to capture 90% of the energy of the acoustic field, but 350 were required to resolve 50% of the flow energy. Clearly, a very large part of the flow, however energetic it might be, simply does not radiate but it is not obvious if this is due to the nature of the source or purely a result of radiation effects.

The radiation effect has been explained in terms of source cancellation[29,30] and by viewing the radiation process as equivalent to the imposition of a spatial filter using a wavenumber criterion. Such an approach has been used by Freund[26] who found that the part of the source which radiates is indeed that part left over after applying an appropriate spatial filter. Similarly, Sinayoko and Agarwal[31] apply a linear convolution filter to decompose the flow into radiating and non-radiating parts.

A question which has attracted relatively little attention in these studies is that of the effect of radial cancellation or, alternatively, the effect of the radial structure of the source. A jet is a roughly circular source with finite axial extent and axial and azimuthal cancellation effects have been studied in the past[29,33] but radial effects have not been considered as important. For example, Freund, in his paper on noise sources in jets[26], says of his filtering operation that "it is not guaranteed to remove all non-radiating components" of the source and that "additional cancellation may occur due to the radial structure of the source". Exact, near- and far-field, analysis of a finite disk source will shed some light on the problem of azimuthal and radial effects which can extend the previous far-field analyses of model problems such as those of Michalke[33] and of Michel[29,30].

The analysis to be presented below attempts to explain some of the features of these problems. Previous work[23–25] has found limits on the information radiated from a tonal circular source, motivated by a desire to understand the ill-conditioning of source identification methods. These limits have been found without recourse to a far-field approximation, making the approach suitable for analysis of general problems. The remainder of this paper contains an extension of the theory to explicitly include the radial source term, and to yield spectral quantities in the acoustic field of random sources.

The first extension, which can be viewed as a generalization of previous work on axisymmetric radiators[32], will help explain radial cancellation effects, which have been studied in jet noise using a far-field formulation[30] but not, to the author's knowledge, in the near field. It will be found that for a given azimuthal order, many different sources radiate identical acoustic fields, differing only by a scaling factor. This result is part of the explanation for the

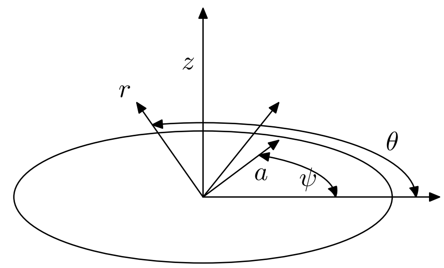


FIG. 1. Coordinate system for disk radiation calculations

ill-conditioning of identification methods and also opens a possible approach to the development of control systems by identifying a class of sources which can give rise to practically identical acoustic fields.

The second extension, to predicting the cross-spectrum between the acoustic pressures radiated by a random source to arbitrary points in the near and/or far field, is an extension of an earlier ring-source model for radiation from random sources characteristic of jets[33]. In this case, it will be found that the cross-spectrum depends on four constants, functions of observer radial separation, which are weighted integrals of the source cross-spectrum.

The results to be presented arise from two different exact theories for radiation from circular sources[24,25,34] which are combined to give a formulation for the information in the acoustic field in terms of radiation functions and weighted integrals of the source term. The implications of the results are discussed in terms of the information content of the acoustic field and with regard to some of the measurement methods used to study noise sources.

II. TONAL DISK SOURCE

The problem is initially formulated as that of calculating the acoustic field radiated by a monopole source distributed over a circular disk. The system for the analysis is shown in Figure 1 with cylindrical coordinates (r, θ, z) for the observer and $(a, \psi, 0)$ for the source. All lengths are non-dimensionalized on disk radius. The field from one azimuthal mode of the

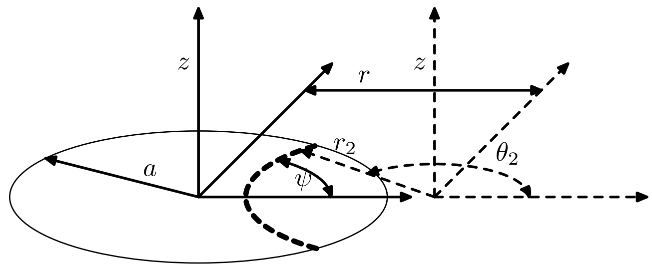


FIG. 2. Transformation to equivalent line source

acoustic source, specified as $s_n(a)\exp\mathrm{j}[n\psi-\omega t]$, is given at an arbitrary position (r,θ,z) by $p_n(k,r,z)\exp\mathrm{j}[n\theta-\omega t]$, where p_n is the Rayleigh integral[23,35]:

$$p_n(k,r,z)=\int_0^1\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{j}(kR'+n\psi)}}{4\pi R'}\,\mathrm{d}\psi s_n(a)a\,\mathrm{d}a, \tag{1}$$
$$R'=\left[r^2+a^2-2ra\cos\psi+z^2\right]^{1/2},$$

where k is non-dimensional wavenumber (Helmholtz number).

A. Equivalent line source expansion

The analysis of the nature of the sound field from an arbitrary disk source is based on a transformation of the disk to an exactly equivalent line source, an approach which has been used to study transient radiation from pistons[3,4], rotor noise[7,8] and source identification methods[23–25].

The transformation to a line source is shown in Figure 2, which shows the new coordinate system (r_2,θ_2,z) centred on a sideline of constant radius r. Under this transformation:

$$p_n(k,r,z)=\int_{r-1}^{r+1}\frac{\mathrm{e}^{\mathrm{j}kR'}}{R'}K(r,r_2)r_2\,\mathrm{d}r_2, \tag{2}$$
$$R'=\left(r_2^2+z^2\right)^{1/2},$$
$$K(r,r_2)=\frac{1}{4\pi}\int_{\theta_2^{(0)}}^{2\pi-\theta_2^{(0)}}\mathrm{e}^{\mathrm{j}n\psi}s_n(a)\,\mathrm{d}\theta_2, \tag{3}$$

for observer positions with $r > 1$, with the limits of integration given by:

$$\theta_2^{(0)} = \cos^{-1}\frac{1 - r^2 - r_2^2}{2rr_2}. \tag{4}$$

Functions of the form of $K(r, r_2)$ have been analyzed in previous work[8] and can be written:

$$K(r, r_2) = \sum_{q=0}^{\infty} u_q(r) U_q(s)(1 - s^2)^{1/2}, \tag{5}$$

where $U_q(s)$ is a Chebyshev polynomial of the second kind, $s = r_2 - r$ and the coefficients $u_q(r)$ are functions of r but not of z. Inserting Equation 5 into Equation 2:

$$p_n(k, r, z) = \sum_{q=0}^{\infty} u_q(r)\mathcal{L}_q(k, r, z), \tag{6}$$

$$\mathcal{L}_q(k, r, z) = \int_{-1}^{1} \frac{\mathrm{e}^{\mathrm{j}kR'}}{R'} U_q(s)(r + s)(1 - s^2)^{1/2}\,\mathrm{d}s, \tag{7}$$

$$R' = \left[(r + s)^2 + z^2\right]^{1/2}. \tag{8}$$

The radiation properties of the integral of Equation 7 have been examined in some detail elsewhere[24,25], giving an exact result for the in-plane case $z = 0$:

$$\mathcal{L}_q(k, r, 0) = \mathrm{j}^q(q + 1)\pi\mathrm{e}^{\mathrm{j}kr}\frac{J_{q+1}(k)}{k}. \tag{9}$$

For large order q, the Bessel function $J_q(k)$ is exponentially small for $k < q + 1$ so that the line source modes with order $q+1 > k$ generate noise fields of exponentially small amplitude. Since the integrals have their maximum in the plane $z = 0$, Equation 9 says that the whole field is of exponentially small amplitude if $q + 1 < k$. This gives an indication of how much of a given source distribution radiates into the acoustic field, near or far.

In previous analyses, two approximations to $\mathcal{L}_q$ have been developed. One is an asymptotic formula valid in the limit $k \to \infty$, derived using the method of stationary phase[24,25]. This will not be required here, but we will make use of the far-field form of Equation 7:

$$\mathcal{L}_q \approx \mathrm{j}^q\pi\frac{\mathrm{e}^{\mathrm{j}kR}}{R}\frac{q + 1}{k\sin\phi}\left[\left(r + \mathrm{j}\frac{q + 2}{k\sin\phi}\right)J_{q+1}(k\sin\phi) - \mathrm{j}J_q(k\sin\phi)\right], \tag{10}$$

where $R=[r^2+z^2]^{1/2}$ and $\phi=\cos^{-1}z/R$.

Given the basic information about the form of the radiated field, there remains to establish the relationship between the radial structure of the source $s_n(a)$ and the line source coefficients $u_q(r)$.

B. Series expansion for spinning sound fields

A recently derived series[34] for the field radiated by a ring source of radius a can be used to find a second expression for the sound radiated by a disk source with arbitrary radial variation:

$$\begin{aligned}R_n &= \int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{j}(kR'+n\psi)}}{4\pi R'}\,\mathrm{d}\psi,\\ &= \mathrm{j}^{2n+1}\frac{\pi}{4}\frac{1}{(aR)^{1/2}}\sum_{m=0}^{\infty}(-1)^m\frac{(2n+4m+1)(2m-1)!!}{(2n+2m)!!}\\ &\times H^{(1)}_{n+2m+1/2}(kR)P^n_{n+2m}(\cos\phi)J_{n+2m+1/2}(ka),\end{aligned}\tag{11}$$

with $H^{(1)}_\nu(x)$ the Hankel function of the first kind of order ν, J_ν the Bessel function of the first kind and p^m_n the associated Legendre function. The observer position is specified in the spherical polar coordinates used in Equation 10.

Multiplication by the radial source term $as_n(a)$ and integration gives an expression for the field radiated by a general source of unit radius and azimuthal order n:

$$\begin{aligned}p_n(k,r,z) &= \mathrm{j}^{2n+1}\frac{\pi}{4}\sum_{m=0}^{\infty}(-1)^m\frac{(2n+4m+1)(2m-1)!!}{(2n+2m)!!}P^n_{n+2m}(\cos\phi)S_{n+2m},\\ S_{n+2m}(k,r,z) &= \int_0^1 s_n(a)J_{n+2m+1/2}(ka)H^{(1)}_{n+2m+1/2}(kR)\left(\frac{a}{R}\right)^{1/2}\mathrm{d}a.\end{aligned}$$

Setting $z=0$ ($\phi=\pi/2$, $R=r$):

$$\begin{aligned}p_n(k,r,0) &= \frac{\mathrm{j}\pi}{4}\sum_{m=0}^{\infty}A_mS_{n+2m},\\ A_m &= \frac{1}{m!}\frac{(2n+4m+1)(2n+2m-1)!!(2m-1)!!}{2^m(2n+2m)!!},\end{aligned}\tag{12}$$

where use has been made of the expression[36]:

$$P_{n+2m}^{n}(0) = \frac{(-1)^{m+n}}{2^m}\frac{(2n+2m-1)!!}{m!}. \tag{13}$$

C. Line source coefficients

The expressions for p_n from section II.A and section II.B are both exact and can be equated to derive a system of equations relating the coefficients $u_q(r)$ to the weighted integrals of the radial source distribution $s_n(a)$:

$$\frac{\mathrm{j}}{4}\sum_{m=0}^{\infty} A_m S_{n+2m} = \mathrm{e}^{\mathrm{j}kr}\sum_{q=0}^{\infty} u_q(r)\mathrm{j}^q(q+1)\frac{J_{q+1}(k)}{k}. \tag{14}$$

Under repeated differentiation, Equation 14 becomes a lower triangular system of linear equations which connects the coefficients $u_q(r)$ and S_{n+2m}:

$$\frac{\mathrm{j}}{4}\sum_{m=0}^{\infty} A_m S_{n+2m}^{(v)} = \sum_{q=0}^{\infty} u_q(r)\mathrm{j}^q(q+1)\left[\mathrm{e}^{\mathrm{j}kr}\frac{J_{q+1}(k)}{k}\right]^{(v)}, \tag{15}$$

where superscript (v) denotes the vth partial derivative with respect to k, evaluated at $k=0$.

Using standard series[36], the products of special functions can be written:

$$\mathrm{e}^{\mathrm{j}kr}\frac{J_{q+1}(k)}{k} = \frac{1}{\mathrm{j}^q}\sum_{t=0}^{\infty}(\mathrm{j}k)^{t+q}E_{t,q}(r), \tag{16}$$

$$E_{t,q}(r) = \frac{1}{2^{q+1}}\sum_{s=0}^{[t/2]}\frac{r^{t-2s}}{4^s s!(s+q+1)!(t-2s)!},$$

where $[t/2]$ is the largest integer less than or equal to $t/2$, and

$$\begin{aligned}\left(\frac{a}{r}\right)^{1/2} H_{n+1/2}^{(1)}(kr)J_{n+1/2}(ka) = {} & \left(\frac{r}{2}\right)^{2n+1}\sum_{t=0}^{\infty}\frac{k^{2t+2n+1}}{t!}\left(-\frac{r^2}{4}\right)^t V_{n,t}(a/r) \\ & -(-1)^n\mathrm{j}\sum_{t=0}^{\infty}\frac{k^{2t}}{t!}\left(-\frac{r^2}{4}\right)^t W_{n,t}(a/r),\end{aligned} \tag{17}$$

with the polynomials $V_{n,t}$ and $W_{n,t}$ given by:

$$V_{n,t}(x) = \sum_{s=0}^{t}\binom{t}{s}\frac{x^{2s+n+1}}{\Gamma(n+s+3/2)\Gamma(t-s+n+3/2)}, \tag{18a}$$

$$W_{n,t}(x) = \sum_{s=0}^{t}\binom{t}{s}\frac{x^{2s+n+1}}{\Gamma(n+s+3/2)\Gamma(t-s-n+1/2)}. \tag{18b}$$

Given the power series, the derivatives at $k = 0$ are readily found:

$$\mathrm{j}^q \frac{\partial^v}{\partial k^v} \left[\mathrm{e}^{\mathrm{j}kr} \frac{J_{q+1}(k)}{k} \right]_{k=0} = \begin{cases} 0, & v < q; \\ \mathrm{j}^v v! E_{v-q,q}(r), & v \geq q. \end{cases} \tag{19a}$$

$$\frac{\partial^v}{\partial k^v} \left[(a/r)^{1/2} H^{(1)}_{n+1/2}(kr) J_{n+1/2}(ka) \right]_{k=0} = \begin{cases} 0, & v = 2v'+1,\ v' < n; \\ \left(\frac{r}{2}\right)^{2n+1} \left(-\frac{r^2}{4}\right)^{v'-n} \frac{v!}{(v'-n)!} V_{n,v'-n}(a/r), & v = 2v'+1,\ v' \geq n; \\ -(-1)^n \mathrm{j} \frac{(2v')!}{v'!} \left(-\frac{r^2}{4}\right)^{v'} W_{n,v'}(a/r), & v = 2v'. \end{cases} \tag{19b}$$

Setting $v = 0, 1, \ldots$ yields an infinite lower triangular system of equations for $u_q(r)$:

$$\mathsf{E}\mathbf{U} = \mathbf{B}, \tag{20}$$

with $\mathbf{U} = [u_0\, u_1\, \ldots]^T$ and the elements of matrix E and vector $\mathbf{B}$ given by:

$$E_{vq} = \begin{cases} \mathrm{j}^v (q+1) v! E_{v-q,q}(r), & q \leq v; \\ 0, & q > v. \end{cases} \tag{21a}$$

$$B_v = \frac{\mathrm{j}}{4} \int_0^1 T_v(r,a) s_n(a)\, \mathrm{d}a \tag{21b}$$

where

$$T_v = (-1)^{n+v'} v! \left(\frac{r}{2}\right)^v \sum_{m=0}^{\infty} A_m \begin{cases} 0 & v = 2v'+1,\ v' < n+2m; \\ \frac{V_{n+2m,v'-n-2m}(a/r)}{(v'-n-2m)!} & v = 2v'+1,\ v' \geq n+2m; \\ -\frac{\mathrm{j}}{v'!} W_{n+2m,v'}(a/r) & v = 2v'. \end{cases} \tag{22}$$

Given a radial source term $s_n(a)$, Equation 20 can be solved to find the coefficients $u_q(r)$ of the equivalent line source modes. Since it is lower triangular, the first few values of u_q can be reliably estimated, although ill-conditioning prevents accurate solution for arbitrary large q.

D. Radiated field

From the relationship between the radial source term and the line source coefficients, some general properties of the acoustic field can be stated. The first result, already shown in previous work[24,25] is that, since the line source modes with $q+1 > k$ generate exponentially small fields, the acoustic field has no more than k degrees of freedom, in the sense that the radiated field is given by a weighted sum of the fields due to no more than k elementary sources. From Equation 20, this result can be extended.

The first extension comes from the fact that $B_{2v+1} \equiv 0$, for $v' < n$, on the right hand side of Equation 20. This means that u_q, $q = 2v' + 1$, is uniquely defined by the lower order coefficients with $q \leq 2v'$. The result is that the acoustic field of azimuthal order n, whatever might be its radial structure, has no more than $k - n$ degrees of freedom, whether in the near or far field.

A second extension comes from examination of Equation 20. The first few entries of the system of equations are:

$$\begin{bmatrix} 1/2 & 0 & 0 & 0 & \cdots \\ r/2 & 1/4 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & 0 & \cdots \end{bmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} B_0 \\ 0 \\ \vdots \end{pmatrix}, \tag{23}$$

resulting in the solution:

$$u_0 = 2B_0; \quad u_1 = -2ru_0 = -4rB_0, \tag{24}$$

so that the ratio of u_0 and u_1 is constant, for arbitrary $s_n(a)$. This means that low frequency sources of the same radius and azimuthal order generate fields which vary only by a scaling factor, since the higher order terms are exponentially small. Again, this result holds in the near and in the far field.

Finally, if we attempt to isolate a source $s_n(a)$ associated with a single line source mode, by setting $u_q \equiv 1$ for some q, with all other $u_q \equiv 0$, we find that the line modes must occur in pairs, since if $u_{2v'} \equiv 1$, $u_{2v'+1} \neq 0$, being fixed by the condition $B_{2v'+1} \equiv 0$, further reducing the number of degrees of freedom or, alternatively, worsening the conditioning of the inverse problem.

E. Comparison to far-field methods

An alternative analysis which is widely used in radiation prediction uses the far field approximations $R' \approx R - a\sin\phi\cos\psi$, $1/R' \approx 1/R$. On this approximation:

$$p_n \approx (-\mathrm{j})^n \frac{\mathrm{e}^{\mathrm{j}kR}}{2R} \int_0^1 J_n(ka\sin\phi) s_n(a) a \,\mathrm{d}a, \tag{25}$$

so that the radiated field is given by a Hankel transform of the radial source, with a dependence on the polar angle ϕ. In some sense, this can also be viewed as fixing a limit on the radiated information as in, for example, the use of ring sources to study coherence effects on jet noise[30,33], or as a spatial filter. The approach suffers, however, from its inability to give information on the structure of the near field which might be of use in understanding such experimental methods as near-field to far-field correlations[37]. The approach presented in this paper gives the radiated field, near and far, as the sum of products of two integrals. The first of these integrals $\mathcal{L}_q$ contains only radiation effects while the second u_q depends only on the source. The source and radiation terms are thus 'uncoupled', simplifying the problem of analysing the radiated field, without needing to make a far-field approximation.

III. RANDOM DISK SOURCE

The second problem considered is that of the noise radiated by a random disk source. This is a general problem for broadband noise from rotating systems and is also a model problem for jet noise, extending the random ring source problem which has been studied previously in order to examine the effects of source coherence on jet noise[33]. In the ring source analysis, the axial and radial extent of the jet were neglected in order to study the effect of the azimuthal structure of the jet on the far-field noise. Here, we develop a model which includes the radial and azimuthal terms in a model which is exact in the near and far fields.

The assumptions made are that the source terms are statistically stationary and that the statistical properties of the source are symmetric about the source axis. It will also be assumed that the non-dimensional wavenumber $k \lesssim 2$, which is a reasonable assumption for the frequency range of maximum noise level for a subsonic jet. The result derived is an expression for the cross-spectrum between the pressure at two points, which reduces to the power spectrum when the points coincide. The expression is quite general and, unlike previous formulae, does not require that the points be in the acoustic far field of the source.

The starting point is an expression for the pressure radiated from a source distributed over a unit disk:

$$p(r, \theta, z, t) = \int_0^1 \int_0^{2\pi} \frac{f(a, \psi, t - R/c)}{4\pi R} a \, \mathrm{d}\psi \, \mathrm{d}a, \tag{26}$$

from which the correlation between p measured at two points (r_1, θ_1, z_1) and (r_2, θ_2, z_2) is:

$$\overline{p(r_1, \theta_1, z_1, t)p(r_2, \theta_2, z_2, t + \tau)} = \frac{1}{(4\pi)^2} \int_0^1 \int_0^{2\pi} \int_0^1 \int_0^{2\pi} \frac{\overline{f(a_1, \psi_1, t - R_1/c) f(a_2, \psi_2, t - R_2/c + \tau)}}{R_1 R_2} a_1 a_2 \, \mathrm{d}\psi_1 \, \mathrm{d}a_1 \, \mathrm{d}\psi_2 \, \mathrm{d}a_2. \tag{27}$$

Fourier transforming to find the cross spectrum between the points:

$$W_{12}(f) = \frac{1}{(4\pi)^2} \int_0^1 \int_0^{2\pi} \int_0^1 \int_0^{2\pi} \frac{\mathrm{e}^{\mathrm{j}k(R_2 - R_1)}}{R_1 R_2} F_{12}(a_1, \psi_1; a_2, \psi_2) a_1 a_2 \, \mathrm{d}\psi_1 \, \mathrm{d}a_1 \, \mathrm{d}\psi_2 \, \mathrm{d}a_2, \tag{28}$$

$$F_{12}(a_1, \psi_1; a_2, \psi_2) = \int_{-\infty}^{\infty} \overline{f(a_1, \psi_1, t) f(a_2, \psi_2, t + \tau)} \mathrm{e}^{\mathrm{j}2\pi f \tau} \, \mathrm{d}\tau, \tag{29}$$

where F_{12} is the correlation between the source at two points (a_1, ψ_1) and (a_2, ψ_2).

On the assumption of axial symmetry, the source correlation can depend only on the angular separation between two points $\psi_2 - \psi_1$, so that F_{12} and W_{12} can be expanded in

Fourier series in azimuth:

$$F_{12}(a_1, \psi_1; a_2, \psi_2) = \sum_{m=-\infty}^{\infty} F_{12}^{(m)}(a_1, a_2)\mathrm{e}^{\mathrm{j}m(\psi_2-\psi_1)},$$
$$W_{12}(r_1, \theta_1, z_1; r_2, \theta_2, z_2) = \sum_{m=-\infty}^{\infty} W_{12}^{(m)}(r_1, z_1; r_2, z_2)\mathrm{e}^{\mathrm{j}m(\theta_2-\theta_1)},$$

with:

$$W_{12}^{(m)}(r_1, \theta_1, z_1; r_2, \theta_2, z_2) = \frac{1}{(4\pi)^2}\int_0^1\int_0^{2\pi}\frac{\mathrm{e}^{-\mathrm{j}(kR_1+m\psi_1)}}{R_1} \times \left[\int_0^1\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{j}(kR_2+m\psi_2)}}{R_2}F_{12}^{(m)}(a_1, a_2)a_2\,\mathrm{d}\psi_2\,\mathrm{d}a_2\right]a_1\,\mathrm{d}\psi_1\,\mathrm{d}a_1. \tag{30}$$

Transforming to the equivalent line source form, as above:

$$\frac{1}{4\pi}\int_0^1\int_0^{2\pi}\frac{\mathrm{e}^{\mathrm{j}(kR_2+m\psi_2)}}{R_2}F_{12}^{(m)}(a_1, a_2)a_2\,\mathrm{d}\psi_2\,\mathrm{d}a_2 = \sum_{q_2=0}^{\infty}u_{q_2}(r_2, a_1)\mathcal{L}_{q_2}(k, r_2, z_2),$$

which results in:

$$W_{12}^{(m)} = \sum_{q_2}\mathcal{L}_{q_2}(k, r_2, z_2)\frac{1}{4\pi}\int_0^1\int_0^{2\pi}\frac{\mathrm{e}^{-\mathrm{j}(kR_1+m\psi_1)}}{R_1}u_{q_2}(r_2, a_1)a_1\,\mathrm{d}a_1\,\mathrm{d}\psi_1,$$
$$= \sum_{q_1}\sum_{q_2}u_{q_1}(r_1, r_2)\mathcal{L}_{q_2}(k, r_2, z_2)\mathcal{L}_{q_1}^{*}(k, r_1, z_1),$$

where $*$ denotes complex conjugation. The coefficients u_{q_1} are found by treating u_{q_2} as the radial source in the (a_1, ψ_1) integral. Up to this point, the analysis is exact but to simplify the development, we introduce the assumption $k < 2$ so that only modes of order 0 and 1 contribute to the acoustic field.

Solving Equation 20 yields:

$$u_0 = 2B_0,$$
$$u_1 = 4(B_1 - rB_0),$$

with:

$$B_0 = \int_0^1 s_m(a)w_m(a/r)\,\mathrm{d}a,$$
$$B_1 = \int_0^1 s_m(a)v_m(a)\,\mathrm{d}a,$$

where:

$$w_m(x) = \frac{1}{2\pi}\sum_{q=0}^{\infty}\frac{1}{q!}\frac{(2m+2q-1)!!(2q-1)!!}{2^q(2m+2q)!!}x^{m+2q+1},$$

$$v_m(x) = \begin{cases} x/2\pi, & m=0; \\ 0, & m\neq 0. \end{cases}$$

The result is that the mth azimuthal component of the cross-spectrum between two field points for $k \lesssim 2$ is given by:

$$\begin{aligned} W_{12}^{(m)} &= \mathcal{L}_0(k,r_2,z_2)\left[u_{00}\mathcal{L}_0^*(k,r_1,z_1) + u_{01}\mathcal{L}_1^*(k,r_1,z_1)\right] \\ &\quad + \mathcal{L}_1(k,r_2,z_2)\left[u_{10}\mathcal{L}_0^*(k,r_1,z_1) + u_{11}\mathcal{L}_1^*(k,r_1,z_1)\right], \end{aligned} \tag{31}$$

where

$$u_{00} = 4\int_0^1\int_0^1 F_{12}^{(m)}(a_1,a_2)w_m(a_1/r_1)w_m(a_2/r_2)\,\mathrm{d}a_1\,\mathrm{d}a_2, \tag{32a}$$

$$u_{01} = 8\int_0^1\int_0^1 F_{12}^{(m)}(a_1,a_2)w_m(a_2/r_2)[v_m(a_1) - r_1 w_m(a_1/r_1)]\,\mathrm{d}a_1\,\mathrm{d}a_2, \tag{32b}$$

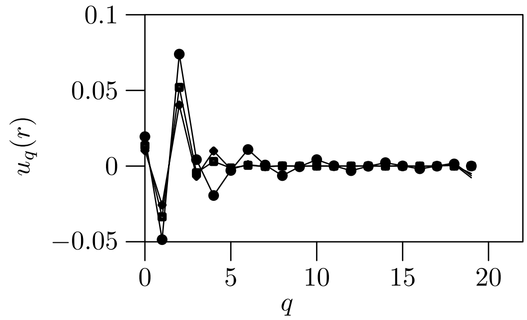
$$u_{10} = 8\int_0^1\int_0^1 F_{12}^{(m)}(a_1,a_2)w_m(a_1/r_1)[v_m(a_2) - r_2 w_m(a_2/r_2)]\,\mathrm{d}a_1\,\mathrm{d}a_2, \tag{32c}$$

$$u_{11} = 16\int_0^1\int_0^1 F_{12}^{(m)}(a_1,a_2)[v_m(a_1) - r_1 w_m(a_1/r_1)][v_m(a_2) - r_2 w_m(a_2/r_2)]\,\mathrm{d}a_1\,\mathrm{d}a_2. \tag{32d}$$

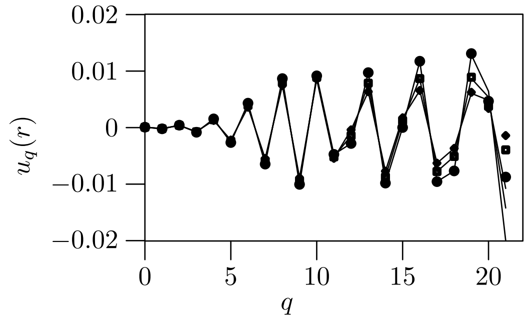
The modal coefficients of the cross-spectrum of a jet noise field, at the wavenumbers of interest in practice, are thus fixed by four coefficients, functions of the radial separations r_1 and r_2, which are weighted integrals of the source cross-spectrum.

IV. RESULTS

To check the analyses presented above, and to consider their implications, some results are presented for tonal and random disk sources.



$n = 2$



$n = 16$

FIG. 3. Line source mode coefficients computed using the method of section II.C (solid lines) and directly from analytical formulae (symbols) for $r = 5/4$, $s = a^\gamma$, $\gamma = 0$ (circles), $\gamma = 2$ (squares) and $\gamma = 4$ (diamonds) for $n = 2$ and 16.

A. Line source coefficient evaluation

The first results are a check on the calculation of the coefficients $u_q(r)$ comparing those computed using Equation 20 and those computed directly from exact closed-form expressions[8] for $K(r, r_2)$ in the case when the radial source term is a monomial in radius $s_n = a^\gamma$. Figure 3 compares the two sets of coefficients for $\gamma = 0, 2, 4$, with the plots terminated at a value of q where the difference between the two sets of results becomes noticeable, $q \approx 20$. This gives an indication of the effect of the ill conditioning of Equation 20. For $q \lesssim 20$, the computed values of u_q are reliable. It is noteworthy that for small q, the coefficients are practically equal for all values of γ so that for low frequency radiation, the

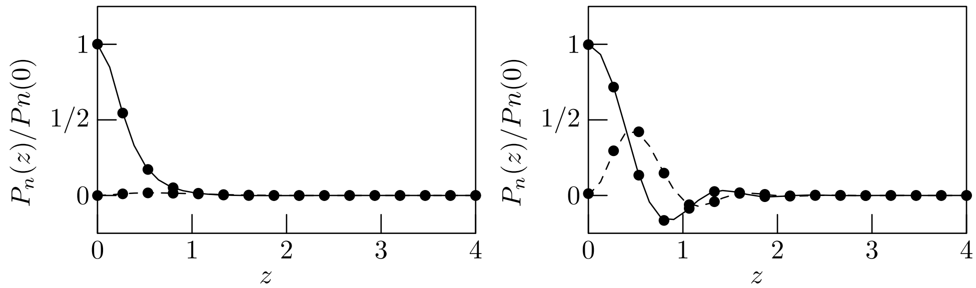


FIG. 4. Acoustic field predicted by full numerical integration (lines) and line source summation (symbols) for $n = 8$, $r = 5/4$. Real part shown solid, imaginary part dashed. Left hand plot: $k = 5$; right hand plot: $k = 9$.

radiated fields will be practically indistinguishable.

B. Tonal radiation from a disk

As a test of the ability to predict radiation from tonal sources, we present data for the acoustic field of a disk source with $n = 8$, $s_n = J_n(\chi_{n1}a)$, where χ_{n1} is the first non-zero root of $J_n(x)$. Full numerical integration and line source calculations have been performed for two wavenumbers, $k = 5$ and $k = 9$, respectively. The first 11 line source modes were used in each case, with the modal coefficients being found from Equation 20. Sample results are shown in Figure 4, with the data scaled on the value at $z = 0$, and it is clear that the line source model gives accurate results, even when only a subset of the modes is used. From these, and other, data, the reliability of the model for tonal sources is confirmed.

C. Low frequency random source

In order to generate data to test the random disk source model, we must assume a form for the source correlation. Michalke[33] gives a form suitable for a ring source which meets the symmetry requirements laid out above. With the addition of radially varying terms, Michalke's expression can be extended:

$$F_{12}(a_1, \psi_1; a_2, \psi_2) = F(a_1)F(a_2)\exp\left[-\frac{(a_1 - a_2)^2}{\beta^2}\right]\exp\left[-\frac{1 - \cos(\psi_1 - \psi_2)}{\alpha^2}\right] \tag{33}$$

with α being an azimuthal length scale and β controlling the correlation in radius. Equation 33 can be interpreted as the product of the local source strengths $F(a_1)$ and $F(a_2)$ with a coherence function, given by the exponentials, which is symmetric in source position and has unit value when the source points coincide.

The azimuthal components of F_{12} can be found from mathematical tables[33,36] as:

$$F_{12}^{(m)} = F(a_1)F(a_2)\exp\left[-\frac{(a_1 - a_2)^2}{\beta^2}\right]\exp\left[-\frac{1}{\alpha^2}\right] I_m(1/\alpha^2) \tag{34}$$

where I_m is a modified Bessel function.

Figure 5 shows sample results for the predicted cross spectrum between pressure at a point $r_1 = 5/4$, $z_1 = 0$ and $r_2 = 5$, $0 \leq z_2 \leq 8$, for a disk source of unit strength. The reference results are the cross-spectra found by full numerical integration of Equation 30. The first comparison is with Equation 31 where the functions $\mathcal{L}_q$ have been evaluated by numerical integration. In the second comparison, the functions $\mathcal{L}_q$ have been evaluated using the exact in-plane result, Equation 9, for $z_1 = 0$, and the far-field approximation, Equation 10, for $r_2 = 5$, $0 \leq z_2 \leq 8$. All data have been scaled on the numerically evaluated cross-spectrum at $z_2 = 0$.

The first obvious point from Figure 5 is the similarity of the cross-spectra, even for quite large variations in the parameter β: changing m changes the form of the radiated field, as might be expected, but changes in the source correlation have little effect on the radiated field. The second point is that the line source approach gives very good results, even for $k = 2$ where, in principle, the approximation used should start to break down. Finally, although computational efficiency is not the primary aim of the method, we note that the line source approach converts the four dimensional integral, Equation 30, required at each

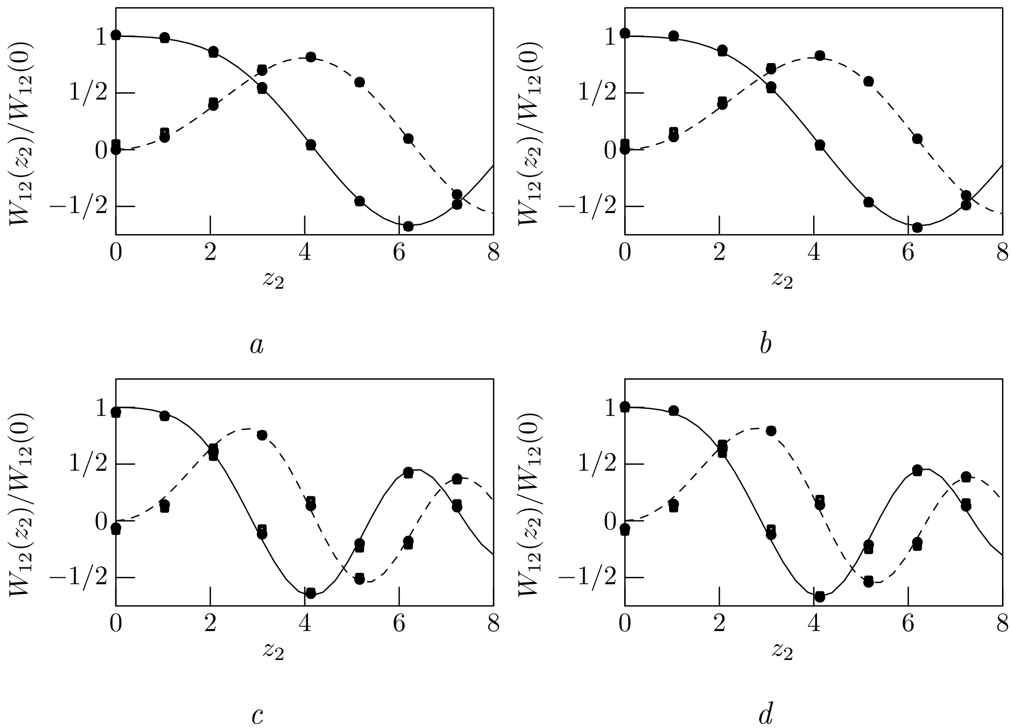


FIG. 5. Cross-spectrum $W_{12}^{(m)}(z_2)$ scaled on $W_{12}^{(m)}(0)$, $r_1 = 5/4$, $z_1 = 0$, $r_2 = 5$. Numerical evaluation shown as solid line (real part) and dashed line (imaginary part); Equation 31 with numerical evaluation of $\mathcal{L}_q$ shown as circles; Equation 31 with far-field approximation shown as squares. Parameters: a: $k = 1$, $m = 0$, $\alpha = 1$, $\beta = 100$; b: $k = 1$, $m = 0$, $\alpha = 3$, $\beta = 0.01$; c: $k = 2$, $m = 1$, $\alpha = 1$, $\beta = 100$; d: $k = 2$, $m = 1$, $\alpha = 3$, $\beta = 0.01$.

field point, into four two-dimensional integrals which are functions of radial separation only, Equation 32, and four one-dimensional integrals $\mathcal{L}_i$, giving a large saving in calculation time.

D. Noise cancellation by an equivalent source

One implication of the results of this paper is that it is not possible to tell different sources apart if, to within a scaling factor, they have same line source coefficients u_q, for those line source modes with $q \leq k$. Even without considering errors from background noise or other causes, this is equivalent to a condition on weighted integrals of the radial source

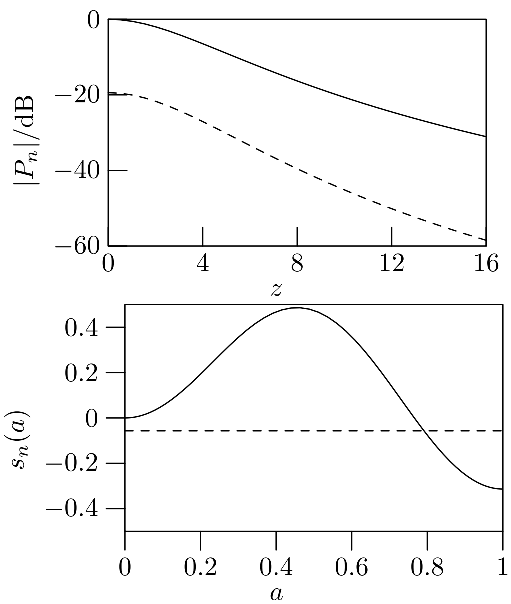


FIG. 6. Cancellation effects for radial source terms with $n = 2$, $k = 1$, $r = 5$: top figure radiated field from original $s_n(a)$ (solid) and modified source $s_n(a) - \zeta s'_n(a)$ (dashed); bottom row source terms $s_n(a)$ (solid) and $\zeta s'_n(a)$ dashed.

s_n. Any sources which yield the same, or nearly the same, integrals B_v for $v < V$, with V a positive integer, in Equation 21, will have indistinguishable acoustic fields for $k < V$.

This conclusion can also be read as a statement about noise cancellation, such as in active noise control. The acoustic field of a given source can be cancelled by any source which has the same set of line source coefficients.

An example of this cancellation is shown in Figure 6. The original field is generated using a source term $s_n(a)$ and the line source coefficients u_q of s_n are calculated. A secondary source term $s'_n(a)$ is generated and its line source coefficients u'_q are computed. The secondary source s'_n is then scaled by a factor $\zeta = u_0/u'_0$. As a test, $s_n = J_n(\chi_{n2}a)$, with χ_{n2} the second extremum of $J_n(x)$, and $s'_n \equiv 1$. The first plot in Figure 6 shows the field due to s_n and that radiated by $s_n - \zeta s'_n$. The large reduction in far-field noise, about 20dB, is clear over the whole range of z shown in the figure.

The source terms are shown in the second plot of Figure 6. The secondary source $\zeta s_n'$ is of much smaller amplitude than s_n even though it generates a nearly-equivalent field: the effect of matching the line source coefficients has been to produce a field which is very similar to that of the original source, even though the source distributions are quite different in form and in amplitude.

In the analysis of Section II.A, it was found that for large line source order q, the acoustic field is exponentially small when $k < q + 1$. In the results shown in Figure 6, $n = 2$ so that the field from sources with $q > 1$ is not exponentially small. Despite this, the cancellation is still effective throughout the far field. In other results for $r \approx 1$, i.e. near the source, it was found that there can be regions where the noise increases slightly, although over most of the range of axial displacement, the noise was still reduced by a large amount and the maximum was reduced by 20dB.

A question which will not be dealt with in detail here is how a controller based on this analysis might be implemented for tonal sources. Given that the cancelling source can have a very simple form, needing only its amplitude adjusted to give cancellation of the primary field, it is possible that quite simple systems might be usable as control inputs, as long as they have the required azimuthal dependence and their frequency can be adjusted to match that of the source. A simple controller would be a set of speakers mounted on a ring in the source plane although the precise form of such a controller is a matter for future work.

V. CONCLUSIONS

The radiation properties of disk sources of arbitrary radial variation have been analyzed to establish the part of the source which radiates into the acoustic field, without recourse to a far field approximation. Limits have been established on the number of degrees of freedom of the part of the source which radiates and the implications of these limits have been discussed for the problems of rotor noise and studies of source mechanisms in jets. The analysis has been developed for tonal and for random sources, with implications for applications in active control of noise from rotors and experimental analysis of jet noise sources.

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