PHD

Left 3-Engel Elements

Hadjievangelou, Anastasia

Award date:
2022

Awarding institution:
University of Bath

Left 3-Engel Elements

submitted by

Anastasia Hadjievangelou

for the degree of Doctor of Philosophy

of the

University of Bath

Department of Mathematical Sciences

2022

Acknowledgements

I would like to thank my supervisor Gunnar Traustason for his continuous support, advice and input not only into this project, but generally throughout my 4 years of PhD research here at the University of Bath. He is one of the people who inspired me in the first place to pursue a PhD in Algebra and has continuously kept doing so. Thank you for sharing with me your knowledge and guidance.

I am grateful to Marialaura Noce for our fruitful collaboration during her 6 months PostDoc at the University of Bath. I thank her for her continuous friendship and kindness to always help young researchers.

I am grateful to the EPSRC and to Mandy Norton for their financial support.

I would like to acknowledge all members of 4 West 2.15, past and present. Their friendship and support has helped me through the difficult parts of my PhD. I am grateful to Paolo, Marco and Giacomo for meeting them and for being able to call them my friends.

Finally, I would like to thank my family for their support, encouragement and investment in my education. They have given me the opportunity to follow what I love and for that I am very grateful.

'Πολέμιον ανθρώποις αυτοί εαυτοίς.'

Contents

Abstract

In this thesis we focus on results concerning left 3-Engel elements. We will first record some basic definitions and elementary results about groups and Lie algebras. We then focus on Engel groups and Engel elements. In particular, we generalise the construction given in [18] by giving an infinite family of examples of a locally finite 2-group G containing a left 3-Engel element x where $\langle x \rangle^G$, the normal closure of x in G, is not nilpotent. The construction is based on a family of Lie algebras that are of interest in their own right given in [21] and make use of a classical theorem of Lucas, regarding when $\binom{m}{n}$ is even. We then proceed to the main result of this thesis. We show that in the case of any odd prime p, we can give an example of a locally finite p-group G containing a left 3-Engel element x where $\langle x \rangle^G$ is not nilpotent.

Chapter 0

Introduction

0.1 Overview

In this thesis we study left 3-Engel elements in locally finite p-groups. Let G be a group and $x \in G$ an element in the group. We call the element x a *left 3-Engel element* in G, if for each $g \in G$ we have $[g,_3 x] := [[[g, x], x], x] = 1$. One can define *right 3-Engel elements* in a similar manner, that is if for each $g \in G$ we have $[x,_3 g] := [[[x, g], g], g] = 1$.

Engel elements and Engel groups are of great interest because the Engel problems share a great similarity to the Burnside problems. We will go through the theory of Engel groups and also many well known results surrounding Engel theory as well as famous open problems. We shall look at the corresponding Burnside problems and understand how they are connected, but moreover how can one use Engel theory to understand the Burnside problems.

The main focus of this thesis is based on a famous result of M. Newell. In [17] he proved that if x is a right 3-Engel element in G then the normal closure of x, meaning $\langle x \rangle^G$ is nilpotent, of class at most 3. The natural question arises whether the analogous result holds for left 3-Engel elements. If you consider the analogue situation for left 2-Engel elements it is easy to show that $\langle x \rangle^G$ is nilpotent, in particular it is abelian. If however x is left 3-Engel element we show in this thesis that the normal closure of x need not to be nilpotent, by giving examples of locally finite p-groups, for both p odd and even, containing a left 3-Engel element x, such that $\langle x \rangle^G$ is not nilpotent. This is in sharp contrast with M Newell's result.

We now include a brief overview of each chapter:

- In Chapter 0 we recall the basic definitions and elementary results about groups and Lie algebras. We give as an example the adjoint homomorphism, which will be used widely in this thesis. We go through the main commutator formulae, which will come useful later in calculations. We proceed by defining nilpotent groups and their nilpotency class and by extension soluble groups, as well as nilpotent and enveloping algebras.

- In Chapter 1 we dive into the realm of Engel Theory. We introduce the reader to Engel groups and Engel elements but also the main problems in Engel Theory. We connect the Engel Problems to Burnside Problems through an interesting correlation between them and proceed by mentioning some major open problems in Engel theory. We will also discover how could one use Engel elements to gain a better understanding of the Burnside problems, in particular groups of exponent 8. We go deeper into 3-Engel and 4-Engel groups while also mentioning what is at the moment known for 5-Engel groups and open problems. Then we explore more details on right and left Engel elements, with emphasis on the latter. We define the Hirsch-Plotkin radical and explain its connection to left Engel elements. Moreover, we explain the advances made towards answering the longstanding open question regarding whether left 3-Engel elements are contained in the Hirsch-Plotkin Radical. Lastly, we give the main motivation behind this thesis and close up by explaining the technique of lifting the Lie algebra to the group.

- In Chapter 2 we give an infinite family of examples that generalise the construction given in [18] of a locally finite 2-group G containing a left 3-Engel element x where $\langle x \rangle^G$ is not nilpotent. The construction is based on a family of Lie algebras that are of interest in their own right given in [21] and make use of a classical theorem of Lucas. We will make use of results from Chapter 1.

- In Chapter 3 we give an example of a locally finite p-group G containing a left 3-Engel element x where $\langle x \rangle^G$ is not nilpotent, for any odd prime p. The odd case turns out to be more involved and the construction in many ways quite different from the $p = 2$ case.

- In Chapter 4 we discuss some open problems and further areas for investigation.

0.2 Main Results

The main results of the thesis are as follows:

- We give an infinite family of locally finite 2-groups G with a left 3-Engel element x, such that the normal closure of x in G, meaning $\langle x \rangle^G$, is not nilpotent. These generalize the construction given in [18].

- The main focus of this thesis is the following result. For any odd prime p, we give an example of a locally finite p-group G containing a left 3-Engel element x, such that $\langle x \rangle^G$ is not nilpotent.

0.3 Preliminaries

In this section we will record some basic definitions and elementary results about groups and Lie algebras. We firstly dive into Lie theory to define the nilpotent Lie algebras. Then we proceed by establishing the main commutator formulae. Lastly, we study the classes of nilpotent and solvable groups. These can be constructed from abelian groups by repeatedly forming extensions, the process by which finite groups are built from simple groups.

0.3.1 Lie Theory

In this subsection we recap the basic vocabulary of Lie algebras. We focus on nilpotent Lie algebras and their nilpotency class which will be used throughout this thesis.

Definition 0.3.1. Let $\mathbb{F}$ be a field. A *Lie algebra over* $\mathbb{F}$ is an $\mathbb{F}$-vector space L, equipped with a bilinear map, the *Lie product*

$$L \times L \to L, \quad (x, y) \mapsto x \cdot y,$$

that is multiplication on the right by y satisfying the following properties:

$$x \cdot x = 0 \text{ for all } x \in L, \tag{L1}$$

$$(x \cdot y) \cdot z + (y \cdot z) \cdot x + (z \cdot x) \cdot y = 0 \text{ for all } x, y, z \in L. \tag{L2}$$

Throughout this thesis we will assume that, when dealing with group commutators or Lie products, these are left normed. Condition (L2) is known as the *Jacobi identity.* Moreover, since the Lie product is bilinear condition (L1) implies

$$x \cdot y = -y \cdot x \text{ for all } x, y \in L.$$

Recall that if we have an associative algebra over $\mathbb{F}$ then we can turn it into a Lie algebra over $\mathbb{F}$ via $[x, y] := xy - yx$. Here $[x, y]$ is sometimes called the *Lie bracket.*

Definition 0.3.2. An *ideal* of a Lie algebra L is defined to be a subspace I of L such that

$$x \cdot y \in I \text{ for all } x \in L, y \in I.$$

Example 0.3.3. An important example of an ideal is the *centre* of L, defined by

$$Z(L) := \{x \in L : x \cdot y = 0 \text{ for all } y \in L\}.$$

Definition 0.3.4. If L_1 and L_2 are Lie algebras over a field $\mathbb{F}$, we say that a map $\phi : L_1 \to L_2$ s a *homomorphism* if ϕ is a linear map and

$$\phi(x \cdot y) = \phi(x) \cdot \phi(y) \text{ for all } x, y \in L_1.$$

Definition 0.3.5. Let V a finite-dimensional vector space over $\mathbb{F}$. We define $gl(V)$ as the set of all linear maps from V to V . This again is a vector space over $\mathbb{F}$ and it becomes a Lie algebra, defined as the *General Linear Lie algebra*, if the Lie bracket is given by the Lie bracket of endomorphisms. In other words if,

$$[x, y] := x \circ y - y \circ x,$$

for all $x, y \in \mathrm{gl}(V)$, where $\circ$ denotes the composition of maps.

Example 0.3.6. Throughout the thesis we will make use of the *adjoint homomorphism*

$$\mathrm{ad} : L \to \mathrm{gl}(L)$$

defined by $(\mathrm{ad}(x))(y) := x \cdot y$ for $x, y \in L$, that is $\mathrm{ad}(x)$ is a multiplication by x on the left. Note also that from the Jacobi identity we have

$$\mathrm{ad}(x \cdot y) = \mathrm{ad}\, x \circ \mathrm{ad}\, y - \mathrm{ad}\, y \circ \mathrm{ad}\, x = [\mathrm{ad}(x), \mathrm{ad}(y)],$$

for all $x, y \in L$. Therefore we see that ad is indeed a Lie homomorphism.

Definition 0.3.7. We define the *lower central series* of a Lie algebra L to be the series with terms

$$L^1 = L' = L \cdot L \text{ and } L^k = L^{k-1} \cdot L \text{ for } k \geqslant 2.$$

Then, $L \supseteq L^1 \supseteq L^2 \supseteq \cdots$. Since the product of ideals is an ideal as a consequence of the Jacobi identity, L^k is even an ideal of L. Notice that $L^k/L^{k+1} \subseteq Z(L/L^{k+1})$.

Definition 0.3.8. The Lie algebra L is said to be *nilpotent* if for some $m \geqslant 1$ we have

$L^m = 0$. In other words, if

$$x_1 \cdot x_2 \cdot \cdots x_{m-1} \cdot y = \mathrm{ad}(x_1)\,\mathrm{ad}(x_2) \cdots \mathrm{ad}(x_{m-1})(y) = 0,$$

for all $x_1, \ldots, x_{m-1}, y \in L$ so that $\mathrm{ad}(x_1) \cdots \mathrm{ad}(x_{m-1}) = 0$.

We now give the definition of an enveloping algebra.

Definition 0.3.9. Let L be a Lie algebra and M an associative algebra. For the purposes of this thesis we want the associative algebra to contain the identity. For M to be an enveloping algebra of L, we require an injective Lie algebra homomorphism $\phi : L \to M$, where M is viewed as a Lie algebra with respect to the Lie bracket and such that M is generated by $\phi(L)$ as an associative algebra. In particular, this means that we require $\phi(xy) = [\phi(x), \phi(y)] = \phi(x)\phi(y) - \phi(y)\phi(x)$, for all $x, y \in L$. The universal enveloping algebra of L, denoted by $U(L)$, is the unique largest such M.

The enveloping algebra $U(L)$ possesses the following universal property.

Definition 0.3.10. The map ϕ extends to a unique unital algebra homomorphism $\hat{\phi} : U(L) \to M$, such that $\phi = \hat{\phi} \circ h$, where $h : L \to U(L)$ is an injective Lie homomorphism. Since $U(L)$ is generated by elements of L, the map $\hat{\phi}$ is uniquely determined by $\hat{\phi}(x_i \cdots x_n) = \phi(x_1) \cdots \phi(x_n)$ for $x_i \in L$.

Remark. Definition 0.3.9 and Definition 0.3.10 are equivalent.

Remark. Consider the special case when the center of L is trivial. In this case we use the embedding $\phi : L \to \mathrm{ad}(L)$, which is an injective Lie algebra homomorphism, since $\mathrm{Ker}\phi = 0$, in such a way that the Lie product is preserved. Thus we have the property

$$\mathrm{ad}(x \cdot y) = [\mathrm{ad}(x), \mathrm{ad}(y)],$$

for all $x, y \in L$.

0.3.2 Commutators and Commutator Subgroups

In this subsection we develop a systematic calculus of commutators. We go through some basic commutator definitions and we establish the main commutator formulae.

Definition 0.3.11. We define recursively commutators of *weight* $1, 2, \ldots$ in variables $x_1, x_2, \ldots$ as formal bracket expressions. The letters $x_1, x_2, \ldots$ are commutators of weight 1; if c_1 and c_2 are commutators of weight w_1 and w_2, then $[c_1, c_2]$ is a commutator of weight $w_1 + w_2$. The *multiweight* of a commutator is the collection of the partial weights in particular variables, which are defined recursively in an obvious way.

Example 0.3.12. The commutator $[[x_1, x_2], x_1]$ is of weight 3 and in particular of weight 1 in x_2 and of weight 2 in x_1.

Definition 0.3.13. A commutator $[\ldots[[a_1, a_2], a_3]\ldots, a_k]$ is called *simple* and is denoted by $[a_1, a_2, \ldots, a_k]$.

Lemma 0.3.14. Let a, b, c be elements in any group G. The following commutator formulae hold:

(a) $a^b = a[a, b]$,

(b) $[a, b]^{-1} = [b, a]$,

(c) $[a, b]^c = [a^c, b^c]$,

(d) $[a, b] = [b, a^{-1}]^a$,

(e) $[ab, c] = [a, c]^b[b, c] = [a, c][a, c, b][b, c]$,

(f) $[a, bc] = [a, c][a, b]^c = [a, c][a, b][a, b, c]$.

It is useful to be able to form commutators of subsets as well as elements. Let $X_1, X_2, \ldots$ be nonempty subsets of a group G.

Definition 0.3.15. We define the *commutator subgroup* of X_1 and X_2 to be

$$[X_1, X_2] = \langle [x_1, x_2] \mid x_1 \in X_1, x_2 \in X_2 \rangle.$$

More generally, let

$$[X_1, \ldots, X_n] = [[X_1, \ldots, X_{n-1}], X_n],$$

where $n \geqslant 2$ and $[X_1] = \langle X_1 \rangle = X_1$.

Definition 0.3.16 (Hall-Witt Identity)**.** For any elements a, b, c in any group G we have

$$[a, b^{-1}, c]^b[b, c^{-1}, a]^c[c, a^{-1}, b]^a = 1.$$

This enables us to have the following important lemma.

Lemma 0.3.17 (Three Subgroup Lemma)**.** Let A, B, C be subgroups and N a normal subgroup in a group G. If $[[A, B], C] \leqslant N$ and $[[B, C], A] \leqslant N$, then $[[C, A], B] \leqslant N$.

0.3.3 Series

Having established the properties of commutators and commutator subgroups that hold in any group, we now go through the definitions of upper and lower series which we will use to define later nilpotent and solvable groups.

Definition 0.3.18. Consider the chain of nested groups

$$1 = G_0 \leqslant G_1 \leqslant G_2 \leqslant \cdots \leqslant G_n = G.$$

We define this to be a *series* of G of length n. If all of the G_i are normal in G, the series is *normal*; if each G_i is a normal subgroup of G_{i+1}, the series is *subnormal*. The sections G_{i+1}/G_i of a subnormal series are the *factors* of the series. The series is *abelian* if each factor G_{i+1}/G_i is abelian.

Definition 0.3.19. If every strictly ascending chain of subgroups

$$G_1 \subset G_2 \subset G_3 \subset \cdots$$

in a group G is finite then G is said to satisfy the *maximal condition* (or *ascending chain condition*). In other words, a group G satisfies the maximal condition if and only if all its subgroups are finitely generated.

Definition 0.3.20. A series

$$1 = G_0 \leqslant G_1 \leqslant G_2 \leqslant \cdots \leqslant G_n = G$$

is *central* if $[G_i, G] \leqslant G_{i+1}$ for all $i = 1, 2, \ldots, n-1$. It follows that then each G_i is normal in G.

0.3.4 Upper and Lower Central Series

In this subsection we introduce a natural way of generating a descending sequence of commutator subgroups of a group, by repeatedly commuting with G.

Definition 0.3.21. The terms of the *lower central series* of a group G,

$$\gamma_1(G) \geqslant \gamma_2(G) \geqslant \cdots$$

are recursively, the commutator subgroups $\gamma_1(G) = G, \gamma_{k+1}(G) = [\gamma_k(G), G]$.

Each $\gamma_k(G)$ is *fully-invariant* in G, that is for any endomorphism ϕ of G we have $\phi(\gamma_k(G)) \subseteq \gamma_k(G)$. Since this holds for all endomorphisms ϕ then it is also true for all automorphisms ψ of G, hence $\gamma_k(G)$ is also a *characteristic subgroup.* In particular, $\gamma_i(G) \geqslant \gamma_{i+1}(G)$, for all i. In general, the lower central series does not reach 1, however it must terminate so there exists a limit ordinal α such that

$$\gamma_\alpha(G) = \bigcap_{\beta<\alpha} \gamma_\beta(G) = \gamma_{\alpha+1}(G) = etc.$$

Definition 0.3.22. The terms of the *upper central series* of a group G

$$1 = \zeta_0(G) \leqslant \zeta_1(G) \leqslant \zeta_2(G) \leqslant \cdots,$$

are defined recursively: $\zeta_1(G) = Z(G)$ is the center of G, and $\zeta_{k+1}(G)$ is the full inverse image of $Z(G/\zeta_k(G))$ in G. In particular, $\zeta_{k+1}(G)/\zeta_k(G) = Z(G/\zeta_k(G))$.

Each $\zeta_k(G)$ is characteristic, but not necessarily fully-invariant in G. This series need not reach G, but if it does the group is then called *hypercentral.*

We write $\gamma_i = \gamma_i(G)$ and $\zeta_i = \zeta_i(G)$ for short throughout the section.

0.3.5 Nilpotent Groups

In this subsection we give the equivalent definitions of nilpotent groups.

Theorem 0.3.23. For a group G, the following are equivalent:

(a) $\gamma_{k+1} = 1$;

(b) G has central series of finite length k;

(c) $[g_1, g_2, \ldots, g_{k+1}] = 1$ for all $g_i \in G$.'

(d) $\zeta_k = G$.

Definition 0.3.24. A group G satisfying the conditions in Theorem 0.3.23 is called *nilpotent of class* $\leqslant k$. The least such number k is called the *nilpotency class* of G.

Example 0.3.25.

1. A nilpotent group of class 0 has order 1, while nilpotent groups of class at most 1 are abelian.

2. The direct product of two nilpotent groups is again nilpotent.

3. All finite p-groups are nilpotent.

Corollary 0.3.26. Suppose that G is a nilpotent group. Then

(a) $\gamma_i > \gamma_{i+1}$ unless $\gamma_i = 1$;

(b) $\zeta_i < \zeta_{i+1}$ unless $\zeta_i = G$.

(c) The nilpotent class of G = the length of the upper central series = the length of the lower central series.

0.3.6 Soluble Groups and Derived Series

In this subsection we give the equivalent definitions of soluble groups.

Definition 0.3.27. The terms of the *derived series* of a group G are defined recursively

$$G = G^{(0)} \geqslant G^{(1)} \geqslant G^{(2)} \geqslant \cdots,$$

where $G^{(1)} = [G, G]$ and $G^{(k+1)} = [G^{(k)}, G^{(k)}]$.

Note that $G^{(k)}$ is fully-invariant, and hence a characteristic subgroup. Also, this series need not reach 1 or even terminate. Of course all the factors $G^{(k)}/G^{(k+1)}$ are abelian groups.

We proceed with the equivalent definitions of soluble groups.

Theorem 0.3.28. For a group G, the following are equivalent:

(a) $G^{(d)} = 1$;

(b) G has a normal series of length d with abelian factors;

(c) G has a subnormal series of length d with abelian factors.

Definition 0.3.29. A group G satisfying the conditions in Theorem 0.3.28 is called *soluble of derived length* $\leqslant d$. The least number d is the *derived length* of G. Groups of derived length 2 are often called *metabelian*.

Remark. Every finite group G has a unique maximal normal soluble subgroup, namely the product S of all normal soluble subgroups, the *soluble radical* of G.

Example 0.3.30. 1. A group G has derived length 0 if and only if it has order 1.

2. Groups with derived length at most 1 are just the abelian groups.

3. The product of two normal soluble subgroups of a group is soluble.

4. Nilpotent groups are clearly soluble, however an example of a non-nilpotent soluble group is S_3, because its center is trivial.

Chapter 1

Engel Elements in Groups

1.1 Engel groups and Engel elements

In this Chapter we dive into the realm of Engel Theory. We introduce Engel groups and Engel elements, while we focus on the main problems in Engel Theory. Let G be a group and $a \in G$ an element in the group. We list the following definitions.

The element a is a *left Engel element* in G, if for each $g \in G$ there exists a non-negative integer $n(g,a)$, possibly depending on g and a, such that

$$[g,_{n(g,a)} a] = [[[g, \underbrace{a], a], \ldots, a]}_{n(g,a)} = 1.$$

If for some positive integer k we have $[g,_k a] = 1$ for all $g \in G$, then a is called a *left k-Engel element* in G. If $n(g,a)$ is independent of g and a then we say that a is a *bounded left Engel element*. In particular, a is bounded by $n(g,a) = n$ and is more specifically a *left n-Engel element* in G. We denote by $L(G)$, $L_n(G)$ and $\bar{L}(G)$ the set of left Engel, left n-Engel and bounded left Engel elements of G, respectively. Clearly, $L_n(G) \subseteq \bar{L}(G)$, for all n. One notices that in the definition of a left Engel element the variable element g is placed on the left hand side of the element a. So similarly, by placing the variable element g on the right hand side of a we can define right Engel elements in G.

An element $a \in G$ is a right Engel element in G, if for each $g \in G$ there exists a non-negative integer $n(g,a)$ such that

$$[[[a, \underbrace{g], g], \ldots, g]}_{n(g,a)} = 1.$$

If for some positive integer k we have $[a,_k g] = 1$ for all $g \in G$, then a is called a *right k-Engel element* in G. If $n(g, a)$ is bounded above by n then we say that a is a *bounded right n-Engel element* in G. We denote by $R(G)$, $R_n(G)$ and $\bar{R}(G)$ the right Engel, right n-Engel and bounded right Engel elements of G, respectively.

Remark. Note that all these subsets containing left or right Engel elements are invariant under automorphisms of G.

Definition 1.1.1. If $L(G) = G$, or equivalently if $R(G) = G$, then G is called an *Engel group*.

Definition 1.1.2. Let $n \in \mathbb{N}$. If $[x,_n y] = 1$ for all $x, y \in G$, then G is called an *n-Engel group*. In other words, in an n-Engel group all elements in G are both left n-Engel and right n-Engel at the same time.

Remark. The following relations hold:

$$L_1(G) \subseteq L_2(G) \subseteq \cdots \subseteq L_n(G) \subseteq \cdots \subseteq L(G) \quad \text{and} \quad \bar{L}(G) = \bigcup_{n\in\mathbb{N}} L_n(G)$$
$$R_1(G) \subseteq R_2(G) \subseteq \cdots \subseteq R_n(G) \subseteq \cdots \subseteq R(G) \quad \text{and} \quad \bar{R}(G) = \bigcup_{n\in\mathbb{N}} R_n(G)$$

Example 1.1.3.

1. Clearly, $L_0(G) = R_0(G) = 1$.
2. $L_1(G) = R_1(G) = Z(G)$. In particular, a group is 1-Engel if and only if it is abelian.
3. Locally nilpotent groups are Engel groups.

We now have the building blocks to dig deeper into the major problems in Engel Theory. One of the major questions is the following.

Question 1.1.4. Is every Engel group locally nilpotent?

This can be seen as the Engel analogue of the General Burnside Problem, which asks if every periodic group is locally finite. In other words, is every finitely generated group, in which every element is of finite order, finite? We thus notice that there is a correlation between the two problems. The General Burnside Problem was posed in 1902 by William Burnside and it has been one of the most influencing problems in group theory. The problem has many variants. One of them is the Burnside Problem which asks whether every group of finite exponent is locally finite. We can raise a similar question in the realm of Engel groups.

Let n be a positive integer.

Question 1.1.5. Is every n-Engel group locally nilpotent?

Positive answers have been given to the question, such as for finite groups (Zorn, 1936) [28], soluble groups (Gruenberg, 1953) [6], groups that satisfy the maximal condition (Baer, 1957) [2], finitely generated residually finite n-Engel Groups (Wilson, 1991) [24], etc. We could even compare the latter with Zelmanov's positive solution of the Restricted Burnside Problem, that asks whether every finitely generated residually finite group of finite exponent is finite [26, 27]. In other words, it asks if there exists a largest finite r-generator group of exponent n, for r and n positive integers. However, Golod groups provide a negative answer to Question 1.1.5 [5].

Remark. Let p be a prime and $d \geqslant 2$ any integer. There exists an (non-nilpotent) infinite d-generator p-group $G_d(p)$, such that every subgroup of $G_d(p)$ of at most $d-1$ generators is nilpotent (finite).

Definition 1.1.6. These groups are called *Golod groups*.

For $d \geqslant 3$ Golod groups are finitely generated Engel groups, but not nilpotent. In other words the examples of Golod groups give for each prime p an infinite 3-generator residually finite p-group with all 2-generator subgroups finite (nilpotent). It is worth mentioning that Golod groups also provide a negative answer to the General Burnside Problem. This follows from the fact that Golod groups are finitely generated infinite torsion groups, with however no uniform bound on the order of their elements.

Today it is known that every n-Engel group is locally nilpotent if $n \leqslant 4$ (Hopkins [13], 1929, $n = 2$; Heineken [12], 1961, $n = 3$; Havas, Vaughan-Lee [11] (and Traustason, 2005 [22]), $n = 4$), yet it is still an open problem whether there exist n-Engel groups, for $n \geqslant 5$, that are not locally nilpotent.

1.2 Structure Results on 3-Engel and 4-Engel Groups

As stated in the previous section it was Heineken who proved in 1961 that 3-Engel groups are locally nilpotent. Moreover, in 2005 from Havas, Vaughan-Lee and Traustason we know that 4-Engel groups are also locally nilpotent. In fact we know quite a bit about the structure of n-Engel groups for $n \leqslant 4$ in addition to the fact that they are locally nilpotent. In this section we state some of the structural results on 3-Engel and 4-Engel groups. We will omit proofs and more complicated details.

1.2.1 3-Engel Groups

We start with Heineken's result on proving global nilpotence for 3-Engel groups that are $\{2, 5\}$-free, that is groups that have no elements of order 2 or 5.

Theorem 1.2.1 (Heineken)**.** Let G be a 3-Engel group that is $\{2, 5\}$-free. Then G is nilpotent of class at most 4.

In general the Engel condition is weaker than the nilpotency condition. Of course in the case of finite groups these two coincide as shown by Zorn, however this can't be generalised. There is a simple general method for constructing non-nilpotent $(p+1)$-Engel p-groups, so 3-Engel 2-groups do not have to be nilpotent. We now give a general construction. Let $A = C_p^\infty$ be an elementary abelian p-group of countably infinite rank. Consider the wreath product

$$G = C_p \text{Wr } C_p^\infty.$$

If we let C_p be generated by b and $a_1, a_2, \ldots$ be a basis for A, then G has an elementary abelian normal subgroup B with basis $\{b^a \mid a \in A\}$ and

$$G = \prod_{a \in A} \langle b^a \rangle \rtimes A.$$

We have

$$\begin{aligned} [b, a_1, a_2, \ldots, a_n] &= b^{(-1+a_1)(-1+a_2)\cdots(-1+a_n)} \\ &= b^{(-1)^n + \ldots + a_1 \cdots a_n} \neq 1 \text{ (for all } n), \end{aligned}$$

for example because the constant $b^{a_1 \cdots a_n}$ has multiplicity 1, so G is not nilpotent. Moreover, for $x, y \in G$ we have $[x, y], y^p \in B$, thus

$$\begin{aligned} [x,_{p+1} y] = [x, y,_p, y] &= [x, y]^{(-1+y)^p} \\ &= [x, y]^{-1+y^p} \\ &= [x, y]^{-1}[x, y], \text{ as } y^p \in B \\ &= 1. \end{aligned}$$

Next we turn our attention to the following result from W. Kappe and L. Kappe.

Theorem 1.2.2 (Kappe, Kappe)**.** Let G be a 3-Engel group, then $\langle x \rangle^G$ is nilpotent of class at most 2, for all $x \in G$.

The primes 2 and 5 that are not covered in Theorem 1.2.1 turn out to be exceptional. For the prime 2 this is because 2 is smaller than the Engel degree 3 and one can show that for a prime $p < n$ there exists a metabelian n-Engel p-group that is non-nilpotent. Only the prime 5 turns however out to be exceptional with respect to solvability. In order to see this we need to show that any 3-Engel 2-group is solvable. After some more reduction work it actually suffices to show the following by Gupta and Weston.

Theorem 1.2.3 (Gupta, Weston)**.** Every 3-Engel group of exponent 4 is solvable.

In fact one can show more than this. Gupta and Newman have given an almost complete description of the variety of 3-Engel groups. In their paper [7] they have proven in particular that every n-generator 3-Engel group is nilpotent of class at most $2n - 1$, which is the best upper bound.

1.2.2 4-Engel Groups

In this subsection we discuss various structure results for locally nilpotent 4-Engel groups.

Proposition 1.2.4 (Traustason)**.** Let G be a locally nilpotent 4-Engel group. Then $\langle x \rangle^G$ is nilpotent of class at most 4, for all $x \in G$.

Havas and Vaughan-Lee's theorem that 4-Engel groups are locally nilpotent implies that Traustason's result applies to all 4-Engel groups. Moreover, Traustason has shown the following.

Theorem 1.2.5 (Traustason)**.** Let G be a 4-Engel p-group with $p \neq 2, 5$. Then $\langle x \rangle^G$ is nilpotent of class at most 3, for all $x \in G$.

Then the following was shown by Vaughan-Lee.

Theorem 1.2.6 (Vaughan-Lee)**.** G is a 4-Engel group if and only if $\langle x \rangle^G$ is 3-Engel, for all $x \in G$.

We now turn our attention to global nilpotence. The general construction of non-nilpotent $(p+1)$-Engel p-groups gives examples of non-nilpotent 4-Engel 3-groups, but Traustason has shown the following result.

Theorem 1.2.7 (Traustason)**.** Let G be a 4-Engel group without $\{2, 3, 5\}$-elements. Then G is nilpotent of class at most 7.

The primes 2,3,5 are exceptional, like the analogous result for 3-Engel groups. The primes 2 and 3 are exceptional because they are less than 4 and 5 is exceptional as it was already exceptional for 3-Engel groups.

We next turn our attention to solvability. The following was shown by Abdollahi and Traustason.

Theorem 1.2.8 (Traustason, Abdollahi)**.** Every 4-Engel 3-group G is solvable.

We end this section with an open problem.

Problem What is the structure of 5-Engel groups?

At present we hardly know anything about them. As mentioned earlier it is still an open question whether or not 5-Engel groups are locally nilpotent. Although this problem is open, it is clear that 5-Engel groups do not satisfy some properties analogous to the properties of 3-Engel and 4-Engel groups. For example, the normal closure of an element in a 5-Engel group need not be 4-Engel. In particular, Gupta and Levin have shown that the normal closure of an element in a 5-Engel group need not be nilpotent [8]. Lastly even though it is known that if $p > 3$ then the normal closure of an element in a finite 5-Engel p-group is nilpotent of bounded class, the situation in 5-Engel 2-groups is still unknown.

1.3 Left Engel Elements

In this section we will explore more details on right and left Engel elements, with emphasis on the latter.

A classical result of Heineken gives the famous relation between left and right Engel elements in an arbitrary group [12]. In particular, it states that in any group G the inverse of a right Engel element is a left Engel element.

Theorem 1.3.1. (Heineken) Let G be a group. Then

1. $R(G)^{-1} \subseteq L(G)$.
2. $R_n(G)^{-1} \subseteq L_{n+1}(G)$.
3. $\bar{R}(G)^{-1} \subseteq \bar{L}(G)$.

Remark. Even though Heineken's result provides a form of relation between left and right Engel elements, we still do not know of other inclusions holding between Engel subsets in an arbitrary group.

We now introduce two subgroups which will play an important role throughout this thesis.

Definition 1.3.2. For an arbitrary group G we define the *Hirsch-Plotkin radical* of G, denoted by $HP(G)$, to be the unique maximal locally nilpotent normal subgroup of G.

Definition 1.3.3. We define the *Baer radical* of G, denoted by $B(G)$, to be the unique maximal normal subgroup of G in which all cyclic subgroups are subnormal.

Definition 1.3.4. We define the *Fitting subgroup* of G, denoted by $Fitt(G)$, to be the subgroup generated by all normal nilpotent subgroups of G. Note that the normal closure of every element of $Fitt(G)$ in G is nilpotent. If there is a bound for the nilpotence class of $\langle x\rangle^G$ and the lowest bound is n then we say that G has *Fitting degree* n.

Remark. Note that

$$Fitt(G) \leqslant B(G) \leqslant HP(G)$$

for any group G. Clearly, if G is finite then $Fitt(G)$ and $HP(G)$ coincide.

It is straightforward to see that any element of the Hirsch-Plotkin radical $HP(G)$ of G is a left Engel element and every element of the Baer radical is a bounded left Engel element. The converse is known to be true for some classes of groups, including solvable groups and finite groups (more generally groups satisfying the maximal condition on subgroups) [2, 6]. The converse is however not true in general as it fails for Golod groups [5], since $L(G_3(p)) = G_3(p) \neq HP(G_3(p))$.

As stated in Robinson's Book: A Course in the Theory of Groups, one of the major goals of Engel theory is to find conditions which will guarantee that $L(G)$ and $\bar{L}(G)$ are subgroups which coincide with the Hirsch-Plotkin radical and the Baer radical respectively. In this thesis we are focused on the question of when does $L(G) = HP(G)$. In fact one sees readily that a left 2-Engel element is always in the Hirsch-Plotkin radical. Moreover the following is true:

Proposition 1.3.5. Let $\langle x\rangle^G$ denote the normal closure of x in G. Then

$$L_2(G) = \{x \in G \mid \langle x\rangle^G \text{ is abelian }\}.$$

Therefore, $L_2(G)$ is contained in $Fitt(G)$ and so also in $B(G)$ and $HP(G)$.

Let us see why this holds. Since x is a left 2-Engel element then x commutes with $[x, y]$, that is x commutes with $x^{-1}x^y$ and so with x^y, for all $y \in G$. Letting $y = ab^{-1}$, for $a, b \in G$, gives that x commutes with $x^{ab^{-1}}$ and so x^b commutes with x^a, which proves the above Proposition.

Moreover, every element in a group of exponent 3 is a left 2-Engel element. In particular, in groups of exponent 3 any two conjugates a, a^b commute and so the fact that $L_2(G) \subseteq HP(G)$ can be seen as the underlying reason why groups of exponent 3 are locally finite (Burnside). Let us briefly see why this is the case. Let a, b be elements in a group G. Then $1 = (ba)^3 = b^3a^{b^2}a^ba$, which implies that

$$a^{b^2}a^ba = 1.$$

Replacing b by b^2 in the above equations gives

$$a^b a^{b^2} a = 1,$$

therefore a and a^b commute or equivalently a and $[b, a] = a^{-b}a$ commute. Thus every group of exponent 3 satisfies the 2-Engel identity

$$[[y, x]x] = 1.$$

Question 1.3.6. Is a left n-Engel element always in the Hirsch-Plotkin radical of G?

The answer turns out to be negative and this is a consequence of the work of Lysenok and Ivanov [14, 16]. They independently show that for sufficiently large k the variety of groups of exponent 2^k is not locally finite. From this one can show that for sufficiently large n there are left n-Engel elements that are not contained in the Hirsch–Plotkin radical. This underlies the close link between left Engel elements and groups of 2-power exponent. Let us understand the details behind this result. Firstly, by the deep results of Ivanov and Lysenok, for some positive integer k there is a finitely generated infinite group G of exponent 2^k. Suppose that k is the least integer with this property, so every finitely generated group of exponent dividing 2^{k-1} is finite. We will make use of the following formula.

Lemma 1.3.7. Let a be any element of G of order 2 and x an arbitrary element of G. Then, for all positive integers m, the following relation holds

$$[x,_m a] = [x, a]^{(-2)^{m-1}}.$$

Proof. We show this by induction on m. Clearly, if $m = 1$ the formula holds. Assume that the formula is true for all values up to $m - 1$. We have

$$\begin{aligned}
[x,_m a] &= [x, a,_{m-1} a] = [x^{-1}a^{-1}xa, a]^{(-2)^{m-2}}, \text{ by induction hypothesis} \\
&= (a^{-1}x^{-1}axa^{-1}x^{-1}a^{-1}xaa)^{(-2)^{m-2}} \\
&= (a^{-1}x^{-1}axa^{-1}x^{-1}ax)^{(-2)^{m-2}}, \text{ since } a \text{ is an element of order 2} \\
&= ([a, x])^{2(-2)^{m-2}} \\
&= [x, a]^{(-2)^{m-1}}.
\end{aligned}$$

□

Thus, every element of order 2 of G lies in $L_{k+1}(G)$. If we assume that $L_{k+1} \subseteq HP(G)$, then $G/HP(G)$ is of exponent dividing 2^{k-1} and so it is finite. Since G is finitely generated, $HP(G)$ is also. But this yields that $HP(G)$ is a periodic finitely generated nilpotent group and so it is finite. It follows that G is finite, a contradiction. Given this, Lysenok showed that for $m = 13$ the group G satisfies the property $L_m(G) \nsubseteq HP(G)$ and thus our smallest such integer k is at most 13.

Turning our attention to groups of exponent 8, let G be such a group and let $a \in G$ satisfy $a^2 = 1$. Then a is a left 4-Engel element in G, because

$$[x,_4 a] = [x,a]^{(-2)^3} = 1.$$

If we suppose that all left 4-Engel elements of a group G of exponent 8 are in $HP(G)$ then for any $x \in G$, we have $x^4 \in HP(G)$. Hence, $G^4 \leqslant HP(G)$, which implies that $G/HP(G)$ is of exponent 4. Using the fact that groups of exponent 4 are locally finite [19] one can also see that G is also locally finite. From this we see that if all 4-Engel elements in a group G of exponent 8 are in the $HP(G)$ then we would have a solution to a longstanding problem of whether groups of exponent 8 are locally finite.

Hence the question which naturally arises is the following:

Question 1.3.8. What is the least positive integer n such that $L_n(G) \nsubseteq HP(G)$?

As mentioned above $L_2(G) \subseteq HP(G)$. It is still a longstanding open question whether $L_3(G) \subseteq HP(G)$. Recently there has been a lot of work done towards solving this problem with two major breakthroughs. We split these results in two subsections going through the details for each one.

1.3.1 Left 3-Engel Elements of Odd Order

In [15] the following result is shown.

Theorem 1.3.9. Let G be any group and let $a \in G$ be a left 3-Engel element of *odd* order. Then a is contained in $HP(G)$.

In order to understand this result we first need to introduce sandwich groups. We first make the observation that an element $a \in G$ is a left 3-Engel element if and only if $\langle a, a^x \rangle$ is nilpotent of class at most 2 for all $x \in G$ [1]. We introduce the following related class of groups.

Definition 1.3.10. Let G be a group. We say that $X \subseteq G$ is a *sandwich set* in G if $\langle x, y^g \rangle$ is nilpotent of class at most 2 for all $x, y \in X$ and $g \in G$. A *sandwich group* is a group G that can be generated by a sandwich set X. The *rank* of a sandwich group G is the smallest possible cardinality of a sandwich set that generates G.

If $a \in G$ is a left 3-Engel element then $H = \langle a \rangle^G$ is a sandwich group and the following statements are equivalent:

(1) For every pair (G, a), where a is a left 3-Engel element in the group G, a is in the $HP(G)$.

(2) Every sandwich group is locally nilpotent.

From the discussion above Theorem 1.3.9 is established by proving that any finitely generated sandwich group $\langle x_1, \ldots, x_r \rangle$, generated by elements of odd orders, that is x_i is of odd order for $i = 1, \ldots, r$, is nilpotent.

1.3.2 Left 3-Engel Elements of Order 2

Proceeding even further to finding an answer to Question 1.3.8 in [20] it is shown that in order to generalise this to left 3-Engel elements of any finite order it suffices to deal only with elements of order 2. This was established by showing the following theorem.

Theorem 1.3.11. Let G be a group and let $x \in G$ be a left 3-Engel element in G of order dividing 60. Suppose furthermore that $\langle x \rangle^G$ has no elements of order $8, 9$ or 25. Then $x \in HP(G)$.

Using Theorem 1.3.9 we can reduce the above Theorem to the following.

Theorem 1.3.12. Let G be a group and let $x \in G$ be a left 3-Engel element in G of order dividing 4. If $\langle x \rangle^G$ has no elements of order 8, then $x \in HP(G)$.

In this subsection we continue our study of left 3-Engel elements started in Subsection 2.3.1 in order to build up to the idea behind Theorem 1.3.11.

Throughout we will make use of the following result shown by A. Abdollahi.

Theorem 1.3.13 (Abdollahi)**.** Let p be any prime and G be a group. If $x \in G$ is a left 3-Engel element and $x^{p^n} = 1$ for some integer $n > 1$, then $\langle x^p \rangle^G$ is soluble of derived length at most $n - 1$ and $x^p \in B(G)$. In particular, $\langle x^p \rangle^G$ is locally nilpotent.

Left 3-Engel elements of finite order. For left 3-Engel elements of finite order some further reduction can be made. Suppose G is a group with a left 3-Engel element x of order $m = p_1^{n_1} \cdots p_r^{n_r}$, where $p_1, \ldots, p_r$ are distinct primes and $n_1, \ldots, n_r$ are positive integers. For $1 \leqslant j \leqslant r$, let $m_j = m/p_j^{n_j}$. Then $m_1, \ldots, m_r$ are coprime. Thus in order to show that $x \in HP(G)$, it suffices to show that $x^{m_1}, \ldots, x^{m_r} \in HP(G)$. So the problem of showing that an element of finite order is in $HP(G)$ reduces to dealing with elements of prime power order. Further reductions can be made. First we recall a standard notion. Let G be a group. For any set π consisting of primes, we say that x is a π-element in G if the order of x only has numbers from π as prime factors. One can show the following result.

Lemma 1.3.14. Let π be a set of primes. Suppose that for all groups G and all primes $p \in \pi$ we have that all left 3-Engel elements of order p in G are contained in $HP(G)$. It then follows that for all groups G, all left 3-Engel elements in G that are π-elements are in $HP(G)$.

Then the problem of showing that all left 3-Engel elements of finite order are in the Hirsch-Plotkin radical thus reduces to only having to consider elements of prime order. Thus dealing with left 3-Engel elements of finite order reduces to working with sandwich groups generated by elements of prime order p. This is because the following are equivalent for any prime p:

(1_p) For every pair (G, a) where a is a left 3-Engel element of order p in G we have that $a \in HP(G)$.

(2_p) Every finitely generated sandwich group generated by elements of order p is nilpotent.

Left 3-Engel elements in groups of finite exponent. Determining whether a left 3-Engel element of finite order is in the Hirsch-Plotkin radical seems a very difficult problem in general. One could thus consider adding further constraints on the group. For example one could require that for the given left 3-Engel element x in G we have that $\langle x\rangle^G$ is of finite exponent. In fact we will consider a weaker condition. Let $p_1^{n_1}, \ldots, p_r^{n_r}$ be non-trivial powers where $p_1, \ldots, p_r$ are distinct primes. Consider the following statement.

$\mathcal{E}(p_1^{n_1}, \ldots, p_r^{n_r})$: For all groups G and all left 3-Engel elements $x \in G$ of order dividing $p_1^{n_1} \cdots p_r^{n_r}$, where $\langle x\rangle^G$ has no elements of order $p_1^{n_1+1}, \ldots, p_r^{n_r+1}$, we have that $x \in HP(G)$.

Remark. Notice that if $\mathcal{E}(p_1^{n_1}, \ldots, p_r^{n_r})$ holds, then it would follow that a left 3-Engel element of G is in $HP(G)$ when $\langle x\rangle^G$ has exponent dividing $p_1^{n_1} \cdots p_r^{n_r}$. Thus in particular all left 3-Engel elements in a group G of exponent dividing $p_1^{n_1} \cdots p_r^{n_r}$ would be in the Hirsch-Plotkin radical.

We will next prove a reduction result that is similar in nature to Lemma 1.3.14. For a given prime p and positive integer n, consider the following statement.

$\mathcal{Q}(p,n):$ For all groups G and all left 3-Engel elements $x \in G$, where x is of order p and $\langle x\rangle^G$ has no element of order p^{n+1}, we have that $x \in HP(G)$.

Proposition 1.3.15. Let $m = p_1^{n_1} \cdots p_r^{n_r}$ be an integer where $p_1, \ldots, p_r$ are distinct primes and $n_1, \ldots, n_r$ are positive integers. Then

$$\mathcal{Q}(p_1, n_1) \wedge \cdots \wedge \mathcal{Q}(p_r, n_r) \implies \mathcal{E}(p_1^{n_1}, \ldots, p_r^{n_r}).$$

Remark. In particular, it follows from last proposition that in order to show that left 3-Engel elements in groups of finite exponent are in the Hirsch-Plotkin radical, it suffices to show that $\mathcal{Q}(p,n)$ holds for all primes and positive integers n.

At this stage we don't even know if for a group G, satisfying the hypothesis in $\mathcal{Q}(p,n)$, we can conclude that $\langle x\rangle^G$ is a p-group. Let us thus consider the following weaker statement.

$\mathcal{R}_G(p,n):$ For all left 3-Engel elements $x \in G$, where x is of order p and $\langle x\rangle^G$ has no element of order p^{n+1}, we have that $\langle x\rangle^G$ is a p-group.

Remark. Of course $\mathcal{R}_G(p,n)$ implies that $\langle x\rangle^G$ is of exponent dividing p^n. The next proposition gives a sufficient condition for $\mathcal{R}_G(p,n)$.

Proposition 1.3.16. Let p be a prime and n a positive integer. Let G be a group with the property that for any left 3-Engel element $x \in G$ of order p we have that $\langle x\rangle^G$ has no element of order p^{n+1}. Now suppose furthermore that for any left 3-Engel element x of order p and any element $g \in \langle X\rangle^G$ of order dividing p^n we have

$$\langle x, x^g, \ldots, x^{g^{p^n-1}}\rangle$$

is nilpotent. Then $\mathcal{R}_G(p,n)$ holds.

Remark. As any sandwich group of rank 3 is nilpotent [25] it follows from the above Proposition that $\mathcal{R}_G(2,1)$ and $\mathcal{R}_G(3,1)$ hold in any group G. Groups of exponent 2 are abelian and from Burnside [4] we know that groups of exponent 3 are locally finite. It thus follows that $\mathcal{Q}_G(2,1)$ and $\mathcal{Q}_G(3,1)$ hold.

From Proposition 1.3.15 it suffices to show that $\mathcal{Q}_G(2,2), \mathcal{Q}_G(3,1)$ and $\mathcal{Q}_G(5,1)$ hold. We have already seen from last Remark that $\mathcal{Q}_G(3,1)$ holds. Moreover, from

Theorem 1.3.9 $\mathcal{Q}_G(5,1)$ holds. Note that Theorem 1.3.11 was shown prior to Theorem 1.3.9. Therefore, it suffices to only show $\mathcal{Q}_G(2,2)$ holds, done in [20].

1.4 Main Motivation

In [17] M. Newell proved that if a is a right 3-Engel element in G then $a \in HP(G)$ and in fact he proved the stronger result that $\langle a\rangle^G$ is nilpotent of class at most 3. The natural question arises whether the analogous result holds for left 3-Engel elements. In [18] it is shown that this is not the case by giving an example of a locally finite 2-group with a left 3-Engel element a such that $\langle a\rangle^G$ is not nilpotent. In [9] we extend this example, of a 2-group, to an infinite family of examples. Moreover in [10] an example is given, for each odd prime p, of a locally finite p-group containing a left 3-Engel element x where $\langle x\rangle^G$ is not nilpotent.

1.5 Lifting the Lie algebra to the Group

Throughout the thesis we make use of a series of steps of lifting a Lie algebra to the group. We start by taking a Lie algebra over $\mathbb{F}$

$$W = \mathbb{F}x_1 + \mathbb{F}x_2 + \cdots + \mathbb{F}x_m, \text{ where } m \in \mathbb{N},$$

with trivial center. Let

$$E = \langle \mathrm{ad}(x_1), \mathrm{ad}(x_2), \ldots, \mathrm{ad}(x_m))\rangle \leqslant End(W).$$

As $Z(W)$ is trivial, the map $\mathrm{ad} : W \to \mathrm{End}(W)$ is injective and thus E is a finite-dimensional associative enveloping algebra of W. We will use W to construct a locally nilpotent Lie algebra over $\mathbb{F}$ of countably infinite dimension. This will then help us to construct a locally finite group G. We now introduce a notation of modified unions of finite subsets of $\mathbb{N}$. We let

$$A \sqcup B = \begin{cases} A \cup B, & \text{if } A \cap B = \emptyset \\ \emptyset, & \text{otherwise} \end{cases}$$

For each non-empty subset A of $\mathbb{N}$ we let W_A be a copy of the vector space W, that is $W_A = \{z_A : z \in W\}$ with addition $z_A + t_A = (z+t)_A$. We then take the direct sum of these

$$W^* = \bigoplus_{\emptyset \neq A \subseteq \mathbb{N}} W_A$$

that we turn into a Lie algebra with multiplication

$$z_A \cdot t_B = (z \cdot t)_{A \sqcup B}$$

when $z_A \in W_A$ and $t_B \in W_B$ and extend linearly on W^*. The interpretation here is that $z_\emptyset = 0$. Notice that W^* has basis

$$\{x_{i_A} : 1 \leqslant i \leqslant m, \emptyset \neq A \subseteq \mathbb{N}\}.$$

Lemma 1.5.1. W^* is locally nilpotent.

Proof. Any finitely generated subalgebra of W^* is contained in some

$$S = \langle x_{1A_1^1}, \ldots, x_{1A_r^1}, x_{2A_1^2}, \ldots, x_{2A_t^2}, \ldots, x_{mA_1^m}, \ldots, x_{mA_l^m} \rangle.$$

Thus it suffices to show that S is nilpotent. Observe first that any Lie product with a repeated entry of x_{i_A} is trivial and thus a non-trivial Lie product of the generators of S can include in total at most $(r + \cdots + t + l)$ such elements. Thus, S is nilpotent of class at most $(r + \cdots + t + l)$. Therefore, W^* is locally nilpotent.

□

In this thesis we make use of a special case of this process. We take the Lie algebra L over $\mathbb{F}$ and let W be any Lie algebra over $\mathbb{F}$ with trivial center, $\mathrm{Z}(W) = 0$. Then take $x \in W$ to retrieve a Lie algebra

$$L^* = W^* \oplus \mathbb{F}x$$

with basis

$$\{x\} \cup \{x_{i_A} : 1 \leqslant i \leqslant m, \emptyset \neq A \subseteq \mathbb{N}\}.$$

Similarly as before we have the following Lemma.

Lemma 1.5.2. L^* is locally nilpotent.

Proof. Any finitely generated subalgebra of L^* is contained in some

$$S = \langle x, x_{1A_1^1}, \ldots, x_{1A_r^1}, x_{2A_1^2}, \ldots, x_{2A_t^2}, \ldots, x_{mA_1^m}, \ldots, x_{mA_l^m} \rangle.$$

Thus it suffices to show that S is nilpotent. Observe first that any Lie product with a repeated entry of x_{i_A} is trivial thus a non-trivial Lie product of the generators of S

can include in total at most $(r + \cdots + t + l)$ such elements. For the purposes of this thesis we will assume that x is a left n-Engel element. Thus have that S is nilpotent of class at most $n(r + \cdots + t + l)$. Therefore, L^* is locally nilpotent.

□

Chapter 2

Left 3-Engel elements in locally finite 2-groups

In this Chapter we give an infinite family of examples that generalise the construction given in [18] of a locally finite 2-group G containing a left 3-Engel element x where $\langle x \rangle^G$ is not nilpotent. The construction is based on a family of Lie algebras that are of interest in their own right given in [21] and make use of a classical theorem of Lucas.

The structure of the Chapter is as follows. In Section 3.1 we construct a family of Lie algebras L and by using the methodology of Chapter 2.3 we construct a locally nilpotent Lie algebra L^* over $\mathbb{F}$. In Section 3.2 we focus on the group extension G of the Lie algebra introduced in Section 3.1 inside $GL(L^*)$ containing $1 + \mathrm{ad}(x)$, where $1 + \mathrm{ad}(x)$ is a left 3-Engel element in G, whose normal closure $\langle 1 + \mathrm{ad}(x) \rangle^G$ is not nilpotent.

Before stating Lucas's Theorem we need some notation.

Let p be a prime and consider non-negative integers m and n written in base p

$$\begin{aligned} m &= m_0 + m_1 p + \cdots + m_{k-1} p^{k-1} + m_k p^k \\ n &= n_0 + n_1 p + \cdots + n_{k-1} p^{k-1} + n_k p^k, \end{aligned}$$

where $0 \leqslant m_0, \ldots, m_k, n_0, \ldots, n_k \leqslant p - 1$. We introduce a partial order $\leqslant_p$, where $n \leqslant_p m$ if and only if $n_i \leqslant m_i$, for $0 \leqslant i \leqslant k$.

Theorem 2.0.1 (Lucas' Theorem)**.** *The binomial coefficient $\binom{m}{n}$ is divisible by p if and only if $n \nleqslant_p m$.*

Remark. Notice that when $p = 2$ we get that the binomial coefficient $\binom{m}{n}$ is odd if and only if $n \leqslant_2 m$.

2.1 The Lie Algebra L

In this section we construct a family of Lie algebras that extend the example given in [21]. The construction makes an interesting use of Lucas' Theorem.

Let $\mathbb{F}$ be the field of order 2 and let $n = 2^m - 2$, for $m \geqslant 2$. Consider the $(n+2)$-dimensional vector space $L = \mathbb{F}v(0) + \mathbb{F}v(1) + \cdots + \mathbb{F}v(n-1) + \mathbb{F}w + \mathbb{F}x$. We equip L with a binary product where

$$\begin{aligned} v(i) \cdot v(j) &= \binom{j+1}{n-i} v(i \oplus j) \\ v(i) \cdot w = w \cdot v(i) &= v(i+1), \quad i = 0, 1, \ldots, n-2 \\ v(n-1) \cdot w = w \cdot v(n-1) &= w \\ v(i) \cdot x = x \cdot v(i) &= 0 \\ w \cdot x = x \cdot w &= v(0), \end{aligned}$$

such that $i \oplus j \equiv i + j \pmod{n-1}$ and $i \oplus j \in \{0, \ldots, n-2\}$, and where $ww = xx = 0$. We then extend the product linearly on L. The next theorem is our first main result.

Theorem 2.1.1. *L is a Lie algebra over $\mathbb{F}$.*

Proof. Let $0 \leqslant i, j, k \leqslant n-1$ and suppose that

$$\begin{aligned} i+1 &= a_0 + 2a_1 + \cdots + 2^{m-1}a_{m-1} \\ j+1 &= b_0 + 2b_1 + \ldots + 2^{m-1}b_{m-1} \\ k+1 &= c_0 + 2c_1 + \cdots + 2^{m-1}c_{m-1} \\ n+1-(i+1) &= (1-a_0) + 2(1-a_1) + \cdots + 2^{m-1}(1-a_{m-1}) \\ n+1-(j+1) &= (1-b_0) + 2(1-b_1) + \cdots + 2^{m-1}(1-b_{m-1}) \\ n+1-(k+1) &= (1-c_0) + 2(1-c_1) + \cdots + 2^{m-1}(1-c_{m-1}), \end{aligned}$$

where $0 \leqslant a_i, b_i, c_i \leqslant 1$ for $0 \leqslant i \leqslant m-1$. In order to show that the product is alternating, it only remains to see that $v(i) \cdot v(i) = 0$ and $v(i) \cdot v(j) = v(j) \cdot v(i)$ (recall that the characteristic is 2). Firstly

$$v(i)\cdot v(i)=\begin{pmatrix} i+1 \\ n+1-(i+1)\end{pmatrix}v(i\oplus i).$$

In order for the product to be zero we know by Lucas' Theorem that we need $n+1-(i+1)\nleqslant_2 i+1$. Assume for contradiction that $(n+1)-(i+1)\leqslant_2 i+1$. Then $1-a_i\leqslant a_i$, for all $0\leqslant i\leqslant m-1$ which implies that $a_0=a_1=\cdots=a_{m-1}=1$. This gives $i+1=1+2+\cdots+2^{m-1}=n+1$ that contradicts $i\leqslant n-1$.

Now, for $v(j)\cdot v(i)=v(i)\cdot v(j)$, we need

$$\begin{pmatrix} i+1 \\ n-j\end{pmatrix}=\begin{pmatrix} j+1 \\ n-i\end{pmatrix}.$$

$$\text{But, }\begin{pmatrix} i+1 \\ n+1-(j+1)\end{pmatrix}\text{ is odd }\iff 1-b_i\leqslant a_i,\quad\text{for all }0\leqslant i\leqslant m-1$$
$$\iff 1-a_i\leqslant b_i,\quad\text{for all }0\leqslant i\leqslant m-1$$
$$\iff \begin{pmatrix} j+1 \\ n+1-(i+1)\end{pmatrix}\text{ is odd.}$$

Having established that the product is alternating we turn to the Jacobi identity . If we have basis elements e,f then, as $e\cdot e=0$ and $e\cdot f=f\cdot e$, we get $(e\cdot e)\cdot f+(e\cdot f)\cdot e+(f\cdot e)\cdot e=2(e\cdot f)\cdot e=0$. Therefore, we only need to deal with the cases when the three basis elements are different. We will divide our anaysis into few cases. Notice first, that any Jacobi relation involving one occurrence of x and two $v(i)$'s, for $0\leqslant i\leqslant n-1$, is clearly 0. There is one remaining type of a Jacobi relation that involves x. This is the one for a triple $(x,w,v(i))$, where $0\leq i\leq n-1$.

First, let $0\leqslant i\leqslant n-2$. We have, $v(i)wx+wxv(i)+xv(i)w=v(i+1)x+v(0)v(i)=\binom{i+1}{n}v(i)=0$, since $i+1\leqslant n-1<n$ and thus $n\nleqslant_2 i+1$. If $i=n-1$, then $v(n-1)wx+wxv(n-1)+xv(n-1)w=wx+v(0)v(n-1)=v(0)+\binom{n}{n}v(0)=0$.

Let us next consider triples of the type $(w,v(i),v(j))$ where $0\leqslant i,j\leqslant n-1$.

Case 1: Let $i = n-1$ and $0 \leqslant j \leqslant n-2$. Then,

$$\begin{aligned} & wv(n-1)v(j) + v(n-1)v(j)w + v(j)wv(n-1) \\ = \;& wv(j) + \binom{j+1}{1} v(j)w + v(j+1)v(n-1) \\ = \;& v(j+1) + (j+1)v(j+1) + \binom{j+2}{1} v(j+1) \\ = \;& 2(j+2)v(j+1) = 0. \end{aligned}$$

Case 2: Let $0 \leqslant i < j \leqslant n-2$. Then,

$$\begin{aligned} & wv(i)v(j) + v(i)v(j)w + v(j)wv(i) \\ = \;& v(i+1)v(j) + \binom{j+1}{n-i} v(i \oplus j)w + v(j+1)v(i) \\ = \;& \binom{j+1}{n-i-1} v(i \oplus (j+1)) + \binom{j+1}{n-i} v(i \oplus (j+1)) + v(i)v(j+1) \\ = \;& \binom{j+1}{n-i-1} v(i \oplus (j+1)) + \binom{j+1}{n-i} v(i \oplus (j+1)) + \binom{j+2}{n-i} v(i \oplus (j+1)) \\ = \;& 2\binom{j+2}{n-i} v(i \oplus (j+1)) = 0, \end{aligned}$$

where the last equality follows from Pascal's Rule.

Finally, consider $v(i)v(j)v(k) + v(j)v(k)v(i) + v(k)v(i)v(j) = \alpha_1 v(i \oplus j \oplus k) + \alpha_2 v(i \oplus j \oplus k) + \alpha_3 v(i \oplus j \oplus k)$, for $0 \leqslant i, j, k \leqslant n-1$. Clearly, if all coefficients α_i are even then the Jacobi identity holds. So, assume without loss of generality that α_1 is odd. Then, as $v(i)v(j) \neq 0$ we get $1 \leqslant a_i + b_i$, for $i = 0, \ldots, m-1$. Then, $(i+1)+(j+1) = (a_0+b_0)+2(a_1+b_1)+\cdots+2^{m-1}(a_{m-1}+b_{m-1}) \geqslant 1+2+\cdots+2^{m-1} = n+1$, so $i + j \geqslant n-1$.

Case 1: Consider the case where $i + j = n-1$. Then, $v(i)v(j)v(k) = v(0)v(k) = \binom{k+1}{n} v(k \oplus 0)$. Notice that $\binom{k+1}{n}$ is odd if and only if $k = n-1$. Hence $k = n-1$ and $v(i)v(j)v(k) = v(0 \oplus (n-1)) = v(0)$. Thus,

$$\begin{aligned} & v(i)v(j)v(k) + v(j)v(k)v(i) + v(k)v(i)v(j) \\ = \;& v(0) + (j+1)v(j)v(i) + (i+1)v(i)v(j) \\ = \;& v(0) + (j+1)v(0) + (i+1)v(0) \\ = \;& (n+2)v(0) = 2^m v(0) = 0, \end{aligned}$$

as required.

Case 2: Consider the case where $i+j \geqslant n$. We want to show that if α_1 is odd then exactly one of α_2, α_3 is odd. We shall consider each term separately. We have

$$\begin{aligned} v(i)v(j)v(k) &= \binom{j+1}{n+1-(i+1)} v(i \oplus j) v(k) \\ &= \binom{j+1}{n+1-(i+1)} \binom{k+1}{n+1-(i+j-n+2)} v(i \oplus j \oplus k), \text{ since } i+j \geqslant n \\ &= \binom{j+1}{n+1-(i+1)} \binom{k+1}{2(n+1)-1-(i+1+j+1)} v(i \oplus j \oplus k). \end{aligned}$$

We assumed that α_1 must be odd, hence we need both binomial coefficients to be odd. We have $\binom{j+1}{n+1-(i+1)}$ is odd if and only if $1 \leqslant a_i + b_i$, for all i. Let t be the smallest index such that $a_t + b_t = 1$. Therefore, $a_0 = \cdots = a_{t-1} = b_0 = \cdots = b_{t-1} = 1$.
In order for $\binom{k+1}{2(n+1)-1-(i+1+j+1)}$ to be odd we must have that $2(n+1) - 1 - (i+1+j+1) \leqslant_2 k+1$. So,

$$\begin{aligned} & 2(n+1) - 1 - (i+1+j+1) \\ = \; & 2(1+2+2^2+\cdots+2^{m-1}) - 1 - ((a_0+b_0) + 2(a_1+b_1) + \cdots + 2^t + \\ & \cdots + 2^{m-1}(a_{m-1}+b_{m-1})) \\ = \; & 2^t - 1 + 2^{t+1}(2 - a_{t+1} - b_{t+1}) + \cdots + 2^{m-1}(2 - a_{m-1} - b_{m-1}) \\ = \; & 1 + 2 + 2^2 + \cdots + 2^{t-1} + 2^{t+1}(2 - a_{t+1} - b_{t+1}) + \cdots \\ & +2^{m-1}(2 - a_{m-1} - b_{m-1}) \\ \leqslant_2 \; & c_0 + 2c_1 + \cdots + 2^{t-1}c_{t-1} + 2^t c_t + 2^{t+1}c_{t+1} + \cdots + 2^{m-1}c_{m-1}, \end{aligned}$$

hence, $c_0 = c_1 = \cdots = c_{t-1} = 1$ and $2 \leqslant a_i + b_i + c_i$, for all $i \geqslant t+1$. Notice that it follows in particular that $1 \leq a_i + c_i, b_i + c_i$ for all i except possibly $i = t$.

Notice that the assumption $2(n+1) - 1 - (i+1+j+1) \leqslant_2 k+1$ implies that $2(n+1) - 1 - (i+1+j+1) \leqslant k+1$. That is $2(n-1) \leqslant i+j+k$. Then we must have that $j+k \geqslant n-1$ and $i+k \geqslant n-1$. If there is an equality, for example $j+k = n-1$, then, by symmetry, we have already shown in Case 1 that the Jacobi identity holds. So, we may assume that $j+k, i+k \geqslant n$. Hence, similarly as above, $v(j)v(k)v(i) = \binom{k+1}{n+1-(j+1)}\binom{i+1}{2(n+1)-1-(k+1+j+1)} v(i \oplus j \oplus k)$ and $v(k)v(i)v(j) = \binom{i+1}{n+1-(k+1)}\binom{j+1}{2(n+1)-1-(k+1+i+1)} v(i \oplus j \oplus k)$.
We have two cases:

(a) $\underline{c_t = 0}$: Assume without loss of generality that $a_t = 0$ and $b_t = 1$. Then, $a_t + c_t = 0 \not\geqslant 1$ and $b_t + c_t = 1$, hence $\binom{k+1}{n+1-(j+1)}$ is odd and $\binom{i+1}{n+1-(k+1)}$ is even.

Notice that this implies that the smallest index s such that $b_s + c_s = 1$ is $s = t$. Hence, $\binom{i+1}{2(n+1)-1-(j+1+k+1)}$ is odd, since $2 \leqslant a_i + b_i + c_i$, for all $i \geqslant t+1$ and $a_0 = \cdots = a_{t-1} = 1$. This shows that the Jacobi identity holds.

(b) $\underline{c_t = 1}$: Assume without loss of generality that $a_t = 0$ and $b_t = 1$. Then, both $a_t + c_t, b_t + c_t \geqslant 1$, hence both binomials coefficients $\binom{k+1}{n+1-(j+1)}$ and $\binom{i+1}{n+1-(k+1)}$ are odd. Now, similarly as in case (a), if s is the smallest index such that $a_s + c_s = 1$, then $s = t$ and so $\binom{j+1}{2(n+1)-1-(i+1+k+1)}$ is odd. It only remains to show that $\binom{i+1}{2(n+1)-1-(j+1+k+1)}$ is even. But, $b_t + c_t = 2$, so the smallest index l such that $b_l + c_l = 1$ is $l \geqslant t+1$. Then, in order for $\binom{i+1}{2(n+1)-1-(j+1+k+1)}$ to be odd we require $a_0 = a_1 = \cdots = a_t = 1$, which contradicts our assumption, hence $\binom{i+1}{2(n+1)-1-(j+1+k+1)}$ must be even and the Jacobi identity holds.

□

Lemma 2.1.2. The Lie algebra L has trivial center.

Proof. Take an element of L say $l = \lambda_{-1}x + \lambda_0 v(0) + \cdots + \lambda_{n-1}v(n-1) + \mu w$, where $\lambda_{-1}, \lambda_0, \ldots, \lambda_{n-1}, \mu \in \mathbb{F}$, that lies in the center of L. Multiplying by x gives $\mu v(0) = 0$, therefore $\mu = 0$. Then, multiplying by w gives $\lambda_{-1}v(0) + \lambda_0 v(1) + \cdots + \lambda_{n-2}v(n-1) + \lambda_{n-1}w = 0$ and therefore $\lambda_{-1} = \cdots = \lambda_{n-1} = 0$. □

Lemma 2.1.3. $W = \mathbb{F}v(0) + \cdots + \mathbb{F}v(n-1) + \mathbb{F}w$ is a simple ideal of L.

Proof. Consider the ideal I generated by y, where $y = \lambda_0 v(0) + \cdots + \lambda_{n-1}v(n-1) + \mu w$ and $\lambda_0, \ldots, \lambda_{n-1}, \mu \in \mathbb{F}$ are not all zero. We first show that $w \in I$. If $\lambda_0 = \cdots = \lambda_{n-1} = 0$, this is clear. If not, take the smallest i such that $\lambda_i \neq 0$, where $0 \leqslant i \leqslant n-1$. Taking y and mulitiplying $n-i$ times by w gives us $\lambda_i w$ that implies that $w \in I$. Having established that $w \in I$ we can multiply it by $x, v(0), ..., v(n-2)$ to see that $v(0), \ldots, v(n-1) \in I$. Hence $I = W$. □

Let $E = \langle \mathrm{ad}(x), \mathrm{ad}(v(0)), \mathrm{ad}(v(1)), \ldots, \mathrm{ad}(v(n-1)), \mathrm{ad}(w) \rangle \leqslant End(L)$. As $Z(L)$ is trivial, E is the associative enveloping algebra of L.

Lemma 2.1.4. The associative enveloping algebra E is finite-dimensional.

Proof. This follows from the fact that $\dim(L) = n+2$, hence $\dim\,(\mathrm{End}(L)) = (n+2)^2$, thus E must be of finite dimension. □

Following the process in Chaper 2.3, we will use L to construct a locally nilpotent Lie algebra over $\mathbb{F}$ of countably infinite dimension. This will then help us to construct a locally finite group G with a left 3-Engel element y where $\langle y \rangle^G$ is not nilpotent. We now introduce a notation that was used in [21] of modified unions of subsets of $\mathbb{N}$. We let

$$A \sqcup B = \begin{cases} A \cup B, & \text{if } A \cap B = \emptyset \\ \emptyset, & \text{otherwise} \end{cases}$$

For each non-empty subset A of $\mathbb{N}$ we let W_A be a copy of the vector space $W = \mathbb{F}v(0)+\cdots+\mathbb{F}v(n-1)+\mathbb{F}w$, that is $W_A = \{z_A : z \in W\}$ with addition $z_A+t_A = (z+t)_A$. We then take the direct sum of these

$$W^* = \bigoplus_{\emptyset \neq A \subseteq \mathbb{N}} W_A$$

that we turn into a Lie algebra with multiplication

$$z_A \cdot t_B = (z \cdot t)_{A \sqcup B}$$

when $z_A \in W_A$ and $t_B \in W_B$ and extend linearly on W^*. The interpretation here is that $z_\emptyset = 0$. Finally, we extend this to the semidirect product with $\mathbb{F}x$

$$L^* = W^* \oplus \mathbb{F}x$$

induced from the action $z_A \cdot x = (z \cdot x)_A$. Notice that L^* has basis

$$\{x\} \cup \{v(0)_A, \ldots, v(n-1)_A, w_A : \emptyset \neq A \subseteq \mathbb{N}\}$$

and that

$$v(i)_A \cdot v(i)_B = w_A \cdot w_B = 0,$$

$$v(i)_A \cdot x = 0, \quad w_A \cdot x = v(0)_A,$$

for all $0 \leqslant i, j \leqslant n-1$ and

$$v(i)_A \cdot v(j)_B = \binom{j+1}{n-i} v(i \oplus j)_{A \sqcup B}, \text{ for all } 0 \leqslant i, j \leqslant n-1,$$

$$v(i)_A \cdot w_B = v(i+1)_{A \sqcup B}, \text{ for all } 0 \leqslant i, j \leqslant n-2$$

$$\text{and } v(n-1)_A \cdot w_B = w_{A \sqcup B}.$$

Lemma 2.1.5. L^* is locally nilpotent.

Proof. Notice that any finitely generated subalgebra of L^* is contained in some

$$S = \langle x, v(0)_{A_1^0}, \dots, v(0)_{A_r^0}, \dots, v(n-1)_{A_1^{n-1}}, \dots, v(n-1)_{A_t^{n-1}}, w_{B_1}, \dots, w_{B_l}\rangle.$$

Thus it suffices to show that S is nilpotent. Observe first that any Lie product with a repeated entry of $v(i)_A$ or w_B is trivial and thus a non-trivial Lie product of the generators of S can include in total at most $r+\cdots+t+l$ such elements. As $v(i)_A \cdot x = (w_B \cdot x)\cdot x = 0$ we have $(z\cdot x)\cdot x = 0$ for all $z \in L^*$, that is x is a left 2-Engel element in L^*. Thus we see that S is nilpotent of class at most $2(r+\cdots+t+l)$, as in Lemma 1.5.2. Therefore, L^* is locally nilpotent. □

2.2 The Group G

For an element $y \in L^*$ we denote by $\mathrm{ad}(y)$ the linear operator on L^* induced by multiplication by y on the right. In this section we find a group G inside $\mathrm{GL}(L^*)$ containing $1+\mathrm{ad}(x)$, where $1+\mathrm{ad}(x)$ is a left 3-Engel element in G, but where $\langle 1+\mathrm{ad}(x)\rangle^G$ is not nilpotent. The next Lemma is a preparation for this.

Lemma 2.2.1. The adjoint linear operator $\mathrm{ad}(x)$ on L^* satisfies:
(a) $\mathrm{ad}(x)^2 = 0$.
(b) $\mathrm{ad}(x)\mathrm{ad}(y)\mathrm{ad}(x) = 0$, for all $y \in L^*$.

Proof. (a) This follows from our earlier observation that $(z\cdot x)\cdot x = 0$ for all $z \in L^*$.
(b) Follows from $w_A \cdot x \cdot v(i)_B \cdot x = v(0)_A \cdot v(i)_B \cdot x = 0, w_A \cdot x \cdot w_B \cdot x = v(0)_A \cdot w_B \cdot x = v(1)_{A\sqcup B} \cdot x = 0$. □

Now let y be any of the generators $x, v(i)_A, w_A$, for $0 \leqslant i \leqslant n-1$. Since $\mathrm{ad}(y)^2 = 0$ for all y it follows that

$$(1+\mathrm{ad}(y))^2 = 1 + 2\mathrm{ad}(y) + \mathrm{ad}(y)^2 = 1.$$

Thus, 1+$\mathrm{ad}(y)$ is an involution in $\mathrm{GL}(L^*)$. Notice also that the following are elementary abelian 2-groups of countably infinite rank.

$$\begin{aligned}
\mathcal{V}_0 &= \langle 1+\mathrm{ad}(v(0)_A) : A \subseteq \mathbb{N}\rangle, \\
\mathcal{V}_1 &= \langle 1+\mathrm{ad}(v(1)_A) : A \subseteq \mathbb{N}\rangle, \\
&\vdots \\
\mathcal{V}_{n-1} &= \langle 1+\mathrm{ad}(v(n-1)_A) : A \subseteq \mathbb{N}\rangle, \\
\mathcal{W} &= \langle 1+\mathrm{ad}(w_A) : A \subseteq \mathbb{N}\rangle
\end{aligned}$$

We will be working with the group $G = \langle 1+\mathrm{ad}(x), \mathcal{V}_0, \mathcal{V}_1, \ldots, \mathcal{V}_{n-1}, \mathcal{W}\rangle$.

Lemma 2.2.2. The following commutator relations hold in G:
(a) $[1+\mathrm{ad}(w_A), 1+\mathrm{ad}(x)] = 1+\mathrm{ad}(v(0)_A)$.
(b) $[1+\mathrm{ad}(v(i)_A), 1+\mathrm{ad}(x)] = 1$.
(c) $[1+\mathrm{ad}(v(i)_A), 1+\mathrm{ad}(v(j)_B)] = 1+\binom{j+1}{n-i}\mathrm{ad}(v(i\oplus j)_{A\sqcup B})$.
(d) $[1+\mathrm{ad}(v(i)_A), 1+\mathrm{ad}(w_B)] = 1+\mathrm{ad}(v(i+1)_{A\sqcup B})$, if $0 \leqslant i \leqslant n-2$.
(e) $[1+\mathrm{ad}(v(n-1)_A), 1+\mathrm{ad}(w_B)] = 1+\mathrm{ad}(w_{A\sqcup B})$.

Proof. (a) We have

$$\begin{aligned}
& [1+\mathrm{ad}(w_A), 1+\mathrm{ad}(x)] \\
=\ & (1+\mathrm{ad}(w_A))\cdot(1+\mathrm{ad}(x))\cdot(1+\mathrm{ad}(w_A))\cdot(1+\mathrm{ad}(x)) \\
=\ & 1+\mathrm{ad}(w_A)\mathrm{ad}(x)+\mathrm{ad}(x)\mathrm{ad}(w_A)+\mathrm{ad}(x)\mathrm{ad}(w_A)\mathrm{ad}(x) \\
=\ & 1+\mathrm{ad}(w_A\cdot x) \\
=\ & 1+\mathrm{ad}(v(0)_A),
\end{aligned}$$

where we have used Lemma 2.2.1. Part (b) is proven similarly. For (c) we have

$$\begin{aligned}
& [1+\mathrm{ad}(v(i)_A), 1+\mathrm{ad}(v(j)_B)] \\
=\ & (1+\mathrm{ad}(v(i)_A))(1+\mathrm{ad}(v(j)_B)(1+\mathrm{ad}(v(i)_A))(1+\mathrm{ad}(v(j)_B)) \\
=\ & 1+\mathrm{ad}(v(i)_A)\mathrm{ad}(v(j)_B)+\mathrm{ad}(v(j)_B)\mathrm{ad}(v(i)_A) \\
=\ & 1+\mathrm{ad}(v(i)_A\cdot v(j)_B) \\
=\ & 1+\binom{j+1}{n-i}\mathrm{ad}(v(i\oplus j)_{A\sqcup B}).
\end{aligned}$$

Parts (d) and (e) are proved similarly.

□

Remark. Notice that as L^* is locally nilpotent it follows from Lemma 2.2.2 that G is locally nilpotent. Next proposition clarifies the structure of G.

Proposition 2.2.3. We have $G = \langle 1+\mathrm{ad}(x)\rangle \mathcal{V}_0 \cdots \mathcal{V}_{n-1}\mathcal{W}$. In particular, for every element $g \in G$ there exists an expression $g = (1+\mathrm{ad}(x))^{\epsilon} \cdot r_0 \cdots r_{n-1} \cdot s$, with $\epsilon \in \{0,1\}, r_0 \in \mathcal{V}_0, \ldots, r_{n-1} \in \mathcal{V}_{n-1}$ and $s \in \mathcal{W}$.

Proof. Suppose that

$$g = g_0(1+\mathrm{ad}(x))g_1 \cdots (1+\mathrm{ad}(x))g_n$$

where $g_0, \ldots, g_n$ are products of elements of the form $1+\mathrm{ad}(v(0)_A), \ldots, 1+\mathrm{ad}(v(n-1)_A)$, $1+\mathrm{ad}(w_A)$. From Lemma 2.2.2 we know that $(1+\mathrm{ad}(w_A))(1+\mathrm{ad}(x)) = (1+\mathrm{ad}(x))(1+\mathrm{ad}(w_A))(1+\mathrm{ad}(v(0)_A))$ and $(1+\mathrm{ad}(x))$ commutes with all products of the form $1+\mathrm{ad}(v(i)_A)$, for $0 \leqslant i \leqslant n-1$. We can thus collect the $(1+\mathrm{ad}(x))$'s to the left starting with the leftmost occurrence. This may introduce more elements of the form $1+\mathrm{ad}(v(0)_A)$, but no new elements $1+\mathrm{ad}(x)$. We thus have that

$$g = (1+\mathrm{ad}(x))^n g_1 \cdots g_m$$

where g_i is of the form $1+\mathrm{ad}(v(j)_A)$, for $0 \leqslant j \leqslant n-1$ or of the form $1+\mathrm{ad}(w_A)$. Notice also that we can assume that $n = \epsilon$, where $\epsilon \in \{0,1\}$. This reduces our problem to the case when $g \in \langle \mathcal{V}_0, \ldots, \mathcal{V}_{n-1}, \mathcal{W}\rangle$. Suppose

$$g = g_1 g_2 \cdots g_n,$$

where the terms g_i are $(1+\mathrm{ad}(v(0)_{A_1})), \ldots, (1+\mathrm{ad}(v(0)_{A_r})), \ldots, (1+\mathrm{ad}(v(n-1)_{B_1})), \ldots,$ $(1+\mathrm{ad}(v(n-1)_{B_s})), (1+\mathrm{ad}(w_{C_1})), \ldots, (1+\mathrm{ad}(w_{C_t}))$ in some order. By Lemma 2.2.2 we have that $(1+\mathrm{ad}(v(j)_B))(1+\mathrm{ad}(v(i)_A)) = (1+\mathrm{ad}(v(i)_A))(1+\mathrm{ad}(v(j)_B))(1+\binom{j+1}{n-i}\mathrm{ad}(v(i\oplus j)_{A\sqcup B}))$ and $(1+\mathrm{ad}(w_B))(1+\mathrm{ad}(v(i)_A) = (1+\mathrm{ad}(v(i)_A))(1+\mathrm{ad}(w_B))(1+\mathrm{ad}(v(i+1)_{A\sqcup B}))$, if $0 \leqslant i \leqslant n-2$ or $(1+\mathrm{ad}(w_B))(1+\mathrm{ad}(v(n-1)_A)) = (1+\mathrm{ad}(v(n-1)_A))(1+\mathrm{ad}(w_B))(1+\mathrm{ad}(w_{A\sqcup B}))$, if $i = n-1$. We can thus collect the terms so that

$$\begin{aligned} g &= (1+\mathrm{ad}(v(0)_{A_1}))\cdots(1+\mathrm{ad}(v(0)_{A_r}))\cdots(1+\mathrm{ad}(v(n-1)_{B_1}))\cdots \\ &\quad (1+\mathrm{ad}(v(n-1)_{B_s}))\cdot(1+\mathrm{ad}(w_{C_1}))\cdots(1+\mathrm{ad}(w_{C_t}))\cdot h_1\cdots h_m, \end{aligned}$$

where h_i are of the form $1+\mathrm{ad}(v(i)_D)$, with $0 \leqslant i \leqslant n-1$, or of the form $1+ad(w_D)$, where D is a modified union of at least two sets from

$$\mathcal{S} = \{A_1, \ldots, A_r, \ldots, B_1, \ldots, B_s, C_1, \ldots, C_t\}.$$

Thus,

$$g = a_0 a_1 \cdots a_{n-1} a h,$$

where $a_i \in \mathcal{V}_i$, for $0 \leqslant i \leqslant n-1$, $a \in \mathcal{W}$ and h is a product of elements of the form $1 + \mathrm{ad}(v(i)_D)$, with $0 \leqslant i \leqslant n-1$, or of the form $1 + ad(w_D)$, where D is a modified union of at least two sets from $\mathcal{S}$.

Repeating this collection process we get

$$g = b_0 b_1 \cdots b_{n-1} b k,$$

where $b_i \in \mathcal{V}_i$, for $0 \leqslant i \leqslant n-1$, $b \in \mathcal{W}$ and k is a product of elements of the form $1 + \mathrm{ad}(v(i)_E)$, where $0 \leqslant i \leqslant n-1$, or of the form $1 + ad(w_E)$, where E is a modified union of at least three sets from $\mathcal{S}$.

Continuing in this manner we conclude that after k steps

$$g = c_0 c_1 \cdots c_{n-1} c f,$$

where $c_i \in \mathcal{V}_i$, for $0 \leqslant i \leqslant n-1$, $c \in \mathcal{W}$ and f is a product of elements of the form $1 + \mathrm{ad}(v(i)_H)$, where $0 \leqslant i \leqslant n-1$, or of the form $1 + ad(w_H)$, where H is a modified union of at least $k+1$ sets from $\mathcal{S}$. However, any modified union of at least $r + \cdots + s + t + 1$ sets from $\mathcal{S}$ is trivial and thus $f = 1$ when $k = r + \cdots + s + t$. This completes the proof. □

Theorem 2.2.4. The element $1 + \mathrm{ad}(x)$ is a left 3-Engel element in G such that $\langle 1 + \mathrm{ad}(x) \rangle^G$ is not nilpotent.

Proof. Showing that $1 + \mathrm{ad}(x)$ is a left 3-Engel element in G is equivalent to showing that $[(1 + \mathrm{ad}(x))^g, 1 + \mathrm{ad}(x)]$ commutes with $1 + \mathrm{ad}(x)$ for all $g \in G$. Let $g = h(1 + \mathrm{ad}(w_{A_1})) \cdots (1 + \mathrm{ad}(w_{A_k}))$ be an arbitrary element in G, where $h \in \langle (1 + \mathrm{ad}(x)) \rangle \mathcal{V}_0 \cdots \mathcal{V}_{n-1}$. We want to show that

$$[(1 + \mathrm{ad}(x))^g, 1 + \mathrm{ad}(x), 1 + \mathrm{ad}(x)] = 1.$$

Let $y \in L^*$. Then

$$(1 + \mathrm{ad}(y))^{1 + \mathrm{ad}(w_A)} = (1 + \mathrm{ad}(w_A))(1 + \mathrm{ad}(y))(1 + \mathrm{ad}(w_A)) = 1 + \mathrm{ad}(y) + \mathrm{ad}(y \cdot w_A).$$

Notice that $(1 + \mathrm{ad}(x))^g = (1 + \mathrm{ad}(x))^{(1 + \mathrm{ad}(w_{A_1})) \cdots (1 + \mathrm{ad}(w_{A_k}))}$, since by Lemma 2.2.2 we know that $1 + \mathrm{ad}(x)$ commutes with all elements of the form $1 + \mathrm{ad}(v(i)_B)$, for

$0 \leqslant i \leqslant n-1$. Then, by induction we obtain that

$$(1+\mathrm{ad}(x))^g = 1+\mathrm{ad}(y)$$

where

$$y = x + \sum_{1\leqslant i\leqslant k} v(0)_{A_i} + \sum_{1\leqslant i<j\leqslant k} v(1)_{A_i\sqcup A_j} + \cdots + \sum_{1\leqslant i(1)<i(2)<\ldots<i(n+1)\leqslant k} w_{A_{i(1)}\sqcup\ldots\sqcup A_{i(n+1)}}.$$

Since $\mathrm{ad}(x)\mathrm{ad}(y)\mathrm{ad}(x)=0$, the commutator of $(1+\mathrm{ad}(x))^g$ with $(1+\mathrm{ad}(x))$ is

$$\begin{aligned}(1+\mathrm{ad}(y))(1+\mathrm{ad}(x))(1+\mathrm{ad}(y))(1+\mathrm{ad}(x)) &= 1+\mathrm{ad}(y)\mathrm{ad}(x)+\mathrm{ad}(x)\mathrm{ad}(y)\\ &\quad +\mathrm{ad}(y)\mathrm{ad}(x)\mathrm{ad}(y).\end{aligned}$$

Then,

$$\begin{aligned}[(1+\mathrm{ad}(x))^g, 1+\mathrm{ad}(x), 1+\mathrm{ad}(x)] &= [(1+\mathrm{ad}(x))(1+\mathrm{ad}(y))]^4, \text{ by Lemma 1.3.7}\\ &= (1+\mathrm{ad}(y)\mathrm{ad}(x)+\mathrm{ad}(x)\mathrm{ad}(y)+\mathrm{ad}(y)\mathrm{ad}(x)\mathrm{ad}(y))^2\\ &= 1,\end{aligned}$$

using the fact that $\mathrm{ad}(x)\mathrm{ad}(y)\mathrm{ad}(x) = \mathrm{ad}(y)^2 = \mathrm{ad}(x)^2 = 0$.

However, the normal closure of $1+\mathrm{ad}(x)$ in G is not nilpotent, as for $A_i = \{i\}$ we have

$$\begin{aligned}&[1+\mathrm{ad}(w_{A_0}), 1+\mathrm{ad}(x), 1+\mathrm{ad}(w_{A_1}), 1+\mathrm{ad}(w_{A_2}), \ldots, 1+\mathrm{ad}(w_{A_n}), \ldots,\\ &\quad 1+\mathrm{ad}(x), 1+\mathrm{ad}(w_{A_{mn+1}}), 1+\mathrm{ad}(w_{A_{mn+2}}), \ldots, 1+\mathrm{ad}(w_{A_{(m+1)n}})]\\ =&\ 1+\mathrm{ad}(w_B),\end{aligned}$$

where $B = A_0 \sqcup A_1 \sqcup \ldots \sqcup A_{(m+1)n} = \{0,1,2,\ldots,(m+1)n\}$. □

One might wonder if the nilpotency class of the subgroup generated by r conjugates is unbounded for the family we have constructed. That turns out not to be the case. Our next aim is to show that the subgroup generated by any r conjugates $(1+\mathrm{ad}(x))^{g_1}, \ldots, (1+\mathrm{ad}(x))^{g_r}$ of $1+\mathrm{ad}(x)$ in G will be nilpotent of r-bounded class.

We first work in a more general setting. For each $e \in E$ and $\emptyset \neq A \subseteq \mathbb{N}$, let

$e(A) \in \mathrm{End}(L^*)$ where

$$\begin{aligned} v(i)_B \cdot e(A) &= (v(i) \cdot e)_{B \sqcup A} \\ w_B \cdot e(A) &= (w \cdot e)_{B \sqcup A} \\ x \cdot e(A) &= (x \cdot e)_A, \end{aligned}$$

for $0 \leqslant i \leqslant n-1$.

Then let $E^* = \langle \mathrm{ad}(x), e(A) : e \in E \text{ and } \emptyset \neq A \subseteq \mathbb{N} \rangle$. As L^* is locally nilpotent, one sees readily that the elements of E^* are nilpotent and thus $1 + E^*$ is a subgroup of $\mathrm{End}(L^*)$.

Despite the fact that the normal closure of $1 + \mathrm{ad}(x)$ in G is not nilpotent, it turns out that the nilpotency class of the subgroup generated by any r conjugates grows linearly with respect to r. In order to see this we must first introduce some more notation. In relation with the r conjugates we let $A_1, A_2, \ldots, A_r$ be any r subsets of $\mathbb{N}$. For each r-tuple $(i1, i2, \ldots, ir)$ of non-negative integers and each $e \in E$ we let

$$e^{(i1,\ldots,ir)} = \sum_{\substack{B_1 \subseteq A_1 \\ |B_1| = i1}} \cdots \sum_{\substack{B_k \subseteq A_r \\ |B_r| = ir}} e(B_1 \sqcup B_2 \sqcup \cdots \sqcup B_r).$$

Notice that

$$e^{(i1,\ldots,ir)} \cdot f^{(j1,\ldots,jr)} = \binom{i1+j1}{i1} \cdots \binom{ir+jr}{ir} (ef)^{(i1+j1,\ldots,ir+jr)}.$$

Consider the r conjugates of $(1 + \mathrm{ad}(x))$ in G. Recall that each conjugate is of the from $(1+\mathrm{ad}(x))^{(1+\mathrm{ad}(w_{C_1}))\cdots(1+\mathrm{ad}(w_{C_j}))}$. Without loss of generality one can assume that each C_k is a singleton set. The following argument also works for the more general case. Let

$$A_1 = \{1, \ldots, k_1\},\ A_2 = \{k_1 + 1, \ldots, k_2\}, \ldots, A_r = \{k_{r-1} + 1, \ldots, k_r\}$$

and

$$e_1 = \mathrm{ad}(v(0)), e_2 = \mathrm{ad}(v(1)), \ldots, e_n = \mathrm{ad}(v(n-1)), e_{n+1} = \mathrm{ad}(w).$$

Then we have seen (see the proof of Theorem 2.2.4) that

$$\begin{aligned}
(1+\mathrm{ad}(x))^{(1+\mathrm{ad}(w_1))\cdots(1+\mathrm{ad}(w_{k_1}))} &= 1+\mathrm{ad}(x)+e_1^{(1,0,\ldots,0)}+\cdots+e_{n+1}^{(n+1,0,\ldots,0)}\\
&\vdots\\
(1+\mathrm{ad}(x))^{(1+\mathrm{ad}(w_{k_{r-1}+1}))\cdots(1+\mathrm{ad}(w_{k_r}))} &= 1+\mathrm{ad}(x)+e_1^{(0,\ldots,0,1)}+\cdots+e_{n+1}^{(0,\ldots,0,n+1)}.
\end{aligned}$$

Let

$$f_{(i,k)} = e_i^{(0,\ldots,0,i,0,\ldots,0)},$$

where i is the k-th coordinate and $1 \leqslant i \leqslant n+1$. Let

$$F = \langle f_{(j,k)} = e_j^{(0,\ldots,0,j,0,\ldots,0)} : 1 \leqslant j \leqslant n+1, \quad 1 \leqslant k \leqslant r\rangle.$$

Our aim is to find an upper bound for the nilpotence class of F. For this we need to understand better the two aspects of multiplying $e^{(i1,\ldots,ir)}$ and $f^{(j1,\ldots,jr)}$. These are

A. Under which conditions the binomial coefficients are non-trivial ;

B. Under which conditions is the Lie product $e_i \cdot e_j$ non trivial.

The next two lemmas will help clarify these questions.

Lemma 2.2.5. If $i+j \geqslant n+2$, then $\binom{i+j}{i}$ is even, where $0 \leqslant i,j \leqslant n+1$.

Proof. Suppose that

$$\begin{aligned}
i &= \alpha_0 + 2\alpha_1 + \ldots + 2^{m-1}\alpha_{m-1}\\
j &= \beta_0 + 2\beta_1 + \ldots + 2^{m-1}\beta_{m-1}.
\end{aligned}$$

We have that $\binom{i+j}{i}$ is odd if and only if $i \leqslant_2 i+j$. The latter happens if and only if there exists no l such that $\alpha_l = \beta_l = 1$, where $0 \leqslant l \leqslant m-1$. In particular, for the binomial coefficient to be non-zero we need $i+j \leqslant 1+2+\cdots+2^{m-1} = 2^m - 1 = n+1$. □

Lemma 2.2.6. If $i+j \leqslant n-1$ and $1 \leqslant i,j \leqslant n$, then $e_i \cdot e_j = 0$.

Proof. As a preparation we first show that $v(i)\cdot v(j) = 0$ if $i+j \le n-2$. To see this let

$$\begin{aligned}
i+1 &= a_0 + 2a_1 + \ldots + 2^{m-1}a_{m-1}\\
j+1 &= b_0 + 2b_1 + \ldots + 2^{m-1}b_{m-1}\\
n+1-(i+1) &= (1-a_0) + 2(1-a_1) + \ldots + 2^{m-1}(1-a_{m-1}),
\end{aligned}$$

where $0 \leqslant a_l, b_l \leqslant 1$ for $0 \leqslant l \leqslant m-1$. Then

$$\binom{j+1}{n+1-(i+1)} \text{ is odd } \iff 1 \leqslant a_l + b_l \text{ for all } 0 \leq l \leq m-1.$$

In particular for $\binom{j+1}{n+1-(i+1)}$ to be odd we need $i+1+j+1 \geqslant 1+2+\cdots+2^{m-1} = 2^m-1 = n+1$. That is we need $i+j \geqslant n-1$. Thus if $i+j \leqslant n-2$ we have $v(i)\cdot v(j) = 0$. Having established this preliminary result we turn to $e_i \cdot e_j$ where $1 \leq i, j \leq n$. Firstly,

$$\begin{aligned} we_ie_j &= wv(i-1)v(j-1) = v(i)v(j-1) \\ &= \binom{j}{n+1-(i+1)} v(i \oplus (j-1)) \end{aligned}$$

and so it immediately follows from the result above that $we_ie_j = 0$, if $i+j-1 \leqslant n-2$, that is if $i+j \leqslant n-1$. Then, for $0 \leqslant k \leqslant n-1$, we have

$$v(k)e_i \cdot e_j = v(k)v(i-1)v(j-1).$$

From our preliminary result above we know that $v(k)v(i-1) = 0$, if $k+(i-1) \leqslant n-2$. We can therefore assume that $k+(i-1) \geqslant n-1$ and hence $k \oplus (i-1) = k+(i-1)-(n-1)$. Therefore,

$$\begin{aligned} v(k)e_i \cdot e_j &= \binom{i}{n+1-(k+1)} v(k+(i-1)-(n-1))v(j-1) \\ &= \binom{i}{n+1-(k+1)} v(k+i-n)v(j-1) \end{aligned}$$

which, by our preliminary result again, is trivial when $k+i-n+j-1 \leqslant n-2$, that is if $k+i+j \leqslant 2n-1$. Given that $0 \leqslant k \leqslant n-1$ we have in particular that the product is trivial when $i+j \leqslant n-1$. Hence, $e_i \cdot e_j = 0$ if $i+j \leqslant n-1$. □

From these two lemmas we get the following result.

Lemma 2.2.7. Let $1 \leq i, j \leq n$. If $f_{(i,k)} \cdot f_{(j,s)}$ is nonzero, then $n \leqslant i+j \leqslant n+1$.

We are now ready for establishing the linear upper bound for the nilpotence class of F.

Lemma 2.2.8. F is nilpotent of a class at most $4r-1$.

Proof. Notice that by Lemma 2.2.5 we have that $f_{(n+1,k)} \cdot f_{(j,s)} = 0$ for any $1 \leq j \leq n+1$. We thus only need to consider products of elements $f_{(i,k)}$ where $1 \leq i \leq n$. Take any such product of even length $2u$ where u is going to be determined later. Suppose the product is

$$(f_{(i_1,k_1)}f_{(j_1,s_1)})\cdots(f_{(i_u,k_u)}f_{(j_u,s_u)}), \tag{2.1}$$

where $1 \leqslant i_1,\dots,i_u,\dots,j_1,\dots,j_u \leqslant n$ and $1 \leqslant k_1,\dots,k_u,\dots,s_1,\dots,s_u \leqslant r$. We want to determine u so that this product becomes 0. From Lemma 2.2.7 we know that for this product to be non-trivial we need $i_l + j_l \geqslant n$, for all $1 \leqslant l \leqslant u$. For $1 \leq l \leq r$, let t_l be the sum of all the lth coordinates of the superfixes of the $2u$ elements in the product. For this to be non-zero we need

$$t_1 + \ldots + t_r = (i_1 + j_1) + \cdots + (i_u + j_u) \geqslant nu.$$

If one of $t_1,\dots,t_r$ is greater than $n+1$ then Lemma 2.2.5 implies that the product is zero. We need to find how big u has to be so that this happens. Notice that the largest value of $t_1,\dots,t_r$ is greater or equal to the mean value and this is at least $\frac{nu}{r}$. Therefore, it suffices that

$$\frac{nu}{r} \geqslant n+2.$$

This holds when $u = 2r$. Hence, $F^{2(2r)} = F^{4r} = 0$. □

From this it is not difficult to obtain a linear upper bound for nilpotence class of the group generated by r conjugates of $1 + \mathrm{ad}(x)$. First we extend the analysis of F to the subalgebra $Q = \langle \mathrm{ad}(x), F\rangle$ of E^*. If no element $f_{(n+1,k)}$ occurs then any product of elements of Q of length $4r+1$ is trivial. Here we are using the fact that $\mathrm{ad}(x)^2 = 0$, Lemma 2.2.8 and the fact that $\mathrm{ad}(x)$ commutes with elements of the form $f_{(i,l)}$ when $1 \leq i \leq n$. Suppose therefore that at least one element $f_{(n+1,k)}$ is involved in a product of elements of Q. If an element of the form $f_{(i,l)}$ included, where $1 \leqslant i \leqslant n$, we can pick $f_{(n+1,k)}$ and $f_{(i,l)}$ that are of closest distance within the product. This reduces things to the following situations:

$$\begin{aligned}
&\cdots f_{(n+1,k)}\,\mathrm{ad}(x) f_{(i,l)}\cdots,\\
&\cdots f_{(i,l)}\,\mathrm{ad}(x) f_{(n+1,k)}\cdots,\\
&\cdots f_{(n+1,k)} f_{(i,l)}\cdots,\\
&\cdots f_{(i,l)} f_{(n+1,k)}\cdots.
\end{aligned}$$

These products are trivial, as $\mathrm{ad}(x)$ commutes with $f_{(i,l)}$ and the products of $f_{(n+1,k)}$ and $f_{(i,l)}$ are trivial by Lemma 2.2.5. We can therefore assume that no element $f_{(i,l)}$ is involved, where $1 \leq i \leq n$. Then we only have a product in $\mathrm{ad}(x)$'s and elements of the form $f_{(n+1,k)}$. By Lemma 2.2.5 and the fact that $\mathrm{ad}(x)^2 = 0$ the product needs to alternate between $\operatorname{ad}(x)$ and elements of the form $f_{(n+1,k)}$. We will make use of the fact that

$$\operatorname{ad}(x) f_{(n+1,k)} = f_{(n+1,k)} \operatorname{ad}(x) + f_{(1,k)}.$$

We have

$$\begin{aligned}
\operatorname{ad}(x) f_{(n+1,k_1)} \operatorname{ad}(x) f_{(n+1,k_2)} &= (f_{(n+1,k_1)} \operatorname{ad}(x) + f_{(1,k_1)}) \operatorname{ad}(x) f_{(n+1,k_2)} \\
&= f_{(n+1,k_1)} \operatorname{ad}(x)^2 f_{(n+1,k_2)} + f_{(1,k_1)} \operatorname{ad}(x) f_{(n+1,k_2)} \\
&= 0 + \operatorname{ad}(x) f_{(1,k_1)} f_{(n+1,k_2)} = 0,
\end{aligned}$$

where the last equality follows from Lemma 2.2.5. Similarly any product of the form $f_{(n+1,k_1)} \operatorname{ad}(x) f_{(n+1,k_2)} \operatorname{ad}(x)$ is 0. To conclude we have seen that $Q^{4r+1} = 0$. Now let H be a subgroup of G generated by any r conjugates $(1 + \operatorname{ad}(x))^{g_1}, \ldots, (1 + \operatorname{ad}(x))^{g_r}$ of $1 + \operatorname{ad}(x)$ in G. Then

$$\gamma_{4r+1}(H) \leqslant 1 + Q^{4r+1} = 1.$$

Thus we have the following result.

Proposition 2.2.9. Let $(1+\operatorname{ad}(x))^{g_1}, \ldots, (1+\operatorname{ad}(x))^{g_r}$ be any r conjugates of $1+\operatorname{ad}(x)$ in G. Then $H = \langle (1 + \operatorname{ad}(x))^{g_1}, \ldots, (1 + \operatorname{ad}(x))^{g_r} \rangle$ is nilpotent of class at most $4r$.

Chapter 3

Locally finite p-groups with a left 3-Engel element whose normal closure is not nilpotent

In this Chapter we give an example of a locally finite p-group G containing a left 3-Engel element x where $\langle x\rangle^G$ is not nilpotent, for any odd prime p. The odd case turns out to be more involved and the construction in many ways quite different from the $p = 2$ case.

The structure of the Chapter is as follows. In Section 4.1 we describe the pair (G, x) that will provide our example and we show directly that x is a left 3-Engel element in G. In order to show that $\langle x\rangle^G$ is not nilpotent we need more work. In Sections 4.2 and 4.3 we construct a pair (L, z) where L is a Lie algebra over $\mathbb{F}_p$, the field of p elements, and where $\mathrm{Id}(z)$ is not nilpotent. The pair (L, z) can be seen as the Lie algebra version of our group construction and in Section 4.4 we build a group H within $\mathrm{End}(L)$ containing $1 + \mathrm{ad}(z)$ where $\langle 1 + \mathrm{ad}(z)\rangle^H$ is not nilpotent and where $(H, 1 + \mathrm{ad}(z))$ is a homomorphic image of (G, x).

3.1 The group G

Let $x, a_1, a_2, \ldots$ be group variables. We recall that a simple commutator in $x, a_1, a_2, \ldots$ is a group word defined recursively as follows: $x, a_1, a_2, \ldots$ are simple commutators and if u, v are simple commutators then $[u, v]$ is a simple commutator. We say that a simple commutator s has multi-weight $(m, e_1, e_2, \ldots)$ in $x, a_1, a_2, \ldots$, if x occurs m times and a_i occurs e_i times in s. The weight of s is then $m + e_1 + e_2 + \cdots$. The following definition will also play a crucial role.

Definition. Let s be a simple commutator of multi-weight $(m, e_1, e_2, \ldots)$ in $x, a_1, a_2, \ldots$. The *type* of s is $t(s) = e_1 + e_2 + \ldots - 2m$.

Remark. If u, v are simple commutators in $x, a_1, a_2, \ldots$ then $t([u, v]) = t(u) + t(v)$. In particular $t([u, a_j]) = t(u) + 1$ and $t([u, x]) = t(u) - 2$.

For a fixed odd prime p, let $G = \langle x, a_1, a_2, \ldots \rangle$ be the freest group satisfying the following relations:

1. $\langle a_i \rangle^G$ is abelian for all $i \geq 1$;
2. $\langle x \rangle^G$ is metabelian;
3. $x^p = a_1^p = a_2^p = \cdots = 1$;
4. if $s \neq x$ is a simple commutator in $x, a_1, a_2, \ldots$, where $t(s) \geq 2$ or $t(s) \leq -2$, then $s = 1$.

Remarks.

- Notice that G is a locally finite p-group. Let us understand why this is true. Firstly, notice that the elementary abelian subgroups $\langle a_i \rangle^G$ are locally finite, for all i. As the product of abelian locally finite subgroups is locally finite then $N := \langle a_1 \rangle^G \cdot \langle a_2 \rangle^G \cdots$ is a locally finite subgroup of G. Thus, since $\langle x \rangle^G$ in G is cyclic, we have that $G/N = \langle xN \rangle$ is of order p. Hence G is (locally finite)-by-finite, so locally finite.
- From (4) it follows that all the $a_i's$ commute, as all simple commutators of the form $[a_i, a_j]$, for $1 \leqslant i, j$ are of type 2.
- One can see that if s is a simple commutator in $x, a_1, a_2, \ldots$, then $[s, x] = 1$ whenever $t(s) \neq 1$. This is something we will make use of later.

Let s be a left-normed commutator of weight $3m + 1$ in $x, a_1, a_2, \ldots$, starting in x. The reader can convince himself/herself that from (4) it follows that such a commutator will be trivial if it is not of the form $[x, a_{j_1}, a_{j_2}, a_{j_3}, x, a_{j_4}, a_{j_5}, \ldots, x, a_{j_{2m}}, a_{j_{2m+1}}]$. We will establish later that there are such non-trivial commutators for any m that implies that $\langle x \rangle^G$ is not nilpotent. However the structure of G is not transparent enough at this stage for us to see this directly. Instead will will use Lie ring methods to lift up this property from an analogous Lie ring setting that we will deal with in the next two sections. On the other hand we can establish directly that x is a left 3-Engel element and this we will do now for the rest of this section. We start with a useful lemma.

Lemma 3.1.1. Let $g \in G$. We can write $g = \alpha\beta\gamma\delta$ where $\alpha, \beta, \gamma, \delta$ are products of simple commutators in $x, a_1, a_2, \ldots$ of types $1, -2, -1, 0$.

Proof. We prove first by induction on $k \geq 1$ that we can write $g = \alpha_k\beta_k\gamma_k\delta_k\epsilon(k+1)$, where $\alpha_k, \beta_k, \gamma_k, \delta_k$ are products of simple commutators in $x, a_1, a_2, \ldots$ of types $1, -2, -1, 0$ and $\epsilon(k+1)$ is a product of commutators of weight $k+1$ or higher. For the induction basis observe that we can write $g = x_1x_2\cdots x_m$ where $x_j \in \{x, a_1, a_2, \ldots\}$ and modulo commutators of weight 2 or higher we can rearrange the product so that $g = \alpha_1\beta_1\epsilon(2)$ and where α_1 is a product of elements from $\{a_1, a_2, \ldots\}$, β_1 is a power of x and $\epsilon(2)$ is a product of simple commutators of weight 2 or higher (here $\gamma_1 = \delta_1 = 1$). For the induction step suppose that $k \geq 2$ and that the inductive claim holds for smaller values of k. Thus we know that we can write $g = \alpha_{k-1}\beta_{k-1}\gamma_{k-1}\delta_{k-1}\epsilon(k)$ where $\alpha_{k-1}, \beta_{k-1}, \gamma_{k-1}, \delta_{k-1}$ are products of simple commutators in $x, a_1, a_2, \ldots$ of types $1, -2, -1, 0$ and where $\epsilon(k)$ is a product of simple commutators of weight k or higher. Working modulo commutators of weight $k+1$ and higher, we can now collect the commutators of weight k so that they appear with the commutator of smaller weight that are of the same type. This gives us the expression $g = \alpha_k\beta_k\gamma_k\delta_k\epsilon(k+1)$ that we wanted. This finishes the proof of the inductive hypothesis. Notice now that we are working with simple commutators in $x_1, x_2, \ldots, x_m$ and as $H = \langle x_1, \ldots, x_m\rangle$ is nilpotent we have that for a large enough k we have $\epsilon(k+1) = 1$. □

Let $g \in G$. We want to show that $[g, x, x, x] = 1$. Let $g = \alpha\beta\gamma\delta$ be an expression for g as in Lemma 2.1. As we remarked above, all simple commutators of type different from 1 commute with x. This means that β, γ, δ commute with x and without loss of generality we can assume that

$$g = b_1 \cdots b_m$$

where $b_1, \ldots, b_m$ are simple commutators in $x, a_1, a_2, \ldots$ of type 1.

Let $H = \langle x, b_1, \ldots, b_m\rangle$. Observe that relations (1) and (2) imply that $\langle b_i\rangle^H$ is abelian and $\langle x\rangle^H$ is metabelian. Notice also that relations (1) and (3) imply that $b_1^p = \cdots = b_m^p = 1$. Finally, suppose that we have a simple commutator c in $x, b_1, \ldots, b_m$ of multi-weight $(l, f_1, f_2, \ldots, f_m)$ in $x, b_1, \ldots, b_m$ then the type of c as a simple commutator in $x, a_1, a_2, \ldots$ is $-2l + f_1t(b_1) + \cdots + f_mt(b_m) = -2l + f_1 + \cdots + f_m$ that is equal to the type of c as a simple commutator in $x, b_1, \ldots, b_m$. Hence relations (4) hold for H with $x, a_1, a_2, \ldots$ replaced by $x, b_1, \ldots, b_m$. It follows from this that there is a homomorphism $\alpha : G \to H$ that maps $x, a_1, \ldots, a_m$ to $x, b_1, \ldots, b_m$ and a_k to 1 for $k \geq m+1$. In particular $[b_1 \cdots b_m, x, x, x] = 1$ would follow from $[a_1 \cdots a_m, x, x, x] = 1$. Thus the problem of showing that x is a left 3-Engel element reduces to showing that

$[a_1 \cdots a_r, x, x, x] = 1$ for all integers $r \geq 1$. Before turning to this we need to introduce some more standard notation.

Definition. Let $y, c_1, c_2, ..., c_m$ be group elements. For any non-trivial subset $I = \{i_1, \ldots, i_k\}$ of $\{1, 2, \ldots, m\}$, where $i_1 < \ldots < i_k$ we denote by $[y, c_I]$ the commutator $[y, c_{i_1}, \ldots, c_{i_k}]$.

Proposition 3.1.2. We have $[a_1 \cdots a_r, x, x, x] = 1$ for all integers $r \geq 1$.

Proof. We prove this by induction on r. For the induction basis simply observe that $t([a_1, x, x, x]) = -5$ and thus $[a_1, x, x, x] = 1$ by relation (4). Now suppose $r \geq 2$ and that the result holds for all smaller values for r. First we see from standard commutator properties that

$$[a_1 \cdots a_r, x] = \left(\prod_{I \subseteq \{2,\ldots,r\}} [a_1, x, a_I] \right) \cdot \left(\prod_{I \subseteq \{3,\ldots,r\}} [a_2, x, a_I] \right) \ldots [a_r, x].$$

Notice that relations (1) imply that the order of factors within each sub-product above does not matter. The RHS has $s = 2^r - 1$ factors. Suppose these are $b_1, \ldots, b_s$ and thus $[a_1 \cdots a_r, x] = b_1 \cdots b_s$. We want to show that $[b_1 \cdots b_s, x, x] = 1$ or equivalently that $[x, b_1 \cdots b_s]$ commutes with x. Expanding again we see that

$$[x, b_1 \cdots b_s] = \prod_{\emptyset \neq I \subseteq \{1,\ldots,s\}} [x, b_I].$$

Notice that each factor is in $(\langle x \rangle^G)'$ and as $\langle x \rangle^G$ is metabelian the order of the factors does not matter. Notice also that every non-trivial factor is of the form $[x, [x, a_{I_1}]^{-1}, \ldots, [x, a_{I_l}]^{-1}]$ where, because of relations (1), the sets $I_1, \ldots, I_l$ are pairwise disjoint. We also have that $\min(I_1) < \min(I_2) < \ldots < \min(I_l)$. Let A be the product of such factors where not all $a_1, a_2, \ldots, a_r$ are included and let B be the product of those factors where each $a_1, a_2, \ldots, a_r$ occurs once. Then

$$[x, b_1 \cdots b_s] = AB.$$

It is not difficult to see by standard arguments that the induction hypothesis implies that A commutes with x. We are thus left with showing that B commutes with x. Now let

$$[x, [x, a_{I_1}], \ldots, [x, a_{I_m}]]^{(-1)^m}$$

be one of the factors of B. This will commute with x if the type is not 1. As $I_1 \cup \cdots \cup$

$I_m = \{1, 2, \ldots, r\}$ we have that the $t([x, [x, a_{I_1}], \ldots, [x, a_{I_m}]]) = r - 2(m+1)$ that is equal to 1 if and only if $r = 2m+3$. We are thus left with showing that $e(m)$ commutes with x where

$$e(m) = \prod_{(I_1, I_2, \ldots, I_m)} [x, [x, a_{I_1}], \ldots, [x, a_{I_m}]].$$

Here the product is taken over all partitions of $\{1, 2, \ldots, 2m+3\}$ into a disjoint union of m non-empty subsets where

$$1 = \min(I_1) < \min(I_2) < \ldots < \min(I_m).$$

As $\langle x \rangle^G$ is metabelian the value of a factor does not change if we permute the terms $[x, a_{I_2}], \ldots, [x, a_{I_m}]$. Notice also that for $[x, [x, a_{I_1}]]$ to be non-trivial we need $|I_1| = 3$. If some I_j has size less than 2 where $2 \leq j \leq m$ then (because we can permute the terms $[x, a_{I_2}], \ldots, [x, a_{I_m}]$ without changing the value) we see that the factor is trivial as $[x, [x, a_{I_1}], [x, a_{I_j}]]$ has type less than -1. For a factor of $e(m)$ to be non-trivial we thus need all of $I_2, I_3, \ldots, I_m$ to have size 2 or 3. Suppose s are of size 2 and t of size 3. Then the we get the equations

$$\begin{aligned} s + t &= m - 1 \\ 2s + 3t &= 2m. \end{aligned}$$

That has the solution $t = 2$ and $s = m - 3$ (in particular $m \geq 3$). This shows that

$$e(m) = \prod_{(I_1, I_2, \ldots, I_m)} [x, [x, a_{I_1}], \ldots, [x, a_{I_m}]]$$

where the product is taken over all partitions of $\{1, 2, \ldots, 2m+3\}$ where $|I_1| = |I_2| = |I_3| = 3$ and $|I_4| = \ldots = |I_m| = 2$ and $1 \in I_1$.

Consider any factor of this product. As $[x, a_{I_4}], \ldots, [x, a_{I_m}]$ commute with x, we have

$$[e(m), x] = \prod_{(I_1, I_2, \ldots, I_m)} [x, [x, a_{I_1}], [x, a_{I_2}], [x, a_{I_3}], x, [x, a_{I_4}], \ldots, [x, a_{I_m}]].$$

It follows from this that in order to show that $[e(m), x] = 1$ for all $m \geq 1$, it suffices to show that $[e(3), x] = 1$.

To see this take any $2 \le i < j < k < r \le 9$. We have

$$1 = [x, a_1, [x, a_i, a_j, a_k, a_r]] = \prod_{(I_1, I_2)} [x, a_{I_1}, x, a_{I_2}].$$

where the product is taken over all partitions of $\{1, i, j, k, r\}$ where $|I_1| = 3$, $|I_2| = 2$ and $1 \in I_1$. From this it follows that

$$e(3) = \prod_{(I_1, I_2, I_3)} [x, [x, a_{I_1}], [x, a_{I_2}], [x, a_{I_3}]]$$

is trivial and thus in particular $[e(3), x] = 1$. This finishes the proof. □

As a consequence we have the following theorem.

Theorem 3.1.3. The element x is a left 3-Engel element in G.

3.2 The Lie algebra L

In this section we will construct a pair (L, z) that will provide us with the Lie algebra analogue of (G, x). Most of this section will be about describing the Lie algebra L and getting some close transparent information about $\mathrm{Id}(z)$, the ideal in L generated by z. This will then be used in Section 4 to prove that $\mathrm{Id}(z)$ is not nilpotent. Throughout the paper p is a fixed odd prime and $\mathbb{F} = \mathbb{F}_p$ is the field of p elements.

We will start with a result that is probably well known although we do not have a reference. For that reason and for the convenience of the reader we will include a proof. We first need some notation. Let $z, c_1, c_2, \ldots$ be some Lie algebra variables. For any non-trivial multi-set $\{c_{i_1}, c_{i_2}, \ldots, c_{i_k}\}$ with $i_1 \le i_2 \cdots \le i_k$ we denote by $[z, c_I]$ the left normed Lie commutator $[z, c_{i_1}, \ldots, c_{i_k}]$.

Proposition 3.2.1. Let E be the largest Lie algebra over $\mathbb{F}$ generated $z, c_1, c_2, \ldots$ where $c_1, c_2, \ldots$ commute. Then $\mathrm{Id}(z)$ is freely generated by $\langle [z, c_I],\ I \subset_{fm} \mathbb{N}^m \rangle$, where I runs through all finite multi-subsets of $\mathbb{N}^m$. Here $\mathbb{N}^m$ is the multi-set variant of the set of natural numbers namely $\{1, 1, \ldots, 2, 2, \ldots, \ldots\}$.

Proof. Let A be the enveloping algebra of E, that is the largest associative algebra generated by $z, c_1, c_2, \ldots$ where $c_1, c_2, \ldots$ commute. Thus E is the Lie sub-algebra generated by $z, c_1, c_2, \ldots$ where the Lie product of $u, v \in A$ is $[u, v] = uv - vu$. In order to prove the proposition it suffices to show that the associative sub-algebra C generated by the set $\{[z, c_I],\, I \subset_{fm} \mathbb{N}^m\}$ is freely generated as an associative algebra by these elements. To see this, pick a variable y_I for each finite multi sub-set of $\mathbb{N}^m$ and consider the free associative algebra B, freely generated by these. Order the generators of A with $c_1 < c_2 < \cdots < z$ and use these to define a lexicographical order on words in these. We also order the generators y_I by

$$y_I < y_J \text{ if and only if } zc_{i_1} \cdots c_{i_r} < zc_{j_1} \cdots c_{j_s}$$

where $I = \{i_1, \ldots, i_r\}$, $J = \{j_1, \ldots, j_s\}$, $i_1 \leq \cdots \leq i_r$ and $j_1 \leq \cdots \leq j_s$. We are here using the lexicographical order above for the displayed inequalities. Notice that if we expand $[z, c_I]$ then the leading term with respect to that lexicographical order is $zc_{i_1} \cdots c_{i_r}$. We will denote the latter by zc_I. We then use this order do define a lexiographical order on words in the y_I's. Now consider the algebra homomorphism from B to A that maps y_I to $[z, c_I]$. In order to show that C is freely generated by the $[z, c_I]$'s it suffices to show that this homomorphism is injective. Thus let y be a non-zero element in B say

$$y = \alpha y_{I_1} \cdots y_{I_r} + \cdots + \beta y_{J_1} \cdots y_{J_s}$$

where $\alpha y_{I_1} \cdots y_{I_r}$ is the leading term of y with respect to the lexicographical ordering on B described above. From the definition of this lexicographical order, we see that

$$\bar{y} = \alpha zc_{I_1} \cdots zc_{I_r} + \cdots + \beta zc_{J_1} \cdots zc_{J_s}$$

has leading term $\alpha zc_{I_1} \cdots zc_{I_r}$ with respect to the lexicographical order on A. But notice that this is also the leading term of the image of y under the homomorphism, namely

$$\tilde{y} = \alpha[z, c_{I_1}] \cdots [z, c_{I_r}] + \cdots + \beta[z, c_{J_1}] \cdots [z, c_{J_s}].$$

This shows that the image of y is non-zero and thus the homomorphism is injective. This finishes the proof. □

Thus $\mathrm{Id}_E(z)$ is a free Lie algebra, freely generated by the $[z, c_I]$'s. For the following we will use a slightly different total order on this set of free generators. If I and J are both non-empty, then $[z, c_I] < [z, c_J]$ as above. However if one of the multi-sets is empty we define the order differently so that z is the largest element. That is $[z, c_I] < z$ whenever I is non-empty. Now consider the largest metabelian quotient, $M(z)$, of $\mathrm{Id}(z)$. This is the free metabelian Lie algebra over $\mathbb{F}$ with free generators the $[z, c_I]$'s. Now it is well known (see for example [3]), that $M(z)$ has a basis consisting of the left-normed Lie commutators

$$[[z, c_{I_1}], [z, c_{I_2}], \ldots, [z, c_{I_m}]]$$

where $m \geq 1$ and $[z, c_{I_1}] > [z, c_{I_2}] \leq \cdots \leq [z, c_{I_m}]$.

We are now getting ready to describe our Lie algebra L. First we consider the largest Lie algebra $F = \langle z, c_1, c_2, \ldots \rangle$ over $\mathbb{F}_p$, such that

1. $\mathrm{Id}(c_i)$ is abelian for $i = 1, 2, \ldots$;

2. $\mathrm{Id}(z)$ is metabelian.

From the analysis above, the following is a basis for $\mathrm{Id}_F(z)$

$$[[z, c_{I_1}], [z, c_{I_2}], \ldots, [z, c_{I_m}]] \tag{3.1}$$

where $m \geq 1$, $I_1, \ldots, I_m$ are pairwise disjoint and $[z, c_{I_1}] > [z, c_{I_2}] \leq \cdots \leq [z, c_{I_m}]$. We define a type of a Lie commutator in a similar way as we did for group commutators in Section 1. This term will play an important role in this section.

Definition. Let s be a simple commutator of multi-weight $(m, e_1, \ldots, e_r)$ in $z, c_1, \ldots, c_r$. The *type* of s is $t(s) = e_1 + \cdots + e_r - 2m$.

Remark. If c and d are simple commutators then $t([c, d]) = t(c) + t(d)$. In particular $t([c, z]) = t(c) - 2$ and $t([c, c_1]) = t(c) + 1$.

We will now construct a Lie algebra L that will be a quotient of F by a certain multi-homogeneous ideal J. Note that since F is multi-graded, L will also be multi-graded. Our multi-homogeneous ideal will be the smallest ideal of F containing the following elements from our basis for $\mathrm{Id}_F(z)$ described above. Let $c = [[z, c_{I_1}], [z, c_{I_2}], \ldots, [z, c_{I_m}]]$ be one of these basis elements.

3. c is in J if $c \neq z$ is a commutator of type either less than -1 or more than 1;

4. c is in J if one of $I_1, I_3, \ldots, I_m$ has size greater than 2.

In this and following section we will determine the structure of J and prove that $\mathrm{Id}_L(z)$ is not nilpotent.

3.2.1 Some detailed analysis about which basis elements are in J

One can see directly that a number of the commutators in (3.1) will be in J as a consequence of conditions (3) and (4). In this section, we will now look more closely at these commutators and see that a number of elements are in J as a less direct consequence. Consider one of the basis elements, $[[z, c_{I_1}], [z, c_{I_2}], \ldots, [z, c_{I_m}]]$, from (3.1) above. Note that, in particular, by condition (3) we have that $[z, c_I]$ is in J when $|I| \geq 4$. In the following we will denote with α the number of I_i of size 1 and β those of size 2, for $i \geq 3$.

Remark. Notice that, since $\mathrm{Id}(z)$ is metabelian, it does not matter how we order $I_3, \ldots, I_m$ as they can be permuted without changing the value of the Lie commutator. In fact, the only constraint is that the smallest i in $I_1 \cup \cdots \cup I_m$ should be in I_2, since $[z, c_{I_1}] > [z, c_{I_2}] \leq \cdots \leq [z, c_{I_m}]$.

Commutators of type -1 where $|I_1|+|I_2| = 3$. In this case we have $t([z, c_{I_1}], [z, c_{I_2}]) = -1$ and $|I_3| + \cdots + |I_m| = 2m - 4$. Notice that none of $I_3, \ldots, I_m$ is empty or of size 1 since otherwise the commutator would be in J by condition (3). Then all commutators are in J except possibly those where

$$|I_1| + |I_2| = 3, |I_3| = \cdots = |I_m| = 2, |I_1| \leq 2.$$

Commutators of type -1 where $|I_1| + |I_2| = 4$. Here $t([z, c_{I_1}], [z, c_{I_2}]) = 0$ and $|I_3| + \cdots + |I_m| = 2m - 5$. As before, none of $I_3, \ldots, I_m$ is empty since otherwise the commutator would be in J by condition (3). Furthermore none of the sets is of size greater than 2 by condition (4). We thus obtain the following system of equations

$$\begin{cases} \alpha + \beta = m - 2 \\ \alpha + 2\beta = 2m - 5 \end{cases}$$

whose solution is $\alpha = 1$ and $\beta = m - 3$. Therefore all commutators are in J except possibly those where

$$|I_1| + |I_2| = 4, |I_3| = 1, |I_4| = \cdots = |I_m| = 2, |I_1| \leq 2.$$

Commutators of type -1 where $|I_1| + |I_2| = 5$. We have $t([z, c_{I_1}], [z, c_{I_2}]) = 1$ and $|I_3| + \cdots + |I_m| = 2m - 6$. First, we consider the case when I_3 is empty. In this case $t([z, c_{I_1}], [z, c_{I_2}], [z, c_{I_3}]) = -1$ and none of $I_4, \ldots, I_m$ are of size 1. Thus they are all of size 2, and all commutators are in J except possibly the cases

$$|I_1| = 2, |I_2| = 3, |I_3| = 0, |I_4| + \cdots + |I_m| = 2.$$

Now it remains to deal with the case for which none of $I_3, \ldots, I_m$ is empty. Using α and β as before, we have

$$\begin{cases} \alpha + \beta = \ \ m - 2 \\ \alpha + 2\beta = 2m - 6 \end{cases}$$

whose solution is $\alpha = 2$ and $\beta = m - 4$. Thus all commutators are in J except possibly the cases where

$$|I_1| = 2, |I_2| = 3, |I_3| = |I_4| = 1, |I_5| = \cdots = |I_m| = 2.$$

Commutators of type 0 where $|I_1| + |I_2| = 3$. In this case none of $I_3, \ldots, I_m$ are empty or of size 1, as otherwise the commutator is in J by condition (3). Also, none are of size 3 by condition (4). Therefore, they must all be of size 2, which implies that the type is -1, that is a contradiction since we supposed that the type is 0.

Commutators of type 0 where $|I_1| + |I_2| = 4$. We have that none of $I_3, \ldots, I_m$ is empty. Using α and β as before, we have

$$\begin{cases} \alpha + \beta = \ \ m - 2 \\ \alpha + 2\beta = 2m - 4 \end{cases}$$

whose solution is $\alpha = 0$ and $\beta = m - 2$. Thus all commutators are in J except possibly those where

$$|I_1| + |I_2| = 4, |I_3| = \cdots = |I_m| = 2, |I_1| \leq 2.$$

Commutators of type 0 where $|I_1| + |I_2| = 5$. First, notice that none of $I_3, \ldots, I_m$ is empty. Indeed, if I_3 is empty then $t([z, c_{I_1}], [z, c_{I_2}], [z, c_{I_3}]) = -1$ and the commutator has type at most -1 if it is non-trivial. We then get a contradiction since we are considering the case when the type is 0. Thus none of $I_3, \ldots, I_m$ is empty. Therefore we have

$$\begin{cases} \alpha + \beta = \ \ m - 2 \\ \alpha + 2\beta = 2m - 5 \end{cases}$$

whose solution is $\alpha = 1$ and $\beta = m - 3$. Therefore all commutators are in J except possibly those where

$$|I_1| = 2, |I_2| = 3, |I_3| = 1, |I_4| = \cdots = |I_m| = 2.$$

Commutators of type 1. Since $|I_3| + \cdots + |I_m| \leq 2m - 4$, we need $|I_1| + |I_2| = 5$. Then all commutators are in J except possibly those where

$$|I_1| = 2, |I_2| = 3, |I_3| = \cdots = |I_m| = 2.$$

In view of the analysis above we come up with the following decomposition of $\mathrm{Id}_F(z)$ as a direct sum of sub spaces.

$$\mathrm{Id}_F(z) = \langle X \setminus Z \rangle \oplus \langle Z \rangle$$

where Z is the list of basis elements with the following partitions $(I_1, \ldots, I_m)$

$$\begin{array}{ll}
[z, c_I], \text{ for } |I| \leq 3 & \\
|I_1| + |I_2| = 3, |I_1| \leq 2, |I_3| = \cdots = |I_m| = 2 & (\zeta_1) \\
|I_1| + |I_2| = 4, |I_1| \leq 2, |I_3| = 1, |I_4| = \cdots = |I_m| = 2 & (\zeta_2) \\
|I_1| = 2, |I_2| = 3, |I_3| = 0, |I_4| = \cdots = |I_m| = 2 & (\zeta_3) \\
|I_1| = 2, |I_2| = 3, |I_3| = |I_4| = 1, |I_5| = \cdots = |I_m| = 2 & (\zeta_4) \\
|I_1| = 2, |I_2| = 3, |I_3| = 1, |I_4| = \cdots = |I_m| = 2 & (\xi_1) \\
|I_1| + |I_2| = 4, |I_1| \leq 2, |I_3| = \cdots = |I_m| = 2 & (\xi_2) \\
|I_1| = 2, |I_2| = 3, |I_3| = \cdots = |I_m| = 2. & (\tau_1)
\end{array}$$

We have seen above that $\langle X \setminus Z \rangle \subseteq J$. Now J is the smallest subspace of F containing $X \setminus Z$ that is invariant under taking Lie commutators with the c_j's and z. In the following we go systematically through all possible scenarios where we need to add new elements to $\langle X \setminus Z \rangle$ in order to get a subspace $Z_0 + \langle X \setminus Z \rangle$, where $Z_0 \leq Z$, that is invariant under Lie commutators with the c_j's. All the new elements will be relations amongst the ζ's, ξ's, and τ's. We will distinguish several cases considering the cardinality of the sets $I_1, \ldots, I_m$. We will then later ensure furthermore that we obtain a subspace that is also invariant under taking Lie commutators with z.

3.2.2 Relations among $\zeta_1, \zeta_2, \zeta_3, \zeta_4$.

We now begin systematically adding elements to $\langle X \setminus Z \rangle$ to ensure that the resulting subspace is invariant under taking Lie commutator with c_k. For this we only need to consider elements of types $-2, -1$ and 0. We start with a basis element u from $X \setminus Z$ that is of type -2. We write $[u, c_k]$ as a linear combination of basis elements in X. If all the components are in $X \setminus Z$, then nothing needs to be added. We thus only need to consider the cases when some of the components are among $\zeta_1, \zeta_2, \zeta_3$ or ζ_4. This means reducing the size of one of $I_1, \ldots, I_m$ for some of $\zeta_1, \ldots, \zeta_4$ by one, to get a basis element u of type -2 and then expand $[u, c_k]$ to get a linear combination of $\zeta_1, \ldots, \zeta_4$ modulo J. This may then give us a new relator to add to $\langle X \setminus Z \rangle$. Below we exhaust all possibilities.

Case A1: $|I_1| = 2, |I_2| = 3, |I_3| = 0, |I_4| = 1, |I_5| = \cdots = |I_m| = 2$**.**

Without loss of generality, we can assume that $I_1 \cup I_2 \cup \cdots \cup I_m \cup \{k\} = \{1, \ldots, 2m-1\}$. Recall that for the basis elements for $\mathrm{Id}_F(z)$ we should have $1 \in I_2$. For this reason we need to deal separately with the cases $k \geq 2$ and $k = 1$.

Consider first the case when $k \geq 2$. Suppose $I_4 = \{i\}$. Here as elsewhere in this section we will be calculating modulo J. We have

$$\begin{aligned} 0 &= [[z, c_{I_1}], [z, c_{I_2}], z, [z, c_i], [z, c_{I_5}], \ldots, [z, c_{I_m}], c_k] \\ &= \zeta_4(I_1, I_2, \{k\}, \{i\}, I_5, \ldots, I_m) + \zeta_3(I_1, I_2, \varnothing, \{i, k\}, I_5, \ldots, I_m). \end{aligned}$$

Therefore

$$\zeta_4(I_1, I_2, \{k\}, \{i\}, I_5, \ldots, I_m) = -\zeta_3(I_1, I_2, \varnothing, \{i, k\}, I_5, \ldots, I_m). \quad \text{(R1)}$$

This means that we need to add the relator

$$\zeta_4(I_1, I_2, \{k\}, \{i\}, I_5, \ldots, I_m) + \zeta_3(I_1, I_2, \varnothing, \{i, k\}, I_5, \ldots, I_m)$$

to $\langle X \setminus Z \rangle$.

The reader can convince himself that the case $k = 1$ does not give us a new relation.

Case A2: $|I_1| = 1, |I_2| = 3, |I_3| = 0, |I_4| = \cdots = |I_m| = 2$**.**

First consider the case when $k \geq 2$ and suppose that $I_1 = \{i\}$. We have

$$\begin{aligned} 0 &= [[z, c_i], [z, c_{I_2}], z, [z, c_{I_4}], \ldots, [z, c_{I_m}], c_k] \\ &= [[z, c_{\{i,k\}}], [z, c_{I_2}], z, [z, c_{I_4}], \ldots, [z, c_{I_m}]] \\ &+ [[z, c_i], [z, c_{I_2}], [z, c_k], [z, c_{I_4}], \ldots, [z, c_{I_m}]] \\ &= \zeta_3(\{i,k\}, I_2, \varnothing, I_4, \ldots, I_m) + \zeta_2(\{i\}, I_2, \{k\}, I_4, \ldots, I_m). \end{aligned}$$

Therefore,

$$\zeta_2(\{i\}, I_2, \{k\}, I_4, \ldots, I_m) = -\zeta_3(\{i,k\}, I_2, \varnothing, I_4, \ldots, I_m). \tag{R2}$$

Again one sees that for $k = 1$, one does not get a new relation.

Case A3: $|I_1| = |I_2| = 2, |I_3| = 0, |I_4| = \cdots = |I_m| = 2$**.**

Consider the case when $k \geq 2$. We have

$$\begin{aligned} 0 &= [[z, c_{I_1}], [z, c_{I_2}], z, [z, c_{I_4}], \ldots, [z, c_{I_m}], c_k] \\ &= [[z, c_{I_1}], [z, c_{I_2 \cup \{k\}}], z, [z, c_{I_4}], \ldots, [z, c_{I_m}]] \\ &+ [[z, c_{I_1}], [z, c_{I_2}], [z, c_k], [z, c_{I_4}], \ldots, [z, c_{I_m}]] \\ &= \zeta_3(I_1, I_2 \cup \{k\}, \varnothing, I_4, \ldots, I_m) + \zeta_2(I_1, I_2, \{k\}, I_4, \ldots, I_m). \end{aligned}$$

Therefore, we get

$$\zeta_2(I_1, I_2, \{k\}, I_4, \ldots, I_m) = -\zeta_3(I_1, I_2 \cup \{k\}, \varnothing, I_4, \ldots, I_m). \tag{R3}$$

We will handle $k = 1$ later.

Case A4: $|I_1| = 2, |I_2| = 1, |I_3| = 1, |I_4| = \cdots = |I_m| = 2$**.**

Let $I_3 = \{i\}$. For the case $k \geq 2$ we get

$$\begin{aligned} 0 &= [[z, c_{I_1}], [z, c_{\{1,k\}}], [z, c_i], [z, c_{I_4}], \ldots, [z, c_{I_m}]] \\ &+ [[z, c_{I_1}], [z, c_1], [z, c_{\{i,k\}}], [z, c_{I_4}], \ldots, [z, c_{I_m}]] \\ &= \zeta_2(I_1, \{1,k\}, \{i\}, I_4, \ldots, I_m) + \zeta_1(I_1, \{1\}, \{i,k\}, I_4, \ldots, I_m) \\ &= -\zeta_3(I_1, \{1,i,k\}, \varnothing, I_4, \ldots, I_m) + \zeta_1(I_1, \{1\}, \{i,k\}, I_4, \ldots, I_m), \end{aligned}$$

where the last equality follows from R3. Therefore,

$$\zeta_1(I_1, \{1\}, \{i,k\}, I_4, \ldots, I_m) = \zeta_3(I_1, \{1,i,k\}, \varnothing, I_4, \ldots, I_m). \tag{R4}$$

We will deal with the case $k = 1$ later.

Case A5: $|I_1| = 1, |I_2| = 2, |I_3| = 1, |I_4| = \cdots = |I_m| = 2$**.**

Consider first the case $k \geq 2$. Let $I_1 = \{i\}, I_3 = \{j\}$ and $k \geq 2$. We have

$$\begin{aligned}
0 &= [[z, c_{\{i,k\}}], [z, c_{I_2}], [z, c_j], [z, c_{I_4}], \ldots, [z, c_{I_m}]] \\
&+ [[z, c_i], [z, c_{I_2 \cup \{k\}}], [z, c_j], [z, c_{I_4}], \ldots, [z, c_{I_m}]] \\
&+ [[z, c_i], [z, c_{I_2}], [z, c_{\{j,k\}}], [z, c_{I_4}], \ldots, [z, c_{I_m}]] \\
&= \zeta_2(\{i,k\}, I_2, \{j\}, I_4, \ldots, I_m) + \zeta_2(\{i\}, I_2 \cup \{k\}, \{j\}, I_4, \ldots, I_m) \\
&+ \zeta_1(\{i\}, I_2, \{j,k\}, I_4, \ldots, I_m) \\
&= -\zeta_3(\{i,k\}, I_2 \cup \{j\}, \varnothing, I_4, \ldots, I_m) - \zeta_3(\{i,j\}, I_2 \cup \{k\}, \varnothing, I_4, \ldots, I_m) \\
&+ \zeta_1(\{i\}, I_2, \{j,k\}, I_4, \ldots, I_m),
\end{aligned}$$

where the last equality follows from R2 and R3. Therefore,

$$\begin{aligned}
\zeta_1(\{i\}, I_2, \{j,k\}, I_4, \ldots, I_m) = \zeta_3(\{i,k\}, I_2 \cup \{j\}, \varnothing, I_4, \ldots, I_m) \\
+ \zeta_3(\{i,j\}, I_2 \cup \{k\}, \varnothing, I_4, \ldots, I_m).
\end{aligned} \tag{R5}$$

We deal with the case $k = 1$ later.

Case A6: $|I_1| = 0, |I_2| = 3, |I_3| = 1, |I_4| = \cdots = |I_m| = 2$**.**

We first look at the case $k \geq 2$. Let $|I_3| = \{i\}$. We have

$$\begin{aligned}
0 &= [[z, c_k], [z, c_{I_2}], [z, c_i], [z, c_{I_4}], \ldots, [z, c_{I_m}]] \\
&+ [z, [z, c_{I_2}], [z, c_{\{i,k\}}], [z, c_{I_4}], \ldots, [z, c_{I_m}]] \\
&= \zeta_2(\{k\}, I_2, \{i\}, I_4, \ldots, I_m) + \zeta_1(\varnothing, I_2, \{i,k\}, I_4, \ldots, I_m) \\
&= -\zeta_3(\{k,i\}, I_2, \varnothing, I_4, \ldots, I_m) + \zeta_1(\varnothing, I_2, \{i,k\}, I_4, \ldots, I_m).
\end{aligned}$$

Therefore,

$$\zeta_1(\varnothing, I_2, \{i,k\}, I_4, \ldots, I_m) = \zeta_3(\{i,k\}, I_2, \varnothing, I_4, \ldots, I_m). \tag{R6}$$

Again the case $k = 1$ does not add anything new.

Notice that it follows from the relations above that modulo J we can write the basis elements of types ζ_1, ζ_2 and ζ_4 as a linear combination of basis elements of type ζ_3. Notice also that from (R6) and the fact that $\mathrm{Id}_F(z)$ is metabelian, it follows that

$$\begin{aligned}
\zeta_3(I_1, I_2, \varnothing, I_4, I_5, \ldots, I_m) &= \zeta_1(\varnothing, I_2, I_1, I_4, I_5, \ldots, I_m) \\
&= \zeta_1(\varnothing, I_2, I_4, I_1, I_5, \ldots, I_m) \\
&= \zeta_3(I_4, I_2, \varnothing, I_1, I_5, \ldots, I_m).
\end{aligned}$$

Therefore we get

$$\zeta_3(I_1, I_2, \varnothing, I_4, I_5, \ldots, I_m) = \zeta_3(I_4, I_2, \varnothing, I_1, I_5, \ldots, I_m). \tag{R7}$$

Case A7: $|I_1| = |I_2| = 1, |I_3| = \cdots = |I_m| = 2$.

Let $I_1 = \{i\}, I_3 = \{j, r\}$ and consider the case where $k \geq 2$. We have

$$\begin{aligned}
0 &= [[z, c_{\{i,k\}}], [z, c_1], [z, c_{\{j,r\}}], [z, c_{I_4}], \ldots, [z, c_{I_m}]] \\
&+ [[z, c_i], [z, c_{\{1,k\}}], [z, c_{\{j,r\}}], [z, c_{I_4}], \ldots, [z, c_{I_m}]] \\
&= \zeta_1(\{i, k\}, \{1\}, \{j, r\}, I_4, \ldots, I_m) + \zeta_1(\{i\}, \{1, k\}, \{j, r\}, I_4, \ldots, I_m \\
&= \zeta_3(\{i, k\}, \{1, j, r\}, \varnothing, I_4, \ldots, I_m) + \zeta_3(\{i, j\}, \{1, k, r\}, \varnothing, I_4, \ldots, I_m) \\
&+ \zeta_3(\{i, r\}, \{1, j, k\}, \varnothing, I_4, \ldots, I_m), \text{ (using R4 and R5).}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\zeta_3(\{i, k\}, \{1, j, r\}, \varnothing, I_4, \ldots, I_m) = &- \zeta_3(\{i, j\}, \{1, k, r\}, \varnothing, I_4, \ldots, I_m) \\
&- \zeta_3(\{i, r\}, \{1, j, k\}, \varnothing, I_4, \ldots, I_m).
\end{aligned} \tag{R8}$$

For $k = 1$ one can check that there is not a new relation.

Case A8: $|I_1| = 0, |I_2| = 2, |I_3| = \cdots = |I_m| = 2$.

Let $I_2 = \{1,i\}, I_3 = \{j,r\}$ and $k \geq 2$. We have

$$\begin{aligned} 0 &= [z,[z,c_{\{1,i,k\}}],[z,c_{I_3}],\ldots,[z,c_m]] + [[z,c_k],[z,c_{\{1,i\}}],[z,c_{I_3}],\ldots,[z,c_{I_m}]] \\ &= \zeta_1(\varnothing,\{1,i,k\},I_3,\ldots,I_m) + \zeta_1(\{k\},\{1,i\},I_3,\ldots,I_m) \\ &= \zeta_3(\{j,r\},\{1,i,k\},\varnothing,I_4,\ldots,I_m) + \zeta_3(\{k,j\},\{1,i,r\},\varnothing,I_4,\ldots,I_m) \\ &+ \zeta_3(\{k,r\},\{1,i,j\},\varnothing,I_4,\ldots,I_m). \end{aligned}$$

Modulo relation R8, we get

$$\zeta_3(\{j,r\},\{1,i,k\},\varnothing,I_4,\ldots,I_m) = \zeta_3(\{i,k\},\{1,j,r\},\varnothing,I_4,\ldots,I_m). \qquad \text{(R9)}$$

The reader can check that here fas well as for A3, A4 and A5 with $k = 1$ one does not obtain any new relations. The same is true for the following remaining cases.

Case A9. $|I_1| = |I_2| = 2, |I_3| = |I_4| = 1, |I_5| = \cdots = |I_m| = 2$.
Case A10: $|I_1| = 2, |I_2| = 3, |I_3| = |I_4| = |I_5| = 1, |I_6| = \cdots = |I_m| = 2$.
Case A11: $|I_1| = 1, |I_2| = 3, |I_3| = |I_4| = 1, |I_5| = \cdots = |I_m| = 2$.
Case A12: $|I_1| = 2, |I_2| = 0, |I_3| = \cdots = |I_m| = 2$.

To summarize we get the following set of equivalent relations for elements of type -1 modulo J.

$$\zeta_4(I_1,I_2,\{i\},\{j\},I_5,\ldots,I_m) = -\zeta_3(I_1,I_2,\varnothing,\{i,j\},I_5,\ldots,I_m) \qquad (E1)$$

$$\zeta_2(I_1,I_2,\{i\},I_4,\ldots,I_m) = -\zeta_3(I_1,I_2\cup\{i\},\varnothing,I_4,\ldots,I_m) \ (|I_1| = |I_2| = 2) \qquad (E2)$$

$$\zeta_2(\{i\},I_2,\{j\},I_4,\ldots,I_m) = -\zeta_3(\{i,j\},I_2,\varnothing,I_4,\ldots,I_m) \qquad (E3)$$

$$\zeta_1(\varnothing,I_2,I_3,I_4,\ldots,I_m) = \zeta_3(I_3,I_2,\varnothing,I_4,\ldots,I_m) \qquad (E4)$$

$$\zeta_1(I_1,\{1\},\{i,j\},I_4,\ldots,I_m) = \zeta_3(I_1,\{1,i,j\},\varnothing,I_4,\ldots,I_m) \qquad (E5)$$

$$\begin{aligned} &\zeta_1(\{i\},I_2,\{j,k\},I_4,\ldots,I_m) = \zeta_3(\{i,j\},I_2\cup\{k\},\varnothing,I_4,\ldots,I_m) \\ &+ \zeta_3(\{i,k\},I_2\cup\{j\},\varnothing,I_4,\ldots,I_m) \end{aligned} \qquad (E6)$$

$$\zeta_3(I_1,I_2,\varnothing,I_4,I_5,\ldots,I_m) = \zeta_3(I_4,I_2,\varnothing,I_1,I_5,\ldots,I_m) \qquad (E7)$$

$$\zeta_3(I_1,I_2\cup\{1\},\varnothing,I_4,\ldots,I_m) = \zeta_3(I_2,I_1\cup\{1\},\varnothing,I_4,\ldots,I_m) \qquad (E8)$$

$$\begin{aligned} &\zeta_3(\{i,k\},\{1,j,r\},\varnothing,I_4,\ldots,I_m) + \zeta_3(\{i,j\},\{1,k,r\},\varnothing,I_4,\ldots,I_m) \\ &+ \zeta_3(\{i,r\},\{1,j,k\},\varnothing,I_4,\ldots,I_m) = 0. \end{aligned} \qquad (E9)$$

Notice that relations E1-E6 tell us that, modulo J, all the basis elements of type -1 in X can be written as linear combinations of elements of type ζ_3. Then E7 and E8 imply that

$$\zeta_3(I_1, I_2, \varnothing, I_4, \ldots, I_m) = \zeta_3(I_1, I_2 \cup \{1\}, \varnothing, I_4, \ldots, I_m)$$

is symmetric in $I_1, I_2, I_4, \ldots, I_m$. Apart from this we have one extra relation E9.

3.2.3 Relations among ξ_1, ξ_2.

We continue the analysis and deal here with $[u, c_k]$ were u is of type -1. One readily sees that if $u \in X \setminus Z$, then $[u, c_k] \in \langle X \setminus Z\rangle$ and no new relators needed. We therefore only need to consider u where u is one of the extra relators that we obtained in Section 3.2.

B1. Consequences of E1. For $k \geq 2$ we have

$$[\zeta_4(I_1, I_2, \{i\}, \{j\}, I_5, \ldots, I_m), c_k] = -[\zeta_3(I_1, I_2, \varnothing, \{i, j\}, I_5, \ldots, I_m), c_k]$$

which implies that

$$\begin{aligned}\xi_2(I_1, I_2, \{j\}, \{i, k\}, I_5, \ldots, I_m) = &- \xi_2(I_1, I_2, \{i\}, \{j, k\}, I_5, \ldots, I_m)\\ &- \xi_2(I_1, I_2, \{k\}, \{i, j\}, I_5, \ldots, I_m).\end{aligned}$$

Therefore,

$$\begin{aligned}\xi_2(I_1, I_2, \{j\}, \{i, k\}, I_5, \ldots, I_m) = &- \xi_2(I_1, I_2, \{i\}, \{j, k\}, I_5, \ldots, I_m)\\ &- \xi_2(I_1, I_2, \{k\}, \{i, j\}, I_5, \ldots, I_m).\end{aligned} \tag{F1}$$

For $k = 1$ one does not get a new relation.

B2. Consequences of E8. For the case $k \geq 2$ we get

$$[\zeta_3(I_1, I_2 \cup \{1\}, \varnothing, I_4, \ldots, I_m), c_k] = [\zeta_3(I_2, I_1, \{1\}, \varnothing, I_4, \ldots, I_m), c_k].$$

Therefore,

$$\xi_2(I_1, I_2 \cup \{1\}, \{k\}, I_4, \ldots, I_m) = \xi_2(I_2, I_1 \cup \{1\}, \{k\}, I_4, \ldots, I_m). \tag{F2}$$

Again for $k = 1$ we do not get any new relation.

B3. Consequences of E7. For the case $k \geq 2$ we get

$$[\zeta_3(I_1, I_2, \varnothing, I_4, I_5, \dots, I_m), c_k] = [\zeta_3(I_4, I_2, \varnothing, I_1, I_5, \dots, I_m), c_k].$$

Therefore,

$$\xi_2(I_1, I_2, \{k\}, I_4, I_5, \dots, I_m) = \xi_2(I_4, I_2, \{k\}, I_1, I_5, \dots, I_m). \tag{F3}$$

For $k = 1$ we do not get anything new.

B4. Consequences of E9. Commuting E_9 with a_s, for the case $s \geq 2$ we get

$$\begin{aligned} \xi_2(\{i,k\}, \{1,j,r\}, \{s\}, I_4, \dots, I_m) = &- \xi_2(\{i,j\}, \{1,k,r\}, \{s\}, I_4, \dots, I_m) \\ &- \xi_2(\{i,r\}, \{1,j,k\}, \{s\}, I_4, \dots, I_m). \end{aligned} \tag{F4}$$

For $k = 1$ we get nothing new.

B5. Consequences of E4. For the case $k \geq 2$ we get

$$[\zeta_1(\varnothing, I_2, I_3, I_4, \dots, I_m), c_k] = [\zeta_3(I_3, I_2, \{k\}, I_4, \dots, I_m), c_k].$$

As a consequence we obtain that

$$\xi_2(\{k\}, I_2, I_3, I_4, \dots, I_m) = \xi_2(I_3, I_2, \{k\}, I_4, \dots, I_m).$$

Therefore,

$$\xi_1(\{i\}, I_2, \{j,k\}, I_4, \dots, I_m) = \xi_2(\{j,k\}, I_2, \{i\}, I_4, \dots, I_m). \tag{F5}$$

Again there is no relation for $k = 1$.

B6. Consequences of E5. For the case $k \geq 2$ we have

$$[\zeta_1(I_1, \{1\}, \{i,j\}, I_4, \dots, I_m), c_k] = [\zeta_3(I_1, \{1,i,j\}, \varnothing, I_4, \dots, I_m), c_k].$$

Then

$$\xi_1(I_1, \{1,k\}, \{i,j\}, I_4, \dots, I_m) = \xi_2(I_1, \{1,i,j\}, \{k\}, I_4, \dots, I_m).$$

Therefore,

$$\xi_1(I_1, \{1, k\}, \{i, j\}, I_4, \dots, I_m) = \xi_2(I_1, \{1, i, j\}, \{k\}, I_4, \dots, I_m). \tag{F6}$$

For $k = 1$ we do not get anything new and the same is true when consider consequences of E2,E3 and E6. From this analysis we get the following set of equivalent relations.

$$\xi_1(\{i\}, I_2, \{j, k\}, I_4, \dots, I_m) = \xi_2(\{j, k\}, I_2, \{i\}, I_4, \dots, I_m) \quad (G1)$$
$$\xi_1(I_1, \{1, k\}, \{i, j\}, I_4, \dots, I_m) = \xi_2(I_1, \{1, i, j\}, \{k\}, I_4, \dots, I_m) \quad (G2)$$
$$\xi_2(I_1, I_2 \cup \{1\}, \{k\}, I_4, \dots, I_m) = \xi_2(I_2, I_1 \cup \{1\}, \{k\}, I_4, \dots, I_m) \quad (G3)$$
$$\xi_2(I_1, I_2, \{k\}, I_4, I_5, \dots, I_m) = \xi_2(I_4, I_2, \{k\}, I_1, I_5, \dots, I_m) \quad (G4)$$
$$\xi_2(\{i, k\}, \{1, j, r\}, \{s\}, I_4, \dots, I_m) + \xi_2(\{i, j\}, \{1, k, r\}, \{s\}, I_4, \dots, I_m) + \xi_2(\{i, r\}, \{1, j, k\}, \{s\}, I_4, \dots, I_m) = 0 \quad (G5)$$
$$\xi_2(I_1, \{1, i, k\}, \{j\}, I_4, \dots, I_m) + \xi_2(I_1, \{1, k, j\}, \{i\}, I_4, \dots, I_m) + \xi_2(I_1, \{1, i, j\}, \{k\}, I_4, \dots, I_m) = 0. \quad (G6)$$

3.2.4 Relations in τ_1.

Here we deal with $[u, c_k]$ where u is of type 0. Again we only get a new relation when u is one of the relators we obtained in Section 3.3.

C1. Consequences of G3. For $k \geq 2$ we have

$$[\xi_2(I_1, I_2 \cup \{1\}, \{i\}, I_4, \dots, I_m), c_k] = [\xi_2(I_2, I_1 \cup \{1\}, \{i\}, I_4, \dots, I_m), c_k].$$

which gives

$$\tau_1(I_1, I_2 \cup \{1\}, \{i, k\}, I_4, \dots, I_m) = \tau_1(I_2, I_1 \cup \{1\}, \{i, k\}, I_4, \dots, I_m).$$

Therefore, we get

$$\tau_1(I_1, I_2 \cup \{1\}, I_3, \dots, I_m) = \tau_1(I_2, I_1 \cup \{1\}, I_3, \dots, I_m). \tag{H1}$$

For $k = 1$ we get nothing new.

C2. Consequences of G4. For $k \geq 2$ we have

$$\tau_1(I_1, I_2, I_3, I_4, \dots, I_m) = \tau_1(I_4, I_2, I_3, I_1, \dots, I_m). \tag{H2}$$

For $k = 1$ we get nothing new.

C3. Consequences of G5. For $k \geq 2$ we have

$$\begin{aligned}\tau_1(\{i,k\}, \{1,j,r\}, I_3, \dots, I_m) = &- \tau_1(\{i,j\}, \{1,k,r\}, I_3, \dots, I_m) \\ &- \tau_1(\{i,r\}, \{1,k,j\}, I_3, \dots, I_m).\end{aligned} \tag{H3}$$

Again for $k = 1$ we get nothing new and the same is true for the consequence of G1, G2 and G6. Thus the only extra relators we get are H1, H2 and H3. We will now see that if we add to $\langle X \setminus Z \rangle$ the subspace generated by the extra relators we have obtained in Sections 3.2, 3.3 and 3.4, then we get an ideal in F.

Lemma 3.2.2. The subspace generated by $X \setminus Z$ and the relators given in E1-E9, G1-G6 and H1-H3 is an ideal in F and thus equal to J.

Proof. Let V be this subspace. We found the relations E1-E9, G1-G6 and H1-H3 by systematically ensuring that we obtained a subspace that is invariant under taking Lie commutators with the c_j's. Notice that if we take a basis element $u = [[z, c_{i_1}], [z, c_{i_2}], \dots, [z, c_{I_m}]]$ from X where $[u, z] \notin V$ then we must have $|I_1|, |I_3|, |I_4|, \dots, |I_m| \leq 2$ and $|I_2| \leq 3$. Thus u has type at most 1. On the other hand if the type of u is less than 1, then $[u, z]$ has type less than -1 and is thus in V. This means that we only need to consider $u \in V$ that is linear combination of elements of type τ_1. In other words we have to commute each relation H1, H2, H3 with z and check that the resulting element is in V. If we commute relation H1 with z, we get

$$[\tau_1(I_1, I_2 \cup \{1\}, I_3, \dots, I_m), z] = [\tau_1(I_2, I_1 \cup \{1\}, I_3, \dots, I_m), z]$$

which implies that

$$\zeta_3(I_1, I_2 \cup \{1\}, \varnothing, I_4, \dots, I_m) = \zeta_3(I_2, I_1 \cup \{1\}, \varnothing, I_4, \dots, I_m),$$

which is relation E8 from Section 3.2.2. Using similar arguments we commute relation H2 with z, in which case we have

$$[\tau_1(I_1, I_2, I_3, I_4, \dots, I_m), z] = [\tau_1(I_4, I_2, I_3, I_1, \dots, I_m), z]$$

which implies

$$\zeta_3(I_1, I_2, \varnothing, I_4, I_5, \ldots, I_m) = \zeta_3(I_4, I_2, \varnothing, I_1, I_5, \ldots, I_m)$$

which is relation E7 from Section 3.2.2. Lastly, commuting relation H3 with z gives

$$\begin{aligned}[][\tau_1(\{i,k\},\{1,j,r\},I_3,\ldots,I_m),z] &= -[\tau_1(\{i,j\},\{1,k,r\},I_3,\ldots,I_m),z] \\ &\quad -[\tau_1(\{i,r\},\{1,k,j\},I_3,\ldots,I_m),z]\end{aligned}$$

which implies

$$\begin{aligned}\zeta_3(\{i,k\},\{1,j,r\},\varnothing,I_4,\ldots,I_m) &= -\zeta_3(\{i,j\},\{1,k,r\},\varnothing,I_4,\ldots,I_m) \\ &\quad -\zeta_3(\{i,r\},\{1,j,k\},\varnothing,I_4,\ldots,I_m)\end{aligned}$$

which is relation E9. This completes the proof.

□

3.3 The ideal $\mathrm{Id}_L(z)$

Let $W = W(m, 1, \overset{2m+1}{\ldots}, 1)$ be the subspace of F generated by all

$$E(I_1, \ldots, I_m) = [[z, c_{I_1}], [z, c_{\{1\}\cup I_2}], [z, c_{I_3}], \ldots, [z, c_{I_m}]]$$

with $I_1 \cup \cdots \cup I_m = \{2, \ldots, 2m+1\}$ and $|I_1| = \cdots = |I_m| = 2$. We know that these elements are linearly independent and we have also seen that

$$N = N(m, 1, \overset{2m+1}{\ldots}, 1) = W(m, 1, \overset{2m+1}{\ldots}, 1) \cap J$$

is generated by two sets of relations

$$E(I_{\sigma(1)}, \ldots, I_{\sigma(m)}) - E(I_1, \ldots, I_m), \text{ where } \sigma \in \mathrm{Sym}\{1, \ldots, m\} \qquad (\mathcal{R}_1)$$

$$\begin{aligned}E(\{i_1,i_2\},\{j_1,j_2\},I_3,\ldots,I_m) = &- E(\{i_1,j_1\},\{i_2,j_2\},I_3,\ldots,I_m) \qquad (\mathcal{R}_2) \\ &- E(\{i_1,j_2\},\{i_2,j_1\},I_3,\ldots,I_m).\end{aligned}$$

Our aim is to show that $\frac{W}{N} \neq 0$, for all $m \geq 1$. Then it will follow that $L = \frac{F}{J}$ has multihomogenous elements of arbitrary weight m in z that are non-zero. As a consequence, we see that $\mathrm{Id}_L(z)$ is not nilpotent.

From now on, we work modulo $(\mathcal{R}_1)$ and thus the order of $I_1, \ldots, I_m$ does not matter for the value of $E(I_1, \ldots, I_m)$. From now on we write $E(\{I_1, \ldots, I_m\})$ for $E(I_1, \ldots, I_m)$ modulo $(\mathcal{R}_1)$. In this way we get an element for each partition $\{I_1, \ldots, I_m\}$ and these are a basis for W modulo $(\mathcal{R}_1)$.

Definition. We say that a basis element $E(\{I_1, \ldots, I_m\})$ has *norm* k if there are exactly k of the I_j's, where $I_j \subseteq \{2, \ldots, m+1\}$. Notice that we then also have exactly k of the I_j's, where $I_j \subseteq \{m+2, \ldots, 2m+1\}$.

Suppose that $k \geq 1$. We can assume $I_1, I_3, \ldots, I_{2k-1} \subseteq \{2, \ldots, m+1\}$ and $I_2, I_4, \ldots, I_{2k} \subseteq \{m+2, \ldots, 2m+1\}$. If $I_1 = \{i_1, i_2\}$ and $I_2 = \{j_1, j_2\}$, then modulo $(\mathcal{R}_2)$ we have

$$\begin{aligned} E(\{\{i_1, i_2\}, \{j_1, j_2\}, I_3, \ldots, I_m\}) = &- E(\{\{i_1, j_1\}, \{i_2, j_2\}, I_3, \ldots, I_m\}) \\ &- E(\{\{i_1, j_2\}, \{i_2, j_1\}, I_3, \ldots, I_m\}). \end{aligned}$$

Notice that the terms on the right hand side have norm $k-1$. We will refer to this as a 1-step decomposition of E through the pair (I_1, I_2) and we denote it $\mathcal{D}_{\{(I_1,I_2)\}}(E(\{I_1, \ldots, I_m\}))$. Suppose now $k \geq 2$. We can continue and apply the 1-step decomposition to the two terms through the pair (I_3, I_4) to get a 2-step decomposition of $E(I_1, \ldots, I_m)$ through $\{(I_1, I_2), (I_3, I_4)\}$ that we denote

$$\mathcal{D}_{\{(I_1,I_2),(I_3,I_4)\}}(E(\{I_1, \ldots, I_m\})).$$

We can continue in this manner through the pairs $(I_1, I_2), \ldots, (I_{2k-1}, I_{2k})$ to get a decomposition into 2^k terms of norm 0, denoted

$$\mathcal{D}_{\{(I_1,I_2),\ldots,(I_{2k-1},I_{2k})\}}(E(\{I_1, \ldots, I_m\})).$$

Remark. To say that an element $E(\{I_1, I_2, \ldots, I_m\})$ has norm 0 is to say that $\{I_1, \ldots, I_m\} = \{(2, \sigma(m+2)), (3, \sigma(m+3)), \ldots, (m+1, \sigma(2m+1))\}$ for some permutation of σ of $m+2, \ldots, 2m+1$. We have seen that, modulo $(\mathcal{R}_1)$ and $(\mathcal{R}_2)$, W is generated by basis elements of norm 0. Let W_0 be the subspace of W generated by these. Thus $W/W \cap J \cong (W_0 + J)/J \cong W_0/W_0 \cap J$.

Notice that the decomposition $\mathcal{D}_{\{(I_1,I_2),\ldots,(I_{2k-1},I_{2k})\}}(E(\{I_1, \ldots, I_m\}))$, does not depend on the order of the pairs in $\{(I_1, I_2), \ldots, (I_{2k-1}, I_{2k})\}$, but it may depend on what pairings are chosen. For each $E(\{I_1, \ldots, I_m\})$ of norm $k \geq 2$, pick one of the decompositions according to some pairing and denote this $E^0(\{I_1, \ldots, I_m\})$. Thus for every element $E(\{I_1,\ldots,I_m\})$ of norm $k \geq 2$, there is a defining relation

$$E(\{I_1,\ldots,I_m\}) = E^0(\{I_1,\ldots,I_m\})$$

modulo $(\mathcal{R}_2)$, where $E^0(\{I_1,\ldots,I_m\})$ is a linear combination of terms of norm 0. Using these defining relations, all relations in $(\mathcal{R}_2)$ become relations in W_0, the subspace of W generated by $E(\{I_1,\ldots,I_m\})$ of norm 0. We next want to understand what these relations in W_0 are. Let $(\mathcal{R}_2)_2$ be the collections of all relations in $(\mathcal{R}_2)$ where none of the $I_5,\ldots,I_m$ is contained in $\{2,\ldots,m+1\}$ or $\{m+2,\ldots,2m+1\}$. Next lemma simplifies our task.

Lemma 3.3.1. Let $E = E(\{I_1,\ldots,I_m\})$ be a basis element of norm k where $I_1, I_3, \ldots, I_{2k-1} \subseteq \{2,\ldots,m+1\}$ and $I_2, I_4, \ldots, I_{2k} \subseteq \{m+2,\ldots,2m+1\}$ modulo $(\mathcal{R}_2)_2$. We have

$$\mathcal{D}_{\{(I_1,I_2),\ldots,(I_{2k-1},I_{2k})\}}(E) = \mathcal{D}_{\{(I_1,I_{\sigma(2)}),\ldots,(I_{2k-1},I_{\sigma(2k)})\}}(E)$$

for any $\sigma \in \mathrm{Sym}\{2,4,\ldots,2k\}$.

Proof. We prove the result by induction on k. If $k=1$ then it is clear, since in this case σ is the identity. For $k=2$ this is a direct consequence of $(\mathcal{R}_2)_2$. Now suppose that $k \geq 3$ and that the result is true for smaller values of k. Without loss of generality, we can suppose $\sigma(2) = 4$. Using the induction hypothesis twice we have

$$\begin{aligned} &\mathcal{D}_{\{(I_1,I_4),(I_3,I_{\sigma(4)}),(I_5,I_{\sigma(6)}),\ldots,(I_{2k-1},I_{\sigma(2k)})\}}(E) \\ &= \mathcal{D}_{\{(I_1,I_4),(I_3,I_2),(I_5,I_6),\ldots,(I_{2k-1},I_{2k})\}}(E) \\ &= \mathcal{D}_{\{(I_1,I_2),(I_3,I_4),(I_5,I_6),\ldots,(I_{2k-1},I_{2k})\}}(E), \end{aligned}$$

as required. ☐

Now take any relation in $(\mathcal{R}_2)$. Without loss of generality, we can assume (after reordering $I_1,\ldots,I_m$) that half of the elements of $I_5 \cup \cdots \cup I_m$ are in $\{2,\ldots,m+1\}$ and half in $\{m+2,\ldots,2m+1\}$. We are using here the fact that the size of $(I_1 \cup I_2) \cap \{2,\ldots,m+1\}$ is between 0 and 4). This requires the further assumption that m must be even. We thus have now a relation of the form:

$$\begin{aligned} E(\{\{i_1,i_2\},\{j_1,j_2\},I_3,\ldots,I_m\}) = &-E(\{\{i_1,j_1\},\{i_2,j_2\},I_3,\ldots,I_m\}) \\ &-E(\{\{i_1,j_2\},\{i_2,j_1\},I_3,\ldots,I_m\}). \end{aligned} \tag{3.1}$$

Suppose exactly k of $I_5,\ldots,I_m$ are subsets of $\{2,\ldots,m+1\}$. Without loss of generality we can suppose that

$$I_5, I_7, \dots, I_{2k+3} \subseteq \{2, \dots, m+1\} \text{ and } I_6, I_8, \dots, I_{2k+4} \subseteq \{m+2, \dots, 2m+1\}.$$

By Lemma 3.3.1, we know that modulo $(\mathcal{R}_2)_2$, relation (3.1) is equivalent to a sum of relations of the type

$$\begin{aligned} E(\{\{i_1,i_2\},\{j_1,j_2\},I_3,I_4,J_5,\dots,J_m\}) &= -E(\{\{i_1,j_1\},\{i_2,j_1\},I_3,I_4,J_5,\dots,J_m\}) \\ &\quad -E(\{\{i_1,j_2\},\{i_2,j_2\},I_3,I_4,J_5,\dots,J_m\}) \end{aligned} \tag{3.2}$$

where no $J_5, \dots, J_m$ is a subset of $\{2, \dots, m+1\}$ or $\{m+2, \dots, 2m+1\}$. Notice that all the relations (2) are in $(\mathcal{R}_2)_2$.

The conclusion is that all the relations in W_0 we are looking for, will be consequence of $(\mathcal{R}_2)_2$. Let $J_5, \dots, J_m$ be fixed and consider the collection of all the relations in $(\mathcal{R}_2)_2$ where $I_5 = J_5 \dots I_m = J_m$.

Let $\bar{E}(\{I_1, I_2, I_3, I_4\}) = E(\{I_1, I_2, I_3, I_4, J_5, \dots, J_m\})$. Suppose $\{2, \dots, m+1\} \setminus (J_5 \cup \dots \cup J_m) = \{i_1, i_2, i_3, i_4, j_1, j_2, j_3, j_4\}$, where $i_1, i_2, i_3, i_4 \in \{2, \dots, m+1\}$ and $j_1, j_2, j_3, j_4 \in \{m+2, \dots, 2m+1\}$. We now go systematically through all possible types of $(\mathcal{R}_2)_2$ relations.

Case 1: $I_3 = \{i_3, j_3\}, I_4 = \{i_4, j_4\}$. We have

$$\begin{aligned} \bar{E}(\{\{i_1,i_2\},\{j_1,j_2\},\{i_3,j_3\},\{i_4,j_4\}\}) &= -\bar{E}(\{\{i_1,j_1\},\{i_2,j_2\},\{i_3,j_3\},\{i_4,j_4\}\}) \\ &\quad -\bar{E}(\{\{i_1,j_2\},\{i_2,j_1\},\{i_3,j_3\},\{i_4,j_4\}\}). \end{aligned}$$

These are defining relations.

Case 2: $I_3 = \{i_3, i_4\}, I_4 = \{j_3, j_4\}$. We have

$$\begin{aligned} \bar{E}(\{\{i_1,i_2\},\{j_1,j_2\},\{i_3,i_4\},\{j_3,j_4\}\}) &= -\bar{E}(\{\{i_1,j_1\},\{i_2,j_2\},\{i_3,i_4\},\{j_3,j_4\}\}) \\ &\quad -\bar{E}(\{\{i_1,j_2\},\{i_2,j_1\},\{i_3,i_4\},\{j_3,j_4\}\}) \\ &= \bar{E}(\{\{i_1,j_1\},\{i_2,j_2\},\{i_3,j_3\},\{i_4,j_4\}\}) \\ &\quad +\bar{E}(\{\{i_1,j_1\},\{i_2,j_2\},\{i_3,j_4\},\{i_4,j_3\}\}) \\ &\quad +\bar{E}(\{\{i_1,j_2\},\{i_2,j_1\},\{i_3,j_3\},\{i_4,j_4\}\}) \\ &\quad +\bar{E}(\{\{i_1,j_2\},\{i_2,j_1\},\{i_3,j_4\},\{i_4,j_3\}\}). \end{aligned}$$

If we instead pair $\{i_1, i_2\}$ and $\{j_3, j_4\}$ we get

$$\begin{aligned}
\bar{E}(\{\{i_1,i_2\},\{j_1,j_2\},\{i_3,i_4\},\{j_3,j_4\}\}) &= -\bar{E}(\{\{i_1,j_3\},\{j_1,j_2\},\{i_3,i_4\},\{i_2,j_4\}\}) \\
&\quad -\bar{E}(\{\{i_1,j_4\},\{j_1,j_2\},\{i_3,i_4\},\{i_2,j_4\}\}) \\
&= \bar{E}(\{\{i_1,j_3\},\{i_2,j_4\},\{i_3,j_1\},\{i_4,j_2\}\}) \\
&\quad +\bar{E}(\{\{i_1,j_3\},\{i_2,j_4\},\{i_3,j_2\},\{i_4,j_1\}\}) \\
&\quad \bar{E}(\{\{i_1,j_4\},\{i_2,j_3\},\{i_3,j_1\},\{i_4,j_2\}\}) \\
&\quad +\bar{E}(\{\{i_1,j_4\},\{i_2,j_3\},\{i_3,j_2\},\{i_4,j_1\}\}).
\end{aligned}$$

From these two pairings we get

$$\begin{aligned}
&\sum_{\sigma\in\mathrm{Sym}\{1,2\},\tau\in\mathrm{Sym}\{3,4\}} \bar{E}(\{\{i_1,j_{\sigma(1)}\},\{i_2,j_{\sigma(2)}\},\{i_3,j_{\tau(3)}\},\{i_4,j_{\tau(4)}\}\}) \\
= &\sum_{\sigma\in\mathrm{Sym}\{1,2\},\tau\in\mathrm{Sym}\{3,4\}} \bar{E}(\{\{i_1,j_{\tau(3)}\},\{i_2,j_{\tau(4)}\},\{i_3,j_{\sigma(1)}\},\{i_4,j_{\sigma(2)}\}\}).
\end{aligned}$$

Case 3: $I_3 = \{j_2, j_3\}, I_4 = \{i_4, j_4\}$. We have

$$\begin{aligned}
0 &= \bar{E}(\{\{i_1,i_2\},\{i_3,j_1\},\{j_2,j_3\},\{i_4,j_4\}\}) + \bar{E}(\{\{i_1,i_3\},\{i_2,j_1\},\{j_2,j_3\},\{i_4,j_4\}\}) \\
&\quad +\bar{E}(\{\{i_2,i_3\},\{i_1,j_1\},\{j_2,j_3\},\{i_4,j_4\}\}) \\
&= -\bar{E}(\{\{i_1,j_2\},\{i_2,j_1\},\{i_3,j_3\},\{i_4,j_4\}\}) - \bar{E}(\{\{i_1,j_3\},\{i_2,j_2\},\{i_3,j_1\},\{i_4,j_4\}\}) \\
&\quad -\bar{E}(\{\{i_1,j_2\},\{i_2,j_1\},\{i_3,j_3\},\{i_4,j_4\}\}) - \bar{E}(\{\{i_1,j_3\},\{i_2,j_1\},\{i_3,j_2\},\{i_4,j_4\}\}) \\
&\quad -\bar{E}(\{\{i_1,j_1\},\{i_2,j_2\},\{i_3,j_3\},\{i_4,j_4\}\}) - \bar{E}(\{\{i_1,j_2\},\{i_2,j_3\},\{i_3,j_2\},\{i_4,j_4\}\}).
\end{aligned}$$

This gives us the relation

$$\sum_{\sigma\in\mathrm{Sym}\{1,2,3\}} \bar{E}(\{\{i_1,j_{\sigma(1)}\},\{i_2,j_{\sigma(2)}\},\{i_3,j_{\sigma(3)}\},\{i_4,j_4\}\}) = 0.$$

By symmetry the case of having $I_3 = \{i_2, i_3\}, I_4 = \{i_4, j_4\}$ works in a similar manner.

Case 4: $I_3 = \{j_1, j_2\}, I_4 = \{j_3, j_4\}$. We have

$$\begin{aligned}
0 &= \bar{E}(\{\{i_1,i_2\},\{i_3,i_4\},\{j_1,j_2\},\{j_3,j_4\}\}) + \bar{E}(\{\{i_1,i_3\},\{i_2,i_4\},\{j_1,j_2\},\{j_3,j_4\}\}) \\
&\quad +\bar{E}(\{\{i_1,i_4\},\{i_2,i_3\},\{j_1,j_2\},\{j_3,j_4\}\}).
\end{aligned}$$

Modulo Case 2, the choice of pairings does not matter, therefore we get

$$\begin{aligned} 0 \;=\; & -\bar{E}(\{\{i_1,j_1\},\{i_2,j_2\},\{i_3,i_4\},\{j_3,j_4\}\}) - \bar{E}(\{\{i_1,j_2\},\{i_3,j_1\},\{i_3,i_4\},\{j_3,j_4\}\}) \\ & -\bar{E}(\{\{i_1,j_1\},\{i_3,j_2\},\{i_2,i_4\},\{j_3,j_4\}\}) - \bar{E}(\{\{i_1,j_2\},\{i_3,j_1\},\{i_2,i_4\},\{j_3,j_4\}\}) \\ & -\bar{E}(\{\{i_1,j_1\},\{i_4,j_2\},\{i_2,i_3\},\{j_3,j_4\}\}) - \bar{E}(\{\{i_1,j_2\},\{i_4,j_1\},\{i_2,i_3\},\{j_3,j_4\}\}). \end{aligned}$$

Cancelling the signs gives

$$\begin{aligned} & \sum_{\sigma\in \mathrm{Sym}\{2,3,4\}} \bar{E}(\{\{i_1,j_1\},\{i_2,j_{\sigma(2)}\},\{i_3,j_{\sigma(3)}\},\{i_4,j_{\sigma(4)}\}\}) \\ & + \sum_{\sigma\in \mathrm{Sym}\{1,3,4\}} \bar{E}(\{\{i_1,j_2\},\{i_2,j_{\sigma(1)}\},\{i_3,j_{\sigma(3)}\},\{i_4,j_{\sigma(4)}\}\}) = 0, \end{aligned}$$

which follows from Case 3. We have seen that $W/W\cap J \cong W_0/W_0\cap J$, as vector spaces, and $W_0\cap J$ is generated by relations

$$\sum_{\sigma\in \mathrm{Sym}\{1,2,3\}} E(\{\{i_1,j_{\sigma(1)}\},\{i_2,j_{\sigma(2)}\},\{i_3,j_{\sigma(3)}\},\{i_4,j_4\},\dots,\{i_m,j_m\}\}) = 0; \qquad (3.3)$$

and

$$\begin{aligned} & \sum_{\sigma\in \mathrm{Sym}\{1,2\},\tau\in \mathrm{Sym}\{3,4\}} E(\{\{i_1,j_{\sigma(1)}\},\{i_2,j_{\sigma(2)}\},\{i_3,j_{\tau(3)}\},\{i_4,j_{\tau(4)}\},\{i_5,j_5\},\dots,\{i_m,j_m\}\}) \\ & = \sum_{\sigma\in \mathrm{Sym}\{1,2\},\tau\in \mathrm{Sym}\{3,4\}} \bar{E}(\{\{i_1,j_{\tau(3)}\},\{i_2,j_{\tau(4)}\},\{i_3,j_{\sigma(1)}\},\{i_4,j_{\sigma(2)}\},\{i_5,j_5\},\dots,\{i_m,j_m\}\}); \end{aligned} \qquad (3.4)$$

where $\{i_1,\dots,i_m\} = \{2,3,\dots,m+1\}$ and $\{j_1,\dots,j_m\} = \{m+2,\dots,2m+1\}$. We show that $W_0\cap J$ is a proper subspace of W_0 by showing that all the relators in (3) and (4) lie in a subspace of co-dimension 1 in W_0.

Let M_0 be the subspace of W_0 generated by all

$$E(\{\{2,\sigma(j_1)\},\dots,\{m+1,\sigma(j_m)\}\}) + E(\{\{2,j_1\},\dots,\{m+1,j_m\}\})$$

with $\{j_1,\dots,j_m\} = \{m+2,\dots,2m+1\}$ and where σ is a transposition in $\mathrm{Sym}\,(\{m+2,\dots,2m+1\})$. Thus modulo M_0 we have $E(\{\{2,\sigma(m+2)\},\dots,\{m+1,\sigma(2m+1)\}\}) = \mathrm{sign}(\sigma)E(\{\{2,m+2\},\dots,\{m+1,2m+1\}\})$. Notice that M_0 is of codimension 1 in W_0 and contains $W_0\cap J$. For (3) this is because S_3 has equally many even and odd permutations and for (4) we can transform the LHS to the RHS using even permutations. Thus $W_0/W_0 \cap J \neq 0$ that implies that $W/W \cap J \neq 0$.

As a consequence we get the following main result of this section.

Theorem 3.3.2. The ideal generated by z in L is non-nilpotent.

3.4 The normal subgroup $\langle x\rangle^G$

In Section 2 we considered the largest locally finite p-group $G = \langle x, a_1, a_2, \dots\rangle$ satisfying the following relations:

1. $\langle a_i\rangle^G$ is abelian for all $i \geq 1$;
2. $\langle x\rangle^G$ is metabelian;
3. $x^p = a_1^p = a_2^p = \cdots = 1$;
4. if $s \neq x$ is a simple commutator in $x, a_1, a_2, \dots$, where $t(s) \geq 2$ or $t(s) \leq -2$, then $s = 1$.

Now we use the Lie algebra L constructed in the Section 3 to get a group $H = \langle 1 + \operatorname{ad}(z), 1+\operatorname{ad}(c_1), 1+\operatorname{ad}(c_2), \dots\rangle$. In this section we will show that H is a homomorphic image of G by proving that the relations (1)-(4) hold in H where $x, a_1, a_2, \dots$ are replaced by $1+\operatorname{ad}(z), 1+\operatorname{ad}(c_1), 1+\operatorname{ad}(c_2), \dots$. We will also use the work in Section 2 and Section 3 to show that $\langle 1+\operatorname{ad}(z)\rangle^H$ is non-nilpotent from which follows that $\langle x\rangle^G$ is non-nilpotent.

Lemma 3.4.1. We have $\operatorname{ad}(z)^2 = 0$ and $\operatorname{ad}(z)\operatorname{ad}(w)\operatorname{ad}(z) = 0$ for any simple Lie product w in $z, c_1, c_2, \dots$.

Proof. Let u be a simple Lie product in $z, c_1, c_2, \dots$. Notice first that if u is non-trivial then the type of $[u, z, z]$ is less than or equal to -3 and therefore $[u, z, z] = 0$. Turning to the second claim we know that $[u, z] = 0$ if u is of type different from 1. We can therefore assume that u is of type 1. Then if w is non-zero, we have that $[u, z, w, z]$ is of type at most -2 and $[u, z, w, z] = 0$. □

Lemma 3.4.2. Let w_H be a simple group commutator in $1 + \mathrm{ad}(z), 1 + \mathrm{ad}(c_1), 1 + \mathrm{ad}(c_2), \ldots$ and let w_L be the corresponding Lie commutator in $z, c_1, c_2, \ldots$. Then

$$w_H = 1 + \mathrm{ad}(w_L).$$

Proof. We show this by using induction on the weight of w_H. If the weight is 1 then this is obvious. Now suppose the weight is $k \geq 2$ and the result holds whenever the weight is smaller. We have $w_H = [u_H, v_H]$, where u_H, v_H are simple commutators of smaller weight in $1+\mathrm{ad}(z), 1+\mathrm{ad}(c_1), 1+\mathrm{ad}(c_2), \ldots$. Let u_L, v_L be the corresponding simple commutators in $z, c_1, c_2, \ldots$. Using the induction hypothesis we have

$$\begin{aligned} w_H =& [u_H, v_H] = (1 + \mathrm{ad}(u_L))^{-1}(1 + \mathrm{ad}(v_L))^{-1}(1 + \mathrm{ad}(u_L))(1 + \mathrm{ad}(v_L)) \\ =& (1 - \mathrm{ad}(u_L))(1 - \mathrm{ad}(v_L))(1 + \mathrm{ad}(u_L))(1 + \mathrm{ad}(v_L)) \text{ (using Lemma 3.4.1)} \\ =& 1 + \mathrm{ad}(u_L)\,\mathrm{ad}(v_L) - \mathrm{ad}(v_L)\,\mathrm{ad}(u_L) \\ =& 1 + \mathrm{ad}([u_L, v_L]) \\ =& 1 + \mathrm{ad}(w_L). \end{aligned}$$

□

Now we show that H satisfies the relations (1)-(4) for G where $x, a_1, a_2, \ldots$ are replaced by $1+\mathrm{ad}(z), 1+\mathrm{ad}(c_1), 1+\mathrm{ad}(c_2), \ldots$. More precisely, we have the following.

Proposition 3.4.3. Let H be as before. Then

1. $\langle 1 + \mathrm{ad}(c_i)\rangle^H$ is abelian for all $i \geq 1$;
2. $\langle 1 + \mathrm{ad}(z)\rangle^H$ is metabelian;
3. $(1 + \mathrm{ad}(z))^p = (1 + \mathrm{ad}(c_1))^p = (1 + \mathrm{ad}(c_2))^p = \cdots = 1$;
4. if $s \neq 1 + \mathrm{ad}(z)$ is a simple commutator in $1 + \mathrm{ad}(z), 1 + \mathrm{ad}(c_1), 1 + \mathrm{ad}(c_2), \ldots$, where $t(s) \geq 2$ or $t(s) \leq -2$, then $s = 1$.

Proof. (1) Let w_H be a simple group commutator in $1+\mathrm{ad}(z), 1+\mathrm{ad}(c_1), 1+\mathrm{ad}(c_2), \ldots$ with a repeated occurrence of $1+\mathrm{ad}(c_i)$. Then $w_H = 1+\mathrm{ad}(w_L)$ and as $\mathrm{Id}(c_i)$ is abelian, $\mathrm{ad}(w_L) = 0$ and thus $w_H = 1$. This implies that $\langle 1 + \mathrm{ad}(c_i)\rangle^H$ is abelian for all $i \geq 1$.
(2) Let u_H, v_H be simple commutators in $1 + \mathrm{ad}(z), 1 + \mathrm{ad}(c_1), 1 + \mathrm{ad}(c_2), \ldots$ where both have at least two occurrences of $1 + \mathrm{ad}(z)$. Then

$$[u_H, v_H] = 1 + \mathrm{ad}([u_L, v_L])$$

and since $\mathrm{Id}(z)$ is metabelian, then $[u_L, v_L] = 0$ and $[u_H, v_H] = 1$. It follows then that $\langle 1 + \mathrm{ad}(z)\rangle^H$ is metabelian.
(3) Clearly $(1 + \mathrm{ad}(z))^p = 1 + p\,\mathrm{ad}(z) = 1$ and $(1 + \mathrm{ad}(c_i))^p = 1 + p\,\mathrm{ad}(c_i) = 1$.
(4) Let $w_H \neq 1 + \mathrm{ad}(z)$ be a simple commutator in $1 + \mathrm{ad}(z), 1 + \mathrm{ad}(c_1), 1 + \mathrm{ad}(c_2), \ldots,$ where $t(w_H) \geq 2$ or $t(w_H) \leqslant -2$. Then, $w_H = 1 + \mathrm{ad}(w_L)$, where w_L is a Lie commutator in $z, c_1, c_2, \ldots,$ where $t(w_L) \geq 2$ or $t(w_L) \leqslant -2$. Here $w_L = 0$ and so $w_H = 1$. This completes the proof. □

Theorem 3.4.4. The normal closure of $1 + \mathrm{ad}(z)$ is not nilpotent.

Proof. Consider

$$\begin{aligned}&[[1 + \mathrm{ad}(z), 1 + \mathrm{ad}(c_1), 1 + \mathrm{ad}(c_2), 1 + \mathrm{ad}(c_3)], [1 + \mathrm{ad}(z), 1 + \mathrm{ad}(c_4), 1 + \mathrm{ad}(c_5)],\\ &\ldots, [1 + \mathrm{ad}(z), 1 + \mathrm{ad}(c_{2m}), 1 + \mathrm{ad}(c_{2m+1})]]\\ &= 1 + \mathrm{ad}([[z, c_1, c_2, c_3], [z, c_4, c_5], \ldots, [z, c_{2m}, c_{2m+1}]]).\end{aligned}$$

Notice that

$$1 + \mathrm{ad}([[z, c_1, c_2, c_3], [z, c_4, c_5], \ldots, [z, c_{2m}, c_{2m+1}]]) \neq 1,$$

since, for example $[[z, c_1, c_2, c_3], [z, c_4, c_5], \ldots, [z, c_{2m}, c_{2m+1}], z]) \neq 0$. Thus, we have shown that there is a simple commutator of arbitrary weight m in $(1 + \mathrm{ad}(z))$ that is non-trivial. This finishes the proof. □

Chapter 4

Future Work and Further Questions

This section discusses some open questions, conjectures and areas of future research.

4.1 Open questions

We return now back to our initial question of what is the least positive integer n for which $L_n(G) \nsubseteq HP(G)$? Through recent calculations on Magma and GAP we have the following conjecture.

Conjecture 4.1.1. There exists a positive integer n and a family of finite 2-groups $(G_i)_{i=0}^{\infty}$ where $x_i \in G_i$ is left 3-Engel and $r_i = \max\{\text{class}\langle x_i, x_i^{g_1}, x_i^{g_2}, \ldots, x_i^{g_n}\rangle\}$ is unbounded.

This conjecture is based on the analogous situation for Lie algebras. Let us understand the conjecture better. Let $S = \langle x_1, \ldots, x_{n+1}\rangle$ be the largest sandwich group generated by the involutions $x_1, \ldots, x_{n+1}$. Then $T = S/\gamma_\infty(S)$ is the largest residually finite quotient of S. Notice that $H_i := \langle x_i, x_i^{g_1}, \ldots, x_i^{g_n}\rangle \leqslant G_i$ is a quotient of T and as $(H_i)_{i=1}^{\infty}$ have unbounded nilpotency class we see that T is not nilpotent. Therefore, if the answer is positive and the conjecture is true then we will have a way of showing that there is a finitely generated residually finite group T containing a left 3-Engel element that is not in the Hirsch-Plotkin radical. Therefore, we would show that $L_3(G) \nsubseteq HP(G)$.

Turning our attention to groups of exponent 8, as mentioned in section 2.3 the following question is still unanswered.

Question 4.1.2. Are groups of exponent 8 locally finite?

As we have already seen answering this question relies on showing that all left 4-Engel elements of a group G of exponent 8 are in $HP(G)$.

Question 4.1.3. Are all left 4-Engel elements in a group of G of exponent 8 in $HP(G)$?

Knowing that all left 3-Engel elements lie in the Hirsch-Plotkin radical would be a stepping stone towards answering Question 5.1.3. In section 2.3 it was shown that for big enough n there exist a group G, containing a left n-Engel element where $\langle x \rangle^G$ is not locally nilpotent. This relied on Ivanov and Lysenok's solution to the Burnside problem for groups of 2-power exponent. However as it stands we don't know what happens in the case of a p-group, where p is odd.

Remark: Recently we have started a new collaboration with Sandro Mattarei from University of Lincoln who is an expert on Lie algebras. Lie algebras are a potential tool to construct the groups that we want. This work will continue in September.

Bibliography

[1] A. Abdollahi, Left 3-Engel elements in groups, *J. Pure Appl. Algebra* **188** (2004), 1-6.

[2] R. Baer, Engelsche Elemente Noetherscher Gruppen, *Math. Ann.* **133** (1957), 256-270.

[3] Y. Bakhturin, Identical relations in Lie algebras, *VNU Science Press*, Utrecht, (1987).

[4] W. Burnside, On an unsettled question in the theory of discontinous groups, *Quart. J. Pure Appl. Math.* **37** (1901), 230-238.

[5] E.S. Golod, On nil-algebras and finitely approximable p-groups, *Izv. Akad. Nauk SSSR Ser. Mat.* **28** (1964), 273-276.

[6] K. W. Gruenberg, The Engel elements of a soluble group, *Illinois J. Math.* **3** (1959), 151-169.

[7] N. D. Gupta and M. F. Newman, Third Engel groups, *Bull. Austral. Math. Soc* **40** (1989), 215-230.

[8] N. D. Gupta and F. Levin, On soluble Engel groups and Lie algebras, *Arch. Math* **34** (1980), 289-295.

[9] Hadjievangelou, A., Traustason, G., Left 3-Engel elements in locally finite 2-groups, *Communications in Algebra* **49 No. 11** (2021), 4869–4882.

[10] Hadjievangelou, A., Noce, M., Traustason, G., Locally finite p-groups with a left 3-Engel element whose normal closure is not nilpotent, *International Journal of Algebra and Computation* **31 No. 1** (2021), 135-160.

[11] Havas, G., Vaughan Lee, M. R., 4-Engel groups are locally nilpotent, *International Journal of Algebra and Computation* **15** (2005), 649-682.

[12] H. Heineken, Eine Bemerkung über engelshe Elemente, *Arch. Math.* **11** (1960), 321.

[13] C. Hopkins, Finite groups in which conjugate operations are commuta- tive, *Am. J. Math.* **51** (1929), 35-41.

[14] S. V. Ivanov, The free Burnside groups of sufficiently large exponents, *Int. J. Algebra and Comp.* **4** (1994), 1-308.

[15] E. Jabara and G. Traustason, Left 3-Engel elements of odd order in groups, *Proc. Am. Soc.* **147** (2019), 1921-1927.

[16] I. G. Lysenok, Infinite Burnside groups of even period, *Izv. Math.* **60** (1996), 453-654.

[17] M. Newell, On right-Engel elements of length three, *Proc. Royal Irish Ac.* **96A**, No.1 (1996), 17-24.

[18] Noce, M., Tracey, G., Traustason, G., A left 3-Engel element whose normal closure is not nilpotent, *Journal of Pure and Applied Algebra* **224 No.3**, (2020), 1092-1101.

[19] I. N. Sanov, Solution of Burnside's Problem for Exponent Four, Leningrad State Univ., *Ann. Maths. Ser.* **10** (1940), 166-170.

[20] G. Tracey and G. Traustason, Left 3-Engel elements in groups of exponent 60, *Int. J. Algebra and Comp.* **28** No. 4 (2018), 673-695.

[21] G. Traustason, Engel Lie-algebras, *Quart. J. Math. Oxford.* Ser. (2) **44** (1993), 355-384.

[22] G. Traustason, 2-generator 4-Engel groups, *International Journal of Algebra and Computation* **15** No.2 (2005), 309-316.

[23] G. Traustason, Left 3-Engel elements in groups of exponent 5, *J. Algebra* **414** (2014), 41-71.

[24] Wilson, J. S., Two-generator conditions for residually finite groups, *Bull. London Math. Soc.* **23** (1991), no. 3, 239-248.

[25] G. Traustason, Left 3-Engel elements in groups of exponent 5, *J. Algebra* **414** (2014), 41-71.

[26] E. I. Zel'manov, The solution of the restricted Burnside problem for groups of odd exponent, *Math. USSR Izvestia* **36** (1991), 41-60.

[27] E. I. Zel'manov, The solution of the restricted Burnside problem for 2-groups, *Mat. Sb.* **182** (1991), 568-592.

[28] M. Zorn, Nilpotency of finite groups, *Bull. Amer. Math. Soc.* **42** (1936), 485-486.