Saturated and primitive smooth compactifications of ball quotients

P. G. Beshkov, A. K. Kasparian, * and G. K. Sankaran†

Let $X_i = (\mathbb{B}/\Gamma_i)'$, $1 \leq i \leq 2$ be smooth toroidal compactifications of quotients $\mathbb{B}/\Gamma_i$ of the complex 2-ball

$$\mathbb{B} = \{(z_1, z_2) \in \mathbb{C}^2 ||z_1|^2 + |z_2|^2 < 1\} = PSU_{2,1}/PS(U_2 \times U_1)$$

by lattices $\Gamma < PU(2,1)$, $D^{(i)} := X_i \setminus (\mathbb{B}/\Gamma_i)$ be the toroidal compactifying divisors and $\rho_i : X_i \rightarrow Y_i$ be compositions of blow downs with exceptional divisors $E(\rho_i)$ onto minimal surfaces $Y_i$. The present note establishes a bijective correspondence between the unramified coverings $f : X_2 \rightarrow X_1$ of degree $d$, which restrict to unramified coverings $f : D^{(2)} \rightarrow D^{(1)}$, $f : E(\rho_2) \rightarrow E(\rho_1)$ of degree $d$ and the unramified coverings $\varphi : Y_2 \rightarrow Y_1$ of degree $d$ of the corresponding minimal models, which restrict to unramified coverings $\varphi : \rho_2(D^{(2)}) \rightarrow \rho_1(D^{(1)})$, $\varphi : \rho_2(E(\rho_2)) \rightarrow \rho_1(E(\rho_1))$ of degree $d$. The aforementioned covering relations among $X_i$ define an artinian partial order $\succeq$ on the set $S$ of the smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$. The maximal elements with respect to $\succeq$ are called saturated and the minimal elements with respect to $\succeq$ are said to be primitive. Our considerations reduce the study of $X \in S$ to the study of the primitive $X \in S$. For an arbitrary totally ordered subset $\{X_\alpha\}_{\alpha \in A} \subset S$, all the minimal models $Y_\alpha$ of $X_\alpha$ have one and a same universal cover and one and a same Kodaira dimension. We discuss the saturated and the primitive $X \in S$ of non-positive Kodaira dimension. The covering relations among the smooth toroidal compactifications $(\mathbb{B}/\Gamma)'$ are studies in Uludag’s [Uludag], Stover’s [Stover], Di Cerbo and Stover’s [DiCerboStover1] and other articles.

Here is a synopsis of the article. Let $\rho_1 : X_1 \rightarrow Y_1$ be a composition of blow downs of a smooth projective surface $X_1$ onto a smooth projective surface $Y_1$. The first section establishes a bijective correspondence between the unramified coverings $f : X_2 \rightarrow X_1$ of degree $d$ and the unramified covering $\varphi : Y_2 \rightarrow Y_1$ of degree $d$ through fibered product commutative diagrams (4) with appropriate compositions of blow downs $\rho_2 : X_2 \rightarrow Y_2$. In order to induce $\varphi : Y_2 \rightarrow Y_1$ by $f : X_2 \rightarrow X_1$, one observes that $\varphi\rho_2$ is the Stein factorization of the proper holomorphic map $\rho_1 f : X_2 \rightarrow Y_1$. If $D^{(i)} \subset X_i$ are (possibly reducible) divisors, which do not contain irreducible components of the exceptional divisors $E(\rho_i)$ of $\rho_i : X_i \rightarrow Y_i$, then $f$ is shown to restrict to an unramified covering $f : D^{(2)} \rightarrow D^{(1)}$ of degree $d$ if and only if $\varphi$ restricts to an unramified covering $\varphi : \rho_2(D^{(2)}) \rightarrow \rho_1(D^{(1)})$ of degree $d$. In particular, if $\rho_1 : X_1 = (\mathbb{B}/\Gamma_1)' \rightarrow Y_1$ is a composition of blow downs of a smooth

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†Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK (masgks@bath.ac.uk).
toroidal compactification $X_1 = (\mathbb{B}/\Gamma_1)'$ onto a minimal surface $Y_1$ then the unramified coverings $f : X_2 = (\mathbb{B}/\Gamma_2)' \to (\mathbb{B}/\Gamma_1)' = X_1$ of degree $d$, which restrict to unramified coverings $f : \mathbb{B}/\Gamma_2 \to \mathbb{B}/\Gamma_2$ of degree $d$ are in a bijective correspondence with the unramified coverings $\varphi : Y_2 \to Y_1$ by appropriate minimal models $Y_2$ of $X_2$. Under the aforementioned correspondence, $f$ restricts to an unramified covering $f : D^{(2)} = X_2 \setminus (\mathbb{B}/\Gamma_2) \to X_1 \setminus (\mathbb{B}/\Gamma_1) = D^{(1)}$ of degree $d$ of the corresponding compactifying divisors if and only if $\varphi$ restricts to an unramified covering $\varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)})$ of degree $d$. In such a way, the presence of a finite unramified cover $X_2 = (\mathbb{B}/\Gamma_2)'$ of $X_1 = (\mathbb{B}/\Gamma_1)'$ can be detected by the means of an arbitrary minimal model $Y_1$ of $X_1$ and its finite unramified covers $Y_2$.

Let $X_2, Y_2$ be smooth projective surfaces and $\rho_2 = \beta_1 \ldots \beta_r : X_2 \to Y_2$ be a compositon of blow downs with exceptional divisors $E(\beta i) \subset \beta_{i+1} \ldots \beta_r(X_2)$. The second section introduces compatibility conditions on the finite unramified coverings $f : X_2 \to f(X_2)$ or $\varphi : Y_2 \to \varphi(Y_2)$ with $\rho_2$ in such a way that the existence of $f : X_2 \to f(X_2)$ to be equivalent to the existence of $\varphi : Y_2 \to \varphi(Y_2)$. In particular, for a smooth toroidal compactification $X_2 = (\mathbb{B}/\Gamma_2)'$ with toroidal compactifying divisor $D^{(2)} = X_2 \setminus (\mathbb{B}/\Gamma_2)$ and a composition of blow downs $\rho_2 = \beta_1 \ldots \beta_r : X_2 \to Y_2$ onto a minimal surface $Y_2$, there exists an unramified covering $f : X_2 \to f(X_2) = X_1$, which is compatible with $\rho_2$ and restricts to an unramified covering $f : \mathbb{B}/\Gamma_2 \to \mathbb{B}/\Gamma_2$ of degree $d$ if and only if there is an unramified covering $\varphi : Y_2 \to \varphi(Y_2) = Y_1$ of a minimal model $Y_1$ of $X_1$, which is compatible with $\rho_2$ and restricts to an unramified covering $\varphi : \rho_2(D^{(2)}) \to \varphi\rho_2(D^{(2)})$ of degree $d$. Moreover, $X_1 = (\mathbb{B}/\Gamma_1)'$ is a smooth toroidal compactification and if $\rho_1 : X_1 \to Y_1$ is a composition of blow downs onto $Y_1$ then $\varphi\rho_2(D^{(2)}) = \rho(D^{(1)})$ for the compactifying divisor $D^{(1)} = X_1 \setminus (\mathbb{B}/\Gamma_1)$ of $\Gamma$. A smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ is primitive if there is no unramified covering $f : X \to f(X)$ of degree $d$, which restricts to an unramified covering $f : \mathbb{B}/\Gamma \to \mathbb{B}/\Gamma$ of degree $d$ and is compatible with some composition of blow downs $\rho : X \to Y$ onto a minimal surface $Y$. Due to the established duality between the finite unramified coverings $f : X \to f(X)$ and $\varphi : Y \to \varphi(Y)$ of one and a same degree, the primitiveness of $X = (\mathbb{B}/\Gamma)'$ can be detected by the properties of $Y$.

The last, third section studies the finite unramified Galois coverings $f : X = (\mathbb{B}/\Gamma)' \to f(X)$ of smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$, which admit a blow down $\beta : X \to Y$ of $n \in \mathbb{N}$ smooth irreducible rational $(-1)$-curves onto a minimal surface $Y$. Di Cerbo and Stover have shown in [DiCerboStover2] that the smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$ with abelian or bi-elliptic minimal model $Y$ have the aforementioned property. We establish that for such $X = (\mathbb{B}/\Gamma)'$ the compatibility of the unramified coverings $\varphi : Y \to \varphi(Y)$ of degree $d$, restricting to unramified coverings $\varphi : \beta(D) \to \varphi\beta(D)$ of degree $d$ with $\beta : X \to Y$ is automatic, as far as $\beta(E(\beta)) = \beta(D)^{\text{sing}}$ coincides with the singular locus of $\beta(D)$. The relative automorphism group $\text{Aut}(Y, \beta(D)) = \text{Aut}(Y, \beta(D), \beta(D)^{\text{sing}})$ admits an isomorphism $\Phi : \text{Aut}(Y, \beta(D)) \to \text{Aut}(X, D)$ onto the relative automorphism group $\text{Aut}(X, D) = \text{Aut}(X, D, E(\beta))$. Let $N(\pi_1(Y))$ be the normalizer of the fundamental group $\pi_1(Y)$ of $Y$ in the biholomorphism group $\text{Aut}(Y)$ of the universal cover $Y$ of $Y$. It is well known that the biholomorphism group $\text{Aut}(Y)$ of $Y$ is the quotient $\text{Aut}(Y) = N(\pi_1(Y))/\pi_1(Y)$. If an unramified covering $\varphi : Y \to \varphi(Y)$ of degree $d$ restricts to an unramified covering $\varphi : \beta(D) \to \varphi\beta(D)$ of degree $d$ then any $g_0 \in N(\pi_1(Y)) \cap \pi_1(\varphi(Y))$ is shown to induce a biholomorphism $g_0 : \beta(D) \to \beta(D)$ and, therefore, a factorization $f = f_0\zeta$ of the associated unramified covering $f : X = (\mathbb{B}/\Gamma)' \to X_0 =$
curves. is a smooth divisor on $X$. In particular, for a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ with abelian minimal model $Y$, we establish that any unramified covering $f : X = (\mathbb{B}/\Gamma)' \to X_0 = (\mathbb{B}/\Gamma_0)'$. The unramified covering $f : X = (\mathbb{B}/\Gamma)' \to X_0 = (\mathbb{B}/\Gamma_0)'$ of degree $d$, which restricts to an unramified covering $f : \mathbb{B}/\Gamma \to \mathbb{B}/\Gamma_0$ of degree $d$, factors through a Galois covering $X \to X/\langle g \rangle$, $g \in \text{Aut}(X, D)$, which restricts to a Galois covering $\mathbb{B}/\Gamma \to \mathbb{B}/\Gamma/\langle g \rangle$. The third section discusses also the saturation and the primitiveness of the smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$ with Kodaira dimension $\kappa(X) = -\infty$, as well as the smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$ with $K3$ or Enriques minimal model.

1 Unramified pull back of a smooth compactification

Lemma 1. Let $M$ be a complex manifold and $N$ be a complex analytic subvariety of $M$ or an open subset of $M$.

(i) If $f : M \to f(M)$ is an unramified covering of degree $d$ then $f : N \to f(N)$ is an unramified covering of degree $d$ exactly when $f : M \setminus N \to f(M) \setminus f(N)$ is an unramified covering of degree $d$.

(ii) Let us suppose that $f : M \to f(M)$ is a holomorphic map onto a complex manifold, $f(N) \cap f(M \setminus N) = \emptyset$ and $f : N \to f(N)$, $f : M \setminus N \to f(M \setminus N)$ are unramified coverings of degree $d$. Then $f : M \to f(M)$ is an unramified covering of degree $d$.

Proof. (i) Let $X := N$ or $X := M \setminus X$. Then $f : X \to f(X)$ is an unramified covering of degree $\deg(f|\chi) = \deg(f|\chi) = d$ exactly when $f^{-1}(f(X)) = X$. If so, then the intersection $f^{-1}(f(M \setminus X)) \cap X = \emptyset$ is empty, whereas $f^{-1}(f(M \setminus X)) = M \setminus X$, the union $f(M) = f(X) \amalg \cdots \amalg f(M \setminus X)$ is disjoint and $f : M \setminus X \to f(M \setminus X) = f(M \setminus X)$ is an unramified covering of degree $d$.

(ii) The union $f(M) = f(N) \amalg \cdots \amalg f(M \setminus N)$ is disjoint, so that $f^{-1}(f(M \setminus N)) = M \setminus N$, $f^{-1}(f(N)) = N$ and $f : M \to f(M)$ is an unramified covering of degree $d$.

□

Lemma 2. Let $f : X \to X'$ be an unramified covering of degree $d$ of smooth projective surfaces.

(i) Suppose that $D = \bigsqcup_{j=1}^k D_j$ is a divisor on $X$ with disjoint smooth irreducible components $D_j$ and $f$ restricts to an unramified covering $f : D \to f(D)$ of degree $d$. Then $f(D) = \bigcup_{j=1}^k f(D_j)$ has smooth irreducible components $f(D_j)$, $f$ restricts to unramified coverings $f : D_j \to f(D_j)$ for all $1 \leq j \leq k$ and $f(D_i) \cap f(D_j) = \emptyset$ for $f(D_i) \neq f(D_j)$.

In particular, $D_j$ are smooth elliptic curves and only if $f(D_j)$ are smooth elliptic curves.

(ii) If $C'$ is a smooth irreducible rational curve on $X'$ then the complete preimage $f^{-1}(C') = \bigsqcup_{i=1}^d C_i$ consists of $d$ disjoint smooth irreducible rational curves $C_i$ and $f$ restricts to isomorphisms $f : C_i \to C'$ for all $1 \leq i \leq d$.

Proof. (i) The unramified covering $f : D \to f(D)$ is a local biholomorphism, so that $f(D)$ is a smooth divisor on $X'$. Thus, all the irreducible components $f(D_j)$ of $f(D)$ are smooth.
curves and \( f(D_i) \cap f(D_j) \neq \emptyset \) requires \( f(D_i) \equiv f(D_j) \). For any \( 1 \leq i \leq k \) let \( J(i) \) be the set of those \( 1 \leq j \leq k \), for which \( f(D_j) \equiv f(D_i) \). Then there exists a subset \( I \subseteq \{1, \ldots, k\} \) with \( \bigcap_{i \in I} J(i) = \{1, \ldots, k\} \) and \( f(D) = \bigcap_{i \in I} f(D_i) \). By the very definition of \( J(i) \), there holds the inclusion \( \bigcap_{j \in J(i)} D_j \subseteq f^{-1}(f(D_i)) \). Since \( f \) restricts to an unramified covering \( f : D \to f(D) \) of degree \( d \), any \( p \in f^{-1}(f(D_i)) \) belongs to \( D_j \) for some \( 1 \leq s \leq k \). Then \( f(p) \in f(D_i) \) specified that \( s \in J(i) \), whereas \( f^{-1}(f(D_i)) \subseteq \bigcap_{j \in J(i)} D_j \) and \( f^{-1}(f(D_i)) = \bigcap_{j \in J(i)} D_j \). Thus, for any \( i \in I \) the morphism \( f \) restricts to an unramified covering \( f : \bigcap_{j \in J(i)} D_j \to f(D_i) \) of degree \( d \). By definition, any \( f(p) \in f(D_i) \) with \( p \in \bigcap_{j \in J(i)} D_j \) has a trivializing neighborhood \( U \) on \( f(D_i) \), whose pull back \( f^{-1}(U) = \bigcap_{q \in f^{-1}(p)} V_q \) is a disjoint union of neighborhoods \( V_q \) of \( q \in f^{-1}(p) \) on \( \bigcap_{j \in J(i)} D_j \) with biholomorphic restrictions \( f : V_q \to U \). For a sufficiently small \( U \) one can assume that \( V_q \subseteq D_j \) for \( q \in D_j \). That is why \( f \) restricts to unramified coverings \( f : D_j \to f(D_j) = f(D_i) \). In particular, \( D_j \) are smooth elliptic curves exactly when \( f(D_j) \) are smooth elliptic curves.

(ii) Let \( f^{-1}(C') = \bigcup_{i=1}^{k} C_i \) be a union of \( k \) irreducible curves \( C_i \), \( d_i := \deg [f|_{C_i}: C_i \to C'] \) and \( \text{Br}(f|_{C_i}) := \{ q \in C' | |f^{-1}(q) \cap C_i| < d_i \} \) be the branch locus of \( f|_{C_i} \) for \( 1 \leq i \leq k \). Any \( \text{Br}(f|_{C_i}) \) is a finite set, as well as the intersection \( \bigcup_{1 \leq i < j \leq k} C_i \cap C_j \) of different irreducible components, so that

\[
\Sigma := \left[ \bigcup_{i=1}^{k} \text{Br}(f|_{C_i}) \right] \cup \left[ \bigcup_{1 \leq i < j \leq k} f(C_i \cap C_j) \right]
\]

is a finite subset of \( C' \). For any \( q \in C' \setminus \Sigma \) one has \( f^{-1}(q) = \bigcup_{i=1}^{k} f^{-1}(q) \cap C_i \), whereas

\[
d = |f^{-1}(q)| = \sum_{i=1}^{k} |f^{-1}(q) \cap C_i| = \sum_{i=1}^{k} d_i.
\]

If \( q_j \in \text{Br}(f|_{C_j}) \) then \( f^{-1}(q_j) = \bigcup_{i=1}^{k} f^{-1}(q_j) \cap C_i \) with \( |f^{-1}(q_j) \cap C_j| < d_j \), so that

\[
d = |f^{-1}(q_j)| \leq \sum_{i=1}^{k} |f^{-1}(q_j) \cap C_i| < \sum_{i=1}^{k} d_i = d.
\]

This is an absurd, justifying \( \text{Br}(f|_{C_j}) = \emptyset \) for all \( 1 \leq j \leq k \). Similarly, for any \( p \in C_i \cap C_j \) there holds

\[
d = |f^{-1}(p)| < \sum_{i=1}^{k} |f^{-1}(p) \cap C_i| = \sum_{i=1}^{k} d_i = d.
\]

The contradiction shows that the irreducible components \( C_i \) of \( f^{-1}(C') \) are disjoint. The unramified coverings \( f|_{C_i}: C_i \to C' \) of the smooth irreducible rational curve \( C' \) are of degree \( d_i = 1 \), due to \( \pi_1(C') = \{1\} \). Therefore \( d = \sum_{i=1}^{k} d_i = k \) and \( f^{-1}(C') = \bigcup_{i=1}^{k} C_i \) consists of \( d \)
disjoint smooth irreducible rational curves with biholomorphic restrictions $f|_{C_i} : C_i \to C'$ for all $1 \leq i \leq d$.

A $(-1)$-curve $L_i$ on a smooth projective surface $Y$ is a smooth irreducible rational curve with self-intersection $L_i^2 = -1$. Throughout, we say that a smooth projective surface $Y$ is minimal if it does not contain a $(-1)$-curve. This is slightly different from the contemporary viewpoint of the Minimal Model Program, which considers a smooth projective surface $Y$ to be minimal if its canonical divisor $K_Y$ is nef (i.e., $K_Y.C \geq 0$ for all effective curves $C \subset Y$). The numerical effectiveness of $K_Y$ excludes the existence of $(-1)$-curves on $Y$. If $Y$ is of Kodaira dimension $\kappa(Y) = -\infty$ then $K_Y$ is not nef, regardless of the presence of $(-1)$-curves on $Y$. That is the reason for exploiting the older, out of date notion of minimality of a smooth projective surface, which requires the non-existence of $(-1)$-curves on $Y$. By a theorem of Castelnuovo (Theorem V.5.7 [Ha]), for any smooth irreducible projective surface $X$ there is a birational morphism $\rho : X \to Y$ onto a minimal smooth projective surface $Y$, which is a composition of blow downs of $(-1)$-curves. If $X$ is of Kodaira dimension $\kappa(X) \geq 0$ then the minimal model $Y$ of $X$ is unique (up to an isomorphism). This is no more true when $X$ is birational to a rational or a ruled surface.

**Lemma 3.** (i) Let $\Bl : X_1 \to Y_1$ be a blow down of a $(-1)$-curve $L_1 \subset X_1$ and $\varphi : Y_2 \to Y_1$ be an unramified covering of degree $d$. Then the fibered product commutative diagram

$$
\begin{array}{ccc}
X_2 := X_1 \times_{Y_1} Y_2 & \xrightarrow{\beta} & Y_2 \\
\downarrow f & & \downarrow \varphi \\
X_1 & \xrightarrow{\Bl} & Y_1 \\
\end{array}
$$

consists of an unramified covering $f : X_2 \to X_1$ of degree $d$ and the blow down $\beta : X_2 \to Y_2$ of the disjoint union $f^{-1}(L_1) = \bigsqcup_{j=1}^{d} L_{1,j}$ of the $(-1)$-curves $L_{1,j}$.

(ii) Let $\rho_1 : \Bl_1 \ldots \Bl_{r-1} \Bl_r : T_r := X_1 \to Y_1 =: T_0$ be a composition of blow downs $\Bl_i : T_i \to T_{i-1}$ of $(-1)$-curves $L_i \subset T_i$ and $\varphi : Y_2 \to Y_1$ be an unramified covering of degree $d$. Then the fibered product commutative diagrams

$$
\begin{array}{ccc}
S_i := T_i \times_{T_{i-1}} S_{i-1} & \xrightarrow{\beta_i} & S_{i-1} \\
\downarrow \varphi_i & & \downarrow \varphi_{i-1} \\
T_i & \xrightarrow{\Bl_i} & T_{i-1} \\
\end{array}
$$

fit into a commutative diagram

$$
\begin{array}{ccc}
S_r \ldots S_i := T_i \times_{T_{i-1}} S_{i-1} & \xrightarrow{\beta_i} & S_{i-1} \ldots S_0 := Y_2 \\
\downarrow f & & \downarrow \varphi_i \downarrow \varphi_{i-1} \\
T_r := X \ldots T_i & \xrightarrow{\Bl_i} & T_{i-1} \ldots T_0 := Y_1 \\
\end{array}
$$
and induce a fibered product commutative diagram

\[
X_2 = X_1 \times_{Y_1} Y_2 \xrightarrow{\rho_2} Y_2
\]
\[
\downarrow \quad \varphi \downarrow
\]
\[
X_1 \xrightarrow{\rho_1} Y_1
\]

(4)

with an unramified covering \( f : X_2 \to X_1 \) of degree \( d \) and a composition \( \rho_2 = \beta_1 \ldots \beta_{r-1} \beta_r : X_2 \to Y_2 \) of blow downs of \( \varphi_i^{-1}(L_i) = \bigcup_{j=1}^{d} L_{i,j} \) for all \( 1 \leq i \leq r \).

**Proof.** (i) By the very definition of a blow down \( \Bl : X_1 \to Y_1 \) of \( L_1 \) to \( \Bl(L_1) = q_1 \in Y_1 \), one has \( X_1 \setminus L_1 = Y_1 \setminus \{ q_1 \} \). Then

\[
X_2 := X_1 \times_{Y_1} Y_2 = [(X_1 \setminus L_1) \times_{Y_1} Y_2] \coprod [L_1 \times_{Y_1} Y_2]
\]
decomposes into the disjoint union of

\[
(X_1 \setminus L_1) \times_{Y_1} Y_2 = \{(x_1, y_2) \mid x_1 = \Bl(x_1) = \varphi(y_2) \} \simeq Y_2 \setminus \varphi^{-1}(q_1) \quad \text{and}
\]
\[
L_1 \times_{Y_1} Y_2 = \{(x_1, y_2) \mid q_1 = \Bl(x_1) = \varphi(y_2) \} = L_1 \times \varphi^{-1}(q_1).
\]

If \( \varphi^{-1}(q_1) = \{ p_{1,j} \mid 1 \leq j \leq d \} \) then \( X_2 \) is the blow up of \( Y_2 \) at \( \{ p_{1,j} \mid 1 \leq j \leq d \} \). Due to \( \Bl f = \varphi \beta \), the exceptional divisor of \( \beta \) is \( \beta^{-1}(\{ p_{1,j} \mid 1 \leq j \leq d \}) = \beta^{-1}(q_1) = (\varphi \beta)^{-1}(q_1) = (\Bl f)^{-1}(q_1) = f^{-1}\Bl^{-1}(q_1) = f^{-1}(L_1) = \bigcup_{j=1}^{d} L_{1,j} \). According to Corollary 17.7.3 (i) from Grothendieck’s [Groth4], \( f : X_2 \to X_1 \) is an unramified covering, since \( \varphi : Y_2 \to Y_1 \) is an unramified covering.

(ii) By an increasing induction on \( 1 \leq i \leq r \), one applies (i) to the fibered product commutative diagrams (2) and justifies (ii).

\[\Box\]

**Lemma 4.** (i) In the notations from Lemma 3 (i) and the fibered product commutative diagram (1), let \( D^{(2)} \) be a (possibly reducible) divisor on \( X_2 \), which does not contain an irreducible component of the exceptional divisor of \( \beta \) and \( D^{(1)} \) be a (possibly reducible) divisor on \( X_1 \), which does not contain the exceptional divisor \( L_1 \) of \( \Bl \). Then the restriction \( f : D^{(2)} \to D^{(1)} \) is an unramified covering of degree \( d = \deg[f : X_2 \to X_1] \) if and only if \( \varphi : \beta(D^{(2)}) \to \Bl(D^{(1)}) \) is an unramified covering of degree \( d \).

(ii) In the notations from Lemma 3 (ii) and the fibered product commutative diagram (4), let \( D^{(2)} \) be a (possibly reducible) divisor on \( X_2 \), which does not contain an irreducible component of the exceptional divisor of \( \rho_2 \) and \( D^{(1)} \) be a (possibly reducible) divisor on \( X_1 \), which does not contain an irreducible component of the exceptional divisor of \( \rho_1 \). Then the restriction \( f : D^{(2)} \to D^{(1)} \) is an unramified covering of degree \( d \) if and only if the restriction \( \varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)}) \) is an unramified covering of degree \( d \).
Proof. (i) If \( f : D^{(2)} \to D^{(1)} \) is an unramified covering of degree \( d \) then \( f^{-1}(D^{(1)} \cap L_1) = f^{-1}(D^{(1)}) \cap f^{-1}(L_1) = D^{(2)} \cap f^{-1}(L_1) \) and the restriction \( f : D^{(1)} \cap f^{-1}(L_1) \to D^{(1)} \cap L_1 \) is an unramified covering of degree \( d \). After denoting \( f^{-1}(L_1) = \prod_{j=1}^{d} L_{1,j} \), \( \beta(L_{1,j}) = p_{1,j} \) and \( \text{Bl}(L_1) = q_1 \), one applies Lemma 1 (i), in order to conclude that

\[ \varphi \equiv f : \beta(D^{(2)}) \setminus \{p_{1,j} | 1 \leq j \leq d\} \equiv D^{(2)} \setminus f^{-1}(L_1) \to D^{(1)} \setminus L_1 \equiv \text{Bl}(D^{(1)}) \setminus \{q_1\} \]

is an unramified covering of degree \( d \). Now, \( \varphi \) restricts to \( \varphi : \{p_{1,j} | 1 \leq j \leq d\} \to \{q_1\} \), so that

\[ \varphi : \beta(D^{(2)}) = \beta(D^{(2)}) \setminus \{p_{1,j} | 1 \leq j \leq d\} \prod \{p_{1,j} | 1 \leq j \leq d\} \to \text{Bl}(D^{(1)}) \setminus \{q_1\} \prod \{q_1\} = \text{Bl}(D^{(1)}) \]

is an unramified covering of degree \( d \) by Lemma 1 (ii).

Conversely, assume that \( \varphi : \beta(D^{(2)}) \to \text{Bl}(D^{(1)}) \) is an unramified covering of degree \( d \). Choose a sufficiently small neighborhood \( V \) of \( q_1 = \text{Bl}(L_1) \) on \( Y_1 \), such that \( \varphi^{-1}(V) = \prod_{j=1}^{d} U_j \) is a disjoint union of neighborhoods \( U_j \) of \( p_{1,j} \), \( 1 \leq j \leq d \) on \( Y_2 \) with biholomorphic restrictions \( \varphi : U_j \to V \) of \( \varphi \). Bearing in mind that \( \text{Bl}_1 : X_1 \to Y_1 \) is the blow up of \( Y_1 \) at \( q_1 \), one decomposes

\[ \text{Bl}(D^{(1)}) = \left[ \text{Bl}(D^{(1)}) \setminus V \right] \prod \left[ \text{Bl}(D^{(1)}) \cap V \right] \] and

\[ D^{(1)} = \left[ \text{Bl}(D^{(1)}) \setminus V \right] \prod \text{Bl}^{-1}(\text{Bl}(D^{(1)}) \cap V). \]

Similarly, \( \beta : X_2 \to Y_2 \) is the blow up of \( Y_2 \) at \( \varphi^{-1}(q_1) = \{p_{1,j} | 1 \leq j \leq d\} \), so that there are decompositions

\[ \beta(D^{(2)}) = \left[ \beta(D^{(2)}) \setminus \varphi^{-1}(V) \right] \prod \left[ \beta(D^{(2)}) \cap \varphi^{-1}(V) \right] \] and

\[ D^{(2)} = \left[ \beta(D^{(2)}) \setminus \varphi^{-1}(V) \right] \prod \beta^{-1}(\beta(D^{(2)}) \cap \varphi^{-1}(V)). \]

According to \( \varphi^{-1}(\text{Bl}(D^{(1)}) \cap V) = \varphi^{-1}(\beta(D^{(1)})) \cap \varphi^{-1}(V) = \beta(D^{(2)}) \cap \varphi^{-1}(V) \), the restriction \( \varphi : \beta(D^{(2)}) \cap \varphi^{-1}(V) \to \text{Bl}(D^{(1)}) \cap V \) is an unramified covering of degree \( d \). Now, Lemma 1 (ii) applies to provide that

\[ f \equiv \varphi : \beta(D^{(2)}) \setminus \varphi^{-1}(V) \to \text{Bl}(D^{(1)}) \setminus V \]

is an unramified covering of degree \( d \). According to Lemma 1 (ii), it suffices to show that

\[ f : \beta^{-1}(\beta(D^{(2)}) \cap \varphi^{-1}(V)) \to \text{Bl}^{-1}(\text{Bl}(D^{(1)}) \cap V) \]

is an unramified covering of degree \( d \), in order to conclude that \( f : D^{(2)} \to D^{(1)} \) is an unramified covering of degree \( d \). To this end, note that

\[ \varphi^{-1}(\text{Bl}(D^{(1)}) \cap V) = \beta(D^{(2)}) \cap \varphi^{-1}(V) = \beta(D^{(2)}) \cap \left( \prod_{j=1}^{d} U_j \right) = \prod_{j=1}^{d} \beta(D^{(2)}) \cap U_j, \]
so that
\[ \varphi : \prod_{j=1}^{d} \left[ \beta(D^{(2)}) \cap U_j \right] \longrightarrow \text{Bl}(D^{(1)}) \cap V \]
is an unramified covering of degree \( d \). Therefore, the biholomorphisms \( \varphi : U_j \to V \) restrict to biholomorphisms \( \varphi : \beta(D^{(2)}) \cap U_j \to \text{Bl}(D^{(1)}) \cap V \). According to \( \varphi(p_{1,j}) = q_1 \), there arise biholomorphisms
\[ \varphi : (\beta(D^{(2)}) \cap U_j) \setminus \{ p_{1,j} \} \longrightarrow (\text{Bl}(D^{(1)}) \cap V) \setminus \{ q_1 \}. \]
By the very definition of a blow up at a point, these induce biholomorphisms
\[ f : \left[ (\beta(D^{(2)}) \cap U_j) \setminus \{ p_{1,j} \} \right] \prod L_{1,j} \longrightarrow \left[ (\text{Bl}(D^{(1)}) \cap V) \setminus \{ q_1 \} \right] \prod L_1 \]
for all \( 1 \leq j \leq d \). Bearing in mind that
\[ \prod_{j=1}^{d} \left\{ (\beta(D^{(2)}) \cap U_j) \setminus \{ p_{1,j} \} \right\} \prod L_{1,j} = \beta^{-1}(\beta(D^{(2)}) \cap \varphi^{-1}(V)), \]
one concludes that \( \varphi \) induces an unramified covering
\[ f : \beta^{-1}(\beta(D^{(2)}) \cap \varphi^{-1}(V)) \longrightarrow \text{Bl}^{-1}(\text{Bl}(D^{(1)}) \cap V) \]
of degree \( d \).

(ii) Along the commutative diagram (3), if \( f : D^{(2)} \to D^{(1)} \) is an unramified covering of degree \( d \) then by a decreasing induction on \( r \geq i \geq 1 \) and making use of (i), one observes that \( \varphi_i : \beta_{i+1} \ldots \beta_r(D^{(2)}) \to \text{Bl}_{i+1} \ldots \text{Bl}_r(D^{(1)}) \) is an unramified covering of degree \( d \), whereas \( \varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)}) \) is an unramified covering of degree \( d \). Conversely, suppose that \( \varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)}) \) is an unramified covering of degree \( d \). Then by an increasing induction on \( 1 \leq i \leq r \) and making use of (i), one concludes that
\[ \varphi_1 : \beta_{i+1} \ldots \beta_r(D^{(2)}) \to \text{Bl}_{i+1} \ldots \text{Bl}_r(D^{(1)}) \]
is an unramified covering of degree \( d \). As a result, \( f : D^{(2)} \to D^{(1)} \) is an unramified covering of degree \( d \).

\[ \square \]

**Corollary 5.** Let \( X_1 = (\mathbb{B}/\Gamma_1) \) be a smooth toroidal compactification, \( \rho_1 : X_1 \to Y_1 \) be a composition of blow downs onto a minimal surface \( Y_1 \), \( \varphi : Y_2 \to Y_1 \) be an unramified covering of degree \( d \) and (4) be the defining commutative diagram of the fibered product \( X_2 = X_1 \times_{Y_1} Y_2 \). Then:

(i) there is a subgroup \( \Gamma_2 \) of \( \Gamma_1 \) of index \([\Gamma_1 : \Gamma_2] = d\), such that \( X_2 = (\mathbb{B}/\Gamma_2)' \) is the toroidal compactification of \( \mathbb{B}/\Gamma_2 \);

(ii) \( f : X_2 \to X_1 \) restricts to unramified coverings \( f : \mathbb{B}/\Gamma_2 \to \mathbb{B}/\Gamma_1 \), respectively, \( f : D^{(2)} := X_2 \setminus (\mathbb{B}/\Gamma_2) \to X_1 \setminus (\mathbb{B}/\Gamma_1) =: D^{(1)} \) of degree \( d \);

(iii) the composition \( \rho_2 : X_2 \to Y_2 \) of blow downs maps onto a minimal surface \( Y_2 \);

(iv) \( \varphi \) restricts to an unramified covering \( \varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)}) \) of degree \( d \).
Proof. By Lemma 3 (ii), the fibered product diagram (4) consists of an unramified covering \( f : X_2 \to X_1 \) of degree \( d \) and a composition \( \rho_2 : X_2 \to Y_2 \) of blow downs. The surface \( Y_2 \) is minimal. Otherwise any \((-1\)-curve \( L'_i \) on \( Y_2 \) maps isomorphically onto a \((-1\)-curve \( \varphi(L'_i) \subset Y_1 \), according to Lemma 2 (ii). That contradicts the minimality of \( Y_1 \) and shows the minimality of \( Y_2 \).

The unramified covering \( f : X_2 \to X_1 = (\mathbb{B}/\Gamma_1)' \) of degree \( d \) restricts to an unramified covering \( f : f^{-1}(\mathbb{B}/\Gamma_1) \to \mathbb{B}/\Gamma_1 \) of degree \( d \). The smoothness of \( \mathbb{B}/\Gamma_1 \) excludes the existence of isolated branch points of the \( \Gamma_1\)-Galois covering \( \zeta_1 : \mathbb{B} \to \mathbb{B}/\Gamma_1 \). However, \( \zeta_1 \) can ramify along divisors and \( \mathbb{B} \) is not the usual universal cover of the complex manifold \( \mathbb{B}/\Gamma_1 \). Nevertheless, \( \mathbb{B} \) is the orbifold universal cover of \( \mathbb{B}/\Gamma_1 \) and the orbifold universal covering map \( \zeta_1 : \mathbb{B} \to \mathbb{B}/\Gamma_1 \) factors through a (possibly ramified) covering \( \zeta_2 : \mathbb{B} \to f^{-1}(\mathbb{B}/\Gamma_1) \) and the covering \( f : f^{-1}(\mathbb{B}/\Gamma_1) \to \mathbb{B}/\Gamma_1 \), i.e., \( \zeta_1 = f \zeta_2 \). Since \( \pi_1^{orb}(\mathbb{B}) = \{1\} \) is a normal subgroup of \( \Gamma_2 := \pi_1^{orb}(f^{-1}(\mathbb{B}/\Gamma)) \), the covering \( \zeta_2 \) is Galois and its Galois group \( \Gamma_2 \) is a subgroup of \( \Gamma_1 = \pi_1^{orb}(\mathbb{B}/\Gamma_1) \) of index \( [\Gamma_1 : \Gamma_2] = d \). In particular, \( f^{-1}(\mathbb{B}/\Gamma_1) = \mathbb{B}/\Gamma_2 \). By Lemma 1 (i), \( f \) restricts to an unramified covering \( f : D^{(2)} := X_2 \setminus (\mathbb{B}/\Gamma_2) \to X_1 \setminus (\mathbb{B}/\Gamma_1) =: D^{(1)} \) of degree \( d \) of the toroidal compactifying divisor \( D^{(1)} = \prod_{j=1}^k D_{j}^{(1)} \) of \( \mathbb{B}/\Gamma_1 \). Note that for any \( 1 \leq j \leq k \) the restriction \( f : f^{-1}(D_j^{(1)}) \to D_j^{(1)} \) is an unramified covering of degree \( d \), whereas a local biholomorphism. Therefore \( f^{-1}(D_j^{(1)}) = \bigcup_{i=1}^{r_j} D_{j,i}^{(2)} \) is smooth and has disjoint smooth irreducible components \( D_{j,i}^{(2)} \). As a result,

\[
D^{(2)} = f^{-1}(D^{(1)}) = \prod_{j=1}^k f^{-1}(D_j^{(1)}) = \prod_{j=1}^k \prod_{i=1}^{r_j} D_{j,i}^{(2)}
\]

has disjoint smooth irreducible components \( D_{j,i}^{(2)} \). By assumption, \( D_j^{(1)} \) are smooth elliptic curves, so that all \( D_{j,i}^{(2)} \) are smooth elliptic curves by Lemma 2 (i). That is why, \( X_2 = (\mathbb{B}/\Gamma_2)' \) is the toroidal compactification of \( \mathbb{B}/\Gamma_2 \). According to Lemma 4 (iii), \( \varphi : Y_2 \to Y_1 \) restricts to an unramified covering \( \varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)}) \) of degree \( d \).

\[ \square \]

Lemma 6. (i) Let \( f : X_2 \to X_1 \) be an unramified covering of degree \( d \) of smooth projective surfaces and \( \text{Bl} : X_1 \to Y_1 \) be a blow down of a \((-1\)-curve \( L_1 \subset X_1 \). Then the Stein factorization \( \varphi \beta \) of \( \text{Bl}f \) consists of the blow down \( \beta : X_2 \to Y_2 \) of \( f^{-1}(L_1) = \bigcup_{j=1}^d L_{1,j} \) and an unramified covering \( \varphi : Y_2 \to Y_1 \) of degree \( d \), so that \( X_2 = X_1 \times_{Y_1} Y_2 \) is the fibered product of \( X_1 \) and \( Y_2 \) over \( Y_1 \).

(ii) Let \( \rho_1 = \text{Bl}_1 \ldots \text{Bl}_r : T_r := X_1 \to Y_1 =: T_0 \) be a composition of blow downs of \((-1\)-curves \( L_i \subset T_i \) and \( f : X_2 \to X_1 \) be an unramified covering of degree \( d \). Then the Stein factorization \( \varphi \rho_2 \) of \( \rho_1 f : X_2 \to Y_1 \) closes the fibered product commutative diagram (4) with the composition \( \rho_2 = \beta_1 \ldots \beta_r : S_r := X_2 \to Y_2 := S_0 \) of the blow downs \( \beta_i : S_i \to S_{i-1} \) of \( \varphi_i^{-1}(L_i) = \bigcup_{j=1}^d L_{i,j} \) for all \( 1 \leq i \leq r \) and an unramified covering \( \varphi : Y_2 \to Y_1 \) of degree \( d \).

Proof. (i) If \( \text{Bl}f = \varphi \beta : X_2 \to Y_1 \) is the Stein factorization of \( \text{Bl}f \) and \( q_1 := \text{Bl}(L_1) \)
then \((Blf)^{-1}(q_1) = f^{-1}Bl^{-1}(q_1) = f^{-1}(L_1) = \prod_{j=1}^{d} L_{1,j}\) has irreducible components \(L_{1,j}\) by Lemma ???. For any \(q \in Y_1 \setminus \{q_1\}\) one has \((Blf)^{-1}(q) = f^{-1}Bl^{-1}(q) = f^{-1}(q)\) of cardinality \(|f^{-1}(q)| = d\). Therefore, the surjective morphism \(\beta : X_2 \to Y_2\) with connected fibres is the blow down of \(L_{1,j}, \forall 1 \leq j \leq d\). According to Lemma 1 (i), the restriction \(f : X_2 \setminus f^{-1}(L_1) \to X_1 \setminus L_1\) is an unramified covering of degree \(d\), since \(f : f^{-1}(L_1) \to L_1\) is an unramified covering of degree \(d\). In such a way, there arises a commutative diagram

\[
\begin{array}{ccc}
X_2 \setminus f^{-1}(L_1) & \xrightarrow{\beta=\text{Id}} & Y_2 \setminus f^{-1}(L_1) \\
| f \downarrow & & \varphi \downarrow \\
X_1 \setminus L_1 & \xrightarrow{\text{Bl=Id}} & Y_1 \setminus \{q_1\}
\end{array}
\]

and \(\varphi : Y_2 \setminus \beta f^{-1}(L_1) \to Y_1 \setminus \{q_1\}\) is an unramified covering of degree \(d\). If \(p_{1,j} := \beta(L_{1,j})\) then \(\beta^{-1} \varphi^{-1}(q_1) = (\varphi \beta)^{-1}(q_1) = (Blf)^{-1}(q_1) = \prod_{j=1}^{d} L_{1,j}\) reveals that \(\varphi^{-1}(q_1) = \{p_{1,j} | 1 \leq j \leq d\}\) consists of \(d\) points and \(\varphi : Y_2 \to Y_1\) is an unramified covering of degree \(d\). By Lemma 3 (i), the fibered product \(X'_2 := X_1 \times_{Y_1} Y_2\) is the blow up of \(Y_2\) at \(\varphi^{-1}(q_1) = \{p_{1,j} | 1 \leq j \leq d\}\), so that \(X'_2 = X_2\).

According to Grothendieck's Corollary 17.7.3 (i) from [Groth4], it suffices to show that \(X'_2 = X_2\), in order to conclude that \(\varphi : Y_2 \to Y_1\) is an unramified covering of degree \(d\). We have justified straightforwardly that \(\varphi : Y_2 \to Y_1\) is an unramified covering of degree \(d\), in order to use it towards the coincidence of \(X_2\) with the fibered product \(X'_2 := X_1 \times_{Y_1} Y_2\).

(ii) is an immediate consequence of the fact that the composition of morphisms with connected fibres has connected fibres.

\[\square\]

**Corollary 7.** Let \(f : X_2 \to X_1 = (\mathbb{B}/\Gamma_1)'\) be an unramified covering of degree \(d\) of a smooth toroidal compactification \(X_1 = (\mathbb{B}/\Gamma_1)\), \(\rho_1 : X_1 \to Y_1\) be a composition of blow downs onto a minimal surface \(Y_1\) and \(D^{(1)} := X_1 \setminus (\mathbb{B}/\Gamma_1)\) be the toroidal compactifying divisor of \(\mathbb{B}/\Gamma_1\). Then:

(i) there exist a composition \(\rho_2 : X_2 \to Y_2\) of blow downs onto a minimal surface \(Y_2\) and an unramified covering \(\varphi : Y_2 \to Y_1\) of degree \(d\), which exhibits \(X_2 = X_1 \times_{Y_1} Y_2\) as a fibered product of \(X_1\) and \(Y_2\) over \(Y_1\);

(ii) there is a subgroup \(\Gamma_2 < \Gamma_1\) of index \([\Gamma_1 : \Gamma_2] = d\), such that \(X_2 = (\mathbb{B}/\Gamma_2)'\) is the toroidal compactification of \(\mathbb{B}/\Gamma_2\) and \(f\) restricts to unramified coverings \(f : \mathbb{B}/\Gamma_2 \to \mathbb{B}/\Gamma_1, f : D^{(2)} := X_2 \setminus (\mathbb{B}/\Gamma_2) \to X_1 \setminus (\mathbb{B}/\Gamma_2) =: D^{(1)}\) of degree \(d\);

(iii) \(\varphi\) restricts to an unramified covering \(\varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)})\) of degree \(d\).

**Proof.** (i) is an immediate consequence of Lemma 6 (ii) and the fact that any unramified cover \(Y_2\) of a minimal surface \(Y_1\) is minimal.

(ii) The unramified covering \(f : X_2 \to X_1 = (\mathbb{B}/\Gamma_1)'\) of degree \(d\) restricts to an unramified covering \(f : f^{-1}(\mathbb{B}/\Gamma_1) \to \mathbb{B}/\Gamma_1\) of degree \(d\). As in the proof of Corollary 5, there is a subgroup \(\Gamma_2 < \Gamma_1\) of index \([\Gamma_1 : \Gamma_2] = d\), such that \(X_2 = (\mathbb{B}/\Gamma_2)'\) is the
toroidal compactification of $\mathbb{B}/\Gamma_2$ and $f$ restricts to unramified coverings $f : \mathbb{B}/\Gamma_2 \to \mathbb{B}/\Gamma_1$, $f : D^{(2)} := X_2 \setminus (\mathbb{B}/\Gamma_2) \to X_1 \setminus (\mathbb{B}/\Gamma_1) := D^{(1)}$ of degree $d$. 

(iii) is an immediate consequence of Lemma 4 (ii).

\[ \square \]

**Definition 8.** A smooth toroidal compactification $X_1 = (\mathbb{B}/\Gamma_1)'$ is saturated if there is no unramified covering $f : X_2 = (\mathbb{B}/\Gamma_2)' \to (\mathbb{B}/\Gamma_1)' = X_1$ of degree $d$, which restricts to an unramified covering $f : \mathbb{B}/\Gamma_2 \to \mathbb{B}/\Gamma_1$ of degree $d$.

Bearing in mind that the fundamental group of a smooth projective variety is a birational invariant, one combines Corollary 5 with Corollary 7 and obtains the following

**Corollary 9.** A smooth toroidal compactification $X_1 = (\mathbb{B}/\Gamma_1)'$ is saturated if and only if one and, therefore, any minimal model $Y_1$ of $X_1$ is simply connected.

### 2 Unramified push forward of a smooth compactification

Let $X_2$ be a smooth projective surface, $\beta : X_2 \to Y_2$ be a blow down with exceptional divisor $E(\beta) = \bigsqcup_{s=1}^{d} L_{1,s}$ and $f : X_2 \to X_1$ be an unramified covering of degree $d$, which restricts to an unramified covering $f : E(\beta) \to f(E(\beta))$ of degree $d$. According to Lemma 2 (ii), $L_1 := f(E(\beta))$ is a $(-1)$-curve on $X_1$. Then Lemma 6 (i) implies that there is a fibered product commutative diagram (1) with the blow down $\text{Bl} : X_1 \to Y_1$ of $L_1$ and an unramified covering $\varphi : Y_2 \to Y_1$ of degree $d$, which shrinks $\beta(E(\beta)) = \{ p_{1,j} := \beta(L_{1,j}) \mid 1 \leq j \leq d \}$ to a point $q_1 \in Y_1$. We say that $\varphi$ is induced by $f$.

Suppose that $\rho_2 = \beta_1 \cdots \beta_r : S_r := X_2 \to Y_2 =: S_0$ is a composition of blow downs $\beta_i : S_i := \beta_{i+1} \cdots \beta_r(S_r) \longrightarrow S_{i-1} := \beta_1 \cdots \beta_r(S_r)$ (5)

with exceptional divisors $E(\beta_i) = \bigsqcup_{s=1}^{d} L_{i,s}$ for all $1 \leq i \leq r$. By a decreasing induction on $r \geq i \geq 1$, let us assume that there is a fibered product commutative diagram

\[
\begin{array}{ccc}
S_r & \xrightarrow{\beta_r} & S_{r-1} \\
| & \nearrow \varphi_r & \searrow \varphi_{r-1} \\
\text{Bl}_r & \xrightarrow{\varphi_r^{-1}} & \varphi_r^{-1}(S_{r-1}) \\
| & \nearrow \varphi_{r+1} & \searrow \varphi_r \\
f(S_r) & \xrightarrow{\varphi_{r+1}} & \varphi_{r+1}(S_{r+1}) \\
\end{array}
\]

\[
\begin{array}{ccc}
\cdots S_{i+1} & \xrightarrow{\beta_{i+1}} & S_i \\
| & \nearrow \varphi_{i+1} & \searrow \varphi_i \\
\text{Bl}_{i+1} & \xrightarrow{\varphi_{i+1}} & \varphi_i(S_i) \\
\end{array}
\]

with fibered product squares $\text{Bl}_j \varphi_j = \varphi_{j-1} \beta_j$, such that $\varphi_j$ restricts to an unramified covering $\varphi_j : E(\beta_j) \to L_j := \varphi_j(E(\beta_j))$ of degree $d$ and $\varphi_{j-1}$ shrinks the set $\beta_j(E(\beta_j)) = \{ p_{j,s} := \beta_j(L_{j,s}) \mid 1 \leq s \leq d \}$ to a point $q_j \in \varphi_{j-1}(S_{j-1})$ for all $r \geq j \geq i + 1$. If $\varphi_i : S_i \to \varphi_i(S_i)$ restricts to an unramified covering $\varphi_i : E(\beta_i) \to L_i := \varphi_i(E(\beta_i))$ of degree $d$ then there is an unramified covering $\varphi_{i-1} : S_{i-1} \to \varphi_{i-1}(S_{i-1})$ of degree $d$, which shrinks $\beta_i(E(\beta_i)) = \{ p_{i,s} := \beta_i(L_{i,s}) \mid 1 \leq s \leq d \}$ to a point $q_i \in S_{i-1}$ and closes the fibered product commutative diagram $\varphi_{i-1} \beta_i = \text{Bl}_i \varphi_i$. Thus, if an unramified covering $f : X_2 \to X_1$
of degree $d$ induces unramified coverings $E(\beta_i) = \prod_{s=1}^{d} L_{i,s} \to L_i$ of degree $d$ for all $1 \leq i \leq r$ then there is an unramified covering $\varphi := \varphi_0 : Y_2 = S_0 \to \varphi_0(S_0) =: Y_1$ of degree $d$, which induces unramified coverings $\beta_i(E(\beta_i)) = \{p_{i,s} := \beta_i(L_{i,s})\mid 1 \leq s \leq d\} \to \{q_i\} \subset \varphi_i^{-1}(S_{i-1})$ of degree $d$ for all $1 \leq i \leq r$.

Conversely, assume that $Y_2$ is a smooth projective surface, $\beta : X_2 \to Y_2$ is a blow down with exceptional divisor $E = \prod_{s=1}^{d} L_{1,s}$ and $\varphi : Y_2 \to Y_1$ is an unramified covering of degree $d$, which shrinks $\beta(E(\beta)) = \{p_{1,s} := \beta(L_{1,s})\mid 1 \leq s \leq d\}$ to a point $q_1 \in Y_1$. According to Lemma 3 (i), there is a fibered product commutative diagram (1), where $\text{Bl}_i : X_1 \to Y_1$ is the blow up of $Y_1$ at $q_1 \in Y_1$ and $f : X_2 \to X_1$ is an unramified covering of degree $d$, which restricts to an unramified covering $f : E(\beta) = \prod_{s=1}^{d} L_{1,s} \to L_1 := \text{Bl}_1^{-1}(q_1)$ of degree $d$. Let $\rho_2 = \beta_1 \ldots \beta_r : S_r := X_2 \to Y_2 =: S_0$ be a composition of blow downs (5) with exceptional divisors $E(\beta_i) = \prod_{s=1}^{d} L_{i,s}$. By an increasing induction on $1 \leq i \leq r$, suppose that

$$
\begin{array}{ccc}
S_1 & \xrightarrow{\beta_i} & S_{i-1} \\
\varphi_i & \downarrow & \varphi_{i-1} \\
\varphi_i(S_1) & \xrightarrow{\text{Bl}_i} & \varphi_{i-1}(S_{i-1})
\end{array} \quad \ldots \quad 
\begin{array}{ccc}
S_1 & \xrightarrow{\beta_1} & S_0 = Y_2 \\
\varphi & \downarrow & \varphi = \varphi_0 \\
\varphi(S_1) & \xrightarrow{\text{Bl}_1} & \varphi(Y_2)
\end{array}
$$

is a fibered product commutative diagram with fibered product squares $\varphi_{j-1}\beta_j = \text{Bl}_j\varphi_j$, such that $\varphi_{j-1}$ restricts to an unramified covering

$$
\varphi_{j-1} : \beta_j(E(\beta_j)) = \{p_{j,s} := \beta_j(L_{j,s})\mid 1 \leq s \leq d\} \to \{q_j\} \subset \varphi_{j-1}(S_{j-1})
$$

of degree $d$ and $\varphi_j$ restricts to an unramified covering

$$
\varphi_j : E(\beta_j) = \prod_{s=1}^{d} L_{j,s} \to \varphi_j(E(\beta_j)) =: L_j
$$

of degree $d$ for all $1 \leq j \leq i$. If $\varphi_i$ restricts to an unramified covering

$$
\varphi_i : \beta_{i+1}(E(\beta_{i+1})) = \{p_{i+1,s} := \beta_{i+1}(L_{i+1,s})\mid 1 \leq s \leq d\} \to \{q_{i+1}\} \subset \varphi_i(S_i)
$$

of degree $d$ then there is an unramified covering

$$
\varphi_{i+1} : S_{i+1} \to \varphi_{i+1}(S_{i+1})
$$

of degree $d$, which restricts to an unramified covering

$$
\varphi_{i+1} : E(\beta_{i+1}) = \prod_{s=1}^{d} L_{i+1,s} \to L_{i+1} := \varphi_{i+1}(E(\beta_{i+1}))
$$
of degree $d$ and closes the fibered product commutative diagram $\phi_i\beta_{i+1} = \text{Bl}_{i+1}\phi_{i+1}$ with the blow down $\text{Bl}_{i+1} : \varphi_{i+1}(S_{i+1}) \to \varphi_i(S_i)$ of $L_{i+1}$. In such a way, if $\varphi : Y_2 \to Y_1$ is an unramified covering of degree $d$, which induces unramified coverings

$$\beta_i(E(\beta_i)) = \{q_{i,s} := \beta_i(L_{i,s}) \mid 1 \leq s \leq d\} \to \{q_i\} \subset \varphi_{i-1}(S_{i-1})$$

of degree $d$ for all $1 \leq i \leq r$ then $f := \varphi_r : X_2 \to f(X_2)$ is an unramified covering of degree $d$, which induces unramified coverings $E(\beta_i) = \prod_{s=1}^{d} L_{i,s} \to L_i$ of degree $d$ for all $1 \leq i \leq r$.

The above considerations justify the following

**Lemma-Definition 10.** Let $X_2$, $Y_2$ be smooth projective surfaces and

$$\rho_2 = \beta_1 \ldots \beta_r : S_r := X_2 \to Y_2 =: S_0$$

be a composition of blow downs (5) with exceptional divisors $E(\beta_i)$ for all $1 \leq i \leq r$. Then the following are equivalent:

(i) there is an unramified covering $f : X_2 \to f(X_2)$ of degree $d$, which induces unramified coverings $E(\beta_i) = \prod_{s=1}^{d} L_{i,s} \to L_i$ of degree $d$ for all $1 \leq i \leq r$;

(ii) there is an unramified covering $\varphi : Y_2 \to \varphi(Y_2)$ of degree $d$, which induces unramified coverings $\beta_i(E(\beta_i)) = \{q_{i,s} = \beta_i(L_{i,s}) \mid 1 \leq s \leq d\} \to \{q_i\} \subset \varphi_{i-1}(S_{i-1})$ of degree $d$ for all $1 \leq i \leq r$.

If there holds one and, therefore, any one of the aforementioned conditions then there is a fibered product commutative diagram (4), where

$$\rho_1 = \text{Bl}_1 \ldots \text{Bl}_r : X_1 := \varphi(X_2) =: \varphi(Y_2) =: Y_1$$

is the composition of blow downs $\text{Bl}_i$ of $L_i$ for all $1 \leq i \leq r$ and we say that $f : X_2 \to f(X_2)$ and $\varphi : Y_2 \to \varphi(Y_2)$ are compatible with $\rho$.

**Corollary 11.** Let $X_2 = (\mathbb{B}/\Gamma_2)'$ be a smooth toroidal compactification and $\rho_2 : X_2 \to Y_2$ be a composition of blow downs onto a minimal surface $Y_2$. If there is an unramified covering $f : X_2 = (\mathbb{B}/\Gamma_2)' \to f(X_2) =: X_1$ of degree $d$, which is compatible with $\rho_2$ and restricts to an unramified covering $f : \mathbb{B}/\Gamma_2 \to f(\mathbb{B}/\Gamma_2)$ of degree $d$ then:

(i) there is a fibered product commutative diagram (4) with an unramified covering $\varphi : Y_2 \to \varphi(Y_2) =: Y_1$ of degree $d$ and a composition of blow downs $\rho_1 : X_1 \to Y_1$ onto a minimal surface $Y_1$;

(ii) there is a lattice $\Gamma_1$ of $\text{Aut}(\mathbb{B}) = PU(2,1)$, containing $\Gamma_2$ as a subgroup of index $[\Gamma_1 : \Gamma_2] = d$, and such that $X_1 = (\mathbb{B}/\Gamma_1)'$ is the toroidal compactification of $\mathbb{B}/\Gamma_1$;

(iii) $\varphi$ restricts to an unramified covering $\varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)})$ of degree $d$, where $D^{(j)} := X_1 \setminus (\mathbb{B}/\Gamma_j)$ are the compactifying divisors of $\mathbb{B}/\Gamma_j$, $1 \leq j \leq 2$.

**Proof.** (i) is an immediate consequence of Lemma 10.

Towards (ii), let us note that the composition $f\zeta_2 : \mathbb{B} \to f(\mathbb{B}/\Gamma_2)$ of the orbifold universal covering $\zeta_2 : \mathbb{B} \to \mathbb{B}/\Gamma_2$ with the unramified covering $f : \mathbb{B}/\Gamma_2 \to f(\mathbb{B}/\Gamma_2)$ is Galois, since $\pi_1^{\text{orb}}(\mathbb{B}) = \{1\}$ is a normal subgroup of $\Gamma_1 := \pi_1^{\text{orb}}(f(\mathbb{B}/\Gamma_2))$. Moreover, $\pi_1^{\text{orb}}(\mathbb{B}/\Gamma_2) = \Gamma_2$ is a subgroup of $\Gamma_1$ of index $[\Gamma_1 : \Gamma_2] = d$ and $f(\mathbb{B}/\Gamma_2) = \mathbb{B}/\Gamma_1$. By Lemma 1 (i), $f : X_2 \to X_1$.
restricts to an unramified covering $f : D^{(2)} = X_2 \setminus (\mathbb{B}/\Gamma_2) \to D^{(1)} := X_1 \setminus (\mathbb{B}/\Gamma_1)$ of degree $d$. The toroidal compactifying divisor $D^{(2)}$ of $\mathbb{B}/\Gamma_2$ has disjoint smooth elliptic irreducible components, so that Lemma 2 (i) applies to provide that $D^{(1)}$ consists of disjoint smooth elliptic irreducible components and $X_1 = (\mathbb{B}/\Gamma_1)'$ is the toroidal compactification of $\mathbb{B}/\Gamma_1$. According to Lemma 4 (ii), that suffices for $\varphi : Y_2 \to Y_1$ to restrict to an unramified covering $\varphi : \rho_2(D^{(2)}) \to \rho_1(D^{(1)})$.

\[
\text{Corollary 12. Let } X_2 = (\mathbb{B}/\Gamma_2)' \text{ be a smooth toroidal compactification, } D^{(2)} := X_2 \setminus (\mathbb{B}/\Gamma_2) \text{ be the compactifying divisor of } \mathbb{B}/\Gamma_2 \text{ and } \rho_2 : X_2 \to Y_2 \text{ be a composition of blow downs onto a minimal surface } Y_2. \text{ If } \varphi : Y_2 \to \varphi(Y_2) \text{ is an unramified covering of degree } d, \text{ which is compatible with } \rho_2 \text{ and restricts to an unramified covering } \varphi : \rho_2(D^{(2)}) \to \varphi\rho_2(D^{(2)}) \text{ of degree } d \text{ then:}
\]

(i) there is a fibered product commutative diagram (4) with an unramified covering $f : X_2 \to f(X_2) =: X_1$ of degree $d$ and a composition of blow downs $\rho_1 : X_1 \to Y_1$ onto a minimal surface $Y_1$;

(ii) there is a lattice $\Gamma_1$ of $\text{Aut}(\mathbb{B}) = PU(2,1)$, containing $\Gamma_2$ as a subgroup of index $[\Gamma_1 : \Gamma_2] = d$ and such that $X_1 = (\mathbb{B}/\Gamma_1)'$ is the toroidal compactification of $\mathbb{B}/\Gamma_1$;

(iii) $f$ restricts to an unramified covering $f : \mathbb{B}/\Gamma_2 \to \mathbb{B}/\Gamma_1$ of degree $d$.

\textbf{Proof.} Lemma 10 justifies (i). According to Lemma 4 (ii), $f$ restricts to an unramified covering $f : D^{(2)} \to f(D^{(2)})$ of degree $d$. Then Lemma 1 (i) applies to provide that $f : X_2 \setminus D^{(2)} = \mathbb{B}/\Gamma_2 \to X_1 \setminus f(D^{(2)})$ is an unramified covering of degree $d$. The proof of Corollary 11 (ii) has established that this is sufficient for the existence of a lattice $\Gamma_1$ of $\text{Aut}(\mathbb{B}) = PU(2,1)$, containing $\Gamma_2$ as a subgroup of index $[\Gamma_1 : \Gamma_2] = d$ and such that $X_1 \setminus f(D^{(2)}) = \mathbb{B}/\Gamma_1$. That justifies (iii). By assumption, $D^{(2)}$ consists of smooth elliptic irreducible components. Therefore $f(D^{(2)})$ has smooth elliptic irreducible components and $X_1 = (\mathbb{B}/\Gamma_1) \coprod f(D^{(2)})$ is the toroidal compactification of $\mathbb{B}/\Gamma_1$. 

\textbf{Definition 13. Let } X = (\mathbb{B}/\Gamma)' \text{ be a smooth toroidal compactification. If there is no unramified covering } f : X \to f(X) \text{ of degree } d, \text{ which restricts to an unramified covering } f : \mathbb{B}/\Gamma \to f(\mathbb{B}/\Gamma) \text{ of degree } d \text{ and is compatible with some composition of blow downs } \rho : X \to Y \text{ onto a minimal surface } Y, \text{ we say that } X = (\mathbb{B}/\Gamma)' \text{ is primitive.}

The Euler characteristic of a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ is a natural number $e(X) = e(\mathbb{B}/\Gamma)$. That is why, there exists a primitive smooth toroidal compactification $X_0 = \mathbb{B}/\Gamma_0$ and a finite sequence

\[
X_n := X \xrightarrow{f_n} X_{n-1} \quad \ldots \quad X_i \xrightarrow{f_i} X_{i-1} \quad \ldots \quad X_1 \xrightarrow{f_1} X_0
\]

of unramified coverings $f_i : X_i = (\mathbb{B}/\Gamma_i)' \to (\mathbb{B}/\Gamma_{i-1})'$ of degree $d_i$ of smooth toroidal compactifications $X_j = (\mathbb{B}/\Gamma_j)'$, which restrict to unramified coverings $f_i : \mathbb{B}/\Gamma_i \to \mathbb{B}/\Gamma_{i-1}$ of degree $d_i$ and are compatible with some compositions of blow downs $\rho_i : X_i \to Y_i$ onto minimal surfaces $Y_i$. Combining Corollary 11 with Corollary 12, one obtains the following
Corollary 14. Let $X = (\mathbb{B}/\Gamma)'$ be a smooth toroidal compactification with toroidal compactifying divisor $D := X \setminus (\mathbb{B}/\Gamma)$. Then $X$ is primitive if and only if no one minimal model $Y$ of $X$ with a composition of blow downs $\rho : X \to Y$ admits an unramified covering $\varphi : Y \to \varphi(Y)$ of degree $d > 1$, which restricts to an unramified covering $\varphi : \rho(D) \to \varphi\rho(D)$ of degree $d$ and is compatible with $\rho$.

Let us suppose that a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ with toroidal compactifying divisor $D := X \setminus (\mathbb{B}/\Gamma)$ admits a blow down $\beta : X \to Y$ of $n \in \mathbb{N}$ smooth irreducible rational $(-1)$-curves onto a minimal surface $Y$ and there is an unramified covering $\varphi : Y \to \varphi(Y)$ of degree $d$, which restricts to unramified coverings $\varphi : \beta(D) \to \varphi\beta(D)$ and $\varphi : \beta(E(\beta)) \to \varphi\beta(E(\beta))$ of degree $d$. Then the Euler number of the smooth surface $\varphi(Y)$ is $e(\varphi(Y)) = \frac{e(Y)}{d} \in \mathbb{Z}$ and the cardinality of $\varphi\beta(E(\beta))$ if $|\varphi\beta(E(\beta))| = \frac{|\beta(E(\beta))|}{d} = \frac{n}{d} \in \mathbb{N}$, so that $d \in \mathbb{N}$ divides $e(Y)$ and $n = |\beta(E(\beta))|$. As a result, $d$ divides the greatest common divisor $\text{GCD}(|\beta(E(\beta))|, e(Y))$.

Note that the compatibility of an unramified covering $\varphi : Y \to \varphi(Y)$ with $\beta : X \to Y$ reduces to $\varphi^{-1}(\varphi\beta(E(\beta)) = \beta(E(\beta))$ and is detected on $Y$. When $\rho = \beta_1 \ldots \beta_r : X \to Y$ is a composition of $r \geq 2$ blow downs, the compatibility of an unramified covering $\varphi : Y \to \varphi(Y)$ of degree $d$ with $\rho$ cannot be traced out on the minimal model $Y$ of $X$ alone. Namely, if $S_0 := Y$, $T_0 := \varphi(Y)$ then in the notations from the commutative diagram (3), the unramified covering $\varphi_1 : S_1 \to T_1$ of degree $d$ may restrict to an unramified covering $\varphi_1 : \beta_2(E(\beta_2)) \to \varphi_1\beta_2(E(\beta_2))$ of degree $d$, but $\varphi_0 := \varphi$ is not supposed to restrict to an unramified covering $\varphi : \beta_1\beta_2(E(\beta_2)) \to \varphi\beta_1\beta_2(E(\beta_2))$ of degree $d$. More precisely, if an irreducible component $L_{1,j}$ of $E(\beta_1)$ intersects $\beta_2(E(\beta_2))$ in at least two points then $|\beta_1\beta_2(E(\beta_2))| < d$ and $\varphi : \beta_1\beta_2(E(\beta_2)) \to \varphi\beta_1\beta_2(E(\beta_2))$ is of degree $< d$.

3 Saturated and primitive smooth compactifications of non-positive Kodaira dimension

Definition 15. Let $X = (\mathbb{B}/\Gamma)'$ and $X_0 = (\mathbb{B}/\Gamma_0)'$ be smooth toroidal compactification. We say that $X$ dominates $X_0$ and write $X \geq X_0$ or $X_0 \preceq X$ if there exist a finite sequence of ball lattices

$$\Gamma_n := \Gamma < \Gamma_{n-1} < \ldots < \Gamma_i < \Gamma_{i-1} < \ldots < \Gamma_1 < \Gamma_0,$$

with smooth toroidal compactifications $X_i = (\mathbb{B}/\Gamma_i)'$ of the corresponding ball quotients $\mathbb{B}/\Gamma_i$ and a finite sequence of unramified coverings

$$X_n := X \xrightarrow{f_n} X_{n-1} \quad \ldots \quad X_i \xrightarrow{f_i} X_{i-1} \ldots \quad X_1 \xrightarrow{f_1} X_0$$

of degree $\deg [f_i : X_i \to X_{i-1}] = |\Gamma_{i-1} : \Gamma_i| = d_i \in \mathbb{N}$, which restrict to unramified coverings $f_i : \mathbb{B}/\Gamma_i \to \mathbb{B}/\Gamma_{i-1}$ of degree $d_i$ and are compatible with some compositions $\rho_i = \beta_{i,1} \ldots \beta_{i,r_i} : X_i \to Y_i$ of blow downs $\beta_{i,j}$ onto minimal surfaces $Y_i$.

It is clear that a smooth toroidal compactification $X = \mathbb{B}/\Gamma$ is saturated if and only if it is maximal with respect to the partial order $\succeq$. Similarly, $X$ is primitive exactly when it is minimal with respect to $\succeq$. Note that the partial order $\succeq$ on the set $S$ of the smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$ is artinian, i.e., any subset $S_0 \subseteq S$ has a minimal element.
$X_o = (\mathbb{B}/\Gamma_o)' \in S_o$. The minimal $X \in S$ are exactly the primitive ones, but the minimal $X_o \in S_o$ are not necessarily primitive, since such $X_o$ is not supposed to be a minimal element of $S$.

The present section discusses the saturated and the primitive smooth toroidal compactifications $X = (\mathbb{B}/\Gamma)'$ of Kodaira dimension $\kappa(X) \leq 0$.

**Proposition 16.** If $X = (\mathbb{B}/\Gamma)'$ is a smooth toroidal compactification of Kodaira dimension $\kappa(X) = -\infty$ then $X$ is a rational surface or $X$ has a ruled minimal model $\pi : Y \to E$ with an elliptic base $E$.

Any smooth rational $X = (\mathbb{B}/\Gamma)'$ is both saturated and primitive.

There is no smooth saturated $X = (\mathbb{B}/\Gamma)'$, whose minimal model is a ruled surface $\pi : Y \to E$ with an elliptic base $E$.

**Proof.** (i) Let $\rho : X = (\mathbb{B}/\Gamma)' \to Y$ be a composition of blow downs onto a minimal surface $Y$ of $\kappa(Y) = -\infty$. Then $Y = \mathbb{P}^2(\mathbb{C})$ is the complex projective plane or $\pi : Y \to E$ is a ruled surface with a base $E$ of genus $g \in \mathbb{Z}_{\geq 0}$. The toroidal compactifying divisor $D := X \setminus (\mathbb{B}/\Gamma) = \prod_{j=1}^{k} D_j$ has disjoint smooth irreducible elliptic components $D_j$. If $g \geq 2$ then the morphisms $\pi \rho : D_j \to E$ map to points $p_j := \pi \rho(D_j) \in E$, so that $\rho(D_j) \subseteq \pi^{-1}(p_j)$ for all $1 \leq j \leq k$. The exceptional divisor $L$ of $\rho : X \to Y$ has finite image $\rho(L) = \{ q_1, \ldots, q_m \}$ on $Y$ and $\rho(L) \subseteq \prod_{i=1}^{m} \pi^{-1}(\pi(q_i))$. Therefore

$$Y' := Y \setminus \left( \prod_{i=1}^{m} \pi^{-1}(\pi(q_i)) \right) \subseteq Y \setminus \rho(L) = X \setminus L$$

and $\rho$ acts identically on $Y'$. Moreover,

$$Y'' := Y' \setminus \left( \prod_{j=1}^{k} \pi^{-1}(p_j) \right) = Y' \setminus \left( \left( \prod_{i=1}^{m} \pi^{-1}(\pi(q_i)) \right) \prod_{j=1}^{k} \pi^{-1}(p_j) \right) \subseteq \mathbb{B}/\Gamma.$$

However, $Y''$ contains (infinitely many) fibres $\pi^{-1}(e) \simeq \mathbb{P}^1(\mathbb{C})$, $e \in E$ of $\pi : Y \to E$ and that contradicts the Kobayashi hyperbolicity of $\mathbb{B}/\Gamma$. In such a way, we have shown that any minimal model $Y$ of a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ of $\kappa(X) = -\infty$ is birational to $\mathbb{P}^2(\mathbb{C})$ or to a minimal ruled surface $\pi : Y \to E$ with an elliptic base $E$.

Any rational $X = (\mathbb{B}/\Gamma)'$ is simply connected and does not admit finite unramified coverings $X_1 \to X$ of degree $d > 1$. That is why $X$ is saturated. Let us suppose that $f : X = (\mathbb{B}/\Gamma)' \to X_0 = (\mathbb{B}/\Gamma_0)'$ is an unramified covering of degree $d > 1$, which is compatible with some composition of blow downs $\rho : X \to Y$ onto a minimal rational surface $Y$ and restricts to an unramified covering $f : \mathbb{B}/\Gamma \to \mathbb{B}/\Gamma_0$ of degree $d$. The Kodaira dimension is preserved under finite unramified coverings, so that $\kappa(X_0) = \kappa(X) = -\infty$. The surface $X_0$ is not simply connected, whereas non-rational. Therefore, there is a composition $\rho_0 : X_0 \to Y_0$ of blow downs onto a ruled surface $\pi_0 : Y_0 \to E_0$ with base $E_0$ of genus $g_0 \in \mathbb{N}$. The surjective morphism $\rho_0 f : X = (\mathbb{B}/\Gamma)' \to Y_0$ induces an embedding $(\rho_0 f)^* : H^{0,1}(Y_0) \to H^{0,1}(X)$. On one hand, the irregularity of $Y_0$ is $h^{0,1}(Y_0) := \dim_{\mathbb{C}} H^{0,1}(Y_0) = g_0 \in \mathbb{N}$. On the other hand, the rational surface $X$ has vanishing irregularity $h^{0,1}(X) = 0$. That contradicts the presence
of a finite unramified covering $f : X \to X_0$ of degree $d > 1$ and shows that any smooth rational toroidal compactification $X = (\mathbb{B}/\Gamma)'$ is prime.

Let $X = (\mathbb{B}/\Gamma)'$ be a smooth toroidal compactification, whose minimal model $Y$ is a ruled surface $\pi : Y \to E$ with an elliptic base $E$. Since $Y$ is birational to $\mathbb{P}^1(\mathbb{C}) \times E$ and the fundamental group is a birational invariant, one has $\pi_1(X) \simeq \pi_1(Y) \simeq \pi_1(E) \simeq (\mathbb{Z}^2, +)$. In particular, $Y$ is not simply connected. According to Corollary 9, $X$ cannot be saturated. 

According to the Enriques-Kodaira classification, there are four types of minimal smooth projective surfaces $Y$ of Kodaira dimension $\kappa(Y) = 0$. These are the abelian and the b-ielliptic surfaces with universal cover $\mathbb{C}^2$, as well as the $K3$ and the Enriques surfaces with $K3$ universal cover. If $\varphi : Y_2 \to Y_1$ is a finite unramified covering of smooth projective surfaces then the Kodaira dimension $\kappa(Y_1) = \kappa(Y_2)$ and the universal covers $\tilde{Y}_2 = \tilde{Y}_2$ coincide. Let $Y_2$ be a smooth projective surface with a fixed point free involution $g_o : Y_2 \to Y_2$ and $\beta : X_2 \to Y_2$ be the blow up of $Y_2$ at a $\langle g_o \rangle$-orbit $\{p_{1,1}, p_{1,2} = g_o(p_{1,1})\} \subset Y_2$. Then by the very definition of a blow up, $g_o$ induces a fixed point free involution $g_1 : X_2 \to X_2$, which leaves invariant the exceptional divisor $E(\beta) = L_{1,1} \coprod L_{1,2}$, $L_{1,i} := \beta^{-1}(p_{1,i})$ of $\beta$ and there is a fibered product commutative diagram (4) with a $\langle g_o \rangle$-Galois covering $\varphi : Y_2 \to Y_1$, a $\langle g_1 \rangle$-Galois covering $f : X_2 \to X_1$ and the blow up $Bl : X_1 \to Y_1$ of $Y_1$ at $\{q_1\} = \varphi(\{p_{1,1}, p_{1,2}\})$. Now, suppose that $\rho_2 = \beta_1 \ldots \beta_r : S_r := X_2 \to Y_2 \to S_0$ is a composition of blow downs with exceptional divisors $E(\beta_i) = L_{i,1} \coprod L_{i,2}$ and $g_o : S_0 \to S_0$ is a fixed point free involution. By an increasing induction on $1 \leq i \leq r$, if $g_{i-1} : S_{i-1} \to S_{i-1}$ is a fixed point free involution, which leaves invariant $\beta_i(E(\beta_i)) = \{p_{i,1}, p_{i,2}\}$ then there is a fixed point free involution $g_i : S_i \to S_i$, which leaves invariant $E(\beta_i) = L_{i,1} \coprod L_{i,2}$. In such a way, if a fixed point free involution $g_0 : S_0 \to S_0$ induces isomorphisms $L_{i,1} \to L_{i,2}$ for all $1 \leq i \leq r$ then there is a fixed point free involution $g_r : S_r \to S_r$ and a fibered product commutative diagram (4) with a $\langle g_o \rangle$-Galois covering $\varphi : Y_2 \to Y_1$, a $\langle g_r \rangle$-Galois covering $f : X_2 \to X_1$ and the composition $p_1 = Bl_1 \ldots Bl_r : X_1 \to Y_1$ of the blow downs of $E(\beta_i)/\langle g_i \rangle = L_i \simeq \mathbb{P}^1(\mathbb{C})$. If $g_o : S_0 \to S_0$ induces isomorphisms $L_{i,1} \to L_{i,2}$ of the irreducible components of $E(\beta_i) = L_{i,1} \coprod L_{i,2}$ for all $1 \leq i \leq r$, we say that $g_o$ is compatible with $\rho_2 = \beta_1 \ldots \beta_r$.

**Proposition 17.** Let $X = (\mathbb{B}/\Gamma)'$ be a smooth toroidal compactification, $D := X \setminus (\mathbb{B}/\Gamma)$ be the toroidal compactifying divisor of $\mathbb{B}/\Gamma$ and $\rho = \beta_1 \ldots \beta_r : X \to Y$ be a composition of blow downs onto a $K3$ surface $Y$. Then:

(i) $X$ is a saturated compactification;

(ii) $X$ is non-primitive exactly when there is a fixed point free involution $g_o : Y \to Y$, which is compatible with $\rho$ and leaves invariant $\rho(D)$;

(iii) if $X$ is non-primitive then there is a fibered product commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\rho} & Y \\
\downarrow f & \xrightarrow{\varphi} & \downarrow \\
X_0 & \xrightarrow{\rho_0} & Y_0
\end{array}
$$

with a primitive smooth toroidal compactification $X_0 = (\mathbb{B}/\Gamma_0)'$, a composition of blow downs.
\( \rho_0 : X_0 \to Y_0 \) onto a minimal Enriques surface \( Y_0 \) and unramified double covers \( f : X \to X_0 \), \( \varphi : Y \to Y_0 \).

**Proof.** (i) is an immediate consequence of \( \pi_1(Y) = \{1\} \), according to Corollary 9.

(ii) and (iii) follow from Corollary 14 and the fact that a minimal projective surface \( Y_0 \) admits an unramified covering \( \varphi : Y \to Y_0 \) by a K3 surface \( Y \) if and only if \( Y_0 \) is the quotient of \( Y \) by a fixed point free involution \( g_0 : Y \to Y \). Such \( Y_0 = Y/\langle g_0 \rangle \) are called minimal Enriques surfaces and do not admit unramified coverings \( \varphi_0 : Y_0 \to \varphi_0(Y_0) \) of degree \( > 1 \).

**Proposition 18.** Let \( X = (\mathbb{B}/\Gamma)' \) be a smooth toroidal compactification and \( \rho : \beta_1 \ldots \beta_r : X \to Y \) be a composition of blow downs onto a minimal Enriques surface \( Y \). Then:

1. \( X \) is a primitive compactification;
2. \( X \) is not saturated;
3. there is an unramified double cover \( f : X_1 = \mathbb{B}/\Gamma_1 \to \mathbb{B}/\Gamma = X \) by a saturated smooth toroidal compactification \( X_1 = (\mathbb{B}/\Gamma_1)' \) with K3 minimal model \( Y_1 \).

**Proof.** (i) is due to the lack of an unramified covering \( \varphi : Y \to \varphi(Y) \) of degree \( d > 1 \).

(ii) follows from \( \pi_1(Y) = (\mathbb{Z}_2, +) \neq \{1\} \).

(iii) is an immediate consequence of the Enriques-Kodaira classification of the smooth projective surfaces.

Let \( X = (\mathbb{B}/\Gamma)' \) be a smooth toroidal compactification with abelian or bi-elliptic minimal model \( Y \). According to Theorem 1.3 from Di Cerbo and Stover’s article [DiCerboStover2], \( X \) can be obtained from \( Y \) by blow up \( \beta : X \to Y \) of \( n \in \mathbb{N} \) points \( p_1, \ldots, p_n \in Y \).

**Proposition 19.** Let \( X = (\mathbb{B}/\Gamma)' \) be a smooth toroidal compactification with a blow down \( \beta : X \to Y \) onto a minimal surface \( Y \) with exceptional divisor \( E(\beta) = \prod_{i=1}^{n} L_i \) and \( D := X \setminus (\mathbb{B}/\Gamma) \) be the toroidal compactifying divisor of \( \mathbb{B}/\Gamma \). Then:

1. \( \beta \) transforms \( E(\beta) \) onto the singular locus \( \beta(E(\beta)) = \beta(D)_{\text{sing}} \) of \( \beta(D) \subset Y \);
2. \( X \) is non-primitive if and only if there is an unramified covering \( \varphi : Y \to \varphi(Y) \) of degree \( d > 1 \), which restricts to an unramified covering \( \varphi : \beta(D) \to \varphi(\beta(D)) \) of degree \( d \);
3. the relative automorphism group \( \text{Aut}(Y, \beta(D)) = \text{Aut}(Y, \beta(D), (\beta(D)_{\text{sing}})) \) admits an isomorphism

\[
\Phi : \text{Aut}(Y, \beta(D)) \to \text{Aut}(X, D)
\]

with the relative automorphism group \( \text{Aut}(X, D) = \text{Aut}(X, D, E(\beta)) \);

4. \( g_0 \in \text{Aut}(Y, \beta(D)) \) is fixed point free if and only if it corresponds to a fixed point free \( g = \Phi(g_0) \in \text{Aut}(X, D) \).

**Proof.** (i) If \( D = \bigcup_{j=1}^{k} D_j \) has irreducible components \( D_j \) then the singular locus of \( \beta(D) \) is

\[
\beta(D)_{\text{sing}} = \bigcup_{j=1}^{k} \beta(D_j)_{\text{sing}} \cup \bigcup_{1 \leq i < j \leq k} \beta(D_i) \cap \beta(D_j).
\]
Since $D_j$ are smooth irreducible elliptic curves, $\beta(D)^{\text{sing}} \subseteq \beta(E(\beta))$. Conversely, any $(-1)$-curve $L_i$ on $X = (\mathbb{B}/\Gamma)'$ intersects $D = \bigcap_{j=1}^{k} D_j$ in at least three points, due to the Kobayashi hyperbolicity of $\mathbb{B}/\Gamma$. In fact, $|L_i \cap F| \geq 4$, according to Theorem 1.1 (2) from Di Cerbo and Stover's article [DiCerboStover2]. Therefore, the multiplicity of $\beta(L_i) = p_i$ with respect to $\beta(D)$ is $\geq 4$ and $p_i \in \beta(D)^{\text{sing}}$. That justifies $\beta(E(\beta)) \subseteq \beta(D)^{\text{sing}}$ and $\beta(E(\beta)) = \beta(D)^{\text{sing}}$.

(ii) By Corollary 14 and (i), $X = (\mathbb{B}/\Gamma)'$ is non-primitive if and only if there is an unramified covering $\varphi : Y \to \varphi(Y)$ of degree $d > 1$, which restricts to unramified coverings $\varphi : \beta(D) \to \varphi(\beta(D))$ and $\varphi : \beta(D)^{\text{sing}} \to \beta(D)^{\text{sing}}$ of degree $d$. Let us observe that any unramified covering $\varphi : \beta(D) \to \varphi(\beta(D))$ of degree $d$ restricts to an unramified covering $\varphi : \beta(D)^{\text{sing}} \to \beta(D)^{\text{sing}}$ of degree $d$, as far as the local biholomorphism $\varphi : \beta(D) \to \varphi(\beta(D))$ preserves the multiplicities of the points with respect to $\beta(D)$ and $\beta(D)^{\text{sing}}$ consists of the points of $\beta(D)$ of multiplicity $\geq 2$.

(iii) If a holomorphic automorphism $g_o : Y \to Y$ restricts to a holomorphic automorphism $g_o : \beta(D) \to \beta(D)$ then $g_o$ preserves the multiplicities of the points with respect to $\beta(D)$ and $\beta(D)^{\text{sing}}$ is $(g_o)$-invariant. That justifies $\text{Aut}(Y, \beta(D)) \leq \text{Aut}(Y, \beta(D), \beta(D)^{\text{sing}})$ and $\text{Aut}(Y, \beta(D)) = \text{Aut}(Y, \beta(D), \beta(D)^{\text{sing}})$.

In order to show the existence of a group isomorphism

$$\Phi : \text{Aut}(Y, \beta(D), \beta(D)^{\text{sing}}) \to \text{Aut}(X, D, E(\beta)),$$

let us pick up $g_o \in \text{Aut}(Y, \beta(D), \beta(D)^{\text{sing}})$ and $Y \setminus E(\beta) = Y \setminus \beta(D)^{\text{sing}}$ is acted by $\Phi(g_o)|_{Y \setminus E(\beta)} := g_o|_{Y \setminus \beta(D)^{\text{sing}}}$. By the very definition of a blow up at a point, the bijection $g_o : \beta(D)^{\text{sing}} \to \beta(D)^{\text{sing}}$ with $g_o(\beta(L_{1,j})) = \beta(L_{1,j})$ induces isomorphisms

$$\Phi(g_o) : L_{1,i} \to L_{1,j}$$

and provides an element $\Phi(g_o) \in \text{Aut}(X, E(\beta))$. After observing that $\Phi(g_o)(D \setminus E(\beta)) = g_o(\beta(D) \setminus \beta(D)^{\text{sing}}) = \beta(D) \setminus \beta(D)^{\text{sing}} = D \setminus E(\beta)$, one concludes that $\Phi(g_o)$ transforms the Zariski closure $D$ of $D \setminus E(\beta)$ onto itself and $\Phi(g_o) \in \text{Aut}(D)$.

The correspondence $\Phi$ is a group homomorphism since $g_o$ and $\Phi(g_o)$ coincide on Zariski open subsets of $Y$, respectively, $X$. Towards the bijectiveness of $\Phi$, let $g \in \text{Aut}(X, D, E(\beta))$ and note that $Y \setminus \beta(D)^{\text{sing}} \subseteq X \setminus E(\beta)$. That allows to define $\phi^{-1}(g)|_{Y \setminus \beta(D)^{\text{sing}}} := g|_{X \setminus E(\beta)}$. The isomorphism $g : E(\beta) \to E(\beta)$ of the exceptional divisor $E(\beta)$ of $\beta$ induces a permutation $\phi^{-1}(g) : \beta(D)^{\text{sing}} \to \beta(D)^{\text{sing}}$ of the finite set $\beta(D)^{\text{sing}}$ and provides an automorphism $\phi^{-1}(g) \in \text{Aut}(Y, \beta(D)^{\text{sing}})$. Bearing in mind that $\phi^{-1}(g)(\beta(D) \setminus \beta(D)^{\text{sing}}) = g(D) \setminus E(\beta) = D \setminus E(\beta) = \beta(D) \setminus \beta(D)^{\text{sing}}$, one concludes that $\phi^{-1}(g) \in \text{Aut}(\beta(D))$ is an automorphism of the Zariski closure $\beta(D)$ of $\beta(D) \setminus \beta(D)^{\text{sing}} = \beta(D)^{\text{smooth}}$.

Note that any automorphism $g \in \text{Aut}(X, D)$ acts on the set of the smooth irreducible rational curves on $X$. Moreover, $g$ preserves the self-intersection number of such a curve and $\langle g \rangle$ acts on the set $E(\beta) = \bigcap_{i=1}^{n} L_i$ of the $(-1)$-curves on $X$. Thus, $g \in \text{Aut}(X, D, E(\beta))$ and $\text{Aut}(X, D) \subseteq \text{Aut}(X, D, E(\beta))$, whereas $\text{Aut}(X, D, E(\beta)) = \text{Aut}(X, D)$.

(iv) If $g \in \text{Aut}(X, D)$ has no fixed points on $X$ then $g_o := \Phi^{-1}(g) \in \text{Aut}(Y, \beta(D))$ restricts to $g_o|_{Y \setminus E(\beta)} = g|_{X \setminus E(\beta)}$ without fixed points. The assumption $g_o(p_i) = p_i = \text{Bl}(L_i)$ for some $1 \leq i \leq n$ implies that $g$ restricts to an automorphism $g : L_i \to L_i$. Any biholomorphism $g \in \text{Aut}(L_i) = \text{Aut}(\mathbb{P}^1(C)) = \text{PGL}(2, \mathbb{C})$ of the projective line $L_i = \mathbb{P}^1(C)$ is a fractional linear transformation and has two fixed points, counted with their
point free is an absurd. In such a way, any fixed point free automorphism of \( n \) and \( \zeta \) is primitive if there is a fixed point free automorphism \( g \in \text{Aut}(Y, \beta(D)) \) has no fixed points on \( Y \).

Conversely, if \( g_0 \in \text{Aut}(Y, \beta(D)) \) has no fixed points on \( Y \) and \( g := \Phi(g_0) \) then the restriction \( g|_X \rightarrow (\mathbb{B}/\Gamma)' \) has no fixed points. If \( g(x) = x \) for some \( x \in E(\beta) = \prod_{i=1}^n L_i \) then \( x \in L_i \) for some \( 1 \leq i \leq n \) and \( g(L_i) = L_i \). As a result, \( g_0 \) fixes \( p_i = \beta(L_i) \in Y \), which is an absurd. In such a way, any fixed point free \( g_0 \in \text{Aut}(Y, \beta(D)) \) corresponds to a fixed point free \( g = \Phi(g_0) \in \text{Aut}(X, D) \).

\[ \square \]

**Proposition 20.** Let \( X = (\mathbb{B}/\Gamma)' \) be a smooth toroidal compactification with toroidal compactifying divisor \( D := X \setminus (\mathbb{B}/\Gamma) \) and a blow down \( \beta : X \rightarrow Y \) of \( n \in \mathbb{N} \) smooth irreducible rational \((-1\)-curves. Then \( \text{Aut}(X, D) \) is a finite group.

**Proof.** By Proposition 19 (iii), \( \text{Aut}(X, D) = \text{Aut}(X, D, E(\beta)) \). Any \( g \in \text{Aut}(X, D) \) acts on \( D = \prod_{j=1}^k D_j \) and induces a permutation of the smooth elliptic irreducible components \( D_1, \ldots, D_k \) of \( D \). In such a way, there arises a representation

\[ \Sigma_1 : \text{Aut}(X, D) \rightarrow \text{Sym}(D_1, \ldots, D_k) = \text{Sym}(k). \]

The image of \( \Sigma_1 \) in the finite group \( \text{Sym}(k) \) is a finite group, so that it suffices to show the finiteness of \( \ker(\Sigma_1) \), in order to conclude that \( \text{Aut}(X, D) \) is a finite group. Similarly, \( \text{Aut}(X, D) = \text{Aut}(X, D, E(\beta)) \) acts on the exceptional divisor \( E(\beta) = \prod_{i=1}^n L_i \) of \( \beta : X \rightarrow Y \) and defines a representation

\[ \Sigma_2 : \text{Aut}(X, D) \rightarrow \text{Sym}(L_1, \ldots, L_n) = \text{Sym}(n). \]

Since \( \Sigma_2(\ker(\Sigma_1)) \) is a finite group, it suffices to show that \( G := \ker(\Sigma_2) \cap \ker(\Sigma_1) \) is a finite group. For any \( 1 \leq i \leq n \), \( 1 \leq j \leq k \) and \( g \in G \), the finite set \( L_i \cap D_j \) is transformed into itself, according to \( g(L_i \cap D_j) \subseteq g(L_i) \cap g(D_j) = L_i \cap D_j \). Therefore, there is a representation

\[ \Sigma_{i,j} : G \rightarrow \text{Sym}(L_i \cap D_j). \]

The image \( \Sigma_{i,j}(G) \) is a finite group, while the kernel \( K_{i,j} := \ker(\Sigma_{i,j}) \) fixes any point \( p \in L_i \cap D_j \) and acts on \( D_j \). It is well known that the holomorphic automorphisms \( \text{Aut}_{p}(D_j) \) of an elliptic curves \( D_j \), which fix a point \( p \in D_j \) form a cyclic group of order 2, 4 or 6. Therefore, \( K_{i,j} \leq \text{Aut}_{p}(D_j), G, \ker(\Sigma_1) \) and \( \text{Aut}(X, D) \) are finite groups.

\[ \square \]

**Definition 21.** A smooth toroidal compactification \( X = (\mathbb{B}/\Gamma)' \) with a blow down \( \beta : X \rightarrow Y \) of \( n \in \mathbb{N} \) smooth irreducible rational \((-1\)-curves onto a minimal surface \( Y \) is Galois non-primitive if there is a fixed point free automorphism \( g \in \text{Aut}(X, D) \setminus \{\text{Id}_X\} \).

Any Galois non-primitive \( X = (\mathbb{B}/\Gamma)' \) is non-primitive, because the \( (g) \)-Galois covering \( \zeta : X \rightarrow \zeta(X) = X/\langle g \rangle \) is unramified and restricts to unramified coverings \( \zeta : \mathbb{B}/\Gamma \rightarrow \zeta(\mathbb{B}/\Gamma) \) and

\[ \zeta : E(\beta) = \prod_{i=1}^n L_i \rightarrow \zeta(E(\beta)) \text{ of degree } |\langle g \rangle| = \text{ord}(g). \]
Note that the presence of an unramified covering \( \varphi : Y \to \varphi(Y) \) implies the coincidence \( \tilde{Y} = \varphi(Y) \) of the universal cover \( \tilde{Y} \) of \( Y \) with the universal cover \( \varphi(Y) \) of \( \varphi(Y) \). The fundamental group \( \pi_1(\varphi(Y)) \) of \( \varphi(Y) \) acts on \( \tilde{Y} \) by biholomorphic automorphisms without fixed points and contains the fundamental group \( \pi_1(Y) \) of \( Y \) as a subgroup of index \( [\pi_1(\varphi(Y)) : \pi_1(Y)] = d \).

**Proposition 22.** Let \( X = (\mathbb{B}/\Gamma)' \) be a smooth toroidal compactification with toroidal compactifying divisor \( D := X \setminus (\mathbb{B}/\Gamma) \), \( \beta : X \to Y \) be a blow down of \( n \in \mathbb{N} \) smooth irreducible rational \((-1)\)-curves to a minimal surface \( Y \) and \( N(\pi_1(Y)) \) be the normalizer of the fundamental group \( \pi_1(Y) \) of \( Y \) in the biholomorphism group \( \text{Aut}(\tilde{Y}) \) of the universal cover \( \tilde{Y} \) of \( Y \). Then \( X \) is Galois non-primitive if and only if there exist a natural divisor \( d > 1 \) of \( \text{GCD}(|\beta(D)_{\text{sing}}|, e(Y)) \in \mathbb{N} \) and an unramified covering \( \varphi : Y \to \varphi(Y) \) of degree \( d \), such that \( \pi_1(\varphi(Y)) \cap N(\pi_1(Y)) \geq \pi_1(Y) \) and \( \varphi : \beta(D) \to \varphi(D) \) is an unramified covering of degree \( d \).

**Proof.** If \( X = (\mathbb{B}/\Gamma)' \) is Galois non-primitive then there exists a fixed point free biholomorphism \( g \in \text{Aut}(X, D) \setminus \{\text{Id}_X\} \) of \( X \). By Proposition 19(iv), \( g \) induces a fixed point free biholomorphism \( g_0 = \Phi^{-1}(g) \in \text{Aut}(Y, \beta(D)) \setminus \{\text{Id}_Y\} \) of \( Y \). The element \( g_0 \) of the finite group \( \text{Aut}(Y, \beta(D)) \) is of finite order \( d \in \mathbb{N} \setminus \{1\} \) and the \( \langle g_0 \rangle \)-Galois coverings \( \zeta : Y \to Y/\langle g_0 \rangle \), \( \zeta : \beta(D) \to \zeta(\beta(D)) \) are unramified and of degree \( d \). The automorphism \( g_0 \) of \( Y \) lifts to an automorphism \( \sigma \in \text{Aut}(\tilde{Y}) \) of the universal cover \( \tilde{Y} \) of \( Y \), which normalizes \( \pi_1(Y) \) and belongs to

\[
\pi_1(\zeta(Y)) = \pi_1(Y/\langle g_0 \rangle) = \pi_1((\tilde{Y}/\pi_1(Y))/\langle \sigma \pi_1(\tilde{Y}) \rangle) = \pi_1((\tilde{Y}/\langle \sigma, \pi_1(Y) \rangle) = \langle \sigma, \pi_1(Y) \rangle.
\]

Conversely, suppose that \( \varphi : Y \to \varphi(Y) \) is an unramified covering of degree \( d > 1 \), which restricts to an unramified covering \( \varphi : \beta(D) \to \varphi(\beta(D)) \) of degree \( d \) and there exists \( \sigma \in [\pi_1(\varphi(Y)) \cap N(\pi_1(Y))] \setminus \pi_1(Y) \). Then \( g_0 := \sigma \pi_1(Y) \in \text{Aut}(Y) = N(\pi_1(Y))/\pi_1(Y) \) is a non-identical biholomorphism \( g_0 : Y \to Y \). Since \( \langle \sigma, \pi_1(Y) \rangle \) is a subgroup of \( \pi_1(\varphi(Y)) \), the unramified covering \( \varphi : Y \to \varphi(Y) \) factors through the \( \langle g_0 \rangle \)-Galois covering \( \zeta : Y \to Y/\langle g_0 \rangle \) and a covering \( \varphi_o : Y/\langle g_0 \rangle \to \varphi(Y) \) along the commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\zeta} & Y/\langle g_0 \rangle \\
\varphi \downarrow & & \downarrow \varphi_o \\
\varphi(Y) & &
\end{array}
\]

The finite coverings \( \zeta : Y \to Y/\langle g_0 \rangle \) and \( \varphi_o : Y/\langle g_0 \rangle \to \varphi(Y) \) are unramified, because their composition \( \varphi = \varphi_o \zeta : Y \to \varphi(Y) \) is unramified. That is why, \( g_0 \) has no fixed points on \( Y \). If \( \beta(D) \not\subseteq Y \), \( g_0 \)-invariant then there is an orbit \( \text{Orb}_{\langle g_0 \rangle}(y_0) \subset Y \) of some \( y_0 \in \beta(D) \) which intersects, both, \( \beta(D) \) and \( Y \setminus \beta(D) \). Therefore, \( \zeta : \beta(D) \to \zeta(\beta(D)) \) has a fibre \( \zeta^{-1}(\langle g_0 \rangle) \) of cardinality \( |\zeta^{-1}(\langle g_0 \rangle)| < \deg(\zeta) = |\langle g_0 \rangle| = \text{ord}(g_0) \) and \( \zeta : \beta(D) \to \zeta(\beta(D)) \) is ramified. As a result, the composition \( \varphi = \varphi_o \zeta : \beta(D) \to \varphi(\beta(D)) \) is ramified. The contradiction shows the \( \langle g_0 \rangle \)-invariance of \( \beta(D) \). According to Proposition 19 (iv), the
fixed point free $g_0 \in \text{Aut}(Y, \beta(D)) \setminus \{\text{Id}_Y\}$ corresponds to a fixed point free $g = \Phi(g_0) \in \text{Aut}(X, D) \setminus \{\text{Id}_X\}$ and $X$ is Galois non-primitive.

\[ \square \]

**Definition 23.** A covering $\varphi : Y \to \varphi(Y)$ by a smooth projective surface $Y$ has Galois factorization if there exist $g_0 \in \text{Aut}(Y) \setminus \{\text{Id}_Y\}$ and a covering $\varphi_0 : Y/\langle g_0 \rangle \to \varphi(Y)$, such that $\varphi = \varphi_0 \zeta$ factors through the $\langle g_0 \rangle$-Galois covering $\zeta : Y \to Y/\langle g_0 \rangle$ and a covering $\varphi_0$ along the commutative diagram (6).

Now, Proposition 22 can be reformulated in the form of the following

**Corollary 24.** Let $X = (\mathbb{B}/\Gamma)'$ be a non-primitive smooth toroidal compactification with toroidal compactifying divisor $D := X \setminus (\mathbb{B}/\Gamma)$, $\beta : X \to Y$ be a blow down of $n \in \mathbb{N}$ smooth irreducible rational $(-1)$-curves onto a minimal surface $Y$ and $\varphi : Y \to \varphi(Y)$ be an unramified covering of degree $d$, which restricts to an unramified covering $\varphi : \beta(D) \to \varphi\beta(D)$ of degree $d$. Then $X$ is Galois non-primitive if and only if $\varphi$ admits a Galois factorization.

**Corollary 25.** (i) Let $X = (\mathbb{B}/\Gamma)'$ be a smooth toroidal compactification with abelian minimal model $Y$. Then $X$ is not saturated and $Y$ is non-primitive if and only if it is Galois non-primitive.

(ii) If $X = (\mathbb{B}/\Gamma)'$ is a smooth toroidal compactification with bi-elliptic minimal model $Y$ then $X$ is not saturated.

**Proof.** (i) Any abelian surface $Y$ has non-trivial fundamental group $\pi_1(Y) \simeq (\mathbb{Z}^2, +)$. According to Corollary 9, that suffices for a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ with abelian minimal model $Y$ to be non-saturated.

By Theorem 1.3 from Di Cerbo and Stover's article [DiCerboStover2], if a smooth toroidal compactification $X = (\mathbb{B}/\Gamma)'$ has abelian minimal model $Y$ then there is a blow down $\beta : X \to Y$ of $n \in \mathbb{N}$ smooth irreducible rational $(-1)$-curves on $X$ onto $Y$. Such $X$ is non-primitive exactly when there exists an unramified covering $\varphi : Y \to \varphi(Y)$ of degree $d > 1$, which restricts to an unramified covering $\varphi : \beta(D) \to \varphi\beta(D)$ of degree $d$. Since $Y$ and $\varphi(Y)$ have one and a same universal cover $\varphi(Y) = \tilde{Y} = \mathbb{C}^2$ and one and a same Kodaira dimension $\kappa(\varphi(Y)) = \kappa(Y) = 0$, the minimal smooth irreducible projective surface $\varphi(Y)$ is abelian or bi-elliptic.

If $\varphi(Y)$ is an abelian surface then its fundamental group $\pi_1(\varphi(Y)) \simeq (\mathbb{Z}^2, +)$ is abelian and $\pi_1(Y) \simeq (\mathbb{Z}^2, +)$ is a normal subgroup of $\pi_1(\varphi(Y))$. As a result, $\varphi : Y \to \varphi(Y)$ is a $\pi_1(\varphi(Y))/\pi_1(Y)$-Galois covering and $Y$ is Galois non-primitive.

Let us suppose that $\varphi(Y)$ is a bi-elliptic surface. According to Bagnara-de Franchis classification of the bi-elliptic surfaces from [BagnaraDeFranchis], there is an abelian surface $A$ and a cyclic subgroup $\langle g \rangle \leq \text{Aut}(A)$ of order $d \in \{2, 3, 4, 6\}$ with a non-translation generator $g \in \text{Aut}(A)$, such that $\varphi(Y) = A/\langle g \rangle$. Let $\text{AffLin}(\mathbb{C}^2) := T(\mathbb{C}^2) \rtimes \text{GL}(2, \mathbb{C})$ be the group of the affine linear transformations of $\mathbb{C}^2 = \tilde{Y} = \varphi(Y) = \tilde{A}$ and

$\mathcal{L} : \text{AffLin}(\mathbb{C}^2) \longrightarrow \text{GL}(2, \mathbb{C})$

be the group homomorphism, associating to $\sigma \in \text{AffLin}(\mathbb{C}^2)$ its linear part $\mathcal{L}(\sigma) \in \text{GL}(2, \mathbb{C})$. Then the fundamental group of $A$ is the maximal translation subgroup

$\pi_1(A) = \pi_1(\varphi(Y)) \cap \ker(\mathcal{L})$

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of $\pi_1(\varphi(Y))$. The translation subgroup $\pi_1(Y) \leq \pi_1(\varphi(Y)) \cap \ker(L)$ of $\pi_1(\varphi(Y))$ is contained in $\pi_1(A)$ and the unramified covering $\varphi : Y \to \varphi(Y)$ factors through unramified coverings $\varphi_1 : Y \to A$ and $\varphi_2 : A \to \varphi(Y)$, along the commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\varphi_1} & A \\
\varphi \downarrow & & \downarrow \varphi_2 \\
\varphi(Y) & & 
\end{array}
$$

The covering $\varphi_1 : Y \to A$ is $\pi_1(A)/\pi_1(Y)$-Galois, so that $\varphi = \varphi_2 \varphi_1$ is a Galois factorization of $\varphi$ for $\pi_1(Y) \leq \pi_1(A)$. In the case of $\pi_1(Y) = \pi_1(A)$, there is an isomorphism $Y \simeq \mathbb{C}^2/\pi_1(Y) \simeq \mathbb{C}^2/\pi_1(A) = A$ and the covering $\varphi : Y \simeq A \to \varphi(Y) = A/\langle g \rangle$ is $\langle g \rangle$-Galois. Thus, $X$ is Galois non-primitive and a co-abelian smooth toroidal compactification $X = (B/\Gamma)'$ is non-primitive if and only if it is Galois non-primitive.

(ii) The fundamental group $\pi_1(Y)$ of a bi-elliptic surface $Y$ is subject to an exact sequence

$$1 \longrightarrow \pi_1(Y) \cap \ker(L) \longrightarrow \pi_1(Y) \longrightarrow \langle g \rangle \longrightarrow 1$$

with a non-translation cyclic subgroup $\langle g \rangle$ of $\text{Aut}(\mathbb{C}^2/\pi_1(Y) \cap \ker(L)) = \text{Aut}(A_o)$ of order 2, 3, 4 or 6. In particular, $Y$ is not simply connected and a smooth toroidal compactification $X = (B/\Gamma)'$ with bi-elliptic minimal model $Y$ is not saturated.

\[\square\]

\textbf{References}


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