Gauge Theory on Manifolds with Conical Asymptotic Geometry

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Matt Turner

Declaration of authorship

I am the author of this thesis, and the work described therein was carried out by myself personally, in collaboration with my supervisors Dr Johannes Nordström and Dr Karsten Matthies, or, where indicated, in collaboration with a fellow doctoral candidate Jakob Stein.

....
Matt Turner

A new start, to make a mark;
a festive block, stumbling in the dark,
pause . . . an island reset.

A new role, hear them sing;
strike one, in solidarity,
the future to confirm.

A new home, moments of pride;
back in lockdown, stayin' alive,
from Eminem to the Bee Gees.

A new collaboration, a crisis;
several shoulders required,
the last four in what follows.

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CONTENTS

LIST OF FIGURES AND TABLES

CHAPTER 1

INTRODUCTION

A connection A on a principal G-bundle over a G_2-manifold (M, φ) is a G_2-instanton if

$$F_A \wedge *\varphi = 0$$

where F_A is the curvature of the connection A.

In this thesis, we consider the problem of constructing G_2-instantons on two types of asymptotically conical manifolds, namely the Bryant-Salamon $S^3 \times \mathbb{R}^4$ and the infinite collection of manifolds $M_{m,n}$ constructed by Foscolo, Haskins and Nordström. Some progress has been made with the former example by Clarke, Lotay and Oliveira, while nothing was known about the existence of such instantons on the manifolds $M_{m,n}$. We firstly construct a 1-parameter family of instantons on the Bryant-Salamon $S^3 \times \mathbb{R}^4$, and hence complete the picture of all invariant instantons on this manifold. Secondly, we construct a 1-parameter family of instantons on $M_{1,1}$ via a dynamical systems argument.

Further, we consider an existence question of G_2-instantons on $S^3 \times \mathbb{R}^4$ with an asymptotically locally conical metric, known as the BGGG metric. Again, some progress has been made in understanding the existence of such instantons on this manifold by Lotay and Oliveira, but there is still a region of initial conditions for which the existence of global solutions was previously unknown. We prove the existence of a boundary of initial conditions running through this unknown region and hence fully characterise the asymptotic behaviour of all complete bounded G_2-instantons on a specific bundle over this manifold.

1.1 Motivation

Let (M, φ) be a G_2-manifold, that is, a Riemannian 7-manifold with a closed and coclosed 3-form φ that induces a metric with holonomy group the exceptional Lie group G_2. Now, let $P \to M$ be a principal G-bundle, where G is assumed to be a compact semisimple Lie group. A connection

A on P is a G_2-instanton if

$$F_A \wedge *\varphi = 0 \tag{1.1}$$

or equivalently, if

$$F_A \wedge \varphi = -*F_A \tag{1.2}$$

where F_A is the curvature of the connection A and $*$ denotes the Hodge star operator with respect to the metric on M induced by φ.

We note that (1.2) is analogous to the notion of an anti-self-dual (ASD) connection A on an oriented Riemannian 4-manifold, defined as satisfying $F_A = -*F_A$. The moduli space of such connections can be used to define numerical invariants for smooth 4-manifolds; this is known as Donaldson Theory. The analogy of G_2-instantons with ASD connections motivates study for using G_2-instantons to construct enumerative invariants of G_2-manifolds, an idea expressed by Donaldson and Thomas in [12], and developed further by Donaldson and Segal in [11]. Hence it is natural to search for examples of G_2-instantons on all known constructions of G_2-manifolds.

G_2-instantons have attracted interest from both the mathematics and theoretical physics communities. A physical motivation for understanding moduli spaces of G_2-manifolds is their application to M-theory, a branch of String Theory which unifies the various versions of Superstring Theory. In one model, the universe is postulated to have eleven dimensions, four of which consist of Minkowski space-time. The remaining seven locally look like a compact G_2-manifold with diameter of order the Planck length.

The goal of this thesis is to provide examples of G_2-instantons on three G_2-manifolds. Lotay and Oliveira surveyed recent results regarding G_2-instantons on non-compact G_2-manifolds in [23]. They also discussed some open problems in the field; this thesis addresses Problem 3.3 (e) in Section 4.2, Problem 4.4 (d) in Section 4.3 and Problem 3.3 (a) in Chapter 5. We refer the reader to [23] for a comprehensive discussion of these open problems (and others) in the wider context of G_2 gauge theory.

The physical motivation for the study of G_2-manifolds described above restricts attention to compact examples. So a natural question would be: why do we care about the non-compact examples of G_2-manifolds that we consider in this thesis? The advantage of the non-compact manifolds we study is their additional structure, which controls their asymptotic geometry. All three manifolds we consider admit either asymptotically conical (AC) or asymptotically locally conical (ALC) geometry. Given a Riemannian manifold (X, h), the Riemannian cone on X is the manifold

$$C(X) = X \times (0, \infty)_t$$

equipped with the metric

$$g_C = dt^2 + t^2 h.$$

A manifold is AC if it is diffeomorphic to a G_2-cone (i.e a cone on a 6-manifold with holonomy G_2) outside of a compact set, and whose metric is in some sense asymptotic to a conical metric. An ALC G_2-manifold (M, φ) is a Riemannian manifold with metric in some sense asymptotic, outside a compact set, to the metric

$$g_{\varphi_\infty} = dt^2 + \ell^2\theta^2 + t^2\pi^* h$$

on $(1,\infty)_t \times \Sigma$, a circle bundle $\pi : \Sigma^6 \to X^5$ over a 6-dimensional cone $C(X)$. Here, h is a metric on X, θ a connection on Σ, ℓ a positive real number and φ_∞ is a conical G_2-structure on $C(\Sigma)$, which we will define in Section 2.2.2.

By imposing AC or ALC geometry on a non-compact G_2-manifold, we have the possibility of gluing the compact part to a compact manifold with a conical singularity to obtain a smooth compact G_2-manifold. The G_2-structure can be glued because the metrics along the conical ends are sufficiently similar; see [22, 27] for examples when the manifolds have asymptotically cylindrical geometry. We now outline the three non-compact AC or ALC G_2-manifolds that we consider in this thesis.

The first example we consider is the Bryant-Salamon AC G_2-manifold $S^3 \times \mathbb{R}^3$. It is one of three AC G_2-structures constructed by Bryant and Salamon in [5]; the other two are defined on the total space of the bundle of anti-self-dual 2-forms on either S^4 or $\mathbb{CP}^2$. The G_2-structure on $S^3 \times \mathbb{R}^4$ forms the AC limit of the so-called $\mathbb{B}_7$ family of complete ALC G_2-metrics constructed by Bogoyavlenskaya [2].The second case that we consider is an ALC member of this family known as the BGGG manifold, constructed explicitly by Brandhuber-Gomis-Gubser-Gukov [4]. The construction of these metrics on the $\mathbb{B}_7$-family can be found in Section 3.2.2.

The third case is the AC limit of the infinite collection of families of ALC G_2-metrics constructed by Foscolo, Haskins and Nordström in [16]. One such family is known as the $\mathbb{C}_7$ family, which was conjectured to exist in [10]. We will refer to the collection of these families as the infinite extension of $\mathbb{C}_7$. Each family lies on a manifold $M_{m,n}$, for coprime positive integers m and n. $M_{m,n}$ is the circle bundle over the canonical bundle on $\mathbb{CP}^1 \times \mathbb{CP}^1$, whose restriction to the zero section has first Chern class $c_1 = (m, -n)$ under the isomorphism between the second cohomology and $\mathbb{Z} \times \mathbb{Z}$. See Section 3.3 for more details about the topology and geometry of these manifolds.

In all three cases, the manifolds admit a cohomogeneity one action of the group $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$. A cohomogeneity one manifold is a Riemannian manifold with an isometric Lie group action whose generic orbits, known as the principal orbits, have codimension one. With this high level of symmetry, we search for $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant G_2-instantons on the AC and ALC manifolds described above; this reduces the system of PDEs given by (1.1) to a system of nonlinear ODEs, which are generally simpler to solve or understand qualitatively.

There are a number of methods for finding solutions to ODEs and we apply several of these in what follows. The choice of method in proving the existence of instantons in each case depends on the nature of the equations. In this thesis, we find solutions using invariant sets for the Bryant-Salamon manifold and a dynamical systems approach for the manifold $M_{1,1}$. We will also comment on how these instantons can be constructed via a gluing procedure of local solutions.

1.2 Main Results

We consider each of the three cases outlined in Section 1.1 individually. In each case, we apply a different method to prove an existence result. The three manifolds lie in two different families of G_2-manifolds with conical asymptotic geometry, namely the $\mathbb{B}_7$ and $\mathbb{C}_7$ families. They are parameterised by $\beta \in (0, \beta_{ac}]$ for some $\beta_{ac} \in \mathbb{R}$; for any value in $(0, \beta_{ac})$, the metric is ALC. As

$\beta \to \beta_{ac}$, the size of the circle fibres tends to infinity and the geometry transitions; indeed, in the limit, the metric is AC. On the other hand, as $\beta \to 0$, the metric collapses to a Calabi Yau metric on an AC 3-fold which is asympototic to the cone $C(X)$ described in Section 1.1. For the $\mathbb{B}_7$ family, this is the Stenzel metric on one of the two small resolutions of the conifold. For the $\mathbb{C}_7$ family, this is a Calabi Yau metric on the canonical line bundle $K_{\mathbb{CP}^1 \times \mathbb{CP}^1}$. For more detail, see Section 2.2 of [16].

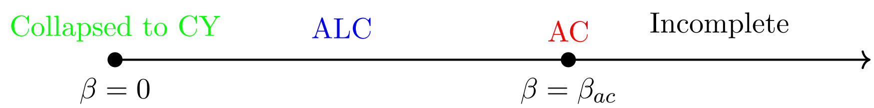

Figure 1.1: Structure of $\mathbb{B}_7$ and $\mathbb{C}_7$ families

We now outline the main results of the thesis for each manifold, and label on Figure 1.1 where each metric lies in the 1-parameter family.

Bryant-Salamon $S^3 \times \mathbb{R}^4$

Recall that the Bryant-Salamon (BS) manifold is an AC metric on $S^3 \times \mathbb{R}^4$ and forms the AC limit of the $\mathbb{B}_7$ family.

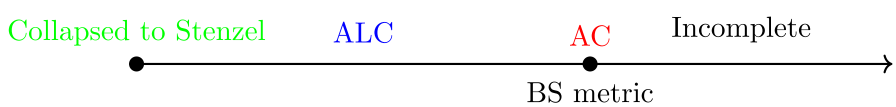

Figure 1.2: Bryant-Salamon manifold in the $\mathbb{B}_7$ family

On the Bryant-Salamon $S^3 \times \mathbb{R}^4$, there are two homogeneous SU(2)-bundles P_1 and P_{id}. Clarke in [7] constructed a 1-parameter family of G_2-instantons on P_1, and Lotay-Oliveira found a single solution on P_{id} in [24]. The family on P_1 bubble off an ASD instanton along the normal bundle to the submanifold $S^3 \times \{0\}$ and converge away from this submanifold to the previously known solution on P_{id}. In Section 4.2, we construct a 1-parameter family of G_2-instantons on the bundle P_{id} which then fully classifies the $\mathrm{SU}(2)^3$-invariant G_2-instantons on $S^3 \times \mathbb{R}^4$ with the BS AC metric.

Theorem A. *There exists a 1-parameter family of* $\mathrm{SU}(2)^3$*-invariant* G_2*-instantons on the bundle* P_{id} *on the* $\mathbb{B}_7$ *Bryant-Salamon* $S^3 \times \mathbb{R}^4$*, paramaterised by* $\gamma \in [-1, 1]$*. The solutions for* $\gamma = \pm 1$ *are flat solutions; for* $\gamma \in (-1, 1)$*, the solutions are asymptotic to the canonical nearly Kähler instanton on* $S^3 \times S^3$ *with its homogeneous nearly Kähler structure. This family on* P_{id} *and the family of Clarke [7] on* P_1 *are the only* $\mathrm{SU}(2)^3$*-invariant* G_2*-instantons on the* $\mathbb{B}_7$ *Bryant-Salamon* $S^3 \times \mathbb{R}^4$*.*

We prove Theorem A using forward invariant sets, i.e. subsets of the phase space for which solutions remain for all time after entering. By using bounded invariant sets, we can find G_2-instantons that exist for all time, and the method gives the limiting behaviour of these solutions. This theorem was proved in collaboration with the doctoral candidate Jakob Stein.

We can view these solutions as the result of gluing two local solutions. The first is a perturbation of one of the flat solutions given by $\gamma = \pm 1$ in a neighbourhood of $S^3 \times \{0\}$. The second is the pull-back of a solution on the cone whose asymptotic limit is the nearly Kähler instanton on $S^3 \times S^3$.

BGGG $S^3 \times \mathbb{R}^4$

Recall that the BGGG manifold is an ALC metric on $S^3 \times \mathbb{R}^4$ and is the only ALC member of the $\mathbb{B}_7$ family that is known explicitly.

Figure 1.3: BGGG manifold in the $\mathbb{B}_7$ family

On the BGGG manifold, we consider the results of Lotay and Oliveira [24] on the bundle P_1. They found a subset of the space of initial conditions which lead to complete bounded G_2-instantons, which is adjacent to a region $\mathcal{S}$ of initial conditions for which the existence of solutions is unknown. We prove that the set of initial conditions that lead to complete bounded solutions is closed, and that the boundary of this set lies in the region $\mathcal{S}$. This characterises the asymptotic behaviour of all complete bounded solutions.

G_2-instantons on ALC G_2-manifolds are in a certain sense asymptotic to Calabi-Yau (CY) monopoles on CY cones; CY monopoles are defined in Section 2.1.3. Indeed, let A be a G_2-instanton on an ALC G_2-manifold (M, φ) asymptotic to a circle bundle $(1,\infty) \times \Sigma \to (1,\infty) \times X$. Recall that the metric on an ALC manifold is asymptotic to

$$g_{\varphi_\infty} = dt^2 + \ell^2\theta^2 + t^2\pi^* h$$

where θ is a connection on Σ, $\ell \in \mathbb{R}$ is positive and φ_∞ is a conical G_2-structure on $C(\Sigma)$. Suppose there is a U(1)-invariant connection

$$A_\infty = a + \ell\Phi \otimes \theta$$

on a principal bundle P over $(1,\infty) \times \Sigma$, where a is a connection pulled back from $(1,\infty) \times X$ and $\Phi \in \Omega^0((1,\infty) \times \Sigma, \mathrm{Ad}(P))$, such that the curvature of A is asymptotic to the curvature of A_∞ outside a compact set. Then Proposition 3 of [24] shows that (a, Φ) is a CY monopole on the CY cone $(1,\infty) \times X$. Associated to any monopole is a quantity known as the mass

$$m = \lim_{t\to\infty} |\Phi|.$$

In what follows, we shall see that all G_2-instantons are asymptotic to model connections A_∞. These instantons will be determined by two functions f^+ and g^+ with the former part being asymptotic to $\ell\Phi \otimes \theta$ and the latter being asymptotic to a. For the model connection A_∞, a is determined by a single value $\tilde{a} \in \mathbb{R}$ and that Φ has values at each point in the 1-dimensional subspace of $\mathfrak{su}(2)$ given by the diagonal matrices. The eigenvalues of Φ in the limit as $t \to \infty$ are equal up to sign, and we let $\tilde{m}$ be the first eigenvalue on the diagonal, a signed value corresponding to m.

On the other hand, abelian instantons are reducible instantons, i.e. are given by the sum of a line bundle and its inverse. They are asymptotic to A_∞ when $a = 0$ and so are determined by Φ. The diagonal nature of the values of Φ corresponds to the decomposition of P into the sum of line bundles. We see in the following result that the behaviour of solutions depends both on $\tilde{m}$ and m.

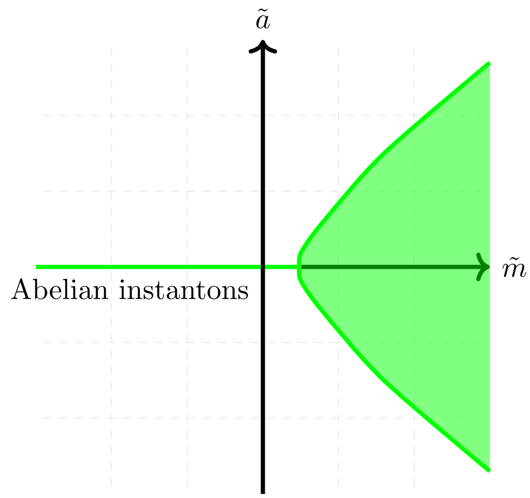

Figure 1.4: A pictorial representation of the initial conditions of all complete bounded solutions on P_1 over the BGGG manifold.

Theorem B. *The region of initial conditions which lead to complete bounded solutions of the G_2-instanton equations on the BGGG manifold is closed. It is formed of the union of a line of abelian instantons and a set with boundary given by the initial conditions of G_2-instantons asymptotic to A_∞ with $\tilde{m} > 0$ and corresponding CY monopoles with mass $m = \frac{1}{2}$, as shown in Figure 1.4. The instantons given by initial conditions on the boundary satisfy $f^+ \to 1$ while f^+ has a limit greater than 1 for the remaining non-abelian solutions; in all cases, g^+ decays in the limit.*

The first set of initial conditions in Theorem B corresponds to abelian instantons which are rigid for each fixed mass m. On the other hand, for each $m \geq \frac{1}{2}$ and $\tilde{m} > 0$, we are able to perturb the abelian solutions in a 1-parameter family while keeping m fixed, as shown in Figure 1.4. Recall that these instantons are determined by two functions f^+ and g^+ with the former part being asymptotic to $\ell\Phi \otimes \theta$ and the latter being asymptotic to a. We will see that when $m = \frac{1}{2}$, the perturbations have g^+ decaying asympotically at a polynomial rate, while when $m > \frac{1}{2}$, it decays exponentially fast.

We prove Theorem B by first continuously parameterising all local solutions near the ALC end; this is the result of Proposition 4.40. These solutions satisfy a singular initial value problem of the form

$$\dot{x} = t^{-2}\Lambda x + t^{-1}Ax + B(x,t).$$

When $m = \frac{1}{2}$, Λ vanishes and the problem is regular; when $m > \frac{1}{2}$, Λ is non-zero and the IVP is irregular. This transition leads to the change in decay rate of g^+. Once we have asymptotic local instantons, we can apply continuous dependence results for these solutions and those near the singular orbit to find a region in the phase space where we can obtain global solutions via a gluing procedure.

AC limit of $\mathbb{C}_7$ family

Recall that the manifold $M_{1,1}$ has a 1-parameter family of G_2-metrics, namely the $\mathbb{C}_7$ family. We consider the AC limit of this family in Chapter 5.

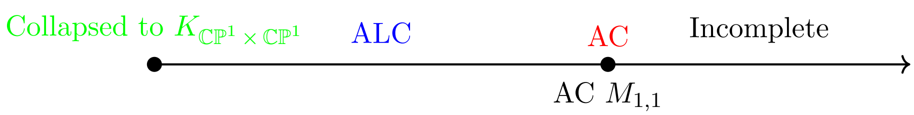

Figure 1.5: BGGG manifold in the $\mathbb{C}_7$ family

We prove the existence of a 1-parameter family of G_2-instantons with gauge group $\mathrm{SU}(2)$ on the AC manifold $M_{1,1}$. $M_{1,1}$ is asymptotic to the G_2-cone whose link is a finite quotient of the nearly Kähler $S^3 \times S^3$. We first show that there is a 1-parameter family of invariant abelian solutions on $M_{1,1}$ and that there is a single value for this parameter for which the solution has bounded curvature; we call this abelian G_2-instanton A^{ab}.

Theorem C. *Consider the AC G_2-manifold $M_{1,1}$ and let A^{ab} be the invariant abelian G_2-instanton with bounded curvature. There is a 1-parameter family of* $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$*-invariant G_2-instantons with full gauge group* $\mathrm{SU}(2)$ *and bounded curvature, which is a small perturbation of A^{ab} near the singular orbit $S^2 \times S^3$.*

While these solutions are not explicit, the construction gives some details about their qualitative behaviour; indeed, the solutions have bounded curvature and are asymptotic to dilation-invariant solutions on the G_2-cone. We prove Theorem C via a dynamical systems approach: we find fixed points and solutions of a reduced system in Section 5.4.2, perturb those solutions to solve the full system and then consider the stable and unstable manifolds of the fixed points to show that we can flow from one perturbed solution to another (see Section 5.4.3).

We can consider the nearly Kähler $S^3 \times S^3$ as the homogeneous manifold $\mathrm{SU}(2)^3/\Delta\,\mathrm{SU}(2)$. There is a unique connection on the tangent bundle whose holonomy is contained in $\mathrm{SU}(3)$ and has skew-symmetric torsion. On any $\mathrm{SU}(2)$-bundle on $S^3 \times S^3$ there is a corresponding canonical connection, which gives a dilation-invariant connection on the cone.

Analogously to the solutions of Theorem A, we can view the solutions of Theorem C as the result of gluing two local solutions. It is clearer in this case how this can be achieved; take the perturbation of A^{ab} as the local solution near the singular orbit, and the pull-back to $M_{1,1}$ of the dilation-invariant solution on the cone given by the canonical connection on the link.

1.3 Plan of the Thesis

We set out some preliminary material in Chapter 2. We define the notions of asympotically conical and asymptotically locally conical manifolds, as well as the concept of a cohomogeneity one action on a manifold. We then describe a type of principal bundle which is compatible with a manifold that admits a cohomogeneity one group action. We see that Theorem 2.5 provides a way of parameterising all possible invariant connections on these bundles. Finally, we derive the G_2-instanton equations as evolution equations, the equations which we later solve to find instantons on the AC limit of the $\mathbb{B}_7$ and $\mathbb{C}_7$ families.

In Chapter 3, we outline topologically the manifolds which are of interest in this thesis, as well as the AC and ALC metrics they admit, constructed in [2, 4, 5, 16]; we describe them all using the notation of [16] for consistency. We then state a classification theorem, before describing how the work of this thesis adds to the gauge theoretic knowledge on manifolds within this classification.

The focus of Chapter 4 is the $\mathbb{B}_7$ family, in which we add to the work of Clarke, Lotay and Oliveira in [7, 24]. We start by proving Theorem A and hence complete the picture of $\mathrm{SU}(2)^3$-invariant G_2-instantons on the Bryant-Salamon AC manifold. In the second half of this chapter, we prove Theorem B, an existence result about instantons on the BGGG manifold in the $\mathbb{B}_7$ family, by constructing local asymptotic solutions and proving results about the properties of the solutions.

In Chapter 5, we turn our attention to the AC members of the $\mathbb{C}_7$ family and its infinite extension. We start by deriving the G_2-instanton equations for a generic member $M_{m,n}$ of this set of AC manifolds. We then find local solutions to these equations near the singular orbit for each coprime positive m, n, before finding flat or abelian connections that extend these local solutions over all of $M_{m,n}$. In Section 5.4, we prove the existence of a 1-parameter family of G_2-instantons with gauge group $\mathrm{SU}(2)$ and bounded curvature on each bundle on $M_{1,1}$ that admits non-abelian instantons. We also see why generalising this result for any coprime positive m, n is more difficult.

Appendix A contains the analysis required to understand the conditions under which certain tensors extend smoothly over the singular orbit. This material is used to prove a number of local existence results throughout Chapters 4 and 5. In Appendix B, we outline some dynamical systems background material, which is required to understand the argument in Section 5.4.2 regarding the perturbation of an autonomous system by a non-autonomous term. This is used to prove the main result of Chapter 5, Theorem 5.28, a more precise formulation of Theorem C.

In summary, the main results of this thesis outlined in Section 1.2 are spread across Chapters 4 and 5, depending on whether they concern members of the $\mathbb{B}_7$ or $\mathbb{C}_7$ families respectively.

CHAPTER 2

PRELIMINARIES

In this chapter, we set out some background material relating to structure on Riemannian manifolds, G_2 geometry and some gauge theoretic concepts on manifolds with a G_2-structure. Firstly, we define Kähler, nearly Kähler and Calabi-Yau manifolds. Then we set out the definitions of G_2-manifolds and, more specifically, AC and ALC G_2-manifolds. We then consider manifolds with an action of a Lie group whose orbits are homogeneous manifolds, and outline the construction of a homogeneous bundle on them. Finally, we define G_2-instantons and see how the equations governing them simplify when we consider G_2-manifolds with high levels of symmetry.

2.1 Structures on Riemannian manifolds

2.1.1 Kähler and Nearly Kähler manifolds

Let M be a complex Riemannian manifold with complex structure J and Hermitian metric g. Let ω be the Hermitian form of g, i.e. $\omega(v,w) = g(Jv,w)$ for all vector fields v, w on M. We say that M is Kähler if $d\omega = 0$ and we call ω the Kähler form.

An SU(3)-structure (ω, Ω) on a 6-manifold Σ is a non-degenerate 2-form ω and a complex volume form Ω satisfying

$$\omega \wedge \operatorname{Re}\Omega = 0, \qquad \frac{1}{6}\omega^3 = \frac{1}{4}\operatorname{Re}\Omega \wedge \operatorname{Im}\Omega\,. \tag{2.1}$$

Such a structure is called half-flat if it also satisfies

$$d\omega \wedge \omega = d\operatorname{Re}\Omega = 0.$$

An SU(3)-structure (ω, Ω) on a 6-manifold Σ is called (strictly) nearly Kähler if

$$d\omega = 3\operatorname{Re}\Omega \quad \text{and} \quad d\operatorname{Im}\Omega = -2\omega^2.$$

We only know of six simply-connected nearly Kähler metrics. Four are homogeneous metrics, constructed by Wolf and Gray [35], on the manifolds S^6, $\mathbb{CP}^3$, the flag manifold $\mathbb{F}_3$ and $S^3 \times S^3$. There are a further two non-homogeneous metrics on S^6 and $S^3 \times S^3$ discovered by Foscolo and Haskins [15]. In this thesis, we will be interested in the homogeneous nearly Kähler manifold $S^3 \times S^3$ and the nearly Kähler structures on finite quotients of $S^3 \times S^3$, constructed by Cortés and Vásquez in [9].

2.1.2 Sasaki-Einstein manifolds

Recall that ALC G_2-manifolds are circle bundles over a cone on a manifold of dimension 5. This manifold will be equipped with a Sasaki-Einstein metric. We now outline the conditions for a metric to be Sasaki-Einstein.

Let α be a 1-form on X. X is known as a contact manifold if the 2-form $\omega = d(t^2\alpha)$ on $C(X)$ is symplectic (i.e. non-degenerate and closed). Note that this means that X must have odd dimension.

Suppose X is a contact manifold. X is Sasakian if $(C(X), g_C)$ is a Kähler manifold with Kähler form ω. Furthermore, if $C(X)$ is Ricci-flat, then X is called Sasaki-Einstein.

We now restrict to the case where X has dimension 5; the following content can be found in [8]. An SU(2)-structure on X can be written as a quadruplet $(\alpha, \omega_1, \omega_2, \omega_3)$ where α is a nowhere vanishing 1-form and the ω_i are 2-forms satisfying

$$\omega_i \wedge \omega_j = \delta_{ij} v, \qquad v \wedge \alpha \neq 0$$

for some nowhere vanishing 4-form v, and

$$\iota_X \omega_1 = \iota_Y \omega_2 \implies \omega_3(X, Y) \geq 0$$

for any vector fields X, Y on X. Such an SU(2)-structure is called hypo if ω_1, $\alpha \wedge \omega_2$ and $\alpha \wedge \omega_3$ are all closed.

Furthermore, an SU(2)-structure (α, ω_i) on X is called hypo-contact if it is hypo and $d\alpha = -2\omega_1$.

An SU(2)-structure (α, ω_i) on X induces an SU(3)-structure (ω, Ω) on $C(X)$ given by

$$\omega = t^2\omega_1 + t\alpha \wedge dt, \qquad \Omega = t^2(\omega_2 + i\omega_3) \wedge (t\alpha + idt).$$

Note that $d\omega = 0$ if and only if $d\alpha = -2\omega_1$. Hence a hypo-contact SU(2)-structure on X induces an SU(3)-structure on $C(X)$ and this cone is Kähler. Hence $(C(X), \alpha)$ is a Sasakian manifold.

2.1.3 Calabi-Yau manifolds and monopoles

We define the terms Calabi-Yau manifold and Calabi-Yau monopole in this section; the definitions can be found in [28] and [30]. A complex manifold Σ is Calabi-Yau if it is a Ricci-flat Kähler manifold with trivial canonical bundle. Write ω for the Kähler form and Ω for the complex volume form on Σ. Note that Calabi-Yau structures are examples of SU(3)-structures.

We restrict to the case where Σ has dimension 6. Let G be a compact, semi-simple Lie group and $P \to \Sigma$ a principal G bundle on Σ. Let A be a connection on Σ and $\Phi \in \Omega^0(\operatorname{Ad} P)$ a section of the adjoint bundle. We say (A, Φ) is a Calabi-Yau monopole if it satisfies

$$F_A \wedge \omega^2 = 0 \quad \text{and} \quad F_A \wedge \operatorname{Re}\Omega = *d_A\Phi$$

where $*$ is the Hodge star of the metric corresponding to (ω, Ω) and d_A is the induced covariant derivative. Then we call Φ the Higgs field of the monopole.

2.2 G_2 Geometry

In this section, we define what it means for a 7-dimensional manifold to have a G_2-structure and then restrict our attention to such manifolds that are non-compact and have conical asymptotic geometry.

2.2.1 G_2-structures

We outline the concept of a G_2-structure on a 7-manifold, following the description of Joyce in [19]. Let $x_1, \ldots, x_7$ be coordinates on $\mathbb{R}^7$ and let

$$\varphi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}$$

on $\mathbb{R}^7$, where $dx_{ijk} = dx_i \wedge dx_j \wedge dx_k$. The stabiliser of φ_0 in $\mathrm{GL}(7, \mathbb{R})$ is the compact, semisimple, simply-connected 14-dimensional Lie group G_2. It also preserves the Euclidean metric $g_0 = dx_1^2 + \cdots + dx_7^2$, the orientation on $\mathbb{R}^7$ and the 4-form $*\varphi_0$, where $*$ is the Hodge star operator determined by g_0.

Let M be an oriented 7-manifold. Let $P_p^3 M \subset \Lambda^3 T_p^* M$ be defined by the property that $\varphi \in P_p^3 M$ if there is an orientation-preserving map $T_pM \to \mathbb{R}^7$ under which φ is mapped to φ_0. Define a 3-form φ on M to be positive if $\varphi|_p \in P_p^3 M$ for every $p \in M$.

To any positive 3-form φ, there is an associated metric g which is mapped to g_0 at each point $p \in M$ under the same isomorphism that maps φ to φ_0. φ determines a principal subbundle of the frame bundle over M which has fibre G_2; hence it yields a G_2-structure on M. We simultaneously refer to φ as the G_2-structure, since it defines a unique G_2-structure on M.

Let (M, φ) be a 7-manifold with a G_2-structure, and let ∇ be the Levi-Civita connection of the associated metric g. Define $\nabla\varphi$ to be the torsion of φ; if it vanishes, we call φ torsion-free. This condition is equivalent to the holonomy group of g being contained in G_2. When φ is torsion-free, we call (M, φ) a G_2-manifold.

As with any G-structure on a manifold for G a subgroup of $\mathrm{GL}(n, \mathbb{R})$, a G_2-structure on M induces a splitting of the bundles of exterior forms into irreducible components. In particular, we have

$$\Lambda^2 T^* M = \Lambda_7^2 \oplus \Lambda_{14}^2,$$

a splitting of the 2-forms on M into irreducible representations of dimension 7 and 14. We shall later see that this splitting gives another definition of a G_2-instanton.

2.2.2 AC and ALC G_2-manifolds

A G_2-manifold (M, φ) is asymptotically conical (AC) if M is complete with one non-compact end where the G_2-structure φ is asymptotic to a conical G_2-structure φ_C on $\mathbb{R}^+_t \times N^6$, for some nearly Kähler manifold (N, ω, Ω). More precisely, there is a compact subset $Y \subset M$, an $R > 1$ and a diffeomorphism $f : (R, \infty) \times N \to M \backslash Y$ such that $f^*(\varphi)$ approaches φ_C with rate $\nu < 0$:

$$|\nabla^j (f^*(\varphi) - \varphi_C)| = O(t^{\nu - j}), \qquad \forall\, j \geq 0$$

on $(R, \infty) \times N$. M has only one end due to the following Cheeger-Gromoll Splitting Theorem.

Theorem 2.2 (Cheeger-Gromoll Splitting Theorem)**.** *If a complete Riemannian manifold M with Ricci curvature $Ric(M) \geq 0$ has a straight line, i.e., a geodesic γ such that $d(\gamma(u), \gamma(v)) = |u - v|$ for all $u, v \in \mathbb{R}$, then it is isometric to a product space $\mathbb{R} \times L$ where L is a Riemannian manifold with $Ric(L) \geq 0$.*

We now turn our attention to Ricci-flat G_2-manifolds with submaximal volume growth of order t^6 and consider metrics described in the literature as asympotically locally conical (ALC). The following material is a summary of content from the introduction of [16] and Section 2.3 of [24]. Such manifolds, under certain decay conditions, are asymptotic to a model metric on a circle bundle over a 6-dimensional cone. We start by constructing this model metric. Let $\pi : \Sigma^6 \to X^5$ be a circle bundle, where (X, h) is a Sasaki-Einstein Riemannian manifold with structure (α, ω_i), θ is a connection on Σ and ℓ a positive real number. Then applying the definition of Section 2.1.2, we see that there is an induced SU(3)-structure (ω_X, Ω_X) on $C(X)$. Then we can construct a U(1)-invariant G_2-structure φ_∞ on the cone $(1, \infty)_t \times \Sigma$ with associated metric

$$g_{\varphi_\infty} = dt^2 + \ell^2 \theta^2 + t^2 \pi^* h.$$

This metric is a Riemannian submersion over $C(X)$, whose circle fibres have constant length $2\pi\ell$. If we take $\ell = t$, the asymptotic geometry transitions and we recover the AC metric described above.

An ALC G_2-manifold (M, φ) is asympototic to such a model metric outside a compact set. More precisely, let $K \subset M$ be compact and let $p : (1, \infty) \times \Sigma \to M \backslash K$ be a diffeomorphism. Then we require that φ satisfies

$$|\nabla^j (\varphi_\infty - p^* \varphi|_{M \backslash K})|_{g_{\varphi_\infty}} = O(t^{\nu - j}) \quad \text{ as } t \to \infty$$

for some $\nu < 0$, $j = 0, 1$ and ∇ the Levi-Civita connection on the cone $(1, \infty) \times \Sigma$. The rate ν determines the structure on $(1, \infty) \times \Sigma$ induced from the torsion-free structure φ on M. Then we can write the model G_2-structure φ_∞ on $C(\Sigma)$ as

$$\varphi_\infty = \ell \theta \wedge \omega_X + \operatorname{Re} \Omega_X + O(t^\nu).$$

Proposition 2.3 (Proposition 2(a), [24])**.** *Let (M, φ) be an ALC G_2-manifold as above with $\nu < 0$. Then the metric $g_C = dt^2 + t^2 h$ on $C(X)$ is Calabi-Yau.*

In this thesis, we will consider one ALC manifold with $\ell = 1$ and rate $\nu = -1$, the BGGG metric of the $\mathbb{B}_7$ family, and a collection of AC manifolds, namely the AC limits of the $\mathbb{B}_7$ family and the infinite extension of the $\mathbb{C}_7$ family.

2.3 Gauge Theory

All the G_2-manifolds of interest in this thesis have a cohomogeneity one group action and we will construct principal bundles on them which are homogeneous when we restrict to each orbit of the action. We will also look to construct G_2-instantons on these manifolds, where we exploit the high level of symmetry to simplify the PDEs governing these special connections to ODEs. We now set out these concepts in more detail.

2.3.1 Homogeneous bundles on cohomogeneity one manifolds

We start by defining cohomogeneity one manifolds; this material can be found, for example, in Chapter 2 of [18] and we will adapt the definitions of Grove and Ziller to our setting throughout this section. Let M be any Riemannian manifold and G be a Lie group. M is a cohomogeneity one manifold if G acts isometrically on M with generic orbits of codimension one. In the following, we consider such manifolds M which are complete irreducible and Ricci flat, with G compact; then M is non-compact with one end and so M/G is a half line $[0, \infty)$. We say that the orbits over points in $(0, \infty)$ are principal orbits, while the orbit corresponding to 0 is called the singular orbit.

We can encode the cohomogeneity one structure of M in its group diagram: we write $K_0 \subset K \subset G$, where K_0 and K are the stabilisers of the G-action on M restricted to the principal orbits and singular orbit respectively. There is a K-representation V of dimension $\dim(K) - \dim(K_0) + 1$ such that K acts transitively on the unit sphere in V with stabiliser K_0; then $M = G \times_K V$.

A principal H-bundle P on a homogeneous manifold G/K is called G-homogeneous if the action of G on G/K lifts to a G-action on P which commutes with the action of H. Such bundles are determined by their isotropy homomorphism, which we now construct via the method of Kobayashi and Nomizu in [21, p105]. Let u_0 be an arbitrary point of P over the identity $x_0 \in G/K$. The isotropy subgroups of the translation action of G on G/K are conjugates on K. Let $k \in K$; then ku_0 is a point in P, which lies in the same fibre as u_0. Thus, we can write $ku_0 = u_0 h$ for some $h \in H$. We define the isotropy homomorphism $\lambda : K \to H$ by $\lambda(k) = h$; it is proved in Section II.11 of [21] that this is indeed a homomorphism.

Conversely, given a homomorphism $\lambda : K \to H$, the associated H-bundle

$$P_\lambda = G \times_{(K,\lambda)} H \tag{2.4}$$

is a G-homogeneous H-bundle on G/K whose isotropy representation is λ. Here K acts on H via $k \cdot h = \lambda(k)^{-1} h$ for $k \in K, h \in H$. The homogeneous manifolds considered in this thesis are reductive, which means that the Lie algebra $\mathfrak{g}$ of G has an Ad_K-invariant splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. This splitting induces a connection, known as the canonical invariant connection, on the bundle $G \to G/K$ whose horizontal space at the identity in G is $\mathfrak{m}$. It induces a corresponding canonical invariant connection on any P_λ, as shown by The in [32], which is determined by the left-invariant translation of $d\lambda \oplus 0 : \mathfrak{k} \oplus \mathfrak{m} \to \mathfrak{h}$. We now refer to the following theorem by Wang [34], for a method of parameterising invariant connections on P_λ.

Theorem 2.5 (Theorem 1, [34]). *There is a 1-1 correspondence between G-invariant connections* *on P_λ and morphisms of K-representations*

$$\Lambda : (\mathfrak{m}, \mathrm{Ad}) \to (\mathfrak{h}, \mathrm{Ad} \circ \lambda). \tag{2.6}$$

Remark 2.7. A G-invariant connection on P_λ can be written as a 1-form $\omega \in \Omega^1(P_\lambda, \mathfrak{h})$ ([32, Section 2.3]), namely

$$\omega_{[g,h]} = \mathrm{Ad}_{h^{-1}} \circ (d\lambda + \Lambda) \circ (\Theta_G)_g + (\Theta'_H)_h$$

where Θ_G is the left-invariant Maurer-Cartan form on G and Θ'_H is the right-invariant Maurer-Cartan form on H. Then the difference between any invariant connection and the canonical invariant connection is determined by the morphism Λ, and the horizontal space of such a connection is determined by $\ker \Lambda$. $\diamondsuit$

Consider a cohomogeneity one manifold $M = G \times_K V$, with principal orbits G/K_0, singular orbit G/K and K-representation V. Let $\lambda_s : K \to H$ be a group homomorphism and let $\lambda_p : K_0 \to H$ be the restriction of λ to K_0. We now construct a principal H-bundle over M whose restriction to each principal orbit is $P_p := P_{\lambda_p}$, the associated bundle to λ_p given by (2.4), and whose restriction to the singular orbit is $P_s := P_{\lambda_s}$. We adapt the construction of Grove and Ziller in [18] of such a principal bundle to the case where the base manifold is not compact. Let $j : K \hookrightarrow G$ be the inclusion map in the group diagram of M. We define a cohomogeneity one manifold P, with group action $H \times G$, as follows. Consider the group diagram

$$K_0 \hookrightarrow K \xrightarrow{(\lambda_s, j)} H \times G$$

and define

$$P = (H \times G) \times_K V.$$

Here K acts on $H \times G$ by $k \cdot (h, g) = (\lambda_s(k)^{-1}h, kg)$. Furthermore, $H \times G$ acts on itself by right multiplication. Then, to give P the structure of a principal H-bundle over M, we need to consider the subaction of $H = H \times \{1\}$ on P. The stabilizers of the $H \times G$ action on elements in the principal orbits (resp. singular orbit) are conjugates of K_0 (resp. K). For all $(h, g) \in H \times G$, we have that

$$H \cap (h, g)K_0(h, g)^{-1} = (h, g)((H \times \{1\}) \cap K_0)(h, g)^{-1}$$

is trivial, and similarly for K; thus the subaction of H on P is free. By comparing the group diagrams, we see that $P/H = M$; the quotient map provides a smooth projection map $\pi : P \to M$ whose fibres are the orbits of the H-action. It follows that P is a principal H-bundle over M, whose restriction to each principal orbit is P_p and whose restriction to the singular orbit is P_s.

2.3.2 G_2-instanton evolution equations

Let (M, φ) be a G_2-manifold and let P be a principal bundle on M. Recall that a connection A on P is a G_2-instanton if

$$F_A \wedge \varphi = -{*F_A} \tag{2.8}$$

where F_A is the curvature of the connection A and $*$ denotes the Hodge star operator with respect to the metric on M induced by φ. Here we write $A \in \Omega^1(M; \mathrm{Ad}\, P)$ by considering the difference with the canonical invariant connection described in Section 2.3.1. There are a number of equivalent ways of defining G_2-instantons; indeed, we can impose the condition

$$F_A \wedge *\varphi = 0$$

which is equivalent to (2.8). Further, we can consider the splitting induced by φ of the exterior forms on M described in Section 2.2.1. Let π_7 denote the projection of a 2-form to Λ^2_7, then a connection A on $P \to M$ is a G_2-instanton if $\pi_7(F_A) = 0$.

We now consider the manifold $M = (0,\infty) \times N$, where N is a 6-manifold admitting a 1-parameter family of half-flat SU(3)-structures (ω, Ω). Recall that an SU(3)-structure (ω, Ω) on N is a pair (ω, Ω) satisfying

$$\omega \wedge \operatorname{Re}\Omega = 0, \qquad \frac{1}{6}\omega^3 = \frac{1}{4}\operatorname{Re}\Omega \wedge \operatorname{Im}\Omega \tag{2.9}$$

and is half-flat if

$$d\omega \wedge \omega = d\operatorname{Re}\Omega = 0.$$

Then M has a G_2-structure $\varphi = dt \wedge \omega + \operatorname{Re}\Omega$ which is torsion-free if it satisfies the Hitchin flow, namely

$$\partial_t \operatorname{Re}\Omega = d\omega, \qquad \partial_t(\omega^2) = -2d\operatorname{Im}\Omega\,.$$

The induced metric has the form $g = dt^2 + g_t$ where g_t is a 1-parameter family of metrics on N.

Lemma 2.10. *The G_2-structure $\varphi = dt \wedge \omega + \operatorname{Re}\Omega$ satisfies the Hitchin flow if and only if (ω, Ω) is nearly Kähler.*

Proof. On the cone, we can write $\Omega = t^3\Omega_N$ and $\omega = t^2\omega_N$, where (ω_N, Ω_N) is an SU(3)-structure on $\{1\} \times N \subset M$. Then

$$3t^2 \operatorname{Re}\Omega_N = \partial_t \operatorname{Re}\Omega = d\omega = t^2 d\omega_N + 2t dt \wedge \omega_N.$$

Fixing a choice of t, we have that $3\operatorname{Re}\Omega_N = d\omega_N$. Similarly, we have that $d\operatorname{Im}\Omega_N = -2\omega_N^2$ and hence (ω_N, Ω_N) is nearly Kähler. □

In summary, if M is a G_2-manifold with conical asymptotic geometry, then in the limit as $t \to \infty$, the SU(3)-structure defining the G_2-structure is nearly Kähler if φ satisfies the Hitchin flow.

Let P be a principal H-bundle on M, then P is a pull-back of a bundle on N. We work in temporal gauge, so we assume that a connection on P is of the form $A = \alpha(t)$, where $\alpha(t)$ is a 1-parameter family of connections on P. Then the curvature of A is given by $F_A = dt \wedge \dot{\alpha} + F_\alpha(t)$, where $F_\alpha(t)$ is the curvature of the connection $\alpha(t)$ on P over N. In this setting, the G_2-instanton equation for the connection A becomes an evolution equation for $\alpha(t)$, given by

$$\dot{\alpha} \wedge \frac{1}{2}\omega^2 - F_\alpha \wedge \operatorname{Im}\Omega = 0, \qquad F_\alpha \wedge \frac{1}{2}\omega^2 = 0. \tag{2.11}$$

The following lemma provides a useful reformulation of the above evolution equations.

Lemma 2.12 (Lemma 1, [23]). *Let $M = (0,\infty) \times N$ be equipped with a torsion-free G_2-structure $\varphi = dt \wedge \omega + \operatorname{Re}\Omega$ as above. Then G_2-instantons are in 1-1 correspondence with 1-parameter families of connections $\{\alpha(t)\}_{t \in I_t}$ solving the evolution equation*

$$J_t \dot{\alpha} = - *_t (F_\alpha \wedge \operatorname{Im}\Omega) \tag{2.13}$$

subject to the constraint

$$\Lambda_t F_\alpha = 0, \tag{2.14}$$

where Λ_t *denotes the metric dual of the operation of wedging with* $\omega(t)$*. Here,* $*_t$ *is the Hodge star operator associated to the* SU(3)*-structure* $(\omega(t), \Omega(t))$*. Moreover,* (2.14) *is compatible with the evolution: if it holds for some* t*, then it holds for all* $t \in (0, \infty)$*.*

When deriving the G_2-instanton equations explicitly as an ODE system, it will be more convenient to use the formulation (2.13). Such a derivation for the G_2-manifolds $M_{m,n}$ is given in Section 5.1. Throughout this thesis, the connections we consider will be invariant under the cohomogeneity one group action on the manifold. This means that the restriction of a connection 1-form in $\Omega^1(M; \mathrm{Ad}\, P)$ to each orbit can be written as a map from the tangent space at a single point to the fibre over that point of the adjoint bundle, which is a copy of the Lie algebra $\mathfrak{h}$ of H. This is the map Λ given in (2.6). The equivariance condition on Λ with respect to each isotropy subgroup is exactly the condition that the corresponding section has values in the adjoint bundle. In summary, a connection A on P will be written throughout this thesis as a section of

$$T^*M|_C \otimes \mathfrak{h}$$

where C is a curve meeting all principal orbits orthogonally, and hence can be parameterised by $t \in [0, \infty)$.

CHAPTER 3

FAMILIES OF G_2-MANIFOLDS

In this chapter, we outline the topology and geometry of the manifolds of interest. We describe the construction of each metric via evolution equations known as the Hitchin Flow. We also state a classification theorem by Foscolo, Haskins and Nordström [16] of all complete $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant G_2-metrics before discussing the progress made to construct G_2-instantons on each member of the classification, including the results of this thesis.

3.1 Evolution Equations

The manifolds described in this chapter all have a cohomogeneity one action of $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$ and the metrics are invariant under this action. This high level of symmetry allows us to reduce the PDEs defining a G_2-structure to ODEs. The resulting equations of this general procedure are known as evolution equations, and in this specific case are known as the 'Hitchin Flow'.

We now outline how this works for a general $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant G_2-metric. Let (ω, Ω) be a 1-parameter family of $\mathrm{SU}(3)$-structures on the 6-manifold N^6, and consider the G_2-structure on the manifold $M = (0, \infty)_t \times N$ given by

$$\varphi = dt \wedge \omega + \mathrm{Re}\,\Omega, \qquad *\varphi = \frac{1}{2}\omega^2 - dt \wedge \mathrm{Im}\,\Omega,$$

so that the induced metric has the form $dt^2 + g_t$. Now φ is torsion-free if and only if φ is closed and coclosed. These conditions are equivalent to the half-flat equations for (ω, Ω) together with the Hitchin flow evolution equations

$$\partial_t \mathrm{Re}\,\Omega = d\,\omega, \qquad \omega \wedge \partial_t\,\omega = -d\,\mathrm{Im}\,\Omega\,. \tag{3.1}$$

We now specialise to the case where $N = S^3 \times S^3 \cong \mathrm{SU}(2)^2$ or a finite quotient thereof. Recall from Lemma 2.10 that the G_2-structure φ on M satisfying (3.1) implies that the $\mathrm{SU}(3)$-structures

$(\omega, \Omega)_t$ on $\{t\} \times N$ are nearly Kähler. Fix a basis of left-invariant 1-forms $e_1, e_2, e_3, e_1', e_2', e_3'$ on $\mathrm{SU}(2) \times \mathrm{SU}(2)$ with the property that

$$de_i = -e_j \wedge e_k, \qquad de_i' = -e_j' \wedge e_k'$$

for (i, j, k) any cyclic permutation of $(1, 2, 3)$. Denote a basis of dual vector fields by E_i, E_i' that satisfy $[E_i, E_j] = E_k$. Consider the maximal torus $T^2 \subset \mathrm{SU}(2) \times \mathrm{SU}(2)$; we may assume T^2 is generated by E_3 and E_3'.

The group of Lie algebra inner automorphisms of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ is $\mathrm{Aut} = \mathrm{SO}(3) \times \mathrm{SO}(3)$; passing to the quotient, we require the subgroup of inner automorphisms which commute with the isotropy representation of K_0. For the $\mathbb{B}_7$ family, K_0 is trivial and this subgroup is the whole of Aut; for $\mathbb{C}_7$ and its infinite extension, K_0 is a finite group and the subgroup of inner automorphisms is $\mathrm{Aut}_T = \mathrm{O}(2) \times \mathrm{O}(2)$, where $\mathrm{O}(2)$ is the subgroup of $\mathrm{SO}(3)$ which fixes E_3. We can now state a proposition of [16], which describes the half-flat structures on the principal orbits of $\mathrm{SU}(2)^2/K_0$ in both cases.

Proposition 3.2 (Prop 3.8, [16])**.** *Fix $p, q \in \mathbb{R}$. Up to the action of* Aut *and* Aut_T*, the forms* $(\omega, \mathrm{Re}\,\Omega)$ *on* $\mathrm{SU}(2)^2/K_0$ *can be written in the form*

$$\omega = \alpha_1 e_1 \wedge e_1' + \alpha_2 e_2 \wedge e_2' + \alpha_3 e_3 \wedge e_3'$$

$$\mathrm{Re}\,\Omega = p e_1 \wedge e_2 \wedge e_3 + q e_1' \wedge e_2' \wedge e_3' + d\,(a_1 e_1 \wedge e_1' + a_2 e_2 \wedge e_2' + a_3 e_3 \wedge e_3').$$

The 2-form ω is non-degenerate if and only if $\alpha_i > 0$ for all $i = 1, 2, 3$. Also, $\mathrm{Re}\,\Omega$ *is actually the real part of a holomorphic volume form if and only if*

$$\Lambda(a_1, a_2, a_3) := a_1^4 + a_2^4 + a_3^4 - 2a_1^2 a_2^2 - 2a_2^2 a_3^2 - 2a_3^2 a_1^2 + 4(p-q) a_1 a_2 a_3 + 2pq(a_1^2 + a_2^2 + a_3^2) + p^2 q^2 < 0.$$

Finally, the choice of normalisation in (2.1) *implies that* $2\alpha_1\alpha_2\alpha_3 = \sqrt{-\Lambda(a_1, a_2, a_3)}$.

Fix $p, q \in \mathbb{R}$. Then a cohomogeneity one G_2-structure on M is a closed $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant 3-form

$$\varphi = p e_1 \wedge e_2 \wedge e_3 + q e_1' \wedge e_2' \wedge e_3' + d\,(a(e_1 \wedge e_1' + e_2 \wedge e_2') + b e_3 \wedge e_3'), \tag{3.3}$$

where d is the differential in 7 dimensions. Set $F(a, b) := -\Lambda(a, a, b)$, then the Hitchin flow equations can be written as

$$\dot{x}_1 = \frac{\partial_a F(y_1, y_2)}{4\sqrt{F(y_1, y_2)}}, \quad \dot{x}_2 = \frac{\partial_b F(y_1, y_2)}{2\sqrt{F(y_1, y_2)}}, \quad \dot{y}_1 = \frac{x_1 x_2}{\sqrt{x_1^2 x_2}}, \quad \dot{y}_2 = \frac{x_1^2}{\sqrt{x_1^2 x_2}}$$

for the functions $x_1 = \dot{a}\dot{b}$, $x_2 = \dot{a}^2$, $y_1 = a$ and $y_2 = b$.

Denote by g_t the metric on $\{t\} \times N$ compatible with the SU(3)-structure $(\omega(t), \Omega(t))$; then the G_2-structure φ on M that solves the Hitchin flow induces the metric $g = dt^2 + g_t$. Madsen and Salamon [25, Chapter 5] give the following explicit formula for g_t:

$$g_t = \frac{a(b-p)}{\dot{a}\dot{b}}(e_1 \otimes e_1 + e_2 \otimes e_2) + \frac{a^2 - bp}{\dot{a}^2} e_3 \otimes e_3 + \frac{a(b+q)}{\dot{a}\dot{b}}(e_1' \otimes e_1' + e_2' \otimes e_2') + \frac{a^2 + bq}{\dot{a}^2} e_3' \otimes e_3'$$
$$- \frac{b^2 + pq}{2\dot{a}\dot{b}}(e_1 \otimes e_1' + e_1' \otimes e_1 + e_2 \otimes e_2' + e_2' \otimes e_2) + \frac{b^2 - 2a^2 - pq}{2\dot{a}^2}(e_3 \otimes e_3' + e_3' \otimes e_3). \tag{3.4}$$

Remark 3.5. Consider the de Rham cohomology class of $\operatorname{Re}\Omega$ and note that $H^3(S^3 \times S^3; \mathbb{R}) \cong \mathbb{R}^2$; indeed, the forms $\operatorname{Re}\Omega$ are parametrised by $(p,q) \in \mathbb{R}^2$. From [25], we see that the $\mathbb{Z}_2$-action which interchanges the two copies of S^3 preserves g_t if and only if the cohomology class $[\operatorname{Re}\Omega]$ satisfies $p+q=0$. In this situation, the metric and evolution equations simplify, and provide an algebraically neater framework for studying G_2-instantons; this is satisfied for the $\mathbb{B}_7$ family and for $M_{1,1}$, but not for $M_{m,n}$ in general. ◊

3.2 The $\mathbb{B}_7$ Family

In this section, we introduce the 1-parameter family $\mathbb{B}_7$ of ALC G_2-manifolds constructed by Bogoyavlenskaya [2] and its AC limit, the Bryant-Salamon manifold M_{BS}. These manifolds are topologically $S^3 \times \mathbb{R}^4$ and have a cohomogeneity one structure with principal orbits $S^3 \times S^3$ and singular orbit S^3.

3.2.1 The manifolds $S^3 \times \mathbb{R}^4$

Topologically, we can view $S^3 \times \mathbb{R}^4$ as a circle bundle over the cone on $S^2 \times S^3$. With this in mind, we have the setup for an ALC manifold, as outlined in Section 2.2.2.

Recall from Section 2.1.2 that a Sasaki-Einstein structure on a 5-manifold X induces an SU(3)-structure on the cone $C(X)$. In order to understand the ALC metrics on $S^3 \times \mathbb{R}^4$, we require a Sasaki-Einstein structure on the 5-manifold $X = S^2 \times S^3$ and the induced SU(3)-structure on $C(X)$. The standard homogeneous Sasaki-Einstein structure on $S^2 \times S^3$ can be described explicitly as follows. Write $S^2 \times S^3 = \mathrm{SU}(2)^2/\Delta\mathrm{U}(1)$, with the coframes e_i, e_i' on $\mathrm{SU}(2)^2$ as defined in Section 3.1. Define the SU(2)-structure (α, ω_i) by

$$\alpha = -\frac{1}{3}(e_1 - e_1'), \qquad \omega_1 = \frac{1}{6}(e_2 \wedge e_3 - e_2' \wedge e_3'),$$

$$\omega_2 = \frac{1}{6}(e_2 \wedge e_3' + e_2' \wedge e_3), \qquad \omega_3 = \frac{1}{6}(e_2' \wedge e_2 + e_3' \wedge e_3).$$

This SU(2)-structure is hypo-contact since $d\alpha = -2\omega_1$, hence (α, ω_i) defines a Sasaki-Einstein structure on $X = S^2 \times S^3$.

Notice that this SU(2)-structure also satisfies

$$d\omega_2 = 3\alpha \wedge \omega_3 \quad \text{and} \quad d\omega_3 = -3\alpha \wedge \omega_2$$

and hence the corresponding SU(3)-structure (ω_X, Ω_X) on $C(X)$, given by

$$\omega_X = t^2\omega_1 + t\alpha \wedge dt, \qquad \Omega_X = t^2(\omega_2 + i\omega_3) \wedge (t\alpha + i dt),$$

is half-flat and $(C(X), g_C)$ is Calabi-Yau.

Let $\theta = \frac{1}{2}(e_1 + e_1')$ be a connection on the circle bundle $S^3 \times S^3 \to S^2 \times S^3$. Then θ is the pull-back of an anti-self-dual (ASD) 2-form on $S^2 \times S^2$, hence is Hermitian Yang-Mills.

$S^3 \times S^3$ can be constructed homogeneously as $\mathrm{SU}(2)^3/\Delta\,\mathrm{SU}(2)$ where $\Delta\,\mathrm{SU}(2) \subset \mathrm{SU}(2)^3$ acts diagonally on the right. There is a unique homogeneous nearly Kähler structure on $S^3 \times S^3$, as shown by Butruille in [6]. The AC examples considered in this thesis will be asymptotic to a G_2-holonomy cone over this nearly Kähler structure, or finite quotients thereof.

$S^3 \times \mathbb{R}^4$ can be constructed homogeneously as

$$\mathrm{SU}(2)^2 \times_{\Delta\,\mathrm{SU}(2)} \mathbb{H}$$

where $\Delta\,\mathrm{SU}(2) \subset \mathrm{SU}(2)^2$ acts diagonally on the right. Here, we are identifying SU(2) with the unit quaternions. There is a natural action of $\mathrm{SU}(2)^3$ on $\mathrm{SU}(2)^2 \times_{\Delta\,\mathrm{SU}(2)} \mathbb{H}$ by left multiplication. This action on $S^3 \times \mathbb{R}^4 \cong \mathrm{SU}(2) \times \mathbb{H}$ is given by

$$(q_1, q_2, q_3) \cdot (p, v) \mapsto (q_1 p \bar{q}_2, q_3 v \bar{q}_1). \tag{3.6}$$

The restriction of this action to $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$ is a cohomogeneity one action on $S^3 \times \mathbb{R}^4$, encoded in the group diagram

$$\Delta\mathrm{U}(1) \subset \Delta\,\mathrm{SU}(2) \times \mathrm{U}(1) \subset \mathrm{SU}(2)^2 \times \mathrm{U}(1).$$

The $\mathbb{B}_7$ family of ALC metrics are invariant under this action, with principal orbits $S^3 \times S^3$ and an associative singular orbit $S^3 \times \{0\}$. The $\mathrm{SU}(2)^3$-action (3.6) also acts with a cohomogeneity one structure on $S^3 \times \mathbb{R}^4$ and the AC limit M_{BS} of the $\mathbb{B}_7$ family is invariant under this action. In the next section, we describe the construction of the entire $\mathbb{B}_7$ family of metrics.

3.2.2 Complete torsion-free G_2-structures on $S^3 \times \mathbb{R}^4$

The AC limit of the $\mathbb{B}_7$ family was constructed by Bryant-Salamon in [5], and the ALC metrics in the family were found by Bogoyavlenskaya in [2]. We set out the construction of the $\mathbb{B}_7$ family given in [16]. We start by describing the local solutions near the singular orbit $S^3 \times \{0\}$.

Proposition 3.7 ([16, Proposition 4.5 (i)])**.** $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$*-invariant G_2-structures on $S^3 \times \mathbb{R}^4$ have the form*

$$\varphi = r_0^3 e_1 \wedge e_2 \wedge e_3 - r_0^3 e_1' \wedge e_2' \wedge e_3' + d(a(e_1 \wedge e_1' + e_2 \wedge e_2') + b e_3 \wedge e_3')$$

for some functions $a, b : [0, \infty) \to \mathbb{R}$. There exists a 2-parameter family of torsion-free G_2-structures φ of this form defined in a neighbourhood of the singular orbit. The family is parameterised by $r_0 > 0$ and $\bar{a}, \bar{b} \in \mathbb{R}$ with $64 r_0 (2\bar{a} + \bar{b}) = 1$ and

$$a = r_0^3 + \tfrac{1}{4} r_0 t^2 + \bar{a} t^4 + O(t^6),$$
$$b = r_0^3 + \tfrac{1}{4} r_0 t^2 + \bar{b} t^4 + O(t^6).$$

Having described local solutions in a neighbourhood of the singular orbit, we follow [16] and consider asymptotic local solutions at infinity for AC ends, in order to prove part (ii) of Theorem 3.9. Their results are given in the following proposition, where we take $p = -q = r_0^3$.

Proposition 3.8 ([16, Proposition 5.3 (ii)])**.** *Let C be the G_2-holonomy cone over the homogeneous nearly Kähler structure on $S^3 \times S^3$ and set $\nu_\infty = \frac{\sqrt{145}+7}{2}$. Then for every $c \in \mathbb{R}$, there exists a unique $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant torsion-free G_2-structure*

$$\varphi = p e_1 \wedge e_2 \wedge e_3 + q e_1' \wedge e_2' \wedge e_3' + d(a(e_1 \wedge e_1' + e_2 \wedge e_2') + b e_3 \wedge e_3')$$

on $(T,\infty) \times S^3 \times S^3$ for some $T > 0$, where the functions $t^{-3}a$ and $t^{-3}b$ admit convergent generalised power series expansions in powers of t^{-3} and $t^{-\nu_\infty}$ satisfying

$$\frac{54}{\sqrt{3}}t^{-3}a = 1 + O(t^{-3}), \qquad \frac{54}{\sqrt{3}}t^{-3}b = 1 + O(t^{-3}), \qquad \frac{54}{\sqrt{3}}t^{-3}(b-a) = ct^{-\nu_\infty} + O(t^{-12}).$$

In particular, the associated metric g_φ has a complete asymptotically conical end as $t \to \infty$ asymptotic to the cone C with rate -3.

We are now ready to state a theorem of [16], which gives the existence of complete torsion-free ALC and AC G_2-structures on M.

Theorem 3.9 ([16, Theorem 6.16 (i) and (ii)])**.** *Fix real $\bar{a} \geq \bar{b}$, and a real number $r_0 > 0$. Let*

$$\varphi = r_0^3 e_1 \wedge e_2 \wedge e_3 - r_0^3 e_1' \wedge e_2' \wedge e_3' + d(a(e_1 \wedge e_1' + e_2 \wedge e_2') + be_3 \wedge e_3')$$

be the (locally defined) $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant torsion-free G_2-structure of Proposition 3.7 closing smoothly on $\mathrm{SU}(2)^2/\Delta\,\mathrm{SU}(2)$ and satisfying

$$a = r_0^3 + \frac{1}{4}r_0 t^2 + \bar{a}t^4 + O(t^6), \qquad b = r_0^3 + \frac{1}{4}r_0 t^2 + \bar{b}t^4 + O(t^6)$$

in a neighbourhood of the singular orbit.

(i) *If $\bar{a} > \bar{b}$, then φ extends to a complete torsion-free $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant ALC G_2-structure on $S^3 \times \mathbb{R}^4$.*

(ii) *If $\bar{a} = \bar{b}$, then φ is the Bryant-Salamon complete torsion-free $\mathrm{SU}(2)^3$-invariant AC G_2-structure on $S^3 \times \mathbb{R}^4$.*

Remark 3.10. In Section 6.5 of [16], the solutions a and b for the ALC G_2-structures of Theorem 3.9 are proved to satisfy

$$\dot{a} > \dot{b} > 0, \qquad a > b > r_0^3, \qquad F > 0.$$

for $t \in (0,\infty)$. ◊

The G_2-structure φ on M induces a metric $g = dt^2 + g_t$ given by:

$$\begin{aligned} g_t = {} & \frac{a(b-r_0^3)}{\dot{a}\dot{b}}(e_1 \otimes e_1 + e_2 \otimes e_2) + \frac{a^2 - br_0^3}{\dot{a}^2} e_3 \otimes e_3 \\ & + \frac{a(b-r_0^3)}{\dot{a}\dot{b}}(e_1' \otimes e_1' + e_2' \otimes e_2') + \frac{a^2 - br_0^3}{\dot{a}^2} e_3' \otimes e_3' \\ & - \frac{b^2 - r_0^6}{2\dot{a}\dot{b}}(e_1 \otimes e_1' + e_1' \otimes e_1 + e_2 \otimes e_2' + e_2' \otimes e_2) + \frac{b^2 - 2a^2 + r_0^6}{2\dot{a}^2}(e_3 \otimes e_3' + e_3' \otimes e_3). \end{aligned} \tag{3.11}$$

The only ALC metric of Theorem 3.9 (i) that is explicitly known is the BGGG metric, which was constructed by Brandhuber, Gomis, Gubser and Gukov [4]. We postpone giving the explicit functions defining the metric until Section 4.3 because we apply a change of notation to work with M_{BGGG}.

3.3 The $\mathbb{C}_7$-family and its Infinite Extension

We now give a comprehensive description of the AC G_2-structures on the manifolds $M_{m,n}$ constructed in [16]. The $\mathbb{C}_7$-family is given by the special case where $m = n = 1$. Topologically, these manifolds are the total space of a circle bundle over the canonical bundle on $\mathbb{CP}^1 \times \mathbb{CP}^1$, whose first Chern class is given by the integers m and n.

3.3.1 The manifolds $M_{m,n}$

For any $m, n \in \mathbb{Z}$, define subgroups $K_{m,n}$ of T^2 by

$$K_{m,n} = \{(e^{i\theta_1}, e^{i\theta_2}) \in T^2 \mid e^{i(m\theta_1+n\theta_2)} = 1\}.$$

We have $K_{m,n} \cong U(1) \times \mathbb{Z}_{\gcd(m,n)}$ via

$$(e^{i\theta}, \zeta) \mapsto (e^{ik\theta}\zeta^r, e^{-ih\theta}\zeta^s),$$

where $m = \gcd(m,n)h$, $n = \gcd(m,n)k$ and $rh + sk = 1$. In particular, if m, n are coprime, then $K_{m,n} \cong U(1)$ via the map $e^{i\theta} \mapsto (e^{in\theta}, e^{-im\theta})$. In this case, we can fix integers $r, s \in \mathbb{Z}$ with $mr + ns = 1$. Then, by the natural embedding of $K_{m,n}$ in $\mathrm{SU}(2)^2$, the left-invariant vector field given by $nE_3 - mE_3'$ generates the $K_{m,n}$-action on $\mathrm{SU}(2)^2$.

Take the canonical line bundle $B = K_{\mathbb{CP}^1 \times \mathbb{CP}^1}$ over $\mathbb{CP}^1 \times \mathbb{CP}^1$; this bundle is of interest since it is an example of an AC Calabi-Yau 3-fold which is asymptotic to a $\mathbb{Z}_2$-quotient of the conifold $C(\Sigma)$, where $\Sigma = \mathrm{SU}(2)^2/\Delta\mathrm{U}(1)$ is endowed with its $\mathrm{SU}(2)\times\mathrm{SU}(2)$-invariant Sasaki-Einstein structure. There is a natural action of $\mathrm{SU}(2)$ on the line bundle $\mathcal{O}(-1) \to \mathbb{CP}^1$ which can be extended to an action of $\mathrm{SU}(2)^2$ on $K_{\mathbb{CP}^1 \times \mathbb{CP}^1}$. Its group diagram is

$$K_{2,-2} \subset T^2 \subset \mathrm{SU}(2) \times \mathrm{SU}(2)$$

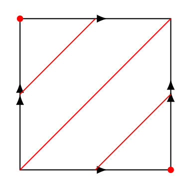

Figure 3.1: $K_{2,-2}$ embedded in T^2

and the tangent cone at infinity is a free $\mathbb{Z}_2$-quotient of the conifold. Here, the choice of stabiliser $K_{2,-2}$ was made in [16] to be consistent with the physics literature.

For coprime integers m and n, we now consider a non-trivial circle bundle $M_{m,n}$ over B, whose restriction to the zero section has first Chern class $c_1 = m[\omega] - n[\omega']$, where $[\omega]$ and $[\omega']$ are generators of the second integral cohomology group of the two $\mathbb{CP}^1$ factors. The natural action of each $\mathrm{SU}(2)$ factor on $\mathcal{O}(-1)$ gives a natural action of $\mathrm{SU}(2)^2$ on the circle bundle $M_{m,n}$. The total space is a simply connected cohomogeneity one 7-manifold, encoded in the group diagram

$$K_{m,n} \cap K_{2,-2} \subset K_{m,n} \subset \mathrm{SU}(2) \times \mathrm{SU}(2).$$

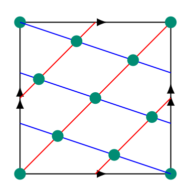

Figure 3.2: $K_{1,3} \cap K_{2,-2}$ embedded in T^2

Note that $K_{m,n} \cap K_{2,-2}$ is isomorphic to $\mathbb{Z}_{2|m+n|}$, where this cyclic group is embedded in the maximal torus $T^2 \subset \mathrm{SU}(2) \times \mathrm{SU}(2)$ via $\zeta \mapsto (\zeta^n, \zeta^{-m})$, for a primitive $2|m+n|$th root ζ. See Figure 2 for an example, when $m = 1$ and $n = 3$: the four corner points are identified, leaving $2|m+n| = 8$ distinct points.

Furthermore, the metrics constructed in Section 3.3.2 will have a cohomogeneity one action of $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$, where the additional $\mathrm{U}(1)$ acts on the circle fibres of $M_{m,n}$ by multiplication. It is easier to describe the manifolds $M_{m,n}$ via an $\mathrm{SU}(2)\times\mathrm{SU}(2)$ cohomogeneity one group action, but the requirement of the additional $\mathrm{U}(1)$-invariance simplifies the equations for an invariant torsion-free G_2-structure.

3.3.2 A complete torsion-free G_2-structure on $M_{m,n}$

We now outline the construction in [16] of complete torsion-free AC G_2-metrics on the manifolds $M_{m,n}$, using the parameter $\beta \in (0, \beta_{ac}]$ as described in Section 1.2. We start by considering $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant solutions which exist in a neighbourhood of the singular orbit Q of $M_{m,n}$, where $Q = (SU(2)\times \mathrm{SU}(2))/K_{m,n} \cong S^2 \times S^3$.

Proposition 3.12 ([16, Proposition 4.5 (iii) and (iv)])**.** *Fix coprime positive $m, n \in \mathbb{Z}$. $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant G_2-structures on $M_{m,n}$ have the form*

$$\varphi_\beta = -m^2 r_0^3 e_1 \wedge e_2 \wedge e_3 + n^2 r_0^3 e_1' \wedge e_2' \wedge e_3' + d(a(e_1 \wedge e_1' + e_2 \wedge e_2') + b e_3 \wedge e_3')$$

for some functions $a, b : [0,\infty) \to \mathbb{R}$. There exists a 2-parameter family of torsion-free G_2-structures φ of this form defined in a neighbourhood of the singular orbit. The family is parameterised by $r_0 \in \mathbb{R}$ and $\beta > 0$ with

$$a = r_0^2 \beta t + \frac{(m+n)\beta^3 - m^{\frac{5}{2}} n^{\frac{5}{2}}}{12\sqrt{mn}\beta^3} t^3 + O(t^5),$$
$$b = mnr_0^3 + \frac{\sqrt{mn}(m+n)r_0}{2\beta} t^2 + \frac{m+n}{96 r_0 \beta^5}\left(7 - 4(m+n)\beta^3\right) t^4 + O(t^6).$$

Having described local solutions in a neighbourhood of the singular orbit, we consider asymptotic local solutions by applying Proposition 3.8 with $p = -m^2 r_0^3$ and $q = n^2 r_0^3$. We are then ready to state the main relevant theorem of [16], which gives existence of a complete torsion-free AC G_2-structure on $M_{m,n}$.

Theorem 3.13 ([16, Theorem 7.1 (ii)])**.** *Fix coprime positive $m, n \in \mathbb{Z}$, and a real number $r_0 > 0$. For $\beta > 0$, let*

$$\varphi_\beta = -m^2 r_0^3 e_1 \wedge e_2 \wedge e_3 + n^2 r_0^3 e_1' \wedge e_2' \wedge e_3' + d(a(e_1 \wedge e_1' + e_2 \wedge e_2') + b e_3 \wedge e_3')$$

be the (locally defined) $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant torsion-free G_2-structure of Proposition 3.12 closing smoothly on $\mathrm{SU}(2)^2/K_{m,n}$ and satisfying

$$a = r_0^3 \beta t + O(t^3), \qquad b = mnr_0^3 + O(t^2)$$

as $t \to 0$. There exists $\beta_{ac} > 0$ such that $\varphi_{\beta_{ac}}$ extends to a complete torsion-free AC G_2-structure asymptotic to the cone over the $\mathbb{Z}_{2(m+n)}$-quotient of the homogeneous nearly Kähler structure on $S^3 \times S^3$ with rate -3.

Remark 3.14. In Section 7.1 of [16], the solutions a and b for the AC G_2-structure of Theorem 3.13 are proved to satisfy

$$b > a > 0, \quad \dot{a} > \dot{b} > 0, \quad b > \max(-m^2 r_0^3, -n^2 r_0^3), \quad b > mnr_0^3,$$
$$4a^2(b + m^2 r_0^3)(b + n^2 r_0^3) > (b^2 - m^2 n^2 r_0^6)^2, \quad ka > \frac{b^2 - m^2 n^2 r_0^6}{\sqrt{(b + m^2 r_0^3)(b + n^2 r_0^3)}} \text{ for } k \in (1,2)$$

for $t \in (0, \infty)$. This is the only information we have about a and b globally, since the G_2-structure is not derived explicitly in the proof of its existence. ◊

The G_2-structure φ_β on $M_{m,n}$ induces a metric $g = dt^2 + g_t$ given by:

$$\begin{aligned} g_t = &\frac{a(b+m^2r_0^3)}{\dot{a}\dot{b}}(e_1 \otimes e_1 + e_2 \otimes e_2) + \frac{a^2+bm^2r_0^3}{\dot{a}^2} e_3 \otimes e_3 \\ &+ \frac{a(b+n^2r_0^3)}{\dot{a}\dot{b}}(e_1' \otimes e_1' + e_2' \otimes e_2') + \frac{a^2+bn^2r_0^3}{\dot{a}^2} e_3' \otimes e_3' \\ &- \frac{b^2-m^2n^2r_0^6}{2\dot{a}\dot{b}}(e_1 \otimes e_1' + e_1' \otimes e_1 + e_2 \otimes e_2' + e_2' \otimes e_2) \\ &+ \frac{b^2-2a^2+m^2n^2r_0^6}{2\dot{a}^2}(e_3 \otimes e_3' + e_3' \otimes e_3). \end{aligned} \tag{3.15}$$

3.4 A Classification Theorem

The classification of complete $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant simply-connected G_2-manifolds is given in the following theorem.

Theorem 3.16 ([16, Theorem 7.3])**.** *Let (M, g) be a complete $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant G_2-metric with M simply-connected. Then (M, g) is isometric to one of the following complete metrics:*

(i) *the complete AC metrics of Theorem 3.13;*

(ii) *Byrant-Salamon's complete $\mathrm{SU}(2)^3$-invariant AC G_2-metric on $S^3 \times \mathbb{R}^4$ given in Theorem 3.9 (ii) - see [5];*

(iii) *the $\mathbb{B}_7$ family of complete $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant ALC G_2-metrics on $S^3 \times \mathbb{R}^4$ given in Theorem 3.9 (i) - see [2] and [4];*

(iv) *the $\mathbb{D}_7$ family[1] of complete $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant ALC G_2-metrics on $S^3 \times \mathbb{R}^4$ - see [16, Theorem 6.17 (i)];*

(v) *if $\beta > \beta_{ac}$ in Theorem 3.13, then φ_β extends to a complete torsion-free ALC G_2-structure asymptotic to a circle bundle over a $\mathbb{Z}_2$-quotient conifold - see [16, Theorem 7.1 (i)].*

Some progress has been made to construct G_2-instantons on these manifolds: Clarke [7], Lotay and Oliveira [24] describe some solutions of the G_2-instanton equations for the Bryant Salamon metric (ii), and some progress is made towards describing the solutions for the family (iii). At the other end to the AC limit of each 1-parameter family $\mathbb{B}_7$, $\mathbb{C}_7$ and its infinite extension, and $\mathbb{D}_7$ is a collapsed metric on an AC Calabi-Yau 6-manifold. Instantons and monopoles on these metrics have been considered by Stein in [30].

Beyond this thesis, there is an abundance of gauge theoretic questions relating to these families. One such question is to understand the relationship between the gauge theory on the ALC

[1] This family was originally conjectured in the physics literature - see [3] and [10].

G_2-metrics in cases (iii), (iv) and (v), and the AC metrics on the collapsed limit. An ALC G_2-instanton has a Calabi-Yau monopole on a Calabi-Yau cone as its limit; hence the results of Stein could be used as a guide for constructing G_2-instantons on ALC manifolds. Another possibility for future work is to further develop the technique used in Chapter 5 to construct instantons on any AC $M_{m,n}$ when m and n are distinct. This is a more challenging problem, for the reasons described throughout Chapter 5, and the methods would require a number of changes to overcome these challenges.

In this thesis, we consider the problem of constructing G_2-instantons on (i) and (ii), as well as proving the existence of a boundary of solutions for the BGGG example in case (iii).

CHAPTER 4

G_2-INSTANTONS ON THE $\mathbb{B}_7$-FAMILY

We now focus our attention on the gauge theory on two members of the $\mathbb{B}_7$ family of metrics, namely the BS AC metric and the BGGG ALC metric. We first construct a 1-parameter family of instantons on the AC example M_{BS}, before proving that the set of complete bounded solutions on a bundle over M_{BGGG} is closed.

4.1 G_2-instanton Equations for the $\mathbb{B}_7$ family

Recall that the $\mathbb{B}_7$ family of G_2-metrics described in Theorem 3.9 have a cohomogeneity one action of $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$. We are interested in constructing principal bundles on a generic member of the $\mathbb{B}_7$ family which are homogeneous when restricted to each orbit. We then write the G_2-instanton equations explicitly for a generic member of the family.

We now describe the possible homogeneous bundles on the orbits of the $\mathbb{B}_7$ family. The additional U(1)-symmetry can be encoded by writing the principal orbits in the form

$$\mathrm{SU}(2)^2 \cong \mathrm{SU}(2)^2 \times \mathrm{U}(1)/\Delta\mathrm{U}(1).$$

Isomorphism classes of homogeneous SU(2)-bundles on the principal orbits are is correspondence with conjugacy classes of isotropy homomorphisms $\lambda_p : \Delta\mathrm{U}(1) \to \mathrm{SU}(2)$. Such classes are parameterised by $k \in \mathbb{Z}$ where

$$\lambda_p^k : e^{i\theta} \mapsto \begin{pmatrix} e^{ik\theta} & 0 \\ 0 & e^{-ik\theta} \end{pmatrix}.$$

The singular orbit can be written in the form

$$\mathrm{SU}(2)^2/\Delta\,\mathrm{SU}(2) \cong (\mathrm{SU}(2)^2 \times \mathrm{U}(1))/(\Delta\,\mathrm{SU}(2) \times \mathrm{U}(1)).$$

In this case, the possible isotropy homomorphisms $\Delta\,\mathrm{SU}(2) \times \mathrm{U}(1) \to \mathrm{SU}(2)$ are

$$\lambda_s^{1,k} : (g, e^{i\theta}) \mapsto \begin{pmatrix} e^{ik\theta} & 0 \\ 0 & e^{-ik\theta} \end{pmatrix}$$

for $k \in \mathbb{Z}$, and

$$\lambda_s^{\mathrm{id},0} : (g, e^{i\theta}) \mapsto g.$$

Take the complement of $\Delta\,\mathfrak{su}(2) \oplus \mathfrak{u}(1)$ to be $\mathfrak{m} = \Delta^-\,\mathfrak{su}(2) \oplus 0$, where $\Delta^-\,\mathfrak{su}(2)$ is the anti-diagonal copy of $\mathfrak{su}(2)$ in $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Any invariant connection on $P_s^{1,k}$ can be written as $d\lambda_s^{1,k} + \Lambda_{1,k}$, where $d\lambda_s^{1,k} = E_3 \otimes k d\theta \in \mathfrak{su}(2) \otimes T_e^*M$ is the canonical invariant connection and $\Lambda_{1,k} \in \mathfrak{su}(2) \otimes T_e^*M$ is given by a morphism of $\Delta\,\mathrm{SU}(2) \times \mathrm{U}(1)$-representations

$$\Lambda_{1,k} : (\mathfrak{m}, \mathrm{Ad}) \to (\mathfrak{su}(2), \mathrm{Ad} \circ \lambda_s^{1,k}).$$

Indeed, we can extend $d\lambda_s^{1,k} + \Lambda_{1,k}$ to an $\mathrm{Ad}(P_s^{1,k})$-valued 1-form by left invariance; the property that $\Lambda_{1,k}$ is a morphism of representations ensures that the values lie in the adjoint bundle. Following the work of Lotay and Oliveira [24], by considering the splitting into irreducibles, Schur's Lemma tells us that $k = 1$ for non-abelian connections and hence there are two bundles P_1 and P_{id}, which are two different extensions over the singular orbit of the unique bundle on $\mathbb{R}^+ \times \mathrm{SU}(2)^2$.

Recall from Section 3.1 that an invariant connection is determined by a section of $T^*M|_C \otimes \mathfrak{su}(2)$, where C is a curve meeting all principal orbits orthogonally, parameterised by $t \in [0, \infty)$. In this way, we can write the G_2-instanton equations for a connection on P_1 or P_{id} over a member of the $\mathbb{B}_7$ family as a system of ODEs of functions $\mathbb{R}_t^+ \to \mathbb{R}$. We reformulate Proposition 8 of [24] in our notation, which gives the $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant G_2-instanton equations on the space of principal orbits for all members of the $\mathbb{B}_7$ family.

Proposition 4.1. *Let $\mathbb{R}^+ \times \mathrm{SU}(2)^2 \cong \mathbb{R}^+ \times \mathrm{SU}(2)^2 \times \mathrm{U}(1)/\Delta\mathrm{U}(1)$ be equipped with one of the AC or ALC G_2-structures φ of Theorem 3.9 determined by the functions a and b. Let A be an $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant G_2-instanton on the $\mathrm{SU}(2)$-bundle over $\mathbb{R}^+ \times \mathrm{SU}(2)^2$ given by the pull-back of P_p^1. Then we can write*

$$A = f(E_1 \otimes e_1 + E_2 \otimes e_2) + f'(E_1 \otimes e_1' + E_2 \otimes e_2') + gE_3 \otimes e_3 + hE_3 \otimes e_3' \tag{4.2}$$

where $f, f', g, h : \mathbb{R}^+ \to \mathbb{R}$ satisfy

$$\begin{aligned}
\frac{2\dot{a}^3\dot{b}^2}{b - r_0^3}\dot{f} &= f'(1-h)(2a^2 - r_0^3(b + r_0^3)) + f'g(b(b + r_0^3) - 2a^2) + f(g - h - 1)a(b - r_0^3), \\
\frac{2\dot{a}^3\dot{b}^2}{b - r_0^3}\dot{f}' &= f(1-g)(2a^2 - r_0^3(b + r_0^3)) + fh(b(b + r_0^3) - 2a^2) + f'(h - g - 1)a(b - r_0^3), \quad (4.3) \\
\frac{2\dot{a}^4\dot{b}}{b - r_0^3}\dot{g} &= (h - f'^2)(2a^2 - r_0^3(b + r_0^3)) - 2ff'a(b - r_0^3) + (g - f^2)(b(b + r_0^3) - 2a^2), \\
\frac{2\dot{a}^4\dot{b}}{b - r_0^3}\dot{h} &= (g - f^2)(2a^2 - r_0^3(b + r_0^3)) - 2ff'a(b - r_0^3) + (h - f'^2)(b(b + r_0^3) - 2a^2).
\end{aligned}$$

This proposition was proved by Lotay and Oliveira by showing that the connection is given by (4.2) on each orbit up to a gauge transformation. Then the G_2-instanton equations are only satisfied if this gauge transformation, seen as a function of $t \in (0, \infty)$, is constant. An analogous proof is given for the derivation of the G_2-instanton equations for the $\mathbb{C}_7$ family and its infinite extension in Proposition 5.6.

4.2 The Bryant-Salamon $S^3 \times \mathbb{R}^4$

The work in Section 4.2 is in collaboration with Jakob Stein.

M_{BS} is $\mathrm{SU}(2)^3$-invariant so we consider $\mathrm{SU}(2)^3$-invariant connections; then $f = g$ and $f' = h$. Writing $f^+ = f + f'$ and $f^- = f - f'$, the instanton equations (4.3) become

$$\begin{aligned} \dot{f}^+ &= \frac{\dot{b}}{b - r_0^3} f^+ \left(\frac{2(b + r_0^3)}{3b + r_0^3} - f^+ \right) + \frac{\dot{b}}{3b + r_0^3} (f^-)^2, \\ \dot{f}^- &= \frac{2\dot{b}}{b - r_0^3} f^- (f^+ - 1). \end{aligned} \tag{4.4}$$

Since M_{BS} is an AC G_2-manifold, a G_2-instanton on M_{BS} will be asymptotic to a G_2-instanton on the cone on $S^3 \times S^3$. The G_2-structure on the cone is given by (3.3), with $p = q = 0$ and

$$a = b = \frac{\sqrt{3}}{54} t^3 + O(t^{-3}).$$

Hence we can calculate the coefficients in the equations (4.4), giving

$$\frac{\dot{b}}{b - r_0^3} = 3t^{-1} + \cdots, \qquad \frac{\dot{b}}{3b + r_0^3} = t^{-1} + \cdots, \qquad \frac{2(b + r_0^3)}{3b + r_0^3} = \frac{2}{3} + \cdots. \tag{4.5}$$

Then the $\mathrm{SU}(2)^3$-invariant G_2-instanton equations (4.4) are asymptotic to the G_2-instanton equations on the cone, which we can calculate using (4.5) to be

$$\begin{aligned} \dot{f}^+ &= \frac{1}{t} \left(f^+(2 - 3f^+) + \left(f^-\right)^2 \right) \\ \dot{f}^- &= \frac{6}{t} \left(f^-(f^+ - 1) \right). \end{aligned} \tag{4.6}$$

The critical points of (4.6) are given by

$$(f^+, f^-) \in \{(0,0), (1, \pm 1), (\tfrac{2}{3}, 0)\}.$$

The first three points yield flat connections and $(\frac{2}{3}, 0)$ corresponds to the unique non-trivial invariant nearly-Kähler instanton on the link of the asymptotic cone - see Section 3.3.3 of [24]. We can transform (4.6) into an autonomous system by applying the time transformation $t = e^\tau$, which yields

$$\begin{aligned} \dot{f}^+ &= f^+(2 - 3f^+) + \left(f^-\right)^2 \\ \dot{f}^- &= 6f^-(f^+ - 1). \end{aligned} \tag{4.7}$$

The phase diagram of this autonomous system is shown in Figure 4.1. The flow behaviour varies between the regions separated by the line $x = 1$ and the hyperbola $y^2 = x(3x - 2)$. The point $(\frac{2}{3}, 0)$ corresponding to the nearly-Kähler instanton is a stable fixed point, while the points $(1, \pm 1)$ and $(0, 0)$ are saddles.

Lemma 4.8. *Solutions to* (4.6) *converging to the asymptotically stable critical point* $(\frac{2}{3}, 0)$ *are in a two-parameter family* $\mu, \nu \in \mathbb{R}$ *for* t *sufficiently large:*

$$f^+ = \tfrac{2}{3} + \mu t^{-2} + O(t^{-3}) \qquad\qquad f^- = \nu t^{-2} + O(t^{-3})$$

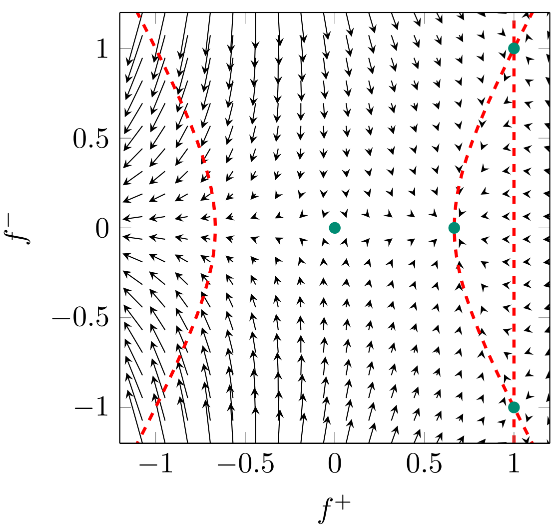

Figure 4.1: The phase diagram of the autonomous system (4.7) on the cone, with regions separated by the line $x = 1$ and the hyperbola $y^2 = x(3x - 2)$, and critical points $(0, 0)$, $(1, \pm 1)$ and $(\frac{2}{3}, 0)$.

Proof. Let $y = (f^+, f^-)$, $y_0 = (\frac{2}{3}, 0)$ and consider the autonomous equations (4.7). The linearisation at y_0 of this system is

$$\begin{pmatrix} 2 - 6f^+ & 2f^- \\ 6f^- & 6(f^+ - 1) \end{pmatrix}\Bigg|_{(\frac{2}{3},0)} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \tag{4.9}$$

and so has a repeated eigenvalue -2. Hence we can find a 2-parameter family of local solutions for sufficiently large t, with $y - y_0 = O(t^{-2})$, as given in the statement of the lemma. □

In what follows, we reformulate the results in [24] which classify the local solutions near the singular orbit that extend smoothly. We then use forward invariant sets to prove the existence of a 1-parameter family of solutions. This will complete the picture of all $\mathrm{SU}(2)^3$-invariant G_2-instantons on the Bryant-Salamon manifold.

4.2.1 Initial conditions

We now consider the conditions for extending the connections of Proposition 4.1 smoothly over the singular orbit. We restate Propositions A.2 and A.3 from Appendix A which are reformulations of Lemma 4 of [24].

Proposition A.2. *On $P_{id} \to M_{BS}$, there is a 1-parameter family of solutions A^γ in a neighbourhood of the singular orbit of the $\mathrm{SU}(2)^3$-invariant G_2-instanton equations of the form*

$$f^+(t) = 1 + \frac{\gamma^2 - 1}{32 r_0^2} t^2 + O(t^4),$$
$$f^-(t) = \gamma + \frac{\gamma(\gamma^2 - 1)}{16 r_0^2} t^2 + O(t^4).$$

Setting $\gamma = \frac{1}{\sqrt{3}} y_0$ gives the family of local solutions in [24, Proposition 6].

Proposition A.3. *On $P_1 \to M_{BS}$, there is a 1-parameter family of local solutions A_δ in a neighbourhood of the singular orbit of the* $\mathrm{SU}(2)^3$*-invariant G_2-instanton equations of the form*

$$f^+(t) = \delta t^2 + O(t^4),$$
$$f^-(t) = 0.$$

Setting $\delta = \frac{1}{2} x_1$ gives the family of local solutions that extend to the global solutions of [24, Theorem 4]. These are the solutions of Clarke constructed in [7] and are proved to be the only irreducible $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant solutions on P_1 in Theorem 7 of [24].

4.2.2 Invariant sets

The $\mathrm{SU}(2)^3$-invariant G_2-instanton equations (4.4) contain an invariant line $\ell = \{f^- = 0\}$. On this line, the equations reduce to

$$\dot{f}^+ = \frac{\dot{b}}{b - r_0^3} f^+ \left(\frac{2(b + r_0^3)}{3b + r_0^3} - f^+ \right) \tag{4.10}$$

which has solutions

$$f^+ = \frac{2(b - r_0^3)}{c_1 (3b + r_0^3)^{\frac{1}{3}} + 3b + r_0^3}. \tag{4.11}$$

If $c_1 = 4^{-\frac{4}{3}}(\delta^{-1} - 16 r_0^2)$, then we get a 1-parameter family A_δ on P_1 extending the local solutions of Proposition A.3. These solutions are exactly those obtained by Clarke; see [7, Theorem 1.1] and [24, Theorem 4].

If $c_1 = -4^{\frac{2}{3}} r_0^2$, then we get a single solution A^0 on P_{id} extending the local solution of Proposition A.2 with $\gamma = 0$. This is the solution found by Lotay and Oliveira in [24, Theorem 5].

The moduli space of invariant G_2-instantons on $\mathrm{SU}(2)$-bundles over M_{BS} is also considered in [24]. As $\delta \to \infty$, the instantons A_δ bubble off an anti-self-dual (ASD) connection along the normal bundle of the associative singular orbit S^3. In addition, Theorem 6 of [24] describes how A_δ converges uniformly with all derivatives to A^0 as $\delta \to \infty$ on any compact set disjoint from the associative S^3.

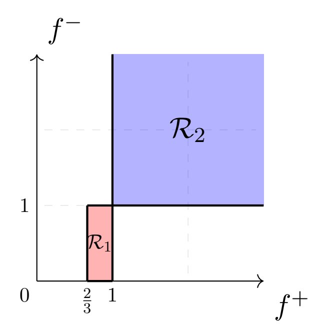

Figure 4.2: $\mathcal{R}_1$ and $\mathcal{R}_2$

We now consider the possibility of extending the rest of the 1-parameter family of local solutions given in Proposition A.2 to instantons on the whole of M_{BS}. We start by defining a couple of forward invariant sets for the $\mathrm{SU}(2)^3$-invariant G_2-instanton equations (4.4), namely

$$\mathcal{R}_1 = \{(f^+, f^-) : \tfrac{2}{3} < f^+ < 1, 0 < f^- < 1\} \subset \mathbb{R}^2,$$
$$\mathcal{R}_2 = \{(f^+, f^-) : f^+ > 1, f^- > 1\} \subset \mathbb{R}^2.$$

The following lemma shows that once a solution enters $\mathcal{R}_1$, it stays inside for all time. Recall that from Remark 3.10 that we have $b > r_0^3 > 0$ and $\dot{b} > 0$ for all $t > 0$.

Lemma 4.12. *The set* $\mathcal{R}_1$ *is forward invariant under* (4.4).

Proof. Suppose $(f^+, f^-) \in \mathcal{R}_1$ at some time t_0. We consider the four cases of the solutions hitting the boundary of $\mathcal{R}_1$.

Case 1 $f^+ = \frac{2}{3}$. Suppose there exists $t_1 > t_0$ such that $f^+(t_1) = \frac{2}{3}$ and that t_1 is the first such time after t_0 that this occurs, then equations (4.4) imply that $\dot{f}^+(t_1) \geq 0$. Since $f^+ \in \mathcal{R}_1$ in (t_0, t_1), then $\dot{f}^+(t_1)$ cannot be positive. If $\dot{f}^+(t_1) = 0$, then $f^-(t_1) = 0$ and we refer to the third case.

Case 2 $f^+ = 1$. Suppose there exists $t_1 > t_0$ such that $f^+(t_1) = 1$ and that this is the first time that happens after t_0. If $f^-(t_1) = 0$, we have the flat solution $(f^+, f^-) = (1, 0)$, otherwise $0 < f^-(t_1) \leq 1$. We have

$$\dot{f}^+(t_1) = \frac{\dot{b}(t_1)}{3b(t_1) + r_0^3}((f^-)^2(t_1) - 1) \leq 0.$$

If $f^- < 1$, then $\dot{f}^+(t_1) < 0$, contradicting t_1 being the first time $f^+ = 1$. If $f^- = 1$, then for t in an interval $(t_1 - \epsilon, t_1)$, $\dot{f}^-(t) > 0$ and hence $f^+(t) > 1$, a contradiction.

Case 3 $f^- = 0$. Suppose there exists $t_1 > t_0$ such that $f^-(t_1) = 0$ where t_1 is the first time this occurs after t_0, then $\dot{f}^-(t_1) = 0$ and

$$\ddot{f}^-(t_1) = \frac{2\dot{b}(t_1)}{b(t_1) - r_0^3}(f^+(t_1) - 1) < 0$$

since $f^+(t_1) < 1$. Hence t_1 is a maximum for f^- which contradicts t_1 being the first time $f^- = 0$.

Case 4 $f^- = 1$. Since $f^- > 0$ and $\frac{2}{3} < f^+ < 1$, then $\dot{f}^- < 0$ and hence f^- is decreasing, and so will be bounded above by $f^- < 1$. Then, for all $t > t_0$, $(f^+, f^-) \in \mathcal{R}_1$.

□

We now show that the set $\mathcal{R}_2$ is also forward invariant.

Lemma 4.13. *The set* $\mathcal{R}_2$ *is forward invariant under* (4.4).

Proof. Suppose $(f^+, f^-) \in \mathcal{R}_2$ at some time t_0. We consider the two cases of the solutions hitting the boundary of $\mathcal{R}_2$.

Case 1 $f^- = 1$. Suppose there exists $t_1 > t_0$ such that $f^-(t_1) = 1$, then

$$\dot{f}^-(t_1) = \frac{2\dot{b}(t_1)}{b(t_1) - r_0^3}(f^+(t_1) - 1) > 0$$

since $f^+(t_1) \geq 1$ and $f^+(t_1) \neq 1$ by uniqueness of solutions. This contradicts $f^-(t) > 1$ for $t_0 < t < t_1$.

Case 2 $f^+ = 1$. Suppose there exists $t_1 > t_0$ such that $f^+(t_1) = 1$, then

$$\begin{aligned}\dot{f}^+(t_1) &= \frac{\dot{b}(t_1)}{b(t_1) - r_0^3}\left(\frac{2(b(t_1) + r_0^3)}{3b(t_1) + r_0^3} - 1\right) + \frac{\dot{b}(t_1)}{3b(t_1) + r_0^3}(f^-)^2(t_1) \\ &> \frac{\dot{b}(t_1)}{b(t_1) - r_0^3}\left(\frac{2(b(t_1) + r_0^3)}{3b(t_1) + r_0^3} - 1\right) + \frac{\dot{b}(t_1)}{3b(t_1) + r_0^3} = 0.\end{aligned}$$

This contradicts $f^+(t) > 1$ for $t_0 < t < t_1$. Hence $(f^+, f^-) \in \mathcal{R}_2$ for all $t > t_0$.

□

We now consider the limiting behaviour of solutions that lie in the forward invariant sets $\mathcal{R}_1$ and $\mathcal{R}_2$. We first see that solutions in $\mathcal{R}_1$ converge to a nearly Kähler instanton on the link.

Lemma 4.14. *Any solution (f^+, f^-) to (4.4) lying in $\mathcal{R}_1$ at some $t > 0$ converges to $(\frac{2}{3}, 0)$ as $t \to \infty$.*

Proof. As described in Section 4.2, the only fixed points of the G_2-instanton equations on the cone are $(f^+, f^-) \in \{(0,0), (\frac{2}{3}, 0), (1, \pm 1)\}$. Since f^- is decreasing in $\mathcal{R}_1$ and the instanton equations depend solely on the monotonic function b of the G_2-structure on M_{BS}, the solution will not display periodic behaviour and will be bounded, hence will converge to a limit point. The only possible such point as $t \to \infty$ is the nearly Kähler instanton on the link given by $(f^+, f^-) = (\frac{2}{3}, 0)$. □

On the other hand, solutions which lie in $\mathcal{R}_2$ do not converge as $t \to \infty$.

Lemma 4.15. *Any solution (f^+, f^-) to (4.4) lying in $\mathcal{R}_2$ at some $t > 0$ cannot converge as $t \to \infty$.*

Proof. In $\mathcal{R}_2$,

$$\dot{f}^- = \frac{2\dot{b}}{b - r_0^3} f^-(f^+ - 1) > 0$$

and so f^- is increasing. The only fixed point of the instanton equations on the cone that lies on the boundary of $\mathcal{R}_2$ is $(1,1)$, which has $f^- = 1$. Thus any solution lying in $\mathcal{R}_2$ after some time T must have $f^- > 1$ for all time and hence cannot converge. □

4.2.3 $\mathrm{SU}(2)^3$-invariant solutions

Recall that there are two SU(2)-bundles on M_{BS}, namely P_1 and P_{id}. We have seen that there is a 1-parameter family A_δ on P_1, constructed by Clarke [7]. We now consider the problem of extending the local solutions on P_{id} of Proposition A.2 to global solutions when $\gamma \neq 0$. There are two flat solutions $A^{\pm 1}$ given by constant functions with values $(f^+, f^-) = (1, \pm 1)$.

For the remaining values of γ, we use the forward invariant sets of the previous section to bound a subset of these local solutions. The following theorem describes which of these solutions are bounded on M_{BS} and classifies all possible bounded $\mathrm{SU}(2)^3$-invariant G_2-instantons on M_{BS}.

Theorem 4.16. *Let M_{BS} be the $\mathbb{B}_7$ Bryant-Salamon $S^3 \times \mathbb{R}^4$. There exists a 1-parameter family A^γ of $\mathrm{SU}(2)^3$-invariant G_2-instantons on the bundle P_{id} on M_{BS}, parameterised by the initial conditions $(f^+, f^-)(0) = (1, \gamma)$ for $\gamma \in [-1, 1]$ and extending the local solutions given in Proposition A.2. A^0 is the solution (4.11) with $c_1 = -4^{\frac{2}{3}} r_0^2$ and $A^{\pm 1}$ are the flat solutions. A^γ for $\gamma \in (-1, 1)$ is asymptotic to the nearly Kähler instanton on the link, given by $(f^+, f^-) = (\frac{2}{3}, 0)$.*

A^γ is gauge equivalent to $A^{-\gamma}$, but not via equivariant gauge transformations. The families A^γ on P_{id} and A_δ on P_1 of Clarke are the only bounded $\mathrm{SU}(2)^3$-invariant G_2-instantons on M_{BS}.

Proof. Proposition A.2 gives solutions on some interval $[0, T)$ with $(f^+, f^-)(0) = (1, \gamma)$ for $\gamma \in [0, 1]$ and $(f^+, f^-)(\epsilon) \in \mathcal{R}_1$ for sufficiently small ϵ. Then Lemma 4.12 implies that these local solutions extend to global solutions which remain in the set $\mathcal{R}_1$ for all $t > \epsilon$. Since these solutions are bounded, the functions must have a finite limit as $t \to \infty$ and Lemma 4.14 implies that this limit is the nearly Kähler instanton on the link $S^3 \times S^3$.

We see from [24, Theorems 4 and 7] that all local solutions in Proposition A.3 on P_1 extend globally and that these are the only $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant solutions. We now consider classifying all bounded $\mathrm{SU}(2)^3$-invariant solutions on P_{id}. By the symmetry $f^- \mapsto -f^-$, we only need to consider the expansions on P_{id} of Proposition A.2 for $\gamma \geq 0$. We have extended all the local solutions for $\gamma \in [0, 1]$. If $\gamma > 1$, then the local solutions satisfy $(f^+, f^-)(\epsilon) \in \mathcal{R}_2$ for sufficiently small ϵ. Lemma 4.13 implies that these solutions remain in $\mathcal{R}_2$ for all $t > \epsilon$ and Lemma 4.15 shows that these solutions are not bounded as $t \to \infty$. Hence, the only bounded solutions for $\gamma \geq 0$ are those for which $\gamma \in [0, 1]$. □

4.3 The BGGG $S^3 \times \mathbb{R}^4$

We now consider the explicit member of the ALC $\mathbb{B}_7$ family known as the BGGG metric. In this section, we prove that the region of initial conditions which lead to complete bounded instantons on P_1 is a closed subset in $\mathbb{R}^2$. We describe this set of initial conditions and analyse properties of the corresponding solutions, including their asymptotics.

4.3.1 The BGGG metric

In order to build on the results of Lotay and Oliveira [24], we outline some of their results and change notation for this section[1]. We write the basis vectors E_i, E'_i of $\mathfrak{su}(2)$ as $\frac{1}{2}T_i$; then $[T_i, T_j] = 2\epsilon_{ijk}T_k$. Split the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ as $\mathfrak{su}(2)^+ \oplus \mathfrak{su}(2)^-$ and write $T_i^\pm = (T_i, \pm T_i)$, which form a basis of $\mathfrak{su}(2)^\pm$ respectively, and a dual basis is given by $\eta_i^\pm = \frac{1}{4}(e_i \pm e'_i)$. The

[1] Note: due to different conventions for the choice of a U(1) basis generator, the following implements the permutation $(1, 3, 2)$ on the indices of T_i from those that appear in [24].

$\mathrm{SU}(2)^2$-invariant SU(3)-structure on $S^3 \times S^3$ is given by

$$\omega = 4 \sum_{i=1}^{3} A_i B_i \eta_i^- \wedge \eta_i^+,$$
$$\operatorname{Re}\Omega = 8B_1B_2B_3\eta_1^- \wedge \eta_2^- \wedge \eta_3^- - 4\sum_{i,j,k} \epsilon_{ijk} A_i A_j B_k \eta_i^+ \wedge \eta_j^+ \wedge \eta_k^-$$

for functions $A_i, B_i : \mathbb{R}_+ \to \mathbb{R}$. With the assumption of the additional U(1)-invariance giving $A_1 = A_2$ and $B_1 = B_2$, the Hitchin flow equations (3.1) transform to ODEs for the functions A_i, B_i, namely

$$\begin{aligned}
\dot{A}_1 &= \frac{1}{2}\left(\frac{B_1^2 + B_3^2 - A_1^2}{B_1 B_3} - \frac{A_3}{A_1}\right), \\
\dot{A}_3 &= \frac{1}{2}\left(\frac{A_3^2}{A_1^2} - \frac{A_3^2}{B_1^2}\right), \\
\dot{B}_1 &= \frac{1}{2}\left(\frac{A_1^2 + B_3^2 - B_1^2}{A_1 B_3} + \frac{A_3}{B_1}\right), \\
\dot{B}_3 &= \frac{A_1^2 + B_3^2 - B_1^2}{A_1 B_1}.
\end{aligned}$$

In order to write the BGGG metric explicitly, we define the implicit coordinate r via

$$t(r) = \int_{\frac{9}{4}}^{r} \frac{\sqrt{(s - \frac{3}{4})(s + \frac{3}{4})}}{\sqrt{(s - \frac{9}{4})(s + \frac{9}{4})}} ds.$$

Then the metric takes the form $g = dt^2 + g_t$ where

$$g_t = \sum_{i=1}^{3} (2A_i)^2 \eta_i^+ \otimes \eta_i^+ + (2B_i)^2 \eta_i^- \otimes \eta_i^-$$

and

$$A_1 = \sqrt{\frac{(r - \frac{9}{4})(r + \frac{3}{4})}{3}}, \quad A_3 = \sqrt{\frac{(r - \frac{9}{4})(r + \frac{9}{4})}{(r - \frac{3}{4})(r + \frac{3}{4})}},$$
$$B_1 = \sqrt{\frac{(r + \frac{9}{4})(r - \frac{3}{4})}{3}}, \quad B_3 = \frac{2r}{3}.$$

Asymptotically, the functions A_i, B_i can be written as Laurent series expansions on some interval

(T, ∞), namely

$$\begin{aligned}
A_1 &= \frac{t}{\sqrt{3}} + \rho + \frac{3\sqrt{3}}{8}t^{-1} + O(t^{-2}),\\
A_3 &= 1 - \frac{9}{4}t^{-2} + O(t^{-3}),\\
B_1 &= \frac{t}{\sqrt{3}} + \rho + \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{8}t^{-1} + O(t^{-2}),\\
B_3 &= \frac{2t}{3} + 2\rho + \frac{\sqrt{3}}{2} + \frac{3}{2}t^{-1} + O(t^{-2}).
\end{aligned}$$

Since $t \sim r$ as $t \to \infty$, we have the following relation from Section 2.2.2 of [24] for the BGGG member of the family:

$$B_3 - \frac{2}{\sqrt{3}}B_1 = \frac{6 - 2\sqrt{3}}{3}\rho + \frac{\sqrt{3} - 2}{2} + O(t^{-1}) = -\frac{1}{2} + O(r^{-1}).$$

Hence, we see that for the BGGG metric, we have $\rho = -\frac{\sqrt{3}}{4}$. We can apply these results to write local expansions for the coefficients of (4.20):

$$\begin{aligned}
\frac{1}{2}\left(\frac{A_3}{B_1^2} - \frac{A_3}{A_1^2}\right) &= -\frac{9}{2}t^{-3} + \gamma_1 t^{-4}, \qquad (4.17)\\
\frac{1}{2}\left(\frac{A_1^2 + B_3^2 + B_1^2}{A_1 B_1 B_3} - \frac{A_3^2 + 2A_1^2}{A_1^2 A_3}\right) &= -1 + \frac{5}{2}t^{-1} + \gamma_2 t^{-2},
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1 &\sim -\frac{243}{16}t^{-1} + O(t^{-3}),\\
\gamma_2 &\sim -\frac{15}{4} + O(t^{-1}),
\end{aligned}$$

using $t \sim r$ as $t \to \infty$.

4.3.2 Existing results

Following [24], we can write a connection on P_1 or P_{id} over M_{BGGG} as

$$\begin{aligned}
A =& A_1 g^+ (T_1 \otimes \eta_1^+ + T_2 \otimes \eta_2^+) + A_3 f^+ T_3 \otimes \eta_3^+ \qquad (4.18)\\
&+ B_1 g^- (T_1 \otimes \eta_1^- + T_2 \otimes \eta_2^-) + B_3 f^- T_3 \otimes \eta_3^-.
\end{aligned}$$

The following proposition proves that on P_1, we need only consider the case where $f^- = g^- = 0$.

Proposition 4.19 ([24, Proposition 9])**.** *Let $X \subset S^3 \times \mathbb{R}^4$ contain the singular orbit $S^3 \times \{0\}$ of the $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-action and be equipped with an $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant G_2-metric. There is a 2-parameter family of $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant G_2-instantons A with gauge group $\mathrm{SU}(2)$ in a neighbourhood of the singular orbit in X extending smoothly over P_1.*

Moreover, any such G_2-instanton A can be written as in (4.18) with $f^- = g^- = 0$ and with f^+ and g^+ solving the ODEs

$$\dot{f}^+ + \frac{1}{2}\left(\frac{A_3}{B_1^2} - \frac{A_3}{A_1^2}\right) f^+ = -(g^+)^2 \tag{4.20}$$
$$\dot{g}^+ + \frac{1}{2}\left(\frac{A_1^2 + B_3^2 + B_1^2}{A_1 B_1 B_3} - \frac{A_3^2 + 2A_1^2}{A_1^2 A_3}\right) g^+ = -f^+ g^+$$

where $f^+ = f_1^+ t + O(t^3)$ and $g^+ = g_1^+ t + O(t^3)$ for $f_1^+, g_1^+ \in \mathbb{R}$.

In the case of the BGGG metric, we can write the coefficients in (4.20) involving the metric functions explicitly in r. We have

$$\frac{A_3}{B_1^2} - \frac{A_3}{A_1^2} = \frac{-9r}{(r-\frac{3}{4})^{\frac{3}{2}}(r+\frac{3}{4})^{\frac{3}{2}}(r+\frac{9}{4})^{\frac{1}{2}}(r-\frac{9}{4})^{\frac{1}{2}}} < 0 \tag{4.21}$$

and is increasing for all $r \geq \frac{9}{4}$, tending to 0;

$$\frac{A_1^2 + B_3^2 + B_1^2}{A_1 B_1 B_3} - \frac{A_3^2 + 2A_1^2}{A_1^2 A_3} = \frac{-2r^5 + 8r^4 - 6r^3 - \frac{243}{16}r^2 + \frac{27297}{1312}r + \frac{2187}{256}}{r(r-\frac{3}{4})^{\frac{1}{2}}(r+\frac{9}{4})^{\frac{1}{2}}(r-\frac{9}{4})^{\frac{3}{2}}(r+\frac{3}{4})^{\frac{3}{2}}} < 0$$

with a turning point near $r = 3.77$ where the function switches from increasing to decreasing before tending to -2 from above.

When $g^+ = 0$, we obtain a 1-parameter family of abelian G_2-instantons with bounded curvature. The following corollary in [24] describes them explicitly.

Lemma 4.22 ([24, Corollary 1(b)]). *Let A be an $\mathrm{SU}(2)^2$-invariant G_2-instanton with bounded curvature and gauge group $\mathrm{U}(1)$ on M_{BGGG}. Then A can be written as $A = A_3^2 x_3 \eta_3^+$ for $x_3 \in \mathbb{R}$.*

Lotay and Oliveira prove some further existence results for instantons on M_{BGGG}. We now state two theorems which describe these results; the first gives a region of initial conditions for which the corresponding local solutions either extend to abelian solutions (i.e. g^+ vanishes) or do not extend to complete bounded solutions on M_{BGGG}.

Theorem 4.23 ([24, Theorem 8]). *Let A be a $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant G_2-instanton with gauge group $\mathrm{SU}(2)$ defined in a neighbourhood of the singular orbit on M_{BGGG} smoothly extending over P_1 as given by Proposition 4.19. If $f_1^+ \leq \frac{1}{2}$, or $g_1^+ \geq 0$ with $g_1^+ \geq f_1^+$, then A extends globally to M_{BGGG} with bounded curvature if and only if A is reducible.*

The next theorem gives an existence result for complete bounded solutions corresponding to a region of initial conditions.

Theorem 4.24 ([24, Theorem 9]). *Let A be an $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant G_2-instanton with gauge group $\mathrm{SU}(2)$ defined in a neighbourhood of the singular orbit of M_{BGGG}, smoothly extending over P_1 as given by Proposition 4.19. If $f_1^+ \geq \frac{1}{2} + g_1^+ > \frac{1}{2}$, then A extends globally to M_{BGGG} with bounded curvature.*

The following lemma states that if (f^+, g^+) is a complete solution to the ODEs, then the sign of g^+ does not change, and hence we only need to consider the case where $g^+ \geq 0$.

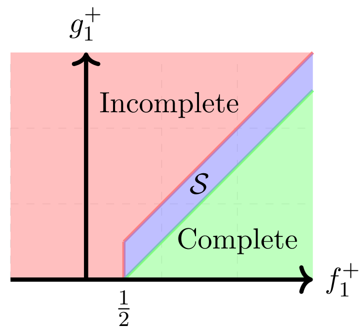

Figure 4.3: The regions of initial conditions of solutions (f^+, g^+) which are unknown (blue), complete with bounded curvature (green), or not (red). The f_1^+-axis gives a 1-parameter family of abelian solutions with bounded curvature.

Lemma 4.25 ([24, Lemma 6])**.** *Let (f^+, g^+) solve (4.20). The sign of g^+ does not change as long as f^+ does not blow up, and if $g^+(t_0) = 0$ for some $t_0 > 0$ or if $g_1^+ = 0$ then $g^+ = 0$.*

The remaining region of interest of initial conditions for f^+ and g^+ is

$$\mathcal{S} = \{(f_1^+, g_1^+) : 0 < f_1^+ - \frac{1}{2} < g_1^+ < f_1^+\}$$

as shown in Figure 4.3. The abelian solution of Theorem 4.23 with initial conditions $(f_1^+, g_1^+) = (\frac{1}{2}, 0)$ is given by $f^+ = A_3$ and $g^+ = 0$; then $f^+ \to 1$ as $t \to \infty$. Any boundary of initial conditions of complete bounded solutions existing in $\mathcal{S}$ will include this point. From Remark 18 of [24], we know that the solutions of Theorem 4.24 have g^+ decaying and f^+ tending to a finite limit $F_\infty \in \mathbb{R}$ as $t \to \infty$.

Recall from Section 1.2 that G_2-instantons on ALC G_2-manifolds are asymptotic to a U(1)-invariant connection

$$A_\infty = a + \ell\Phi \otimes \theta$$

on a principal bundle P over $(1, \infty) \times \Sigma$, and that (a, Φ) is a CY monopole on the CY cone $(1, \infty) \times X$. The BGGG metric has $\ell = 1$ and $\theta = 2\eta_3^+$, while the notation from Section 1.1 transforms to $\tilde{m} = f_1^+$ and $\tilde{a} = g_1^+$. The mass of the monopole is

$$m = \lim_{t\to\infty} |\Phi|$$

so writing A as in (4.18), we have

$$m = \lim_{t\to\infty} \left| \frac{A_3}{2} f^+ T_3 \right| = \frac{F_\infty}{2}.$$

4.3.3 Properties of solutions

In order to understand solutions to the G_2-instanton equations (4.20) and prove Theorem B, we want to study their asymptotic behaviour.

Lemma 4.26. *Suppose $f_1^+ > 0$. If $f^+(t_0) < 0$ for some $t_0 > 0$, then $f^+(t) < 0$ for all $t \geq t_0$. In particular, if $F_\infty = \lim_{t\to\infty} f^+$ exists, then $f^+ > 0$ for all $t > 0$ and $F_\infty \geq 1$.*

Proof. Suppose there exists some $t_0 > 0$ such that $f^+(t_0) \leq 0$, then

$$\dot{f}^+(t_0) = -(g^+)^2(t_0) - \frac{1}{2}\left(\frac{A_3}{B_1^2} - \frac{A_3}{A_1^2}\right) f^+(t_0) \leq 0.$$

Then f^+ is decreasing and so stays non-positive for all $t \geq t_0$.

We have

$$-(g^+)^2 = \dot{f}^+ - \frac{1}{2}\left(\frac{A_3}{B_1^2} - \frac{A_3}{A_1^2}\right) f^+ \to 0$$

so g^+ must decay in the limit. The second equation gives

$$\frac{d}{dt}\log g^+ = -\frac{1}{2}\left(\frac{A_1^2 + B_3^2 + B_1^2}{A_1 B_1 B_3} - \frac{A_3^2 + 2A_1^2}{A_1^2 A_3}\right) - f^+ \to 1 - F_\infty; \tag{4.27}$$

then g^+ decays in the limit only if $1 - F_\infty \leq 0$, and hence $F_\infty \geq 1$. □

The function g^+ decays exponentially if $F_\infty > 1$ and at a slower rate if $F_\infty = 1$. We will see that for each fixed $F_\infty \geq 1$, there is a 1-parameter family of solutions with g^+ decaying with either polynomial rate if $F_\infty = 1$ or with exponential rate if $F_\infty > 1$. The following lemma describes the decay rate of the function g^+ for a complete bounded solution (f^+, g^+).

Lemma 4.28. *Suppose (f^+, g^+) is a complete bounded solution of (4.20). We can write*

$$g^+ = \exp((1 - F_\infty)t)t^{-\frac{5}{2}}h(t) \tag{4.29}$$

for some function h satisfying

$$\dot{h} = h(F_\infty - f^+ - \gamma_2 t^{-2}) = -h(k + \gamma_2)t^{-2}. \tag{4.30}$$

Here, γ_2 is defined in (4.21) and we define k to be the bounded function satisfying

$$f^+ = F_\infty + t^{-2}k(t).$$

Proof. From (4.27), we have

$$\frac{d}{dt}\log g^+ \leq (1 - \epsilon)(1 - F_\infty)$$

on an interval $t \in [T, \infty)$ for sufficiently large $T > 0$ and for some $\epsilon > 0$. Hence

$$g^+ \exp((1 - \epsilon)(F_\infty - 1)t) \tag{4.31}$$

is a bounded function. Write the first equation of (4.20) as

$$\dot{f}^+ + (-\tfrac{9}{2}t^{-3} + \gamma_1 t^{-4})f^+ = -(g^+)^2 \tag{4.32}$$

and let

$$u(t) = \int_t^\infty (-\tfrac{9}{2}t^{-3} + \gamma_1 t^{-4})dt.$$

Then asymptotically, we have $u(t) \sim ct^{-2}$ for some $c \in \mathbb{R}$. In this notation, we can rewrite (4.32) as

$$\frac{d}{dt}(f^+e^u) = -e^u(g^+)^2$$

which asymptotically has solutions

$$f^+(t) = F_\infty e^{-u(t)} + e^{-u(t)} \int_t^\infty e^{u(t)}(g^+(t))^2 dt.$$

The integrand is bounded since (4.31) is bounded; therefore, the first term dominates the second term asymptotically. For large t, we have $f^+(t) \sim F_\infty(1 - u(t)) \sim F_\infty(1 - ct^{-2})$. Together with (4.17), we see that a series expansion of f^+ for large t is of the form, for some bounded function k,

$$f^+ = F_\infty + t^{-2}k(t). \tag{4.33}$$

Substituting this expansion into equation (4.27) and integrating yields

$$\log g^+ = (1 - F_\infty)t - \tfrac{5}{2} \log t + O(t^{-1}))$$

and hence we can write

$$g^+ = \exp((1 - F_\infty)t)t^{-\frac{5}{2}}h(t)$$

for some function h satisfying

$$\dot{h} = h(F_\infty - f^+ - \gamma_2 t^{-2}) = -h(k + \gamma_2)t^{-2}.$$

□

Hence $\frac{d}{dt} \log h$ is $O(t^{-2})$, hence integrable, and decays as $t \to \infty$. Then h is bounded for sufficiently large t.

Now suppose (f^+, g^+) is a solution to (4.20) such that $f^+ \to F_\infty > 1$ as $t \to \infty$. Let

$$f^+_{\text{ab}} = F_\infty A_3, \qquad g^+_{\text{ab}} = 0$$

be the abelian solution with $f^+_{\text{ab}} \to F_\infty$. Finally, let

$$\hat{f}^+ = \tfrac{1}{f^+_{\text{ab}}}(f^+ - f^+_{\text{ab}}) \quad \text{and} \quad \hat{g}^+ = g^+.$$

Then $\hat{f}^+$ decays as $t \to \infty$ and satisfies

$$\frac{d}{dt}\hat{f}^+ = -\tfrac{1}{f^+_{\text{ab}}}(\hat{g}^+)^2. \tag{4.34}$$

Hence we can integrate (4.34) which yields

$$\hat{f}^+(t) = -\int_t^\infty \tfrac{1}{f^+_{\text{ab}}}(\hat{g}^+)^2 = O(t^{-5}\exp(2(1 - F_\infty)t))$$

and so, on some interval (T, ∞) for sufficiently large $T \in \mathbb{R}$, we can write

$$f^+ = f^+_{\text{ab}} + o(t^{-\frac{5}{2}}\exp((1 - F_\infty)t)). \tag{4.35}$$

We now prove a couple of lemmas which describe properties of solutions (f^+, g^+) to the G_2-instantons equations and compare two distinct solutions. We first try perturbing the initial conditions in the direction of the complete solutions with bounded curvature. These lemmas will be useful when we consider a sequence of solutions whose initial conditions move in the direction of $\mathcal{S}$.

Lemma 4.36. *Suppose $(\hat{f}^+, \hat{g}^+)$ and (f^+, g^+) are two distinct solutions such that*

$$f_1^+ \geq \hat{f}_1^+ > 0 \qquad \text{and} \qquad \hat{g}_1^+ \geq g_1^+ > 0.$$

Then $f^+ > \hat{f}^+$ and $\hat{g}^+ > g^+$ for all $t > 0$.

Proof. Suppose that $f^+ = \hat{f}^+$ at some time, and let $t_0 > 0$ be the first time this happens after $t = 0$. Then

$$\frac{d}{dt}(f^+ - \hat{f}^+)(t_0) = (\hat{g}^+)^2(t_0) - (g^+)^2(t_0).$$

Since $f^+ - \hat{f}^+$ is positive in a neighbourhood of $t = 0$, then $\frac{d}{dt}(f^+ - \hat{f}^+)(t_0) \leq 0$, and hence

$$\hat{g}^+(t_0) \leq g^+(t_0) \tag{4.37}$$

as $g^+, \hat{g}^+ \geq 0$. On the other hand, we have

$$\frac{d}{dt}(g^+ - \hat{g}^+)(\epsilon) = 4(g_1^+ - \hat{g}_1^+) + [3\hat{g}_1^+(\hat{f}_1^+ - f_1^+) + C(g_1^+ - \hat{g}_1^+)]\epsilon^2 + O(\epsilon^3) < 0$$

for a function C depending on the metric functions and sufficiently small $\epsilon > 0$, i.e. $g^+ < \hat{g}^+$ in a neighbourhood of $t = 0$. Thus, for (4.37) to hold, there must exist $0 < t \leq t_0$ such that $g^+(t) = \hat{g}^+(t)$. Suppose t_1 is the first such t; t_1 and t_0 must be distinct by uniqueness of solutions. Then

$$\frac{d}{dt}(g^+ - \hat{g}^+)(t_1) = g^+(t_1)(\hat{f}^+(t_1) - f^+(t_1)) \geq 0.$$

Hence $\hat{f}^+(t_1) \geq f^+(t_1)$. This contradicts t_0 being the first time that $f^+ - \hat{f}^+$ is non-positive. Thus $f^+ > \hat{f}^+$ for all $t > 0$ and hence $\hat{g}^+ > g^+$ for all $t > 0$. □

We now see that any solution (f^+, g^+) with an initial condition satisfying the assumptions of Lemma 4.36 for a fixed complete solution $(\hat{f}^+, \hat{g}^+)$ is itself convergent, and the limit points of these two solutions are distinct.

Lemma 4.38. *With the assumptions of Lemma 4.36, if $\hat{f}^+ \to \hat{F}_\infty$, then f^+ also converges to a finite limit F_∞ and $\hat{F}_\infty < F_\infty$.*

Proof. Lemma 4.36 tells us that if we start at a complete solution anywhere in the space of positive initial conditions, then choosing another set of initial conditions satisfying the inequalities of Lemma 4.36, we still have a complete solution with f^+ tending to a finite limit $F_\infty \geq 1$. We can see this by supposing that the solution $(\hat{f}^+, \hat{g}^+)$ is complete while the solution (f^+, g^+) is not. Lemma 4.36 tells us that f^+ must go to positive infinity in the limit and further that any solution with initial conditions in the direction of the region of complete solutions must have the same property. However, if we continue to move in this direction, we will eventually hit the space of complete solutions, which have a finite limit F_∞. This leads to a contradiction.

Clearly we have $\hat{F}_\infty \leq F_\infty$. Now suppose that $\hat{F}_\infty = F_\infty$; then the function $f^+ - \hat{f}^+$ must be a positive function that tends to zero. From (4.21), we have

$$\frac{d}{dt}(f^+ - \hat{f}^+) = -\frac{1}{2}\left(\frac{A_3}{B_1^2} - \frac{A_3}{A_1^2}\right)(f^+ - \hat{f}^+) + (\hat{g}^+ - g^+)(\hat{g}^+ + g^+) > 0$$

for all $t > 0$ and so the function $f^+ - \hat{f}^+$ is an increasing function. This contradicts the assumption that it tends to zero. Thus the limits must be distinct. □

4.3.4 The existence of a boundary of solutions

We now state the main result of Section 4.3 which fully characterises the asymptotic behaviour of all complete bounded solutions to (4.20). Recall that solutions (f^+, g^+) have initial conditions (f_1^+, g_1^+).

Theorem 4.39. *The region of initial conditions which lead to complete bounded solutions is closed. It is the union of $\{g_1^+ = 0, f_1^+ < \frac{1}{2}\}$ and a closed set with boundary given by a graph of a continuous function $f_1^+ \mapsto g_1^+(f_1^+)$ which lies in the region $\mathcal{S}$ and its reflection in the f_1^+-axis. The boundary passes through the point $(f_1^+, g_1^+) = (\frac{1}{2}, 0)$ corresponding to the abelian solution with $g = 0, F_\infty = 1$, and corresponds to solutions with $\lim_{t\to\infty} f^+ = F_\infty = 1$.*

The proof of this theorem involves a number of steps which we now outline.

- We start by parameterising local solutions near the singular orbit by writing $s = t^{-1}$; Proposition 4.40 fully characterises this family of solutions, namely that $f^+ \to F_\infty$ and g^+ decays exponentially with rate $1 - F_\infty$ as $t \to \infty$, and the family depends continuously on both parameters.
- In Section 4.3.3, we showed that as we vary the initial conditions f_1^+ and g_1^+ in the direction of $\mathcal{S}$, the solutions satisfy inequalities outlined in Lemma 4.36 for all time, including in the limit as $t \to \infty$. This allows us to construct a monotonic sequence of initial conditions converging in $\mathcal{S}$. The sequence corresponds to solutions, each of which have an asymptotic limit F_∞ for f^+. Hence, we also have a sequence of real numbers $(F_\infty)_n$ that converges to some $F_\infty \geq 1$.
- Finally we consider two solutions, the first corresponding to the limit of the sequence of initial conditions described above, and the second given by the limit of the sequence $(F_\infty)_n$. We use the continuity result of Proposition 4.40 to show that these solutions coincide globally. This global solution is the limit of the aforementioned sequence of solutions, and hence we show that the set of complete bounded solutions is closed, with a boundary lying in $\mathcal{S}$ that corresponds to solutions with $F_\infty = 1$, as shown in Figure 4.4.

The first step in the proof is to parameterise the local asymptotic solutions near $s = 0$. The following proposition shows that there is a 2-parameter family of such solutions; we delay the proof until Section 4.3.5.

Proposition 4.40. *There is a 2-parameter family of smooth solutions to* (4.20) *on* (T, ∞) *for some large* $T \in \mathbb{R}$, *parameterised by* $F_\infty := \lim_{t\to\infty} f^+(t) \in [1, \infty)$ *and by* $\lambda \in \mathbb{R}$ *such that*

$$g^+ = e^{(1-F_\infty)t}\left[\lambda t^{-\frac{5}{2}} + O(t^{-\frac{7}{2}})\right]. \tag{4.41}$$

The family depends continuously on compact intervals on the parameters F_∞ *and* λ.

Additionally, in what follows, we will use the Theorem of Continuous Dependence on Initial Conditions on a closed interval, so we state it now.

Theorem 4.42 (Continuous Dependence on Initial Conditions, Theorem 12.VI [33])**.** *Let* J *be a compact interval with* $t_0 \in J$ *and let* $y_0(t)$ *be a solution of the initial value problem*

$$y' = f(t, y) \qquad \text{in } J, \qquad y(t_0) = \eta.$$

Suppose $f(t, y)$ *is Lipschitz continuous with respect to* y *on some open set containing* (t, y_0) *for* $t \in J$. *Then the solution* y_0 *depends continuously on the initial conditions. More precisely, for every* $\epsilon > 0$, *there is* $\delta > 0$ *such that if* $|\zeta - \eta| < \delta$, *then every solution* z *of the initial value problem*

$$z' = f(t, z), \qquad z(t_0) = \zeta$$

exists in all of J *and satisfies*

$$|z(t) - y_0(t)| < \epsilon \qquad \text{in } J.$$

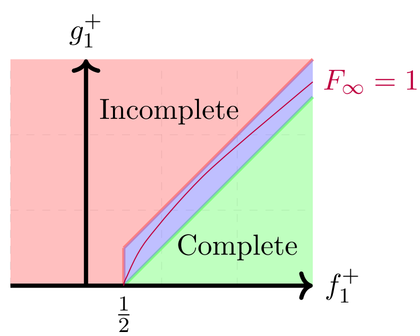

Figure 4.4: The boundary of the region of initial conditions that lead to complete bounded solutions.

We prove the first part of Theorem 4.39 by showing that the limit of every sequence of complete bounded solutions is also a complete bounded solution, and hence the set of complete bounded solutions is closed.

Suppose that $(f^+, g^+)_n$ is a sequence of complete bounded solutions to (4.20) with initial conditions $(f_1^+, g_1^+)_n$ satisfying $(f_1^+)_n > (f_1^+)_{n+1}$ and $(g_1^+)_n < (g_1^+)_{n+1}$. Denote by $(F_\infty)_n$ the sequence whose n^{th} term is $\lim_{t\to\infty}(f^+)_n$ and let $(\lambda)_n$ be the value of λ for each $(g^+)_n$, written in the form (4.41). Then Lemma 4.38 implies that $(F_\infty)_n > (F_\infty)_{n+1}$ so $(F_\infty)_n$ is decreasing and hence has a finite limit greater than or equal to 1. $(\lambda)_n$ is a sequence of positive real numbers and we see from (4.20) that it must be bounded since f^+ and $\dot{f}^+$ are both bounded. Hence, we can pass to a convergent subsequence, which we will also call $(F_\infty, \lambda)_n$, and only consider this sequence for the remainder of the argument.

We now compare two local solutions. The first is defined on (T_1, ∞) and is the solution of Proposition 4.40 given by (F^1_∞, λ^1) where

$$F^1_\infty = \lim_{n\to\infty} (F_\infty)_n = \lim_{n\to\infty} \lim_{t\to\infty} (f^+)_n \qquad \text{and} \qquad \lambda^1 = \lim_{n\to\infty} \lim_{t\to\infty} \left(e^{((F_\infty)_n - 1)t} t^{\frac{5}{2}} g^+ \right)$$

while the second is the solution (f^+_∞, g^+_∞) of Proposition 4.19, defined on $[0, T_2)$ and given by

$$(f^+_1)_\infty = \lim_{n\to\infty} (f^+_1)_n \qquad \text{and} \qquad (g^+_1)_\infty = \lim_{n\to\infty} (g^+_1)_n.$$

Proposition 4.43. f^+_∞ *extends to a solution on* $[0, \infty)$ *with* $\lim_{t\to\infty} f^+_\infty = F^1_\infty$.

Proof. Let z_1 be the f^+-component of the solution given by the end conditions (F^1_∞, λ^1) and $z_2 = f^+_\infty$. Choose some large time T that lies in the interval of existence of z_1.

Since the sequence $(f^+_1)_n$ of initial conditions converges, then for every $\delta > 0$, there exists N such that for all $n > N$,

$$|(f^+_1)_n - (f^+_1)_\infty| < \delta.$$

Together with Theorem 4.42, for every $\epsilon > 0$, there exists δ_1 and hence N_1 such that for all $n > N_1$,

$$|(f^+_1)_n - (f^+_1)_\infty| < \delta_1 \implies |(f^+)_n(T) - (f^+)_\infty(T)| = |(f^+)_n(T) - z_2(T)| < \frac{\epsilon}{2}.$$

Since the sequence $(F_\infty)_n$ converges, then for every $\delta > 0$, there exists N such that for all $n > N$,

$$|(F_\infty)_n - F^1_\infty| < \delta.$$

By continuous dependence on F_∞ from Proposition 4.40, for every $\epsilon > 0$, there exists δ_2 and hence N_2 such that for all $n > N_2$,

$$|(F_\infty)_n - F^1_\infty| < \delta_2 \implies |(f^+)_n(T) - z_1(T)| < \frac{\epsilon}{2}.$$

Hence, for all $\epsilon > 0$ and all $n > \max\{N_1, N_2\}$,

$$|z_1(T) - z_2(T)| \leq |z_1(T) - (f^+)_n(T)| + |(f^+)_n(T) - z_2(T)| < \epsilon.$$

So $z_1(T) = z_2(T)$ and hence the solutions z_1 and z_2 coincide on $[T, \infty)$. Therefore f^+_∞ is defined on $[0, \infty)$ and $\lim_{t\to\infty} f^+_\infty = F^1_\infty$. □

We have shown that if we have a sequence of complete bounded solutions whose initial conditions $(f^+_1, g^+_1)_n$ satisfy $(f^+_1)_n > (f^+_1)_{n+1}$ and $(g^+_1)_n < (g^+_1)_{n+1}$, then the limit of this sequence is also a complete bounded solution. This shows that the region of initial conditions which lead to complete bounded solutions is closed.

It remains to show that the boundary corresponds to solutions with $F_\infty = 1$. We do this by showing that the set of initial conditions leading to complete bounded solutions with $F_\infty > 1$

is open. Consider the map sending an open subset of $\mathbb{R}^2$ parameterised by $F_\infty > 1$ and $\lambda \in \mathbb{R}$ to the evaluation of the corresponding solutions of Proposition 4.40 at some large time T. This map must be injective; indeed, if two values of parameters were mapped to the same point at T, this would contradict uniqueness of solutions when running the ODE forward in time starting at T.

Proposition 4.40 states that this map is continuous; we can apply the Invariance of Domain Theorem which states that a continuous, injective map from an open subset of $\mathbb{R}^n$ is a homeomorphism onto its image. Thus both maps which send either the initial conditions or the end conditions to the value of the corresponding solution at some large T are homeomorphisms. By composing the latter with the inverse of the former, we see that the set of initial conditions that lead to complete bounded solutions with $F_\infty > 1$ is open.

Finally, we claim that the boundary is a graph of the function $f_1^+ \mapsto g_1^+(f_1^+)$. Suppose that there are two points on the boundary with the same value of f_1^+; then one has a smaller value of g_1^+ and hence by Lemma 4.38, the solution has a larger value of F_∞. This contradicts these initial conditions being on the boundary of those which lead to complete bounded solutions. So the boundary can be described as a graph of a continuous function $f_1^+ \mapsto g_1^+(f_1^+)$. Hence, subject to proving Proposition 4.40, we have proved Theorem 4.39.

Remark 4.44. Recall that these G_2-instantons correspond to a CY monopole with mass

$$m = \lim_{t\to\infty} \left| \frac{A_3}{2} f^+ T_1 \right| = \frac{F_\infty}{2}.$$

Hence we can describe the region of initial conditions which lead to complete bounded solutions as the union of a line of abelian solutions and a set with boundary given by the initial conditions of G_2-instantons asymptotic to the connection A_∞, defined in Section 1.2, with $f_1^+ > 0$ and corresponding CY monopoles with mass $m = \frac{1}{2}$, as shown in Figure 1.4.

The set of initial conditions in this union corresponding to abelian instantons give solutions which are rigid for each fixed mass m. On the other hand, for each $m \geq \frac{1}{2}$ and $f_1^+ > 0$, we are able to perturb the abelian solutions in a 1-parameter family of G_2-instantons with corresponding CY monopoles of the same mass. When $m = \frac{1}{2}$, the perturbations have g^+ decaying asympotically at a polynomial rate, while for $m > \frac{1}{2}$, it decays exponentially fast. ♢

4.3.5 Parameterising end solutions

In this section, we prove Proposition 4.40, a parameterisation of the end solutions, i.e. local solutions on (T, ∞) for some $T \in \mathbb{R}$; we first outline the proof.

- We start with the case of $F_\infty = 1$, since it is a simpler situation and we can apply the usual singular initial value problem arguments to yield a 1-parameter family of local solutions.
- We then consider the case when F_∞ is any real number and see that the ODEs take the form of an irregular singular IVP (ISIVP). For each F_∞, there is an abelian solution with $g_1^+ = 0$ given in Lemma 4.22.

- For a fixed $F_\infty > 1$, we find in Proposition 4.53 a 1-parameter family of solutions which differ from the abelian solution by an exponentially small function and show that this 2-parameter family gives the only local solutions to the ISIVP. This family is characterised by the properties that $f^+ \to F_\infty$ and g^+ decays exponentially with rate $1 - F_\infty$ as $t \to \infty$; the family depends continuously on both parameters.

For general $F_\infty \in \mathbb{R}$, we set up the problem as a singular initial value problem using the formulation of Section 4.3.2 and, in particular, equations (4.33) and (4.29). Write

$$f^+(t) = F_\infty - t^{-2}(\tfrac{9}{4}F_\infty + X(t)) \qquad \text{and} \qquad g^+(t) = t^{-\frac{5}{2}}Y(t)$$

and let $x = (X, Y)^T \in \mathbb{R}^2$. Then for $s = \frac{1}{t}$, we can rewrite (4.20) as

$$\begin{aligned} \frac{dX}{ds} &= -2s^{-1}X - F_\infty\gamma_1 - (\tfrac{81}{8}F_\infty + \tfrac{9}{2}X + Y^2)s + \gamma_1(X + \tfrac{9}{2}F_\infty)s^2 \\ \frac{dY}{ds} &= s^{-2}(F_\infty - 1)Y + (\gamma_2 - \tfrac{9}{4}F_\infty - X)Y \end{aligned}$$

or more concisely,

$$\frac{dx}{ds} = s^{-2}\Lambda x + s^{-1}Ax + B_{F_\infty}(x, s) \tag{4.45}$$

where

$$\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & F_\infty - 1 \end{pmatrix} \text{ and } A = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Case: $\mathbf{F_\infty = 1}$

The case where $F_\infty = 1$ is exceptional in the sense that $\Lambda = 0$ and so the singular IVP is called regular. In this case, g^+ only has a polynomial decay rate. We will use the following theorem which provides local solutions to such singular IVPs.

Theorem 4.46 ([16, Theorem 4.3])**.** *Consider the singular initial value problem*

$$\dot{y} = \frac{1}{t}M_{-1}(y) + M(t, y), \qquad y(0) = y_0,$$

where y takes values in $\mathbb{R}^k$, $M_{-1} : \mathbb{R}^k \to \mathbb{R}^k$ is a smooth function of y in a neighbourhood of y_0 and $M : \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^k$ is smooth in t, y in a neighbourhood of $(0, y_0)$. Assume that

(i) $M_{-1}(y_0) = 0$*;*

(ii) $hId - d_{y_0}M_{-1}$ *is invertible* $\forall\, h \in \mathbb{N}$*,* $h \geq 1$*.*

Then there exists a unique solution $y(t)$ in a sufficiently small neighbourhood of 0. Furthermore, y depends continuously on y_0 satisfying (i) and (ii).

The following proposition gives a generalised power series expansion for solutions (f^+, g^+) on some interval (T, ∞).

Proposition 4.47. *Let $P_1 \to M_{BGGG}$ be the principal bundle corresponding to the trivial isotropy homomorphism. Let A be a G_2-instanton on P_1 and suppose the function $f^+ \to 1$ as $t \to \infty$. Then there exists a 1-parameter family of local solutions to the G_2-instanton ODEs on $(T, \infty) \times S^3 \times S^3$ for some $T > 0$. In particular, there exists $\lambda \in \mathbb{R}$ such that we can write*

$$
\begin{aligned}
f^+ &= 1 - \frac{9}{4}t^{-2} + O(t^{-4}) \\
g^+ &= \lambda t^{-\frac{5}{2}} + O(t^{-\frac{7}{2}}).
\end{aligned}
$$

Proof. We write

$$
f^+(t) = 1 - t^{-2}X(t) \qquad \text{and} \qquad g^+(t) = t^{-\frac{5}{2}}Y(t)
$$

and change variables $s = \frac{1}{t}$. Then the ODEs transform to

$$
\begin{aligned}
\frac{dX}{ds} &= \frac{1}{s}\left(\frac{9}{2} - 2X(s)\right) + O(1) \\
\frac{dY}{ds} &= O(1)
\end{aligned}
$$

with $X(0) = \frac{9}{4}$ and $Y(0) = \lambda \in \mathbb{R}$. This system is of the form

$$
\dot{y} = \frac{1}{s}M_{-1}(y) + M(s, y)
$$

with $M_{-1}(y(0)) = 0$ and $d_{y(0)}M_{-1} = \mathrm{diag}(-2, 0)$. Hence, for all $h \in \mathbb{N}_{\geq 1}$, $h\mathrm{Id} - d_{y(0)}M_{-1}$ is invertible. Thus, by Theorem 4.46, there exists a unique solution in a neighbourhood of $s = 0$. □

Case: General $\mathbf{F}_\infty$

When $F_\infty \neq 1$, Λ is non-zero and (4.45) is known as an irregular singular initial value problem (ISIVP). We have seen that g^+ decays exponentially if and only if $F_\infty > 1$, so we only consider this case. For each $F_\infty < 1$, we get a single bounded solution, the abelian instanton described in Lemma 4.22.

Proposition 4.40 parameterises the local solutions via two parameters F_∞ and λ. In this section, we shall construct these solutions, defined on some small interval, and write them as

$$
x_{F_\infty,\lambda} : (0, \epsilon) \to \mathbb{R}^2 .
$$

When $\lambda = 0$, we reproduce the abelian solutions of Lemma 4.22 by setting $x_3 = F_\infty$. From the explicit description of this family, we see that it depends continuously on the parameter F_∞.

We now find the remaining end solutions by adding a further parameter. For a fixed $F_\infty > 1$, consider the abelian solution $x_{F_\infty,0} = (X, 0)^T$ given by

$$
X = -\tfrac{9}{4}F_\infty + t^2 F_\infty(1 - A_3)
$$

and write

$$
\bar{x} = x_{F_\infty,0} + \exp((1 - F_\infty)s^{-1})z \tag{4.48}
$$

bearing in mind that (4.29) tells us that if $\bar{x}$ is a solution, then z converges. We consider parameterising solutions by the initial condition $z(0) = (0, \lambda)^T$ for $\lambda \in \mathbb{R}$, and so we write

$$z = \begin{pmatrix} 0 \\ \lambda \end{pmatrix} + w.$$

Then $\bar{x}(0) = 0$ and $\bar{x}$ solves (4.45) if $w = (Z, W)^T$ solves

$$\frac{dw}{ds} = s^{-2}\tilde{\Lambda}w + s^{-1}\tilde{A}w + \tilde{B}_\lambda(w, s) \tag{4.49}$$

where

$$\tilde{\Lambda} = \begin{pmatrix} 1 - F_\infty & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\tilde{B}_\lambda(w, s) = \begin{pmatrix} (\gamma_1 s^2 - \frac{9}{2}s)Z - \exp((1 - F_\infty)s^{-1})s(W + \lambda)^2 \\ (\gamma_2 + s^{-2}F_\infty(A_3 - 1) - \exp((1 - F_\infty)s^{-1})Z)(W + \lambda) \end{pmatrix}.$$

For $\lambda = 0$, $w = 0$ is a solution to (4.49) and we obtain the abelian solution $\bar{x} = x_{F_\infty,0}$. We will now prove that for each $\lambda \in \mathbb{R}$, we get a unique solution w to (4.49).

Define the linear operators $R_Z : C^0((0, \epsilon), \mathbb{R}) \to C^1((0, \epsilon), \mathbb{R})$, parameterised by $\mu \geq 0$, by

$$R_Z(c; \mu) = s^{-2}\exp(\mu s^{-1})\int_0^s r^2 \exp(-\mu r^{-1})c(r)dr. \tag{4.50}$$

This choice of operator was inspired by discussions with, and the work of, Mark Haskins and Johannes Nordström. Let $Z = R_Z(c; F_\infty - 1)$, then on $(0, \frac{\epsilon}{2})$, Z solves

$$\frac{dZ}{ds} = (1 - F_\infty)s^{-2}Z(s) - 2s^{-1}Z(s) + c(s).$$

Hence R_Z is a good model for finding a solution to (4.49). We need some control on the operator in order to prove Proposition 4.53; the next lemma gives that control.

Lemma 4.51. *The operator $R_Z : C^0((0, \epsilon), \mathbb{R}) \to C^1((0, \epsilon), \mathbb{R})$ is bounded with respect to the C^0-operator norm for all $\mu \geq 0$; indeed, $\|R_Z\|_{op} \leq \frac{\epsilon}{3}$. $\|c\|_{C^k}$ controls $\|s^{-2}R_Z(c; \mu)\|_{C^k}$ for $\mu \neq 0$.*

Proof. We first prove the claim that the operator norm of R_Z is bounded. For $\mu \geq 0$, we have

$$\|R_Z(c; \mu)\|_{C^0} \leq \left\| s^{-2}\int_0^s r^2 \exp(\mu(s^{-1} - r^{-1}))dr \right\|_{C^0} \|c\|_{C^0} \leq \tfrac{\epsilon}{3}\|c\|_{C^0}$$

since $s^{-1} - r^{-1} \leq 0$ on $(0, s)$, so $\|R_Z\|_{\text{op}} \leq \frac{\epsilon}{3}$.

Now consider controlling $s^{-2}R_Z$ and let $u = r^{-1} - s^{-1}$. Then

$$s^{-2}R_Z(c; \mu)(s) = \int_0^\infty \exp(-\mu u)c(\tfrac{s}{1+su})(1 + su)^{-4}du.$$

By considering $r(u, s) = s(1 + su)^{-1}$, we have

$$\frac{\partial^i r}{\partial s^i} = i!(-u)^{i-1}(1 + su)^{-i-1}$$

which is bounded for small s by a polynomial in u. Let $C(u,s) = c(\frac{s}{1+su})(1+su)^{-4}$, then

$$\frac{\partial^i}{\partial s^i} C(u,s) = -\sum_{j=0}^{i} c^{(j)}(r) \sum_{k=0}^{i-j} C_k^{i,j} \left(\frac{\partial r}{\partial s}, \dots, \frac{\partial^i r}{\partial s^i} \right) (1+su)^{k+j-i-4} u^{i-k-j}$$

where $C_k^{i,j}$ is a polynomial in the partial derivatives of r with respect to s. Thus, for small s, $\frac{\partial^i}{\partial s^i} C(u,s)$ is bounded by $Q(u)\|c\|_{C^k}$ where $Q(u)$ is a polynomial in u. Provided that μ is positive, then the integral converges. So if $\mu \neq 0$, we have shown that $\|s^{-2} R_Z(c;\mu)\|_{C^k}$ is bounded by $\|c\|_{C^k}$. □

As well as the above regularity result for R_Z, we also require that the operator is continuous with respect to μ. That is the result of the next lemma.

Lemma 4.52. *The operator R_Z is continuous with respect to $\mu > 0$ and is continuous from the right as $\mu \to 0$.*

Proof. Suppose $\mu, \tilde{\mu} > 0$ and $\tilde{\mu} = \mu + \alpha$ for some $\alpha > 0$. Then

$$R_Z(c;\mu) - R_Z(c;\tilde{\mu}) = s^{-2} \int_0^s r^2 \big(\exp(\mu(s^{-1} - r^{-1})) - \exp(\tilde{\mu}(s^{-1} - r^{-1})) \big) c(r) dr.$$

Let $x = r^{-1} - s^{-1}$ and define the function

$$g(x) = \exp(-\mu x) - \exp(-(\mu + \alpha)x)$$

which has a maximum at

$$\bar{x} = \frac{1}{\alpha} \log \left(1 + \frac{\alpha}{\mu} \right).$$

Then using, for $x > 0$,

$$\log(x) \geq \frac{x-1}{x},$$

we have

$$\bar{x} \geq \frac{1}{\mu + \alpha}.$$

Hence

$$-\mu \bar{x} \leq \frac{-\mu}{\mu + \alpha} \leq -\frac{1}{2}$$

for sufficiently small α. Thus, we have

$$\begin{aligned} g(x) \leq g(\bar{x}) &= \exp(-\mu \bar{x})(1 - \exp(-\alpha \bar{x})) \\ &= \left(\frac{\alpha}{\mu + \alpha} \right) \exp(-\mu \bar{x}) \\ &\leq \exp(-\tfrac{1}{2}) \left(\frac{\alpha}{\mu + \alpha} \right) \xrightarrow{\alpha \to 0} 0 \end{aligned}$$

for sufficiently small α. Hence

$$|R_Z(c;\mu) - R_Z(c;\tilde{\mu})| = \left| s^{-2} \int_0^s r^2 g(r^{-1} - s^{-1}) c(r) dr \right|$$

is controlled by $\alpha = \tilde{\mu} - \mu$, giving continuity of R_Z away from $\mu = 0$.

Now suppose that $\mu = 0$ and $\tilde{\mu} > 0$. Then

$$R_Z(c;0) - R_Z(c;\tilde{\mu}) = s^{-2}\int_0^s r^2\big(1 - \exp(\tilde{\mu}(s^{-1} - r^{-1}))\big)c(r)dr$$

and we have

$$r^2(1 - \exp(\tilde{\mu}(s^{-1} - r^{-1}))) < r^2\tilde{\mu}(r^{-1} - s^{-1}) = \tilde{\mu}(r - r^2 s^{-1})$$

for $0 \le r < s$. Thus

$$|R_Z(c;0) - R_Z(c;\tilde{\mu})| < \left| s^{-2}\int_0^s \tilde{\mu}(r - r^2 s^{-1})c(r)dr \right| \le \tfrac{1}{6}\tilde{\mu}\|c\|_{C^0}$$

and hence $|R_Z(c;0) - R_Z(c;\tilde{\mu})|$ is controlled by $\tilde{\mu}$, giving continuity of R_Z from the right as the parameter $\mu \to 0$. □

Now that we have some control on the operators R_Z, we next find a further 1-parameter family of solutions for each fixed $F_\infty > 1$. The family is constructed by adding exponentially small terms to the abelian solution $x_{F_\infty,0}$ of Lemma 4.22.

Proposition 4.53. *For each $F_\infty > 1$ and each $\lambda \in \mathbb{R}$,* (4.45) *has a smooth solution*

$$x_{F_\infty,\lambda} = x_{F_\infty,0} + \exp((1 - F_\infty)s^{-1})z$$

on $(0,\epsilon)$, for sufficiently small $\epsilon > 0$, and $z(s) \to (0,\lambda)^T$ as $s \to 0$.

Proof. Let $D := \frac{d}{ds} - s^{-2}\tilde{\Lambda} - s^{-1}\tilde{A}$; then $R = (R_Z(\cdot\,;F_\infty - 1), \int_0^s \cdot\, dt)$ is a right inverse of D. Recall that we write

$$z = \begin{pmatrix} 0 \\ \lambda \end{pmatrix} + w$$

for $\lambda \in \mathbb{R}$. To solve the full system (4.49), we seek w such that

$$Dw = \tilde{B}_\lambda(w,s)$$

where $\tilde{B}_\lambda$ is uniformly Lipschitz in w and so satisfies

$$\|\tilde{B}_\lambda(w_1,s) - \tilde{B}_\lambda(w_2,s)\| \le C\|w_1 - w_2\|$$

for $s \in (0,\epsilon)$ and Lipschitz constant C. There is a well-defined map

$$F : C^0 \mapsto C^0,\ y \mapsto s^{-1}R(\tilde{B}_\lambda(sy,s))$$

and any fixed point of this map gives a solution $w = sy$ to (4.49). Then

$$\|F(y_1) - F(y_2)\|_{C^0} \le \frac{C\epsilon^2}{3}\|y_1 - y_2\|_{C^0}$$

and choosing ϵ small enough yields a contraction. The Contraction Mapping Theorem then gives a fixed point of the map F, and hence a solution to the ODE.

We prove that such a solution is smooth via a bootstrapping argument. Suppose that w is C^k, then

$$\begin{aligned}\|w\|_{C^{k+1}} &= \|w\|_{C^0} + \|w'\|_{C^k} \\ &\leq \|w\|_{C^0} + \|(1-F_\infty)s^{-2}Z\|_{C^k} + \|2s^{-1}Z\|_{C^k} + \|\tilde{B}_\lambda(w,s)\|_{C^k}.\end{aligned}$$

$\tilde{B}_\lambda(w,s)$ is C^k, so by considering $\tilde{B}_\lambda(w,s)$ for a fixed w, we can apply Lemma 4.51 to see that $\|s^{-2}Z\|_{C^k}$ and $\|s^{-1}Z\|_{C^k}$ are both bounded by $\|\tilde{B}_\lambda(w,s)\|_{C^k}$. Hence the C^k solution w is indeed C^{k+1}. As $\tilde{B}_\lambda$ is smooth then by induction, the C^0 solution $w = sy$ is also smooth. □

Proposition 4.54. *Any solution of* (4.20) *with $f^+ \to F_\infty$ as $t \to \infty$ is given by $x_{F_\infty,\lambda}$ for some $\lambda \in \mathbb{R}$.*

Proof. Recall that $w = (Z, W)^T$. We note that $RDw = (Z(s), W(s) - W(0))^T$ and hence R is also a left inverse of D when $W(0) = 0$. We know from (4.35) that any solution to (4.49) must satisfy $Z(0) = 0$ since $f^+ - f^+_{\mathrm{ab}}$ decays faster than $t^{-\frac{5}{2}}\exp((1-F_\infty)t^{-1})$. Hence, for initial conditions $w(0) = 0$, the operator R is both a left and a right inverse of D. Thus, for each $F_\infty > 1$ and $\lambda \in \mathbb{R}$, D is invertible and the above solution $w = sy$ is the unique solution to (4.49). □

Remark 4.55. We can also prove that we have found all end solutions via a gauge fixing argument. The operators R_Z can be extended to a 2-parameter family, parameterised by $\mu, \nu \in \mathbb{R}$, and defined as

$$R_Z(c;\mu,\nu) = \begin{cases} s^{-\nu}\exp(\mu s^{-1}) \displaystyle\int_0^s r^\nu \exp(-\mu r^{-1}) c(r)dr & \text{if } \mu \geq 0 \\ -s^{-\nu}\exp(\mu s^{-1}) \left(\displaystyle\int_s^\infty r^\nu \exp(-\mu r^{-1})\rho(r)c(r)dr - \int_0^\infty r^\nu \rho(r)c(r)dr \right) & \text{if } \mu \leq 0 \end{cases} \tag{4.56}$$

where ρ is a cut off function defined to be $\rho = 1$ on $[0, \frac{\epsilon}{2}]$ and $\rho = 0$ on $[\epsilon, \infty)$. Then, for a fixed $\lambda \in \mathbb{R}$, any solution w of (4.49) also solves the linear ODE

$$\frac{dx}{ds} = s^{-2}\tilde{\Lambda}x + s^{-1}\tilde{A}x + \Delta_\lambda(w(s), s)x \tag{4.57}$$

where $\Delta_\lambda : \mathbb{R}^2 \times \mathbb{R} \to \mathrm{Mat}_{2\times 2}$ is smooth. Fixing $\lambda = 0$, we can show that there exists a smooth map $G : I \to \mathrm{Mat}_{2\times 2}$ with $G(0) = \mathrm{Id}$ such that (4.57) is equivalent to

$$\frac{dy}{ds} = s^{-2}\tilde{\Lambda}y + s^{-1}\tilde{A}y, \qquad y(s) = G(s)w(s). \tag{4.58}$$

By eliminating the non-singular term, we can solve the equation explicitly. The general solution to (4.58) on some interval $(0, \epsilon)$ is

$$y(s) = \begin{pmatrix} 0 \\ \xi \end{pmatrix} + \exp((F_\infty - 1)s^{-1})s^{-2} \begin{pmatrix} \eta \\ 0 \end{pmatrix}$$

for $\eta, \xi \in \mathbb{R}$. The map G satisfies

$$\frac{dG}{ds} = s^{-2}\,\mathrm{ad}_{\tilde{\Lambda}}\, G + s^{-1}\,\mathrm{ad}_{\tilde{A}}\, G + G\Delta_0 \tag{4.59}$$

or more explicitly

$$\frac{dG}{ds} = s^{-2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & F_\infty - 1 & 0 & 0 \\ 0 & 0 & 1-F_\infty & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G + s^{-1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} G + G\Delta_0$$

where we let $\mathrm{Mat}_{2\times 2} \cong \mathbb{R}^4$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a,c,b,d)^T$. Then a solution to this equation is given by the operator

$$\left(\int_0^s \cdot\, dt,\ R_Z(\cdot\,; 1-F_\infty, 2),\ R_Z(\cdot\,; F_\infty - 1, -2),\ \int_0^s \cdot\, dt\right).$$

The general solution to (4.49) is

$$w(s) = G(s)^{-1}\left(\begin{pmatrix} 0 \\ \xi \end{pmatrix} + \exp((F_\infty - 1)s^{-1})s^{-2}\begin{pmatrix} \eta \\ 0 \end{pmatrix}\right)$$

for each $\xi, \eta \in \mathbb{R}$. Write

$$G(s)^{-1} = \begin{pmatrix} a(s) & b(s) \\ c(s) & d(s) \end{pmatrix};$$

then $a(s), d(s) \to 1$ and $b(s), c(s) \to 0$ as $s \to 0$. We can write

$$\bar{x} = x_{F_\infty,0} + \xi \exp\left((1-F_\infty)s^{-1}\right)\begin{pmatrix} b(s) \\ d(s) \end{pmatrix} + \eta s^{-2}\begin{pmatrix} a(s) \\ c(s) \end{pmatrix}$$

and we can see that $\bar{x}$ blows up as $s \to 0$ unless $\eta = 0$. Thus we have a 1-parameter family of solutions depending on $\xi \in \mathbb{R}$ and $\bar{x}$ corresponds to the solution $x_{F_\infty,\xi}$ under the gauge transformation G. ◊

Finally, we want to show that the solutions depend continuously on the parameters.

Proposition 4.60. *For $F_\infty \geq 1$ and $\lambda \in \mathbb{R}$, the map sending the pair (F_∞, λ) to the solutions $x_{F_\infty,\lambda}$ of Proposition 4.47 and Proposition 4.53 is C^0-continuous on compact intervals.*

Proof. We first fix $\lambda \in \mathbb{R}$ and show that the map is continuous with respect to F_∞. The abelian solutions $x_{F_\infty,0}$ are given explicitly in Theorem 4.22 and hence are continuous with respect to F_∞. We have also shown in Lemma 4.52 that R_Z is continuous with respect to F_∞, and hence so is z. Then the function $\exp((1-F_\infty)t)z$ is continuous with respect to F_∞ in any compact interval for large times t. This is sufficient to see that $x_{F_\infty,\lambda} = x_{F_\infty,0} + \exp((1-F_\infty)t)z$ for any fixed $\lambda \in \mathbb{R}$ is continuous with respect to F_∞.

We now allow $\lambda \in \mathbb{R}$ to vary and show that the map is uniformly continuous with respect to λ for F_∞ in a compact set. For $F_\infty > 1$ and $\epsilon > 0$ sufficiently small, let w_λ denote the solution from Proposition 4.53 corresponding to $\lambda \in \mathbb{R}$, and for $F_\infty = 1$, $w_\lambda = \lambda O(t^{-1})$. Let λ, λ' be such that $|\lambda - \lambda'| < \delta$ for some $\delta > 0$.

$$\begin{aligned}\|x_{F_\infty,\lambda} - x_{F_\infty,\lambda'}\|_{C^0} &= \left\|\exp\left((1-F_\infty)s^{-1}\right)\left(\begin{pmatrix} 0 \\ \lambda - \lambda' \end{pmatrix} + w_\lambda - w_{\lambda'}\right)\right\|_{C^0} \\ &\leq |\lambda - \lambda'| + \|w_\lambda - w_{\lambda'}\|_{C^0}\end{aligned}$$

For $F_\infty = 1$, we have $w_\lambda - w_{\lambda'} = (\lambda - \lambda')O(t^{-1})$ while for $F_\infty > 1$,

$$w_\lambda - w_{\lambda'} = R(\tilde{B}_\lambda(w_\lambda, s) - \tilde{B}_{\lambda'}(w_{\lambda'}, s)).$$

Then, since we know $\tilde{B}_\lambda$ is uniformly Lipschitz with Lipschitz constant C from the proof of Proposition 4.53, we have

$$\begin{aligned}\|\tilde{B}_\lambda(w_\lambda, s) - \tilde{B}_{\lambda'}(w_{\lambda'}, s)\|_{C^0} &\leq \|\tilde{B}_\lambda(w_\lambda, s) - \tilde{B}_\lambda(w_{\lambda'}, s)\|_{C^0} + \|\tilde{B}_\lambda(w_{\lambda'}, s) - \tilde{B}_{\lambda'}(w_{\lambda'}, s)\|_{C^0} \\ &\leq C\|w_\lambda - w_{\lambda'}\|_{C^0} + \|\tilde{B}_\lambda(w_{\lambda'}, s) - \tilde{B}_{\lambda'}(w_{\lambda'}, s)\|_{C^0}\end{aligned}$$

and

$$\tilde{B}_\lambda(w_{\lambda'}, s) - \tilde{B}_{\lambda'}(w_{\lambda'}, s) = (\lambda' - \lambda)\begin{pmatrix} \exp\left((1 - F_\infty)s^{-1}\right)s(2W + \lambda + \lambda') \\ X + \exp\left((1 - F_\infty)s^{-1}\right)Z - \gamma_2 + \frac{9}{4}F_\infty \end{pmatrix}$$

is controlled by $|\lambda - \lambda'| < \delta$.

By Lemma 4.51, R_Z has operator norm bounded by $\frac{\epsilon}{3}$ on $C^0((0, \epsilon), \mathbb{R})$. Hence

$$\begin{aligned}\|w_\lambda - w_{\lambda'}\|_{C^0} &\leq \tfrac{\epsilon}{3}\|\tilde{B}_\lambda(w_\lambda, s) - \tilde{B}_{\lambda'}(w_{\lambda'}, s)\|_{C^0} \\ &\leq \tfrac{C\epsilon}{3}\|w_\lambda - w_{\lambda'}\|_{C^0} + \tfrac{\epsilon}{3}\|\tilde{B}_\lambda(w_{\lambda'}, s) - \tilde{B}_{\lambda'}(w_{\lambda'}, s)\|_{C^0}\end{aligned}$$

and so

$$\|w_\lambda - w_{\lambda'}\|_{C^0} \leq \frac{\epsilon}{3 - C\epsilon}\|\tilde{B}_\lambda(w_{\lambda'}, s) - \tilde{B}_{\lambda'}(w_{\lambda'}, s)\|_{C^0}.$$

Thus, for sufficiently small $\epsilon > 0$, $\|w_\lambda - w_{\lambda'}\|_{C^0}$ is controlled by $|\lambda - \lambda'| < \delta$ for all $F_\infty \geq 1$ and in turn, so is $\|x_{F_\infty,\lambda} - x_{F_\infty,\lambda'}\|_{C^0}$. Since this bound is uniform for F_∞ in a compact set, we see that the map $(F_\infty, \lambda) \mapsto x_{F_\infty,\lambda}$ is continuous on compact sets. □

Propositions 4.47 and 4.53 yield smooth local solutions to (4.45) for each $F_\infty \geq 1$ and $\lambda \in \mathbb{R}$. We have now continuously parametrised all local solutions with $F_\infty \geq 1$ and proved Proposition 4.40.

CHAPTER 5

G_2-INSTANTONS ON $M_{m,n}$

In this chapter, we consider constructing G_2-instantons on the $\mathbb{C}_7$ family and its infinite extension. We start by writing the instanton equations for $M_{m,n}$. We next find a general form for the invariant connections on the principal orbits, and consider the conditions required for these connections to extend over the singular orbit. Finally, we find a 1-parameter family of solutions in the case where $m = n = 1$, and describe why the method employed cannot easily be generalised to when m and n are any positive coprime integers. The material in this section, in the specific case where $m = n = 1$, forms the substance of the preprint [26] (joint work with Johannes Nordström and Karsten Matthies). In this thesis, we are restricting our attention to the AC limits of the $\mathbb{C}_7$ family and its infinite extension; recall that these families are parameterised by $\beta \in (0, \beta_{ac}]$ and that the value β_{ac} gives the AC metric. For notational ease, we shall simply write β throughout Chapter 5 when we are referring to the value β_{ac}.

5.1 G_2-instanton Evolution Equations for $M_{m,n}$

We saw in Section 2.3.2 that any G_2-instanton on the manifold $M = (0, \infty) \times N$, defined as a 1-parameter family of connections $\alpha(t)$ on the principal orbits N, must satisfy the equation

$$J_t \dot{\alpha} = - *_t (F_\alpha \wedge \operatorname{Im} \Omega),$$

subject to the constraint $\Lambda_t F_\alpha = 0$. We now derive these equations when $N = \mathrm{SU}(2)^2 / \mathbb{Z}_{2(m+n)}$.

We have a formula for $\operatorname{Im}\Omega$ from [16], namely

$$\begin{aligned} 2\dot{a}^2\dot{b}\operatorname{Im}\Omega \;=\;& (2a^2b + m^2r_0^3(2a^2+b^2-m^2n^2r_0^6))\,e_{123} \\ &+ (2a^2b + n^2r_0^3(2a^2+b^2-m^2n^2r_0^6))\,e_{1'2'3'} \\ &- (a(b^2+m^2n^2r_0^6) + 2abn^2r_0^3)(e_{12'3'} + e_{23'1'}) \\ &+ (b(b^2-2a^2-m^2n^2r_0^6) - 2a^2n^2r_0^3)\,e_{31'2'} \\ &+ (b(b^2-2a^2-m^2n^2r_0^6) - 2a^2m^2r_0^3)\,e_{3'12} \\ &- (a(b^2+m^2n^2r_0^6) + 2abm^2r_0^3)(e_{1'23} + e_{2'31}) \end{aligned}$$

where e_{123} denotes $e_1\wedge e_2\wedge e_3$, $e_{1'2'3'}$ denotes $e_1'\wedge e_2'\wedge e_3'$ and so on. The coefficients on the right hand side of this equation appear frequently in the derivation and depend only on the G_2-structure on $M_{m,n}$, so we choose to rename them as follows:

$$\begin{aligned} \Phi_m &= 2a^2b + m^2r_0^3(2a^2+b^2-m^2n^2r_0^6), & \Phi_n &= 2a^2b + n^2r_0^3(2a^2+b^2-m^2n^2r_0^6), \\ \Psi_m &= b(b^2-2a^2-m^2n^2r_0^6) - 2a^2m^2r_0^3, & \Psi_n &= b(b^2-2a^2-m^2n^2r_0^6) - 2a^2n^2r_0^3, \\ \chi_m &= a(b^2+m^2n^2r_0^6) + 2abm^2r_0^3, & \chi_n &= a(b^2+m^2n^2r_0^6) + 2abn^2r_0^3. \end{aligned}$$

Therefore we may write

$$2\dot{a}^2\dot{b}\operatorname{Im}\Omega = \Phi_m\,e_{123} + \Phi_n\,e_{1'2'3'} + \Psi_m\,e_{3'12} + \Psi_n\,e_{31'2'} - \chi_m(e_{1'23}+e_{2'31}) - \chi_n(e_{12'3'}+e_{23'1'}).$$

We now turn our attention to F_α. As described in Section 2.3.1, any isotropy homomorphism $\lambda_p^j : \mathbb{Z}_{2(m+n)} \to G$ given by $\lambda_p^j(\zeta) = \operatorname{diag}(\zeta^j, \zeta^{-j})$ for $j \in \{0,1,\dots,2(m+n)-1\}$, has an associated $\mathrm{SU}(2)^2$-homogeneous G-bundle

$$P_p^j = \mathrm{SU}(2)^2 \times_{(\mathbb{Z}_{2(m+n)},\lambda_j)} G.$$

A connection on P_p^j can be pulled back to the trivial bundle $\mathrm{SU}(2)^2\times G$, so the set of connections on P_p^j can be viewed as a subset of those on the trivial bundle. Hence, we can write such a connection as a 1-form on $\mathrm{SU}(2)^2$ with values in $\mathfrak{g}$, satisfying the condition given in Theorem 2.5. The canonical invariant connection α^{can} has horizontal space $\mathfrak{m} = \mathfrak{su}(2)\oplus\mathfrak{su}(2)$, so its connection 1-form as an element of $\Omega^1(\mathrm{SU}(2)^2,\mathfrak{g})$ is zero. Then Theorem 2.5 tells us that any other $\mathrm{SU}(2)^2$-invariant connection differs from α^{can} by a morphism of $\mathbb{Z}_{2(m+n)}$-representations $\Lambda : (\mathfrak{su}(2)\oplus\mathfrak{su}(2),\operatorname{Ad}) \to (\mathfrak{g},\operatorname{Ad}\circ\lambda)$. We can extend such a Λ by left-invariance to $\mathrm{SU}(2)^2$, which leads to a 1-form with values in $\mathfrak{g}$. This 1-form can be written as

$$\alpha = \sum_{i=1}^{3}\alpha_i\otimes e_i + \sum_{i=1}^{3}\alpha_i'\otimes e_i',$$

with each α_i, α_i' lying in a different subspace of $\mathfrak{g}$ so that Λ is $\mathbb{Z}_{2(m+n)}$-equivariant. Then on M, the most general $\mathrm{SU}(2)^2$-invariant connection on P_p^j can be written as

$$\alpha(t) = \sum_{i=1}^{3}\alpha_i(t)\otimes e_i + \sum_{i=1}^{3}\alpha_i'(t)\otimes e_i',$$

where α_i, α_i' take values in the subspaces of $\mathfrak{g}$ described above.

A quick calculation gives

$$F_\alpha = \sum_{i=1}^{3}[\alpha_i, \alpha_i']\, e_{ii'} + \sum_{i=1}^{3}(([\alpha_j, \alpha_k] - \alpha_i)\, e_{jk} + ([\alpha_j', \alpha_k'] - \alpha_i')\, e_{j'k'}) \\ + \sum_{i=1}^{3}([\alpha_j, \alpha_k']\, e_{jk'} + [\alpha_j', \alpha_k]\, e_{j'k}) \tag{5.1}$$

where (i, j, k) is a cyclic permutation of (1,2,3).

To calculate the right hand side of (2.13), we now compute

$$\begin{aligned} 2\dot{a}^2\dot{b}F_\alpha \wedge \operatorname{Im}\Omega \ =\ & \Big(([\alpha_2,\alpha_3]-\alpha_1)\Phi_n + [\alpha_2',\alpha_3]\chi_n - [\alpha_2,\alpha_3']\Psi_n - ([\alpha_2',\alpha_3']-\alpha_1')\chi_m\Big)e_{231'2'3'} \\ & - \Big(([\alpha_3,\alpha_1]-\alpha_2)\Phi_n + [\alpha_3,\alpha_1']\chi_n - [\alpha_3',\alpha_1]\Psi_n - ([\alpha_3',\alpha_1']-\alpha_2')\chi_m\Big)e_{131'2'3'} \\ & + \Big(([\alpha_1,\alpha_2]-\alpha_3)\Phi_n + ([\alpha_1',\alpha_2]+[\alpha_1,\alpha_2'])\chi_n + ([\alpha_1',\alpha_2']-\alpha_3')\Psi_m\Big)e_{121'2'3'} \\ & + \Big(([\alpha_2',\alpha_3']-\alpha_1')\Phi_m + [\alpha_2,\alpha_3']\chi_m - [\alpha_2',\alpha_3]\Psi_m - ([\alpha_2,\alpha_3]-\alpha_1)\chi_n\Big)e_{1232'3'} \\ & - \Big(([\alpha_3',\alpha_1']-\alpha_2')\Phi_m + [\alpha_3',\alpha_1]\chi_m - [\alpha_3,\alpha_1']\Psi_m - ([\alpha_3,\alpha_1]-\alpha_2)\chi_n\Big)e_{1231'3'} \\ & + \Big(([\alpha_1',\alpha_2']-\alpha_3')\Phi_m + ([\alpha_1,\alpha_2']+[\alpha_1',\alpha_2])\chi_m + ([\alpha_1,\alpha_2]-\alpha_3)\Psi_n\Big)e_{1231'2'}. \end{aligned}$$

For the left hand side of (2.13), we note that the complex structure J_t is such that

$$J_t e_i^{(\prime)} = -*\left(e_i^{(\prime)} \wedge \frac{1}{2}\omega^2\right).$$

Thus, we may simply compare coefficients of $F_\alpha \wedge \operatorname{Im}\Omega$ and $\dot{\alpha} \wedge \frac{1}{2}\omega^2$, while keeping track of signs. This leads to the following general ODEs for the $\alpha_i(t), \alpha_i'(t)$

$$\begin{aligned} 2\dot{a}^3\dot{b}^2\dot{\alpha}_1 &= (\alpha_1' - [\alpha_2',\alpha_3'])\Phi_m - [\alpha_2,\alpha_3']\chi_m + [\alpha_2',\alpha_3]\Psi_m + ([\alpha_2,\alpha_3]-\alpha_1)\chi_n, \\ 2\dot{a}^3\dot{b}^2\dot{\alpha}_2 &= (\alpha_2' - [\alpha_3',\alpha_1'])\Phi_m - [\alpha_3',\alpha_1]\chi_m + [\alpha_3,\alpha_1']\Psi_m + ([\alpha_3,\alpha_1]-\alpha_2)\chi_n, \\ 2\dot{a}^4\dot{b}\dot{\alpha}_3 &= (\alpha_3' - [\alpha_1',\alpha_2'])\Phi_m - ([\alpha_1,\alpha_2'] + [\alpha_1',\alpha_2])\chi_m + (\alpha_3 - [\alpha_1,\alpha_2])\Psi_n, \\ 2\dot{a}^3\dot{b}^2\dot{\alpha}_1' &= (\alpha_1 - [\alpha_2,\alpha_3])\Phi_n - [\alpha_2',\alpha_3]\chi_n + [\alpha_2,\alpha_3']\Psi_n + ([\alpha_2',\alpha_3']-\alpha_1')\chi_m, \\ 2\dot{a}^3\dot{b}^2\dot{\alpha}_2' &= (\alpha_2 - [\alpha_3,\alpha_1])\Phi_n - [\alpha_3,\alpha_1']\chi_n + [\alpha_3',\alpha_1]\Psi_n + ([\alpha_3',\alpha_1']-\alpha_2')\chi_m, \\ 2\dot{a}^4\dot{b}\dot{\alpha}_3' &= (\alpha_3 - [\alpha_1,\alpha_2])\Phi_n - ([\alpha_1',\alpha_2] + [\alpha_1,\alpha_2'])\chi_n + (\alpha_3' - [\alpha_1',\alpha_2'])\Psi_m. \end{aligned} \tag{5.2}$$

Finally, we have the constraint $\Lambda_t F_\alpha = 0$, or equivalently $F_\alpha \wedge \frac{1}{2}\omega^2 = 0$ which yields the equation

$$\dot{a}\dot{b}([\alpha_1,\alpha_1'] + [\alpha_2,\alpha_2']) + \dot{a}^2[\alpha_3,\alpha_3'] = 0. \tag{5.3}$$

We now restrict to the case of $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant connections. The stabiliser of the $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-action on $M_{m,n}$ away from the singular orbit of the $\mathrm{SU}(2)^2$-action is a subgroup $K_0 \subset T^2 \times \mathrm{U}(1) \subset \mathrm{SU}(2)^2 \times \mathrm{U}(1)$, the image of

$$\mathbb{Z}_2 \times \mathrm{U}(1) \to T^2 \times \mathrm{U}(1) : (\xi, e^{i\theta}) \hookrightarrow (\xi e^{i\theta}, e^{i\theta}, \xi^{-m} e^{-i(m+n)\theta})$$

where $\xi \in \mathbb{Z}_2 \subset \mathrm{U}(1)$. Hence, the principal orbits can be written as

$$\mathrm{SU}(2)^2/\mathbb{Z}_{2(m+n)} \cong \mathrm{SU}(2)^2 \times \mathrm{U}(1)/K_0 \tag{5.4}$$

where this isomorphism is given by $(g_1, g_2) \mapsto (g_1, g_2, 1)$. Isomorphism classes of homogeneous SU(2)-bundles on these principal orbits are in correspondence with conjugacy classes of isotropy homomorphisms (see Section 2.3.1). Therefore, we need to consider isotropy homomorphisms $K_0 \to \mathrm{SU}(2)$ which, up to conjugacy, will be of the form

$$\lambda_p^{l,k}(\xi, e^{i\theta}) = \begin{pmatrix} \xi^l e^{ik\theta} & 0 \\ 0 & \xi^{-l} e^{-ik\theta} \end{pmatrix}$$

for $l \in \{0, 1\}$ and $k \in \mathbb{Z}$.

Following the proof of Proposition 8 of [24], we take the complement of the isotropy algebra $\mathfrak{u}(1)$ to be $\mathfrak{m} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus\ 0$. The canonical invariant connection on the bundle

$$P_p^{l,k} = (\mathrm{SU}(2)^2 \times \mathrm{U}(1)) \times_{(K_0, \lambda_p^{l,k})} \mathrm{SU}(2)$$

is given by $d\lambda_p^{l,k} = E_3 \otimes k d\theta$, where θ is the periodic coordinate on U(1). Theorem 2.5 states that any other invariant connection on $P_p^{l,k}$ can be written as $d\lambda_p^{l,k} + \Lambda_{l,k}$, where $\Lambda_{l,k}$ is the left invariant extension to $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$ of a morphism of K_0-representations

$$\Lambda_{l,k} : (\mathfrak{m}, \mathrm{Ad}) \to (\mathfrak{su}(2), \mathrm{Ad} \circ \lambda_p^{l,k}). \tag{5.5}$$

The adjoint action of $\mathbb{Z}_2$ on the Lie algebra is trivial, so we decompose $\mathfrak{m}$ and $\mathfrak{su}(2)$ into irreducibles of U(1)-representations. We have $\mathfrak{m} = (\mathbb{R} \oplus \mathbb{C}_2) \oplus (\mathbb{R} \oplus \mathbb{C}_2) \oplus 0$ and $\mathfrak{su}(2) = \mathbb{R} \oplus \mathbb{C}_{2k}$, so we see that other irreducible invariant connections exist only when $k = 1$ by Schur's Lemma. Therefore, there are only 2 distinct bundles, parametrised by $l \in \{0, 1\}$, over the principal orbits which admit irreducible invariant connections.

The isomorphism (5.4) induces bundle isomorphisms between P_p^j and $P_p^{l,k}$ when $j + l(m+n) + mk \in 2(m+n)\mathbb{Z}$. When $k = 1$, this is exactly when $l = 0$, $j = -m$ or when $l = 1$, $j = n$. Indeed, P_p^j is the bundle $P_p^{l,k}$ where we forget the additional U(1)-action. Appendix A describes the possible extensions of the bundles P_p^j over the singular orbit for the $\mathrm{SU}(2)^2$-action. We denote the pull-back of the two bundles $P_p^{0,1}$ and $P_p^{1,1}$ to $\mathbb{R}^+ \times (\mathrm{SU}(2)^2 \times \mathrm{U}(1))/K_0$ by P_p^{-m} and P_p^n respectively in what follows.

The singular orbit is of the form $\mathrm{SU}(2)^2/K_{m,n}$, where the $K_{m,n}$-action is generated by $nE_3 - mE_3'$. Therefore, we can consider a new basis for the maximal torus T^2, given by

$$nE_3 - mE_3', \qquad rE_3 + sE_3'.$$

Here, the $r, s \in \mathbb{Z}$ were chosen in Section 3.3.1 to satisfy $mr + ns = 1$; we choose an integral basis in order to easily apply the results of Appendix A on the smooth extension of connections over the singular orbit. The dual left-invariant 1-forms are $se_3 - re_3'$ and $me_3 + ne_3'$ respectively, and we formulate expressions for connections on $M_{m,n}$ using this new basis.

5.2 G_2-instanton Equations

5.2.1 $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant ODEs

We begin by finding all possible connections over the principal orbits on the various homogeneous bundles defined in Section 5.1, and then derive the explicit G_2-instanton equations for these invariant connections.

Proposition 5.6. *Let m, n be distinct positive coprime integers. Let A be an $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant G_2-instanton on $\mathbb{R}^+ \times \mathrm{SU}(2)^2/\mathbb{Z}_{2(m+n)} \cong \mathbb{R}^+ \times \mathrm{SU}(2)^2 \times \mathrm{U}(1)/K_0 \subset M_{m,n}$ with gauge group $\mathrm{SU}(2)$. Then A, up to gauge transformation, takes the following form:*

$$A = f(E_1 \otimes e_1 + E_2 \otimes e_2) + f'(E_1 \otimes e_1' + E_2 \otimes e_2') + g(E_3 \otimes (se_3 - re_3')) + h(E_3 \otimes (me_3 + ne_3')) \quad (5.7)$$

with $f, f', g, h : \mathbb{R}^+ \to \mathbb{R}$ satisfying,

$$\begin{aligned}
2\dot{a}^3\dot{b}^2\dot{f} &= f'(1 - nh + rg)\Phi_m - f(nh - rg)\chi_m + f'(sg + mh)\Psi_m + f(sg + mh - 1)\chi_n, \\
2\dot{a}^3\dot{b}^2\dot{f}' &= f(1 - sg - mh)\Phi_n - f'(sg + mh)\chi_n + f(nh - rg)\Psi_n + f'(nh - rg - 1)\chi_m, \\
2\dot{a}^4\dot{b}\dot{g} &= f^2(m\Phi_n - n\Psi_n) + (f')^2(m\Psi_m - n\Phi_m) + 2ff'(m\chi_n - n\chi_m) \\
&\quad + g(ns\Psi_n + mr\Psi_m - ms\Phi_n - nr\Phi_m) \\
&\quad + h(n^2\Phi_m - m^2\Phi_n - mn\Psi_m + mn\Psi_n), \\
2\dot{a}^4\dot{b}\dot{h} &= -f^2(s\Phi_n + r\Psi_n) - (f')^2(s\Psi_m + r\Phi_m) - 2ff'(s\chi_n + r\chi_m) \\
&\quad + g(rs\Psi_n - rs\Psi_m + s^2\Phi_n - r^2\Phi_m) \\
&\quad + h(nr\Phi_m + ms\Phi_n + ns\Psi_m + mr\Psi_n).
\end{aligned} \quad (5.8)$$

Proof. Recall that on each principal orbit, we have the bundles $P_{j,1}$ which correspond to the isotropy homomorphisms $\lambda_{j,1}$. Linear maps $\Lambda_{j,1}$ given by (5.5) are morphisms of K_0-representations if they are equivariant. Any invariant connection on $P_{j,1}$ differs from the canonical invariant connection $d\lambda_{j,k}$ by the left-invariant extension of such a morphism $\Lambda_{j,1}$. Such a connection on each $P_{j,1}$ is given, up to gauge transformation, by

$$A = E_3 \otimes d\theta + f(E_1 \otimes e_1 + E_2 \otimes e_2) + f'(E_1 \otimes e_1' + E_2 \otimes e_2') + g(E_3 \otimes (se_3 - re_3')) + h(E_3 \otimes (me_3 + ne_3'))$$

where f, f', g, h are constants. Recall that the bundles $P_{0,1}$, $P_{1,1}$ over $\mathrm{SU}(2)^2 \times \mathrm{U}(1)/K_0$ are isomorphic to P_{-m}, P_n over $\mathrm{SU}(2)^2/\mathbb{Z}_{2(m+n)}$ respectively. Thus any $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant connection on P_n or P_{-m} over $\mathbb{R}^+ \times \mathrm{SU}(2)^2/\mathbb{Z}_4$ in temporal gauge has the form

$$A = \gamma\Big(f(E_1 \otimes e_1 + E_2 \otimes e_2) + f'(E_1 \otimes e_1' + E_2 \otimes e_2') + g(E_3 \otimes (se_3 - re_3')) + h(E_3 \otimes (me_3 + ne_3'))\Big)\gamma^{-1}$$

where f, f', g, h are functions $\mathbb{R}^+ \to \mathbb{R}$, $\gamma : \mathbb{R}^+ \to \mathrm{SU}(2)$. The presence of the function γ is due to the dependence on t of the gauge transformations on each principal orbit. We now see that requiring A to be a G_2-instanton forces the function γ to be constant, and hence the choice of gauge transformation does not depend on t. Using the substitutions

$$\begin{aligned}
&\alpha_1 = \gamma f E_1 \gamma^{-1}, \quad \alpha_2 = \gamma f E_2 \gamma^{-1}, \quad \alpha_3 = \gamma(sg + mh)E_3\gamma^{-1}, \\
&\alpha_1' = \gamma f' E_1 \gamma^{-1}, \quad \alpha_2' = \gamma f' E_2 \gamma^{-1}, \quad \alpha_3' = \gamma(nh - rg)E_3\gamma^{-1},
\end{aligned}$$

the general ODEs (5.2) transform to ODEs in f, f', g, h and γ. For example, the ODE for α_1 becomes

$$2\dot{a}^3\dot{b}^2\dot{f}+[\gamma^{-1}\dot{\gamma},E_1]f \;=\; f'(1-nh+rg)\Phi_m - f(nh-rg)\chi_m + f'(sg+mh)\Psi_m + f(sg+mh-1)\chi_n.$$

Hence, $[\gamma^{-1}\dot{\gamma},E_1]f=0$ and since this type of term appears in each ODE and we assume $A\neq 0$, then we must have $\dot{\gamma}=0$, and hence we can write A as

$$A = f(E_1\otimes e_1+E_2\otimes e_2)+f'(E_1\otimes e_1'+E_2\otimes e_2')+g(E_3\otimes(se_3-re_3'))+h(E_3\otimes(me_3+ne_3')),$$

with f, f', g, h satisfying the ODEs given in the statement of the proposition. The constraint (5.3) is trivially satisfied by this choice of gauge. □

Remark 5.9. The functions g and h depend on the choice of integers r and s. Indeed, if g and h are the functions defining A in the basis given by (r,s) and $\bar{g}$, $\bar{h}$ are the functions defining A in the basis given by $(\bar{r},\bar{s})$, then they must satisfy

$$g=\bar{g} \qquad \text{and} \qquad h=(\bar{s}r-\bar{r}s)g+\bar{h}.$$

◊

5.2.2 Extending over the singular orbit

We now consider the conditions for extending the connections of Section 5.2.1 over the singular orbit. Here, we apply the results of Appendix A to write local solutions around the singular orbit as power series expansions. Near $t=0$, the instanton equations (5.8) take the form of a singular initial value problem and so we will apply Theorem 4.46 in this section.

We study the conditions for a connection A of the form (5.7) to extend smoothly over the singular orbit

$$\mathrm{SU}(2)^2/K_{m,n}\cong(\mathrm{SU}(2)^2\times\mathrm{U}(1))/K$$

where $K\cong\mathrm{U}(1)^2$ is the stabiliser of the $\mathrm{SU}(2)^2\times\mathrm{U}(1)$-action on the singular orbit and is embedded in $\mathrm{SU}(2)^2\times\mathrm{U}(1)$ by

$$(e^{i\psi},e^{i\theta})\hookrightarrow(e^{i\psi}e^{i\theta},e^{i\theta},e^{-mi\psi}e^{-(m+n)i\theta}).$$

We want to find the possible $\mathrm{SU}(2)^2\times\mathrm{U}(1)$-homogeneous bundles over the singular orbit which are extensions of the bundles on $\mathbb{R}^+\times\mathrm{SU}(2)^2/\mathbb{Z}_{2(m+n)}$ found in Section 5.1. Such bundles are parameterised by homomorphisms $K\to\mathrm{SU}(2)$, which up to conjugacy are of the form

$$\lambda_s^{\nu,\mu}(e^{i\psi},e^{i\theta})=\begin{pmatrix} e^{i(\nu\psi+\mu\theta)} & 0\\ 0 & e^{-i(\nu\psi+\mu\theta)}\end{pmatrix},$$

for $\nu,\mu\in\mathbb{Z}$, $\lambda_s^{\nu,\mu}\circ\iota=\lambda_p^{l,1}$ for $l\in\{0,1\}$; here, $\iota: K_0\to K$ is the map given by the obvious inclusion $\mathbb{Z}_2\hookrightarrow U(1)$ on the first factor and the identity on the second. Then we must have $\mu=1$ while $\nu\equiv l \mod 2$.

To apply the results of Appendix A, it is easier to consider extending smoothly the $\mathrm{SU}(2)^2$-homogeneous bundles over the singular orbit and to forget the extra U(1)-action. The $\mathrm{SU}(2)^2$-homogeneous bundles P_s^j over $\mathrm{SU}(2)^2/K_{m,n}$ correspond to isotropy homomorphisms $\lambda_s^j(e^{i\theta})=$

$\mathrm{diag}(e^{ij\theta}, e^{-ij\theta})$. The bundle isomorphism induced by (5.4) extends to the singular orbit, with $P_s^{\nu,1}$ isomorphic to P_s^j if $j = (m+n)\nu - m$. Then we must have $j \equiv n, -m \mod 2(m+n)$, where P_s^j is an extension of the bundles P_p^n and P_p^{-m} respectively over the principal orbits. Then as in Section 2.3.1, we have bundles P_j over $M_{m,n}$ whose restriction to the principal orbits is P_p^n or P_p^{-m} and whose restriction to the singular orbit is P_s^j.

Using $A^{\mathrm{can}} = jE_3 \otimes (se_3 - re_3')$ as a reference connection, we can write any $\mathrm{SU}(2)^2$-invariant connection as an $\mathrm{SU}(2)^2$-invariant element of $\Omega^1(\mathrm{Ad}\, P_j)$. In what follows, we apply the results of Appendix A to the 1-form

$$A - A^{\mathrm{can}} = f(E_1 \otimes e_1 + E_2 \otimes e_2) + f'(E_1 \otimes e_1' + E_2 \otimes e_2') + (g - j)(E_3 \otimes (se_3 - re_3')) + h(E_3 \otimes (me_3 + ne_3')).$$

The vanishing at $t = 0$ of the coefficient of $se_3 - re_3'$ in all cases forces $g(0) = j$.

Proposition 5.10. *Let $Y \subset M_{m,n}$ contain the singular orbit $S^3 \times S^3/K_{m,n}$, and be endowed with the $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant AC holonomy G_2-metric of Theorem 3.13. There is a 2-parameter family of $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant G_2-instantons A_j, all with gauge group $\mathrm{SU}(2)$ in a neighbourhood of the singular orbit in Y, extending smoothly over P_j, for $j \equiv n, -m \mod 2(m+n)$ and $j \neq n, -m$. Furthermore, A_j can be written in the form (5.7), where f, f', g and h fall into one of the following cases.*

- *Let $j \geq n$ and write $j = (m+n)\nu - m$ for $\nu \in \mathbb{Z}_+$, then for $h_0, f_0 \in \mathbb{R}$, we have*

$$\begin{aligned} f(t) &= f_0 t^{\nu-1} + O(t^{\nu+1}), \\ f'(t) &= \frac{2\beta^3(s - h_0 + (r-s)\nu) - m^{\frac{3}{2}} n^{\frac{3}{2}}(\nu - 1)}{4m\nu r_0 \beta^2} f_0 t^\nu + O(t^{\nu+2}), \\ g(t) &= j + O(t^2), \\ h(t) &= h_0 + O(t^2). \end{aligned}$$

- *Let $j \leq -m$ and write $j = -(m+n)\nu + n$ for $\nu \in \mathbb{Z}_+$, then for $h_0, f_0' \in \mathbb{R}$, we have*

$$\begin{aligned} f(t) &= \frac{2\beta^3(r - h_0 + (s-r)\nu) - m^{\frac{3}{2}} n^{\frac{3}{2}}(\nu - 1)}{4n\nu r_0 \beta^2} f_0' t^\nu + O(t^{\nu+2}), \\ f'(t) &= f_0' t^{\nu-1} + O(t^{\nu+1}), \\ g(t) &= j + O(t^2), \\ h(t) &= h_0 + O(t^2). \end{aligned}$$

Proof. We only prove the case $j \geq n$; the other case is proved similarly. By Lemma A.4, we can write

$$f(t) = u_f(t) t^{\nu-1}, \qquad f'(t) = u_{f'}(t) t^\nu, \qquad g(t) = u_g(t), \qquad h(t) = u_h(t).$$

Then the G_2-instanton equations in a neighbourhood of the singular orbit take the form

$$\begin{aligned}
\dot{u}_f &= \left(\frac{u_g+m}{m+n}-\nu\right)u_f t^{-1}+O(1),\\
\dot{u}_{f'} &= \left(\left(-\tfrac{u_g+m}{m+n}-\nu\right)u_{f'}+\frac{m^{\frac{3}{2}}n^{\frac{3}{2}}(n-u_g)+2\beta^3(1-(m+n)u_h+u_g(r-s))}{2m(m+n)r_0\beta^2}u_f\right)t^{-1}+O(1),\\
\dot{u}_g &= O(t),\\
\dot{u}_h &= O(t).
\end{aligned}$$

Thus the system has the form of the IVP in Theorem 4.46 if we let $X=(u_f,u_{f'},u_g,u_h)$ and

$$\begin{aligned}
M_{-1}(X)=\Bigg(&\left(\frac{u_g+m}{m+n}-\nu\right)u_f,\\
&\left(\left(-\frac{u_g+m}{m+n}-\nu\right)u_{f'}+\frac{m^{\frac{3}{2}}n^{\frac{3}{2}}(n-u_g)+2\beta^3(1-(m+n)u_h+u_g(r-s))}{2m(m+n)r_0\beta^2}u_f\right),0,0\Bigg).
\end{aligned}$$

Then, setting $M_{-1}(X(0))=0$, we have

$$\begin{aligned}
u_f(0) &= f_0,\\
u_{f'}(0) &= \frac{2\beta^3(s-h_0+(r-s)\nu)-m^{\frac{3}{2}}n^{\frac{3}{2}}(\nu-1)}{4m\nu r_0\beta^2}f_0,\\
u_g(0) &= (m+n)\nu-m,\\
u_h(0) &= h_0.
\end{aligned}$$

We have $\det(h\mathrm{Id}-dM_{-1}(X(0)))=h^3(h+2\nu)\neq 0$ for $h\in\mathbb{N}_{>1}$, thus Theorem 4.46 gives a real analytic solution to the singular IVP. Then, there is a 2-parameter family of solutions, which are parameterised by h_0 and f_0, as in the statement of the proposition. □

Having obtained and applied the results of Appendix A, we no longer need an integral basis of coframes on $M_{m,n}$ and we can take any $r,s\in\mathbb{R}$ such that $mr+ns=1$. When $m=n=1$, the most obvious choice is $r=s=\frac{1}{2}$ and we work with this basis whenever we specialise to the manifold $M_{1,1}$.

In Section 5.3, we outline some elementary solutions to the $\mathrm{SU}(2)^2\times\mathrm{U}(1)$-invariant G_2-instanton equations. We then, in Section 5.4, apply a dynamical systems approach to find solutions on $M_{1,1}$ with full gauge group. These solutions are constructed by flowing from a perturbation of an abelian solution close to the singular orbit to a perturbation of a solution on the cone.

5.3 Elementary Solutions

We now consider global solutions to the G_2-instantons which extend the local solutions of the previous section across the whole of $M_{m,n}$. We start by considering elementary solutions which are either flat or abelian. The former are not gauge equivalent to the trivial connection under invariant gauge transformations; such transformations $g:\mathbb{R}^+\to\mathrm{SU}(2)$ are constant on each principal orbit. The latter are connections whose gauge group reduces to U(1); as the system of ODEs decouples, we can give a 1-parameter family of explicit abelian solutions on each bundle P_j. One special member of this family will prove useful in Section 5.4 for finding solutions with full gauge group SU(2).

5.3.1 Invariant flat connections

Invariant flat connections $A = \alpha(t)$ are given by choices of values for the constant functions f, f', g, h. Then $\dot{\alpha} = 0$ and hence $F_A = dt \wedge \dot{\alpha} + F_{\alpha(t)} = F_{\alpha(t)}$. So we require $F_{\alpha(t)} = 0$ on each principal orbit.

Proposition 5.11. *The connections on $M_{m,n}$ given by*

$$A_n^{\pm} = \pm(E_1 \otimes e_1 + E_2 \otimes e_2) + E_3 \otimes e_3 \quad \text{on the bundle } P_n,$$
$$A_{-m}^{\pm} = \pm(E_1 \otimes e_1' + E_2 \otimes e_2') + E_3 \otimes e_3' \quad \text{on the bundle } P_{-m}$$

are G_2-instantons.

Proof. We can take the constant solutions $(f, f', g, h) = (\pm 1, 0, n, r), (0, \pm 1, -m, s)$ to the ODE system (5.8). Subsituting the corresponding values of α_i, α_i' into (5.1) yields $F_\alpha = 0$. □

Lemma 5.12. *The connections of Proposition 5.11, together with the canonical invariant connection, are the only invariant flat G_2-instantons on $M_{m,n}$.*

Proof. Suppose

$$A = f(E_1 \otimes e_1 + E_2 \otimes e_2) + f'(E_1 \otimes e_1' + E_2 \otimes e_2') + g(E_3 \otimes (se_3 - re_3')) + h(E_3 \otimes (me_3 + ne_3'))$$

is an invariant flat connection on P_j, i.e. f, f', g and h are constant. Consider first the case $f = f' = 0$, then by considering the curvature form (5.1), we see that $sg + mh = 0$ and $rg - nh = 0$, and the only solution is $g = h = 0$, which corresponds to the canonical invariant connection.

Suppose instead that $f \neq 0$, then (5.1) implies that $nh - rg = 0$, $sg + mh = 1$ and hence that $f = \pm 1$, $f' = 0$, $g = n$ and $h = r$. These correspond exactly to the connections on P_n of Proposition 5.11. Similarly, if $f' \neq 0$, then we must have $f = 0$, $f' = \pm 1$, $g = -m$ and $h = s$, giving the connections $A_{-m}^{\pm}$ on P_{-m}. □

Remark 5.13. We note that some of these solutions are gauge equivalent under an $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant gauge transformation. Indeed, the element

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \mathrm{SU}(2).$$

defines a group action of SU(2) on itself by conjugation. The derivative of this action is the adjoint representation, which acts on $\mathfrak{su}(2)$ by

$$E_1 \mapsto -E_1, \qquad E_2 \mapsto -E_2, \qquad E_3 \mapsto E_3.$$

The G_2-instanton equations (5.8) are invariant under this gauge transformation and A_j^+ and A_j^- are gauge equivalent. ♢

Remark 5.14. The instantons of Proposition 5.11 do not depend on the functions a and b, since the coefficients of the Φ and Ψ vanish independently of each other. Hence they also define invariant flat G_2-instantons on $M_{m,n}$ equipped with the torsion free ALC G_2-structures given in case (v) of Theorem 3.16. ♢

5.3.2 Abelian solutions

Abelian solutions are given by morphisms

$$\Lambda : (\mathbb{R} \oplus \mathbb{C}_{2n}) \oplus (\mathbb{R} \oplus \mathbb{C}_{-2m}) \to \mathbb{R}$$

of $\mathbb{Z}_{2(m+n)}$-representations; such maps must vanish on the complex components, corresponding to $f = f' = 0$ in (5.7). Applying this to the equations (5.8) yields

$$\begin{aligned} 2\dot{a}^4\dot{b}\dot{g} &= g(ns\Psi_n + mr\Psi_m - ms\Phi_n - nr\Phi_m) \\ &\quad + h(n^2\Phi_m - m^2\Phi_n - mn\Psi_m + mn\Psi_n), \\ 2\dot{a}^4\dot{b}\dot{h} &= g(rs\Psi_n - rs\Psi_m + s^2\Phi_n - r^2\Phi_m) \\ &\quad + h(nr\Phi_m + ms\Phi_n + ns\Psi_m + mr\Psi_n). \end{aligned} \tag{5.15}$$

We see that for general coprime positive m and n, it is not easy to see how to find explicit solutions to these equations. This is the first obstruction to finding solutions on the infinite extension of the $\mathbb{C}_7$ family using the method that follows for $M_{1,1}$.

We consider solutions to the abelian G_2-instanton equations from a more topological point of view. Note that the curvature form $F_A = dA$ is closed (this is a local condition), and also coclosed since

$$d^*F_A = *d*F_A = -*\, d(F_A \wedge \varphi) = -*\, (dF_A \wedge \varphi + F_A \wedge d\varphi) = 0$$

if A is a G_2-instanton. In particular, F_A is a harmonic 2-form.

When M is a compact G_2-manifold, every class in $H^2(M;\mathbb{R})$ has a harmonic representative; when M is non-compact, this is not necessarily true. Proposition 4.57 of [20] states that if $\mathcal{H}^2_{L^2}$ is the space of L^2-integrable closed and coclosed 2-forms on M, then

$$\mathcal{H}^2_{L^2} \cong H^2_{cs}(M;\mathbb{R}).$$

Poincaré Duality tells us that for $M_{m,n}$, this space is 1-dimensional and so we would expect to find an L^2-integrable abelian instanton A on each U(1)-bundle on $M_{m,n}$ which extends a local solution in the following 2-parameter family.

Proposition 5.16. *There is a 2-parameter family of G_2-instantons on each* SU(2)*-bundle over $(T,\infty) \times K_0 \subset M_{m,n}$, parameterised by $\mu, \nu \in \mathbb{R}$, such that the functions f, f', g and h in (5.7) admit convergent power series expansions in t^{-1} satisfying*

$$f = \mu t^{-6} + O(t^{-7}), \quad f' = -\mu t^{-6} + O(t^{-7}), \quad g = \nu(m+n)t^{-6} + O(t^{-7}), \quad h = \nu(r-s)t^{-6} + O(t^{-7}).$$

A has gauge group SU(2) *unless $\mu = 0$ and then the gauge group reduces to U(1).*

Proof. We change variables to $w = t^{-1}$. Then, we can describe the singular IVP in the usual way; we have

$$\begin{aligned}
\frac{dF}{dw} &= \frac{1}{w}\Big(-4F' - 4F\Big) + O(1)\\
\frac{dF'}{dw} &= \frac{1}{w}\Big(-4F - 4F'\Big) + O(1)\\
\frac{dG}{dw} &= \frac{1}{w}\Big(-4G(1-(ms+nr)) - 4H(n^2-m^2)\Big) + O(1)\\
\frac{dH}{dw} &= \frac{1}{w}\Big(-4H(1+ms+nr) - 4G(s^2-r^2)\Big) + O(1).
\end{aligned}$$

Taking $y_0 = (\mu, -\mu, \nu(m+n), \nu(r-s))$ ensures that $M_{-1}(y_0) = 0$ while $d_{y_0}M_{-1}$ has eigenvalues 0 and -8. Hence, for each μ and ν, there is a unique solution to the IVP by Theorem 4.46. $\square$

When $m = n = 1$ and $r = s = \frac{1}{2}$, we notice that the abelian G_2-instanton equations are uncoupled. Hence, we solve them in terms of the functions a and b, with initial conditions given by Proposition 5.10. The general solutions for g and h are

$$g(t) = \frac{4jr_0^6}{(b+r_0^3)^2}, \tag{5.17}$$

$$h(t) = h_0 \exp\left(\int_0^t \frac{2\dot{b}(b-r_0^3)}{4a^2-(b-r_0^3)^2}d\tau\right). \tag{5.18}$$

By taking $h_0 = 0$ on P_j, we get a particular solution

$$A^{\mathrm{ab}} = \frac{1}{2}gE_3 \otimes (e_3 - e_3') \tag{5.19}$$

where g is given by (5.17). This solution extends the local solution of Proposition 5.16 with $\mu = 0$ and $\nu = 1944jr_0^6$, so both g and h decay at infinity and the limit of the connection A is the canonical invariant connection. A^{ab} will play a role in the construction of G_2-instantons with gauge group SU(2) in Section 5.4.

5.4 Solutions via a Dynamical Systems Approach

We now construct solutions to the G_2-instanton equations which have full gauge group SU(2) and are perturbations of an abelian solution near the singular orbit. We begin by rescaling time so that, using the expansions of a and b given in Proposition 3.8, we can rewrite (5.8) as a sum of autonomonous and non-autonomous parts. Denote $z = (f, f', g, h)$. Rescaling time as $t(\tau) = \exp(\tau)$ yields $\frac{dz}{d\tau} = \dot{z}\frac{dt}{d\tau} = \dot{z}\exp(\tau)$; hence, we can write

$$\frac{dz}{d\tau} = F(z) + G(z,\tau), \tag{5.20}$$

where

$$F(z) = \begin{pmatrix} 2f'(2-(m+2n)h+(2r-s)g)+2f((r+s)g+(m-n)h-1) \\ 2f(2-(2m+n)h+(r-2s)g)-2f'((n-m)h-(r+s)g-1) \\ 2f^2(2m+n)-2(f')^2(2n+m)+4ff'(m-n)-2g(1+2(ms+nr))+4h(n^2-m^2) \\ 2f^2(r-2s)+2(f')^2(s-2r)-4ff'(r+s)+4g(s^2-r^2)+2h(2(ms+nr)-1) \end{pmatrix} \tag{5.21}$$

and with the non-autonomous part of (5.20) satisfying

$$\lim_{\tau\to\infty} \exp(\tau)G(z,\tau) = 0 \tag{5.22}$$

$$\lim_{\tau\to\infty} \exp(\tau)D_1G(z,\tau) = 0. \tag{5.23}$$

5.4.1 Autonomous dynamics and steady states

We start by analysing the dynamical systems behaviour of the truncated autonomous ODE

$$\dot{z} = F(z) \tag{5.24}$$

with flow $\Phi(\cdot,\tau)$. Subsequently, we then show how certain solutions of the autonomous ODE persist in the full system (5.20).

The system (5.24) has steady states

$$z_0 = (0,0,0,0), \quad z_+ = \left(\frac{1}{3}, \frac{1}{3}, \frac{n-m}{3}, \frac{r+s}{3}\right), \quad z_- = \left(-\frac{1}{3}, -\frac{1}{3}, \frac{n-m}{3}, \frac{r+s}{3}\right).$$

The fixed points z_+ and z_- are sent to each other by the gauge transformation described in Remark 5.13. We calculate the respective linearisations at the steady states and their eigenvalues and eigenvectors; the results are given in Table 5.1.

The sets of eigenvalues together show that all steady states are hyperbolic saddles, which implies the existence of respective stable and unstable manifolds, tangential to the stable and unstable eigenspaces. For z_0, both the stable and unstable manifolds are two dimensional. For z_+ and z_-, the stable manifold is three dimensional and the unstable manifold is one dimensional.

5.4.2 Constituent solutions as building blocks

We now consider a couple of solutions to the full system which we use as building blocks for a gluing process, to form solutions which initially move towards the fixed point z_0 before travelling off to one of the fixed points $z_\pm$. We start by describing the subspaces of the 4-dimensional phase space which are invariant under the flow, and the solutions which lie inside these invariant spaces.

The plane $\Pi_1 = \{f = f' = 0\}$ is invariant under both (5.20) and (5.24). Solutions which live in this plane for all time are abelian. The line $\ell_1 = \text{span}\{(0,0,m+n,r-s)\} \subset \Pi_1$ is invariant under (5.24) but not under (5.20) unless $m = n = 1$. This is another obstruction to finding solutions on the infinite extension of the $\mathbb{C}_7$ family using the method that follows.

Fixed Point z	$DF(z)$	Eigenvalues	Eigenvectors
z_0	$\begin{pmatrix} -2 & 4 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ 0 & 0 & -2(1+2(ms+nr)) & 4(n^2-m^2) \\ 0 & 0 & 4(s^2-r^2) & 2(2(ms+nr)-1) \end{pmatrix}$	$\{-6,-6,2,2\}$	$\begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ m+n & 0 & n-m & 0 \\ r-s & 0 & r+s & 0 \end{pmatrix}$
z_+	$\begin{pmatrix} -2 & 2 & 2r & -2n \\ 2 & -2 & -2s & -2m \\ 4m & -4n & -2(1+2(ms+nr)) & 4(n^2-m^2) \\ -4s & -4r & 4(s^2-r^2) & 2(2(ms+nr)-1) \end{pmatrix}$	$\{4,-8,-2,-2\}$	$\begin{pmatrix} -1 & -1 & m & m-n+s(r+s) \\ -1 & 1 & n & -r(r+s) \\ 2(n-m) & 2(m+n) & 0 & n(m-n)+r+s \\ 2(r+s) & 2(r-s) & 1 & r(m-n) \end{pmatrix}$
z_-	$\begin{pmatrix} -2 & 2 & -2r & 2n \\ 2 & -2 & 2s & 2m \\ -4m & 4n & -2(1+2(ms+nr)) & 4(n^2-m^2) \\ 4s & 4r & 4(s^2-r^2) & 2(2(ms+nr)-1) \end{pmatrix}$	$\{4,-8,-2,-2\}$	$\begin{pmatrix} 1 & 1 & -m & n-m-s(r+s) \\ 1 & -1 & -n & r(r+s) \\ 2(n-m) & 2(m+n) & 0 & n(m-n)+r+s \\ 2(r+s) & 2(r-s) & 1 & r(m-n) \end{pmatrix}$

Table 5.1: The linearisation, eigenvalues and corresponding matrix of eigenvectors for each fixed point z_0, $z_\pm$.

Another invariant plane of (5.24) is $\Pi_2 = \{f = f', (n-m)h = (r+s)g\}$. In this plane, the autonomous equation (5.24) reduces to

$$\begin{aligned}\dot{f} = \dot{f}' &= 2f - \frac{6}{r+s}hf,\\ \dot{g} &= 2(m-n)\left(3f^2 - \frac{1}{r+s}h\right),\\ \dot{h} &= 2(h - 3(r+s)f^2).\end{aligned}$$

We note that Π_2 contains the steady states z_0 and $z_\pm$, as well as all of their unstable manifolds. Within this plane, the lines $\ell_\pm = \{f = f' = \pm\frac{1}{r+s}h\}$ are also invariant under (5.24), which then reduce to the single equation

$$\dot{h} = 2h - \frac{6}{r+s}h^2. \tag{5.25}$$

There is an explicit solution

$$h(\tau) = \frac{(r+s)\exp(2\tau)}{1+3\exp(2\tau)}$$

and up to time translation, this is the unique solution with image $(0, \frac{r+s}{3})$. It yields a solution for (5.24), namely

$$z(\tau) = (f, f', g, h)(\tau) = \left(\pm\frac{(r+s)\exp(2\tau)}{1+3\exp(2\tau)}, \pm\frac{(r+s)\exp(2\tau)}{1+3\exp(2\tau)}, 0, \frac{(r+s)\exp(2\tau)}{1+3\exp(2\tau)}\right). \tag{5.26}$$

Up to time translation, this is the unique solution lying in $\ell_\pm$ that satisfies

$$z(\tau) \xrightarrow{\tau\to-\infty} z_0 \text{ and } z(\tau) \xrightarrow{\tau\to\infty} z_\pm.$$

It also lies in the intersections $W^u(z_0) \cap W^s(z_+)$ and $W^u(z_0) \cap W^s(z_-)$ respectively. For $\tau \to \infty$, the stable manifolds $W^s(z_\pm)$ are tangential to their respective stable eigenspaces; each of these spaces, together with the unstable eigenvector contained in the plane Π_2, span the whole of $\mathbb{R}^4$. At any point $z(\tau)$ in (5.26),

$$T_{z(\tau)}W^u(z_0) = \Pi_2,$$

while the tangent space of $W^s(z_\pm)$ at $z_\pm$ is the direct sum of $\ell_\pm$ and a plane transverse to Π_2. If the tangent space to $W^s(z_\pm)$ contains Π_2 at any point on the flow line, then it will contain Π_2 everywhere along the flow line. With these observations, we see that the intersections $W^u(z_0) \cap W^s(z_+)$ and $W^u(z_0) \cap W^s(z_-)$ are transversal along $z(\tau)$ for all τ.

We now look to perturb the solutions (5.26) of the autonomous system to solutions of the full non-autonomous system. In this setting, the stable and unstable manifolds now depend on the choice of initial time; indeed, for each t_0, we have a stable manifold of initial conditions which flow towards the fixed point as $t \to \infty$ and an unstable manifold of initial conditions which flow to the fixed point as $t \to -\infty$. In order to perturb the solutions of the autononomous system, we show that such a perturbation will not affect the existence of the stable manifold of $z_\pm$ for sufficiently large start times $t_0 = T$ and that these perturbed stable manifolds will still intersect the unstable manifold of z_0 in the same way for large T.

The construction of the local stable manifolds from a contraction mapping argument in Theorem 9.3 of [31] shows that small enough perturbations will not change the existence of the stable manifold. Indeed, adding the non-autonomous function G to the existing contraction still yields a contraction for a sufficiently large start time; see Appendix B for more details. This contraction will have a unique fixed point that defines the stable manifold which will still be tangent to the same stable eigenspace.

From the local stable manifolds of $z_\pm$, it only takes a finite time to reach the intersection points constructed above. By smooth dependence on initial conditions and perturbations, we have that the perturbed stable manifolds $W^s_{t_0}(z_\pm)$ are C^1-close to the original stable manifolds near $z_\pm$ for all large $t_0 = T$, due to (5.22) and (5.23).

Remark 5.27. The solutions (5.26) solve the G_2-instanton equations on the G_2-cone, when we translate back into the original time. We remark that it is feasible that a global solution could be constructed by gluing the pull-back of this solution on the cone by the diffeomorphism giving the AC structure to a perturbation of the abelian solution (5.19). However, we instead apply in the following section a dynamical systems argument to obtain such a global solution. ♢

5.4.3 Global solutions which extend smoothly over the singular orbit

Having given a number of reasons why the case $m = n = 1$ is special compared to when m and n are general coprime positive integers, we now restrict our attention to constructing instantons on $M_{1,1}$. We state the main theorem which gives the existence of solutions to the G_2-instanton equations with gauge group SU(2) that are formed using the building blocks from Section 5.4.2.

Theorem 5.28. *Let $M_{1,1}$ be the AC G_2-manifold of Theorem 3.13 and let A^{ab} be the unique abelian solution* (5.19) *on each bundle P_j over $M_{1,1}$ for $j \in \mathbb{Z}$ odd. There is a function $(-\epsilon, \epsilon) \to \mathbb{R}, f_0 \mapsto h_0(f_0)$ such that the unique local solution with initial condition $(f_0, h_0(f_0))$ given by Proposition 5.10, which is a small perturbation of A^{ab} near the singular orbit $S^2 \times S^3$, extends to a global solution. Thus we obtain a 1-parameter family of* $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$*-invariant G_2-instantons with full gauge group* SU(2) *and bounded curvature.*

We prove this theorem via the following argument.

(i) We show that a 2-parameter family of initial conditions is mapped at some time T_1 under the flow to a disc that is transverse to the stable manifold of z_0, by considering the linearisation of the ODE system around the abelian solution A^{ab}.

(ii) We also show that such a disc is mapped at some time $T_2 > T_1$ under the flow to be C^1 close to the unstable manifold of z_0 and hence intersects the stable manifolds of $z_\pm$ along a curve.

(iii) Together this yields a 1-parameter family of global solutions.

In what follows, we choose to swap the order in which we prove the theorem. Stage 1 will address part (ii) of the argument on the interval (T_2, ∞) while Stage 2 concerns part (i) on the interval $(0, T_1)$. We will use the following schematic diagram, shown in Figure 5.2, to illustrate the components of the solutions (iii).

The diagram shows a solution (in purple) tending to z_+ as $t \to \infty$, evolving with time from left to right. The first part of the solution is shown to be a perturbation of the abelian solution A^{ab}, shown in the diagram as a straight line from the initial conditions to z_0. The solution is seen to bypass the point z_0 before tending to the fixed point z_+.

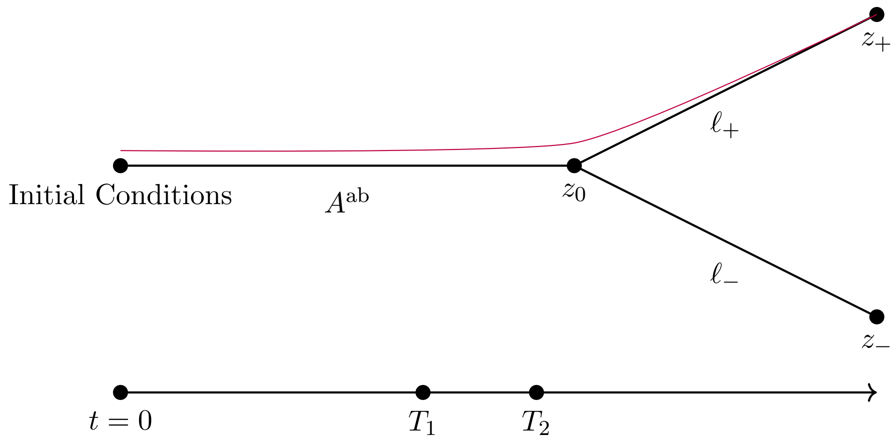

Figure 5.2: Schematic diagram showing the components of the 1-parameter family of solutions of Theorem 5.28.

Stage 1

As before, we only consider the case where $j \geq 1$. We show that there are solutions to (5.20), which initially closely follow (5.19) and then travel along perturbed solutions of (5.26). We appeal to the Inclination Lemma, which gives a convergence result for transversal manifolds.

Lemma 5.29 (The Inclination Lemma, [29] Lemma 7.1)**.** *Let B^s, B^u be balls contained in the local stable and unstable manifolds, respectively, of a hyperbolic fixed point 0; set $V = B^s \times B^u$. Consider a point q in the local stable manifold, and a disc D^u of the same dimension as the local unstable manifold which is transversal to the local stable manifold at the point q. Let D^u_t be the connected component of $V \cap \Phi(t, D^u)$ to which $\Phi(t, q)$ belongs. Given $\epsilon > 0$, there exists $T \in \mathbb{R}$ such that if $t > T$, then D^u_t is ϵ C^1-close to B^u.*

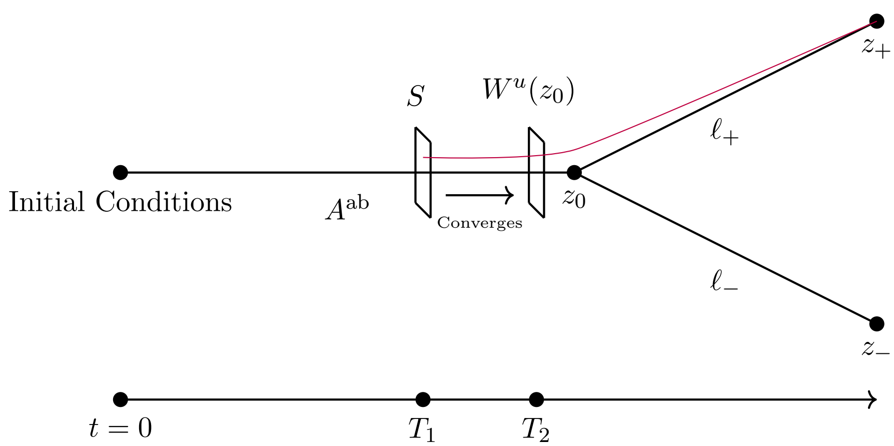

Figure 5.3: Schematic diagram showing the transverse manifold S converging to the unstable manifold of z_0.

By Lemma 5.29, we know that the forward solution to the autonomous system (5.24) of any transversal manifold S to $W^s(z_0)$ at a point in $\ell_1 \subset W^s(z_0)$ at $t = T_1$ converges locally in the C^1 topology to $W^u(z_0)$ for large times $t > T_2$. This is shown schematically in Figure 5.3.

Since the perturbation G in (5.20) is sufficiently small, the arguments of Section 5.4.2 apply, and we see that S still intersects the stable manifolds of $z_\pm$ for the full system at sufficiently large times. Then all intersection points have the desired property that the forward solutions converge to $z_\pm$.

Essentially, we see that as S evolves with the flow, its intersection with each of the stable manifolds $W^s(z_\pm)$ is a 1-dimensional subset; this subset tends to a segment of the corresponding line $\ell_\pm$ as $\tau \to \infty$. With this in mind, we now transform back to the original time t.

Stage 2

The final step is to show that the 2-parameter family of initial conditions which gives a smooth extension over the singular orbit at time $t = 0$ is mapped at time $t = T$ to a 2-dimensional submanifold S which is transverse to the stable manifold of z_0, as shown in Figure 5.4.

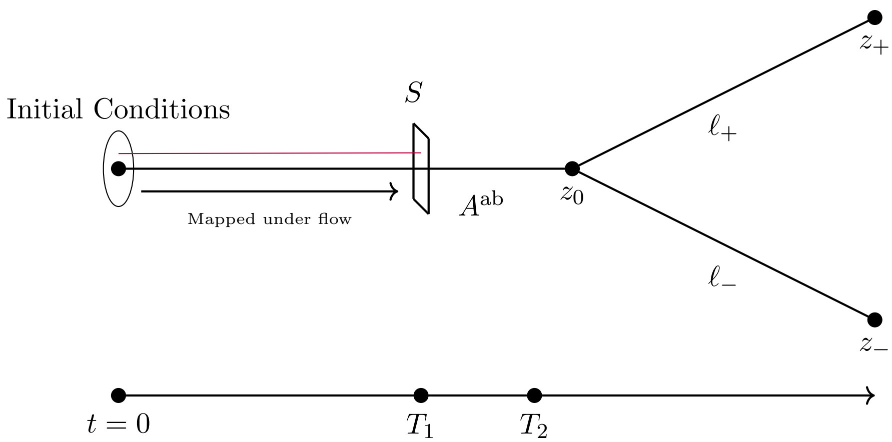

Figure 5.4: Schematic diagram showing a 2-dimensional subspace of initial conditions flowing to a 2-dimensional transverse manifold at $t = T_1$.

To study perturbations with respect to the initial data, we linearise the system around A^{ab}; for notational ease, we write

$$\widetilde{\Phi}_1 := \frac{\Phi_1}{2\dot{a}^3\dot{b}^2} \qquad \text{and} \qquad \widehat{\Phi}_1 := \frac{\Phi_1}{2\dot{a}^4\dot{b}}$$

and the same for Ψ_1 and χ_1.

The linearisation of the ODEs at $f = f' = h = 0$ is

$$A(t) = D_2(F+G)(t,(0,0,g,0)) = \begin{pmatrix} \widetilde{\chi}_1(g-1) & \widetilde{\Phi}_1 + \frac{1}{2}g(\widetilde{\Phi}_1 + \widetilde{\Psi}_1) & 0 & 0 \\ \widetilde{\Phi}_1 - \frac{1}{2}g(\widetilde{\Phi}_1 + \widetilde{\Psi}_1) & -\widetilde{\chi}_1(g+1) & 0 & 0 \\ 0 & 0 & \widehat{\Psi}_1 - \widehat{\Phi}_1 & 0 \\ 0 & 0 & 0 & \widehat{\Phi}_1 + \widehat{\Psi}_1 \end{pmatrix}.$$

The space of possible initial conditions on each bundle P_j is 2-dimensional, parametrised by f_0 and h_0, where such local solutions are given by the expansions given in Proposition 5.10, namely for $j = 2\nu - 1$ and $\nu \in \mathbb{Z}_+$

$$\begin{aligned} f(t) &= f_0 t^{\nu-1} + O(t^{\nu+1}), \\ f'(t) &= \frac{\beta^3(1-2h_0)+1-\nu}{4\nu r_0 \beta^2} f_0 t^{\nu} + O(t^{\nu+2}), \\ g(t) &= 2\nu - 1 + O(t^2), \\ h(t) &= h_0 + O(t^2). \end{aligned} \tag{5.30}$$

Given $Z_0 = (f_0, h_0) \in \mathbb{R}^2$, let $Z(t, Z_0)$ be a solution on P_j of (5.8) with initial conditions determined by Z_0 at $t = 0$. Following the results of Amman in Chapter 9 of [1], we consider a two-dimensional perturbation of the initial conditions of A^{ab}, which forms a disc in the phase space. The following lemma shows that under the flow, this disc becomes transverse to the stable manifold of the fixed point z_0. In particular, the disc contains a curve of initial conditions which is mapped to a subset of the intersection $W^u(z_0) \cap W^s_T(z_0)$.

Lemma 5.31. *Let $N(t, w) := D_2 Z(t, 0)w$ for $w \in \mathbb{R}^2$. For large times T, the image of the linear map $w \mapsto N(T, w)$ is transverse to the tangent space of $W^s(z_0)$.*

Proof. We show that $N(t, w)$ is orthogonal to both of the stable eigenvectors for the fixed point z_0, namely

$$(0, 0, 1, 0) \qquad \text{and} \qquad (1, -1, 0, 0).$$

Firstly, since g is independent of h, we see that $N(t, (0, 1))$ is proportional to the vector $(0, 0, 0, 1)$ for all t, and hence is orthogonal to $W^s(z_0)$. Next, we consider $w = (1, 0)$; we claim that $N(t, w)$ lies in the region

$$\mathcal{R} = \{(f, f', g, h) : f \geq f', f \geq -2f'\} \cup \{(f, f', g, h) : f \leq f', f \leq -2f'\}.$$

In fact, we now show that this condition is preserved by the linearised flow, hence it is enough to prove the claim for some $t > 0$.

First consider the boundary $f + 2f' = 0$ of $\mathcal{R}$ given by the vector $(2, -1)$ in the (f, f')-plane. Let B be the submatrix of A given by the first two columns and rows. Then applying B to $(2, -1)$ gives the direction of the flow at any time t at that point. We apply the cross product and take the z-component, i.e.

$$\left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} \times B \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right)_z = 3\widetilde{\Phi}_1 + \frac{1}{2} g(8\widetilde{\chi}_1 - 5(\widetilde{\Phi}_1 + \widetilde{\Psi}_1)).$$

We have that $\Phi_1 > 0$, $g > 0$ and $8\chi_1 - 5(\Phi_1 + \Psi_1) = (b + r_0^3)^2(8a - 5(b - r_0^3))$. By the conditions on a and b outlined in Remark 3.14, the metric functions a, b satisfy $ka - b + r_0^3 > 0$ for $k \in (1, 2)$, hence the z-component of this vector is positive. The right-hand rule for the cross product implies that the flow must be in the direction indicated in Figure 5.5.

On the other hand, taking the vector $(1, 1)$ gives

$$\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} \times B \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)_z = -g(\widetilde{\Phi}_1 + \widetilde{\Psi}_1 + 2\widetilde{\chi}_1) < 0$$

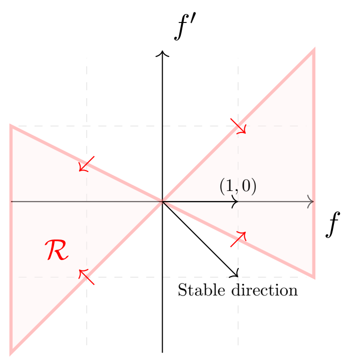

Figure 5.5: The directions of flow of (5.8) along the lines $f' = f$ and $2f' = -f$.

since $g > 0$ and $\Phi_1 + \Psi_1 + 2\chi_1 = (b + r_0^3)^2(b - r_0^3 + 2a) > 0$ by the conditions on a and b given in Remark 3.14. Hence the flow must be in the direction indicated in Figure 5.5. We have proved the claim that if $N(t,(1,0))$ lies in the region $\mathcal{R}$ for some $t > 0$, then it does for all time.

In fact, for any Z_0 with $h_0 = 0$ and f_0 sufficiently small and for small $t > 0$, we have

$$N(t,(1,0)) = \frac{\partial}{\partial f_0} Z(t,(f_0,0)) = t^{\nu-1}(1,0,0,0) + O(t^\nu).$$

So there exists $\epsilon > 0$ such that $N(\epsilon,(1,0)) \in \mathcal{R}$, and hence $N(t,(1,0)) \in \mathcal{R}$ for all time. We have proved that $N(T,(1,0))$ is orthogonal to $W^s(z_0)$ for large T and so the image of the linear map $w \mapsto N(T,w)$ is transverse to the stable manifold for large times T. □

In particular, we can choose initial conditions which flow to the 1-dimensional intersection of the transversal with the stable manifolds of the fixed points $z_\pm$ and hence obtain a 1-parameter family of global solutions. Each member of the 1-parameter family of solutions with limit z_+ is gauge equivalent to a corresponding member of the family tending to z_- under the $\mathrm{SU}(2)^2 \times \mathrm{U}(1)$-invariant gauge transformation of Remark 5.13. Hence there is a 1-parameter family of solutions on each bundle up to gauge transformation given by this construction.

Lemma 5.32. *We can parameterise this family of solutions by f_0.*

Proof. Indeed, $N(T,(0,1))$ is proportional to $v = (0,0,0,1)$ and v is invariant under both the linearisation $A(t)$ and the flow. However, $v \notin T_{z_\pm}W^s(z_\pm)$ and hence $N(T,(0,1))$ is not tangent to $W^s(z_\pm)$. So in a small neighbourhood of the origin in the (f_0,h_0)-plane, the projection to the f_0-axis of the curve of initial conditions corresponding to this family of perturbations is injective. □

To summarise, we have shown that a 2-dimensional perturbation of the initial condition of the abelian solution A^{ab} flows to a manifold transverse to the stable manifold of the fixed point z_0. By applying the Inclincation Lemma, we see that this transversal manifold then flows to become C^1-close to the unstable manifold of z_0 and hence lies in the intersection with the stable manifold of $z_\pm$. Since the non-autonomous part of the system is sufficiently small at large enough times, we can see that a curve of solutions in this intersection flows into the fixed points $z_\pm$ as $t \to \infty$. This curve of solutions exists for all $t \geq 0$ for f_0 in some interval $(-\epsilon,\epsilon)$, giving a 1-parameter family of global solutions. This completes the proof of Theorem 5.28.

5.4.4 Asymptotics of the solutions

The fixed point z_+ defines a connection on the nearly Kähler $(S^3 \times S^3)/\mathbb{Z}_4$, namely

$$A_\infty = \frac{1}{3}\sum_{i=1}^{3} E_i \otimes (e_i + e_i').$$

We can consider $S^3 \times S^3$ as the homogeneous manifold $G/K = \mathrm{SU}(2)^3/\Delta\,\mathrm{SU}(2)$ and let e_i'' be a coframe on the third SU(2)-factor. The nearly Kähler metric is induced from a multiple of the Cartan-Killing form

$$B(X,Y) = \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}(X)\mathrm{ad}(Y)), \qquad \forall X,Y \in \mathfrak{g}.$$

The subspace $\mathfrak{k}$ of $\mathfrak{g}$ is generated by $E_i + E_i' + E_i''$ for $i = 1,2,3$. Let $\mathfrak{m}$ be an orthogonal complement to $\mathfrak{k}$ with respect to B, then the canonical invariant connection on G/K is given by

$$A = \frac{1}{3}\sum_{i=1}^{3}(E_i + E_i' + E_i'') \otimes (e_i + e_i' + e_i'')$$

on the SU(2)-bundle P_λ over G/K, where $\lambda : \Delta\,\mathrm{SU}(2) \to \mathrm{SU}(2)$. Pulling A back to $\mathrm{SU}(2)^2$ via the inclusion map $\mathrm{SU}(2)^2 \to \mathrm{SU}(2)^3$ yields the connection A_∞. The tangent bundle of G/K can be identified with the vector bundle associated to $G \to G/K$ via the representation

$$\rho_{\mathfrak{m}} : K \to \mathrm{GL}(\mathfrak{m}).$$

Then A induces a connection on $T(G/K)$ which has holonomy contained in SU(3) and skew-symmetric torsion. Theorem 10.1 of [17] tells us that such a connection is unique, and hence it must coincide with the so-called canonical Hermitian connection. This connection is given by the formula

$$\nabla - \frac{1}{2}J(\nabla J)$$

where ∇ is the Levi-Civita connection and J is the almost complex structure on G/K. In summary, the limiting connection of the solutions in Theorem 5.28 is the canonical connection on the nearly Kähler manifold $(S^3 \times S^3)/\mathbb{Z}_4$.

CHAPTER 6

SUMMARY AND FURTHER STUDY

We saw in Chapter 1 that manifolds with conical asymptotic geometry are of interest since they could be used to desingularise certain compact G_2-manifolds. Further, recall that compact G_2-manifolds are a contributing component to M-theory, a physical framework of the universe postulated by physicists in 1995. Therefore, understanding G_2-manifolds in this context is of current interest to both mathematicians and physicists. We outlined that the study of G_2-instantons on these manifolds could provide evidence for the existence of an invariant, a tool for distinguishing G_2-manifolds, which was conjectured in [11, 12].

In this thesis, we have considered examples of G_2-instantons on three different G_2-manifolds, each with either asymptotically conical or locally asymptotically conical geometry. In each case we have used a different method to construct solutions and we can read off certain properties of the instantons from the equations.

- For the Bryant-Salamon $S^3 \times \mathbb{R}^4$, we have found a 1-parameter family of solutions which, together with the family constructed by Clarke [7], provide a complete moduli space of solutions in the $\mathrm{SU}(2)^3$-invariant case. The solutions are either flat or converge to the nearly Kähler instanton on $S^3 \times S^3$ along the conical end.
- Another member of the B_7-family is the ALC manifold known as the BGGG manifold. In this case, we have formally characterised all possible local solutions along the ALC end. Furthermore, we have proven the existence of a boundary in the space of initial conditions for which solutions are complete. While this boundary is not known explicitly, we have shown that solutions which have initial conditions on this boundary have certain limiting behaviour which differs from the other complete solutions.
- Finally, we have considered the AC member of the C_7-family. In this case, we constructed a 1-parameter family of solutions using a dynamical systems approach. We first perturb an abelian solution near the singular orbit and show that this perturbation converges to the unstable manifold of a fixed point of the system. Further, it intersects the stable manifold of another fixed point of the system along a curve. This yields a 1-parameter family of

complete solutions which converge to this limit point, a nearly Kähler instanton on a finite quotient of $S^3 \times S^3$.

We now set out some ideas for future work on these three examples. The goal in each case is to fully understand the moduli space of G_2-instantons. This would enable us to investigate the validity of the idea of Donaldson, Thomas and Segal for a G_2-manifold invariant. We now outline some questions in each case.

- For the Bryant-Salamon $S^3 \times \mathbb{R}^4$, we have found a 1-parameter family of solutions on each bundle. Lotay and Oliveira [24] describe bubbling and limiting behaviour of Clarke's family [7], linking the two families in the moduli space. We now want to understand whether there are any further solutions which have weaker symmetry than these $\mathrm{SU}(2)^3$-invariant instantons. We hope to use deformation theory, set out by Driscoll [13], to prove that unobstructed solutions are, in fact, $\mathrm{SU}(2)^3$-invariant.
- In the case of the BGGG manifold, we have now characterised solutions on one of the two possible bundles. While the equations are more complicated on the other bundle, Proposition 4.40 still applies since both bundles look the same asymptotically. Hence, the next step is to understand which of the local solutions of [24] near the singular orbit flow into these end solutions.
- Finally, we have considered the AC member of the C_7-family, but we would like to understand solutions on the infinite family of AC manifolds $M_{m,n}$. We have outlined in Chapter 5 why we cannot simply generalise the argument for $M_{1,1}$ when m and n are distinct. Further work would involve looking for modifications of the dynamical systems argument to apply to every $M_{m,n}$. On the other hand, if we can formalise Remark 5.27 of a gluing procedure and find abelian solutions on each $M_{m,n}$, we may be able to find solutions using this method.

In summary, we have used three different methods of finding solutions to ODEs to construct local or global G_2-instantons on three examples of G_2-manifolds with conical asymptotic geometry. We have provided a significant contribution to the hunt for G_2-instantons and used a novel methodology which could be applied in future work.

APPENDIX A

GAUGE THEORY NEAR THE SINGULAR ORBIT

Let H be the gauge group of a principal bundle P on M, parameterised by isotropy homomorphisms $\lambda : K \to H$. Let Q be the singular orbit of M, under a cohomogeneity one action given by the group diagram

$$K_0 \subset K \subset G.$$

Denote by $\mathfrak{h}$ the Lie algebra of H; then K acts on $\mathfrak{h}$ by $\mathrm{Ad} \circ \lambda$. We consider G-invariant elements of $\Omega^1(\mathrm{Ad}\, P)$. The invariance property means that such a 1-form is determined by its restriction to the fibre of $G \times_K V$ over the point $q \in Q$. Hence, it is given by a K-equivariant map $L : V \to W^* \otimes \mathfrak{h}$. Fix $v_0 = 1 \in S^1 \subset V$, then by K-equivariance, L is uniquely determined by the curve $t \mapsto L(tv_0)$ whose image lies in the subspace of K_0-invariant sections of $\Omega^1(\mathrm{Ad}\, P)$. Indeed, for each t, the restriction of such a 1-form to the corresponding orbit can be written as a map from the tangent space at a single point to the fibre over that point of the adjoint bundle, which is a copy of $\mathfrak{h}$. This is the map Λ given in (2.6). The equivariance condition on Λ with respect to each isotropy subgroup is exactly the condition that the corresponding section has values in the adjoint bundle.

To determine which invariant connections on P extend smoothly over Q, we want to consider the evaluation at v_0 of homogeneous K-invariant polynomials $\phi : V \to \mathrm{Hom}(W, \mathfrak{h}) \cong W^* \otimes \mathfrak{g}$ of minimal degree. Then Lemma 1.1 of [14] implies that the corresponding connection extends over the singular orbit Q if and only if the Taylor series expansion around zero of the coefficient of each component of the representation $W^* \otimes \mathfrak{h}$ has the same parity as the corresponding homogeneous equivariant polynomial ϕ and the first non-zero term has the same degree as the ϕ. We now see what these conditions mean practically for connections on bundles with gauge group $H = \mathrm{SU}(2)$.

A.1 $\mathbb{B}_7$ family

In this section, we re-derive the results of Appendix A.2 of [24] in our notation. Let $G = \mathrm{SU}(2)^2$ and $K = \Delta\, \mathrm{SU}(2)$; then the singular orbit is $Q = S^3 = G/K$ of the cohomogeneity one manifold

$M = S^3 \times \mathbb{R}^4 = G \times_K V$ where $V \cong \mathbb{H}$. Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Given a point $q \in Q$, identify T_qQ with $\mathfrak{p} \cong \mathrm{im}(\mathbb{H})$ and T_qM with $\mathfrak{p} \oplus V$.

Then TQ can be constructed homogeneously as

$$TQ = G \times_K \mathfrak{p}$$

where K acts trivially on $\mathfrak{p}$. Let $e_i^{\pm} = \frac{1}{2}(e_i \pm e_i')$. Thus e_1^-, e_2^-, e_3^- defines a coframe on $\mathfrak{p}$.

If t, x_1, x_2, x_3 are Euclidean coordinates on V, then we see from [16] that along the ray $t \in \mathbb{H} \cong V$, we have

$$e_i^+ = 2t^{-1}dx_i, \qquad i = 1, 2, 3.$$

Thus a 1-form with values in $\mathfrak{h}$ can be written as

$$\begin{aligned} b &= \sum_{i=1}^{3} b_i \otimes e_i + b_i' \otimes e_i' \\ &= \sum_{i=1}^{3} (b_i + b_i') \otimes e_i^+ + (b_i - b_i') \otimes e_i^- \\ &= \sum_{i=1}^{3} 2t^{-1}(b_i + b_i') \otimes dx_i + (b_i - b_i') \otimes e_i^-. \end{aligned} \tag{A.1}$$

$T(NQ)$ is modelled on $W = \mathrm{im}(\mathbb{H}) \oplus \mathbb{H}$ with K acting by

$$(a, a) \cdot (p, q) = (ap\bar{a}, q\bar{a}), \qquad (a, a) \in K, (p, q) \in W.$$

Given a map $\mu : K \to H$, we pull back $P_\mu = G \times_{(K,\mu)} \mathfrak{h}$ back to $S^3 \times \mathbb{R}^4$, which determines the extension. Here, K acts on $\mathfrak{h}$ via $\mathrm{Ad} \circ \mu$. We search for a basis of $\mathrm{Hom}(W, \mathfrak{h})$ given by evaluation at 1 of homogeneous K-equivariant polynomial $\mathbb{H} \to \mathrm{Hom}(W, \mathfrak{h})$. Such homogeneous polynomials $x \mapsto \phi(x)$ must satisfy

$$\phi(x)(p, q) = \mathrm{Ad} \circ \mu(x) \phi(1)(\bar{x}px, qx)$$

for $x \in \mathrm{SU}(2)$ and $(p, q) \in W$.

When $H = \mathrm{SU}(2)$, μ is either trivial or the identity map. Then $\mathfrak{h} = \mathfrak{su}(2) \cong \mathrm{im}(\mathbb{H})$ and $\mathrm{Ad} \circ \mu$ is trivial or the adjoint action respectively. We consider each case separately.

$\mu = \mathbf{id}$

In the case where μ is the identity map, K acts on $\mathfrak{h} = \mathfrak{su}(2)$ by $\mathrm{Ad}(x)q = xq\bar{x}$. Also,

$$\mathrm{Hom}(W, \mathrm{im}(\mathbb{H})) \cong (\mathrm{im}(\mathbb{H}) \otimes \mathrm{im}(\mathbb{H})^*) \oplus (\mathrm{im}(\mathbb{H}) \otimes \mathbb{H}^*).$$

As in the case of $H = \mathrm{U}(1)$, we treat each summand of separately.

We begin with homogeneous polynomials $\mathbb{H} \to \mathrm{im}(\mathbb{H}) \otimes \mathrm{im}(\mathbb{H})^*$ given by $x \mapsto \psi(x)$ with $\psi(x)(p) = x(\psi(1)(\bar{x}px))\bar{x}$ for $x \in \mathrm{SU}(2)$. Consider the degree 4 polynomials $\psi(x)(p) = \langle p, x\ell_1\bar{x}\rangle x\ell_2\bar{x}$ for $\ell_1, \ell_2 \in \{i, j, k\}$; this corresponds under evaluation at 1 to $E_i \otimes e_j^-$ for $i, j = 1, 2, 3$. There is also the constant polynomial, which corresponds to $\sum_{i=1}^3 E_i \otimes e_i^-$.

For homogeneous polynomials $\mathbb{H} \to \mathrm{im}(\mathbb{H}) \otimes \mathbb{H}^*$, we have degree 1 polynomials $\psi_1(x)(q) = -\bar{q}\bar{x} + \langle q, x\rangle$ and $\psi_\ell(x)(q) = \bar{q}\ell\bar{x} - \langle q, x\ell\rangle$ for $\ell \in \{i, j, k\}$. These polynomials correspond under evaluation at 1 to

$$\begin{gathered} E_1 \otimes dx_1 + E_2 \otimes dx_2 + E_3 \otimes dx_3, \quad E_1 \otimes dt - E_3 \otimes dx_2 + E_2 \otimes dx_3, \\ E_2 \otimes dt + E_3 \otimes dx_1 - E_1 \otimes dx_3, \quad E_3 \otimes dt - E_2 \otimes dx_1 + E_1 \otimes dx_2. \end{gathered}$$

We also have the degree 3 polynomials $\psi(x)(q) = \langle \ell_1 x, q\rangle x\ell_2\bar{x}$ which correspond to $E_j \otimes dx_i$ and $E_j \otimes dt$.

With this in mind, we rewrite (A.1) firstly with coefficients $b_i = \sum_{j=1}^3 b_{ij}E_j$, $b_i' = \sum_{j=1}^3 b_{ij}'E_j$ in $\mathrm{SU}(2)$, and then as

$$\begin{aligned} b = &\sum_{i=1}^3 ((b_{11} - b_{11}')E_i \otimes e_i^- + (b_{ii} - b_{11} - b_{ii}' + b_{11}')(E_i \otimes e_i^-) + \sum_{j \neq i}(b_{ij} - b_{ij}')E_j \otimes e_i^-) \\ &+ 2t^{-1}\sum_{i=1}^3 ((b_{11} + b_{11}')E_i \otimes dx_i + (b_{ii} - b_{11} + b_{ii}' - b_{11}')(E_i \otimes dx_i) + \sum_{j \neq i}(b_{ij} + b_{ij}')E_j \otimes dx_i). \end{aligned}$$

Then the b_{ij} are even with $b_{ij}(0) = 0$, while $2t^{-1}b_{11}'$ is odd of degree 1, $2t^{-1}(b_{ii}' - b_{11}')$ is odd of degree 3 and $2t^{-1}b_{ij}'$ is odd of degree 3 for distinct i, j. In summary, we have

- $b_{ii} = O(1)$ and $b_{ii}' = O(1)$ with $b_{ii}(0) = -b_{ii}'(0)$; i.e. there exists $\alpha, \beta, \gamma \in \mathbb{R}$ such that
$$b_{ii}(t) = \alpha + \beta t^2 + O(t^4) \;\text{ and }\; b_{ii}'(t) = -\alpha + \gamma t^2 + O(t^4)$$
- $b_{ij}' = O(t^4)$ and $b_{ij} = O(t^4)$ for distinct i, j.

$\mu = \mathbf{1}$

In the case where μ is trivial, we have degree 1 polynomials corresponding to the 1-forms $E_j \otimes dx_i$ and $E_j \otimes dt$, and degree 0 polynomials corresponding to 1-forms $E_j \otimes e_i^-$. Thus, the $\mathrm{SU}(2)$-valued coefficients of a general 1-form b satisfy the properties that b_{ij}, b_{ij}' are even and $b_{ij}(0) = b_{ij}'(0) = 0$.

In the $\mathrm{SU}(3)^3$-invariant setting, we have the following families of local solutions.

Proposition A.2. *On P_{id}, there is a 1-parameter family of local solutions to the* $\mathrm{SU}(2)^3$*-invariant G_2-instanton equations of the form*

$$\begin{aligned} f^+(t) &= 1 + \frac{\gamma^2 - 1}{32r_0^2}t^2 + O(t^4), \\ f^-(t) &= \gamma + \frac{\gamma(\gamma^2 - 1)}{16r_0^2}t^2 + O(t^4). \end{aligned}$$

Proof. The canonical invariant connection on P_{id} has the form $A^{\mathrm{can}} = \frac{1}{2}\sum_{i=1}^{3} E_i \otimes (e_i + e_i')$, and so we apply the smooth extension results to the form

$$\begin{aligned} A - A^{\mathrm{can}} =& \tfrac{1}{2}(f^+ - 1)(E_1 \otimes (e_1 + e_1') + E_2 \otimes (e_2 + e_2') + E_3 \otimes (e_3 + e_3')) \\ &+ \tfrac{1}{2} f^-(E_1 \otimes (e_1 - e_1') + E_2 \otimes (e_2 - e_2') + E_3 \otimes (e_3 - e_3')). \end{aligned}$$

We set this up as a singular initial value problem in the usual way. Let

$$f^+ = 1 + X(t)t^2, \qquad f^- = \gamma + Y(t)t^2$$

with $y = (X, Y)$ and

$$y_0 = \left(\frac{\gamma^2 - 1}{32 r_0^2}, \frac{\gamma(\gamma^2 - 1)}{16 r_0^2} \right).$$

Then we can write the G_2-instanton equations as

$$\dot{y} = \frac{1}{t} M_{-1}(y) + M(t, y), \qquad y(0) = y_0$$

where $M_{-1}(y_0) = 0$ and

$$d_{y_0} M_{-1} = \begin{pmatrix} -4 & 0 \\ 4\gamma & -2 \end{pmatrix}.$$

Then $h\mathrm{Id} - d_{y_0} M_{-1} = (h + 2)(h + 4) \neq 0$ for all $h \in \mathbb{Z}_{>0}$, yielding a unique local solution of the form given in the statement of the proposition. □

Proposition A.3. *On P_1, there is a 1-parameter family of local solutions to the* $\mathrm{SU}(2)^3$*-invariant G_2-instanton equations of the form*

$$\begin{aligned} f^+(t) &= \delta t^2 + O(t^4), \\ f^-(t) &= 0. \end{aligned}$$

Proof. On P_1, the canonical invariant connection vanishes as an element of $\Omega^1(M; \mathrm{Ad}(P_1))$. Hence, we apply the results of Section A to the 1-form

$$\begin{aligned} A - A^{\mathrm{can}} =& \tfrac{1}{2} f^+(E_1 \otimes (e_1 + e_1') + E_2 \otimes (e_2 + e_2') + E_3 \otimes (e_3 + e_3')) \\ &+ \tfrac{1}{2} f^-(E_1 \otimes (e_1 - e_1') + E_2 \otimes (e_2 - e_2') + E_3 \otimes (e_3 - e_3')). \end{aligned}$$

In this case, we have that all defining functions are even and $f^+(0) = f^-(0) = 0$.

We set this up as a singular initial value problem in the usual way. Let

$$f^+ = X(t)t^2, \qquad f^- = Y(t)t^2$$

with $y = (X, Y)$ and

$$y_0 = (\delta, 0).$$

Then we can write the G_2-instanton equations as

$$\dot{y} = \frac{1}{t} M_{-1}(y) + M(t, y), \qquad y(0) = y_0$$

where $M_{-1}(y_0) = 0$ and

$$d_{y_0}M_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & -6 \end{pmatrix}.$$

Then $h\mathrm{Id} - d_{y_0}M_{-1} = h(h+6) \neq 0$ for all $h \in \mathbb{Z}_{>0}$, yielding a unique local solution of the form given in the statement of the proposition.

The vanishing of f^- is from uniqueness of solutions. □

A.2 $\mathbb{C}_7$ family

For the G_2-manifold $M_{m,n}$, the singular orbit is $Q = S^3 \times S^3/K_{m,n}$; we now determine the conditions for an invariant connection to extend smoothly over Q. This singular orbit is a circle bundle over $D = S^2 \times S^2 = \mathrm{SU}(2)^2/T^2$, i.e. $Q = \mathrm{SU}(2)^2 \times_{T^2} S^1$, and $M_{m,n}$ is the total space of a $\mathbb{C} \times S^1$-bundle over D, i.e. $M_{m,n} = \mathrm{SU}(2)^2 \times_{T^2} (S^1 \times \mathbb{C})$. Here, T^2 acts on S^1 with weight (m,n), and on $\mathbb{C}$ with weight $(2,-2)$. Since m,n are coprime, $K_{m,n} \cong \mathrm{U}(1)$ is embedded in T^2 by $e^{i\theta} \mapsto (e^{in\theta}, e^{-im\theta})$.

Define $\mathfrak{p}$ by $\mathfrak{su}(2) \oplus \mathfrak{su}(2) = \mathrm{Lie}(K_{m,n}) \oplus \mathfrak{p}$, then given a point $q \in Q$, we can identify T_qQ with $\mathfrak{p}$ and T_qM with $\mathfrak{p} \oplus V$. As a $K_{m,n}$-representation, $\mathfrak{p}$ is induced by the T^2-representation $\mathbb{R} \oplus \mathbb{C}_{2,0} \oplus \mathbb{C}_{0,2}$, and so as a real representation, $\mathfrak{p} = \mathbb{R} \oplus \mathbb{R}^2_{2n} \oplus \mathbb{R}^2_{2m}$. Let t, x be coordinates on $V \cong \mathbb{R}^2$, then we see from [16] that the coframe

$$se_3 - re_3' = \frac{1}{(m+n)t}dx;$$

also, recall that e_1, e_2 and e_1', e_2' are coframes on $\mathbb{R}^2_{2n}$ and $\mathbb{R}^2_{2m}$ respectively, while $me_3 + ne_3'$ generates the trivial $\mathbb{R} \subset \mathfrak{p}$.

A tubular neighbourhood of Q in $M_{m,n}$ is equivariantly diffeomorphic to a neighbourhood of the zero section in the vector bundle $\mathrm{SU}(2)^2 \times_{K_{m,n}} V \to Q$, where V is the irreducible 2-dimensional real representation of $K_{m,n}$ with weight $2(m+n)$. In other words, the tubular neighbourhood is modelled on $W = V \oplus \mathfrak{p} = \mathbb{R}^2_{2(m+n)} \oplus \mathbb{R} \oplus \mathbb{R}^2_{2n} \oplus \mathbb{R}^2_{2m}$. By identifying $\mathbb{R}^2 \cong \mathbb{C}$, the action of $\zeta \in K_{m,n} \cong \mathrm{U}(1)$ on $(z_1, y, z_2, z_3) \in W$ is given by

$$\zeta \cdot (z_1, y, z_2, z_3) = (\zeta^{2(m+n)}z_1, y, \zeta^{2n}z_2, \zeta^{-2m}z_3).$$

The action of $K_{m,n}$ on $\mathfrak{g} = \mathfrak{su}(2)$ is given by $\mathrm{Ad} \circ \lambda_j$ on the bundle P_j. We use the matrix basis of $\mathfrak{su}(2)$ given by

$$E_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad E_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Irreducible invariant connections are determined by a morphism of $\mathbb{Z}_{2(m+n)}$-representations

$$\Lambda : (\mathbb{R} \oplus \mathbb{C}_{2n}) \oplus (\mathbb{R} \oplus \mathbb{C}_{-2m}) \to \mathbb{R} \oplus \mathbb{C}_{2j}$$

only if $j \equiv n \mod 2(m+n)$ or $j \equiv -m \mod 2(m+n)$. Since $\mathbb{C}_{2n} = \mathbb{C}_{-2m}$ as $\mathbb{Z}_{2(m+n)}$-representations, there are no restrictions on Λ for these choices of j.

Lemma A.4. *Consider the manifold $M_{m,n}$. Then the invariant connection over the principal orbits*

$$A = A_{12}(E_1\otimes e_1+E_2\otimes e_2)+A'_{12}(E_1\otimes e'_1+E_2\otimes e'_2)+A_{rs}(E_3\otimes(se_3-re'_3))+A_{mn}(E_3\otimes(me_3+ne'_3))$$

on P_j with $j \equiv n \mod 2(m+n)$ or $j \equiv -m \mod 2(m+n)$ extends to a smooth section of $\Omega^1(\operatorname{Ad} P_j)$ if and only if A_{rs}, A_{mn} are even, with $A_{rs}(0) = 0$, and we have

$$\begin{cases} A_{12} \text{ even and } O(t^{|\frac{j-n}{m+n}|}), A'_{12} \text{ odd and } O(t^{|\frac{j+m}{m+n}|}) & \text{for } j \equiv n \mod 2(m+n); \\ A_{12} \text{ odd and } O(t^{|\frac{j-n}{m+n}|}), A'_{12} \text{ even and } O(t^{|\frac{j+m}{m+n}|}) & \text{for } j \equiv -m \mod 2(m+n). \end{cases}$$

Proof. Firstly, note that $(\operatorname{Ad}\circ\lambda_j)(e^{i\theta})E_3 = E_3$. Then, considering the subspace $V \subset W$, we can define an equivariant polynomial $V \to \operatorname{Hom}(V, \mathfrak{su}(2))$ as

$$\phi_\ell(x)(z_1) = \langle \ell x, z_1\rangle E_3 \text{ for } \ell \in \{1, i\}.$$

These are equivariant homogeneous degree 1 polynomials which correspond upon evaluation at $v_0 = 1$ to the 1-forms dt for $\ell = 1$ and dx for $\ell = i$. Then, the coefficient of $dx = (m+n)t(se_3-re'_3)$ must be odd and be of order $O(t)$. Hence, the coefficient A_{rs} of $se_3 - re'_3$ must be even and of order $O(t^2)$.

On the other hand, for the trivial part of $\mathfrak{p}$ which is generated by $me_3+ne'_3$, we take the constant polynomial which maps to the identity in $\operatorname{Hom}(V, \mathfrak{su}(2))$. Then since the action of $K_{m,n}$ is trivial, the equivariance condition is satisfied. Thus the coefficient A_{mn} of $me_3 + ne'_3$ must be even.

Consider the bundle P_j with $j \equiv n, -m \mod 2(m+n)$, then the representation $(\mathbb{R}^2_{2n})^* \otimes \mathfrak{su}(2)$ of $K_{m,n}$ has weight $2(j-n)$. The space of homogeneous polynomials of degree d on V is isomorphic to $\operatorname{Sym}^d V$, and $K_{m,n}$ acts on $\operatorname{Sym}^d V$ with weight $2d(m+n)$. Then the smallest d such that the corresponding degree d polynomial is equivariant is $d = \frac{j-n}{m+n}$. Similarly, the smallest d such that the corresponding degree d polynomial with values in $(\mathbb{R}^2_{-2m})^* \otimes \mathfrak{su}(2)$ is equivariant is $d = \frac{j+m}{m+n}$.

In conclusion, if $j \geq 0$, then on P_j, we have that A_{12} is $O(t^{\frac{j-n}{m+n}})$ and has the same parity as $\frac{j-n}{m+n}$, while A'_{12} is $O(t^{\frac{j+m}{m+n}})$ and has the same parity as $\frac{j+m}{m+n}$.

Finally, if $j < 0$, a similar calculation shows, on P_j, that A_{12} is $O(t^{\frac{-j+n}{m+n}})$ and the same parity as $\frac{-j+n}{m+n}$, while A'_{12} is $O(t^{\frac{-j-m}{m+n}})$ and the same parity as $\frac{-j-m}{m+n}$. □

APPENDIX B

ADDITIONAL DYNAMICAL SYSTEMS THEORY

The material in this section can be found in [31]; it is designed to give the necessary background theory in dynamical systems used in the proof of Theorem 5.28. More precisely, we set out the conditions for which a perturbation of an autonomous system by a non-autonomous term still yields a stable manifold.

Consider the non-autonomous system

$$\dot{x}(t) = A(t)x(t) + g(t). \tag{B.1}$$

We can produce an ansatz for the solution of such an equation, known as a Variation of Constants, which takes the form

$$x(t) = \Pi(t, t_0)c(t), \qquad c(t_0) = x(t_0) = x_0.$$

Here, Π is called the principal matrix solution of the system

$$\begin{cases} \dot{\Pi}(t, t_0) = A(t)\Pi(t, t_0) \\ \Pi(t_0, t_0) = \mathrm{Id}. \end{cases}$$

Substituting this ansatz solution yields the integral equation

$$c(t) = x_0 + \int_{t_0}^{t} \Pi(t_0, s)g(s)ds$$

and hence we have the following theorem.

Theorem B.2. *The solution of* (B.1) *corresponding to the initial condition* $x(t_0) = x_0$ *is*

$$x(t) = \Pi(t, t_0)x_0 + \int_{t_0}^{t} \Pi(t, s)g(s)ds.$$

Now we consider the autonomous system

$$\dot{x} = A(x - x_0) + g(x)$$

where $g(x)$ is the non-linear part. Assume $x_0 = 0$ is a hyperbolic fixed point. Then we can apply Theorem B.2 to rewrite the equation as

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-r)A}g(x(r))dr.$$

Denote by $x_\pm$ the projection $P^\pm$ of $x(0)$ onto the stable and unstable subspaces. We want to understand the condition on $x(0) = x_+ + x_-$ for which $x(t)$ remains bounded for all time. Let $P(t)$ be the function defined by P^+ for $t > 0$ and $-P^-$ for $t \leq 0$. Then Teschl writes x_- as a function of x_+ and hence rewrites $x(t)$ as $x(t) = K(x)(t)$ where

$$K(x)(t) = e^{tA}x_+ + \int_0^\infty e^{(t-r)A}P(t-r)g(x(r))dr.$$

We now specialise to the equation of interest (5.20), namely

$$\frac{dz}{d\tau} = F(z) + G(z,\tau).$$

Let t_0 be the initial time and temporarily set $G = 0$. Then we can write $z(t) = K(z)(t)$ where

$$K(z)(t) = e^{(t-t_0)DF(0)}z_+ + \int_{t_0}^\infty e^{(t-r)DF(0)}P(t-r)(F(z(r)) - DF(0)z(r))dr.$$

With this setup, we can apply Theorem 9.3 of [31]. The result is that for all t_0, and all z_+ in a small enough neighbourhood of 0 within the stable eigenspace, K is a contraction in the space of bounded functions $C_0^k([t_0,\infty),\mathbb{R}^4)$ and has a fixed point $\psi(t, z_+)$ by the contraction principle. Thus the stable manifold is given as $z_+ + P^-\psi(t_0, z_+)$.

We now adapt the proof of Theorem 9.3 of [31] in the case where G is non-zero. Then the function $K(z)$ becomes

$$K_G(z)(t) = e^{(t-t_0)DF(0)}z_+ + \int_{t_0}^\infty e^{(t-r)DF(0)}P(t-r)(F(z(r)) - DF(0)z(r) + G(z(r),r))dr.$$

By (5.22), for t_0 large enough, we have

$$|G(z(r),r)| \leq ce^{-t_0} \tag{B.3}$$

can be made arbitrarily small and G is locally Lipschitz. Then,

$$\begin{aligned} &\|K_G(x) - K_G(y)\| \\ &\quad = \sup_{t\geq 0}\left|\int_{t_0}^\infty e^{(t-r)DF(0)}P(t-r)(F(x) - F(y) - DF(0)(x-y) + (G(x,r) - G(y,r)))dr\right| \\ &\quad \leq C\sup_{t\geq 0}\int_{t_0}^\infty e^{-\alpha_0|t-r|}(|F| + |G|)dr \end{aligned}$$

where α_0 is less than the minimum real part of the eigenvalues of $DF(0)$. Then, since the Jacobian of $F - DF(0)$ vanishes at 0, we have $|F| \leq \epsilon|x-y|$; together with (B.3), we see that K_G is a contraction for a large enough choice of t_0. By the contraction principle, we again have a fixed point which defines the stable manifold at the fixed point.

Hence, a small perturbation of an autonomous system by a non-autonomous term does not affect the existence of a stable manifold at the fixed point.

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