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ENGEL GROUPS

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Abstract

We give a survey on Engel groups with particular emphasis on the development during the last two decades.

0   Introduction

We define the n-Engel word $e_n(x, y)$ as follows $e_0(x, y) = x$, $e_{n+1}(x, y) = [e_n(x, y), y]$. We say that a group G is an Engel group if for each pair of elements $a, b \in G$ we have $e_n(a, b) = 1$ for some positive integer $n = n(a, b)$. If n can be chosen independently of a, b then G is an n-Engel group.

One can also talk about Engel elements. An element $a \in G$ is said to be a left Engel element if for all $g \in G$ there exists a positive integer $n = n(g)$ such that $e_n(g, a) = 1$. If instead one can for all $g \in G$ choose $n = n(g)$ such that $e_n(a, g) = 1$ then a is said to be a right Engel element. If in either case we can choose n independently of g then we talk about left n-Engel or right n-Engel element respectively.

So to say that a is left 1-Engel or right 1-Engel element is the same as saying that a is in the center and a group G is 1-Engel if and only if G is abelian. Every group that is locally nilpotent is an Engel group. Furthermore for any group G we have that all the elements of the locally nilpotent radical are left Engel elements and all the elements in the hyper-center are right Engel elements. For certain classes of groups the converse is true and the locally nilpotent radical and the hyper-center can be characterised as being the set of the left Engel and right Engel elements respectively. There was in particular much activity in this area in the 50's and the 60's.

In this paper we will for the most part omit discussion on Engel elements and focus instead on Engel groups. In particular we are interested in the varieties of n-Engel groups. Our survey is by no means meant to be complete. The main aim is to discuss some central results obtained in the last two decades on n-Engel groups as well as giving some general background to these. The material is organised as follows. As the Engel groups and the Burnside problems are closely related and both originate from the 1901 paper of W. Burnside [7], our survey begins with the work of Burnside on 2-Engel groups. Apart from this the first main result on Engel groups is Zorn's Theorem that tells us that any finite Engel group is nilpotent. In section 2 and 3 we look at a number of generalisations of this. In section 4 we

look into the structure of n-Engel groups. The main open question here is whether n-Engel groups are locally nilpotent and we first demonstrate that if this is not the case then there must exist a finitely generated non-abelian simple n-Engel group. For the remainder of Section 4 we describe some results on the structure of locally nilpotent n-Engel groups. Sections 5 and 6 are then devoted to 3-Engel and 4-Engel groups. We end the survey by looking at some recent generalisations of Engel groups. Although this is a survey, some of the material is new. In particular much of our treatment of 3-Engel groups differs from the original one. The proof of the local nilpotence of 3-Engel groups is new and we have made use of Lie ring methods to give a shorter proof of another of Heineken's central results (Theorem 5.6). Our hope is that this survey will be a useful starting point for a graduate student entering this area and whenever possible we have tried to include proofs when they are short.

1   Origin and early results

> ... Although some useful facts have been brought to light about Engel groups by K. W. Gruenberg and also by the attempts on the Burnside problem, the word problem for E_n remains unsolved for $m > 2$. Problems such as these still seem to present a formidable challenge to the ingenuity of algebraists. In spite of, or perhaps because of, their relatively concrete and particular character, they appear, to me at least, to offer an amiable alternative to the ever popular pursuit of abstractions.   (P. Hall [26], 1958)

As indicated by P. Hall's paragraph above, the theory of Engel groups and the Burnside problems are closely related. The common origin is the famous 1901 paper [7] of W. Burnside where he formulated the Burnside problems. In particular he asked whether a finitely generated group of bounded exponent must be finite. Burnside proves that this is the case for groups of exponent 3 and observes that these groups have the property that any two conjugates a, a^b commute. Let us see briefly why this is the case. Let $a, b \in G$. Then $1 = (ba)^3 = b^3 a^{b^2} a^b a$ which implies that

$$a^{b^2} a^b a = 1. \tag{1}$$

Replacing b by b^2 in (1) gives

$$a^b a^{b^2} a = 1 \tag{2}$$

From (1) and (2) it is clear that a and a^b commute or equivalently that a and $[b, a] = a^{-b} a$ commute. Thus every group of exponent 3 satisfies the 2-Engel identity

$$[[y, x], x] = 1.$$

It is clear that every finitely generated abelian periodic group is finite and it seems natural to think that the fact that groups of exponent 3 satisfy this weak commutativity property is the reason that they are locally finite. It comes thus hardly as a surprise that Burnside wrote a sequel to this paper [8] a year later where he singles out this weak commutativity property. This paper seems to have received

surprisingly little attention, being the first paper written on Engel groups. In this paper Burnside proves that any 2-Engel group satisfies the laws

$$[x, y, z] = [y, z, x] \tag{3}$$
$$[x, y, z]^3 = 1. \tag{4}$$

In particular every 2-Engel group without elements of order 3 is nilpotent of class at most 2. Burnside failed however to observe that these groups are in general nilpotent of class at most 3, although he proved (in modern terminology) that any periodic 2-Engel group is locally nilpotent. It was C. Hopkins [29] that seems to have been the first to show that the class is at most 3. So any 2-Engel group also satisfies

$$[x, y, z, t] = 1. \tag{5}$$

Hopkins also observes that (3)-(5) characterize 2-Engel groups. This transparent description of the variety of 2-Engel groups is usually attributed to Levi [33], although his paper appears much later.

Of course this settles the study of 2-Engel groups no more than knowing that the variety of abelian groups is characterised by the law $[x, y] = 1$ settles the study of abelian groups. For example, the following well known problems raised by Caranti [37] still remain unsolved.

Problem 1. (a) Let G be a group of which every element commutes with all its endomorphic images. Is G nilpotent of class at most 2?

(b) Does there exist a finite 2-Engel 3-group of class three such that $\text{Aut}\, G = \text{Aut}_c G \cdot \text{Inn}\, G$ where $\text{Aut}_c G$ is the group of central automorphisms of G?

2 Zorn's Theorem and some generalisations

Moving on from the work of Burnside and Hopkins on 2-Engel groups the first main general result on Engel groups is Zorn's Theorem [63].

Theorem 2.1 *(Zorn) Every finite Engel group G is nilpotent.*

To prove this one argues by contradiction and taking G to be a minimal non-nilpotent Engel group one first makes use of a well known result of Schmidt that G must be solvable and then the shortest way of finishing the proof is to apply a result that appeared later, namely Gruenberg's Theorem (see Theorem 2.2).

Thus the Engel condition is a generalised nilpotence property. It is not difficult to find examples that show that in general it is weaker than nilpotence. For example for any given prime p the standard wreath product

$$G(p) = C_p \,\text{wr}\, C_p^\infty$$

is a $(p+1)$-Engel p-group that is non-nilpotent. This example also shows that for any given n and prime $p < n$ there exists a non-nilpotent metabelian n-Engel p-group. As the group

$$H(p) = C_p \operatorname{wr} C_p^n$$

is nilpotent of class n it is also clear that for a given n the nilpotence class of a finite n-Engel p-groups is not bounded when $p < n$ and this is even true under the further assumption that the group is metabelian.

The question that now arises is to what extent can Zorn's Theorem be generalised. The most natural question to ask here is whether one can replace "finite" by "finitely generated". One neccessary condition for nilpotence of a finitely generated group is residual finiteness. However we have the famous examples of Golod [16] that give for each prime p an infinite 3-generator residually finite p-group with all 2-generator subgroups finite. In particular these groups are non-nilpotent Engel groups. On the positive side we have that some other necessary conditions, namely solvability [18] and the max condition [3], are sufficient.

Theorem 2.2 *(Gruenberg) Every finitely generated solvable Engel group is nilpotent.*

Theorem 2.3 *(Baer) Every Engel group satisfying the max condition is nilpotent.*

So according to Theorem 2.3 we have that if every subgroup of the Engel group G is finitely generated then it must be nilpotent. Let us see briefly why Theorem 2.2 holds. For the proof we will apply the notion of a group being restrained. This is a very useful property introduced by Kim and Rhemtulla [32].

Definition A group G is said to be *restrained* if

$$\langle a \rangle^{\langle b \rangle}$$

is finitely generated for all $a, b \in H$.

Notice that in every n-Engel group $\langle a \rangle^{\langle b \rangle}$ is generated by $a, a^b, a^{b^2}, \ldots a^{b^{n-1}}$ so every n-Engel group is restrained. We next prove an elementary but very useful lemma [32].

Lemma 2.4 *Let G be a finitely generated restrained group. If H is a normal subgroup of G such that G/H is cyclic, then H is finitely generated.*

Proof As G is finitely generated we have $G = \langle h_1, \ldots, h_r, g \rangle$ with $h_1, \ldots, h_r \in H$. Then

$$H = \langle h_1, \ldots, h_r \rangle^G \cdot \langle g^m \rangle$$

where m is the order of gH in G/H. So H is generated by g^m and

$$\langle h_1 \rangle^{\langle g \rangle} \cup \cdots \cup \langle h_r \rangle^{\langle g \rangle}.$$

As G is restrained each subset in this union is finitely generated. Hence H is finitely generated. $\square$

From this we get the following easy corollary.

Lemma 2.5 *Let G be a finitely generated restrained group. Then G' is finitely generated.*

From this one can easily prove Gruenberg's Theorem. By a well known result of P. Hall we have that if N is normal subgroup of G such that $G/[N, N]$ and N are nilpotent, it follows that G is nilpotent. We use this to prove Gruenberg's Theorem by induction on the derived length of G. Of course this is obvious when G is abelian. For the induction step suppose that G has derived length $r > 1$. By the lemma above we have that $N = [G, G]$ is a finitely generated subgroup of derived length $r - 1$ and thus nilpotent by the induction hypothesis. By P. Hall's result we now only need to show that $G/[N, N]$ is nilpotent but it is an easy exercise to show that a finitely generated metabelian Engel group is nilpotent.

Remark. There is a generalisation of Baer's Theorem due to Peng [42] that says that it suffices to have the max condition on the abelian subgroups. Further generalisations of Baer's and Gruenberg's theorem can be found in [10,11].

We end this section by mentioning another generalisation of Zorn's Theorem that is analogous to a theorem of Burnside on periodic linear groups. Burnside proved that any linear group of bounded exponent is finite. The corresponding result for Engel groups is a theorem of Suprenenko and Garščuk [15].

Theorem 2.6 *Any linear Engel group is nilpotent.*

By the examples of Golod there are finitely generated Engel groups that are not nilpotent. The following problem is the most important open question on Engel groups. We will come back to it later.

Problem 2. Is every finitely generated n-Engel group nilpotent?

3   More recent generalisations of Zorn's Theorem

Many of the results proved in the last two decades rely on Zel'manov's solution to the restricted Burnside problem that is based on some deep results on Engel Lie rings.

3.1 Engel Lie rings

These are the Lie algebra analogs of Engel groups. We say that a Lie algebra L is an Engel Lie algebra if for each $u, v \in L$ there exists an integer $n = n(u, v)$ such

$$u \underbrace{v \cdots v}_{n} = 0.$$

(We are using here the left bracketing convention). If n can be chosen independently of u, v then we say that L is an n-Engel Lie algebra. The connection between Engel groups and Engel Lie algebras is established through the associated Lie ring. Consider the lower central series

$$G = G_1 \geq G_2 \geq \ldots$$

where $G_{i+1} = [G_i, G]$. It is well known that $[G_i, G_j] \leq G_{i+j}$. Let $L_i = G_i/G_{i+1}$ and consider the abelian group

$$L(G) = L_1 \oplus L_2 \oplus \cdots$$

that we turn into a Lie ring by first letting $aG_{i+1} \cdot bG_{j+1} = [a, b] \cdot G_{i+j+1}$ for $aG_{i+1} \in L_i$ and $bG_{j+1} \in L_j$ and then extending linearly to the whole of L. The standard commutator identities including the Hall-Witt identity imply that $L(G)$ is a Lie ring.

Suppose now that G is an n-Engel group and let $y, x_1, \ldots, x_n$ be variables. Expanding

$$1 = [y, \underbrace{x_1 \cdots x_n, \ldots, x_1 \cdots x_n}_{n}]$$

gives

$$1 = \Big(\prod_{\sigma \in S_n} [y, x_{\sigma(1)}, \ldots, x_{\sigma(n)}] \Big) \cdot z$$

where z is a product of commutators involving all of $y, x_1, \ldots, x_n$ and at least one of them twice. Take any elements $v = bG_{j+1} \in L_j$, $u_1 = a_1 G_{i_1+1} \in L_{i_1}$, ..., $u_n = a_n G_{i_n+1} \in L_{i_n}$. Then for any $\sigma \in S_n$ we have that the product $vu_{\sigma(1)} \cdots u_{\sigma(n)}$ is in $L_{j+i_1+\cdots+i_n}$. Furthermore

$$
\begin{aligned}
\sum_{\sigma \in S_n} vu_{\sigma(1)} \cdots u_{\sigma(n)} &= \prod_{\sigma \in S_n} [b, a_{\sigma(1)}, \ldots, a_{\sigma(n)}] G_{j+i_1+\cdots+i_n+1} \\
&= 1 \cdot G_{j+i_1+\cdots+i_n+1} \\
&= 0.
\end{aligned}
$$

As $L(G)$ is generated by $L_1 \cup L_2 \cup \cdots$ as an abelian group, multilinearity gives us that $L(G)$ satisfies the "linearised n-Engel identity"

$$\sum_{\sigma \in S_n} yx_{\sigma(1)} \cdots x_{\sigma(n)} = 0. \tag{6}$$

Notice that when the characteristic of L is not divisible by any of the primes $p \leq n$ then the linearised Engel identity is equivalent to the n-Engel identity $yx^n = 0$. In the general situation if we take any $v = bG_{j+1} \in L_j$ and $u = aG_{i+1} \in L_i$ then $vu^n \in L_{j+ni}$ and as

$$vu^n = [b, \underbrace{u, \ldots, u}]G_{j+ni+1} = 1 \cdot G_{j+ni+1} = 0,$$

it follows that $L(G)$ satisfies

$$vu^n = 0 \text{ if } v \in L \text{ and } u \in L_i \text{ for some integer } i \geq 1. \qquad (7)$$

The relevance of all this comes from the following two celebrated theorems of Zel'manov [58,60,61] that have had profound impact on the theory of Engel groups as we will see later.

Theorem Z1 *Let $L = \langle a_1, \ldots, a_r \rangle$ be a finitely generated Lie ring and suppose that there exist positive integers s, t such that*

$$\sum_{\sigma \in Sym(s)} xx_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(s)} = 0 \qquad (i)$$

$$xy^t = 0 \qquad (ii)$$

for all $x, x_1, \ldots, x_s \in L$ and all Lie products y of the generators $a_1, a_2, \ldots, a_r$.

Then L is nilpotent.

Therem Z2 *Any torsion free n-Engel Lie ring is nilpotent.*

The first of these theorems was the main ingredient in Zel'manov's solution to the restricted Burnside problem for groups of prime power exponent and before we leave this section we discuss some analogs to the Burnside problems in the theory of Engel groups. Let us start by stating the main questions that originated from Burnside's paper.

b1) The general Burnside problem. Is every finitely generated periodic group finite?

b2) The Burnside problem. Let n be a given positive integer. Is every finitely generated group of exponent n finite?

b3) The restricted Burnside problem. Let r and n be given positive integers. Is there a largest finite r-generator group of exponent n?

These have the following analogs for Engel groups:

e1) The general local nilpotence problem. Is every finitely generated Engel group nilpotent?

e2) The local nilpotence problem. Let n be a given positive integer. Is every finitely generated n-Engel group nilpotent?

e3) The restricted local nilpotence problem Let r and n be given positive integers. Is there a largest nilpotent r-generator n-Engel group?

As we have seen the answer to e1 is negative like the answer to b1 and in both cases Golod's examples provide the counterexamples. Theorem Z1 provides a positive answer to both b3 and e3 and we will deal with e3 in the next section. There is however a difference between b2 and e2 in that whereas there are well known counter examples to b2 no such are known for problem e2. We will have a closer look at this later.

3.2 Some consequences for Engel groups

From theorems Z1 and Z2 we are now going to derive strong consequences for Engel groups. We start with a consequence of Theorem Z2 on torsion free n-Engel groups that is due to Zel'manov [59].

As a preparation we first observe that the nilpotence class of a torsion free n-Engel Lie ring is n-bounded and in fact more is true. Let F be the relatively free Lie ring satisfying the linearised n-Engel identity on countably many free generators $x_1, x_2, \ldots$ and let I be the torsion ideal of F. By Theorem Z2 the Lie ring F/I is nilpotent of class say $m = m(n)$. Now if L is any torsion free Lie ring satisfying the linearised n-Engel identity and $u_1, \ldots, u_{m+1}$ any $m + 1$ elements in L, there is a homomorphism $\phi : F \to L$ that maps x_i to u_i for $i = 1, \ldots, m + 1$ and the remaining generators to 0. Now $x_1 \cdots x_{m+1} \in I$, say $kx_1 \cdots x_{m+1} = 0$. It follows that $ku_1 \cdots u_{m+1} = 0$. This shows that any Lie ring satisfying the linearised n-Engel identity satisfies the law $kx_1 \cdots x_{m+1} = 0$ and in particular if L is a torsion-free n-Engel Lie ring, we must have that L is nilpotent of class at most m. Let $\pi(n)$ be the set of all prime divisors of k.

Theorem 3.1 *There exists a finite set of primes $\pi = \pi(n)$ so that any locally nilpotent n-Engel group G without π-elements is nilpotent of n-bounded class. In particular any torsion-free n-Engel group that is locally nilpotent is nilpotent of n-bounded class.*

Proof Let $m = m(n), k = k(n)$ and $\pi = \pi(n)$ be as above and let $a_1, \ldots, a_{m+1}$ be any $m + 1$ elements of G. Let $H = \langle a_1, \ldots, a_{m+1} \rangle$ and consider the descending central series

$$H = \gamma_1(H) \geq \gamma_2(H) \geq \cdots.$$

Let $L(H)$ be the associated Lie ring. Then as we have seen above $L(H)$ satisfies the linearised n-Engel identity. By what we have observed above $L = L(H)$ satisfies

the law $kx_1 \cdots x_{m+1} = 0$. I claim that the class of H is at most m. We argue by contradiction and suppose that the class is $c > m$. As $kL^c = 0$ it follows that $\gamma_c(H)^k \leq \gamma_{c+1}(H) = 1$. But by our assumption G has no elements of order dividing k. Hence $\gamma_c(H) = 1$ which contradicts the assumption that the class is c. This shows that $\gamma_{m+1}(H) = 1$ and as $a_1, \ldots, a_{m+1}$ where arbitrary it follows that G has class at most m. $\square$

We now move on to a second application that makes use of both Theorem Z1 and Z2 and gives a positive solution to problem e3. This is a theorem of Wilson [56]. Our proof is based on arguments from [5].

Theorem 3.2 *Any finitely generated residually nilpotent n-Engel group is nilpotent.*

Proof It suffices to show that there exists a positive integer $l(r, n)$ such that any nilpotent r-generator n-Engel group is nilpotent of class at most $l(r, n)$. Let $G = \langle a_1, \ldots, a_r \rangle$ be any nilpotent r-generator n-Engel group and let $L = L(G)$ be the associated Lie ring. Now $L = L(G)$ is generated by $u_1 = a_1 G_2, \ldots, u_r = a_r G_2$ and satisfies conditions (i) and (ii) of Theorem Z1. It follows that L is nilpotent of class at most l and thus that G is nilpotent of class at most l. $\square$

We mention without proof three other strong results that rely on Theorems Z1 and Z2. The first two [57,38] are strong generalisations of Zorn's Theorem for Engel groups without the assumption that the Engel degree is bounded. The second theorem is a generalisation of the first.

Theorem 3.3 *(Wilson, Zel'manov) Every profinite Engel group is locally nilpotent.*

Theorem 3.4 *(Medvedev) Every compact Engel group is locally nilpotent.*

The last result is on ordered groups. We say that a pair $(G, \leq)$ is an ordered group if $\leq$ is a total order on G and

$$a \leq b \Rightarrow xay \leq xby$$

for all $a, b, x, y \in G$. If we only assume the weaker condition

$$a \leq b \Rightarrow ay \leq by$$

for all $a, b, y \in G$ the group is said to be a right ordered group. Clearly every ordered and right ordered group is torsion-free and so it follows from Theorem 3.1 that a right ordered n-Engel group is locally nilpotent if and only if it is nilpotent. It is known that any torsion-free nilpotent group is orderable that this is also a sufficient condition follows from the following result of Kim and Rhemtulla [31].

Theorem 3.5 *Every orderable n-Engel group is nilpotent.*

The following is still an open question.

Problem 3. Is every right orderable n-Engel group nilpotent?

This is known to be case for 4-Engel groups [34,35]. Before leaving this section we mention another recent result of H. Smith [46]. This has to do with another generalised nilpotence property, namely subnormality. When G is n-Engel this turns out to be a sufficient condition for nilpotence.

Theorem 3.6 *(H. Smith) An n-Engel group with all subgroups subnormal is nilpotent.*

4   The structure of n-Engel groups

In this section we discuss the structure of n-Engel groups. The main question is whether n-Engel groups need to be locally nilpotent. We begin by proving a result that shows that if for a given n there exists a finitely generated n-Engel group that is not nilpotent, there must exist a finitely generated non-abelian simple n-Engel group. This result is well known among specialists in this area but as there doesn't seem to be a proof in the literature we include one here.

Theorem 4.1 *Let G be a finitely generated n-Engel group that is non-nilpotent. There exists a finitely generated section S of G that is simple non-abelian.*

Proof Let $R = \cap_{i=1}^{n} \gamma_i(G)$ be the nilpotent residual of G. By Gruenberg's theorem R is equal to the solvable residual. In particular R has no proper solvable quotient. By Wilson's theorem we know that G/R is nilpotent and as R was the solvable residual we know that $R = G^{(r)}$ for some positive integer r. By Lemma 2.5 we know then that R is finitely generated. As G is non-solvable, R is non-nilpotent. We claim that R has a maximal normal subgroup S with respect to R/S being non-nilpotent. Let $S_1 \subseteq \ldots \subseteq S_i \subseteq \ldots$ be an ascending chain of normal subgroups of R for which R/S_i is non-nilpotent. Let $S_0 = \cup_i S_i$. We claim that R/S_0 is non-nilpotent. We argue by contradiction and suppose that R/S_0 is nilpotent of class m. Equivalently all left-normed commutators in the generators of R of weight $m + 1$ are in S_0. But there are only finitely many such commutators and so they would all be contained in some S_i. This however gives the contradiction that R/S_i is nilpotent. Hence R/S_0 is non-nilpotent. By Zorn's Lemma R has a maximal normal subgroup S with respect to R/S being non-nilpotent. We finish the proof by showing that R/S is simple. Otherwise there would be a normal subgroup T of R lying strictly between S and R. By maximality of S we must then have that R/T is nilpotent. But we had already seen that R has no proper nilpotent quotient. Hence no such T exists and R/S is simple. $\square$

Remark. Notice that the situation here is however different from the Burnside problem. By Baer's Theorem every n-Engel group that satisfies the max condition is nilpotent. Thus there are no n-Engel Tarsky monster and the structure of a

simple n-Engel group, if it exists, is likely to be complicated. This is perhaps the underlying reason why it seems so difficult to solve the local nilpotence question for n-Engel groups. This question remains open except for $n \leq 4$. We will discuss the local nilpotence of 3-Engel and 4-Engel groups later.

As every finitely generated n-Engel group that is residually nilpotent is nilpotent it follows that the locally nilpotent n-Engel groups form a subvariety. We will study this variety for the remainder of this section. We start by proving a result that strengthens Theorem 3.1

Proposition 4.2 *There exist numbers $l = l(n)$ and $m = m(n)$ such that the law*

$$[x_1, x_2, \ldots, x_{m+1}]^l = 1$$

holds in all locally nilpotent n-Engel groups.

Proof By Theorem 3.1, every torsion free locally nilpotent n-Engel group is nilpotent of bounded class, say $m = m(n)$. Let F be the free n-Engel group on $m + 1$ generators, say $x_1, \ldots, x_{m+1}$. Let $R = \cap_{i=1}^{\infty} \gamma_i(F)$. It is clear that F/R is residually nilpotent and as every finitely generated nilpotent group is residually finite it follows that F/R is residually finite. By Theorem 3.2 and Theorem 2.1, F/R is then nilpotent. Let T/R be the torsion group of F/R. Now F/T is a torsion free nilpotent n-Engel group and by the remark at the beginning of the proof, F/T is thus nilpotent of class at most m. So $[x_1, \ldots, x_{m+1}]^l \in R$ for some positive integer $l = l(n)$. Now let G be any locally nilpotent n-Engel group and let $g_1, \ldots, g_{m+1} \in G$. There is a homomorphism $\phi : F \to G$, $\quad \phi(x_i) = g_i$, $\quad i = 1, 2, \ldots, m + 1$. As $\langle g_1, \ldots, g_{m+1} \rangle = \phi(F)$ is nilpotent, we have that $R \leq \mathrm{Ker}\,(\phi)$. Hence $1 = \phi([x_1, \ldots, x_{m+1}]^l) = [g_1, \ldots, g_{m+1}]^l$. $\square$

We will use this proposition to prove two strong structure results for locally nilpotent n-Engel groups. The first one is due to Burns and Medvedev [5] and can be derived from the proposition and Theorem 3.2.

Theorem 4.3 *(Burns, Medvedev) There exist positive integers m and r such that for any locally nilpotent n-Engel group G we have*

$$\gamma_{m+1}(G)^r = \{1\}.$$

Proof Let m and l be as in the proposition. Let F be the relatively free nilpotent n-Engel group on $m + 2$ generators $x_1, \ldots, x_{m+2}$. Let $H = \langle [x_1, \ldots, x_{m+1}], x_{m+2} \rangle$. Now $H'/[H', H']$ is abelian of exponent dividing l by Proposition 4.2. If H' has class c it follows that H' has exponent diving l^c. Take large enough integer e such that $f = \binom{l^e}{k}$ is divisible by l^c for $k = 1, \ldots, c$.

Now let $g = a_1 \cdots a_t$ be any product of left normed commutators of weight $m + 1$. We prove by induction on t that $g^f = 1$. For $t = 1$ this follows from the proposition.

Now suppose that $t \geq 2$ and that the result holds when t has a smaller value. Let $h = a_2 \cdots a_t$. We apply the Hall-Petrescu identity to see that

$$a_1^f h^f = (a_1 h)^f w_2^{\binom{f}{2}} w_3^{\binom{f}{3}} \cdots w_c^{\binom{f}{c}}$$

By the induction hypothesis the left hand side is trivial and by the choice of f we have that all $w_i^{\binom{f}{i}}$ are trivial as well. Hence $g^f = 1$. This finishes the inductive proof and hence $\gamma_{m+1}(G)^f = 1$. $\square$

The second result is recent and due to Crosby and Traustason [9]. It generalises another result of Burns and Medvedev. It says that modulo the hyper-centre, every locally nilpotent n-Engel group is of bounded exponent.

Theorem 4.4 *(Crosby, T) There exists positive integers e and m so that any locally nilpotent n-Engel group satisfies the law*

$$[x^e, x_1, \ldots, x_m] = 1.$$

Proof Let l and m be as in Proposition 4.2. Now let G be the relatively free nilpotent n-Engel group on $m + 1$ generators. Suppose that the nilpotence class of G is $r = r(n)$. Notice that $r \geq m$. We finish the proof by showing by reverse induction on c that G satisfies the law

$$[x^{l^{r-m-c}}, x_1, \ldots, x_m, y_1, \ldots, y_c] = 1$$

for $c = 0, \ldots, r - m$. Letting $c = 0$ then gives us the theorem with $e = l^{r-m}$.

We turn to the inductive proof. The statement is clearly true for $c = r - m$. Now suppose that $0 \leq c \leq r - m - 1$ and that the result holds for larger values of c in $\{0, 1, \ldots, r - m\}$. By the induction hypothesis we have that $x^{l^{r-m-c-1}}$ is in the $(c + m + 1)$st centre. Hence

$$
\begin{aligned}
[x^{l^{r-m-c}}, x_1, \ldots, x_m, y_1, \ldots, y_c] &= [(x^{l^{r-m-c-1}})^l, x_1, \ldots, x_m, y_1, \cdots, y_c] \\
&= [x^{l^{r-m-c-1}}, x_1, \ldots, x_m, y_1, \ldots, y_c]^l \\
&= 1.
\end{aligned}
$$

This finishes the proof. $\square$

Remark. Burns and Medvedev had proved that there exist integers $e = e(n)$ and $m = m(n)$ such that for any locally nilpotent n-Engel group G we have that G^e is nilpotent of class at most m. The result above shows that we can choose e and m such that G^e is always in the mth centre.

We next turn to the structure of locally finite n-Engel p-groups. The following results are due to Abdollahi and Traustason [1]. We have seen that in general

there is no upper bound for the nilpotency class of n-Engel p-groups for a given n and p. The situation is different when one restricts oneself to powerful p-groups. We remind the reader of some definitions. Let G be a finite p-group. If p is odd then G is said to be powerful if $[G, G] \leq G^p$ and if $p = 2$ then G is powerful if $[G, G] \leq G^4$. We also need the notion of powerful embedding. Let H be a subgroup of G. If p is odd then H is said to be powerfully embedded in G if $[H, G] \leq H^p$ and if $p = 2$ then we require instead that $[H, G] \leq H^4$.

Now we list some of the properties that we will be using. Let G be a powerful p-group. If a subgroup H is powerfully embedded in G then H^p is also powerfully embedded. We also have that $(G^{p^i})^{p^j} = G^{p^{i+j}}$. Furthermore, if G is generated by $x_1, \ldots, x_d$ then G^p is and generated by $x_1^p, \ldots, x_d^p$. It follows that if G is generated by elements of order dividing p^m then G has exponent dividing p^m. We also have that the terms of the lower central series are powerfully embedded in G.

Theorem 4.5 *(Abdollahi, T) There exists a positive integer $s = s(n)$ such that any powerful n-Engel p-group is nilpotent of class at most s.*

Proof By Proposition 4.2, $[g_1, \ldots, g_{m+1}]^l = 1$ for all $g_1, \ldots, g_{m+1} \in G$, where m and l are the integers given in Proposition 4.2. Suppose that $v = v(n)$ is the largest exponent of the primes that appear in the decomposition of l. Then $[g_1, \ldots, g_{m+1}]^{p^v} = 1$. So $\gamma_{m+1}(G)$ is generated by elements of order dividing p^v. But $\gamma_{m+1}(G)$ is powerfully embedded in G and therefore it follows that $\gamma_{m+1}(G)$ is powerful and has exponent dividing p^v.

On the other hand, since $\gamma_{m+1}(G)$ is powerfully embedded in G, we have $[\gamma_{m+1}(G), G] \leq \gamma_{m+1}(G)^p$ if p odd, and $[\gamma_{m+1}(G), G] \leq \gamma_{m+1}(G)^4$, if $p = 2$. Using some basic properties of powerful groups we see inductively that

$$[\gamma_{m+1}(G), {}_vG] \leq \gamma_{m+1}(G)^{p^v} = 1.$$

Hence, G is nilpotent of class at most $s(n) = m + v$. □

Building on this result one obtains [1] the following result. We omit the proof here.

Theorem 4.6 *(Abdollahi, T) Let p a prime and let $r = r(p, n)$ be the integer satisfying $p^{r-1} < n \leq p^r$. Let G be a locally finite n-Engel p-group.*

(a) If p is odd, then G^{p^r} is nilpotent of n-bounded class.
(b) If $p = 2$ then $(G^{2^r})^2$ is nilpotent of n-bounded class.

Remark. The r given in Theorem 4.6 is close to be the best lower bound. Suppose that n, p are such that $r = r(p, n) \geq 2$ and

$$G = \mathbb{Z}_p \mathrm{wr} \oplus_{i=1}^\infty \mathbb{Z}_{p^{r-1}}.$$

Then G is a metabelian n-Engel p-group of exponent p^r such that $G^{p^{r-2}}$ is not nilpotent. Therefore, if t is the least non-negative integer such that G^{p^t} is nilpotent, for all locally finite n-Engel p-groups G, then we have $t \in \{r-1, r\}$ if p is odd and $t \in \{r-1, r, r+1\}$ if $p = 2$.

We now turn to some other related structural properties of n-Engel groups. By Gruenberg's theorem all solvable n-Engel groups are locally nilpotent and we thus have the following implications for n-Engel groups:

$$G \text{ nilpotent} \implies G \text{ solvable} \implies G \text{ locally nilpotent.}$$

By Theorem 3.1, we know that there are only finitely many primes that we need to exclude in order to have global nilpotence for a locally nilpotent n-Engel group. The answer to the following questions is therefore essential for clarifying the structure of locally nilpotent n-Engel groups:

Question 1. Which primes need to be excluded to have nilpotence?
Question 2. Which primes need to be excluded to have solvability?

Remark. As we have seen above, for each prime $p < n$ there exists a metabelian n-Engel p-group that is non-nilpotent. Under the assumption that these primes are excluded we however have that nilpotence and solvability are equivalent [19] (see also [23] for a close analysis of the metabelian case).

Theorem 4.7 *(Gruenberg) Let G be a solvable n-Engel group with derived length d. If G has no elements of prime order $p < n$ then G is nilpotent of class at most $(n+1)^{d-1}$.*

Let us next consider another natural question. Recall that G is said to be a Fitting group if $\langle x \rangle^G$ is nilpotent for all $x \in G$. (That the class is m means that every commutator of the form $[..., \underbrace{x, ..., x, ..., x}_{m+1}, ...]$ is trivial). If there is a bound for the nilpotence class of $\langle x \rangle^G$ and the lowest bound is n then we say that G has Fitting degree n.

Question 3. Are all locally nilpotent n-Engel groups Fitting groups?

It is clear that a group G is 2-Engel if and only if $\langle x \rangle^G$ is abelian for all $x \in G$ and by a well known result of L.C. Kappe and W. Kappe [30] that we will come back to later we have that G is 3-Engel if and only if $\langle x \rangle^G$ is nilpotent of class at most 2 for all $x \in G$. Every Fitting group with Fitting degree n is an n-Engel group and as we have just seen the converse is true for $n = 2$ and $n = 3$. However this fails for larger values of n. In [22] Gupta and Levin gave an example of a 4-Engel group that has Fitting degree 4 they also gave the following example that in particular shows that there are 5-Engel groups that are not Fitting groups.

Example. Let p be any prime greater than or equal to 3. Let G be the free nilpotent group of class 2 and exponent p that is of countable rank and let $\mathbb{Z}_p G$ be the group ring over the integers modulo p. Let M_p be the multiplicative group of 2×2 matrices over $\mathbb{Z}_p G$ of the form

$$\begin{pmatrix} g & 0 \\ r & 1 \end{pmatrix}, \quad g \in G, \ r \in \mathbb{Z}_p G.$$

One can check that M_p is a $(p+2)$-Engel group that is not a Fitting group.

More recently [50] it has been established that 4-Engel groups are Fitting groups of Fitting degree at most 4 we will come back to this result later when we discuss 4-Engel groups. In view of the examples of Gupta and Levin the following question arises.

Problem 4. Let G be an n-Engel p-group where $p > n$. Is G a Fitting group?

5   3-Engel groups

5.1   The local nilpotence

It was Heineken [28] who proved in 1961 that 3-Engel are locally nilpotent. We give here a short new proof based on some arguments that were used to deal with 4-Engel groups [52]. The idea is to minimize the use of commutator calculus and try to apply arguments based on the symmetry of the problem and to make use of the Hirsch-Plotkin radical. Another short proof can also be found in [39].

It is not difficult to show that it suffices to show that any 3-generator 3-Engel group is nilpotent. In fact we will prove later a stronger result, namely the analogous result for 4-Engel groups (see Proposition 6.1). So we are left with the three-generator groups. We first prove two useful lemmas. The first one shows that within the class of n-Engel groups the locally nilpotent radical has the radical property.

Lemma 5.1 *Let G be an n-Engel group and let R be the Hirsch-Plotkin radical of G. Then the Hirsch-Plotkin radical of G/R is trival.*

Proof Let S/R be the Hirsch-Plotkin radical of G/R. It suffices to show that S is locally nilpotent. Let H be a finitely generated subgroup of S. Then $H/(H \cap R) \cong HR/R$ is nilpotent and thus solvable of derived length, say m. As H is restrained we have by Lemma 2.5 that that $H^{(m)}$ is finitely generated subgroup of $H \cap R$ and thus nilpotent. Therefore H is solvable and thus nilpotent by Gruenberg's Theorem. $\square$

Lemma 5.2 *Let G be a 3-Engel group and $d, c, x \in G$. If d commutes with c and c^x then d commutes with any element in $\langle c \rangle^{\langle x \rangle}$.*

Proof   It follows from $1 = [c, x, x, x]$ that

$$\langle c \rangle^{\langle x \rangle} = \langle c, c^x, c^{x^2} \rangle.$$

So it suffices to show that d commutes with c^{x^2}. But this follows from

$$
\begin{aligned}
1 &= [c, dx, dx, dx] \\
&= [c, x, dx, dx] \\
&= [c, x, x, dx] \\
&= [c, x, x, d]^x.
\end{aligned}
$$

So d commutes with $[c, x, x] = c^{-x} c c^{-x} c^{x^2}$ and thus c^{x^2}. $\square$

Lemma 5.3 *Every two-generator 3-Engel group is nilpotent.*

Proof   Let $G = \langle a, b \rangle$ be a two-generator 3-Engel group and let R be the Hirsch-Plotkin radical. We know from Lemma 5.1 that G/R has a trivial Hirsch-Plotkin radical. Replacing G by G/R we can thus assume that G has a trivial Hirsch-Plotkin radical and the aim is then to show that G is trivial. As G is three-Engel we have that $[a, a^b]$ commutes with both a and a^b and $\langle a, a^b \rangle$ is nilpotent of class at most 2. By last lemma $Z(\langle a, a^b \rangle)$ is contained in the centre of $\langle a \rangle^G$ which is a abelian normal subgroup of G and thus trivial. Hence $\langle a, a^b \rangle$ is a nilpotent subgroup with a trivial centre and therefore trivial itself. It follows that $a = 1$ and G is cyclic and again as the Hirsch-Plotkin radical is trivial it follows that $G = \{1\}$. $\square$

Theorem 5.4 *(Heineken) Every 3-Engel group is locally nilpotent.*

Proof   As we said previously it suffices to show that any 3-generator 3-Engel group is nilpotent. Thus let $G = \langle x, y, z \rangle$ be a 3-Engel group. As before we let $R(G)$ the Hirsch-Plotkin radical of G. We argue as in the proof of Lemma 5.3 and replacing G by $G/R(G)$ we can assume that $R(G) = 1$. We want to show that $G = 1$. Consider the following setting:

$$
\begin{aligned}
H &= \langle xy^{-1}, yz^{-1} \rangle & Z(H) &\ni c \\
\langle c, c^x \rangle & & Z(\langle c, c^x \rangle) &\ni d.
\end{aligned}
$$

Note that

$$c^x = c^y = c^z.$$

By Lemma 5.2 we have that

$$C_G(\langle c, c^x \rangle) = C_G(\langle c \rangle^{\langle x \rangle})$$

which shows that $C_G(\langle c, c^x \rangle) = C_G(\langle c, c^y \rangle) = C_G(\langle c, c^z \rangle)$ is normalised by G. Thus in particular d^g commutes with c for all $g \in G$ or equivalently d commutes with c^g for all $g \in G$. Hence $d \in Z(\langle c \rangle^G)$ and thus trivial. Thus

$$Z(\langle c, c^x \rangle) = 1 \Rightarrow \langle c, c^x \rangle = 1 \Rightarrow Z(H) = 1 \Rightarrow H = 1 \Rightarrow G = \langle x \rangle \Rightarrow G = 1. \quad \square$$

5.2   Other structure results

We start by proving global nilpotence for 3-Engel groups that are $\{2,5\}$-free. The short proof given here differs from Heineken's original argument [28] and is based on Lie methods. We start with an easy preliminary lemma.

Lemma 5.5 *Let L be a 2-Engel Lie ring. Then $3L^3 = 0$ and $L^4 = 0$.*

Proof   Let $x, x_1, x_2, x_3 \in L$. The 2-Engel identity implies the linearised 2-Engel identity

$$xx_1x_2 + xx_2x_1 = 0,$$

or equivalently

$$xx_2x_1 = -xx_1x_2.$$

We use this to derive the result. Firstly

$$xx_1x_2 = -x_1xx_2 = x_1x_2x = -x(x_1x_2) = -2xx_1x_2$$

that gives $3L^3 = 0$. Secondly

$$xx_1x_2x_3 = -x_3(xx_1x_2) = x_3x_2(xx_1) = -xx_1(x_3x_2) = 2xx_1x_2x_3$$

that gives $xx_1x_2x_3 = 0$ and thus $L^4 = 0$. $\square$.

Theorem 5.6 *(Heineken) Let G be a 3-Engel group that is $\{2,5\}$-free. Then G is nilpotent of class at most 4.*

Proof   Let G be any finitely generated $\{2,5\}$-free 3-Engel group. We know already that G is nilpotent and thus residually a finite p-group, $p \notin \{2,5\}$. We can thus assume that G is a finite 3-Engel p-group where $p \neq 2,5$. Let L be the associated Lie ring then, as we have seen previously, L satisfies the linearized 3-Engel identity. For $x, y \in L$ we let $X = \operatorname{ad} x$ and $Y = \operatorname{ad} y$. Since $p \neq 2$ we have that

$$XY^2 + YXY + Y^2X = 0. \tag{8}$$

For any $a \in L$ we also have

$$\begin{aligned}
0 &= -xay^2 - xyay - xy^2a \\
&= axy^2 + a(xy)y + a(xy^2) \\
&= 3axy^2 - 3ayxy + ay^2x,
\end{aligned}$$

and therefore

$$3XY^2 - 3YXY + Y^2X. \tag{9}$$

We first deal with the case when $p = 3$. Replace L by $\tilde{L} = L/3L$. Now as the characteristic of $\tilde{L}$ is 3 we have $Y^2X = 0$. Consider the subspace

$$I = \operatorname{Sp}\langle uv^2 | u, v \in \tilde{L}\rangle.$$

It is not difficult to see that I is an ideal. The quotient algebra $\tilde{L}/I$ is an 2-Engel algebra and thus nilpotent of class at most 3 by Lemma 5.5. As $\tilde{L}$ is centre-by-2-Engel it follows that $\tilde{L}$ is nilpotent of class at most 4. Now L is graded and this therefore implies that $L^5 \leq 3L^5 \leq 3^2 L^5 \leq ... = 0$ and L is nilpotent of class at most 4. We can thus assume that char $L \neq 2, 3, 5$. From (8) and (9) we have

$$XY^2 = 2YXY \quad \text{and} \quad Y^2 X = -3YXY. \tag{10}$$

It also follows that $3XY^2 = -2Y^2X$. If we interchange X and Y in (10) we get

$$YX^2 = 2XYX \quad \text{and} \quad X^2 Y = -3XYX. \tag{11}$$

Now multiply (10) by X on the left and (11) on the right by Y. We then get

$$X^2 Y^2 = 2XYXY \quad \text{and} \quad X^2 Y^2 = -3XYXY.$$

It follows that $5X^2Y^2 = 0$ so $X^2Y^2 = 0$ since $p \neq 5$. As 2-Engel Lie algebras of characteristic $p \neq 3$ are nilpotent of class at most 2 it follows that that $X_1 X_2 Y^2 = 0$. Using $3XY^2 = -2Y^2X$ we get from this

$$4Y^2 X_1 X_2 = -9 X_1 X_2 Y^2 = 0$$

and it follows that $Y_1 Y_2 X_1 X_2 = 0$ and $L^5 = 0$. $\square$

We next turn to the Fitting property of 3-Engel groups [30]. The proof that we sketch comes from [24].

Theorem 5.7 *(Kappe, Kappe). Let G be a 3-Engel group then $\langle x \rangle^G$ is nilpotent of class at most 2 for all $x \in G$.*

Proof In order to prove the theorem it suffices to show that $[a^b, a, a^c] = 1$ for all $a, b, c \in G$. It therefore suffices to work with 3-generator groups $G = \langle a, b, c \rangle$. By Heineken's result every 3-Engel group is nilpotent and detailed analysis shows that it is nilpotent of class at most 5 (see for example [24] for details). It can now be checked using this detailed analysis that $[a^b, a, a^c] = [a[a, b], a, a[a, c]]$ is trivial. $\square$

The primes 2 and 5 that are not covered by Theorem 5.6 turn out to be exceptional. For the prime 2 this is simply because 2 is smaller than the Engel degree 3 and we had remarked earlier that for a prime $p < n$ there exists a metabelian n-Engel p-group that is non-nilpotent. Only the prime 5 turns however out to be exceptional with respect to solvability. In order to see this we need to show that any 3-Engel 2-group is solvable [21]. By Theorem 4.5 it suffices to show that every 3-Engel group of exponent 4 is solvable [25].

Theorem 5.8 *(Gupta, Weston). Every 3-Engel group of exponent 4 is solvable.*

Proof Let G be a 3-Engel group of exponent 4 and let $x, y \in G$. By Theorem 5.7 we have that
$$[x^2, x^{2y}] = [x, x^y]^4 = 1.$$

Hence $\langle x^2 \rangle^G$ is abelian for all $x \in G$. This implies that G^2 is generated by elements $a_1, a_2, \ldots$ where for each i, $\langle a_i \rangle^G$ is abelian. As G/G^2 is abelian it suffices to show that G^2 is solvable. Let x_1, x_2, x_3 be any three of the generators. Using the 3-Engel identity we have for all $x \in G^2$

$$\begin{aligned} 1 &= [x, x_1 x_2 x_3, x_1 x_2 x_3, x_1 x_2 x_3] \\ &= [[x, x_1], [x_2, x_3]] \cdot [[x, x_2], [x_3, x_1]] \cdot [[x, x_3], [x_1, x_2]]. \end{aligned}$$

Now let y be any element in G^2. Replacing x by xy gives

$$1 = [[x, x_1, y], [x_2, x_3]] \cdot [[x, x_2, y], [x_3, x_1]] \cdot [[x, x_3, y], [x_1, x_2]]. \tag{12}$$

Equipped with this identity and the Jabobi identity we will now see that G^2 is centre-by-metabelian. In order to see this, let x_1, x_2, x_3, x_4, x_5 be any five of the generators of G^2. We have

$$\begin{aligned} 1 &= [x_5, x_1 x_2 x_3 x_4, x_1 x_2 x_3 x_4, x_1 x_2 x_3 x_4, x_1 x_2 x_3 x_4] \\ &= [[x_5, x_1, x_2], [x_3, x_4]] \cdot [[x_5, x_2, x_1], [x_3, x_4]] \\ &\quad [[x_5, x_1, x_3], [x_2, x_4]] \cdot [[x_5, x_3, x_1], [x_2, x_4]] \\ &\quad [[x_5, x_1, x_4], [x_2, x_3]] \cdot [[x_5, x_4, x_1], [x_2, x_3]] \\ &\quad [[x_5, x_2, x_3], [x_1, x_4]] \cdot [[x_5, x_3, x_2], [x_1, x_4]] \\ &\quad [[x_5, x_2, x_4], [x_1, x_3]] \cdot [[x_5, x_4, x_2], [x_1, x_3]] \\ &\quad [[x_5, x_3, x_4], [x_1, x_2]] \cdot [[x_5, x_4, x_3], [x_1, x_2]] \quad \text{(by the Jacobi identity)} \\ &= [[x_1, x_2, x_5], [x_3, x_4]] \cdot [[x_1, x_3, x_5], [x_2, x_4]] \cdot [[x_1, x_4, x_5], [x_2, x_3]] \\ &\quad [[x_2, x_3, x_5], [x_1, x_4]] \cdot [[x_2, x_4, x_5], [x_1, x_3]] \cdot [[x_2, x_1, x_5], [x_3, x_4]] \\ &\quad [[x_1, x_2, x_5], [x_3, x_4]] \cdot [[x_3, x_4, x_5], [x_1, x_2]] \quad \text{(by (12) and the Jacobi identity)} \\ &= [[x_1, x_2], [x_3, x_4], x_5]. \end{aligned}$$

Hence the result. $\square$

In fact one can show more than this. Gupta and Newman [24] have shown that any 5-torsion free 3-Engel group satisfies the identity $[[x_1, x_2, x_3], [x_4, x_5], x_6] = 1$ and so in particular they are solvable of derived length at most 3.

To show that the prime 5 is exceptional with respect to solvability is much harder to prove. This had been conjectured by Macdonald and Neumann [36] and was confirmed by Bachmuth and Mochizuki [2] using ring theoretic methods.

We leave the 3-Engel groups by mentioning again the article [24] by Gupta and Newman. This paper almost completes the description of the variety of 3-Engel groups. The authors in particular get a normal from theorem for the relatively free 3-Engel group of infinite countable rank without elements of order 2. They prove also that every n-generator 3-Engel group is nilpotent of class at most $2n-1$ which is the best upper bound, that the fifth term of the lower central series of a 3-Engel

group has exponent dividing 20 (again the best possible result) and they show that the subgroup generated by fifth powers satisfies the law $[[x_1, x_2, x_3], [x_4, x_5], x_6]$.

Problem 5. To obtain a normal form theorem for the relatively free 3-Engel group of infinite countable rank.

6   4-Engel groups

6.1   The local nilpotence theorem

In this section we give a broad outline of the proof of the local nilpotence theorem. The proof is quite intricate and spread over a few papers. This makes the complete proof very long and technical. The aim here is only to present the broad outline of the proof in linear order skipping over most of the technical details.

Let G be a finitely generated 4-Engel group. We want to show that G is nilpotent. In fact it is quite easy to see that it suffices to show that all three generator 4-Engel groups are nilpotent. Let us see why this is the case. The short proof presented here comes from [52].

Proposition 6.1 *A 4-Engel group is locally nilpotent if and only if all its 3-generator subgroups are nilpotent.*

Proof One inclusion is obvious. Suppose now that G is a 4-Engel group with all its 3-generator subgroups nilpotent. Let $a, b, c \in G$. By our assumption $H = \langle a, b, c \rangle$ is nilpotent and one can read from a polycyclic presentation of the free nilpotent 3-generator 4-Engel group [41] that $[[a, b, b, b], [a, b, b, b]^c] = 1$. It follows that $\langle [a, b, b, b] \rangle^G$ is abelian for all $a, b \in G$ and thus contained in the Hirch-Plotkin radical R of G. It follows that G/R is a 3-Engel group and thus locally nilpotent. But by Lemma 5.1, the Hirsch-Plotkin radical of G/R is trivial. Hence $G/R = \{1\}$ and $G = R$ and thus locally nilpotent. $\square$

Even proving that the 2-generator 4-Engel groups are nilpotent turned out to be a major obstacle. Some good initial progress was however made because of the following much weaker result. This was first proved only for torsion groups [48] and then later for torsion-free groups [34] by Maj and Longobardi. Two proofs of the general result appeared about the same time. Vaughan-Lee gave a computer proof using the Knuth-Bendix algorithm (unpublished) and a machine free proof appeared in 1997 [49].

Proposition 6.2 *Let G be a 4-Engel group and let $a, b \in G$. Then $\langle a, a^b \rangle$ is metabelian and nilpotent of class at most 4.*

We skip over the proof which although not very long is quite intricate. Let us see how this proposition can be applied to make some progress on the local nilpotence problem. The following two lemmas come from [48].

Lemma 6.3 *Let G be a 4-Engel group. The 2-elements form a subgroup.*

Proof Suppose that $a^{2^i} = 1$. We then have

$$(ab)^{2^i} = a^{-2^i}(ab)^{2i} = b^{a^{2^i-1}}b^{a^{2^i}-2}\cdots b^a b.$$

Since $\langle u, u^x \rangle$ is nilpotent for all $u, x \in G$, we have that $u^x u$ is a 2-element whenever u is a 2-element. Therefore $b^a b$ is a 2-element and then also $b^{a^3}b^{a^2}b^a b = (b^a b)^{a^2}(b^a b)$. By induction we get that

$$b^{a^{2^i-1}}b^{a^{2^i-2}}\cdots b^a b = (ab)^{2^i}$$

is a 2-element and hence ab is a 2-element. $\square$

Lemma 6.4 *Let G be a 4-Engel group. Suppose that $a^{p^i} = 1$ where p is a prime and either $i \geq 2$ and p is odd or $i \geq 3$ and $p = 2$. Then*

$$[a^{p^{i-1}}, a^{p^{i-1}b}] = 1$$

for all $b \in G$.

Proof From the 4-Engel identity we have that $1 = [a, [a, [a, a^b]]]$. Now $\langle a, a^b \rangle$ is metabelian and it follows from this that

$$[a, a^b]^{a^2} = [a, a^b]^{2a-1}.$$

To deduce the result from this are straightforward calculations, we refer to [48] for the details. Notice also that when $p \neq 2, 3$ this is a consequence of $\langle a, a^b \rangle$ being a regular p-group as p is greater than the class. $\square$

From this we can deduce the following main ingredient towards the proof of the local nilpotence result.

Proposition 6.5 *Let G be a 4-Engel p-group. If $p = 2$ or $p = 3$ then G is locally finite. For $p \geq 5$ we have that G/R is of exponent dividing p where R is the locally nilpotent radical.*

Proof Consider $H = G/R$. First assume that p is an odd prime number. Let $a \in G$. We claim that the order of $\bar{a} \in H$ divides p. We argue by contradiction and suppose that $\bar{a}$ has order p^i for some $i \geq 2$. By last lemma $\langle \bar{a}^{p^{i-1}} \rangle^H$ is abelian and thus contained in the Hirsch-Plotkin radical of H. But by Lemma 5.1, this radical is trivial. Hence we in particular get the contradiction that $\bar{a}^{p^{i-1}} = 1$. In the case when $p = 3$ we know that groups of exponent 3 are locally finite so we conclude that all 4-Engel 3-groups are locally finite. This leaves us with the case $p = 2$. Arguing in the same manner as for the odd case we see from the last lemma that G/R has exponent dividing 4. As groups of exponent 4 are known to be locally finite [44] it follows from this and Lemma 5.1 that G is locally finite. $\square$

In fact with a bit more work one can deduce from Proposition 6.2 that in any 4-Engel group the torsion elements form a subgroup T and that $T/Z(T)$ is a direct product of p-groups [48]. However next result proved about decade later [52] gives us a stronger result.

Proposition 6.6 *All* 2-*generator* 4-*Engel groups are nilpotent.*

Dealing with this major obstacle turned out to give one of the main tools for obtaining the local nilpotence result. The proof is similar in spirit to the proof that we gave of the local nilpotence of 3-Engel groups although it is much more complicated. The proof although relatively short, is quite intricate and tricky and we only give the general idea. Let $G = \langle x, y \rangle$ be a 2-generator 4-Engel group. Replacing G with G/R, where R is the Hirsch-Plotkin radical, one can assume that G has a trivial Hirsch-Plotkin radical and the aim is to show that if follows that G itself is trivial. As xy^{-1} and $y^{-1}x$ are conjugates it follows from Proposition 6.2 that $H = \langle xy^{-1}, y^{-1}x \rangle$ is nilpotent. Let $c \in Z(H)$, then $c^x = c^y$ and $c^{x^{-1}} = c^{y^{-1}}$. This symmetrical property of c turns out to be very useful and one is able after several reduction steps to show that c must be in the Hirsch-Plotkin radical of G. Hence $c = 1$. But then H is a nilpotent group with a trivial centre and thus trivial itself. This implies that $x = y$ and G is cyclic. As the Hirsch-Plotkin radical of G is trivial it follows that G is trivial.

From this we get the following easy corollaries.

Proposition 6.7 *Let G be a* 4-*Engel group. The torsion elements form a subgroup that is a direct product of p-groups. If R is the Hirsch-Plotkin radical of G then the torsion subgroup of G/R is a direct product of groups of prime exponent $p \neq 2, 3$.*

Lemma 6.8 *Let G be a* 4-*Engel group and $a, b \in G$. Then $\langle a, a^b \rangle$ is nilpotent of class at most* 3.

The proposition reduces the local nilpotence question for 4-Engel groups to groups that are either of prime exponent $p \neq 2, 3$ or torsion free. The lemma can be read off from the polycyclic structure of the free two-generator 4-Engel group [41] that we now know is nilpotent. Reducing the bound 4 to the correct bound 3 turned out to be significant for finishing the proof of the local nilpotence theorem.

We saw earlier on that in order to prove that 4-Engel groups are locally nilpotent it suffices to show that all three-generator 4-Engel groups are nilpotent. In fact this is not proved directly. Instead one obtains the weaker result that any subgroup generated by three conjugates is nilpotent. Before we discuss the proof of this result we introduce a notation. We say that a three-generator group $G = \langle a, b, c \rangle$ is of type (r, s, t) if

$$\langle a, b \rangle \text{ is nilpotent of class at most } r$$
$$\langle a, c \rangle \text{ is nilpotent of class at most } s$$
$$\langle b, c \rangle \text{ is nilpotent of class at most } t.$$

By Lemma 6.8, every 4-Engel group generated by three conjugates is of type $(3, 3, 3)$. The idea is roughly speaking to use induction on the complexity of the type. The next lemma provides the induction basis but is also another important tool to obtain other results.

Lemma 6.9 *Let $G = \langle a, b, c \rangle$ be a 4-Engel group of type $(1, 2, 3)$. Then G is nilpotent.*

This was first proved for 4-Engel groups of exponent 5 using coset enumeration [54]. The general result was first proved with an aid of a computer [27] using the Knuth Bendix procedure. A short machine proof was later given in [53]. With the aid of the machinery obtained so far one can prove the following [54,27].

Proposition 6.10 *Let G be a 4-Engel group that is either of exponent 5 or without $\{2, 3, 5\}$-elements and let $a, x, y \in G$. Then $\langle a, a^x, a^y \rangle$ is nilpotent.*

The proof is very technical and tricky and we skip over it here. We describe only the outline. The proof consists of few steps.

Step 1. $\langle a, a^{a^{a^x}}, a^{a^y} \rangle$ and $\langle a, a^{a^{a^x}}, (a^{a^{a^x}})^{a^y} \rangle$ are nilpotent.

Step 2. $\langle a, a^{a^{a^x}}, a^y \rangle$ is nilpotent.

Step 3. $\langle a, a^{a^x}, a^{a^y} \rangle$ and $\langle a, a^{a^x}, (a^{a^x})^{a^y} \rangle$ are nilpotent.

Step 4. $\langle a, a^{a^x}, a^y \rangle$ is nilpotent.

Step 5. $\langle a, a^x, a^y \rangle$ is nilpotent.

Notice that the groups in Step 1 are of type $(1, 2, 3)$ and therefore nilpotent by last lemma. The group in Step 2 is of type $(1, 3, 3)$, those in Step 3 of type $(2, 2, 3)$, the one in Step 4 of type $(2, 3, 3)$ and finally the one in Step 5 of type $(3, 3, 3)$. The proof of each step consists of clever commutator calculus building on the previous steps to obtain nilpotence. When the nilpotence has been established one can get a precise information about the structure using either machine or hand calculations. With the aid of last proposition one can then prove the following [54,27].

Proposition 6.11 *Let G be a 4-Engel group with a trivial Hirsch-Plotkin radical. Suppose that either G is of exponent 5 or without $\{2, 3, 5\}$-elements. Let $a, x, c \in G$. Then*
$$[c, [x, a, a, a], [x, a, a, a], [x, a, a, a]] = 1.$$

We again skip over the proof which makes use of several commutator identities that hold in the free 4-Engel group of rank 2. The final key step is the following result [27].

Proposition 6.12 *Let G be a 4-Engel group with trivial Hirsch-Plotkin radical, and let $a \in G$. Suppose that if a_1, a_2 are any conjugates of a in G, then $\langle a_1, a_2 \rangle$ has class at most 2. Then the normal closure of a in G is locally nilpotent.*

Let us see how we can now deduce the local nilpotence of 4-Engel groups from these results. First let G be a 4-Engel group of exponent 5 and let R be the Hirsch-Plotkin radical of G. Replacing G by G/R we can suppose that G has a trivial Hirsch-Plotkin radical. Let $x, a \in G$. By Proposition 6.11 we have that any two conjugates of $[x, a, a, a]$ in G generate a subgroup that is nilpotent of class at most 2. By Proposition 6.12 it follows that the normal closure of $[x, a, a, a]$ is locally nilpotent and therefore trivial as the Hirsch-Plotkin radical was trivial. It follows that G is a 3-Engel group and therefore locally nilpotent by Heineken's result. Hence G is trivial. Now we move on to the general case. Let G be a 4-Engel group. As 4-Engel groups of exponent 5 are locally finite we can deduce from Proposition 6.7 that G/R has no elements of order $2, 3$ or 5. Now the same argument as before using Proposition 6.11 and 6.12 shows that G/R is trivial.

6.2 Other structure results

In this section we will discuss various structure results for locally nilpotent 4-Engel groups. Our knowledge today is pretty good and almost on a level with our knowledge on locally nilpotent 3-Engel groups. We start with the question of global nilpotence [48].

Theorem 6.13 *Let G be a 4-Engel group without $\{2, 3, 5\}$-elements. Then G is nilpotent of class at most 7.*

Like the analogous result for 3-Engel groups, this can be proved using Lie ring methods. It suffices to show that any finite 4-Engel p-group G, where $p \neq 2, 3, 5$ is nilpotent of class at most 7. Let $L(G)$ be the associated Lie ring of G. As $p > 4$ we have that $L(G)/pL(G)$ is a 4-Engel Lie algebra over the field of p elements. But any 4-Engel Lie algebra over a field of characteristic $p \notin \{2, 3, 5\}$ is known to be nilpotent of class at most 7 [17,47]. $\square$

Remark The primes $2, 3, 5$ are genuinely exceptional. The primes 2 and 3 are exceptional because they are less than 4 and 5 is exceptional as it was already exceptional for 3-Engel groups. By considering a power-commutator presentation of the relatively free 4-Engel group on three generators [41], one can see that 7 is the best upper bound for the class.

We next turn to solvability. As 5 was exceptional for 3-Engel groups it remains so for 4-Engel groups. The structure of 4-Engel groups of exponent 5 is studied in quite some detail in [40]. In particular the authors obtain a normal form theorem for the relatively free group in this variety.

Groups of exponent 4 are known to be center-by-4-Engel and by a well known result of Razmyslov [44] there are non-solvable groups of exponent 4. Thus we have that

2 is an exceptional prime as well with respect to solvability. This leaves out the prime 3. That 4-Engel 3-groups are solvable was proved by Abdollahi and T [1].

Theorem 6.14 *Every* 4-*Engel* 3-*group* G *is solvable.*

We don't include the proof here as it is quite technical. The general strategy is the same as for the analogous result for 3-Engel groups although the details are much harder. Notice that as $3 < 4 \leq 3^2$ it follows from Theorem 4.5 that G^9 is nilpotent. One can therefore first reduce the problem to groups of exponent 9.

As we mentioned in section 4 there are 5-Engel groups that are not Fitting groups. There remains the question whether 4-Engel groups are Fitting groups. The best possible result would be if the Fitting degree was always at most 3. By the example of Gupta and Levin mentioned earlier, we know however that this is not the case. However 4-Engel groups are always Fitting groups of degree at most 4 [50].

Theorem 6.15 *Let* G *be a* 4-*Engel group. Then* G *is a Fitting group of Fitting degree at most* 4. *Furthermore if* G *has no* $\{2,5\}$ *elements then* G *has Fitting degree at most* 3.

Although there are 4-Engel groups with Fitting degree greater than 3 there is another way of characterizing 4-Engel groups in terms of the normal closures of elements. Notice that a group G is a 3-Engel group if and only if the normal closure of every element is 2-Engel. Vaughan-Lee [55] has shown recently that the analog holds for 4-Engel groups. It should be noted that this property fails to hold for 5-Engel groups.

Theorem 6.16 *(Vaughan-Lee)* G *is a* 4-*Engel group if and only if* $\langle x \rangle^G$ *is* 3-*Engel for all* $x \in G$.

We leave this section with two challenging problems. M. Newell has obtained a remarkable generalisation of Heineken's local nilpotence theorem for 3-Engel groups by showing that in any group G, the right 3-Engel elements belong to the locally nilpotent radical. Whether the analog holds for 4-Engel groups is an open problem.

Problem 6. Let G be a group. Do the right 4-Engel elements belong to the locally nilpotent radical of G? Do the left 3-Engel elements belong to the locally nilpotent radical of G?

What about the structure of 5-Engel groups? At present we hardly know anything about them.

Problem 7. Describe the structure of 5-Engel groups?

7   Generalisations

As we mentioned in the the beginning, the theory of Engel groups and the Burnside problems are closely related and in this last section we look at some recent generalised settings. First we will discuss what we call generalised Burnside varieties. These are natural generalisations of the Burnside varieties and the Engel varieties and share many properties with these. There are two types of these that we will refer to as the strong generalised Burnside varieties and the weak generalised Burnside varieties where the latter include the former. Then we will discuss a further generalisation where instead of varieties we work with certain classes of groups satisfying weaker conditions.

7.1   Generalised Burnside varieties

Let $\mathcal{V}$ be a variety of groups. It is well known that the following are equivalent:

A1) For each positive integer r the class of all nilpotent r-generator groups
     in $\mathcal{V}$ is r-bounded.
A2) Every finitely generated group G in $\mathcal{V}$ that is residually nilpotent is
     nilpotent.
A3) The locally nilpotent groups in $\mathcal{V}$ form a subvariety.

For example the Burnside variety $\mathcal{B}_n$ of groups of exponent n and the n-Engel variety $\mathcal{E}_n$ of n-Engel groups satisfy these conditions.

Let us call a variety that satisfies A1)-A3) a strong generalised Burnside variety. Let C be the infinite cyclic group and for any positive integer n, let C_n be the cyclic group of order n. That the wreath products $C \operatorname{wr} C_n$ and $C_n \operatorname{wr} C$ play a crucial role for describing generalised Burnside varieties is already apparent in the work of J. Groves [20] that was later taken on by G. Endimioni [12,13] and F. Point [43] who studied what the latter refers to as Milnor identities and correspond to the generalised Burnside varieties. Building on the work of F. Point and G. Endimioni the following transparent description of these varieties was obtained in [51] (see also [14] for another proof) using again Zel'manov's results one Engel Lie-rings.

Theorem 7.1 *Let $\mathcal{V}$ be a variety. The following are equivalent.*

(1) $\mathcal{V}$ is a strong generalised Burnside variety.
(2) The groups $C_p \operatorname{wr} C$ and $C \operatorname{wr} C_p$ do not belong to $\mathcal{V}$ for any prime p.

The reason why this criteria is transparent is that the wreath products in (2) are metabelian groups with a simple structure. If one has a clear description of the laws that the variety $\mathcal{V}$ satisfies then it is quite straightforward to check if these groups satisfy these laws. For example, as none of these groups is periodic the Burnside varieties are generalised Burnside varieties and as none of these groups

satisfy an Engel identity the n-Engel varieties are also included.

Many of the results for n-Engel groups described above have analogs for generalised Burnside varieties. For example we have the following result of G. Endimioni [13] that gives another characterisation of generalised Burnside varieties. In the following, $\bar{\mathcal{B}}_e$ is the variety of locally nilpotent groups of exponent e and $\mathcal{N}_c$ is the variety of groups that are nilpotent of class at most c.

Theorem 7.2 *Let $\mathcal{V}$ be a variety. The following are equivalent.*

(1) $\mathcal{V}$ is a strong generalised Burnside variety.
(2) There exist positive integers c, e such that all locally nilpotent groups in
$\mathcal{V}$ are both in $\mathcal{N}_c\bar{\mathcal{B}}_e$ and $\bar{\mathcal{B}}_e\mathcal{N}_c$.

Remark. In view of Therorem 4.4 one can choose the integers c, e such that the second variety in (2) can be replaced by the variety of groups satisfying $[x^e, x_1, \ldots, x_c]$.

This theorem preceded Theorem 7.1 but can also be derived from it and the some of the structure theorems on n-Engel groups.

Proof $(2)\Rightarrow(1)$. As none of the wreath products $C_p \operatorname{wr} C$ are nilpotent-by-torsion and none of the wreath products $C \operatorname{wr} C_p$ are torsion-by-nilpotent it follows from Theorem 7.1 that the variety $\mathcal{W} = \mathcal{V} \cap \mathcal{N}_c\bar{\mathcal{B}}_e \cap \bar{\mathcal{B}}_e\mathcal{N}_c$ is a generalised Burnside variety. So the locally nilpotent groups of $\mathcal{W}$ form a variety. But by our assumption these are also the locally nilpotent groups of $\mathcal{V}$. We thus conclude from A3) that $\mathcal{V}$ is a generalised Burnside variety.

$(1)\Rightarrow(2)$. Let $\mathcal{W}$ be the variety consisting of the locally nilpotent groups of $\mathcal{V}$ and let F be the free 2-generator group in $\mathcal{W}$. Then F is nilpotent and thus in particular a n-Engel group for some positive integer n. It follows that all the locally nilpotent groups in $\mathcal{V}$ are n-Engel and (2) thus follows from Theorems 4.3 and 4.4. $\square$

There are related types of varieties satisfying a weaker condition that have been studied by several authors. Let us call a variety $\mathcal{V}$ a *weak generalised Burnside variety* if it satisfies the following equivalent conditions.

B1) For each positive integer r there exist positive integers $c(r)$ and $e(r)$
so that all finite r-generator groups in $\mathcal{V}$ are in $\mathcal{N}_{c(r)}\bar{\mathcal{B}}_{e(r)}$.
B2) Every finitely generated group G in $\mathcal{V}$ that is residually finite is
nilpotent-by-finite.
B3) The locally nilpotent-by-finite groups in $\mathcal{V}$ form a subvariety.

The following Theorem is due to G. Endimioni [13] although not stated explicitly there in this form.

Theorem 7.3 *(Endimioni) Let $\mathcal{V}$ be a variety. The following are equivalent.*

(1) $\mathcal{V}$ is a weak generalised Burnside variety.
(2) The group $C_p \, wr \, C$ does not belong to $\mathcal{V}$ for any prime p.

We also have the analogous criteria to Theorem 7.2 also due to G. Endimioni. It should be noted that Burns and Medvedev have independently arrived at similar results in [6]. Notice that the following theorem tells us that the $c = c(r)$ and $e = e(r)$ in B1) above can be chosen independently of r.

Theorem 7.4 *Let $\mathcal{V}$ be a variety. The following are equivalent.*

(1) $\mathcal{V}$ is a weak generalised Burnside variety.
(2) There exist positive integers c, e such that all locally nilpotent groups in $\mathcal{V}$ are in $\mathcal{N}_c \bar{\mathcal{B}}_e$.

There is an interesting open question related to strong generalised varieties. This is motivated by a result of Zel'manov [62] who has shown that for any given prime p the variety $\bar{\mathcal{B}}_p$, consisting of the locally finite groups of exponent p, is finitely based. It has been observed by G. Endimioni (written correspondence) that the following are equivalent:

(1) The variety $\overline{\mathcal{B}}_n$ of all locally nilpotent groups of exponent n is finitely based for all positive integers n.
(2) The variety $\overline{\mathcal{E}}_n$ of all locally nilpotent n-Engel groups is finitely based for all positive integers n.
(3) If a strong generalised Burnside variety $\mathcal{V}$ is finitely based then so is the variety $\overline{\mathcal{V}}$ consisting of all locally nilpotent groups in $\mathcal{V}$.

Proof (2)$\Rightarrow$ (3). Suppose that the free 2-generator group of $\overline{\mathcal{V}}$ is n-Engel. Then $\overline{\mathcal{V}} = \mathcal{V} \cap \overline{\mathcal{E}}_n$ and thus finitely based as both $\mathcal{V}$ and $\overline{\mathcal{E}}_n$ are finitely based.

(3)$\Rightarrow$ (1). Clear since $\overline{B}_n$ is a strong generalised Burnside variety.

(1)$\Rightarrow$ (2). From the work of Burns and Medvedev we have that there are positive integers $e(n)$ and $c(n)$ such that any group G in $\overline{\mathcal{E}}n$ satisfies $\gamma_{c(n)}(G^{e(n)}) = \{1\}$. As the variety $\overline{\mathcal{B}}_{e(n)}$ is finitely based the same is true of the variety of all groups that are (class $c(n)$)-by-$\overline{\mathcal{B}}_{e(n)}$. As $\overline{\mathcal{E}}_n$ is the intersection of this variety and $\mathcal{E}_n$ it is finitely based. $\square$.

Notice that the equivalence above also demonstrates again how the Burnside varieties and the Engel varieties are closely linked.

Problem 8. Is it true that for every strong generalised Burnside variety $\mathcal{V}$ that is finitely based, we have that the variety $\overline{\mathcal{V}}$ of all the locally nilpotent groups in $\mathcal{V}$ is also finitely based?

Remark. In particular if all n-Engel groups are locally nilpotent then the answer is yes.

7.2   Generalised Engel groups

Some of the results discussed in section 7.1 can be generalised even further. First we need some definitions and notations.

Let G be any group. For $a, t \in G$, let $H = H(a, t) = \langle a \rangle^{\langle t \rangle}$ and

$$A(a, t) = H/[H, H].$$

Then $A(a, t)$ is an abelian section of G. Let $E(a, t)$ be the ring of all endomorphisms of $A(a, t)$. Notice that t induces an endomorphism on $A(a, t)$ by conjugation.

Definition. Let $I \trianglelefteq \mathbb{Z}[x]$. We say that G is an I-group if

$$a^{f(t)} = 0$$

in $A(a, t)$ for all $a, t \in G$ and for all $f \in I$.

For example any n-Engel group is an $\mathbb{Z}[x](x-1)^n$-group.

If G is any group then the set of polynomials f, such that $a^{f(t)} = 0$ in $A(a, t)$ for all $a, t \in G$, form an ideal $I(G)$. There is therefore a unique maximal ideal I such that G is an I-group. We say that two groups H and G are $\mathbb{Z}[x]$-equivalent if $I(H) = I(G)$. We now turn to the generalisations of some of the results of section 7.1. For each prime number p let f_p be the irreducible polynomial $f_p = x^{p-1} + x^{p-2} + \cdots + 1$. The following theorems come from [51]. The first one builds on a work of A. Shalev [45] on collapsing groups (see also [4]).

Theorem 7.5 *Let $f \in \mathbb{Z}[x]$ such that f is not divisible by any prime p. Then there exists positive integers $c(f)$ and $e(f)$ such that*

$$\gamma_{c(f)}(G^{e(f)}) = \{1\}$$

for any $\mathbb{Z}[x]f$-group that is nilpotent-by-finite.

Theorem 7.6 *Let $f \in \mathbb{Z}[x]$ such that f is neither divisible by p nor f_p for all primes p. For each positive integer r there exists a positive integer $c(r, f)$ such that*

$$\gamma_{c(r,f)}(G) = \{1\}$$

for any nilpotent r-generator $\mathbb{Z}[x]f$-group in G.

Theorem 7.7 *Let $f \in \mathbb{Z}[x]$ such that f is neither divisible by p nor f_p for all primes p. There exist positive integers $c(f)$ and $e(f)$ such that*

$$[G^{e(f)}, {}_c G] = (\gamma_{c(f)}(G))^{e(f)} = \{1\}$$

for any nilpotent $\mathbb{Z}[x]f$-group in G.

Proof It follows from Theorem 7.6 that all the nilpotent $\mathbb{Z}[x]f$-group G are $c(2, f)$-Engel and the rest now follows from Theorems 4.3 and 4.4. $\square$

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