Decay of Hankel singular values of analytic control systems

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Abstract

We show that control systems with an analytic semigroup and control and observation operators that are not too unbounded have a Hankel operator that belongs to the Schatten class S_p for all positive p. This implies that the Hankel singular values converge to zero faster than any polynomial rate. This in turn implies fast convergence of balanced truncations. As a corollary, decay rates for the eigenvalues of the controllability and observability gramians are also provided. Applications to the heat equation and a plate equation are given.

1 Introduction

The Hankel singular values σ_k of a transfer function $\mathsf{G} \in H^\infty$ capture how well the transfer function can be approximated by stable transfer functions of smaller McMillan degree. For all stable transfer functions G_n of McMillan degree n we have

$$\sigma_{n+1} \leq \|\mathsf{G} - \mathsf{G}_n\|_{H^\infty},$$

and there always exist stable transfer functions G_n of McMillan degree n (for example those resulting from balanced truncation [7]) such that

$$\|\mathsf{G} - \mathsf{G}_n\|_{H^\infty} \leq 2 \sum_{k=n+1}^{\infty} \sigma_k.$$

We are interested here in the case where G is irrational (and thus has infinite McMillan degree). The above estimates show that a necessary condition for convergence of finite McMillan degree approximations is compactness of the Hankel operator of G and that a sufficient condition is nuclearity (i.e. membership of the trace class) of the Hankel operator of G. These two conditions are discussed

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in [4]. However, the estimates provide more information: if we have a convergence rate of the Hankel singular values, then these provide a convergence rate for the (e.g. balanced truncation) approximations. For delay equations this issue is treated in detail in [6]. For discrete-time infinite-dimensional systems this is treated in [3]. For applications in random matrix theory the decay rate of Hankel singular values (under for control theory applications unreasonably strict assumptions) is studied in [2]. For finite-dimensional systems an investigation of the decay of Hankel singular values was carried out in [1], where also the decay of the eigenvalues of the systems gramians is studied.

In this article we show that for control systems with an analytic semigroup and control and observation operators that are not too unbounded, the Hankel singular values satisfy

$$\sum_{k=1}^{\infty} \sigma_k^p < \infty,$$

(i.e. the Hankel operator is in the Schatten class S_p) for all $p > 0$. This implies that $n^q \sigma_n \to 0$ for all $q > 0$, so that the Hankel singular values of such systems decay very rapidly. The proof of this fact is based on the characterization of Schatten class Hankel operators in terms of their transfer function belonging to a certain Besov space which was proven independently by Peller [13] and Semmes [15] for the case $0 < p < 1$ most relevant here (and earlier by Peller [12] and [11] for the case $p \geq 1$).

In Section 2 we precisely state this characterization theorem and provide the needed definitions. In Section 3 we then state exactly which control systems we study and give some partial differential equation examples. The main result and its proof follow in Section 4. As corollaries in that section we also obtain results on the decay rate of the eigenvalues of the systems gramians.

2 Bergman spaces, Besov spaces and the characterization theorem

The Bergman space $A^p(\mathbb{C}_0^+; \mathscr{B})$ with $p > 0$, $\mathbb{C}_0^+$ the open right half complex plane and $\mathscr{B}$ a Banach space consists of the analytic functions $f : \mathbb{C}_0^+ \to \mathscr{B}$ that satisfy

$$\int_0^\infty \int_{-\infty}^\infty \|f(x+iy)\|_{\mathscr{B}}^p \, dy \, dx < \infty.$$

The Bergman kernel (for the right half-plane) is defined on $\mathbb{C}_0^+ \times \mathbb{C}_0^+$ as

$$K(z,w) := \frac{1}{(z+\bar{w})^2}. \tag{1}$$

The weighted Bergman space $A^{p,r}(\mathbb{C}_0^+; \mathscr{B})$ with $p > 0$ and $r > -\frac{1}{2}$ consists of those analytic functions $f : \mathbb{C}_0^+ \to \mathscr{B}$ that satisfy

$$\int_0^\infty \int_{-\infty}^\infty \|f(x+iy)\|_{\mathscr{B}}^p \, K(x+iy, x+iy)^{-r} \, dy \, dx < \infty,$$

or equivalently

$$\int_0^\infty \int_{-\infty}^\infty \|f(x+iy)\|_{\mathscr{B}}^p \, x^{2r} \, dy \, dx < \infty,$$

(see Semmes [15]). The Besov space $B^p(\mathbb{C}_0^+;\mathscr{B})$ consists of the analytic functions $f : \mathbb{C}_0^+ \to \mathscr{B}$ that satisfy

$$f^{\left(\frac{2}{p}\right)} \in A^p(\mathbb{C}_0^+;\mathscr{B}).$$

Here $f^{\left(\frac{2}{p}\right)}$ is a fractional derivative. Equivalently (without having to deal with fractional derivatives) the Besov space $B^p(\mathbb{C}_0^+;\mathscr{B})$ consists of the analytic functions $f : \mathbb{C}_0^+ \to \mathscr{B}$ that satisfy

$$f^{(n)} \in A^{p,\frac{np}{2}-1}(\mathbb{C}_0^+;\mathscr{B}),$$

for some integer $n > \frac{1}{p}$ (equivalently: for all $n > \frac{1}{p}$). See e.g. Semmes [15] or Peller [13, page 490] for this definition of Besov spaces.

Remark 1. For future reference we note that with K the Bergman kernel from (1), we have for fixed $w \in \mathbb{C}_0^+$ that

$$K(\cdot, w)^\eta \in B^p(\mathbb{C}_0^+),$$

for all $\eta > 0$ and all $p > 0$. This is (up to notation) the same statement as [15, Lemma 5]. Defining for $\eta > 0$ and $w \in \mathbb{C}_0^+$ the function $f : \mathbb{C}_0^+ \to \mathbb{C}$ as

$$f(z) := K(z, w)^\eta,$$

we then see that any analytic function $g : \mathbb{C}_0^+ \to \mathscr{B}$ with the property that for all $n \in \mathbb{N}$ there exists a $M_n > 0$ such that for all $s \in \mathbb{C}_0^+$

$$\|g^{(n)}(s)\|_{\mathscr{B}} \leq M_n |f^{(n)}(s)|, \tag{2}$$

belongs to $B^p(\mathbb{C}_0^+;\mathscr{B})$ for all $p > 0$.

Recall that the Schatten class S_p consists of those operators whose singular values form an $\ell^p(\mathbb{N})$ sequence. Also recall that the Hardy space $H^2(\mathbb{C}_0^+;\mathscr{H})$ with $\mathscr{H}$ a separable Hilbert space consists of the analytic functions $f : \mathbb{C}_0^+ \to \mathscr{H}$ that satisfy

$$\sup_{x>0} \int_{-\infty}^\infty \|f(x+iy)\|_{\mathscr{H}}^2 \, dy < \infty,$$

and carries the obvious inner-product. By taking non-tangential limits, $H^2(\mathbb{C}_0^+;\mathscr{H})$ can be identified with a closed subspace of $L^2(i\mathbb{R};\mathscr{H})$. Similarly, $H^2(\mathbb{C}_0^-;\mathscr{H})$ can be identified with the orthogonal complement of $H^2(\mathbb{C}_0^+;\mathscr{H})$ in $L^2(i\mathbb{R};\mathscr{H})$. The Hardy space $H^\infty(\mathbb{C}_0^+;\mathscr{B})$ consists of all bounded analytic functions $\mathbb{C}_0^+ \to \mathscr{B}$ and carries the obvious norm. By taking non-tangential limits, $H^\infty(\mathbb{C}_0^+;\mathscr{B})$ can be identified with a closed subspace of $L^\infty(i\mathbb{R};\mathscr{B})$. We recall that a function in $H^\infty(\mathbb{C}_0^+;\mathcal{L}(\mathscr{U},\mathscr{Y}))$ –using the above identification– induces by multiplication a bounded operator from $L^2(i\mathbb{R};\mathscr{U})$ to $L^2(i\mathbb{R};\mathscr{Y})$. The Hankel operator associated to this $H^\infty(\mathbb{C}_0^+;\mathcal{L}(\mathscr{U},\mathscr{Y}))$ function is obtained by restricting this multiplication operator to $H^2(\mathbb{C}_0^-;\mathscr{H})$ and projecting onto $H^2(\mathbb{C}_0^+;\mathscr{H})$ (where again the above identifications are used).

The following theorem that follows from Peller [13] and Semmes [15] characterizes Schatten class Hankel operators in terms of their transfer functions.

Theorem 2. *Let $\mathscr{U}$ and $\mathscr{Y}$ be separable Hilbert spaces at least one of which is finite-dimensional and let $p > 0$. The function $\mathsf{G} \in H^\infty(\mathbb{C}_0^+;\mathcal{L}(\mathscr{U},\mathscr{Y}))$ has a S_p Hankel operator if and only if $\mathsf{G} \in B^p(\mathbb{C}_0^+;\mathcal{L}(\mathscr{U},\mathscr{Y}))$.*

Proof. In Semmes [15] and Peller [13] the case where $\mathscr{U} = \mathscr{Y} = \mathbb{C}$ can be found. Peller [14, Corollary 6.9.4] includes the general case of separable Hilbert spaces $\mathscr{U}$ and $\mathscr{Y}$. This is for the disc case, but as in [13, page 490], the half-plane case can be reduced to the disc case. The condition in [14, Corollary 6.9.4] is that $\mathsf{G} \in B^p(\mathbb{C}_0^+;S_p(\mathscr{U},\mathscr{Y}))$. This only leaves to show that $\mathsf{G} \in B^p(\mathbb{C}_0^+;\mathcal{L}(\mathscr{U},\mathscr{Y}))$ is equivalent to $\mathsf{G} \in B^p(\mathbb{C}_0^+;S_p(\mathscr{U},\mathscr{Y}))$ when either $\mathscr{U}$ or $\mathscr{Y}$ is finite-dimensional.

Let $n \in \mathbb{N}$ and denote the singular values of $\mathsf{G}^{(n)}(s)$ (ordered by magnitude) by $\mu_k(s)$. At most $m := \min\{\dim\mathscr{U},\dim\mathscr{Y}\}$ of these are nonzero. We have

$$\mu_1(s)^p \le \sum_{k=1}^{m} \mu_k(s)^p \le m\mu_1(s)^p.$$

It follows that

$$\|\mathsf{G}^{(n)}(s)\|^p_{\mathcal{L}(\mathscr{U},\mathscr{Y})} \le \|\mathsf{G}^{(n)}(s)\|^p_{S_p(\mathscr{U},\mathscr{Y})} \le m\|\mathsf{G}^{(n)}(s)\|^p_{\mathcal{L}(\mathscr{U},\mathscr{Y})}.$$

The result then follows from the definition of Besov space. □

3 Analytic control systems

We recall that associated to a strongly continuous semigroup on a Hilbert space $\mathscr{X}$ there is a scale of fractional power Hilbert spaces $\mathscr{X}_\gamma$ with $\gamma \in \mathbb{R}$. See e.g. Staffans [16, Section 3.9], Pazy [10, Section 2.6], Engel and Nagel [5, Section 2.5].

In this article we consider dynamical systems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

where A generates an exponentially stable analytic semigroup, $B \in \mathcal{L}(\mathscr{U},\mathscr{X}_\beta)$, $C \in \mathcal{L}(\mathscr{X}_\alpha,\mathscr{Y})$, $D \in \mathcal{L}(\mathscr{U},\mathscr{Y})$ with $\alpha - \beta < 1$ and at least one of $\mathscr{U}$ and $\mathscr{Y}$ is finite-dimensional. More information on such system can be found in e.g. Staffans [16, Section 5.7] where among other things it is shown that the transfer function is well-defined by the formula $\mathsf{G}(s) = C(sI - A)^{-1}B + D$. In Curtain–Sasane [4] it is shown that the Hankel operator of such a system (with the additional condition that *both* $\mathscr{U}$ and $\mathscr{Y}$ are finite-dimensional) is in S_1. In this article we will show –by a completely different method– that the Hankel operator of such a system is in fact in the Schatten class S_p for all $p > 0$.

We give a couple of example of such systems.

Consider the one-dimensional heat equation with Neumann (heat flux) control and Dirichlet (temperature) observation at one end and zero Dirichlet condition at the other end (to ensure that the system is exponentially stable):

$$w_t(t,\xi) = w_{\xi\xi}(t,\xi), \quad w_\xi(t,0) = u(t), \quad w(t,1) = 0, \quad y(t) = w(t,0).$$

We choose as state space $\mathscr{X} = L^2(\Omega)$ with $\Omega = (0,1)$. We then have

$$\mathscr{X}_1 = \{f \in W^{2,2}(\Omega) : f'(0) = 0, \quad f(1) = 0\},$$

(where $W^{2,2}$ denotes the standard Sobolev space) and the Laplacian with this domain generates an exponentially stable analytic semigroup. It follows from Grisvard [8] that

$$\begin{aligned}
\mathscr{X}_\gamma &= \{f \in W^{2\gamma,2}(\Omega) : f'(0) = 0, \quad f(1) = 0\}, \quad && \gamma \in \left(\frac{3}{4}, 1\right), \\
\mathscr{X}_\gamma &= \{f \in W^{2\gamma,2}(\Omega) : f(1) = 0\}, && \gamma \in \left(\frac{1}{4}, \frac{3}{4}\right), \\
\mathscr{X}_\gamma &= W^{2\gamma,2}(\Omega), && \gamma \in \left(0, \frac{1}{4}\right).
\end{aligned}$$

Since the trace operator is continuous $W^{\gamma,2}(\Omega) \to W^{\gamma-\frac{1}{2},2}(\partial\Omega)$ we have $C \in \mathcal{L}(\mathscr{X}_\alpha, \mathbb{C})$ for all $\alpha > \frac{1}{4}$. Since $B = C^*$, it follows that for any $\beta < -\frac{1}{4}$ we have $B \in \mathcal{L}(\mathbb{C}, \mathscr{X}_\beta)$. So all the conditions are satisfied.

If Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary consisting of two nonempty parts Γ_1 and Γ_2 such that $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$ (e.g. Ω an annulus with Γ_1 the inner boundary and Γ_2 the outer boundary) then the above analysis still holds when we control and observe on Γ_1 and have zero Dirichlet condition on Γ_2. To obtain a finite-dimensional output space we could observe the average temperature on Γ_1 instead of the point-wise temperature. This gives the system

$$w_t = \Delta w, \quad \frac{\partial w}{\partial n}|_{\Gamma_1}(t) = u(t), \quad w|_{\Gamma_2}(t) = 0, \quad y(t) = \int_{\Gamma_1} w(t,\xi)\, d\xi,$$

with state space $L^2(\Omega)$, input space $L^2(\Gamma_1)$ and output space $\mathbb{C}$.

Consider the heat equation on a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary Γ with Dirichlet control and observation of the average temperature over the whole of the domain:

$$w_t = \Delta w, \quad w|_\Gamma(t) = u(t), \quad y(t) = \int_\Omega w(t,\xi)\, d\xi.$$

Then A (being the Dirichlet Laplacian) again generates an exponentially stable analytic semigroup on $L^2(\Omega)$, $\alpha = 0$, β can be chosen to be any number strictly smaller than $-\frac{3}{4}$ if we choose the input space to be $L^2(\Gamma)$ and the output space is one-dimensional. So the assumptions are again satisfied.

We can also consider the case where there is a point source inside the domain:

$$w_t(t,\xi) = \Delta w(t,\xi) + \delta_{\xi_0}(\xi)u(t), \quad w|_\Gamma(t) = 0, \quad y(t) = \int_\Omega w(t,\xi)\,d\xi.$$

Then A (being the Dirichlet Laplacian) again generates an exponentially stable analytic semigroup in $L^2(\Omega)$ and $\alpha = 0$. The control operator δ_{ξ_0} belongs to $W^{k,2}(\Omega)$ for $k > \frac{n}{2}$ when $\Omega \subset \mathbb{R}^n$ so that $B \in \mathcal{L}(\mathbb{C}, \mathscr{X}_\beta)$ for any $\beta < -\frac{n}{4}$. It follows that the assumptions are satisfied provided that $n = 1, 2, 3$. In the cases $n = 1, 2$ we can even allow for Dirichlet observation on part of the boundary (which gives $\alpha > \frac{1}{4}$) where instead of the zero Dirichlet condition we then impose zero Neumann condition (these parts Γ_1 and Γ_2 should satisfy the conditions above; e.g. be the inner and outer boundary of an annulus):

$$w_t(t,\xi) = \Delta w(t,\xi) + \delta_{\xi_0}(\xi)u(t), \quad \frac{\partial w}{\partial n}|_{\Gamma_1}(t) = 0 \quad w|_{\Gamma_2}(t) = 0, \quad y(t) = w|_{\Gamma_1}(t).$$

The output space in this case is $L^2(\Gamma_1)$ and the input space is again one-dimensional.

Apart from classical parabolic PDEs such as the heat equation considered above also strongly damped beam and plate equations give rise to analytic systems. As a case in point we consider a hinged Euler-Bernoulli plate with interior point control and boundary velocity observation (but there are many other possibilities) on a domain $\Omega \subset \mathbb{R}^2$ with smooth boundary Γ:

$$\begin{aligned} w_{tt}(t,\xi) - \Delta w_t(t,\xi) + \Delta^2 w(t,\xi) &= \delta_{\xi_0}(\xi)u(t), \\ w|_\Gamma(t) &= 0, \\ \Delta w|_\Gamma(t) &= 0, \\ y(t) &= w_t|_\Gamma(t). \end{aligned}$$

The state space (finite energy space) is now $[H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega)$. The input space is one-dimensional and the output space is $L^2(\Gamma)$. We can choose any $\alpha > \frac{1}{4}$ and $\beta < -\frac{1}{2}$ (see [9, Section 3.4]). So the conditions are again satisfied.

4 The main result

The following is the main result of this article.

Theorem 3. *Assume that A generates an exponentially stable analytic semigroup, $B \in \mathcal{L}(\mathscr{U}, \mathscr{X}_\beta)$, $C \in \mathcal{L}(\mathscr{X}_\alpha, \mathscr{Y})$, $D \in \mathcal{L}(\mathscr{U}, \mathscr{Y})$ with $\alpha - \beta < 1$ and that at least one of $\mathscr{U}$ and $\mathscr{Y}$ is finite-dimensional. Then the Hankel operator of this system is in S_p for all $p > 0$.*

Proof. By the generation theorem for analytic semigroup (see e.g. [5, Theorem II.4.6], [10, Theorem 2.5.2] or [16, Theorem 3.10.6]) we have: for all $\omega > \omega_A$ (ω_A being the growth bound of the semigroup) there exists a $C > 0$ such that for all $s \in \mathbb{C}_\omega^+$ (the open right half plane consisting of elements with real part larger than ω):

$$\|(sI-A)^{-1}\| \le \frac{C}{|\omega - s|}.$$

By exponential stability we can choose $\omega < 0$. It follows that for all $\delta \ge 0$ there exists a $M > 0$ such that for all $s \in \mathbb{C}_\omega^+$

$$\|(sI-A)^{-\delta}\| \le \frac{M}{|\omega - s|^\delta}. \tag{3}$$

We have for $n \in \mathbb{N}$ and fixed $\gamma \in \mathbb{C}_\omega^+$

$$\begin{aligned}\tfrac{1}{n!}\|\mathsf{G}^{(n)}(s)\|_{\mathcal{L}(\mathscr{U},\mathscr{Y})} &= \|C(sI-A)^{-n-1}B\|_{\mathcal{L}(\mathscr{U},\mathscr{Y})} \\ &\le \|C(\gamma I-A)^{-\alpha}\|_{\mathcal{L}(\mathscr{X},\mathscr{Y})}\ \|(\gamma I-A)^{\alpha}(sI-A)^{-\alpha}\|_{\mathcal{L}(\mathscr{X})} \\ &\quad \|(sI-A)^{-n-1+\alpha-\beta}\|_{\mathcal{L}(\mathscr{X})} \\ &\quad \|(\gamma I-A)^{-\beta}(sI-A)^{\beta}\|_{\mathcal{L}(\mathscr{X})}\ \|(\gamma I-A)^{\beta}B\|_{\mathcal{L}(\mathscr{U},\mathscr{X})}.\end{aligned}$$

Using Lemma 4 and the assumptions $B \in \mathcal{L}(\mathscr{U},\mathscr{X}_\beta)$, $C \in \mathcal{L}(\mathscr{X}_\alpha,\mathscr{Y})$ it follows that there exists a $Q > 0$ such that

$$\|\mathsf{G}^{(n)}(s)\|_{\mathcal{L}(\mathscr{U},\mathscr{Y})} \le Q\|(sI-A)^{-n-1+\alpha-\beta}\|_{\mathcal{L}(\mathscr{X})}.$$

Using (3) then gives that there exists a $P > 0$ such that

$$\|\mathsf{G}^{(n)}(s)\|_{\mathcal{L}(\mathscr{U},\mathscr{Y})} \le \frac{P}{|\omega - s|^{n+1-\alpha+\beta}},$$

which with $\eta := \frac{1-\alpha+\beta}{2} > 0$ and $w := -\omega \in \mathbb{C}_0^+$ is exactly condition (2). It follows from Remark 1 that $\mathsf{G} \in B^p(\mathbb{C}_0^+;\mathcal{L}(\mathscr{U},\mathscr{Y}))$ for all $p > 0$. It then follows from Theorem 2 that the Hankel operator is in S_p for all $p > 0$. □

The following lemma was used in the proof of Theorem 3.

Lemma 4. *Let A be the generator of an analytic semigroup with growth bound ω_A and let $\omega > \omega_A$ and $\gamma \in \mathbb{C}_\omega^+$ (the open right half plane of elements with real part larger than ω). Then there exists a $M > 0$ such that for all $s \in \mathbb{C}_\omega^+$ and all $\delta \in [0,1]$:*

$$\|(\gamma I-A)^{\delta}(sI-A)^{-\delta}\| \le M.$$

Proof. We have for $\gamma, s \in \mathbb{C}_{\omega_A}^+$ (using e.g. the explicit expression from Pazy [10, Section 2.6, (6.4) and (6.16)]) that

$$(\gamma I-A)^{\delta}(sI-A)^{-\delta} = \left[(\gamma I-A)(sI-A)^{-1}\right]^{\delta},$$

so that it is sufficient to prove the estimate for $\delta = 1$.

For all $\gamma, s \in \rho(A)$ there holds

$$(\gamma I-A)(sI-A)^{-1} = 1 + (\gamma - s)(sI-A)^{-1}, \tag{4}$$

and by the generation theorem for analytic semigroups we have that for all $\sigma > \omega_A$ there exists a $K > 0$ such that for all $s \in \mathbb{C}_\sigma^+$

$$\|(sI - A)^{-1}\|_{\mathscr{X}} \leq \frac{K}{|\sigma - s|}. \tag{5}$$

Combining (4) and (5) gives

$$\|(\gamma I - A)(sI - A)^{-1}\|_{\mathscr{X}} \leq 1 + K \left| \frac{\gamma - s}{\sigma - s} \right|. \tag{6}$$

This upper bound is uniformly bounded in s on $\mathbb{C}_\omega^+$ for any $\omega > \sigma$, so that the result follows. □

As corollaries, we obtain the following results regarding the singular values (which equal the eigenvalues) of the controllability and observability gramians.

Corollary 5. *Assume that A generates an exponentially stable analytic semigroup, $B \in \mathcal{L}(\mathscr{U}, \mathscr{X}_\beta)$ with $-\beta < \frac{1}{2}$ and that $\mathscr{U}$ is finite-dimensional. Then the controllability gramian of this system is in $S_p(\mathscr{X})$ for all $p > 0$.*

Proof. Consider the system with output space $\mathscr{Y} := \mathscr{X}$, $C = I$ and $D = 0$. This system satisfies the conditions of Theorem 3 so that its Hankel operator is in S_p for all $p > 0$. The conditions imposed also ensure [16, Theorem 5.7.3] that this system is $L^2(0, \infty)$ well-posed on this state space. In particular, its observability and controllability gramians are bounded operators on $\mathscr{X}$. Since $C = I$ this system is exactly observable so that the observability gramian is coercive [17, Remark 6.1.4], and since the observability gramian is also bounded it is a similarity transformation on $\mathscr{X}$. The singular values of the Hankel operator equal the square roots of the singular values of the product $L_B L_C$. Since L_C is a similarity transformation on $\mathscr{X}$ we have that

$$\|L_C^{-1}\| \ \sigma_k(L_B L_C) \leq \sigma_k(L_B) \leq \|L_C\|^{-1} \ \sigma_k(L_B L_C),$$

so that it follows that

$$\|L_C^{-1}\| \ \sigma_k^2 \leq \sigma_k(L_B) \leq \|L_C\|^{-1} \ \sigma_k^2,$$

with σ_k the k-th singular value of the Hankel operator. Using this inequality and the fact that the Hankel operator is in S_p for all $p > 0$, it follows that L_B is in $S_p(\mathscr{X})$ for all $p > 0$. □

Corollary 6. *Assume that A generates an exponentially stable analytic semigroup, $C \in \mathcal{L}(\mathscr{X}_\alpha, \mathscr{Y})$ with $\alpha < \frac{1}{2}$ and that $\mathscr{Y}$ is finite-dimensional. Then the observability gramian of this system is in $S_p(\mathscr{X})$ for all $p > 0$.*

Proof. This follows by duality from Corollary 5. □

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