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OPTIMAL INPUT-OUTPUT STABILIZATION OF INFINITE-DIMENSIONAL DISCRETE TIME-INVARIANT LINEAR SYSTEMS BY OUTPUT INJECTION*

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Abstract. We study the optimal input-output stabilization of discrete time-invariant linear systems in Hilbert spaces by output injection. We show that a necessary and sufficient condition for this problem to be solvable is that the transfer function has a left factorization over H-infinity. Another equivalent condition is that the filter Riccati equation (of an arbitrary realization) has a solution (in general, unbounded and even nondensely defined). We further show that after renorming the state space in terms of the inverse of the smallest solution of the filter Riccati equation, the closed-loop system is not only input-output stable but also strongly internally $*$-stable.

Key words. Riccati equation, linear quadratic optimal control, input-output stabilization, output injection, infinite-dimensional system, left factorization

AMS subject classifications. 47N70, 47A48, 47A20, 47A56, 47A62, 93B28, 93C55, 93D15

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1. Introduction.

This is the second in a series of articles dealing in a novel way with the quadratic cost minimization problem for infinite-dimensional time-invariant linear systems in discrete and continuous time. In the first article [17] we investigated the discrete-time full information infinite-horizon linear quadratic problem, and here we will study a deterministic version of the discrete-time infinite-horizon Kalman filtering problem. In the forthcoming third part we will combine the results from these first two articles to examine coprime factorization and dynamic stabilization. The continuous-time equivalents will be the subject of subsequent articles.

1.1. Stabilization by output injection.

In [17] we studied the linear dynamical system in discrete *future time* defined by

$$x_{n+1} = Ax_n + Bu_n, \quad y_n = Cx_n + Du_n, \quad n \in \mathbb{Z}^+; \qquad x_0 = z, \tag{1.1}$$

where $A\colon \mathscr{X} \to \mathscr{X}$, $B\colon \mathscr{U} \to \mathscr{X}$, $C\colon \mathscr{X} \to \mathscr{Y}$, and $D\colon \mathscr{U} \to \mathscr{Y}$ are bounded linear operators; $\mathscr{X}$, $\mathscr{U}$, and $\mathscr{Y}$ are Hilbert spaces; and $\mathbb{Z}^+$ is the set of nonnegative integers. Here we'll study the same system in discrete *past time*:

$$\begin{aligned} x_{n+1} = Ax_n + Bu_n, \quad y_n = Cx_n + Du_n, \quad n \in \mathbb{Z}^-; \qquad x_0 = z,\\ \exists N \in \mathbb{Z}^+ : x_n = 0 = u_n \quad \forall n \le -N, \end{aligned} \tag{1.2}$$

where $\mathbb{Z}^-$ is the set of negative integers and the last line indicates that the sequences have support bounded to the left, which is a sufficient condition for the difference equations to make sense in past time.

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A classical problem is to modify the properties of these systems by using either *state feedback* or *output injection*. State feedback was studied in [17], and here we focus on output injection. Output injection is of importance since it naturally occurs in observer design. In the case of state feedback one chooses the control u to be given by $u_n = Kx_n + v_n$, where $K\colon \mathscr{X} \to \mathscr{U}$ is bounded linear (state feedback) operator, which in the future time setting results in the closed-loop state feedback system:

$$\begin{aligned} x_{n+1} &= (A+BK)x_n + Bv_n, && n \in \mathbb{Z}^+, \\ y_n &= (C+DK)x_n + Dv_n, && n \in \mathbb{Z}^+, \\ u_n &= Kx_n + v_n, && n \in \mathbb{Z}^+, \\ x_0 &= z. \end{aligned} \tag{1.3}$$

Here the input u of the original system plays the role of one of the two outputs of the closed-loop system, and the new input sequence to the closed-loop system is v. In the case of output injection we relax the output equation in (1.2) by allowing a nonzero error term, $w_n := Cx_n + Du_n - y_n$, and injecting a multiple Hw_n of this term back into the state equation in (1.2), where $H\colon \mathscr{Y} \to \mathscr{X}$. Thus, we now treat y like an input. This leads to the following closed-loop output injection system:

$$\begin{aligned} x_{n+1} &= (A-HC)x_n + (B-HD)u_n + Hy_n, && n \in \mathbb{Z}^-, \\ w_n &= Cx_n + Du_n - y_n, && n \in \mathbb{Z}^-, \\ x_0 &= z, \\ \exists N &\in \mathbb{Z}^+ : x_n = 0 = u_n = y_n \quad \forall n \le -N. \end{aligned} \tag{1.4}$$

One typical goal is to make this closed-loop system stable, or at least input-output stable, in the sense that the mapping from the two input sequences u and y to the output sequence w is bounded from $\ell^2(\mathbb{Z}^-;\mathscr{U}\times\mathscr{Y})$ to $\ell^2(\mathbb{Z}^-;\mathscr{Y})$. In the optimal version of this problem one not only requires this input-output map to be bounded but to have the smallest possible norm.

A solution can be found in the following way by means of an optimal control problem if we assume for the moment (to avoid some technical issues to be discussed later) $\mathscr{X}$, $\mathscr{U}$, and $\mathscr{Y}$ to be finite-dimensional and the system $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to be minimal.

For each $z \in \mathscr{X}$ we look for the infimum of $\sum_{n=-\infty}^{-1}(\|y_n\|^2_{\mathscr{Y}} + \|u_n\|^2_{\mathscr{U}})$ over all input sequences u in (1.2) with finite support for which the final state satisfies $x_0 = z$. It is possible to find unique ℓ^2-sequences u and y and a corresponding sequence x tending to zero as $n \to -\infty$ satisfying the first line of (1.2) which minimizes $\sum_{n=-\infty}^{-1}(\|y_n\|^2_{\mathscr{Y}} + \|u_n\|^2_{\mathscr{U}})$ within this class of solutions. The optimal cost of a given final state $z \in \mathscr{X}$ can be written in the form $\langle P^{-1}z, z\rangle_{\mathscr{X}}$ for some bounded nonnegative self-adjoint invertible operator P, and the operator H is given by $H = -(APC^* + BD^*)S_P^{-1}$, where $S_P = I + DD^* + CPC^*$. The operator P is the minimal nonnegative self-adjoint solution of the so-called filter Riccati equation:

$$APA^* - P + BB^* - (APC^* + BD^*)(I + DD^* + CPC^*)^{-1}(CPA^* + DB^*) = 0.$$

The output injection H that we get in this way is optimal even in a stronger sense: if we replace w_n in (1.4) by Ww_n for some bounded linear operator $W\colon \mathscr{Y} \to \mathscr{W}$, then it is not only still true that the same output injection operator minimizes the ℓ^2 operator norm from the pair $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ to Ww but also the ℓ^2 to ℓ^∞ operator norm from the pair $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ to Ww as well as the ℓ^2 to $\mathscr{W}$ norm of the operator from $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ to Ww_{-1}. The optimal norm of all these operators is equal to the norm of $(WS_pW^*)^{1/2}$.

Since the ℓ^2 operator norm is the same as the H^∞ norm of the transfer function, the above is a (very special) H^∞ problem. The $\left[\begin{smallmatrix} d \\ u \end{smallmatrix}\right]$ to Ww_{-1} norm gives the ℓ^2 norm of the impulse response, which equals the H^2 norm of the transfer function, so that we also solve a (very special) H^2 problem.

In stochastic control theory the system (1.4) with this particular choice of H is known as the Kalman filter, and x_0 in (1.4) is interpreted as the minimal variance estimate of the state at time zero based on past values of $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ of a stochastic version of (1.2). We refer the reader to Green and Limebeer [9, section 5.3], Kwakernaak and Sivan [14, Chapter 6], or Kailath [10, section 7.2] for a discussion of the stochastic interpretation of (1.4). The first two of these references also contain more information about the importance of output injection in observer design and, consequently, its importance in dynamic stabilization by output feedback through the separation principle.

The transfer function of the system (1.4) has an additional interesting property. Let us denote the different transfer functions $u \mapsto y$, $y \mapsto w$, and $u \mapsto w$ of the systems (1.2) and (1.4) by, respectively,

$$\begin{aligned}
G_{u,y}(z) &= zC(I - zA)^{-1}B + D,\\
G_{y,w}(z) &= zC(I - z(A - HC))^{-1}H - I,\\
G_{u,w}(z) &= zC(I - z(A - HC))^{-1}(B - HD) + D.
\end{aligned}$$

Then all of these are defined in a neighborhood of the origin and satisfy $G_{u,y}(z) = -G_{y,w}(z)^{-1}G_{u,w}(z)$ in this neighborhood. The input-output stability of (1.4) implies that both $G_{y,w}$ and $G_{u,w}$ are stable transfer functions. This connects the Kalman filtering problem to the factorization approach to systems theory: the transfer function of the optimal closed-loop system (1.4) gives a left factorization of the transfer function of the original system (1.2).

In this article we will show that, even in the infinite-dimensional nonminimal case, once the necessary condition that the transfer function $G_{u,y}$ has a left factorization over H^∞ of the unit disc is satisfied, the procedure outlined above to obtain an optimal output injection operator can be slightly modified so that it always applies. However, the operator P and the output injection operator H need no longer be bounded, their domains need not be dense in the original state space, and the "natural" state space of the optimal closed-loop system need not coincide with the state space of the original system. In the infinite-dimensional case this may happen even in the case where the original system is minimal. The solution of the filter Riccati equation can be used to renorm the state space, and with this new state space the closed-loop system is not only input-output stable but also internally stable in an appropriate sense.

We continue this introduction in section 1.2 with some finite-dimensional nonminimal examples, and we return to the infinite-dimensional case in section 1.3. In particular, there we give some simple infinite-dimensional examples that indicate possible applications of the theory. We compare our results to the existing literature in section 1.4, and in section 1.5 we give a brief outline of the remainder of the article.

1.2. Two finite-dimensional examples. Already in the finite-dimensional case the above solution is a bit too restrictive. Minimality of the system is not needed. Without any change in the argument it can be replaced by the following weaker *state coercive past cost condition*: There exists a finite constant M such that all solutions of (1.2) satisfy $\|z\|^2_{\mathscr{X}} \le M^2 \sum_{n=-\infty}^{-1} (\|y_n\|^2_{\mathscr{Y}} + \|u_n\|^2_{\mathscr{U}})$. However, even this weaker condition is unduly restrictive: it is possible to slightly modify the solution so that no assumption whatsoever of this type is needed in the finite-dimensional case. This is due to the fact that the problem is essentially an input-output problem, and we may at the outset replace the original system by a minimal realization of the same transfer function. In light of the infinite-dimensional case it is, however, instructive not to do this but to stay with the original realization even when it does not satisfy the state coercive past cost condition. In this case the operator P and therefore the output injection operator H are only defined on a subspace of the state space. The state space of the closed-loop system equals this subspace of the state space of the original system. We illustrate this with two simple examples.

We first return to the state coercive past cost condition mentioned earlier. A simple example where this condition is not satisfied is

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1, 0 \end{bmatrix}, \quad D = 0.$$

This can be seen as follows. For any natural number $n > 1$, the input sequence $\{u_k^n\}_{k \in \mathbb{Z}^-}$ defined by $u_k^n = 2^{k+1}\delta_{-n,k}$ (delta is the Kronecker delta) reaches the state $x_0 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Since the second component is unobservable, the infimum of the cost to reach $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ is clearly zero, but $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] \neq 0$ so that the state coercive past cost condition does not hold. We consider the filter Riccati equation

$$APA^* - P + BB^* - (APC^* + BD^*)(I + DD^* + CPC^*)^{-1}(CPA^* + DB^*) = 0$$

for this system. It is easily computed that the unique solution is the indefinite matrix

$$\frac{1}{3}\begin{bmatrix} 3 & 3 \\ 3 & 1 \end{bmatrix}$$

so that the filter Riccati equation has no nonnegative self-adjoint everywhere defined solution. It does have a nonnegative definite self-adjoint solution defined on a proper subspace of the state space, namely, the identity on the subspace spanned by $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. By a (possibly not everywhere defined) solution we mean a self-adjoint operator P with domain $D(P)$ such that

$$\langle \left[APA^* - P + BB^* - (APC^* + BD^*)(I + DD^* + CPC^*)^{-1}(CPA^* + DB^*)\right] x, x\rangle$$

is equal to zero for all $x \in D(P)$. For a finite-dimensional system, a nonnegative self-adjoint solution, with $D(P)$ equal to the orthogonal complement of the unobservable subspace, always exists. The corresponding output injection operator H is defined by the same formula as in the minimal case, followed by the orthogonal projection onto the range of P (the operator C^* maps into the domain of P so that those equations are well-defined).

The output injection corresponding to the indefinite solution of the filter Riccati equation is $H = \left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right]$ (but this output feedback is not relevant for the problem at hand). The output injection operator derived from the (not everywhere defined) nonnegative self-adjoint solution of the filter Riccati equation is in this case the zero operator $\mathscr{Y} \to R(P)$, where $R(P)$ is the image of P which in the example is the subspace of the state space spanned by $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$.

The optimal closed-loop system is only defined on the subspace $R(P)$. This is the system with $A = 0$, $B = [1, 0]$, $C = 1$, and $D = [0, -1]$. Note that this optimal closed-loop system is exponentially stable. In the finite-dimensional case this will always be true. In the infinite-dimensional case the optimal closed-loop system will not necessarily be exponentially stable, but it will be stable in a weaker sense (Theorem 7.2).

Later in this article we will consider the following *output coercive past cost condition*: There exists a finite constant M such that all solutions of (1.2) satisfy $\|Cz\|_{\mathscr{Y}}^2 \leq M^2 \sum_{n=-\infty}^{-1} (\|y_n\|_{\mathscr{Y}}^2 + \|u_n\|_{\mathscr{U}}^2)$. This condition becomes a triviality for finite-dimensional systems: it is always satisfied.

The main problem with the above example was that it was unobservable, which led to some final states having zero cost. It is also interesting to see what happens when we have an uncontrollable system, which can lead to some final states having infinite cost. A particular example is

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1, 1 \end{bmatrix}, \quad D = 0.$$

For this system the subspace spanned by $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ is unreachable. The smallest nonnegative self-adjoint solution of the filter Riccati equation is $P = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. The optimal cost for reaching the state z is $\langle P^{-1}z, z\rangle$, and this illustrates that the subspace spanned by $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ (the orthogonal complement of the reachable subspace) has infinite cost. In contrast to the unobservable case, we could now in principle use the same formulas as in the minimal case for the optimal closed-loop system, but the natural state space for the optimal closed-loop system is, as before, $R(P)$, which in this case is the subspace spanned by $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ (the reachable subspace).

1.3. The infinite-dimensional case. We saw earlier that the transfer function of the optimal closed-loop system provides a left factorization of the transfer function of the original system. As we will show in this article, this carries over to the infinite-dimensional case. In the finite-dimensional case every transfer function has a left factorization, but this is not true in the infinite-dimensional case. The function $\sqrt{1+2z}$ is not meromorphic on the unit disc (it has an essential singularity in $z = \frac{-1}{2}$), and therefore, it does not have a left factorization over H^∞ of the unit disc. From its Taylor series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1-2n)(n!)^2 2^n} z^n,$$

we can, however, construct a realization on the state space $\ell^2(\mathbb{Z}_{\geq 1})$ as

$$(Ax)_n = 4x_{n+1}, \quad (Bu)_n = \frac{(-1)^n (2n)!}{(1-2n)(n!)^2 8^n} u, \quad Cx = 4x_1, \quad D = 1.$$

So this is an example of a transfer function that does not have a left factorization over H^∞. No output injection operator that is stabilizing in any meaningful sense exists for such a transfer function.

The "correct" state space assumption on the original system is the output coercive past cost condition: There exists a finite constant M such that all solutions of (1.2) satisfy $\|Cz\|_{\mathscr{Y}}^2 \leq M^2 \sum_{n=-\infty}^{-1} (\|y_n\|_{\mathscr{Y}}^2 + \|u_n\|_{\mathscr{U}}^2)$. This condition is satisfied for any realization of a transfer function that has a left factorization over H^∞ of the unit disc, and its being satisfied for some realization implies that the transfer function has such a left factorization (Theorem 6.10).

When we replace the state space of the original system (which is assumed to satisfy the output coercive past cost condition) with the natural state space for its optimal closed-loop system (which is the closure in an appropriate norm of the range of the operator P), then the original system with this state space does satisfy the state coercive past cost condition (Theorem 3.11). Finding a state space on which the state coercive past cost condition is satisfied is therefore not a prerequisite for the solution of the optimal output injection problem, but it is a *consequence* of this solution. The optimal closed-loop system with this renormed state space is internally stable in a suitable sense (Theorem 7.2).

We'll next illustrate these facts by means of three infinite-dimensional examples, which due to space constraints are relatively simple. They are primarily intended to highlight some of the specific features of the theory and to point out some directions of where to look for possible applications, without any claim of completeness.

Consider the following formal set of (identical) scalar difference equations indexed by $k \in \mathbb{Z}^+$:

$$x_{n+1}^k = \frac{1}{2}x_n^k + u_n^k, \quad y_n^k = x_n^k, \quad n \in \mathbb{Z}^+.$$

We can write this as an abstract system,

$$x_{n+1} = Ax_n + Bu_n, \quad y_n = Cx_n,$$

where

$$A = \frac{1}{2}I_{\mathscr{X}}, \quad B = I_{\mathcal{L}(\mathscr{U},\mathscr{X})}, \quad C = I_{\mathcal{L}(\mathscr{X},\mathscr{Y})},$$

and $I_{\mathcal{L}(\mathscr{H},\mathscr{K})}$ denotes the inclusion operator of $\mathscr{H}$ into $\mathscr{K}$. Here $\mathscr{X}$, $\mathscr{U}$, $\mathscr{Y}$ are Hilbert sequences spaces (e.g., weighted ℓ^2 spaces). For well posedness of this system (i.e., boundedness of the operators), it is obviously necessary and sufficient to choose $\mathscr{U} \subset \mathscr{X} \subset \mathscr{Y}$ with continuous inclusions. It is not difficult to see, given $\mathscr{U}$ and $\mathscr{Y}$ such that $\mathscr{U} \subset \mathscr{Y}$ with continuous inclusion, that for any state space $\mathscr{X}$ wedged in between the input and output space, the state coercive past cost condition is satisfied. Note that when $\mathscr{U}$ is strictly contained in $\mathscr{Y}$, then there are infinitely many such spaces. The state space with the strongest possible norm so that the system is well posed and the state coercive past cost condition is satisfied is of particular interest (in this case—of course, up to similarity—it equals $\mathscr{U}$). This will be studied in a more abstract setting at the end of section 3 (in terms of what is there called the completed $\mathcal{I}_p$-compression).

Next we treat two examples that are slightly outside the scope of this article since they also use some results that will only be presented in later articles in this series. However, they are important for the motivation of the present article, and the ideas (if maybe not all the details) are hopefully simple enough for the reader to follow.

Let us consider the wave equation on the interval $[0, 1]$ with some rather special boundary conditions as follows:

$$w_{tt} = w_{\xi\xi}, \quad w_t(t, 0) - w_x(t, 0) = u(t), \quad w(t, 1) = 0, \quad y(t) = w_t(t, 0) + w_x(t, 0).$$

At this moment we will treat this as a formal equation, and we do not yet choose a state space. Using the D'Alembert solution, it can be easily calculated that $y(t) = u(t - 2)$ for $t \geq 2$, and if we assume zero initial conditions, then $y(t) = 0$ for $t \in [0, 2]$. So the output is just a delayed version of the input (due to the specific boundary conditions). It follows that the transfer function is e^{-2s}. Since this transfer function is itself in H^∞ of the right half-plane, it trivially has a left factorization over H^∞ of the right half-plane. Using, e.g., the Cayley transform between continuous time and discrete time (details will be given in [16]) and the results in this article, it follows that there exists a state space on which the state coercive past cost condition is satisfied. Not surprisingly, the finite energy space $W^{1,2}(0,1) \times L^2(0,1)$ is such a state space. If we equip this space with the energy norm, then the optimal cost operator both for future time (treated in [17]) and for past time is the identity (in future time the optimal control is clearly zero, and the cost is simply equal to the energy of the initial condition; in past time the optimal input is chosen zero for $t < -2$, and on $[-2, 0]$ it is chosen so as to reach the given final state; the required cost is the energy of this final state, the corresponding optimal output is equal to zero). It follows that the energy space is—up to similarity—the only state space on which both the state coercive past cost condition and the finite future cost condition hold. So from a control theory point of view, this is the only reasonable state space to consider the formal wave equation on to make it into a properly posed problem.

Finally, we consider the heat equation on the interval $[0, 1]$ with the Neumann boundary control at one end and the Dirichlet observation at the same end:

$$w_t = w_{\xi\xi}, \quad w_\xi(t, 0) = u(t), \quad w(t, 1) = 0, \quad y(t) = w(t, 0).$$

The transfer function is easily seen to be in H^∞ of the right half-plane so that it trivially has a left factorization over H^∞. As above, it follows that there exists a state space on which the state coercive past cost condition is satisfied. Actually, there are many state spaces on which both the state coercive past cost condition and the finite future cost condition are satisfied. Any of the spaces $W^{s,2}(0, 1)$ with $s \in (-\frac{1}{2}, \frac{1}{2})$ will do (essentially because the system is well posed and exponentially stable on all of these spaces). So this heat equation example is very different in nature from the above wave equation example: now there are infinitely many nonisomorphic state spaces that are perfectly reasonable from a control theory perspective.

1.4. The literature. The stabilization by output injection problem is of course classical since it appears naturally in observer and filter designs. The stochastic finite-dimensional version goes back to Kalman [11] and Kalman and Bucy [12]. There are many deterministic estimation problems that also have the Kalman filter as their solution and therefore can be interpreted as deterministic versions of the Kalman filtering problem. The best known is probably the H^2 output estimation problem (see, e.g., [7] for the finite-dimensional case). Another deterministic interpretation of the Kalman filter (which differs from our approach) is given in Willems [20], where also the history of this issue is further addressed. The problem that we study is, however, really a special H^∞ problem instead of the H^2 problem. Due to the special nature of this H^∞ problem, it can be solved using the filter Riccati equation instead of the more complicated H^∞ Riccati equation. In essence the special case of the H^∞ problem treated here is the output estimation equivalent (and the problem studied in [17] was the full information equivalent) of the special output feedback H^∞ problem from Glover and McFarlane [8]. Their solution was also based on the control and filter Riccati equation instead of the H^∞ Riccati equations. Our approach is radically different from the one in [8] since our solution is based directly on the linear quadratic optimal control problem, whereas [8] is based on the solution of the Nehari problem.

In the infinite-dimensional case the *final state* estimation problem was treated for continuous-time deterministic systems in Weiss and Rebarber [19]. This problem was solved there solely by duality; the optimal control problem on the negative time axis that we consider here was not explicitly treated, and in particular, the coercive cost conditions were not identified.

Unbounded solutions to Riccati equations seem to have first been systematically studied in Da Prato and Delfour [5, 6]. We also mention the work of Arov, Kaashoek, and Pik [1] and Arov and Staffans [2] on unbounded solutions of the Kalman–Yakubovich–Popov inequality in discrete and continuous time, respectively.

1.5. Outline of the article. In section 2 we recall some basic definitions concerning discrete time-invariant infinite-dimensional systems. The open-loop final state optimal control problem mentioned above is solved in section 3, and in section 4 the results on the initial state optimal control problem from [17] are reviewed. Duality is treated in section 5 and then applied to the optimal control problems in section 6. Finally, in section 7 the closed-loop final state optimal control problem is treated.

2. Discrete-time systems. In this section we collect definitions and known results on discrete-time systems that are needed in this article. A collection of bounded operators A, B, C, D as before will be called a *node* and will often be denoted as $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$.

We first associate some operators on sequences spaces to the dynamical systems (1.1) and (1.2). In the following definition $\ell^p_c(\mathbb{Z}^-;\mathscr{H})$ with $1 \leq p \leq \infty$ is the subspace of $\ell^p(\mathbb{Z}^-;\mathscr{H})$ consisting of sequences with compact support, and $\mathfrak{s}(\mathbb{Z}^+;\mathscr{H})$ is the space of all sequences $\mathbb{Z}^+ \to \mathscr{H}$.

DEFINITION 2.1.

- *The* input map $\mathcal{B} : \ell^p_c(\mathbb{Z}^-;\mathscr{U}) \to \mathscr{X}$ *is the map that sends* $\{u_n\}_{n\in\mathbb{Z}^-}$ *to* z*:*

$$\mathcal{B}u = \sum_{k=0}^{\infty} A^k B u_{-k-1}.$$

- *The* output map $\mathcal{C} : \mathscr{X} \to \mathfrak{s}(\mathbb{Z}^+;\mathscr{Y})$ *is the map that sends* z *to* $\{y_n\}_{n\in\mathbb{Z}^+}$*:*

$$(\mathcal{C}z)_n = CA^n z.$$

- *The* future input-output map $\mathcal{D} : \mathfrak{s}(\mathbb{Z}^+;\mathscr{U}) \to \mathfrak{s}(\mathbb{Z}^+;\mathscr{Y})$ *is the map that sends* $\{u_n\}_{n\in\mathbb{Z}^+}$ *to* $\{y_n\}_{n\in\mathbb{Z}^+}$*:*

$$(\mathcal{D}u)_n = \sum_{k=0}^{n-1} CA^k B u_k + D u_n.$$

- *The* past input-output map $\mathcal{D}_- : \ell^p_c(\mathbb{Z}^-;\mathscr{U}) \to \ell^p_c(\mathbb{Z}^-;\mathscr{Y})$ *is the map that sends* $\{u_n\}_{n\in\mathbb{Z}^-}$ *to* $\{y_n\}_{n\in\mathbb{Z}^-}$*:*

$$(\mathcal{D}_- u)_n = \sum_{k=-\infty}^{n-1} CA^{n-1-k} B u_k + D u_n.$$

If we equip $\ell^p_c(\mathbb{Z}^-;\mathscr{U})$ with its natural LF topology and $\mathfrak{s}(\mathbb{Z}^+;\mathscr{H})$ with its natural Fréchet topology, then the above maps are continuous.

DEFINITION 2.2. *A state* z *is called* finite-time reachable *if there exist sequences* u*,* x*,* y *such that* (1.2) *holds. The set of finite-time reachable states is denoted by* Ξ_-*. If the closure of* Ξ_- *equals* $\mathscr{X}$*, then the node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *is called* controllable.

Note that $\Xi_- = R(\mathcal{B})$.

DEFINITION 2.3. *A state* z *is called* unobservable *if for initial condition* z *and zero input* u*, the output* y *of the system* (1.1) *is zero. The set of unobservable states is denoted by* $\mathscr{N}$*. The node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *is called* observable *if* $\mathscr{N} = \{0\}$*.*

Note that $\mathscr{N} = N(\mathcal{C})$.

DEFINITION 2.4. *The node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *is called*

- *exponentially stable if* $r(A) < 1$ *(spectral radius),*
- *strongly stable if* $\lim_{k\to\infty} A^k z = 0$ *for all* $z \in \mathscr{X}$,
- *strongly* $*$*-stable if* $\lim_{k\to\infty} A^{*k} z^* = 0$ *for all* $z^* \in \mathscr{X}'$,
- *input stable if* $\mathcal{B}$ *extends to a bounded operator* $\ell^2(\mathbb{Z}^-;\mathscr{U}) \to \mathscr{X}$,
- *output stable if* $\mathcal{C}$ *is a bounded operator* $\mathscr{X} \to \ell^2(\mathbb{Z}^+;\mathscr{Y})$,
- *input-output stable if* $\mathcal{D}$ *restricts to a bounded operator* $\ell^2(\mathbb{Z}^+;\mathscr{U}) \to \ell^2(\mathbb{Z}^+;\mathscr{Y})$,
- *strongly internally stable if it is strongly stable, input stable, output stable, and input-output stable,*
- *strongly internally* $*$*-stable if it is strongly* $*$*-stable, input stable, output stable, and input-output stable,*

Exponential stability implies strong internal stability and strong internal $*$-stability, but the converse is not true. As announced in the introduction, by changing the norm in the state space, we will be able to make the optimal closed-loop system strongly internally $*$-stable, but it will in general not be exponentially stable.

3. The final state optimal control problem. In this section we investigate the open-loop final state optimal control problem introduced in the introduction. The synthesis of the optimal control as an output injection is considered in section 7.

We first define some spaces and operators that allow us to rephrase the open-loop final state optimal control problem in a form appropriate for the application of a standard optimization technique (the orthogonal projection lemma).

For a finite-time reachable state z define

$$\mathscr{W}_c(z) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \ell^2_c(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U}) : \exists x \text{ such that (1.2) holds} \right\}$$

the set of compactly supported input-output trajectories with z as final state. Further define

$$\mathscr{G}_c := \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \ell^2_c(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U}) : \exists x, z \text{ such that (1.2) holds} \right\}$$

the set of compactly supported input-output trajectories. Note that $\mathscr{G}_c$ is the inverse graph of the past input-output map.

Define the operator

$$\mathcal{J}_c : \mathscr{G}_c \to \mathscr{X}, \qquad \mathcal{J}_c \begin{bmatrix} y \\ u \end{bmatrix} = z,$$

where u, y, and z are related by (1.2); i.e., $\mathcal{J}_c$ maps a compactly supported input-output trajectory to the corresponding final state.

Further define the operator

$$\Gamma_p : \mathscr{G}_c \to \mathfrak{s}(\mathbb{Z}^+;\mathscr{Y}), \qquad \Gamma_p = \mathcal{C}\mathcal{J}_c,$$

that maps a compactly supported input-output trajectory on $\mathbb{Z}^-$ to the corresponding output on $\mathbb{Z}^+$ when the input is chosen to be zero on $\mathbb{Z}^+$. Finally, define the set of stable past input-output trajectories $\mathscr{G}$ as the closure of $\mathscr{G}_c$ in $\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})$. Note that $\mathscr{G}$ is the closure of the inverse graph of the past input-output map considered as an unbounded operator $\ell^2(\mathbb{Z}^-;\mathscr{U}) \to \ell^2(\mathbb{Z}^-;\mathscr{Y})$.

To obtain a satisfactory theory for the final state optimal control problem mentioned in the introduction, it is crucial to extend Γ_p to $\mathscr{G}$. For that we make the following assumption.

DEFINITION 3.1. *A node satisfies the* output coercive past cost condition *if there exists an* $M > 0$ *such that for all* $z \in \Xi_-$ *and all* $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{W}_c(z)$

$$\|Cz\|_{\mathscr{Y}} \leq M \left\| \begin{bmatrix} y \\ u \end{bmatrix} \right\|_{\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})}.$$

A stronger condition (which ensures that $\mathcal{J}_c$ extends to $\mathscr{G}$) is the following.

DEFINITION 3.2. *A node satisfies the* state coercive past cost condition *if there exists an* $M > 0$ *such that for all* $z \in \Xi_-$ *and all* $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{W}_c(z)$

$$\|z\|_{\mathscr{X}} \leq M \left\| \begin{bmatrix} y \\ u \end{bmatrix} \right\|_{\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})}.$$

Remark 3.3. The output coercive past cost condition is equivalent to $C\mathcal{J}_c : \mathscr{G}_c \to \mathscr{Y}$ extending to a bounded operator $\mathscr{G} \to \mathscr{Y}$ and is equivalent to $\Gamma_p : \mathscr{G}_c \to \mathfrak{s}(\mathbb{Z}^+;\mathscr{Y})$ extending to a bounded operator $\mathscr{G} \to \mathfrak{s}(\mathbb{Z}^+;\mathscr{Y})$, where this latter space is equipped with its natural Fréchet space topology. The state coercive past cost condition is equivalent to $\mathcal{J}_c : \mathscr{G}_c \to \mathscr{X}$ extending to a bounded operator $\mathscr{G} \to \mathscr{X}$.

Not only $\mathcal{J}_c$ but also its closure will play an important role. This closure will allow us to interpret the notion of final state for some noncompactly supported input-output trajectories. In general, $\mathcal{J}_c$ need not be a closable operator. However, the closure of the graph of $\mathcal{J}_c$ always defines a closed linear relation (or multivalued operator) which we will denote by $\mathcal{J}$. The development of a satisfactory theory does not hinge on $\mathcal{J}$ being single-valued. Multivaluedness of $\mathcal{J}$ relates to ill posedness of the dynamical system defined on $\mathbb{Z}^-$ when the compact support assumption is not made. We recall some basic definitions regarding multivalued operators.

DEFINITION 3.4. *A multivalued operator (or relation)* $T : \mathscr{H}_1 \to \mathscr{H}_2$ *is a subspace* $\mathscr{V}_T$ *of* $\mathscr{H}_1 \times \mathscr{H}_2$*. The operator* T *is called closed when the subspace* $\mathscr{V}_T$ *is closed. We have the following for the domain, kernel, range and multivalued part of* T*, respectively:*

$$\begin{aligned} D(T) &= \{h_1 \in \mathscr{H}_1 : \exists h_2 \text{ such that } (h_1,h_2) \in \mathscr{V}_T\}, \\ N(T) &= \{h_1 \in \mathscr{H}_1 : (h_1,0) \in \mathscr{V}_T\}, \\ R(T) &= \{h_2 \in \mathscr{H}_2 : \exists h_1 \text{ such that } (h_1,h_2) \in \mathscr{V}_T\}, \\ M(T) &= \{h_2 \in \mathscr{H}_2 : (0,h_2) \in \mathscr{V}_T\}. \end{aligned}$$

LEMMA 3.5. *If the node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *satisfies the output coercive past cost condition, then* $M(\mathcal{J}) = N(\mathcal{C})$.

Proof. We denote the closure of Γ_p by $\overline{\Gamma}_p$. Since $\overline{\Gamma}_p = \mathcal{C}\mathcal{J}$, we have $z \in M(\mathcal{J})$ if and only if $\mathcal{C}z \in M(\overline{\Gamma}_p)$. Since under the output coercive past cost condition $\overline{\Gamma}_p$ is single-valued, $M(\overline{\Gamma}_p) = \{0\}$, and it follows that $z \in M(\mathcal{J})$ if and only if $\mathcal{C}z = 0$; i.e., $M(\mathcal{J}) = N(\mathcal{C})$. ∎

LEMMA 3.6. *If the node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *is observable and satisfies the output coercive past cost condition, then* $\mathcal{J}_c$ *is a closable operator.*

Proof. From Lemma 3.5 and observability it follows that $M(\mathcal{J}) = \{0\}$ so that $\mathcal{J}$ is single-valued. Hence $\mathcal{J}_c$ has a closed extension that is a single-valued operator, so it is a closable operator. ∎

We define $\Xi_p := R(\mathcal{J})$. This space has the interpretation of finite cost reachable elements in the state space $\mathscr{X}$. For $z \in \Xi_p$ we define

$$\mathscr{W}(z) := \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in D(\mathcal{J}) : \mathcal{J} \begin{bmatrix} y \\ u \end{bmatrix} = z \right\}$$

the set of stable input-output trajectories with z as final state. The final state optimal control problem consists of finding the element of minimal norm in $\mathscr{W}(z)$. To solve that problem we utilize the following well-known orthogonal projection lemma.

LEMMA 3.7. *Let $\mathscr{H}$ be a Hilbert space and $\mathscr{K}$ be a nonempty closed subspace of $\mathscr{H}$. Define, for $h_0 \in \mathscr{H}$, the affine set*

$$\mathscr{K}(h_0) := \{h \in \mathscr{H} : h = h_0 + k \text{ for some } k \in \mathscr{K}\}.$$

Then there exists a unique $h_{\min} \in \mathscr{K}(h_0)$ such that

$$\|h_{\min}\| = \min_{h \in \mathscr{K}(h_0)} \|h\|.$$

The vector $h_{\min}$ is characterized by the fact that $\mathscr{K}(h_0) \cap (\mathscr{H} \ominus \mathscr{K}) = \{h_{\min}\}$.

Proof. A proof can be found in many books, e.g., [13, section 3.2]. ∎

Applying this orthogonal projection lemma to our problem gives the following.

LEMMA 3.8. *For any $z \in \Xi_p$ the space $\mathscr{W}(z)$ has a unique element of minimal norm which is characterized by the fact that it is in $\ell^2(\mathbb{Z}^-; \mathscr{Y} \times \mathscr{U}) \ominus \mathscr{W}(0)$.*

Proof. Apply the orthogonal projection lemma (Lemma 3.7) with $\mathscr{H} = \ell^2(\mathbb{Z}^-; \mathscr{Y} \times \mathscr{U})$ and $\mathscr{K} = \mathscr{W}(0)$. Then $\mathscr{W}(z)$ takes the role of $\mathscr{K}(h_0)$. We have that $\mathscr{W}(0) = \mathcal{N}(\mathcal{J})$ is a closed subspace since $\mathcal{J}$ is closed and $\mathscr{W}(z)$ is nonempty since $z \in \Xi_p$. ∎

We define the set $\mathscr{G}_{\text{opt}} := D(\mathcal{J}) \ominus \mathscr{W}(0)$. This set has the interpretation of the set of all optimal input-output trajectories with a final state in $\mathscr{X}$. We restrict $\mathcal{J}$ to this set to obtain the (possibly multivalued) operator

$$\mathcal{J}_r : \mathscr{G}_{\text{opt}} \to \mathscr{X}.$$

This operator is clearly closed, injective, and has range Ξ_p. We further define

$$\mathcal{I}_p : \Xi_p \subset \mathscr{X} \to \mathscr{G}_{\text{opt}}, \qquad \mathcal{I}_p = \mathcal{J}_r^{-1},$$

the closed operator that maps a final state to the corresponding optimal input-output trajectory. We note that $N(\mathcal{I}_p) = M(\mathcal{J}_r) = M(\mathcal{J})$ so that $\mathcal{I}_p$ is not injective if $\mathcal{J}$ is multivalued (i.e., in that case two different final states have the same optimal input-output trajectories). Note that

$$\mathcal{I}_p \mathcal{J} = P_{\ell^2(\mathbb{Z}^-; \mathscr{Y} \times \mathscr{U}) \ominus \mathscr{W}(0)}|_{D(\mathcal{J})}, \tag{3.1}$$

the orthogonal projection onto $\ell^2(\mathbb{Z}^-; \mathscr{Y} \times \mathscr{U}) \ominus \mathscr{W}(0)$, since both equal the map that sends an input-output trajectory to the optimal input-output trajectory with the same final state.

For the final state optimal control problem only the subspace Ξ_p of finite cost final states is of importance; the rest of the state space should be ignored. The norm in the state space is also not relevant in the final state optimal control problem. There is a more natural seminorm on Ξ_p associated with the final state optimal control problem. On Ξ_p we define the seminorm

$$\|z\|_p := \|\mathcal{I}_p z\|_{\ell^2(\mathbb{Z}^-; \mathscr{Y} \times \mathscr{U})}.$$

Note that since $\mathcal{I}_p$ is a closed operator, the associated nonnegative symmetric sesquilinear form $\langle \mathcal{I}_p z_1, \mathcal{I}_p z_2\rangle_{\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})}$ in $\mathscr{X}$ is closed.

The following lemma shows that the space Ξ_p and the seminorm on it interact well with the node $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$ if the output coercive past cost condition is satisfied.

LEMMA 3.9. *If the node $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$ satisfies the output coercive past cost condition, then it maps $\Xi_p \times \mathscr{U}$ into $\Xi_p \times \mathscr{Y}$, and its restriction to $\Xi_p \times \mathscr{U}$ is bounded with respect to the seminorm $\|\cdot\|_p$.*

Proof. The operator B obviously maps into Ξ_-, the space of finite-time reachable states. Since $\Xi_- \subset \Xi_p$, certainly B maps into Ξ_p. Since the input u defined by $u_{-1} = v$, $u_k = 0$ for $k < -1$ reaches Bv, we have

$$\|Bv\|_p^2 = \|\mathcal{I}_p Bv\|^2_{\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})} \le \|v\|^2_{\mathscr{U}} + \|Dv\|^2_{\mathscr{Y}}.$$

So B is bounded with respect to the seminorm $\|\cdot\|_p$. Note that the output coercive past cost condition was not used for this.

For $z \in \Xi_p$ we have that for all $\left[\begin{smallmatrix} y \\ u\end{smallmatrix}\right] \in \mathscr{W}(z)$

$$\|Cz\|_{\mathscr{Y}} = \left\| \left(\Gamma_p \begin{bmatrix} y \\ u \end{bmatrix} \right)_0 \right\|_{\mathscr{Y}} \le M \left\| \begin{bmatrix} y \\ u \end{bmatrix} \right\|_{\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})},$$

where we have used that $\Gamma_p : \mathscr{G} \to \mathfrak{s}(\mathbb{Z}^+;\mathscr{Y}\times\mathscr{U})$ is a bounded operator. In particular, the above holds for the element of minimal norm in $\mathscr{W}(z)$: $\left[\begin{smallmatrix} y \\ u\end{smallmatrix}\right] = \mathcal{I}_p z$. This shows that C is bounded with respect to the seminorm $\|\cdot\|_p$.

If $z = \mathcal{J}\left[\begin{smallmatrix} y \\ u\end{smallmatrix}\right]$, then Az is the image under $\mathcal{J}$ of the trajectory obtained by shifting $\left[\begin{smallmatrix} y \\ u\end{smallmatrix}\right]$ one place to the left and adding $\left[\begin{smallmatrix} Cz \\ 0\end{smallmatrix}\right]$ at the last position. It follows that $Az \in \Xi_p$ if $z \in \Xi_p$ and that

$$\|\mathcal{I}_p Az\|^2_{\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})} \le \|\mathcal{I}_p z\|^2_{\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})} + \|Cz\|^2_{\mathscr{Y}}$$

or, equivalently, that

$$\|Az\|_p^2 \le \|z\|_p^2 + \|Cz\|^2_{\mathscr{Y}}.$$

Using that C is bounded with respect to the seminorm $\|\cdot\|_p$, it follows that A is bounded with respect to this seminorm. $\square$

If $\mathcal{I}_p$ is injective, then $\|\cdot\|_p$ is a norm, and Lemma 3.9 shows that we can extend the restriction of the node to $\Xi_p \times \mathscr{U}$ to a node whose state space is the completion of Ξ_p under the norm $\|\cdot\|_p$. This resulting node will be called the *completed $\mathcal{I}_p$-compression* of the node $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$. Its significance will be clear from Theorems 3.11 and 7.2 below. In general (when $\mathcal{I}_p$ is not necessarily injective), the situation is slightly more complicated but essentially the same. On the set $P_{\mathscr{X}\ominus\mathscr{N}}\Xi_p$, the seminorm $\|\cdot\|_p$ is in fact a norm since $\mathscr{N} = N(\mathcal{I}_p)$ by Lemma 3.5. Lemma 3.9 shows that, under the output coercive past cost condition, the conditions of [17, Theorem B.14] are satisfied so that the completed $\mathcal{I}_p$-compression of the node $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$ exists. This has state space $\mathscr{X}_p$, the completion under the $\|\cdot\|_p$ norm of $P_{\mathscr{X}\ominus\mathscr{N}}\Xi_p$. In case $\mathcal{I}_p$ is injective these two procedures obviously coincide.

Note that $\mathscr{X}_p$ need not be contained in the state space $\mathscr{X}$.

The completed $\mathcal{I}_p$-compression has $\mathcal{J}_c$, $\mathcal{J}$, and $\mathcal{I}_p$ operators which we will denote by $\mathcal{J}_c^{\mathscr{X}_p}$, $\mathcal{J}^{\mathscr{X}_p}$, and $\mathcal{I}_p^{\mathscr{X}_p}$, respectively.

LEMMA 3.10. *If the node $\left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$ satisfies the output coercive past cost condition, then $\mathcal{J}_c = P_{\mathscr{X}\ominus\mathscr{N}}\mathcal{J}^{\mathscr{X}_p}|_{\mathscr{G}_c}$.*

Proof. This follows directly from [17, Theorem B.16]. ☐

THEOREM 3.11. *The completed $\mathcal{I}_p$-compression satisfies the state coercive past cost condition and is observable. The operator $\mathcal{I}_p^{\mathscr{X}_p}$ is a unitary map onto its range. The operator $\mathcal{J}^{\mathscr{X}_p}$ is a partial isometry with kernel $\mathscr{W}(0)$.*

Proof. Using (3.1) we have the following for $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_c$:

$$\begin{aligned}\left\| \mathcal{J}_c^{\mathscr{X}_p} \begin{bmatrix} y \\ u \end{bmatrix} \right\|_p &= \left\| \mathcal{I}_p^{\mathscr{X}_p} \mathcal{J}_c^{\mathscr{X}_p} \begin{bmatrix} y \\ u \end{bmatrix} \right\|_{\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})} \\ &= \left\| P_{\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})\ominus N(\mathcal{J}^{\mathscr{X}_p})} \begin{bmatrix} y \\ u \end{bmatrix} \right\|_{\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})} \\ &\le \left\| \begin{bmatrix} y \\ u \end{bmatrix} \right\|_{\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})},\end{aligned}$$

which shows that $\mathcal{J}_c^{\mathscr{X}_p} : \mathscr{G}_c \to \mathscr{X}_p$ is bounded. It follows that the state coercive past cost condition is satisfied.

We just showed that $\mathcal{J}_c^{\mathscr{X}_p}$ has a single-valued bounded extension to $\mathscr{G}$, and since the state coercive past cost condition is satisfied (which implies the output coercive past cost condition), it follows from Lemma 3.5 that $N(\mathcal{C}^{\mathscr{X}_p}) = \{0\}$ so that the completed $\mathcal{I}_p$-compression is indeed observable.

By definition of norms, $\mathcal{I}_p^{\mathscr{X}_p}$ is an isometry, so it is a unitary map onto its range.

By definition of the norm and (3.1), we have for $g \in D(\mathcal{J}^{\mathscr{X}_p})$

$$\|\mathcal{J}^{\mathscr{X}_p} g\|_p = \|\mathcal{I}_p^{\mathscr{X}_p} \mathcal{J}^{\mathscr{X}_p} g\|_{\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})} = \|P_{\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})\ominus\mathscr{W}(0)} g\|_{\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})},$$

which implies that $\mathcal{J}^{\mathscr{X}_p}$ is a partial isometry with kernel $\mathscr{W}(0)$. ☐

Remark 3.12. Define $\mathcal{I}_{p,-} := \mathcal{I}_p|_{\Xi_-}$. Similarly, as for the completed $\mathcal{I}_p$-compression, it can be shown that the completed $\mathcal{I}_{p,-}$-compression is well-defined. It has as state space $\mathscr{X}_{p,-}$, the completion of $P_{\mathscr{X}\ominus\mathscr{N}}\Xi_-$ under the norm $\|\cdot\|_p$, and is controllable, observable, and satisfies the state coercive past cost condition.

4. Recap of the initial state optimal control problem. In this section we review the relevant results from [17], which by the duality theory of the next two sections will lead to synthesis of the optimal output injection in section 7.

The system under study in this section is the initial state problem (1.1) with the associated cost function $\sum_{n=0}^{\infty} \|u_n\|^2 + \|y_n\|^2$. For $z \in \mathscr{X}$, define

$$\mathscr{V}(z) := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \ell^2(\mathbb{Z}^+;\mathscr{U}\times\mathscr{Y}) : \exists x \text{ such that (1.1) holds} \right\}$$

the set of stable input-output trajectories with z as initial state. Further define Ξ_f as the subspace of $\mathscr{X}$ consisting of those z for which $\mathscr{V}(z)$ is nonempty. This is the subspace of finite future cost states (denoted by Ξ_+ in [17]).

The orthogonal projection lemma guarantees that, for $z \in \Xi_f$, $\mathscr{V}(z)$ has a unique element of minimal norm. This provides us with a closed operator $\mathcal{I}_f : \Xi_f \subset \mathscr{X} \to \ell^2(\mathbb{Z}^+;\mathscr{U}\times\mathscr{Y}) \ominus \mathscr{V}(0)$, the future minimizing cost operator, which maps a finite cost initial state to the corresponding optimal input-output trajectory.

DEFINITION 4.1. *The finite future incremental cost condition is the condition $B\mathscr{U} \subset \Xi_f$ (equivalently, $\Xi_- \subset \Xi_f$). The finite future cost condition is the condition $\Xi_f = \mathscr{X}$.*

If the finite future incremental cost condition holds, then the operator

$$\Gamma_f := \mathcal{I}_f \mathcal{B} : \ell^2_c(\mathbb{Z}^-;\mathscr{U}) \to \ell^2(\mathbb{Z}^+;\mathscr{U}\times\mathscr{Y})$$

that maps a compactly supported past input to the corresponding optimal future input-output trajectory is well-defined and bounded. If the finite future cost condition holds, then $\mathcal{I}_f : \mathscr{X} \to \ell^2(\mathbb{Z}^+;\mathscr{U}\times\mathscr{Y})$ is bounded.

On Ξ_f define the seminorm

$$\|z\|_f := \|\mathcal{I}_f z\|_{\ell^2(\mathbb{Z}^+;\mathscr{U}\times\mathscr{Y})}.$$

LEMMA 4.2. *If the node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *satisfies the finite future incremental cost condition, then it maps* $\Xi_f \times \mathscr{U}$ *into* $\Xi_f \times \mathscr{Y}$*, and its restriction to* $\Xi_f \times \mathscr{U}$ *is bounded with respect to the seminorm* $\|\cdot\|_f$*.*

Proof. This follows from [17, Lemma 4.8] with $q = q_f$. ◻

Similarly to $\mathscr{X}_p$, the state space $\mathscr{X}_f$ is defined as the completion under the $\|\cdot\|_f$ norm of $P_{\mathscr{X}\ominus\mathscr{N}}\Xi_f$. The restriction of the node mentioned in Lemma 4.2 extends continuously to a node with this state space. That node is called the completed $\mathcal{I}_f$-compression (the completed q_f-compression in [17]), and it satisfies the finite future cost condition and is observable. Similarly, we can define $\mathcal{I}_{f,-} := \mathcal{I}_f|_{\Xi_-}$ and the completed $\mathcal{I}_{f,-}$-compression (the completed q_f^--compression in [17]) which has as state space $\mathscr{X}_{f,-}$, the completion of $P_{\mathscr{X}\ominus\mathscr{N}}\Xi_-$ under the norm $\|\cdot\|_f$, satisfies the finite future cost condition and is both controllable and observable.

Note that $\mathscr{X}_f$ and $\mathscr{X}_{f,-}$ need not be contained in the state space $\mathscr{X}$.

Remark 4.3. Denote the $\mathcal{I}_f$ operator of the completed $\mathcal{I}_f$-compression by $\mathcal{I}_f^{\mathscr{X}_f}$; then $\mathcal{I}_f^{\mathscr{X}_f}$ is an isometry onto its range (which equals the closure of the range of the $\mathcal{I}_f$ operator of the original node that the completed $\mathcal{I}_f$-compression was constructed from). The inverse of $\mathcal{I}_f^{\mathscr{X}_f}$ (defined on the range of $\mathcal{I}_f^{\mathscr{X}_f}$) is a unitary map that sends the optimal input-output trajectory to the initial state.

The following is the standard control algebraic Riccati equation rewritten in a way (using sesquilinear forms) that easily allows for unbounded solutions.

DEFINITION 4.4. *The triple* (q, s, K) *is called a (nonnegative) solution of the* control Riccati equation *of the node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *if*

1. *q is a closed nonnegative symmetric sesquilinear form in $\mathscr{X}$ whose domain satisfies $AD(q) \subset D(q)$, $B\mathscr{U} \subset D(q)$;*
2. *s is a bounded nonnegative symmetric sesquilinear form on $\mathscr{U}$;*
3. *$K : D(q) \to \mathscr{U}$ is a linear operator;*
4. *for all $z \in D(q)$, $u \in \mathscr{U}$ we have*

$$q(Az+Bu, Az+Bu)+\|Cz+Du\|^2_{\mathscr{Y}}+\|u\|^2_{\mathscr{U}} = q(z,z)+s(Kz-u, Kz-u). \tag{4.1}$$

The solution is called classical *when* $D(q) = \mathscr{X}$*.*

Remark 4.5. In part one [17] of our series we gave several equivalent formulations of the control Riccati equation. Among them is the following in terms of operators instead of sesquilinear forms. The triple (Q, S, K) is called a (nonnegative) solution of the *operator control Riccati equation* of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ if

1. Q is a closed nonnegative self-adjoint operator in $\mathscr{X}$ whose domain satisfies $AD(Q^{1/2}) \subset D(Q^{1/2})$, $B\mathscr{U} \subset D(Q^{1/2})$;
2. S is a bounded nonnegative self-adjoint operator on $\mathscr{U}$;
3. $K : D(Q^{1/2}) \to \mathscr{U}$ is a linear operator;

4. for all $z \in D(Q^{1/2})$, $u \in \mathscr{U}$ we have

$$\|Q^{1/2}(Az+Bu)\|^2_{\mathscr{X}} + \|Cz+Du\|^2_{\mathscr{Y}} + \|u\|^2_{\mathscr{U}} = \|Q^{1/2}z\|^2_{\mathscr{X}} + \|S(Kz-u)\|^2_{\mathscr{U}}.$$

The solution is called *classical* when $D(Q) = \mathscr{X}$.

To discuss transfer functions, we use the following notation: H^∞ denotes the Hardy space of uniformly bounded holomorphic functions, and $\mathbb{D}$ denotes the unit disc. The transfer function of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is defined in a neighborhood of zero by $zC(I-zA)^{-1}B + D$. A node is called a *realization* of a holomorphic function defined in a neighborhood of zero if that function is the transfer function of the node. We note that any holomorphic function defined in a neighborhood of zero has a realization (in fact, it has infinitely many).

DEFINITION 4.6. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic at the origin. A function* $\left[\begin{smallmatrix} \mathsf{M} \\ \mathsf{N} \end{smallmatrix}\right] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))$ *is called a* right factorization *of* G *if* $\mathsf{M}(z)$ *is invertible for all* z *in a neighborhood of the origin and* $\mathsf{G}(z) = \mathsf{N}(z)\mathsf{M}(z)^{-1}$ *in a neighborhood of the origin.*

THEOREM 4.7. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic at the origin, and let* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *be a realization of* G. *The following are equivalent conditions.*

- $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *satisfies the finite future incremental cost condition.*
- *The control Riccati equation of* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *has a (nonnegative self-adjoint) solution.*
- G *has a right factorization.*

Under these equivalent conditions, the triple (q_f, s_f, K_f), *defined by*

$$\begin{aligned} q_f(z_1, z_2) &:= \langle \mathcal{I}_f z_1, \mathcal{I}_f z_2 \rangle_{\ell^2(\mathbb{Z}^+; \mathscr{U} \times \mathscr{Y})}, \\ s_f(u, v) &:= \langle u, v \rangle_{\mathscr{U}} + \langle Du, Dv \rangle_{\mathscr{Y}} + q_f(Bu, Bv), \\ K_f z &= P_{\mathscr{U}} (\mathcal{I}_f z)_0, \end{aligned}$$

is the smallest nonnegative self-adjoint solution of the control Riccati equation. Here $P_{\mathscr{U}}$ *is the canonical projection* $\mathscr{U} \times \mathscr{Y} \to \mathscr{U}$.

Proof. This follows from [17, Theorem 6.3] combined with [17, Theorem 3.14]. ∎

We consider the closed-loop system

$$\begin{aligned} x_{n+1} &= (A + BK)x_n + BEw_n, && n \in \mathbb{Z}^+, \\ y_n &= (C + DK)x_n + DEw_n, && n \in \mathbb{Z}^+, \\ u_n &= Kx_n + Ew_n, && n \in \mathbb{Z}^+, \\ x_0 &= z, \end{aligned} \tag{4.2}$$

where $E \colon \mathscr{W} \to \mathscr{U}$ is a bounded linear operator and $K : D(K) \subset \mathscr{X} \to \mathscr{U}$ is a linear operator with a domain that is A-invariant and that contains the image of B. For such a K, the map from $\{w_n\}_{n \in \mathbb{Z}^+}$ to $\{\left[\begin{smallmatrix} u_n \\ y_n \end{smallmatrix}\right]\}_{n \in \mathbb{Z}^+}$ in (4.2) (with $z = 0$) is well-defined on the sequences with compact support.

THEOREM 4.8. *Assume that the finite future incremental cost condition holds. Then* K_f *minimizes both the* $\mathcal{L}(\ell^1(\mathbb{Z}^+, \mathscr{W}), \ell^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y}))$ *and the* $\mathcal{L}(\ell^2(\mathbb{Z}^+, \mathscr{W}), \ell^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y}))$ *norm of the map from* $\{w_n\}_{n \in \mathbb{Z}^+}$ *to* $\{\left[\begin{smallmatrix} u_n \\ y_n \end{smallmatrix}\right]\}_{n \in \mathbb{Z}^+}$ *in* (4.2) *(with* $z = 0$*), where* K *ranges over all linear maps* $D(K) \subset \mathscr{X} \to \mathscr{U}$ *with a domain that is* A*-invariant and that contains the image of* B. *The operator* K_f *also minimizes the* $\mathcal{L}(\mathscr{W}, \ell^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y}))$ *norm of the map* $w_0 \to \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right]$ *over the same set of feedback operators.*

These minimum norms all equal the square root of $\sup_{\|v\|=1} s_f(Ev, Ev)$.

Proof. The statements on the $\mathcal{L}(\ell^1(\mathbb{Z}^+, \mathscr{W}), \ell^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y}))$ and $\mathcal{L}(\ell^2(\mathbb{Z}^+, \mathscr{W}), \ell^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y}))$ norms follow directly from [17, Theorem 5.1]. The statement about the $\mathcal{L}(\mathscr{W}, \ell^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y}))$ norm follows from slightly adapting the proof of [17, Theorem 5.1] as follows: the lower bound proof remains unchanged, and in the upper bound proof v now has to be chosen as $v_0 = Ew_0$ and $v_k = 0$ for $k > 0$. ◻

Remark 4.9. The operator underlying the closed-loop system (4.2) is generally not a node with state space $\mathscr{X}$ (because K is generally not bounded on $\mathscr{X}$), but it does become a node with state space $\mathscr{X}_f$ once we replace $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ with its completed $\mathcal{I}_f$-compression. This resulting closed-loop node equals what is called the completed q_f-compression of the graph closed-loop node in [17, Theorem 5.3]. Hence, by that theorem, it is strongly internally stable.

5. Duality of discrete-time systems. In this section we reconsider duality for discrete-time systems in order to investigate the duality between the initial state and final state optimal control problems in the next section. The duality that we consider is somewhat nonstandard (e.g., we identify $\ell^2(\mathbb{Z}^-)$ with $\ell^2(\mathbb{Z}^+)$ instead of with $\ell^2(\mathbb{Z}^-)$ itself). This is to make the duality between the initial state and final state optimal control problems work. See Remark 5.3 for a further elaboration on this issue.

We consider the adjoint of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ as an operator from $\mathscr{X}' \times \mathscr{Y}'$ to $\mathscr{X}' \times \mathscr{U}'$, where $\mathscr{H}'$ denotes the dual space of the Hilbert space $\mathscr{H}$. A dual space is not identified with the Hilbert space itself unless this is explicitly stated. We denote the duality product between $\mathscr{H}$ and its dual $\mathscr{H}'$ by $\langle \cdot, \cdot \rangle_{\mathscr{H}, \mathscr{H}'}$ and consider this to be linear in the $\mathscr{H}$ component and antilinear in the $\mathscr{H}'$ component.

The dynamical system that we associate to $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]^*$ is the following initial state problem:

$$x^*_{n+1} = A^* x^*_n + C^* y^*_n, \quad u^*_n = B^* x^*_n + D^* y^*_n, \quad n \in \mathbb{Z}^+; \qquad x^*_0 = z^* \tag{5.1}$$

with state space $\mathscr{X}'$, input space $\mathscr{Y}'$, and output space $\mathscr{U}'$. Throughout we apply the theory from section 4 to this dual system. To indicate the distinction between spaces associated to the primal system and the dual system, we often use the subscript d, so, e.g., $\mathscr{V}_d(z^*)$ is the space introduced in the beginning of section 4 consisting of stable input-output trajectories but now *for the dual system* with initial state z^*.

We define the weighted ℓ^p spaces

$$\ell^p_r(\mathbb{Z}^-; \mathscr{U}) = \{u : \mathbb{Z}^- \to \mathscr{U} : (r^{-n} u_n)_{n \in \mathbb{Z}^-} \in \ell^p(\mathbb{Z}^-; \mathscr{U})\},$$
$$\ell^p_r(\mathbb{Z}^+; \mathscr{Y}) = \{y : \mathbb{Z}^+ \to \mathscr{Y} : (r^{-n} y_n)_{n \in \mathbb{Z}^+} \in \ell^p(\mathbb{Z}^+; \mathscr{Y})\}.$$

Any continuous linear functional on $\ell^2_r(\mathbb{Z}^-; \mathscr{H})$ is of the form

$$\sum_{n=0}^{\infty} \langle h^-_{-n-1}, h^+_n \rangle_{\mathscr{H}, \mathscr{H}'} \tag{5.2}$$

for some $h^+ \in \ell^2_r(\mathbb{Z}^+; \mathscr{H}')$, and any such h^+ through the expression (5.2) gives rise to a continuous linear functional on $\ell^2_r(\mathbb{Z}^-; \mathscr{H})$. Similarly, any continuous antilinear functional on $\ell^2_r(\mathbb{Z}^+; \mathscr{H}')$ is of the form (5.2) for some $h^- \in \ell^2_r(\mathbb{Z}^-; \mathscr{H})$, and any such h^- gives rise to a continuous antilinear functional. So we may treat $\ell^2_r(\mathbb{Z}^-; \mathscr{H})$, and $\ell^2_r(\mathbb{Z}^+; \mathscr{H}')$ as each other's duals. With some abuse of notation, we denote the duality product by

$$\langle h^-, h^+ \rangle_{\ell^2(\mathscr{H})} := \sum_{n=0}^{\infty} \langle h^-_{-n-1}, h^+_n \rangle_{\mathscr{H}, \mathscr{H}'}. \tag{5.3}$$

The dual of $\ell^1_r(\mathbb{Z}^+;\mathscr{H})$ can similarly be identified with $\ell^\infty_r(\mathbb{Z}^-;\mathscr{H}')$ through the expression (5.2). The dual of the subspace $\ell^\infty_{r,0}(\mathbb{Z}^-;\mathscr{H})$ of $\ell^\infty_r(\mathbb{Z}^-;\mathscr{H})$ consisting of those sequences h such that $\lim_{k\to-\infty} r^{-k}h_k = 0$ can be identified with $\ell^1_r(\mathbb{Z}^+;\mathscr{H}')$ through (5.2).

If $r > r(A)$, the spectral radius of A, then the input map extends to a bounded operator $\ell^2_r(\mathbb{Z}^-;\mathscr{U}) \to \mathscr{X}$, and the output map is a bounded operator $\mathscr{X} \to \ell^2_r(\mathbb{Z}^+;\mathscr{Y})$. The input map $\mathcal{B}$ of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is adjoint to the output map $\mathcal{C}_d$ of the dual node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]^*$ in the sense that

$$\langle \mathcal{B}u, z^*\rangle_{\mathscr{X},\mathscr{X}'} = \langle u, \mathcal{C}_d z^*\rangle_{\ell^2(\mathscr{U})}.$$

The past input-output map of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ extends to a bounded operator $\ell^2_r(\mathbb{Z}^-;\mathscr{U}) \to \ell^2_r(\mathbb{Z}^-;\mathscr{Y})$, and the future input-output map of the dual node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]^*$ restricts to a bounded operator $\ell^2_r(\mathbb{Z}^+;\mathscr{Y}') \to \ell^2_r(\mathbb{Z}^+;\mathscr{U}')$. With the above identification of dual spaces, these operators are adjoints. Similarly, the restriction of the future input-output map of the adjoint node to a bounded operator $\ell^1_r(\mathbb{Z}^+;\mathscr{Y}') \to \ell^2_r(\mathbb{Z}^+;\mathscr{U}')$ and the extension of the past input-output map to a bounded operator $\ell^2_r(\mathbb{Z}^-;\mathscr{U}) \to \ell^\infty_{r,0}(\mathbb{Z}^-;\mathscr{Y})$ are adjoint operators with the above identification of dual spaces.

The following lemma characterizes duality in terms of trajectories without explicit reference to the node. It can be derived from [3, Lemma 4.6], but for the reader's convenience we include a direct proof.

LEMMA 5.1. *If* $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_c$, $z = \mathcal{J}_c\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$, *and* $\begin{bmatrix} y^* \\ u^* \end{bmatrix}$ *is a trajectory of* (5.1) *with initial condition* z^*, *then*

$$\langle z, z^*\rangle_{\mathscr{X},\mathscr{X}'} = \left\langle \mathcal{R}\begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y^* \\ u^* \end{bmatrix}\right\rangle_{\ell^2(\mathscr{Y}\times\mathscr{U})}, \tag{5.4}$$

where the operator $\mathcal{R}$ *is defined by*

$$\mathcal{R} : \ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U}) \to \ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U}), \qquad \mathcal{R}\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} -y \\ u \end{bmatrix},$$

and we have used the duality (5.3). *Conversely, if* $\left[\begin{smallmatrix} y^* \\ u^* \end{smallmatrix}\right] \in \mathfrak{s}(\mathbb{Z}^+;\mathscr{Y}'\times\mathscr{U}')$ *and* $z^* \in \mathscr{X}'$ *satisfy* (5.4) *for all* $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_c$ *with* $z = \mathcal{J}_c\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$, *then* $\left[\begin{smallmatrix} y^* \\ u^* \end{smallmatrix}\right]$ *is a trajectory of* (5.1) *with initial condition* z^*.

Proof. The first part of the lemma simply follows by substitution. The converse follows by iteratively applying the assumption as follows. Apply (5.4) with $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ the sequence that is zero everywhere except at position -1 where it equals $\left[\begin{smallmatrix} Dv \\ v \end{smallmatrix}\right]$ with $v \in \mathscr{U}$ arbitrary. It follows that

$$\langle v, u_0^* - D^*y_0^*\rangle_{\mathscr{U},\mathscr{U}'} = \left\langle \begin{bmatrix} -Dv \\ v \end{bmatrix}, \begin{bmatrix} y_0^* \\ u_0^* \end{bmatrix}\right\rangle = \langle Bv, z^*\rangle = \langle v, B^*z^*\rangle,$$

which, since v was arbitrary, implies $u_0^* = B^*z^* + D^*y_0^*$. Taking the element of $\mathscr{G}_c$ whose second component is zero everywhere except at position -2 where it equals v and whose first component is the corresponding output (i.e., Dv at position -2, CBv at position -1, and zero elsewhere) gives $u_1^* = B^*A^*z^* + D^*y_1^* + B^*C^*y_0^*$. Continuing in this fashion shows that $\left[\begin{smallmatrix} y^* \\ u^* \end{smallmatrix}\right]$ is an input-output trajectory of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]^*$ with initial condition z^* as desired. ∎

We will also use the following adjoint of $\mathcal{R}$:

$$\mathcal{R}^* : \ell^2(\mathbb{Z}^+; \mathscr{Y}' \times \mathscr{U}') \to \ell^2(\mathbb{Z}^+; \mathscr{Y}' \times \mathscr{U}'), \qquad \mathcal{R}^* \begin{bmatrix} y^* \\ u^* \end{bmatrix} = \begin{bmatrix} -y^* \\ u^* \end{bmatrix}.$$

Since we do not identify the dual of a Hilbert space with itself, a subspace $\mathscr{V}$ of a Hilbert space $\mathscr{H}$ has two orthogonal subspaces: the subspace $\mathscr{V}^\perp \subset \mathscr{H}'$ of continuous linear functionals on $\mathscr{H}$ that are zero on the subspace $\mathscr{V}$ and the subspace $\mathscr{H} \ominus \mathscr{V}$ that is the orthogonal complement of $\mathscr{V}$ in the sense that $\mathscr{V} \oplus (\mathscr{H} \ominus \mathscr{V}) = \mathscr{H}$. In this article we use both of these notions of orthogonal subspace and use the notations $\perp$ and $\ominus$ as above to distinguish these two notions.

Remark 5.2. We identify the dual G' of a closed subspace G of H by the corresponding subspace of H' so that $\langle g, g' \rangle_{G,G'} = \langle g, g' \rangle_{H,H'}$ for all $g \in G$ and $g' \in G'$. Under this identification $G' = H' \ominus G^\perp$ and $G^\perp = (H \ominus G)'$. In particular, we have $\mathscr{G}' = \ell^2(\mathbb{Z}^+; \mathscr{Y}' \times \mathscr{U}') \ominus \mathscr{G}^\perp$.

Remark 5.3. The above duality setup is somewhat nonstandard. We would, in principle, lose nothing by using the standard duality setup (i.e., identifying the dual of a Hilbert space with the Hilbert space itself and sequences spaces on $\mathbb{Z}^+$ with sequence spaces on $\mathbb{Z}^+$) which has dual systems running backward in time. However, in [16] we will consider an optimal control problem on $\mathbb{Z}$ where the state has to pass through a target state x_0 at $n = 0$, and this problem naturally breaks down into a final state problem on $\mathbb{Z}^-$ and an initial state problem on $\mathbb{Z}^+$. This is our main reason for wanting to study final state systems defined on $\mathbb{Z}^-$ and therefore to identify the dual of an initial state system defined on $\mathbb{Z}^+$ to be a final state system defined on $\mathbb{Z}^-$. The Kalman filter is also most naturally posed on $\mathbb{Z}^-$, which is another reason for treating the equivalent optimal output injection problem on $\mathbb{Z}^-$ as well. Not identifying the state space $\mathscr{X}$ with its dual is appropriate since we want to identify the "natural state space" $\mathscr{X}_p$ on which to consider the final state problem as the dual of the "natural state space" $\mathscr{X}_{f,d}$ on which to consider the initial state problem for the dual system (Lemma 6.6). The only reason for not identifying $\mathscr{U}$ and $\mathscr{Y}$ with their respective duals is consistency.

6. Duality between the optimal control problems. The next lemma relates the spaces $\mathscr{G}$ and $\mathscr{V}_d(0)$ of stable past and future input-output trajectories.

LEMMA 6.1. *We have $\mathcal{R}\mathscr{G} = \mathscr{V}_d(0)^\perp$.*

Proof. It immediately follows from (5.4) with $z^* = 0$ that $\mathcal{R}\mathscr{G}_c \perp \mathscr{V}_d(0)$. By continuity of $\mathcal{R}$, we conclude that $\mathcal{R}\mathscr{G} \perp \mathscr{V}_d(0)$ so that $\mathcal{R}\mathscr{G} \subset \mathscr{V}_d(0)^\perp$.

We now prove that $(\mathcal{R}\mathscr{G}_c)^\perp \subset \mathscr{V}_d(0)$, which through $\mathscr{V}_d(0)^\perp \subset (\mathcal{R}\mathscr{G}_c)^{\perp\perp} = \overline{\mathcal{R}\mathscr{G}_c} = \mathcal{R}\overline{\mathscr{G}_c} = \mathcal{R}\mathscr{G}$ gives the desired other inclusion. So assume that $\left[\begin{smallmatrix} y^* \\ u^* \end{smallmatrix}\right]$ is orthogonal to $\mathcal{R}\mathscr{G}_c$. Then $\left[\begin{smallmatrix} y^* \\ u^* \end{smallmatrix}\right]$ satisfies (5.4) for all $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_c$ with $z^* = 0$, and it follows from Lemma 5.1 that $\left[\begin{smallmatrix} y^* \\ u^* \end{smallmatrix}\right]$ is a trajectory of the dual system with initial condition zero. Since by assumption $\left[\begin{smallmatrix} y^* \\ u^* \end{smallmatrix}\right] \in \ell^2(\mathbb{Z}^+; \mathscr{Y}' \times \mathscr{U}')$, we have that it is an element of $\mathscr{V}_d(0)$ as desired. $\square$

LEMMA 6.2. *We have $\mathcal{R}^*(\ell^2(\mathbb{Z}^+; \mathscr{Y}' \times \mathscr{U}') \ominus \mathscr{V}_d(0)) = \mathscr{G}'$.*

Proof. Denote the identification map implicit in (5.3) by $\mathfrak{I} : \ell^2(\mathbb{Z}^-; \mathscr{Y} \times \mathscr{U}) \to \ell^2(\mathbb{Z}^+; \mathscr{Y}' \times \mathscr{U}')$ (i.e., if we would identify $\mathscr{Y}$ and $\mathscr{U}$ with their respective duals, then it is simply the reflection). Then it is easily seen that $\mathfrak{I}^*\mathcal{R}^*\mathfrak{I}\mathcal{R} = I$. From Lemma 6.1 it follows that $\mathcal{R}\mathscr{G} = \mathscr{V}_d(0)^\perp$ so that $\mathfrak{I}\mathcal{R}\mathscr{G} = \ell^2(\mathbb{Z}^+; \mathscr{Y}' \times \mathscr{U}') \ominus \mathscr{V}_d(0)$. We conclude that $\mathfrak{I}^*\mathcal{R}^*(\ell^2(\mathbb{Z}^+; \mathscr{Y}' \times \mathscr{U}') \ominus \mathscr{V}_d(0)) = \mathscr{G}$ from which the result follows using that $\mathfrak{I}^*$ is an isomorphism from $\mathscr{G}'$ onto $\mathscr{G}$. $\square$

LEMMA 6.3. *The input-output trajectory to final state map $\mathcal{J}_c$ for the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ and the future minimizing operator $\mathcal{I}_{f,d}$ of its dual node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]^*$ are related by*

$$\mathcal{J}_c^* = \mathcal{R}^* \mathcal{I}_{f,d} \tag{6.1}$$

as unbounded operators $\mathscr{X}' \to \mathscr{G}'$.

Proof. The basic duality relationship (5.4) with $z^* \in \Xi_{f,d}$ and $\left[\begin{smallmatrix} y^* \\ u^* \end{smallmatrix}\right] = \mathcal{I}_{f,d} z^*$ gives for $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_c$ as follows:

$$\left\langle \mathcal{J}_c \begin{bmatrix} y \\ u \end{bmatrix}, z^* \right\rangle_{\mathscr{X},\mathscr{X}'} = \left\langle \mathcal{R} \begin{bmatrix} y \\ u \end{bmatrix}, \mathcal{I}_{f,d} z^* \right\rangle_{\ell^2(\mathscr{Y}\times\mathscr{U})}.$$

By Lemma 6.2 we have $\mathcal{R}^*\mathcal{I}_{f,d} z^* \in \mathscr{G}'$, so the above can be rewritten as

$$\left\langle \mathcal{J}_c \begin{bmatrix} y \\ u \end{bmatrix}, z^* \right\rangle_{\mathscr{X},\mathscr{X}'} = \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, \mathcal{R}^*\mathcal{I}_{f,d} z^* \right\rangle_{\mathscr{G},\mathscr{G}'},$$

which shows that $\mathcal{J}_c$ and $\mathcal{R}^*\mathcal{I}_{f,d}$ are adjoint to each other. So we still need only to show $D(\mathcal{I}_{f,d}) \supset D(\mathcal{J}_c^*)$. By definition

$$D(\mathcal{J}_c^*) = \Bigg\{ z^* \in \mathscr{X}' : \exists \begin{bmatrix} y^\$ \\ u^\$ \end{bmatrix} \in \mathscr{G}' \text{ such that } \forall \begin{bmatrix} y \\ u \end{bmatrix} \in \mathscr{G}_c \\ \left\langle \mathcal{J}_c \begin{bmatrix} y \\ u \end{bmatrix}, z^* \right\rangle_{\mathscr{X},\mathscr{X}'} = \left\langle \begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y^\$ \\ u^\$ \end{bmatrix} \right\rangle_{\mathscr{G},\mathscr{G}'} \Bigg\}.$$

Using Lemma 6.2 it follows that

$$D(\mathcal{J}_c^*) = \Bigg\{ z^* \in \mathscr{X}' : \exists \begin{bmatrix} y^\# \\ u^\# \end{bmatrix} \in \ell^2(\mathbb{Z}^+; \mathscr{Y}' \times \mathscr{U}') \ominus \mathscr{V}_d(0) \text{ such that } \forall \begin{bmatrix} y \\ u \end{bmatrix} \in \mathscr{G}_c \\ \left\langle \mathcal{J}_c \begin{bmatrix} y \\ u \end{bmatrix}, z^* \right\rangle_{\mathscr{X},\mathscr{X}'} = \left\langle \mathcal{R} \begin{bmatrix} y \\ u \end{bmatrix}, \begin{bmatrix} y^\# \\ u^\# \end{bmatrix} \right\rangle_{\ell^2(\mathscr{Y}\times\mathscr{U})} \Bigg\}.$$

Using Lemma 5.1 it follows from the equality in the domain definition that $\left[\begin{smallmatrix} y^\# \\ u^\# \end{smallmatrix}\right]$ is a trajectory of the dual node for initial condition z^*. So $z^* \in D(\mathcal{J}_c^*)$ has finite cost, and so $z^* \in D(\mathcal{I}_{f,d})$. Hence $\mathcal{J}_c^* = \mathcal{R}^*\mathcal{I}_{f,d}$. ∎

THEOREM 6.4. *The node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ satisfies the output coercive past cost condition if and only if its dual node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]^*$ satisfies the finite future incremental cost condition.*

Proof. The output coercive past cost condition implies that $C\mathcal{J}_c : \mathscr{G}_c \to \mathscr{Y}$ extends to a bounded operator $\mathscr{G} \to \mathscr{Y}$. By [18, Theorem 13.2], its adjoint equals $\mathcal{J}_c^* C^*$. It follows that $\mathcal{J}_c^* C^*$ is a bounded operator $\mathscr{Y}' \to \mathscr{G}'$. In particular, the range of C^* is contained in the domain of $\mathcal{J}_c^*$, which by Lemma 6.3 equals $\Xi_{f,d}$. This is exactly the finite future incremental cost condition for the dual node.

Conversely, assume that the range of C^* is contained in $\Xi_{f,d}$. The operator $\mathcal{J}_c^* C^*$ is closed as it is the adjoint of the densely defined operator $C\mathcal{J}_c$ (we again use that $(C\mathcal{J}_c)^* = \mathcal{J}_c^* C^*$ by [18, Theorem 13.2]). By the range assumption, $\mathcal{J}_c^* C^*$ is defined on all of $\mathscr{Y}'$ so that by the closed graph theorem it is a bounded operator $\mathscr{Y}' \to \mathscr{G}'$. Since $(C\mathcal{J}_c)^*$ is a bounded (and everywhere defined) operator, the operator $C\mathcal{J}_c$ is closable, and its closure is the bounded operator $((C\mathcal{J}_c)^*)^*$. Thus, $C\mathcal{J}_c$ extends to a bounded operator $\mathscr{G} \to \mathscr{Y}$. So the output coercive past cost condition holds. ∎

In the next proposition, we again use the weighted ℓ^2 space duality from (5.2).

LEMMA 6.5. *The map* $\Gamma_p : \mathscr{G}_c \to \mathfrak{s}(\mathbb{Z}^+;\mathscr{Y})$ *for the node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *and the map* $\Gamma_{f,d} : \ell^2_c(\mathbb{Z}^-;\mathscr{Y}') \to \mathscr{V}_d(0)$ *for the dual node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]^*$ *are related by*

$$\Gamma_p^* = \mathcal{R}^*\Gamma_{f,d}.$$

Proof. Using Lemma 6.3 we have

$$\Gamma_p^* = (\mathcal{C}\mathcal{J}_c)^* = \mathcal{J}_c^*\mathcal{C}^* = \mathcal{R}^*\mathcal{I}_{f,d}\mathcal{B}_d = \mathcal{R}^*\Gamma_{f,d}. \qquad \square$$

As the next lemma shows, the dual space of the state space of the completed $\mathcal{I}_p$-compression can be identified with the state space of the completed $\mathcal{I}_f$-compression of the dual node of the completed $\mathcal{I}_p$-compression.

LEMMA 6.6. *Any bounded linear functional on* $\mathscr{X}_p$ *can be identified with an element* z^* *of* $(\mathscr{X}_p)_{f,d}$ *through*

$$\langle z, z^*\rangle_{\mathscr{X}_p,\mathscr{X}_p'} = \left\langle \mathcal{I}_p^{\mathscr{X}_p} z, \mathcal{R}^*\mathcal{I}_{f,d}^{(\mathscr{X}_p)_{f,d}} z^*\right\rangle_{\ell^2(\mathscr{Y}\times\mathscr{U})}. \tag{6.2}$$

This duality is with respect to the pivot space $\mathscr{X}$ *in the sense that*

$$\langle z, z^*\rangle_{\mathscr{X}_p,\mathscr{X}_p'} = \langle z, z^*\rangle_{\mathscr{X},\mathscr{X}'}$$

if $z \in \mathscr{X}_p \cap \mathscr{X}$ *and* $z^* \in (\mathscr{X}_p)_{f,d} \cap \mathscr{X}'$. *Moreover, this duality is norm-preserving in the sense that* $\|z^*\|_{(\mathscr{X}_p)_{f,d}}$ *equals the* $\mathscr{X}_p'$ *norm of the corresponding functional.*

Proof. It is easily seen that for a given $z^* \in (\mathscr{X}_p)_{f,d}$, the expression (6.2) defines a bounded linear functional on $\mathscr{X}_p$, so it remains to prove the converse. By the duality between $\ell^2(\mathbb{Z}^-;\mathscr{Y}\times\mathscr{U})$ and $\ell^2(\mathbb{Z}^+;\mathscr{Y}'\times\mathscr{U}')$ and the identification of $\mathscr{X}_p$ with $R(\mathcal{I}_p^{\mathscr{X}_p})$, it follows that any linear functional on $\mathscr{X}_p$ must be of the form

$$\left\langle \mathcal{I}_p^{\mathscr{X}_p} z, v\right\rangle_{\ell^2(\mathscr{Y}\times\mathscr{U})}$$

for some $v \in \ell^2(\mathbb{Z}^+;\mathscr{Y}'\times\mathscr{U}')$. We have $R(\mathcal{I}_p^{\mathscr{X}_p}) = D(\mathcal{J}_r^{\mathscr{X}_p}) = D(\mathcal{J}^{\mathscr{X}_p}) \ominus N(\mathcal{J}^{\mathscr{X}_p})$, so we may assume that $v \in N(\mathcal{J}^{\mathscr{X}_p})^\perp$. So $v \in R(\mathcal{J}^{\mathscr{X}_p *})$ since $\mathcal{J}^{\mathscr{X}_p}$ as a partial isometry has closed range. Applying Lemma 6.3 to the completed $\mathcal{I}_p$-compression (and using that $\mathcal{J}_c^* = \mathcal{J}^*$) then gives the result.

That the duality is with respect to $\mathscr{X}$ follows from (5.4) with $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] = \mathcal{I}_p^{\mathscr{X}_p} z$ and $\left[\begin{smallmatrix} y^* \\ u^* \end{smallmatrix}\right] = \mathcal{I}_{f,d}^{(\mathscr{X}_p)_{f,d}} z^*$. The norm preservation follows from the fact that $\mathcal{I}_{f,d}^{(\mathscr{X}_p)_{f,d}}$, $\mathcal{R}$, and $\mathcal{I}_p^{\mathscr{X}_p}$ are isometries. $\square$

LEMMA 6.7. *The operator* $\mathcal{J}^{\mathscr{X}_p}$ *is a coisometry.*

Proof. According to Lemma 6.3, this is equivalent to showing that $\mathcal{I}_{f,d} : \mathscr{X}_p' \to \ell^2(\mathbb{Z}^+;\mathscr{Y}'\times\mathscr{U}')$ is an isometry. By the identification of the dual space in Lemma 6.6, this, in turn, is equivalent to $\mathcal{I}_{f,d} : (\mathscr{X}_p)_{f,d} \to \ell^2(\mathbb{Z}^+;\mathscr{Y}'\times\mathscr{U}')$ being an isometry, which is true by definition of the norm in $(\mathscr{X}_p)_{f,d}$. $\square$

The filter Riccati equation of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is simply the control Riccati equation of the dual node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]^*$.

DEFINITION 6.8. *The triple* (p, r, T) *is called a (nonnegative) solution of the* filter Riccati equation *of the node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *if*

1. p *is a closed nonnegative symmetric sesquilinear form in* $\mathscr{X}$ *whose domain satisfies* $A^*D(p) \subset D(p)$, $C^*\mathscr{Y} \subset D(p)$;
2. r *is a bounded nonnegative symmetric sesquilinear form on* $\mathscr{Y}$;

3. *$T : D(p) \to \mathscr{Y}$ is a linear operator;*
4. *for all $z \in D(p)$, $y \in \mathscr{Y}$ we have*

$$p(A^*z+C^*y, A^*z+C^*y)+\|B^*z+D^*y\|^2_{\mathscr{U}}+\|y\|^2_{\mathscr{Y}}=p(z,z)+r(Tz-y,Tz-y). \tag{6.3}$$

The solution is called classical *when $D(p) = \mathscr{X}$.*

Remark 4.5 can be applied to the filter Riccati equation to obtain an operator version instead of a sesquilinear form version of the filter Riccati equation. Similarly, [17, Appendix A] can be applied to obtain other equivalent forms.

Definition 6.9. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic at the origin. A function $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y} \times \mathscr{U}, \mathscr{Y}))$ is called a* left factorization *of G if $\tilde{\mathsf{M}}(z)$ is invertible for all z in a neighborhood of the origin and $\mathsf{G}(z) = \tilde{\mathsf{M}}(z)^{-1}\tilde{\mathsf{N}}(z)$ in a neighborhood of the origin.*

Theorem 6.10. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic at the origin, and let $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a realization of G. The following are equivalent conditions.*

- *$\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ satisfies the output coercive past cost condition.*
- *The filter Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ has a (nonnegative self-adjoint) solution.*
- *G has a left factorization.*

Under these equivalent conditions, the filter Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ has the smallest (nonnegative self-adjoint) solution (p_p, T_p, r_p) with domain $\Xi_{f,d}$.

Proof. According to Theorem 6.4, the output coercive past cost condition implies that the dual node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ satisfies the finite future incremental cost condition. Theorem 4.7 then shows that the dual node has a solution to its control Riccati equation. This implies that the original node has a solution to its filter Riccati equation.

If the node has a solution to its filter Riccati equation, then the dual node has a solution to its control Riccati equation. It follows from Theorem 4.7 that the transfer function of the dual node has a right factorization: $\mathsf{G}_d(z) = \mathsf{N}(z)\mathsf{M}(z)^{-1}$. Realizing that the transfer function of the original node G and that of the dual node G_d are related by $\mathsf{G}_d(z) = \mathsf{G}(\bar{z})^*$, we obtain $\mathsf{G}(z) = \tilde{\mathsf{M}}(z)^{-1}\tilde{\mathsf{N}}(z)$ with $\tilde{\mathsf{M}}(z) := \mathsf{M}(\bar{z})^*$ and $\tilde{\mathsf{N}}(z) := \mathsf{N}(\bar{z})^*$. So the transfer function of the original node has a left factorization.

Assuming that the transfer function of the original node has a left factorization, it is easily seen, as above, that the transfer function of the dual node has a right factorization. It follows from Theorem 4.7 that the dual node satisfies the finite future incremental cost condition. Theorem 6.4 then shows that the original node satisfies the output coercive past cost condition.

Existence of the smallest solution follows from the existence of the smallest solution (q_f, K_f, s_f) of the control Riccati equation of the dual node. ∎

Remark 6.11. As mentioned in the introduction, the transfer function of the closed-loop system (1.4) provides the left factorization of the transfer function of the open-loop system. This factorization is, in fact, weakly coprime (see Mikkola [15]), but it may not always be strongly coprime (i.e., the Bézout coprime). This question is treated in detail in the forthcoming (part three of this series of articles) [16].

7. The optimal output injection problem. We consider the closed-loop system

$$\begin{aligned} x_{n+1} &= (A-HC)x_n + (B-HD)u_n + Hy_n, \qquad && n \in \mathbb{Z}^-, \\ w_n &= WCx_n + WDu_n - Wy_n, && n \in \mathbb{Z}^-, \\ x_0 &= z, \\ \exists N \in \mathbb{Z}^- &: x_n = 0 = u_n \quad \forall n \le -N, \end{aligned} \tag{7.1}$$

where $W : \mathscr{Y} \to \mathscr{W}$ is a given bounded linear operator and $H : \mathscr{Y} \to \mathscr{X}_e$ is a bounded linear operator. Here $\mathscr{W}$ is a Hilbert space, and $\mathscr{X}_e$ is a Hilbert space that contains $P_{\mathscr{X}\ominus\mathscr{N}}\Xi_p$ as a dense subspace and is such that the restriction of the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ to $P_{\mathscr{X}\ominus\mathscr{N}}\Xi_p \times \mathscr{U}$ extends continuously to a bounded linear operator from $\mathscr{X}_e \times \mathscr{U}$ to $\mathscr{X}_e \times \mathscr{Y}$.

THEOREM 7.1. *Assume that* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *satisfies the output coercive past cost condition. Define* $H_p : \mathscr{Y} \to \mathscr{X}_p$ *by*

$$H_p y = \mathcal{J}^{\mathscr{X}_p} P_{\mathscr{G}} g, \tag{7.2}$$

where $g_{-1} = \left[\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right]$ *and* $g_n = 0$ *for* $n < -1$ *and* $P_{\mathscr{G}}$ *is the orthogonal projection* $\ell^2(\mathbb{Z}^-; \mathscr{Y} \times \mathscr{U}) \to \mathscr{G}$.

Then H_p *minimizes both the* $\mathcal{L}(\ell^2(\mathbb{Z}^-, \mathscr{Y} \times \mathscr{U}), \ell^\infty(\mathbb{Z}^-, \mathscr{W}))$ *and the* $\mathcal{L}(\ell^2(\mathbb{Z}^-, \mathscr{Y} \times \mathscr{U}), \ell^2(\mathbb{Z}^-, \mathscr{W}))$ *norm of the map from* $\{\left[\begin{smallmatrix} y_n \\ u_n \end{smallmatrix}\right]\}_{n\in\mathbb{Z}^-}$ *to* $\{w_n\}_{n\in\mathbb{Z}^-}$ *in* (7.1)*, where* H *ranges over all linear maps* $\mathscr{Y} \to \mathscr{X}_e$ *with* $\mathscr{X}_e$ *a Hilbert space that contains* $P_{\mathscr{X}\ominus\mathscr{N}}\Xi_p$ *as a dense subspace and is such that the restriction of the node* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *to* $P_{\mathscr{X}\ominus\mathscr{N}}\Xi_p \times \mathscr{U}$ *extends continuously to a node with* $\mathscr{X}_e$ *as state space. The operator* H_p *also minimizes the* $\mathcal{L}(\ell^2(\mathbb{Z}^-, \mathscr{Y} \times \mathscr{U}), \mathscr{W})$ *norm of the map* $\{\left[\begin{smallmatrix} y_n \\ u_n \end{smallmatrix}\right]\}_{n\in\mathbb{Z}^-} \mapsto w_{-1}$.

Denote the smallest nonnegative self-adjoint solution of the filter Riccati equation of $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *by* (p_p, T_p, r_p)*. Then the minimum norms mentioned above all equal the square root of* $\sup_{\|h\|=1} r_p(W^*h, W^*h)$.

Proof. The assumption that $P_{\mathscr{X}\ominus\mathscr{N}}\Xi_p \subset \mathscr{X}_e$ implies that $P_{\mathscr{X}\ominus\mathscr{N}}\Xi_- \subset \mathscr{X}_e$ and so ensures that the extended node has the same transfer function as the original node. In particular, it too satisfies the output coercive past cost condition (this follows from Theorem 6.10). So the dual of the extended node satisfies the finite future incremental cost condition by Theorem 6.4.

We have that $K := -H^*$ is a bounded operator from $\mathscr{X}_e'$ to $\mathscr{Y}'$ and that $E := W^*$ is a bounded operator from $\mathscr{W}'$ to $\mathscr{Y}'$. Obviously, the domain of K (which equals the whole state space $\mathscr{X}_e'$) is A^*-invariant and contains the range of C^*. The adjoint of the closed-loop system (7.1) considered as a dynamical system on $\mathbb{Z}^+$ then is

$$\begin{aligned} x_{n+1}^d &= (A^* + C^*K)x_n^d + C^*Ew_n^*, && n \in \mathbb{Z}^+, \\ u_n^* &= (B^* + D^*K)x_n^d + D^*Ew_n^*, && n \in \mathbb{Z}^+, \\ -y_n^* &= Kx_n^d + Ew_n^*, && n \in \mathbb{Z}^+, \\ x_0^d &= z^*; \end{aligned} \tag{7.3}$$

i.e., it is the closed-loop system (4.2) of the adjoint node of the extended node with an additional minus sign in the third equation.

From the above it follows that the search over H in the optimal output injection problem translates to the search over K in the optimal feedback problem.

By the discussion in section 5, the input-output maps of (7.1) and (7.3) are adjoints when both are considered $\ell^2 \to \ell^2$ and also when considered $\ell^1 \to \ell^2$ and $\ell^2 \to \ell^\infty$, respectively. Also the maps $\{\left[\begin{smallmatrix} y_n \\ u_n \end{smallmatrix}\right]\}_{n\in\mathbb{Z}^-} \mapsto w_{-1}$ and $w_0^* \mapsto \{\left[\begin{smallmatrix} y_n^* \\ u_n^* \end{smallmatrix}\right]\}_{n\in\mathbb{Z}^+}$ are adjoints.

From the fact that the operator norm of an operator equals that of its adjoint and Theorem 4.8, it follows that the square root of $\sup_{\|h\|=1} r_p(W^*h, W^*h)$ is a lower bound for all three operator norms considered and that this lower bound is reached for $H = -K_{f,d}^*$. The formula for H_p follows once we show that $H_p = -K_{f,d}^*$ as operators $\mathscr{Y} \to \mathscr{X}_p$, which we now obtain.

We have, for $y \in \mathscr{Y}$ and $z^* \in (\mathscr{X}_p)_{f,d} = \mathscr{X}_p'$ (this equality of spaces follows from Lemma 6.6),

$$\langle H_p y, z^* \rangle_{\mathscr{X}_p, \mathscr{X}_p'} = \langle \mathcal{J}^{\mathscr{X}_p} P_{\mathscr{G}} g, z^* \rangle_{\mathscr{X}_p, \mathscr{X}_p'},$$

which by Lemma 6.6 equals

$$\left\langle \mathcal{I}_p^{\mathscr{X}_p} \mathcal{J}^{\mathscr{X}_p} P_{\mathscr{G}} g, \mathcal{R}^* \mathcal{I}_{f,d}^{(\mathscr{X}_p)_{f,d}} z^* \right\rangle_{\ell^2(\mathscr{Y} \times \mathscr{U})}.$$

By (3.1) the above equals

$$\left\langle P_{\ell^2(\mathbb{Z}^-; \mathscr{Y} \times \mathscr{U}) \ominus N(\mathcal{J}^{\mathscr{X}_p})} P_{\mathscr{G}} g, \mathcal{R}^* \mathcal{I}_{f,d}^{(\mathscr{X}_p)_{f,d}} z^* \right\rangle_{\ell^2(\mathscr{Y} \times \mathscr{U})}.$$

By Lemma 6.3 we can omit the projection onto the orthogonal complement of the kernel in this last formula. By Lemma 6.2 and Remark 5.2 we have $\mathcal{R}^* \mathcal{I}_{f,d}^{(\mathscr{X}_p)_{f,d}} z^* \in \ell^2(\mathbb{Z}^+; \mathscr{Y}' \times \mathscr{U}') \ominus \mathscr{G}^\perp$ so that the projection onto $\mathscr{G}$ may also be omitted. Hence

$$\langle H_p y, z^* \rangle_{\mathscr{X}_p, \mathscr{X}_p'} = \langle \mathcal{R} g, \mathcal{I}_{f,d}^{(\mathscr{X}_p)_{f,d}} z^* \rangle_{\ell^2(\mathscr{Y} \times \mathscr{U})} = -\langle y, K_{f,d} z^* \rangle_{\mathscr{Y}, \mathscr{Y}'},$$

where the last equality holds by definition of $K_{f,d}$ and $\mathcal{R}$. This proves that H_p given by (7.2) is indeed the optimal output injection. $\square$

The formula (7.2) shows that the optimal output injection $H_p y$ for the node $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is the final state of the dynamical system associated to that node for some input defined on $\mathbb{Z}^-$ in terms of y.

THEOREM 7.2. *Assume that* $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ *satisfies the output coercive past cost condition. Then the closed-loop system of the completed* $\mathcal{I}_p$*-compression with the optimal output injection is strongly internally* $*$*-stable.*

Proof. The proof of Theorem 7.1 shows that the dual of the closed-loop system of the completed $\mathcal{I}_p$-compression with the optimal output injection is the closed-loop system of a completed $\mathcal{I}_f$-compression with the optimal state feedback. From Remark 4.9 it follows that this latter system is strongly internally stable. The result immediately follows. $\square$

Remark 7.3. Many of the operators defined here are closely related to analogous operators introduced in [4]. More precisely, the completed $\mathcal{I}_p$-compression is a passive observable and backward conservative input/state/output system if we equip its output space $\mathscr{Y}$ with the (equivalent) inner product induced by the quadratic form r in Definition 6.8 and use $\mathscr{U} \times \mathscr{Y}$ as the input space. With regard to this system, the coisometry $\mathcal{J}^{\mathscr{X}_p}$ coincides with the input map $\mathfrak{B}_{\Sigma_{i/s/o}}$ in [4, section 10], the isometry $\mathcal{I}_{f,d}$ is the output map of the adjoint system, and the two Hankel operators Γ_p and $\Gamma_{f,d}$ can be interpreted as compressions of the past/future map Γ_Σ in [4] and its adjoint.

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