SIAM J. MATH. ANAL.
Vol. 55, No. 5, pp. 5221–5259

LOCALIZED QUANTITATIVE ESTIMATES AND POTENTIAL BLOW-UP RATES FOR THE NAVIER–STOKES EQUATIONS*

TOBIAS BARKER†

In memory of my Grandfather Maurice (1930–2022)

Abstract. We show that if v is a smooth suitable weak solution to the Navier–Stokes equations on $B(0,4)\times(0,T_*)$, which possesses a singular point $(x_0,T_*)\in B(0,4)\times\{T_*\}$, then for all $\delta>0$ sufficiently small, one necessarily has $\limsup_{t\uparrow T_*}\frac{\|v(\cdot,t)\|_{L^3(B(x_0,\delta))}}{(\log\log\log(\frac{1}{(T_*-t)^{\frac14}}))^{\frac{1}{1129}}}=\infty$. This *local* result improves on the corresponding *global* result recently established by Tao [in *Nine Mathematical Challenges: An Elucidation*, American Mathematical Society, Providence, RI, 2021, pp. 149–193]. The proof is based on a quantification of the *qualitative* local result of Escauriaza, Seregin, and Šverak [*Uspekhi Mat. Nauk*, 58 (2003), pp. 3–44]. In order to prove the required localized quantitative estimates, we show that in certain settings, one can quantify a *qualitative* truncation/localization procedure introduced by Neustupa and Penel [in *Applied Nonlinear Analysis*, Kluwer/Plenum, New York, 1999, pp. 391–402]. After performing the quantitative truncation procedure, the remainder of the proof hinges on a physical space analogue of Tao's breakthrough strategy, established by Barker and Prange [*Comm. Math. Phys.*, 385 (2021), pp. 717–792].

Key words. Navier–Stokes equations, local quantitative estimates, local blow-up rates, critical norms

MSC codes. 35Q30, 35K55, 35B44, 35B65

DOI. 10.1137/22M1527179

1. Introduction.

In [23], Leray showed that for any square integrable weakly divergence-free initial data, there exists an associated finite-energy solution[1] $v:\mathbb{R}^3\times(0,\infty)\to\mathbb{R}^3$ of the three-dimensional Navier–Stokes equations

$$\partial_t v-\Delta v+v\cdot\nabla v+\nabla p=0,\quad \operatorname{div} v=0,\quad v(x,0)=v_0(x)\quad\text{in}\quad \mathbb{R}^3\times(0,\infty). \tag{1}$$

The above equations are invariant under the *Navier–Stokes rescaling*

$$(v_\lambda(x,t),p_\lambda(x,s),v_{0\lambda}(x)):=(\lambda v(\lambda x,\lambda^2t),\lambda^2p(\lambda x,\lambda^2t),\lambda v_0(\lambda x))\quad\text{with }\lambda>0. \tag{2}$$

After almost 90 years, it remains unknown if finite-energy solutions of (1), with sufficiently smooth initial data, remain smooth for all times.

This paper concerns *localized quantitative behavior* for potential blow-up solutions of (1) in terms of the L^3 norm, which is invariant with respect to the rescaling for the initial data (2) and is referred to as a *critical norm.*

*Received by the editors October 6, 2022; accepted for publication (in revised form) March 27, 2023; published electronically September 28, 2023.
https://doi.org/10.1137/22M1527179
†Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK (tb2130@bath.ac.uk).
[1]For a definition of finite-energy solutions, see subsection 2.1.3.

The first quantitative blow-up rates for the Navier–Stokes equations were established by Leray. In [23], Leray showed that if a finite-energy solution of the Navier–Stokes equations first loses smoothness at $T_* > 0$, then for $3 < p \leq \infty$,

$$\|v(\cdot,t)\|_{L^p(\mathbb{R}^3)} \geq \frac{C(p)}{(T_*-t)^{\frac{1}{2}(1-\frac{3}{p})}} \quad \text{for every } 0 \leq t < T_*. \tag{3}$$

This can be deduced from the fact that for $3 < p \leq \infty$, a finite-energy solution of (1) (with sufficiently smooth initial data) stays smooth on $(0,T)$ with

$$T \geq C_p \|v_0\|_{L^p(\mathbb{R}^3)}^{-\frac{2}{1-\frac{3}{p}}}. \tag{4}$$

The behavior of the critical L^3 norm of v is more subtle, and the above argument cannot be used to deduce that it necessarily diverges near a potential blow-up time. As observed in [24], this obstruction also occurs for critical norms of other evolution equations with a scaling symmetry (see also [5]).

The first *qualitative* blow-up result for the critical L^3 norm of the Navier–Stokes equations was shown in a celebrated paper by Escauriaza, Seregin, and Šverák in [12]. In [12], a contradiction argument using backward uniqueness and unique continuation for parabolic operators was used to show that if a finite-energy solution on $(0,\infty)$ first loses smoothness at T_*, then for any singular point[2] (x_0, T_*) and any $\delta > 0$, one necessarily has

$$\limsup_{t\uparrow T_*} \|v(\cdot,t)\|_{L^3(B(x_0,\delta))} = \infty. \tag{5}$$

This follows from a regularity criterion established in the same paper. Namely, for a suitable weak solution[3] on $B(0,1)\times(-1,0)$, we have the qualitative regularity criterion that

(6)
$$v \in L^\infty(-1,0;L^3(B(0,1))) \Rightarrow v \in L^\infty(B(0,r)\times(-r^2,0)) \text{ for all sufficiently small } r>0.$$

Since [12], there have been many extensions regarding qualitative potential blow-up behavior of critical norms for the Navier–Stokes equations. See [33], [32], [15], [39], [1], and [2]. After Escauriaza, Seregin, and Šverák's result, the subtle question of the divergence of critical norms near a potential blowup was examined for other supercritical partial differential equations. See [24], [26], [25], and [38], for example.

With the exception of [24], the aforementioned results regarding critical norms are based on contradiction arguments and are hence purely qualitative. To the best of the author's knowledge, the first quantitative blow-up rates for critical norms of a supercritical partial differential equation was obtained by Merle and Raphaël in [24]. In [24], a logarithmic blow-up rate was obtained for the critical norms of radial solutions of the L^2 supercritical nonlinear Schrödinger equation. In the breakthrough paper [37], Tao established the first quantitative blow-up rate for the critical L^3 norm for finite-energy solutions of the Navier–Stokes equations (1). In particular, Tao showed that if a finite-energy solution on $(0,\infty)$ first loses smoothness at T_*, then one necessarily has

[2]For the definition of a singular point, see subsection 2.1.4.
[3]For a definition of suitable weak solutions, see subsection 2.1.3.

$$\limsup_{t\uparrow T_*}\frac{\|v(\cdot,t)\|_{L^3(\mathbb{R}^3)}}{(\log\log\log(\frac{1}{T_*-t}))^c}=\infty. \tag{7}$$

Here, $c>0$ is a universal constant. Tao's result is a consequence of the following quantitative estimate. Namely, for a smooth finite-energy solution to the Navier–Stokes equations on $\mathbb{R}^3\times(-1,0)$, we have that

$$\|v\|_{L^\infty(-1,0;L^3(\mathbb{R}^3))}\leq M\Rightarrow \|v\|_{L^\infty(-\frac{1}{2},0;L^\infty(\mathbb{R}^3))}\leq \exp\exp\exp(M^C), \tag{8}$$

where C is a positive universal constant. The above quantitative blow-up rate and estimate have been subsequently extended and generalized in [7], [30], [31], and [16].

In light of Escauriaza, Seregin, and Šverák's blow-up criterion (5) and regularity criterion (6), it is natural to ask the following questions:

1. Can Escauriaza, Seregin, and Sverák's regularity criterion (6) be quantified?
2. Can we obtain quantiative blow-up rates for the critical L^3 norm in any local neighborhood of a singular point?

Question 2 is also of potential interest in the numerical search for potentially singular solutions of the Navier–Stokes equations (1). In [17], the growth of the localized L^3 norm is investigated for certain numerical solutions for the three-dimensional axisymmetric solutions of the Navier–Stokes equation. In this paper, we provide an answer to questions 1 and 2 by means of the following results.

1.1. Main results. Our first result quantifies Escauriaza, Seregin, and Šverák's regularity criterion (6), thus providing a positive answer to question 1 in the previous subsection.

THEOREM 1. *Suppose that (v,p) is a smooth solution to the Navier–Stokes equations on $B(0,4)\times(-16,0]$ and that (v,p) is a suitable weak solution on $Q(0,4):=B(0,4)\times(-16,0)$.*
Furthermore, suppose that $\mathcal{M}\in[\mathcal{M}_0,\infty)$ is such that

- $\|v\|_{L^\infty_t L^3_x(Q(0,4))}\leq\mathcal{M}$ *and*
- $\|p\|^{\frac{3}{2}}_{L^{\frac{3}{2}}_{x,t}(Q(0,4))}\leq\mathcal{M}$.

Here, $\mathcal{M}_0$ is a sufficiently large universal constant.

Then we conclude that

$$\|v\|_{L^\infty_{x,t}(B(0,\frac{1}{2})\times(-e^{-e^{e^{\mathcal{M}^{1128}}}},0))}\leq e^{e^{e^{\mathcal{M}^{1128}}}}. \tag{9}$$

As a consequence of the quantitative estimate (9), we obtain quantitative blow-up rates of the critical L^3 norm in any local neighborhood of a singular point. This improves Tao's necessary condition (7) and quantifies Escauriaza, Seregin, and Šverák's qualitative necessary condition (5).

THEOREM 2. *Suppose that (v,p) is a smooth suitable weak solution to the Navier–Stokes equations on $B(0,4)\times(0,T_*)$.*
Furthermore, suppose that

- $v\in L^\infty_{t,loc}([0,T_*);L^\infty(B(0,4)))$ *and*
- *there exists $x_0\in B(0,4)$ such that for all $r\in(0,\mathrm{dist}(x_0,\partial B(0,4))$*

$$v\notin L^\infty_{x,t}(B(x_0,r)\times(T_*-r^2,T_*)).$$

Then under the above assumptions, we conclude that for all $\delta\in(0,\mathrm{dist}(x_0,\partial B(0,4))$, we have

$$\limsup_{t\uparrow T_*} \frac{\|v(\cdot,t)\|_{L^3(B(x_0,\delta))}}{\left(\log\log\log\left(\frac{1}{(T_*-t)^{\frac14}}\right)\right)^{\frac{1}{1129}}}=\infty. \tag{10}$$

1.2. Comparison to previous literature and novelty of our approach. As previously mentioned, Tao's global quantitative potential blow-up rate (7) relies on the quantitative estimate (8), which is achieved via a strategy[4] using quantitative Carleman inequalities for parabolic differential inequalities. A key part of Tao's strategy is using that a smooth solution $v:\mathbb{R}^3\times[0,1]\to\mathbb{R}^3$ with *global bound*

$$\|v\|_{L^\infty_t L^3_x(\mathbb{R}^3\times(0,1))}\leq M \tag{11}$$

possesses space-time regions of regularity, which can be quantified explicitly in terms of M.

An elementary approach to obtaining quantitative regions of space-time regularity in this setting was provided in the subsequent paper [7]. In [7], it was used that the global functions (v,p) satisfying (11) also satisfy

$$\int\limits_{\frac12}^{1}\int\limits_{B(0,\exp(M^C))}|v|^{\frac{10}{3}}+|p|^{\frac53}\,dxdt\lesssim M^{O(1)}. \tag{12}$$

Here, C is a positive universal constant. In particular, the estimate (12) occurs for an integral over a region with exponentially more spatial scales compared to the right-hand-side bound. This then allows the use of the pigeonhole principle and Caffarelli, Kohn, and Nirenberg's ε-regularity criterion [10] to obtain that v is quantitatively bounded in quantitative space-time annuli.

In the context of a smooth suitable weak solution (v,p) on $B(0,4)\times(-16,0]$ with

$$\|v\|_{L^\infty_t L^3_x(B(0,4)\times(-16,0))}\leq\mathcal{M}$$

($\mathcal{M}$ is as in Theorem 1), (v,p) does not satisfy (12) but instead obeys the Morrey bound

$$\sup_{(x_0,t_0)\in B(0,1)\times(-1,0],0<r\leq1}\frac{1}{r^{\frac53}}\int\limits_{-r^2}^{0}\int\limits_{B(0,r)}|v|^{\frac{10}{3}}+|p|^{\frac53}\,dxdt\lesssim\mathcal{M}^{O(1)}. \tag{13}$$

In contrast to the global setting (12), (13) is integrated over a region that is related to the right-hand-side bound by a power law. This prevents us applying the same arguments as in the global case ([37] and [7]) to obtain quantitative bounds for v on quantitative space-time annuli. This represents a major block when attempting to apply the previous strategies ([37] and [7]) for producing quantitative estimates to the local setting of suitable weak solutions.

A systematic bridge between global regularity results and local regularity results was established by Neustupa and Penel in [27] in the context of establishing a one-component regularity criterion for the Navier–Stokes equations, which we will refer to as a "truncation procedure". This bridge relies on the following fact for suitable weak solutions. Namely, if (v,p) is a suitable weak solution on $B(0,1)\times(-1,0)$, then there exists $0<r_1<r_2<1$ and $0<\delta_1<1$ such that

$$v\in L^\infty_{x,t}(B(0,r_2)\setminus\overline{B(0,r_1)}\times(-\delta_1,0)). \tag{14}$$

[4]For a detailed description of Tao's strategy, we refer the reader to [37] and [7].

This relies on the celebrated fact (proven in [10]) that the parabolic one-dimensional Hausdorff dimension of the singular set is zero. Therefore, (14) is purely qualitative.[5]

Having established (14), in [27], an appropriate cutoff function Φ is chosen, and the Bogovskiĭ operator [9] is utilized to "truncate" the original suitable weak solution v to obtain a global function $u:\mathbb{R}^3\times(-\delta_1,0)\to\mathbb{R}^3$ possessing the following properties:

- u agrees with v on $B(0,r_1)\times(-\delta_1,0)$;
- u solves the Navier–Stokes equations with compactly supported forcing $f:\mathbb{R}^3\times(-\delta_1,0)\to\mathbb{R}^3$;
- f belongs to subcritical spaces with respect to the Navier–Stokes rescaling for the forcing.[6]

The fact that f belongs to subcritical spaces means that it does not interfere with the regularity properties of u, and one can then obtain a priori estimates for the global function u. The main difficulty with applying Neustupa and Penel's above truncation procedure to obtain the localized quantitative estimates in Theorem 1 is as follows. Neustupa and Penel's truncation procedure cannot be quantified in the general case since it relies on the qualitative property (14) coming from Caffarelli, Kohn, and Nirenberg's qualitative result [10].

To overcome the aforementioned difficulties in producing the local quantitative estimates in Theorem 1, we show that under the assumptions of Theorem 1, Neustupa and Penel's truncation procedure can be fully quantified (Proposition 4). A key part of quantifying the above truncation procedure is the application of local-in-space smoothing near the initial time for the Navier–Stokes equations, which was initiated in Jia and Šverák's seminal work [18] and subsequently extended in [6], [19]–[20], and [4].

Having quantified Neustupa and Penel's truncation procedure, we get a solution to the Navier–Stokes equations $V:\mathbb{R}^3\times(-2,0)\to\mathbb{R}^3$ with forcing $F:\mathbb{R}^3\times(-2,0)\to\mathbb{R}^3$ such that V has quantitatively controlled norms and F has quantitatively controlled (negligible) subcritical norms. This allows us to apply energy methods to V, and then using ε-regularity in a similar way to [7], we can obtain quantitative space-time regions of space-time regularity for V. This then allows us to apply the strategy for producing quantitative estimates in [7][7] to V instead of v, which enables us to obtain Theorem 1. In order to apply the strategy in [7], some bookkeeping is required to ensure that the support in space of the forcing F does not intersect with the space-time regions where the quantitative Carleman inequalities are applied to $\Omega=\nabla\times V$. For diagrams illustrating the physical space strategy in [7], we recommend [7, Figure 1] and [8, Figures 4–5].

1.3. Extensions and open problems. Tao's quantitative blow-up rate for the L^3 norm of the Navier–Stokes equations near a blow-up time has been improved in [7], [30]–[31], and [16]. In [7], it was shown that if $v:\mathbb{R}^3\times(0,T_*)\to\mathbb{R}^3$ is a smooth solution to the Navier–Stokes equations on $\mathbb{R}^3\times(0,T_*)$, which satisfies the *global* bound

$$\|v\|_{L^\infty(0,T_*;L^{3,\infty}(\mathbb{R}^3))}\leq M \tag{15}$$

and possesses a singular point at $(x,t)=(0,T_*)$, then

[5] After this paper was submitted, quantitative analogues of (14) were established in [22].

[6] For the notion of subcritical forcing, see subsection 2.1.4.

[7] This is a physical space analogue of Tao's strategy in [37]. For a diagram illustrating Tao's strategy, we recommend [37, Figure 1].

$$\int\limits_{B(0,R)} |v(x,t)|^3 dx \geq \exp(-\exp(M^C)) \log\left(\frac{R^2 M^{-C}}{T_* - t}\right) \tag{16}$$

for all t sufficiently close to T_*. Here, C is a positive universal constant. In [30], it was shown that if $v : \mathbb{R}^3 \times (0,\infty) \to \mathbb{R}^3$ is a smooth finite-energy axisymmetric solution, which first loses smoothness at T_*, then

$$\limsup_{t \uparrow T_*} \frac{\|v(\cdot,t)\|_{L^3(\mathbb{R}^3)}}{\left(\log\log\left(\frac{1}{T_*-t}\right)\right)^c} = \infty. \tag{17}$$

Here, c is a positive universal constant. See also [29] for a subsequent strengthening of (17).

Using the methodology of this paper, we expect that these results can be extended by being localized. For the blow-up rate (16), it seems plausible that the global assumption (15) can be replaced by the local assumption

$$\|v\|_{L^\infty(T_*-\delta^2,T_*;L^{3,\infty}(B(0,\delta)))} \leq M \quad \text{for } \delta \in (0, T_*^{\frac{1}{2}}).$$

We also expect that for axisymmetric solutions, the blow-up rate (17) can be refined to obtain

$$\limsup_{t \uparrow T_*} \frac{\|v(\cdot,t)\|_{L^3(B(x_0,\delta))}}{\left(\log\log\left(\frac{1}{T_*-t}\right)\right)^c} = \infty \quad \text{for } \delta > 0.$$

Here, $x_0 = (0,0,z_0)$ is any potential singular point at time T_*. We also expect that similar localized refinements apply to the global results obtained in [31] and [16].

In [3], Neustupa and Penel's *qualitative* truncation procedure was used to show that if $v : \mathbb{R}^3 \times (-1,0) \to \mathbb{R}^3$ is a smooth finite-energy solution on $\mathbb{R}^3 \times (-1,0)$, then for any $\delta > 0$,

$$\sup_n \|v(\cdot, t_{(n)})\|_{L^3(B(0,\delta))} = M \quad \text{with} \quad t_{(n)} \uparrow 0 \tag{18}$$

implies that the space-time origin is not a singular point. It remains unclear how to quantify the regularity criterion (18). In particular, the following remains open.

Open Problem: Under the assumption (18), does there exist a function $F : \{(a_{(n)})_{n\in\mathbb{N}} : a_{(n)} \in (-1,0)\} \times [0,\infty) \times [0,\infty) \to [0,\infty)$ such that

$$|v(0,0)| \leq F((t_{(n)})_{n\in\mathbb{N}}, M, \delta)? \tag{19}$$

The difficulty in obtaining (19) is that under the assumption (18), v does not necessarily belong to critical Morrey spaces, which is currently needed in this paper to quantify Neustupa and Penel's truncation procedure.

2. Preliminaries.

2.1. Notation.

2.1.1. Universal constants, vectors, and domains. We use the notation $X \lesssim Y$, which means that there exists a positive universal constant C such that $X \leq CY$. We will also use the notation $X \lesssim_q Y$, which means that there exists a positive constant $C(q)$ depending on q only such that $X \leq C(q)Y$. Sometimes in this paper, we will write $X \lesssim Y \leq Z$. This means that there exists a positive universal constant C such that $X \leq CY \leq Z$. In this paper, we will sometimes refer to a quantity (for example, $\mathcal{M}$) being "sufficiently large," which should be understood as $\mathcal{M}$ being larger than some universal constant that can (in principle) be specified.

For a vector a, a_i denotes the ith component of a. For $(x,t)\in\mathbb{R}^4$ and $r>0$, we denote $B(x,r):=\{y\in\mathbb{R}^3:|y-x|<r\}$ and $Q((x,t),r):=B(x,r)\times(t-r^2,t)$. When $(x,t)=(0,0)$, we will sometimes write $Q(0,r):=Q((0,0),r)$. We denote the average of a function f over the ball $B(x,r)$ by

$$(f)_{B(x,r)}:=\frac{1}{|B(0,r)|}\int\limits_{B(x,r)} f dy.$$

Here, $|B(0,r)|$ denotes the Lebesgue measure of the ball $B(0,r)$.
For $a,\, b\in\mathbb{R}^3$, we write $(a\otimes b)_{\alpha\beta}=a_\alpha b_\beta$, and for $A,\, B\in M_3(\mathbb{R})$, $A:B=A_{\alpha\beta}B_{\alpha\beta}$. Here and in the whole paper, we use Einstein's convention on repeated indices. For $F:\Omega\subseteq\mathbb{R}^3\to\mathbb{R}^3$, we define $\nabla F\in M_3(\mathbb{R})$ by $(\nabla F(x))_{\alpha\beta}:=\partial_\beta F_\alpha$.

For $\lambda\in\mathbb{R}$, $\lfloor\lambda\rfloor$ denotes the greatest integer less than λ. Furthermore, $\lceil\lambda\rceil$ denotes the smallest integer greater than λ.

2.1.2. Sobolev spaces, Bochner spaces, and norms. For $\mathcal{O}\subseteq\mathbb{R}^3$, $k\in\mathbb{N}$ and $1\le p\le\infty$, $W^{k,p}(\mathcal{O})$ denotes the Sobolev space with differentiability k and integrability p.

If X is a Banach space with norm $\|\cdot\|_X$, then $L^s(a,b;X)$, with $a<b$ and $s\in[1,\infty)$, will denote the usual Banach space of strongly measurable X-valued functions $f(t)$ on (a,b) such that

$$\|f\|_{L^s(a,b;X)}:=\left(\int\limits_a^b\|f(t)\|_X^s dt\right)^{\frac{1}{s}}<+\infty.$$

The usual modification is made if $s=\infty$. Let $C([a,b];X)$ denote the space of continuous X valued functions on $[a,b]$ with usual norm. In addition, let $C_w([a,b];X)$ denote the space of X valued functions, which are continuous from $[a,b]$ to the weak topology of X.

Let $\mathcal{O}\subseteq\mathbb{R}^n$ and $a,b\in\mathbb{R}$. Sometimes we will denote $L^p(a,b;L^q(\mathcal{O}))$ and $L^p(a,b;W^{k,q}(\mathcal{O}))$ by $L^p_tL^q_x(\mathcal{O}\times(a,b))$ and $L^p_tW^{k,q}_x(\mathcal{O}\times(a,b))$. For the case $p=q$, we will often write $L^p_{x,t}(\mathcal{O}\times(a,b))$ to denote $L^p(a,b;L^p(\mathcal{O}))$. For convenience, we will often use the following notation for energy-type norms:

$$\|f\|_{L^\infty_tL^2_x\cap L^2_t\dot{H}^1_x(\mathcal{O}\times(a,b))}:=\Big(\|f\|^2_{L^\infty(a,b;L^2(\mathcal{O}))}+\|\nabla f\|^2_{L^2(a,b;L^2(\mathcal{O}))}\Big)^{\frac{1}{2}}. \tag{20}$$

2.1.3. Finite-energy solutions and suitable weak solutions. We say that v is a finite-energy solution on (T_1,T) if $v\in C_w([T_1,T];L^2_\sigma(\mathbb{R}^3)\cap L^2(T_1,T;\dot{H}^1(\mathbb{R}^3))$ is a distributional solution to (1) and satisfies the *global energy inequality*

$$\|v(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+\int\limits_{T_1}^t\int\limits_{\mathbb{R}^3}|\nabla v(x,s)|^2dxds\le\|v(\cdot,T_1)\|^2_{L^2(\mathbb{R}^3))}\quad\forall t\in[T_1,T]. \tag{21}$$

Here, $L^2_\sigma(\mathbb{R}^3):=\overline{\{\phi\in C_0^\infty(\mathbb{R}^3):\operatorname{div}\phi=0\}}^{\|\cdot\|_{L^2(\mathbb{R}^3)}}$.

Let $\mathcal{O}\subseteq\mathbb{R}^3$. We say that (v,p) is a *suitable weak solution* to the Navier–Stokes equations in $\mathcal{O}\times(T_1,T)$ if it fulfills the properties described in [34, Definition 6.1]. One such property we will refer to in this manuscript is the *local energy inequality* for such (v,p). In particular, $(v,p)\in C_w([T_1,T];L^2_x(\mathcal{O}))\cap L^2_t\dot{H}^1_x(\mathcal{O}\times(T_1,T))\times L^{\frac{3}{2}}_{x,t}(\mathcal{O}\times(T_1,T))$ is said to satisfy the local energy inequality in $\mathcal{O}\times(T_1,T)$ if the inequality

(22)

$$\int_{\mathcal{O}}|v(x,t)|^2\phi(x,t)dx+2\int_{T_1}^{t}\int_{\mathcal{O}}|\nabla v|^2\phi dxds\leq\int_{T_1}^{t}\int_{\mathcal{O}}|v|^2(\partial_t\phi+\Delta\phi)+(|v|^2+2p)v\cdot\nabla\phi dxds$$

holds for almost every $t\in(T_1,T)$ and all $0\leq\phi\in C_0^\infty(\mathcal{O}\times(T_1,\infty))$.

2.1.4. Singular points and criticality. Let $\mathcal{O}\subseteq\mathbb{R}^3$. Let (v,p) be a suitable weak solution to the Navier–Stokes equations on $\mathcal{O}\times(T_1,T)$. We say that $z_0=(x_0,t_0)\in\overline{\mathcal{O}}\times(T_1,T]$ is a *singular point* if

$$v\notin L^\infty_{x,t}((B(x_0,r)\cap\mathcal{O})\times(t_0-r^2,t_0)) \tag{23}$$

for all $r>0$ sufficiently small.

Consider the Navier–Stokes equations with forcing f:

$$\partial_t v-\Delta v+v\cdot\nabla v+\nabla p=f,\quad \operatorname{div}v=0. \tag{24}$$

This is invariant with respect to the scaling

(25)

$$(v_\lambda(x,t),p_\lambda(x,s),v_{0\lambda}(x)):=(\lambda v(\lambda x,\lambda^2t),\lambda^2p(\lambda x,\lambda^2t),\lambda^3f(\lambda x,\lambda^2t))\quad\text{with }\lambda>0.$$

We say that the scalar quantity $G(v,p,f)$ is *critical* or *scale-invariant* if $G(v_\lambda,p_\lambda,f_\lambda)=G(v,p,f)$ for all $\lambda>0$. The quantity $G(v,p,f)$ is said to be *supercritical* if there exists an $\alpha>0$ such that $G(v_\lambda,p_\lambda,f_\lambda)=\lambda^{-\alpha}G(v,p,f)$ for all $\lambda>0$. Finally, $G(v,p,f)$ is said to be *subcritical* if there exists an $\beta>0$ such that $G(v_\lambda,p_\lambda,f_\lambda)=\lambda^{\beta}G(v,p,f)$ for all $\lambda>0$. We will sometimes use phrases such as "v/f belongs to critical/supercritical /subcritical spaces." This means that v/f belongs to a space-time Lebesgue space, whose norm is critical/supercritical/subcritical.

2.2. Bogovskiĭ operator on an annulus.

LEMMA 1 (Bogovskiĭ operator on an annulus). *Let $R\geq1$ and define the annulus $A(R):=\{x\in\mathbb{R}^3:R<|x|<2R\}$.*

There exists a linear operator $B:C^\infty_{0,\mathrm{avg}}(A(R))\to C_0^\infty(A(R))$ satisfying (denote $w=Bg$) the equation

$$\begin{cases}\mathit{div}\,w=g & \mathit{in}\ A(R),\\ w\big|_{\partial A(R)}=0 & \mathit{on}\ \partial A(R).\end{cases} \tag{26}$$

Here and in the following, avg *denotes zero spatial average.*

Let $k\in\mathbb{N}_0$ and $1<p<\infty$. Then for all $g\in C^\infty_{0,\mathrm{avg}}(A(R))$,

$$\|\nabla^{k+1}w\|_{L^p(A(R))}\leq C(k,p)\|g\|_{W^{k,p}(A(R))}. \tag{27}$$

Hence, B extends uniquely to a bounded linear operator $B:\mathring{W}^{k,p}_{\mathrm{avg}}(A(R))\to\mathring{W}^{k+1,p}(A(R))$ solving (26)*, where* $^\circ$ *denotes the closure of test functions.*

Let $I\subset\mathbb{R}$ be an open interval and $g\in L_1(I;L_{p,\mathrm{avg}}(A(R)))$. Consider the linear operator B defined by applying the above operator at almost every time.

If $\partial_t g \in L_1(I; L_p(A(R)))$, then B commutes with the time derivative:

$$\partial_t B(g) = B(\partial_t g). \tag{28}$$

For the time-independent assertions, see [9] and [14, Theorem III.3.3]. The fact that the constant in (27) is independent of $R \geq 1$ can be seen by rescaling $w_R(x) = w(Rx)$ and $g_R(x) = Rg(Rx)$ and then applying the Bogovskiĭ operator on $A(1)$. To prove (28), one may use the finite difference operator $D_t^h \varphi = (\varphi + \varphi(\cdot + h))/h$ and take $h \to 0^+$. See [14, Exercise III.3.7].

3. Local-in-space smoothing and localized backward propagation of vorticity concentration.

3.1. Local-in-space smoothing for suitable weak solutions. In [7], Barker and Prange used a strategy to produce quantitative estimates based on (i) local-in-space smoothing for a short time [7, Theorem 5.1] and (ii) backward propagation of vorticity concentration [7, Lemma 3.1]. In [7], these concepts and statements are formulated in a global context (i.e., the velocity field is defined on $\mathbb{R}^3$ for each time). In order to prove the localized quantitative estimate in Theorem 1, we require analogues of these concepts but in a local context rather than a global context.

It is already known that local-in-space short time smoothing is essentially a local phenomenon that applies in a local context (see [7, Theorem 3] and [19, Theorem 1.1]). We use Theorem 1.1, Remark 1.2, and Theorem 3.1 of [20], which are formulated in a convenient way for our purposes. Here is the statement below, which is essentially taken from [20].

PROPOSITION 1 (local-in-space short time smoothing [20, Theorem 1.1, Remark 1.2, and Theorem 3.1]).
There exists positive universal constants c_0 and ϵ_* such that the following holds true. Let $M \geq 1$, and define $T_0 := c_0 M^{-18}$. Suppose that (v, p) is a suitable weak solution to the Navier–Stokes equations on $B(0,4) \times (0, T_0)$ such that

$$\lim_{t \to 0^+} \|v(\cdot, t) - v_0\|_{L^2(B(0,4))} = 0 \quad \text{with } v_0 \in L^2(B(0,4)). \tag{29}$$

Furthermore, suppose that

$$\|v\|^2_{L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1 (B(0,4)\times(0,T_0))} + \|p\|_{L^{\frac{3}{2}}_{x,t}(B(0,4)\times(0,T_0))} \leq M \tag{30}$$

and

$$\|v_0\|_{L^3(B(0,3))} \leq \epsilon_*. \tag{31}$$

Then the above hypothesis implies that v satisfies the quantitative bound

$$|\nabla^j v(x,t)| \lesssim_j t^{-\frac{j+1}{2}}, \quad \text{where} \quad j = 0, 1 \ldots \quad \text{and} \quad (x,t) \in B(0,2) \times (0, T_0). \tag{32}$$

Remark 3. Note that in [20], (32) is stated and proven for $j = 0$ by showing that for certain $r > 0$ and x_0, we have

$$\frac{1}{r^2} \int_0^{r^2} \int_{B(x_0,r)} |v|^3 + |p|^{\frac{3}{2}} dx dt \leq \varepsilon_{CKN}$$

and then applying Caffarelli, Kohn, and Nirenberg's celebrated ε-regularity criterion [10]. It is well known that the ε-regularity criteria in [10] gives higher derivative estimates via parabolic smoothing. Thus, the proof of [20, Theorem 1.1] also gives (32) for all $j \in \mathbb{N}$.

Remark 4. From [20, Theorem 3.1], the hypothesis that (v,p) is a suitable weak solution on $B(0,4)\times(0,T_0)$ can be replaced by the hypothesis that v is a suitable weak solution on $B(0,4)\times(0,T)$ for any given $T>0$. Under this change of hypothesis, the estimate (32) instead holds for $(x,t)\in B(0,2)\times(0,\min(T,T_0))$.

Remark 5 (removing the pressure). From [21, Theorem 16], a version of Proposition 1 is stated without any assumed bound on the pressure. Such a statement does not obviously improve our main results due to the appearance of the pressure in the quantitative truncation procedure (Proposition 4).

Now we state a corollary which is analogous to the global result of [7, Theorem 5.1] (which in turn is based on ideas in [18]) but in the local setting.

COROLLARY 1 (local-in-space short time smoothing, with initial data locally in L^6). *There exists a positive universal constant c_1 such that the following holds true.*
Let M and N be sufficiently large, and define $T_1 := c_1 M^{-18} N^{-52}$. Suppose that (v,p) is a suitable weak solution to the Navier–Stokes equations on $B(0,4)\times(0,T_1)$ such that

$$\lim_{t\to 0^+}\|v(\cdot,t)-v_0\|_{L^2(B(0,4))}=0, \quad \textit{with } v_0\in L^2(B(0,4)). \tag{33}$$

Furthermore, suppose that

$$\|v\|^2_{L^\infty_t L^2_x\cap L^2_t \dot{H}^1_x(B(0,4)\times(0,T_1))}+\|p\|_{L^{\frac{3}{2}}_{x,t}(B(0,4)\times(0,T_1))}\leq M \tag{34}$$

and

$$\|v_0\|_{L^6(B(0,3))}\leq N. \tag{35}$$

Then the above hypotheses imply that v satisfies the quantitative bound

$$|\nabla^j v(x,t)|\lesssim_j t^{-\frac{j+1}{2}}, \quad \textit{where} \quad j=0,1\dots \quad \textit{and} \quad (x,t)\in B(0,2)\times(0,T_1). \tag{36}$$

Proof. Take $\mu\in(0,\frac{1}{3}]$, which will be fixed later. Throughout, we fix $x_0\in B(0,2\mu^{-1})$. Note that

$$B(x_0,3)\subseteq B(0,3\mu^{-1}) \quad \text{and} \quad B(x_0,4)\subseteq B(0,4\mu^{-1}). \tag{37}$$

Now let us consider the rescaled functions $v_\mu : B(0,4\mu^{-1})\times(0,T_1\mu^{-2})\to\mathbb{R}^3$ and $p_\mu : B(0,4\mu^{-1})\times(0,T_1\mu^{-2})\to\mathbb{R}$ defined by

$$(v_\mu(x,t),p_\mu(x,t)) := (\mu v(\mu x,\mu^2 t),\mu^2 p(\mu x,\mu^2 t)). \tag{38}$$

Let us also define $v_{\mu 0} : B(0,4\mu^{-1})\to\mathbb{R}^3$ by

$$v_{\mu 0}(x) := \mu v_0(\mu x). \tag{39}$$

Using the properties (34)–(35), we infer that

(40)
$$\|v_\mu\|^2_{L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(B(x_0,4)\times(0,T_1\mu^{-2}))} + \|p_\mu\|_{L^{\frac{3}{2}}_{x,t}(B(x_0,4)\times(0,T_1\mu^{-2}))} \leq \hat{M} := 3\mu^{-\frac{4}{3}}M \quad \text{and}$$

$$\|v_{\mu 0}\|_{L^3(B(x_0,3))} \leq C_{leb} N \mu^{\frac{1}{2}}. \tag{41}$$

Here, $C_{leb} \in (0,\infty)$ is a universal constant coming from the continuous embedding $L^6(B(x_0,3)) \hookrightarrow L^3(B(x_0,3))$. In view of (41), we now take

$$\mu := \min\left\{\frac{\epsilon_*^2}{C_{leb}^2 N^2}, \frac{1}{3}\right\} = \frac{\epsilon_*^2}{C_{leb}^2 N^2} \tag{42}$$

for N sufficiently large. This gives that

$$\|v_{\mu 0}\|_{L^3(B(x_0,3))} \leq \epsilon_* \text{ and } \hat{M} = C_{univ} N^{\frac{8}{3}} M. \tag{43}$$

Now we take c_1 in Corollary 1 to be an appropriate universal constant such that

$$T_1 \mu^{-2} = c_0 \hat{M}^{-18}. \tag{44}$$

Here, c_0 is as in Proposition 1. Then (43)–(44) and a translation in space allow us to apply Proposition 1 to get

$$|\nabla^j v_\mu(x,t)| \lesssim_j t^{-\frac{j+1}{2}}, \quad \text{where} \quad j = 0,1\ldots \quad \text{and} \quad (x,t) \in B(x_0,2)\times(0,T_1\mu^{-2}).$$

Recalling that $x_0 \in B(0,2\mu^{-1})$ is arbitrary, we undo the Navier–Stokes rescaling (38) to finally infer that

$$|\nabla^j v(x,t)| \lesssim_j t^{-\frac{j+1}{2}}, \quad \text{where} \quad j = 0,1\ldots \quad \text{and} \quad (x,t) \in B(0,2)\times(0,T_1). \qquad \square$$

Remark 6. It follows from Remark 4 and the above proof that the hypothesis that (v,p) is a suitable weak solution on $B(0,4)\times(0,T_1)$ can be replaced by the hypothesis that v is a suitable weak solution on $B(0,4)\times(0,T)$ for any given $T>0$. Under this change of hypothesis, the estimate (36) instead holds for $(x,t) \in B(0,2)\times(0,\min(T,T_1))$.

Let us now state another corollary to Proposition 1 which concerns local-in-space smoothing of solutions with locally square integrable vorticity.

COROLLARY 2 (local-in-space short time smoothing with initial vorticity locally in L^2). *There exists a positive universal constant c_2 such that the following holds true. Let M be sufficiently large, and define $T_2 := c_2 M^{-44}$. Suppose that (v,p) is a suitable weak solution to the Navier–Stokes equations on $B(0,4)\times(0,T_2)$ such that*

$$\lim_{t\to 0^+} \|v(\cdot,t) - v_0\|_{L^2(B(0,4))} = 0 \quad \textit{with } v_0 \in W^{1,2}(B(0,4)). \tag{45}$$

Furthermore, suppose that

$$\|v\|^2_{L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(B(0,4)\times(0,T_2))} + \|p\|_{L^{\frac{3}{2}}_{x,t}(B(0,4)\times(0,T_2))} \leq M \tag{46}$$

and that for $\omega_0 = \nabla \times v_0$,

$$\|\omega_0\|^2_{L^2(B(0,\frac{7}{2}))} \leq M. \tag{47}$$

Then the above hypotheses imply that v satisfies the quantitative bound

$$|\nabla^j v(x,t)| \lesssim_j t^{-\frac{j+1}{2}}, \quad \textit{where} \quad j = 0,1\ldots \quad \textit{and} \quad (x,t) \in B(0,2)\times(0,T_2). \tag{48}$$

Proof. Recall that from the Biot–Savart law, $-\Delta v_0 = \nabla\times\omega_0$. Using this and elliptic regularity theory gives

$$\|\nabla v_0\|^2_{L^2(B(0,3))} \lesssim \|\omega_0\|^2_{L^2(B(0,\frac{7}{2}))} + \|v_0\|^2_{L^2(B(0,\frac{7}{2}))}.$$

Applying the Sobolev embedding theorem then gives

$$\|v_0\|_{L^6(B(0,3))} \lesssim \|\nabla v_0\|_{L^2(B(0,3))} + \|v_0\|_{L^2(B(0,3))} \lesssim \|\omega_0\|_{L^2(B(0,\frac{7}{2}))} + \|v_0\|_{L^2(B(0,\frac{7}{2}))}.$$

Using this and (46)–(47), we get that

$$\|v_0\|_{L^6(B(0,3))} \leq C_{univ} M^{\frac{1}{2}}. \tag{49}$$

Here, C_{univ} is a positive universal constant. Taking $N = C_{univ}M^{\frac{1}{2}}$ and taking c_2 to be an appropriate universal constant, we get the desired conclusion by directly applying Corollary 1. ∎

Remark 7. As was the case in Remark 6, the hypothesis that (v,p) is a suitable weak solution on $B(0,4)\times(0,T_2)$ can be replaced by the hypothesis that v is a suitable weak solution on $B(0,4)\times(0,T)$ for any given $T>0$. Under this change of hypothesis, the estimate (48) instead holds for $(x,t)\in B(0,2)\times(0,\min(T,T_2))$.

3.2. Local behavior of vorticity implies localized quantitative estimates. The strategy to prove the local estimates in Theorem 1 is a localized version of the strategy in [7]. As in [7], we rely on connecting the behavior of the vorticity on spatial scales with quantitative estimates of the velocity field (see [7, Lemmas 3.1–3.2]). However, unlike in [7], in order to prove Theorem 1, we require such a connection to be in a purely local setting. This is given by the following proposition.

Proposition 2. *Let M be sufficiently large. Suppose that (v,p) is a smooth solution to the Navier–Stokes equations on $B(0,1)\times(-1,0]$ satisfying*

$$\sup_{0<r\leq 1}\left\{\frac{1}{r}\|v\|^2_{L^\infty_t L^2_x\cap L^2_t\dot{H}^1_x(Q(0,r))} + \frac{1}{r^{\frac{4}{3}}}\|p\|_{L^{\frac{3}{2}}_{x,t}(Q(0,r))}\right\} \leq M. \tag{50}$$

Furthermore, suppose that there exists $t\in(-M^{-48},0)$ such that the vorticity $\omega=\nabla\times v$ satisfies

$$\int\limits_{B(0,M^{24}(-t)^{\frac{1}{2}})} |\omega(x,t)|^2 dx \leq \frac{M^{-22}}{(-t)^{\frac{1}{2}}}. \tag{51}$$

Then the above assumptions imply that

$$|\nabla^j v(x,s)| \lesssim_j (s-t)^{-\frac{j+1}{2}}, \quad \textit{where} \quad j=0,1\ldots \quad \textit{and} \quad (x,s)\in B(0,M^{23}(-t)^{\frac{1}{2}})\times(t,0]. \tag{52}$$

Proof. First, note that (50) implies that for M sufficiently large,

$$\frac{1}{4M^{23}(-t)^{\frac{1}{2}}}\|v\|^2_{L^\infty_t L^2_x\cap L^2_t\dot{H}^1_x(Q(0,4M^{23}(-t)^{\frac{1}{2}}))} + \frac{1}{(4M^{23}(-t)^{\frac{1}{2}})^{\frac{4}{3}}}\|p\|_{L^{\frac{3}{2}}_{x,t}(Q(0,4M^{23}(-t)^{\frac{1}{2}}))} \leq M. \tag{53}$$

Take $\lambda := M^{23}(-t)^{\frac{1}{2}}$, and let us consider the rescaled functions $v_\lambda : B(0,\lambda^{-1}) \times (0, M^{-46}) \to \mathbb{R}^3$ and $p_\lambda : B(0,\lambda^{-1}) \times (0, M^{-46}) \to \mathbb{R}$ defined by

$$(v_\lambda(y,s), p_\lambda(y,s)) := (\lambda v(\lambda y, \lambda^2 s + t), \lambda^2 p(\lambda y, \lambda^2 s + t)). \tag{54}$$

Under this rescaling, $B(0,\lambda^{-1}) \supset B(0,4)$, and, furthermore, (51) and (53) imply that for M sufficiently large, we have

$$\|v_\lambda\|^2_{L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(B(0,4)\times(0,M^{-46}))} + \|p_\lambda\|_{L^{\frac{3}{2}}_{x,t}(B(0,4)\times(0,M^{-46}))} \leq 4^{\frac{4}{3}} M \tag{55}$$

and

$$\int\limits_{B(0,\frac{7}{2})} |\omega_\lambda(y,0)|^2 dy \leq M \leq 4^{\frac{4}{3}} M. \tag{56}$$

Now (55)–(56) allows us to apply Corollary 2 and Remark 7 with M replaced by $\hat{M} := 4^{\frac{4}{3}} M$. If M is sufficiently large, this gives that

$$|\nabla^j v_\lambda(y,s)| \lesssim_j s^{-\frac{j+1}{2}}, \quad \text{where} \quad j = 1\ldots \quad \text{and} \quad (y,s) \in B(0,2) \times (0, M^{-46}].$$

We then undo the Navier–Stokes rescaling (54) to finally infer the desired conclusion (52) for the original functions (v,p). ∎

3.3. Localized backward propagation of vorticity concentration. As previously mentioned, backward propagation plays a central role in the strategy to produce quantitative estimates in [7]. The relevant statement is [7, Lemma 3.1], which is in a global context. Now we state and prove a localized version of [7, Lemma 3.1] which will play a crucial role in proving Theorem 1.

PROPOSITION 3. *Let M be sufficiently large. Suppose that (v,p) is a smooth solution to the Navier–Stokes equations on $B(0,1) \times (-1,0]$ satisfying*

$$\sup_{0<r\leq 1} \left\{ \frac{1}{r} \|v\|^2_{L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(Q(0,r))} + \frac{1}{r^{\frac{4}{3}}} \|p\|_{L^{\frac{3}{2}}_{x,t}(Q(0,r))} \right\} \leq M. \tag{57}$$

Furthermore, suppose that t and $t' \in (-M^{-48}, 0)$ are such that

$$\frac{-t'}{-t} \leq M^{-50} \tag{58}$$

and

$$\int\limits_{B(0,M^{24}(-t')^{\frac{1}{2}})} |\omega(x,t')|^2 dx > \frac{M^{-22}}{(-t')^{\frac{1}{2}}}. \tag{59}$$

Then the above assumptions imply that

$$\int\limits_{B(0,M^{24}(-t)^{\frac{1}{2}})} |\omega(x,t)|^2 dx > \frac{M^{-22}}{(-t)^{\frac{1}{2}}}. \tag{60}$$

Proof. Assume that the contrapositive to (60) holds. Then by Proposition 2, we have that there exists a positive universal constant C_{univ} such that

$$|\omega(x,s)| \leq \frac{C_{univ}}{-t}, \quad \text{where} \quad (x,s)\in B(0,M^{23}(-t)^{\frac{1}{2}})\times(\tfrac{t}{2},0]. \tag{61}$$

Note that the assumption (58) implies that

$$B(0,M^{24}(-t')^{\frac{1}{2}})\times\{t'\}\subset B(0,M^{23}(-t)^{\frac{1}{2}})\times(\tfrac{t}{2},0].$$

This together with (61) implies that for M sufficiently large, we have

$$\int\limits_{B(0,M^{24}(-t')^{\frac{1}{2}})} |\omega(x,t')|^2 dx \leq \frac{M^{74}}{(-t')^{\frac{1}{2}}}\Big(\frac{-t'}{-t}\Big)^2. \tag{62}$$

Now using the assumption (58) gives that

$$\int\limits_{B(0,M^{24}(-t')^{\frac{1}{2}})} |\omega(x,t')|^2 dx \leq \frac{M^{-26}}{(-t')^{\frac{1}{2}}},$$

which contradicts (59). ∎

4. Quantitative truncation procedure.

4.1. Local scale-invariant Morrey bounds. A key part of showing Theorem 1 is showing that the solution in Theorem 1 can be truncated to produce a solution of the Navier–Stokes equations with forcing. It is crucial that the forcing is compactly supported and that the subcritical norms of the forcing are quantified and sufficiently small. A central part of this quantitative truncation procedure involves using that the solution in Theorem 1 has a scale-invariant Morrey bound, which we demonstrate in this subsection.

LEMMA 2. *Suppose that (v,p) is a suitable weak solution to the Navier–Stokes equations on $Q(0,4)$. Furthermore, suppose that for some $\mathcal{M}\geq 1$, we have*

$$\|v\|_{L^\infty_t L^3_x(Q(0,4))}\leq \mathcal{M}. \tag{63}$$

Then the above assumptions imply that

$$\sup_{z_0=(x_0,t_0)\in B(0,1)\times(-1,0],\ 0<r\leq 1}\left\{\frac{1}{r^2}\int\limits_{Q(z_0,r)}|p|^{\frac{3}{2}}dxdt+\frac{1}{r}\|v\|^2_{L^\infty_t L^2_x\cap L^2_t\dot{H}^1_x(Q(z_0,r))}\right\} \tag{64}$$
$$\lesssim \mathcal{M}^3+\|p\|^{\frac{3}{2}}_{L^{\frac{3}{2}}_{x,t}(Q(0,4))}.$$

Proof. If (v,p) is a suitable weak solution on $Q(0,4)$, then we define the following dimensionless quantities for $\rho\in(0,4]$. Namely,

$$A(v,\rho):=\frac{1}{\rho}\|v\|^2_{L^\infty_t L^2_x(Q(0,\rho))},\quad E(v,\rho):=\frac{1}{\rho}\|\nabla v\|^2_{L^2(Q(0,\rho))}, \tag{65}$$
$$C(v,\rho)=\frac{1}{\rho^2}\|v\|^3_{L^3(Q(0,\rho))},$$

$$D(p,\rho) := \frac{1}{\rho^2}\|p\|^{\frac{3}{2}}_{L^{\frac{3}{2}}_{x,t}(Q(0,\rho))} \quad \text{and} \quad \mathcal{E}(v,p,\rho) := A(v,\rho) + E(v,\rho) + D(p,\rho). \tag{66}$$

Next, observe that by using spatial translations, it is sufficient to show that the following holds true. Namely, if (v,p) is a suitable weak solution on $Q(0,2)$ with

$$\|v\|_{L^\infty_t L^3_x(Q(0,2))} \leq \mathcal{M} \ (\mathcal{M} \geq 1), \tag{67}$$

then

$$\sup_{0<r\leq 1} \mathcal{E}(v,p,r) \lesssim \mathcal{M}^3 + \|p\|^{\frac{3}{2}}_{L^{\frac{3}{2}}_{x,t}(Q(0,2))}. \tag{68}$$

From (67) and the local energy inequality (22), we observe that in order to show (68), it is sufficient to prove that

$$\sup_{0<r\leq 1} D(p,r) \lesssim \mathcal{M}^3 + \|p\|^{\frac{3}{2}}_{L^{\frac{3}{2}}_{x,t}(Q(0,2))}. \tag{69}$$

From [35, page 847, equation (47)], we have that for any $\theta \in (0,1)$ and $0 < r \leq 1$,

$$D(p,\theta r) \leq C_{univ}(\theta D(p,r) + \theta^{-2} C(v,r)).$$

Here, C_{univ} is a positive universal constant. Thus, using (67), we obtain that for any $\theta \in (0,1)$ and $0 < r \leq 1$,

$$D(p,\theta r) \leq C_{univ}(\theta D(p,r) + \theta^{-2}\mathcal{M}^3).$$

Fixing the universal constant $\theta \in (0,1)$ such that $C_{univ}\theta \leq \frac{1}{2}$, one deduces that

$$D(p,\theta r) \leq \frac{1}{2} D(p,r) + c(\theta)\mathcal{M}^3 \quad \forall r \in (0,1]. \tag{70}$$

Iterating (70) gives that

$$D(p,\theta^k) \leq \frac{1}{2^k} D(p,1) + 2c(\theta)\mathcal{M}^3 \quad \text{for} \quad k = 0,1,2\ldots. \tag{71}$$

Take any $r \in (0,1]$. There exists $k \in \{0,1,2\ldots\}$ such that $\theta^{k+1} < r \leq \theta^k$. From this, we see that

$$D(p,r) \leq \frac{1}{\theta^2} D(p,\theta^k) \leq \frac{1}{\theta^2}\left(\frac{1}{2^k} D(p,1) + 2c(\theta)\mathcal{M}^3\right).$$

Hence,

$$\sup_{0<r\leq 1} D(p,r) \lesssim D(p,1) + \mathcal{M}^3$$

as required. □

4.2. Quantitative truncation procedure: Main statement and proof. A crucial part in proving Theorem 1 is the quantification of a qualitative localization procedure introduced in [27], when the solution has an assumed scale-invariant bound. This is given by the proposition below.

Proposition 4. *Suppose that (v,p) is a smooth solution to the Navier–Stokes equations on $B(0,4)\times(-16,0]$.*
Furthermore, suppose that $\mathcal{M}\in[\mathcal{M}_0,\infty)$ is such that

$$\|v\|_{L^\infty_t L^3_x(Q(0,4))}\leq\mathcal{M}\quad \text{and} \tag{72}$$

$$\|p\|^{\frac{3}{2}}_{L^{\frac{3}{2}}_{x,t}(Q(0,4))}\leq\mathcal{M}. \tag{73}$$

Here, $\mathcal{M}_0$ is a sufficiently large universal constant. Then the above assumptions imply the following.
There exists functions $V:\mathbb{R}^3\times[-2,0]\to\mathbb{R}^3,\ P:\mathbb{R}^3\times[-2,0]\to\mathbb{R},$ and $R\geq 6e^{e^{2\mathcal{M}}}$ such that

(74)
$$(V(x,t),P(x,t))\equiv(\lambda v(\lambda x,\lambda^2 t),\lambda^2 p(\lambda x,\lambda^2 t))\ \text{on}\ B(0,R)\times[-2,0]\quad(\lambda=e^{-e^{2\mathcal{M}}}e^{-\mathcal{M}^5})$$

and

$$\sup_{0<r\leq 1}\left\{\frac{1}{r^2}\int\limits_{Q(0,r)}|P|^{\frac{3}{2}}\,dxdt+\frac{1}{r}\|V\|^2_{L^\infty_t L^2_x\cap L^2_t\dot{H}^1_x(Q(0,r))}\right\}\lesssim\mathcal{M}^3. \tag{75}$$

The functions (V,P) are smooth on $\mathbb{R}^3\times[-2,0)$ and $B(0,R)\times[-2,0]$. Furthermore, (V,P) solves the Navier–Stokes equations with forcing F on $\mathbb{R}^3\times(-2,0)$ such that for every $t\in(-2,0)$,

$$\text{supp } F(\cdot,t)\subseteq B(0,2R)\setminus\overline{B(0,R)}\quad\text{and} \tag{76}$$

$$\text{supp } V(\cdot,t)\subseteq B(0,2R). \tag{77}$$

Additionally, V can be decomposed as $V=V^{(1)}+V^{(2)}$, where $V^{(i)}:\mathbb{R}^3\times[-2,0]\to\mathbb{R}^3$ ($i\in\{1,2\}$) satisfy

$$\|V^{(1)}\|_{L^\infty_t L^3_x(\mathbb{R}^3\times(-2,0))}=\|V\|_{L^\infty_t L^3_x(B(0,R)\times(-2,0))}\leq\mathcal{M}\quad\text{and}\quad\|V^{(2)}\|_{L^\infty_t L^{\frac{10}{3}}_x(\mathbb{R}^3\times(-2,0))}\leq e^{-e^{\mathcal{M}}}. \tag{78}$$

The forcing term F satisfies the estimates

$$\|F\|_{L^{\frac{3}{2}}_t L^\infty_x(\mathbb{R}^3\times(-2,0))},\ \|F\|_{L^{\frac{3}{2}}_t L^{\frac{11}{6}}_x(\mathbb{R}^3\times(-2,0))}\leq e^{-e^{\mathcal{M}}}. \tag{79}$$

Proof. **Step 1: Localized quantitative regions of regularity via local-in-space smoothing**
First, note that under the assumptions of Proposition 4, we can apply Lemma 2 to infer that for $\mathcal{M}_0$ sufficiently large, we have

(80)
$$\sup_{z_0=(x_0,t_0)\in B(0,1)\times(-1,0],\ 0<r\leq 1}\left\{\frac{1}{r^2}\int\limits_{Q(z_0,r)}|p|^{\frac{3}{2}}\,dxdt+\frac{1}{r}\|v\|^2_{L^\infty_t L^2_x\cap L^2_t\dot{H}^1_x(Q(z_0,r))}\right\}\lesssim\mathcal{M}^3.$$

Next, we rescale the solution (v,p):

(81)
$$(v_{\lambda_{(1)}}(x,t),p_{\lambda_{(1)}}(x,t)):=(\lambda_{(1)}v(\lambda_{(1)}x,\lambda_{(1)}^2 t),\lambda_{(1)}^2 p(\lambda_{(1)}x,\lambda_{(1)}^2 t))\quad\text{with }\lambda_{(1)}:=e^{-\mathcal{M}^5}.$$

Then $(v_{\lambda_{(1)}}, p_{\lambda_{(1)}})$ is a smooth solution to the Navier–Stokes equations on $B(0, 4e^{\mathcal{M}^5})\times(-16e^{2\mathcal{M}^5}, 0]$. From (72)–(73), we see that

$$\|v_{\lambda_{(1)}}\|_{L^\infty_t L^3_x(Q(0,4e^{\mathcal{M}^5}))} \le \mathcal{M}^3 \quad \text{and} \tag{82}$$

$$\|p_{\lambda_{(1)}}\|_{L^{\frac{3}{2}}_{x,t}(Q(0,4e^{\mathcal{M}^5}))} \le \mathcal{M}^{\frac{2}{3}} e^{\frac{4}{3}\mathcal{M}^5}. \tag{83}$$

Furthermore, from (80), we have

$$\sup_{z_0=(x_0,t_0)\in B(0,e^{\mathcal{M}^5})\times(-e^{2\mathcal{M}^5},0],\ 0<r\le e^{\mathcal{M}^5}} \left\{ \frac{1}{r^2}\int\limits_{Q(z_0,r)} |p_{\lambda_{(1)}}|^{\frac{3}{2}}\,dxdt + \frac{1}{r}\|v_{\lambda_{(1)}}\|^2_{L^\infty_t L^2_x \cap L^2_t \dot{H}^1_x(Q(z_0,r))} \right\} \lesssim \mathcal{M}^3. \tag{84}$$

Note that for $\mathcal{M}_0$ sufficiently large, we have that $4^{\mathcal{M}^4+3} \le e^{\mathcal{M}^5}$. Thus,

$$\sum_{k=2}^{\lceil \mathcal{M}^4\rceil+1} \int\limits_{4^k<|x|\le 4^{k+1}} |v_{\lambda_{(1)}}(x,-\mathcal{M}^{-95})|^3 dx \le \int\limits_{B(0,e^{\mathcal{M}^5})} |v_{\lambda_{(1)}}(x,-\mathcal{M}^{-95})|^3 dx \le \mathcal{M}^3.$$

Using the pigeonhole principle, there exists $k_0 \in \{2,\dots\lceil\mathcal{M}^4\rceil+1\}$ such that

$$\int\limits_{A_{k_0}} |v_{\lambda_{(1)}}(x,-\mathcal{M}^{-95})|^3 dx \le \frac{1}{\mathcal{M}} \le \epsilon_* \quad \text{with} \quad A_{k_0} := \{x : 4^{k_0} < |x| \le 4^{k_0+1}\}. \tag{85}$$

Here, $\mathcal{M}_0$ is sufficiently large, and ϵ_* is as in (31) (Proposition 1). Now we fix any

$$y_0 \in \{x : 4^{k_0}+4 \le |x| \le 4^{k_0+1}-4\} \subset A_{k_0} \subset B(0, e^{\mathcal{M}^5}).$$

Then (84) implies that we have

$$\frac{1}{16}\int\limits_{-16}^{0}\int\limits_{B(y_0,4)} |p_{\lambda_{(1)}}|^{\frac{3}{2}}\,dxdt + \frac{1}{4}\|v_{\lambda_{(1)}}\|^2_{L^\infty_t L^2_x\cap L^2_t\dot{H}^1_x(B(y_0,4)\times(-\frac{1}{16},0))} \lesssim \mathcal{M}^3.$$

This and (85) imply that for $\mathcal{M}_0$ sufficiently large, we have

$$\begin{gathered} \|v_{\lambda_{(1)}}\|^2_{L^\infty_t L^2_x\cap L^2_t\dot{H}^1_x(B(y_0,4)\times(-\mathcal{M}^{-95},0))} + \|p_{\lambda_{(1)}}\|_{L^{\frac{3}{2}}_{x,t}(B(y_0,4)\times(-\mathcal{M}^{-95},0))} \le \mathcal{M}^5 \\ \text{and} \quad \int\limits_{B(y_0,3)} |v_{\lambda_{(1)}}(x,-\mathcal{M}^{-95})|^3 dx \le \epsilon_*. \end{gathered} \tag{86}$$

This allows us to apply Proposition 1 and Remark 4 with M replaced by $\mathcal{M}^5$. Thus, one deduces that for $\mathcal{M}_0$ sufficiently large,

$$\|\nabla^j v_{\lambda_{(1)}}\|_{L^\infty_{x,t}(B(y_0,2)\times(-\mathcal{M}^{-96},0))} \le \mathcal{M}^{50j+50} \quad \text{for } j=0,1,2.$$

Since this holds for any y_0 such that $B(y_0,4)\subset A_{k_0}$, we infer that

$$\|\nabla^j v_{\lambda_{(1)}}\|_{L^\infty_{x,t}(\{4^{k_0}+2<|x|<4^{k_0+1}-2\}\times(-\mathcal{M}^{-96},0))} \le \mathcal{M}^{50j+50} \quad \text{for } j=0,1,2. \tag{87}$$

Step 2: Pressure and time derivative estimates inside the quantitative region of regularity

Next, we want to use (83) and (87) to gain higher regularity of $p_{\lambda_{(1)}}$ and $\partial_t v_{\lambda_{(1)}}$ inside the region $\{x: 4^{k_0}+3<|x|<4^{k_0+1}-3\}\times(-\mathcal{M}^{-96},0)$.

Let the cutoff function $\varphi\in C^\infty(\mathbb{R}^3;[0,1])$ be such that

$\varphi\equiv 1$ on $\{x: 4^{k_0}+\frac{5}{2}<|x|<4^{k_0+1}-\frac{5}{2}\}$,
supp $\varphi\subset\{x: 4^{k_0}+2<|x|<4^{k_0+1}-2\}$, and
for $j\in\mathbb{N}$ $\quad\|\nabla^j\varphi\|_{L^\infty(\mathbb{R}^3)}\le C_j$.

Here, C_j is a positive constant depending only on j. Let us also recall from the previous step that $A_{k_0}:=\{x: 4^{k_0}<|x|\le 4^{k_0+1}\}\subset B(0,e^{\mathcal{M}^5})$. Thus,

$$|A_{k_0}|\lesssim e^{3\mathcal{M}^5}. \tag{88}$$

Here, $|A_{k_0}|$ denotes the Lebesgue measure of A_{k_0}.

For every $s\in(-\mathcal{M}^{-96},0)$, we decompose the pressure

$$p_{\lambda_1}(x,s)=p_{\mathrm{Riesz},\lambda_{(1)}}(x,s)+p_{\mathrm{h},\lambda_{(1)}}(x,s)\quad\text{with}$$
$$p_{\mathrm{Riesz},\lambda_{(1)}}(x,s)=\mathcal{R}_i\mathcal{R}_j(\varphi(v_{\lambda_{(1)}})_i(v_{\lambda_{(1)}})_j(\cdot,s)).$$

Here, R_i denote Riesz transforms, and the Einstein summation convention has been adopted. Let us note that this decomposition implies that for every $s\in(-\mathcal{M}^{-96},0)$, $p_{\mathrm{h},\lambda_{(1)}}(\cdot,s)$ is harmonic in $\{x: 4^{k_0}+\frac{5}{2}<|x|<4^{k_0+1}-\frac{5}{2}\}$.

First, let us estimate $p_{\mathrm{Riesz},\lambda_{(1)}}$ and $\nabla p_{\mathrm{Riesz},\lambda_{(1)}}$. By the Calderón–Zygmund theory, Hölder's inequality, and (87)–(88), we infer the following for $\mathcal{M}_0$ sufficiently large. Namely, for any $q\in(1,\infty)$ and $s\in(-\mathcal{M}^{-96},0)$,

$$\begin{aligned}\|p_{\mathrm{Riesz},\lambda_{(1)}}(\cdot,s)\|_{L^q(\mathbb{R}^3)}&\lesssim_q\|v_{\lambda_{(1)}}(\cdot,s)\|^2_{L^{2q}(4^{k_0}+2<|x|<4^{k_0+1}-2)}\\&\lesssim\|v_{\lambda_{(1)}}(\cdot,s)\|^2_{L^\infty(4^{k_0}+2<|x|<4^{k_0+1}-2)}|A_{k_0}|^{\frac{1}{q}}\\&\lesssim_q\mathcal{M}^{100}e^{3\mathcal{M}^5}\le e^{4\mathcal{M}^5}.\end{aligned}\tag{89}$$

Similarly, for any $q\in(1,\infty)$ and $s\in(-\mathcal{M}^{-96},0)$, we obtain

$$\begin{aligned}&\|\nabla p_{\mathrm{Riesz},\lambda_{(1)}}(\cdot,s)\|_{L^q(\mathbb{R}^3)}\\&\quad\lesssim_q\|v_{\lambda_{(1)}}(\cdot,s)\|^2_{L^{2q}(4^{k_0}+2<|x|<4^{k_0+1}-2)}+\||v_{\lambda_{(1)}}||\nabla v_{\lambda_{(1)}}|(\cdot,s)\|_{L^q(4^{k_0}+2<|x|<4^{k_0+1}-2)}\\&\quad\lesssim_q\|v_{\lambda_{(1)}}(\cdot,s)\|^2_{L^\infty(4^{k_0}+2<|x|<4^{k_0+1}-2)}|A_{k_0}|^{\frac{1}{q}}\\&\qquad+\|v_{\lambda_{(1)}}(\cdot,s)\|_{L^\infty(4^{k_0}+2<|x|<4^{k_0+1}-2)}\|\nabla v_{\lambda_{(1)}}(\cdot,s)\|_{L^\infty(4^{k_0}+2<|x|<4^{k_0+1}-2)}|A_{k_0}|^{\frac{1}{q}}\\&\quad\lesssim\mathcal{M}^{150}e^{3\mathcal{M}^5}\le e^{4\mathcal{M}^5}.\end{aligned}\tag{90}$$

From (89)–(90) and the Sobolev embedding theorem, we have that

$$\|p_{\mathrm{Riesz},\lambda_{(1)}}\|_{L^{\frac{3}{2}}_tL^\infty_x(\mathbb{R}^3\times(-\mathcal{M}^{-96},0))}\lesssim e^{4\mathcal{M}^5}.\tag{91}$$

Furthermore, for all $q\in(1,\infty)$, we have

$$\|p_{\mathrm{Riesz},\lambda_{(1)}}\|_{L^{\frac{3}{2}}_tL^q_x(\mathbb{R}^3\times(-\mathcal{M}^{-96},0))}+\|\nabla p_{\mathrm{Riesz},\lambda_{(1)}}\|_{L^{\frac{3}{2}}_tL^q_x(\mathbb{R}^3\times(-\mathcal{M}^{-96},0))}\lesssim_q e^{4\mathcal{M}^5}.\tag{92}$$

Next, we focus on estimating $p_{\mathrm{h},\lambda_{(1)}}$ and $\nabla p_{\mathrm{h},\lambda_{(1)}}$. Let us recall that for every $s\in(-\mathcal{M}^{-96},0)$, $p_{\mathrm{h},\lambda_{(1)}}(\cdot,s)$ is harmonic in $\{x: 4^{k_0}+\frac{5}{2}<|x|<4^{k_0+1}-\frac{5}{2}\}$. Let

$x_0 \in \{x : 4^{k_0} + 3 < |x| < 4^{k_0+1} - 3\}$, and take $s \in (-\mathcal{M}^{-96}, 0)$. From properties of harmonic functions and the pressure decomposition, we infer that

$$|p_{\mathrm{h},\lambda_{(1)}}(x_0,s)| + |\nabla p_{\mathrm{h},\lambda_{(1)}}(x_0,s)| \lesssim \|p_{\mathrm{h},\lambda_{(1)}}(\cdot,s)\|_{L^{\frac{3}{2}}(B(x_0,\frac{1}{2}))}$$
$$\lesssim \|p_{\mathrm{Riesz},\lambda_{(1)}}(\cdot,s)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} + \|p_{\lambda_{(1)}}(\cdot,s)\|_{L^{\frac{3}{2}}(B(2e^{\mathcal{M}^5}))}.$$

Thus, for every $s \in (-\mathcal{M}^{-96}, 0)$,

$$\|p_{\mathrm{h},\lambda_{(1)}}(\cdot,s)\|_{L^\infty(4^{k_0}+3<|x|<4^{k_0+1}-3)} + \|\nabla p_{\mathrm{h},\lambda_{(1)}}(\cdot,s)\|_{L^\infty(4^{k_0}+3<|x|<4^{k_0+1}-3)}$$
$$\lesssim \|p_{\mathrm{Riesz},\lambda_{(1)}}(\cdot,s)\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} + \|p_{\lambda_{(1)}}(\cdot,s)\|_{L^{\frac{3}{2}}(B(2e^{\mathcal{M}^5}))}.$$

Using this, (83), and (92), we get that

(93)
$$\|p_{\mathrm{h},\lambda_{(1)}}\|_{L_t^{\frac{3}{2}}L_x^\infty(\{4^{k_0}+3<|x|<4^{k_0+1}-3\}\times(-\mathcal{M}^{-96},0))}$$
$$+ \|\nabla p_{\mathrm{h},\lambda_{(1)}}\|_{L_t^{\frac{3}{2}}L_x^\infty(\{4^{k_0}+3<|x|<4^{k_0+1}-3\}\times(-\mathcal{M}^{-96},0))} \lesssim e^{4\mathcal{M}^5}.$$

This, Hölder's inequality, and (88) imply that for all $q \in (1,\infty)$,

(94)
$$\|p_{\mathrm{h},\lambda_{(1)}}\|_{L_t^{\frac{3}{2}}L_x^q(\{4^{k_0}+3<|x|<4^{k_0+1}-3\}\times(-\mathcal{M}^{-96},0))}$$
$$+ \|\nabla p_{\mathrm{h},\lambda_{(1)}}\|_{L_t^{\frac{3}{2}}L_x^q(\{4^{k_0}+3<|x|<4^{k_0+1}-3\}\times(-\mathcal{M}^{-96},0))} \lesssim_q e^{7\mathcal{M}^5}.$$

Combining (91)–(94) yields that

(95)
$$\|p_{\lambda_{(1)}}\|_{L_t^{\frac{3}{2}}L_x^\infty(\{4^{k_0}+3<|x|<4^{k_0+1}-3\}\times(-\mathcal{M}^{-96},0))} \lesssim e^{4\mathcal{M}^5}$$

and that for all $q \in (1,\infty)$,

(96)
$$\|p_{\lambda_{(1)}}\|_{L_t^{\frac{3}{2}}L_x^q(\{4^{k_0}+3<|x|<4^{k_0+1}-3\}\times(-\mathcal{M}^{-96},0))}$$
$$+ \|\nabla p_{\lambda_{(1)}}\|_{L_t^{\frac{3}{2}}L_x^q(\{4^{k_0}+3<|x|<4^{k_0+1}-3\}\times(-\mathcal{M}^{-96},0))} \lesssim_q e^{7\mathcal{M}^5}.$$

Finally, let us estimate $\partial_t v_{\lambda_{(1)}}$. Using $(v_{\lambda_{(1)}}, p_{\lambda_{(1)}})$ is a smooth solution to the Navier–Stokes equations on $B(0, 4e^{\mathcal{M}^5}) \times (-16e^{2\mathcal{M}^5}, 0]$, we see that the estimates (87) and (96) (together with Hölder's inequality) imply that for all $q \in (1,\infty)$,

(97)
$$\|\partial_t v_{\lambda_{(1)}}\|_{L_t^{\frac{3}{2}}L_x^q(\{4^{k_0}+3<|x|<4^{k_0+1}-3\}\times(-\mathcal{M}^{-96},0))} \lesssim_q e^{7\mathcal{M}^5}.$$

Step 3: Bogovskiĭ operator estimates
Let $k_0 \in \{2, \dots \lceil \mathcal{M}^4 \rceil + 1\}$ be as in the previous steps, and define $R_{k_0} := 4^{k_0} + 3$. Note that

(98)
$$6 \le R_{k_0} \le e^{\mathcal{M}^5},\ A(R_{k_0}) := \{x : R_{k_0} < |x| < 2R_{k_0}\} \subset \{x : 4^{k_0} + 3 < |x| < 4^{k_0+1} - 3\}, \text{ and}$$
$$|A(R_{k_0})| \lesssim e^{3\mathcal{M}^5}.$$

Let the cutoff function $\Phi \in C^\infty(\mathbb{R}^3; [0,1])$ be such that
$\Phi \equiv 1$ on $B(0, R_{k_0})$,
supp $\Phi \subset B(0, 2R_{k_0})$, and
for $j \in \mathbb{N}$ $\quad \|\nabla^j \Phi\|_{L^\infty(\mathbb{R}^3)} \lesssim_j R_{k_0}^{-j}$.

Then for every $t\in(-\mathcal{M}^{-96},0]$, we see that

$$v_{\lambda_{(1)}}(x,t)\cdot\nabla\Phi(x)\in C_0^{\infty}(A(R_{k_0})). \tag{99}$$

Furthermore, using that $v_{\lambda_{(1)}}$ is divergence-free, we see that for every $t\in(-\mathcal{M}^{-96},0]$,

$$\begin{aligned}\int\limits_{A(R_{k_0})} v_{\lambda_{(1)}}(x,t)\cdot\nabla\Phi(x)dx &= \int\limits_{A(R_{k_0})} \mathrm{div}\,(v_{\lambda_{(1)}}(x,t)\Phi(x))dx\\ &= \int\limits_{\partial B(0,2R_{k_0})} v_{\lambda_{(1)}}\Phi\cdot dS - \int\limits_{\partial B(0,R_{k_0})} v_{\lambda_{(1)}}\Phi\cdot dS\\ &= -\int\limits_{\partial B(0,R_{k_0})} v_{\lambda_{(1)}}\cdot dS = -\int\limits_{B(0,R_{k_0})} \mathrm{div}\, v_{\lambda_{(1)}}dx = 0.\end{aligned} \tag{100}$$

Using this, we can apply the Bogovskiĭ operator in Lemma 1 to $-v_{\lambda_{(1)}}\cdot\nabla\Phi$ on $A(R_{k_0})$ for each $t\in(-\mathcal{M}^{-96},0]$, which we denote by $w_{\lambda_{(1)}}:=B(-v_{\lambda_{(1)}}\cdot\nabla\Phi)$. Furthermore, by (99), we have that for every $t\in(-\mathcal{M}^{-96},0]$,

$$w_{\lambda_{(1)}}(\cdot,t)\in C_0^{\infty}(A(R_{k_0})). \tag{101}$$

Let us now estimate the spatial derivatives of $w_{\lambda_{(1)}}:=B(-v_{\lambda_{(1)}}\cdot\nabla\Phi)$. Using (98) and (87) allows us to apply (27) of Lemma 1 to get

$$\|\nabla w_{\lambda_{(1)}}\|_{L^{\infty}_t W^{2,4}_x(A(R_{k_0})\times(-\mathcal{M}^{-96},0))}\lesssim \|v_{\lambda_{(1)}}\cdot\nabla\Phi\|_{L^{\infty}_t W^{2,4}_x(A(R_{k_0})\times(-\mathcal{M}^{-96},0))}\le e^{\mathcal{M}^5}.$$

Using this, (98)–(101), and Poincaré's inequality gives that

$$\|w_{\lambda_{(1)}}\|_{L^{\infty}_t W^{3,4}_x(A(R_{k_0})\times(-\mathcal{M}^{-96},0))}\le e^{3\mathcal{M}^5}. \tag{102}$$

This and the Sobolev embedding theorem gives that

$$\|w_{\lambda_{(1)}}\|_{L^{\infty}_t W^{2,\infty}_x(A(R_{k_0})\times(-\mathcal{M}^{-96},0))}\lesssim e^{3\mathcal{M}^5}. \tag{103}$$

Thus, by Hölder's inequality, we have

$$\|w_{\lambda_{(1)}}\|_{L^{\frac{3}{2}}_t W^{2,\infty}_x(A(R_{k_0})\times(-\mathcal{M}^{-96},0))}\lesssim e^{3\mathcal{M}^5}. \tag{104}$$

By (103), (98), and Hölder's inequality, we get

$$\|w_{\lambda_{(1)}}\|_{L^{\infty}_t L^{\frac{10}{3}}_x(A(R_{k_0})\times(-\mathcal{M}^{-96},0))}\le e^{4\mathcal{M}^5} \tag{105}$$

and

$$\|w_{\lambda_{(1)}}\|_{L^{\frac{3}{2}}_t W^{2,\frac{11}{6}}_x(A(R_{k_0})\times(-\mathcal{M}^{-96},0))}\lesssim e^{\frac{51}{11}\mathcal{M}^5}. \tag{106}$$

Let us now estimate the time derivative of the Bogovskiĭ operator $w_{\lambda_{(1)}}:=B(-v_{\lambda_{(1)}}\cdot\nabla\Phi)$. For every $t\in(-\mathcal{M}^{-96},0)$, we see that

$$\partial_t v_{\lambda_{(1)}}(x,t)\cdot\nabla\Phi(x)\in C_0^{\infty}(A(R_{k_0})).$$

Using this, Lemma 1 (specifically, (28)), and (97), we see that for every $t\in(-\mathcal{M}^{-96},0)$,

$$\partial_t w_{\lambda_{(1)}}(\cdot,t)\in C_0^\infty(A(R_{k_0})) \tag{107}$$

and

$$\|\nabla\partial_t w_{\lambda_{(1)}}\|_{L_t^{\frac32}L_x^4(A(R_{k_0})\times(-\mathcal{M}^{-96},0))}\lesssim \|\partial_t v_{\lambda_{(1)}}\cdot\nabla\Phi\|_{L_t^{\frac32}L_x^4(A(R_{k_0})\times(-\mathcal{M}^{-96},0))}\lesssim e^{7\mathcal{M}^5}. \tag{108}$$

Noting (98) and (107)–(108), we apply Poincaré's inequality to get

$$\|\partial_t w_{\lambda_{(1)}}\|_{L_t^{\frac32}W_x^{1,4}(A(R_{k_0})\times(-\mathcal{M}^{-96},0))}\lesssim e^{8\mathcal{M}^5}. \tag{109}$$

Hence, the Sobolev embedding theorem gives

$$\|\partial_t w_{\lambda_{(1)}}\|_{L_t^{\frac32}L_x^{\infty}(A(R_{k_0})\times(-\mathcal{M}^{-96},0))}\lesssim e^{8\mathcal{M}^5}. \tag{110}$$

Observing (98) and (110), we can apply Hölder's inequality to conclude that

$$\|\partial_t w_{\lambda_{(1)}}\|_{L_t^{\frac32}L_x^{\frac{11}{6}}(A(R_{k_0})\times(-\mathcal{M}^{-96},0))}\lesssim e^{\frac{106}{11}\mathcal{M}^5}. \tag{111}$$

Step 4: Truncating the solution and rescaling to conclude
Let Φ and $w_{\lambda_{(1)}}$ be as in the previous step. Let us now define the truncated solution

$$\begin{aligned} u_{\lambda_{(1)}}(x,t)&:=\Phi v_{\lambda_{(1)}}(x,t)+w_{\lambda_{(1)}}(x,t)\quad\text{and}\quad \pi_{\lambda_{(1)}}(x,t):=\Phi p_{\lambda_{(1)}}(x,t)\\ &\quad\text{for}\quad (x,t)\in\mathbb{R}^3\times(-\mathcal{M}^{-96},0]. \end{aligned}\tag{112}$$

From Lemma 1, properties of the cutoff function Φ, and (101), we deduce the following. Namely, $u_{\lambda_{(1)}}$ is divergence-free and that for every $t\in(-\mathcal{M}^{-96},0]$, $u_{\lambda_{(1)}}$ and $\pi_{\lambda_{(1)}}$ are smooth with compact support in $B(0,2R_{k_0})$. Furthermore, we see that $(u_{\lambda_{(1)}},\pi_{\lambda_{(1)}})$ are smooth on $\mathbb{R}^3\times(-\mathcal{M}^{-96},0)$ and $B(0,R_{k_0})\times[-\mathcal{M}^{-96},0]$. Furthermore, $(u_{\lambda_{(1)}},\pi_{\lambda_{(1)}})$ solve the forced Navier–Stokes equations in $\mathbb{R}^3\times(-\mathcal{M}^{-96},0)$ with forcing term given by

$$\begin{aligned} f_{\lambda_{(1)}}&:=-\Delta\Phi v_{\lambda_{(1)}}-2\nabla\Phi\cdot\nabla v_{\lambda_{(1)}}+\Phi v_{\lambda_{(1)}}\cdot\nabla\Phi v_{\lambda_{(1)}}+(\Phi^2-\Phi)v_{\lambda_{(1)}}\cdot\nabla v_{\lambda_{(1)}}\\ &\quad+(\partial_t-\Delta)w_{\lambda_{(1)}}+\Phi v_{\lambda_{(1)}}\cdot\nabla w_{\lambda_{(1)}}+w_{\lambda_{(1)}}\cdot\nabla(\Phi v_{\lambda_{(1)}})+w_{\lambda_{(1)}}\cdot\nabla w_{\lambda_{(1)}}+\nabla\Phi p_{\lambda_{(1)}}. \end{aligned}\tag{113}$$

Using properties of the cutoff function Φ, (87), (95)–(96), (103)–(104), (106), and (110)–(111), we see that

$$\|f_{\lambda_{(1)}}\|_{L_t^{\frac32}L_x^{\infty}(A(R_{k_0})\times(-\mathcal{M}^{-96},0))}+\|f_{\lambda_{(1)}}\|_{L_t^{\frac32}L_x^{\frac{11}{6}}(A(R_{k_0})\times(-\mathcal{M}^{-96},0))}\le e^{12\mathcal{M}^5}. \tag{114}$$

From properties of the cutoff function Φ and (101), we see that for every $t\in(-\mathcal{M}^{-96},0)$,

$$\operatorname{supp} f_{\lambda_{(1)}}(\cdot,t)\subseteq B(0,2R_{k_0})\setminus\overline{B(0,R_{k_0})}\text{ and }\operatorname{supp} u_{\lambda_{(1)}}(\cdot,t)\subseteq B(0,2R_{k_0})\text{ with }R_{k_0}\ge 6. \tag{115}$$

From (82), properties of the cutoff function Φ, and (105), one deduces that $u_{\lambda_{(1)}}=\Phi v_{\lambda_{(1)}}+w_{\lambda_{(1)}}$ with

$$\|\Phi v_{\lambda_{(1)}}\|_{L^\infty_t L^3_x(\mathbb{R}^3\times(-\mathcal{M}^{-96},0))} = \|u_{\lambda_{(1)}}\|_{L^\infty_t L^3_x(B(0,R_{k_0})\times(-\mathcal{M}^{-96},0))} \leq \mathcal{M} \quad \text{and} \tag{116}$$

$$\|w_{\lambda_{(1)}}\|_{L^\infty_t L^{\frac{10}{3}}_x(\mathbb{R}^3\times(-\mathcal{M}^{-96},0))} \leq e^{4\mathcal{M}^5}. \tag{117}$$

From properties of the cutoff function Φ and (101), we see that

$$\begin{gathered}(u_{\lambda_{(1)}}(x,t),\pi_{\lambda_{(1)}}(x,t)) = (v_{\lambda_{(1)}}(x,t),p_{\lambda_{(1)}}(x,t)) = (\lambda_{(1)}v(\lambda_{(1)}x,\lambda_{(1)}^2 t),\lambda_{(1)}^2 p(\lambda_{(1)}x,\lambda_{(1)}^2 t))\\ \text{on } B(0,R_{k_0})\times[-\mathcal{M}^{-96},0]\supseteq B(0,6)\times[-\mathcal{M}^{-96},0].\end{gathered} \tag{118}$$

Using this, together with (84) and (98), gives that

$$\sup_{0<r\leq\mathcal{M}^{-48}}\left\{\frac{1}{r^2}\int\limits_{Q(0,r)}|\pi_{\lambda_1}|^{\frac{3}{2}}\,dxdt + \frac{1}{r}\|u_{\lambda_{(1)}}\|^2_{L^\infty_t L^2_x\cap L^2_t\dot{H}^1_x(Q(z_0,r))}\right\}\lesssim\mathcal{M}^3. \tag{119}$$

We now define the functions $V:\mathbb{R}^3\times[-2,0]\to\mathbb{R}^3$, $F:\mathbb{R}^3\times[-2,0]\to\mathbb{R}^3$, and $P:\mathbb{R}^3\times[-2,0]\to\mathbb{R}$ by

$$\begin{aligned}&(V(x,t),P(x,t),F(x,t))\\ &\quad := (\lambda_{(2)}u_{\lambda_{(1)}}(\lambda_{(2)}x,\lambda_{(2)}^2 t),\lambda_{(2)}^2\pi_{\lambda_{(1)}}(\lambda_{(2)}x,\lambda_{(2)}^2 t),\lambda_{(2)}^3 f_{\lambda_{(1)}}(\lambda_{(2)}x,\lambda_{(2)}^2 t))\\ &\quad \text{with} \quad \lambda_{(2)} := e^{-e^{2\mathcal{M}}}.\end{aligned} \tag{120}$$

We also define

$$R := R_{k_0}\lambda_{(2)}^{-1}. \tag{121}$$

With these definitions, it is easy to see that V and P are smooth on $\mathbb{R}^3\times[-2,0)$. Additionally, by (118), we see that V and P are smooth on $B(0,R)\times[-2,0]$. Furthermore, the properties (114)–(119) for $(u_{\lambda_{(1)}},\pi_{\lambda_{(1)}},f_{\lambda_{(1)}})$ and R_{k_0} imply the desired properties for (V,P,F) and R stated in Proposition 4. ☐

5. Quantitative regions of regularity for the truncated solution.

5.1. Quantitative epochs of regularity for the truncated solution.

PROPOSITION 5. *There exists $\mathcal{M}_0$ sufficiently large such that the following holds true. Let (v,p), (V,P), F, and R be as in the statement of Proposition* 4. *Let $0<T_1\leq 1$, and define*

$$I := (-T_1,-\tfrac{T_1}{2})\subset[-1,0]\subset[-2,0].$$

Then the above assumptions imply that there exists an epoch of regularity $I'\subset I$, which is a closed interval, such that

$$|I'| = \mathcal{M}^{-46}|I| \text{ and} \tag{122}$$

$$\sup_{(x,t)\in B(0,\frac{R}{2})\times I'}|\nabla^j V(x,t)|\lesssim\frac{1}{\mathcal{M}}|I'|^{-\frac{j+1}{2}} \quad \text{for } j=0,1,2. \tag{123}$$

Proof. **Step 1: Preliminary estimates**
The properties of (V,P,F) and R (as stated in Proposition 4) are also satisfied for the rescaled quantities

$$\begin{aligned}&(V_{T_1}(x,t),P_{T_1}(x,t),F_{T_1}(x,t))\\&\quad:=(T_1^{\frac12}V(T_1^{\frac12}x,T_1t),T_1P(T_1^{\frac12}x,T_1t),T_1^{\frac32}F(T_1^{\frac12}x,T_1t))\quad\text{and}\quad R_{T_1}:=\frac{R}{T_1^{\frac12}}\end{aligned}$$

with any $0<T_1\le 1$. Hence, without loss of generality, we can take $T_1=1$ ($I:=(-1,-\frac12)$) in Proposition 5.

On $\mathbb{R}^3\times(-2,0)$, we have

$$V(\cdot,t)=L(F)(\cdot,t)+B(V,V)(\cdot,t). \tag{124}$$

Here,

$$L(F)(\cdot,t):=e^{(t+2)\Delta}V(\cdot,-2)+\int_{-2}^{t}e^{(t-s)\Delta}\mathbb{P}F(\cdot,s)ds\text{ and} \tag{125}$$

$$B(A,B)(\cdot,t):=-\int_{-2}^{t}e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(A\otimes B)(\cdot,s)ds.$$

Here, $e^{t\Delta}$ denotes the heat semigroup on $\mathbb{R}^3$, and $\mathbb{P}$ denotes the projection onto divergence-free vector fields in $\mathbb{R}^3$. Using $V=V^{(1)}+V^{(2)}$ on $\mathbb{R}^3\times[-2,0]$, (78), and standard estimates of the heat semigroup yields

$$\sup_{t\in[-1,0]}\|e^{t\Delta}V(\cdot,-2)\|_{L^q(\mathbb{R}^3)}\lesssim_q\mathcal{M}\quad\forall q\in[\tfrac{10}{3},\infty]. \tag{126}$$

Using standard estimates of the heat semigroup and Leray projector, together with Hölder's inequality, yields that for $t\in[-2,0)$,

$$\begin{aligned}&\Big\|\int_{-2}^{t}e^{(t-s)\Delta}\mathbb{P}F(\cdot,s)ds\Big\|_{L^{\frac{11}{6}}(\mathbb{R}^3)}\lesssim\|F\|_{L_t^{\frac32}L_x^{\frac{11}{6}}(\mathbb{R}^3\times(-2,0))}\text{ and}\\&\Big\|\int_{-2}^{t}e^{(t-s)\Delta}\mathbb{P}F(\cdot,s)ds\Big\|_{L^{\infty}(\mathbb{R}^3)}\lesssim\int_{-2}^{t}\frac{\|F(\cdot,s)\|_{L^5(\mathbb{R}^3)}}{(t-s)^{\frac{3}{10}}}ds\lesssim\|F\|_{L_t^{\frac32}L_x^{5}(\mathbb{R}^3\times(-2,0))}.\end{aligned} \tag{127}$$

Using (126)–(127) and (79) gives that for $\mathcal{M}_0$ sufficiently large,

$$\sup_{t\in[-1,0)}\|L(F)(\cdot,t)\|_{L^q(\mathbb{R}^3)}\lesssim_q\mathcal{M}\quad\forall q\in[\tfrac{10}{3},\infty]. \tag{128}$$

Let us now estimate the kinetic energy of $B(V,V)(\cdot,t)$, which we will now denote by

(129)
$$W(\cdot,t):=B(V^{(1)},V^{(1)})(\cdot,t)+B(V^{(1)},V^{(2)})(\cdot,t)+B(V^{(2)},V^{(1)})(\cdot,t)+B(V^{(2)},V^{(2)})(\cdot,t)$$

for convenience. From [28], it is known that $e^{t\Delta}\mathbb{P}\nabla\cdot$ has an associated convolution kernel K_{ijk} ($i,j,k\in\{1,2,3\}$). Furthermore, from [36], we have

$$|K(x,t)|\lesssim\frac{1}{(|x|^2+t)^2}\quad\text{for}\quad(x,t)\in\mathbb{R}^3\times(0,\infty). \tag{130}$$

Using (130), (78), and Young's convolution inequality gives that for $t\in(-2,0)$,

$$\|B(V^{(1)},V^{(1)})(\cdot,t)\|_{L^2(\mathbb{R}^3)}\lesssim\int\limits_{-2}^{t}\frac{1}{(t-s)^{\frac34}}\|V^{(1)}(\cdot,s)\|^2_{L^3(\mathbb{R}^3)}ds\lesssim\mathcal{M}^2, \tag{131}$$

$$\begin{aligned}&\|B(V^{(1)},V^{(2)})(\cdot,t)\|_{L^2(\mathbb{R}^3)}\\&\quad\lesssim\int\limits_{-2}^{t}\frac{1}{(t-s)^{\frac{7}{10}}}\|V^{(1)}(\cdot,s)\|_{L^3(\mathbb{R}^3)}\|V^{(2)}(\cdot,s)\|_{L^{\frac{10}{3}}(\mathbb{R}^3)}ds\lesssim\mathcal{M}e^{-e^{\mathcal{M}}},\quad\text{and}\end{aligned} \tag{132}$$

$$\|B(V^{(2)},V^{(2)})(\cdot,t)\|_{L^2(\mathbb{R}^3)}\lesssim\int\limits_{-2}^{t}\frac{1}{(t-s)^{\frac{13}{20}}}\|V^{(2)}(\cdot,s)\|^2_{L^{\frac{10}{3}}(\mathbb{R}^3)}ds\lesssim e^{-2e^{\mathcal{M}}}. \tag{133}$$

Combining (131)–(133) yields that

$$\sup_{t\in(-2,0)}\|W(\cdot,t)\|_{L^2(\mathbb{R}^3)}\lesssim\mathcal{M}^2. \tag{134}$$

Step 2: Energy estimates for W and higher integrability of V
Observe that the smooth function W satisfies the following equation on $\mathbb{R}^3\times(-2,0)$:

$$\begin{aligned}&\partial_tW-\Delta W+W\cdot\nabla W+W\cdot\nabla L(F)+L(F)\cdot\nabla W+L(F)\cdot\nabla L(F)+\nabla\Pi=0,\\&\operatorname{div}W=0,\ \text{and}\ W(\cdot,-2)=0.\end{aligned} \tag{135}$$

Performing a standard energy estimate on (135) yields that for $t\in[-1,0)$,

$$\begin{aligned}\|W(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+2\int\limits_{-1}^{t}\int\limits_{\mathbb{R}^3}|\nabla W(x,s)|^2dxds&=\|W(\cdot,-1)\|^2_{L^2(\mathbb{R}^3)}\\&+2\int\limits_{-1}^{t}\int\limits_{\mathbb{R}^3}(W+L(F))\otimes L(F):\nabla W\,dxds.\end{aligned} \tag{136}$$

Using this, Hölder's inequality, and Young's algebraic inequality yields that for $t\in[-1,0)$,

$$\begin{aligned}\|W(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+\int\limits_{-1}^{t}\int\limits_{\mathbb{R}^3}|\nabla W(x,s)|^2dxds&\lesssim\|W(\cdot,-1)\|^2_{L^2(\mathbb{R}^3)}\\&+\int\limits_{-1}^{t}\int\limits_{\mathbb{R}^3}|W|^2|L(F)|^2+|L(F)|^4dxds.\end{aligned} \tag{137}$$

By (128) and (134), we deduce from (137) that

$$\sup_{t\in[-1,0)}\|W(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}+\int\limits_{-1}^{0}\int\limits_{\mathbb{R}^3}|\nabla W(x,s)|^2dxds\lesssim\mathcal{M}^6. \tag{138}$$

Using this, the Sobolev embedding theorem, and the pigeonhole principle, we see that there exists $t_1\in(-1,-\frac34)$ such that

$$\|W(\cdot,t_1)\|_{L^6(\mathbb{R}^3)}\lesssim\mathcal{M}^3.$$

Combining this with (128) gives that

$$\|V(\cdot,t_1)\|_{L^6(\mathbb{R}^3)} \lesssim \mathcal{M}^3.$$

Using this estimate for $V(\cdot,t_1)$ and (79), one deduces that

$$\sup_{t\in[t_1,0)} \left\| e^{(t-t_1)\Delta}V(\cdot,t_1) + \int_{t_1}^{t} e^{(t-s)\Delta}\mathbb{P}F(\cdot,s)ds \right\|_{L^6(\mathbb{R}^3)} \lesssim \mathcal{M}^3. \tag{139}$$

Using this and applying [13, Theorem 3.2], we see that for $\mathcal{M}$ sufficiently large, there exists a solution $\bar{V}:\mathbb{R}^3\times[t_1,t_1+C_{univ}\mathcal{M}^{-12}]\to\mathbb{R}^3$ to

(140)
$$\bar{V}(\cdot,t) = e^{(t-t_1)\Delta}V(\cdot,t_1) + \int_{t_1}^{t} e^{(t-s)\Delta}\mathbb{P}F(\cdot,s)ds - \int_{t_1}^{t} e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(\bar{V}\otimes\bar{V})(\cdot,s)ds$$

with

$$\sup_{s\in[t_1,t_1+C_{univ}\mathcal{M}^{-12}]} \|\bar{V}(\cdot,s)\|_{L^6(\mathbb{R}^3)} \lesssim \mathcal{M}^3. \tag{141}$$

Here, C_{univ} is a positive universal constant. Since V is smooth on $\mathbb{R}^3\times[-t_1,0)$ and with compact support (77), we have that for $\mathcal{M}$ sufficiently large,

$$V \in L^\infty_t L^6_x(\mathbb{R}^3\times(t_1,t_1+C_{univ}\mathcal{M}^{-12})).$$

Furthermore, V also satisfies the same equation (140) as $\bar{V}$. So by [13, Theorem 3.3], we infer that $\bar{V}\equiv V$ on $\mathbb{R}^3\times[t_1,t_1+C_{univ}\mathcal{M}^{-12}]$. Hence, from (141), we obtain

$$\sup_{s\in[t_1,t_1+C_{univ}\mathcal{M}^{-12}]} \|V(\cdot,s)\|_{L^6(\mathbb{R}^3)} \lesssim \mathcal{M}^3. \tag{142}$$

Step 3: Higher differentiability for V via ε-regularity

The pressure P can be decomposed as

$$P = P_{V\otimes V} + P_F \quad \text{with} \quad P_{V\otimes V} := \mathcal{R}_i\mathcal{R}_j(V_iV_j) \quad \text{and} \quad P_F := (\Delta)^{-1}(\operatorname{div} F). \tag{143}$$

Using (79), (142), and Calderón–Zygmund bounds gives that

$$\begin{aligned} &\|P_{V\otimes V}\|_{L^\infty_t L^3_x(\mathbb{R}^3\times(t_1,t_1+C_{univ}\mathcal{M}^{-12}))} \lesssim \mathcal{M}^6 \text{ and} \\ &\|\nabla P_F\|_{L_t^{\frac{3}{2}} L_x^{\frac{11}{6}}(\mathbb{R}^3\times(t_1,t_1+C_{univ}\mathcal{M}^{-12}))} \lesssim e^{-e^{\mathcal{M}}}. \end{aligned} \tag{144}$$

Next, we fix $x\in B(0,\frac{R}{2})$, $r:=\mathcal{M}^{-\frac{13}{2}}$, and $t\in[t_1+\frac{3}{2}\mathcal{M}^{-13},t_1+2\mathcal{M}^{-13}]$. Using the properties of R in Proposition 4 and (74), we see that

$$\begin{aligned} &B(x,r)\times(t-r^2,t)\subseteq B(0,R)\times(t_1,t_1+C_{univ}\mathcal{M}^{-12}), \\ &(V,P) \text{ solve the Navier-Stokes equations in } B(x,r)\times(t-r^2,t). \end{aligned} \tag{145}$$

Using (142), (145), and Hölder's inequality, one infers that

$$\frac{1}{r^2}\int_{t-r^2}^{t}\int_{B(x,r)} |V(y,s)|^3 dyds \lesssim r^{\frac{3}{2}} \sup_{s\in[t_1,t_1+C_{univ}\mathcal{M}^{-12}]} \|V(\cdot,s)\|^3_{L^6(\mathbb{R}^3)} \lesssim \mathcal{M}^{-\frac{3}{4}}. \tag{146}$$

The same arguments applied to $P_{V\otimes V}$ yield

$$\frac{1}{r^2}\int\limits_{t-r^2}^{t}\int\limits_{B(x,r)} |P_{V\otimes V}(y,s)-(P_{V\otimes V})_{B(x,r)}(s)|^{\frac{3}{2}}dyds \lesssim \mathcal{M}^{-\frac{3}{4}}. \tag{147}$$

Next, using (144), (145), Poincaré's inequality, and Hölder's inequality gives

$$\begin{aligned}\frac{1}{r^2}\int\limits_{t-r^2}^{t}\int\limits_{B(x,r)} |P_F(y,s)-(P_F)_{B(x,r)}(s)|^{\frac{3}{2}}dyds &\lesssim \frac{1}{r^{\frac{1}{2}}}\int\limits_{t-r^2}^{t}\int\limits_{B(x,r)}|\nabla P_F|^{\frac{3}{2}}dyds\\ &\lesssim r^{\frac{1}{22}}\int\limits_{t-r^2}^{t}\Bigg(\int\limits_{B(x,r)}|\nabla P_F|^{\frac{11}{6}}dy\Bigg)^{\frac{9}{11}}ds \lesssim \mathcal{M}^{-\frac{13}{44}}e^{-\frac{3}{2}e^{\mathcal{M}}}.\end{aligned} \tag{148}$$

Combining (146)–(148) gives

$$\frac{1}{r^2}\int\limits_{t-r^2}^{t}\int\limits_{B(x,r)} |V(y,s)|^3+|P(y,s)-(P)_{B(x,r)}(s)|^{\frac{3}{2}}dyds \lesssim \mathcal{M}^{-\frac{3}{4}}+\mathcal{M}^{-\frac{13}{44}}e^{-\frac{3}{2}e^{\mathcal{M}}}. \tag{149}$$

Using (145), Caffarelli, Kohn, and Nirenberg's ε-regularity criterion [10], and [34, Lemma 6.1], we infer from (149) that for $\mathcal{M}$ sufficiently large,

$$|\nabla^j V(x,t)| \lesssim_j r^{-(j+1)} \lesssim \mathcal{M}^{\frac{39}{2}} \quad \text{for} \quad j=0,1,2.$$

Thus, we have

$$\sup_{B(0,\frac{R}{2})\times[t_1+\frac{3}{2}\mathcal{M}^{-13},t_1+2\mathcal{M}^{-13}]} |\nabla^j V| \lesssim \mathcal{M}^{\frac{39}{2}} \quad \text{for} \quad j=0,1,2. \tag{150}$$

Let

$$I' := [t_1+\tfrac{3}{2}\mathcal{M}^{-13}, t_1+\tfrac{3}{2}\mathcal{M}^{-13}+\tfrac{1}{2}\mathcal{M}^{-46}] \subset [t_1+\tfrac{3}{2}\mathcal{M}^{-13}, t_1+2\mathcal{M}^{-13}]. \tag{151}$$

Then $|I'|=\frac{1}{2}\mathcal{M}^{-46}=|I|\mathcal{M}^{-46}$. From this and (150), we get

$$\sup_{B(0,\frac{R}{2})\times I'} |\nabla^j V| \lesssim \mathcal{M}^{\frac{39}{2}} \lesssim \frac{1}{M}|I'|^{-\frac{j+1}{2}} \quad \text{for} \quad j=0,1,2$$

as required. ∎

5.2. Quantitative annuli of regularity for the truncated solution. In order to show the existence of quantitative annuli of regularity for the truncated solution V, we will make use of the following proposition taken from [7].

Proposition 6 ([7, Proposition 6.4]). *There exists a universal constant $\varepsilon_1^* \in (0,1)$ such that if (v,p) is a suitable weak solution to the Navier–Stokes equations on $Q(0,1)$ and for some $\varepsilon_1 \le \varepsilon_1^*$*

$$\int\limits_{Q(0,1)} |v|^3+|p|^{\frac{3}{2}}dxdt \le \varepsilon_1, \tag{152}$$

then one concludes that for $j=0,1$,

(153) $$\nabla^j v \in L^\infty(Q(0,\tfrac{1}{4})) \;\; \textit{with} \;\; \|\nabla^j v\|_{L^\infty(Q(0,\frac{1}{4}))} \lesssim \varepsilon_1^{\frac{1}{3}}$$

and

(154) $$\nabla \omega \in L^\infty(Q(0,\tfrac{1}{4})) \;\; \textit{with} \;\; \|\nabla \omega\|_{L^\infty(Q(0,\frac{1}{4}))} \lesssim \varepsilon_1^{\frac{1}{3}}.$$

PROPOSITION 7. *Fix any $\mu \geq 1$, and let $\mathcal{M}$, (v,p), (V,P), F, and R be as in Proposition* 4. *Let $\Omega := \nabla \times V$. There exists a positive constant $\lambda_0(\mu)$ such that if $\mathcal{M} \geq \lambda_0(\mu)$, then the following holds true. In particular, for any $T_1 \in (0,1]$ and*

(155) $$T_1^{\frac{1}{2}} \leq R_0 \leq T_1^{\frac{1}{2}} 6e^{e^{2\mathcal{M}}} e^{-4\mu\mathcal{M}^{11\mu+22}},$$

there exists R_2 such that

(156) $$2R_0 \leq R_2 \leq 2R_0 e^{\mu\mathcal{M}^{11\mu+22}} \leq R,$$

(157) $$\|\nabla^j V\|_{L^\infty_{x,t}(R_2<|x|<\frac{R_2\mathcal{M}^{11\mu}}{4} \times (-\frac{T_1}{16},0))} \lesssim \mathcal{M}^{-3\mu} T_1^{-\frac{j+1}{2}} \quad \textit{for } j=0,1$$

and

(158) $$\|\nabla \Omega\|_{L^\infty_{x,t}(R_2<|x|<\frac{R_2\mathcal{M}^{11\mu}}{4} \times (-\frac{T_1}{16},0))} \lesssim \mathcal{M}^{-3\mu} T_1^{-\frac{3}{2}}.$$

Proof. **Step 1: Preliminary estimates**
From now on, we take $\mathcal{M} \geq \mathcal{M}(\mu)$ such that

$$1 \leq 6e^{e^{2\mathcal{M}}} e^{-4\mu\mathcal{M}^{11\mu+22}} \quad \forall \mathcal{M} \geq \mathcal{M}(\mu).$$

For identical reasons as described in the proof of Proposition 5, we may assume without loss of generality that $T_1 = 1$ by a scaling argument. As in the proof of Proposition 5, we decompose $V : \mathbb{R}^3 \times [-2,0] \to \mathbb{R}^3$ as

$$V(\cdot,t) = W(\cdot,t) + L(F)(\cdot,t) \quad \text{for} \quad t \in (-2,0).$$

Here, $L(F)$ is defined in (125), and W is defined in (129). We decompose the pressure $P : \mathbb{R}^3 \times [-2,0] \to \mathbb{R}$ as

(159) $$P(\cdot,t) = P_{V\otimes V}(\cdot,t) + P_F(\cdot,t) \quad \text{for} \quad t \in (-2,0).$$

Here, $P_{V\otimes V}$ and P_F are as in (143).

Using (128), (138), and the Sobolev embedding theorem gives

(160) $$\int_{-1}^{0}\int_{\mathbb{R}^3} |V(y,s)|^{\frac{10}{3}}\, dyds \lesssim \mathcal{M}^{10}.$$

Using this and Calderón–Zygmund estimates, we have

(161) $$\int_{-1}^{0}\int_{\mathbb{R}^3} |P_{V\otimes V}(y,s)|^{\frac{5}{3}}\, dyds \lesssim \mathcal{M}^{10}.$$

Thus, for $\mathcal{M}$ sufficiently large, we have

$$\int_{-1}^{0}\int_{\mathbb{R}^3} |V(y,s)|^{\frac{10}{3}} + |P_{V\otimes V}(y,s)|^{\frac{5}{3}}dyds \leq \lambda := \mathcal{M}^{11}. \tag{162}$$

Using (79) and Calderón–Zygmund estimates also gives that

$$\|\nabla P_F\|_{L_t^{\frac{3}{2}}L_x^{\frac{11}{6}}(\mathbb{R}^3\times[-1,0])} \lesssim e^{-e^{\mathcal{M}}}. \tag{163}$$

Step 2: Finding a good scale with pigeonholing arguments
Now we fix any $\mu \geq 1$ and R_0 such that

$$1 \leq R_0 \leq 6e^{e^{2\mathcal{M}}}e^{-4\mu\mathcal{M}^{11\mu+22}}. \tag{164}$$

From (162), we see that

$$\sum_{k=0}^{\infty}\int_{-1}^{0}\int_{\{x:\lambda^{k\mu}R_0<|x|<\lambda^{\mu(k+1)}R_0\}} |V(y,s)|^{\frac{10}{3}} + |P_{V\otimes V}(y,s)|^{\frac{5}{3}}dyds \leq \lambda := \mathcal{M}^{11}.$$

By the pigeonhole principle, there exists

$$k_0 \in \{0,1,\ldots\lfloor\lambda^{\mu+1}\rfloor\} \tag{165}$$

such that

$$\int_{-1}^{0}\int_{\{x:\lambda^{k_0\mu}R_0<|x|<\lambda^{\mu(k_0+1)}R_0\}} |V(y,s)|^{\frac{10}{3}} + |P_{V\otimes V}(y,s)|^{\frac{5}{3}}dyds \leq \lambda^{-\mu} = \mathcal{M}^{-11\mu}. \tag{166}$$

Step 3: Concluding with ε-regularity
Next, we define

$$R_1 := \lambda^{k_0\mu}R_0. \tag{167}$$

From now on, we impose the restriction $\mathcal{M} \geq \max(\mathcal{M}_0, \mathcal{M}(\mu), 5^{\frac{1}{11\mu}})$, and we define

$$A := \{x : R_1 + 1 < |x| < \lambda^{\mu}R_1 - 1\}. \tag{168}$$

This, (166), and Hölder's inequality gives that

(169)
$$\sup_{x_*\in A}\int_{-1}^{0}\int_{B(x_*,1)} |V(y,s)|^3 + |P_{V\otimes V}(y,s) - (P_{V\otimes V})_{B(x_*,1)}(s)|^{\frac{3}{2}}dyds \lesssim \lambda^{-\frac{9\mu}{10}} \leq \lambda^{-\frac{9\mu}{11}}.$$

Combining this with (163) and Poincaré's inequality gives that for $\mathcal{M}$ sufficiently large, there exists a positive universal constant C_{univ} such that

$$\begin{aligned}\sup_{x_*\in A}\int_{-1}^{0}\int_{B(x_*,1)} |V(y,s)|^3 + |P(y,s) - (P)_{B(x_*,1)}(s)|^{\frac{3}{2}}dyds &\lesssim \mathcal{M}^{-9\mu} \\ + e^{-\frac{3}{2}e^{\mathcal{M}}} &\leq C_{univ}\mathcal{M}^{-9\mu}.\end{aligned} \tag{170}$$

Thus, for

$$\mathcal{M} \geq \lambda_0(\mu) := \max\Big(\mathcal{M}_0, \mathcal{M}(\mu), 5^{\frac{1}{11\mu}}, \Big(\frac{C_{univ}}{\varepsilon_1^*}\Big)^{\frac{1}{9\mu}}\Big), \tag{171}$$

we have

$$\sup_{x_*\in A}\int_{-1}^{0}\int_{B(x_*,1)} |V(y,s)|^3 + |P(y,s) - (P)_{B(x_*,1)}(s)|^{\frac{3}{2}}\, dyds \leq C_{univ}\mathcal{M}^{-9\mu} \leq \varepsilon_1^*. \tag{172}$$

Here, $\mathcal{M}_0$ is a sufficiently large universal constant, and ε_1^* is as in Proposition 6. Using (165) and (167), we see that

$$R_0 \leq R_1 \leq R_0\lambda^{\mu\lambda^{\mu+1}} \leq R_0 e^{\mu\lambda^{\mu+2}} = R_0 e^{\mu\mathcal{M}^{11\mu+22}} \text{ and } \lambda^{\mu}R_1 \leq R_0 e^{2\mu\lambda^{\mu+2}} = R_0 e^{2\mu\mathcal{M}^{11\mu+22}}. \tag{173}$$

Using this and (164) gives that

$$\{x : R_1 < |x| < \lambda^{\mu}R_1\} \subset B(0, 6e^{e^{2\mathcal{M}}}) \subseteq B(0,R). \tag{174}$$

Here, R is as in Proposition 4. Combining (174) and (76), we deduce that (V,P) solves the Navier–Stokes equations on $\{x : R_1 < |x| < \lambda^{\mu}R_1\} \times (-1,0)$. This and (172) allow us to apply Proposition 6 to get that for $j = 0,1$,

$$\|\nabla^j V\|_{L^{\infty}(A\times(-\frac{1}{16},0))} \lesssim \mathcal{M}^{-3\mu} \quad \text{and} \quad \|\nabla\Omega\|_{L^{\infty}(A\times(-\frac{1}{16},0))} \lesssim \mathcal{M}^{-3\mu}. \tag{175}$$

From (164), (171), and (173), we see that for $R_2 := 2R_1$,

$$R_1 + 1 \leq R_2 < \frac{\lambda^{\mu}R_2}{4} \leq \lambda^{\mu}R_1 - 1 \quad \text{and} \quad 2R_0 \leq R_2 \leq 2R_0 e^{\mu\mathcal{M}^{11\mu+22}}. \tag{176}$$

We then see that (175) implies that for $j = 0,1$,

$$\begin{aligned} &\|\nabla^j V\|_{L^{\infty}_{x,t}(R_2<|x|<\frac{R_2\mathcal{M}^{11\mu}}{4}\times(-\frac{1}{16},0))} \lesssim \mathcal{M}^{-3\mu} \text{ and} \\ &\|\nabla\Omega\|_{L^{\infty}_{x,t}(R_2<|x|<\frac{R_2\mathcal{M}^{11\mu}}{4}\times(-\frac{1}{16},0))} \lesssim \mathcal{M}^{-3\mu} \end{aligned} \tag{177}$$

as required. ∎

6. Proof of Theorem 1. We will show that the assumptions of Theorem 1 imply that

$$\sup_{t\in[-e^{-e^{e^{\mathcal{M}^{1126}}}},0]} |v(0,t)| \leq e^{e^{e^{\mathcal{M}^{1126}}}}. \tag{178}$$

A translation and scaling argument (replacing $\mathcal{M}$ with $\mathcal{M}' = \frac{64}{49}\mathcal{M}$) then allows us to deduce the desired estimate (9) in Theorem 1.

From now on, we will consider (V,P), F, and R as in Proposition 4, which are derived from (v,p). Recall from Proposition 4 that

$$V(x,t) = \lambda v(\lambda x, \lambda^2 t) \quad \text{on} \quad B(0,R)\times[-2,0] \quad \text{and} \quad \|V\|_{L^{\infty}_t L^3_x(B(0,R)\times(-2,0))} \leq \mathcal{M} \tag{179}$$

with

$$R \geq 6e^{e^{2\mathcal{M}}} \quad \text{and} \quad \lambda = e^{-e^{2\mathcal{M}}} e^{-\mathcal{M}^5}. \tag{180}$$

Recall that (V,P) are smooth on $B(0,R)\times[-2,0]$, solve the Navier–Stokes equations on $B(0,R)\times[-2,0]$, and satisfy (75). Hence, we see that (V,P) obey the hypothesis (50) of Proposition 2 with M replaced by $\mathcal{M}^4$. Using (179)–(180) and Proposition 2, we see that in order to prove (178), it suffices to show the following for $\Omega:=\nabla\times V$. Namely,

$$\int\limits_{B(0,\mathcal{M}^{96}(-t_{\mathcal{M}})^{\frac12})} |\Omega(x,t_{\mathcal{M}})|^2dx \leq \frac{\mathcal{M}^{-88}}{(-t_{\mathcal{M}})^{\frac12}} \text{ with} \tag{181}$$

$$-\,t_{\mathcal{M}}:=\frac{\mathcal{M}^{-401}}{16}\exp\{-2\mathcal{M}^{1126}\exp\exp(\mathcal{M}^{1125})\}.$$

In order to show this, we assume that there exists

$$0<-t_0'<\frac{\mathcal{M}^{-401}}{8} \tag{182}$$

such that

$$\int\limits_{B(0,\mathcal{M}^{96}(-t_0')^{\frac12})} |\Omega(x,t_0')|^2\,dx > \frac{\mathcal{M}^{-88}}{(-t_0')^{\frac12}}. \tag{183}$$

We then show that this necessarily implies that

$$-t_0'\geq \frac{\mathcal{M}^{-401}}{8}\exp\{-2\mathcal{M}^{1126}\exp\exp(\mathcal{M}^{1125})\}, \tag{184}$$

which then gives (181).

Assuming (183), we see that Proposition 3 then implies that

$$\int\limits_{B(0,\mathcal{M}^{96}(-t'')^{\frac12})} |\Omega(x,t'')|^2\,dx > \frac{\mathcal{M}^{-88}}{(-t'')^{\frac12}} \tag{185}$$

at any well-separated backward time

$$t''\in[-\mathcal{M}^{-192},t_0'] \quad\text{such that}\quad \mathcal{M}^{200}t_0'>t''. \tag{186}$$

The rest of the proof relies on quantitative Carleman inequalities [37, Propositions 4.2–4.3].[8] These are the tools used to transfer the concentration information (185) from time t'' to time 0 and from the small scales $B(0,\mathcal{M}^{96}(-t'')^{\frac12})$ to larger scales, which are smaller than R in Proposition 4.

The proof of Theorem 1 using quantitative Carleman inequalities closely follows [7], which is a physical space analogue of Tao's strategy [37]. In order to not to fully repeat parts of the proof in [7], we provide a detailed sketch. This allows the reader with [7] in hand to recover all the details. Throughout the detailed sketch, C denotes a positive universal constant, which may vary from line to line unless otherwise specified.

Step 1: Quantitative unique continuation. The purpose of this step is to prove the estimate

$$T_1^{\frac12}e^{-\frac{C\mathcal{M}^{147}R_*^2}{T_1}} \lesssim \int\limits_{-T_1}^{-\frac{T_1}{2}}\int\limits_{B(0,2R_*)\setminus B(0,R_*/2)} |\Omega(x,t)|^2\,dxdt \tag{187}$$

for all T_1 and R_* such that

[8]For reasons relating to notation, we will refer to [7, Propositions B.1–B.2], which are directly taken from Tao's paper [37, Propositions 4.2–4.3] but adopt different notation.

$$2\mathcal{M}^{200}(-t_0')<T_1<\mathcal{M}^{-192} \quad \text{and} \quad R\geq e^{e^{\mathcal{M}}}\geq R_*\geq \mathcal{M}^{100}\Big(\frac{T_1}{2}\Big)^{\frac12}. \tag{188}$$

Denote $I_1:=(-T_1,-\frac{T_1}{2})$ and $T_1:=-t_0''$. We see that (188) implies that (186) is satisfied for every $t\in(-T_1,-\frac{T_1}{2})$. Hence, (185) is satisfied for every $t\in(-T_1,-\frac{T_1}{2})$. Using that $(-T_1,-\frac{T_1}{2})\subset[-1,0]\subset[-2,0]$, we can apply Proposition 5. This implies that there exists an epoch of regularity $I_1''=[t_1''-T_1'',t_1'']\subset I_1$ such that

$$T_1''=|I_1''|=\mathcal{M}^{-46}|I_1|=\mathcal{M}^{-46}\tfrac{T_1}{2} \tag{189}$$

and for $j=0,1,2$,

$$\|\nabla^jV\|_{L^\infty_{x,t}(B(0,3e^{e^{2\mathcal{M}}})\times I_1'')}\lesssim\frac{1}{\mathcal{M}}|I_1''|^{\frac{-(j+1)}{2}}=\frac{1}{\mathcal{M}}(T_1'')^{\frac{-(j+1)}{2}}. \tag{190}$$

Let $T_1''':=\frac34T_1''$ and $s''\in[t_1''-\frac{T_1''}{4},t_1'']$. Let $x_1\in\mathbb{R}^3$ be such that $e^{e^{\mathcal{M}}}\geq|x_1|\geq\mathcal{M}^{100}(\frac{T_1}{2})^{\frac12}$, and let $r_1:=\mathcal{M}^{50}|x_1|\geq\mathcal{M}^{150}(\frac{T_1}{2})^{\frac12}$. Notice that for $\mathcal{M}$ large enough,

$$r_1:=\mathcal{M}^{50}|x_1|\geq\mathcal{M}^{150}\Big(\frac{T_1}{2}\Big)^{\frac12}\geq\mathcal{M}^{53}(\mathcal{M}^{96}(-t_0'')^{\frac12}) \quad \text{and} \quad r_1^2\geq 4000T_1'''. \tag{191}$$

Notice that

$$B(x_1,r_1)\subset B(0,\mathcal{M}^{51}|x_1|)\subset B(0,3e^{e^{2\mathcal{M}}}). \tag{192}$$

Using the above facts, we can apply the Carleman inequality corresponding to quantitative unique continuation [7, Proposition B.2] on the cylinder $\mathcal{C}_1=\{(x,t)\in\mathbb{R}^3\times\mathbb{R}: t\in[0,T_1'''],\ |x|\leq r_1\}$. We apply this to the function $w:\mathcal{C}_1\to\mathbb{R}^3$, defined for all $(x,t)\in\mathcal{C}_1$ by

$$w(x,t):=\Omega(x_1+x,s''-t),$$

with

$$r=r_1,\quad S=S_1:=T_1''',\quad C_{Carl}=\tfrac{16}{3},\quad \overline{s}_1=\frac{T_1'''}{20000},\quad \text{and}\quad \underline{s}_1=\mathcal{M}^{-150}T_1'''\leq\frac{T_1'''}{10000}.$$

This gives

$$Z_1\lesssim e^{-\frac{r_1^2}{500\overline{s}_1}}X_1+(\overline{s}_1)^{\frac32}\Big(\frac{e\overline{s}_1}{\underline{s}_1}\Big)^{\frac{Cr_1^2}{\overline{s}_1}}Y_1, \tag{193}$$

where

$$X_1:=\int\limits_{s''-T_1'''}^{s''}\int\limits_{B(x_1,\mathcal{M}^{50}|x_1|)}((T_1''')^{-1}|\Omega|^2+|\nabla\Omega|^2)\,dxds,$$

$$Y_1:=\int\limits_{B(x_1,\mathcal{M}^{50}|x_1|)}|\Omega(x,s'')|^2(\underline{s}_1)^{-\frac32}e^{-\frac{|x-x_1|^2}{4\underline{s}_1}}\,dx,$$

$$Z_1:=\int\limits_{s''-\frac{T_1'''}{10000}}^{s''-\frac{T_1'''}{20000}}\int\limits_{B(x_1,\frac{\mathcal{M}^{50}|x_1|}{2})}((T_1''')^{-1}|\Omega|^2+|\nabla\Omega|^2)e^{-\frac{|x-x_1|^2}{4(s''-s)}}\,dxds.$$

By (191), we have

$$B(0,\mathcal{M}^{96}(-s)^{\frac12})\subset B(0,\mathcal{M}^{96}(-t_0'')^{\frac12})\subset B(0,|x_1|)\subset B(x_1,2|x_1|)\subset B\Big(x_1,\frac{\mathcal{M}^{50}|x_1|}{2}\Big)$$

for all $s\in[s''-\frac{T_1'''}{10000},s''-\frac{T_1'''}{20000}]$ and for $\mathcal{M}$ sufficiently large. Combining this with (185) gives

$$Z_1\gtrsim\int\limits_{s''-\frac{T_1'''}{10000}}^{s''-\frac{T_1'''}{20000}}\int\limits_{B(0,\mathcal{M}^{96}(-s)^{\frac12})}(T_1''')^{-1}|\Omega(x,s)|^2\,dxds\,e^{-\frac{C|x_1|^2}{T_1'''}}\gtrsim\mathcal{M}^{-111}(T_1'')^{-\frac12}e^{-\frac{C|x_1|^2}{T_1''}}.$$

Using (192), we use the quantitative regularity (190) to obtain

$$e^{-\frac{r_1^2}{500\overline{s}_1}}X_1\lesssim(T_1'')^{-2}\mathcal{M}^{150}|x_1|^3e^{-\frac{C\mathcal{M}^{100}|x_1|^2}{T_1''}}\lesssim(T_1'')^{-\frac12}e^{-\frac{C\mathcal{M}^{100}|x_1|^2}{T_1''}}$$

and

$$(\overline{s}_1)^{\frac32}\Big(\frac{e\overline{s}_1}{\underline{s}_1}\Big)^{\frac{Cr_1^2}{\overline{s}_1}}Y_1\lesssim\ \mathcal{M}^{225}e^{\frac{C\mathcal{M}^{101}|x_1|^2}{T_1''}}\int\limits_{B(x_1,\frac{|x_1|}{2})}|\Omega(x,s'')|^2\,dx$$
$$+\mathcal{M}^{225}(T_1'')^{-\frac12}e^{-\frac{C\mathcal{M}^{150}|x_1|^2}{T_1''}}.$$

Gathering these bounds and combining with (193) yields

$$\mathcal{M}^{-111}(T_1'')^{-\frac12}e^{-\frac{C|x_1|^2}{T_1''}}\lesssim(T_1'')^{-\frac12}e^{-\frac{C\mathcal{M}^{100}|x_1|^2}{T_1''}}$$
$$+\mathcal{M}^{225}e^{\frac{C\mathcal{M}^{101}|x_1|^2}{T_1''}}\int\limits_{B(x_1,\frac{|x_1|}{2})}|\Omega(x,s'')|^2\,dx+\mathcal{M}^{225}(T_1'')^{-\frac12}e^{-\frac{C\mathcal{M}^{150}|x_1|^2}{T_1''}}.$$

Using (189) and $e^{e^{\mathcal{M}}}\geq|x_1|\geq\mathcal{M}^{100}(\frac{T_1}{2})^{\frac12}$, we see that for $\mathcal{M}$ sufficiently large,

$$\mathcal{M}^{-336}(T_1'')^{-\frac12}e^{-\frac{C\mathcal{M}^{101}|x_1|^2}{T_1''}}\lesssim\int\limits_{B(x_1,\frac{|x_1|}{2})}|\Omega(x,s'')|^2\,dx.$$

Hence, for all $s''\in[t_1''-\frac{T_1''}{4},t_1'']$, for all $e^{e^{\mathcal{M}}}\geq|x_1|\geq\mathcal{M}^{100}(\frac{T_1}{2})^{\frac12}$,

$$\int\limits_{B(x_1,\frac{|x_1|}{2})}|\Omega(x,s'')|^2\,dx\gtrsim\mathcal{M}^{-336}(T_1'')^{-\frac12}e^{-\frac{C\mathcal{M}^{101}|x_1|^2}{T_1''}}=\mathcal{M}^{-336}(T_1'')^{-\frac12}e^{-\frac{2C\mathcal{M}^{147}|x_1|^2}{T_1}}.$$

Let $e^{e^{\mathcal{M}}}\geq R_*\geq\mathcal{M}^{100}(\frac{T_1}{2})^{\frac12}$ and $x_1\in\mathbb{R}^3$ be such that $|x_1|=R_*$. Integrating in time over $[t_1''-\frac{T_1''}{4},t_1'']$ yields the claim (187) of Step 1.

Step 2: Quantitative backward uniqueness. The goal of this step and Step 3 below is to prove the claim

$$T_2^{-\frac12}\exp\big(-\exp(\mathcal{M}^{1123})\big)\lesssim\int\limits_{B\big(0,\frac{3}{16}\mathcal{M}^{1100}R_2\big)\setminus B(0,2R_2)}|\Omega(x,0)|^2\,dx\tag{194}$$

for all $8\mathcal{M}^{401}(-t_0') < T_2 \le 1$ and $\mathcal{M}$ sufficiently large. Here, R_0 and R_2 are as in (196)–(198). This is the key estimate for Step 4 below and the proof of Theorem 1.

Let T_1 and T_2 be such that

$$8\mathcal{M}^{401}(-t_0') < T_2 \le 1 \quad \text{and} \quad T_1 := \frac{T_2}{4\mathcal{M}^{201}}. \tag{195}$$

Let

$$R_0 := K^\sharp (T_2)^{\frac{1}{2}} \tag{196}$$

for a universal constant $K^\sharp \ge 1$ to be chosen sufficiently large below. In particular, it is chosen in Step 3 such that (214) holds, which makes it possible to absorb the upper bound (213) of X_3 in the left-hand side of (212). From (196), we see that for $\mathcal{M}$ sufficiently large, we have that

$$T_2^{\frac{1}{2}} \le R_0 \le T_2^{\frac{1}{2}} 6e^{e^{2\mathcal{M}}} e^{-400\mathcal{M}^{1122}}.$$

By Proposition 7, for $\mathcal{M} \ge \lambda_0(100)$, there exists a scale

$$2R_0 \le R_2 \le 2R_0 e^{100\mathcal{M}^{1122}} \tag{197}$$

and a good cylindrical annulus

$$\mathcal{A}_2 := \Big\{ R_2 < |x| < \tfrac{\mathcal{M}^{1100} R_2}{4} \Big\} \times \Big(-\frac{T_2}{16}, 0 \Big) \tag{198}$$

such that for $j = 0, 1$,

$$\|\nabla^j V\|_{L^\infty(\mathcal{A}_2)} \le \bar{C}_j \mathcal{M}^{-300} T_2^{-\frac{j+1}{2}}, \quad \|\nabla \Omega\|_{L^\infty(\mathcal{A}_2)} \le \bar{C}_2 \mathcal{M}^{-300} T_2^{-\frac{3}{2}}. \tag{199}$$

Let

$$\widetilde{\mathcal{A}}_2 := \Big\{ 4R_2 \le |x| \le \frac{\mathcal{M}^{1100} R_2}{16} \Big\} \times \Big[-\frac{T_2}{\mathcal{M}^{201}}, 0 \Big] \subseteq B(0,R) \times \Big[-\frac{T_2}{\mathcal{M}^{201}}, 0 \Big], \tag{200}$$

and choose $\mathcal{M}$ sufficiently large such that $\bar{C}_j \mathcal{M}^{-300} \le 1$ $(j = 0,1)$ and $\bar{C}_2 \mathcal{M}^{-300} \le 1$. Using (200), we see that (V, P) solve the Navier–Stokes equations on $\widetilde{\mathcal{A}}_2$.

The above facts allow us to apply the Carleman inequality corresponding to quantitative backward uniqueness [7, Proposition B.1] to the function $w : \widetilde{\mathcal{A}}_2 \to \mathbb{R}^3$ defined for all $(x,t) \in \widetilde{\mathcal{A}}_2$ by

$$w(x,t) = \Omega(x, -t)$$

with

$$r_- := 4R_2, \quad r_+ := \tfrac{1}{16}\mathcal{M}^{1100} R_2, \quad S = S_2 := \frac{T_2}{\mathcal{M}^{201}}, \quad \text{and} \quad C_{Carl} = \mathcal{M}^{201}.$$

This gives

$$Z_2 \lesssim e^{-\frac{C\mathcal{M}^{1100}(R_2)^2}{T_2}} \Big(X_2 + e^{\frac{C\mathcal{M}^{2200}(R_2)^2}{T_2}} Y_2 \Big), \tag{201}$$

where

$$X_2 := \int\limits_{-\frac{T_2}{\mathcal{M}^{201}}}^{0} \int\limits_{r_- \leq |x| \leq r_+} e^{\frac{4|x|^2}{T_2}} (\mathcal{M}^{201} T_2^{-1} |\Omega|^2 + |\nabla \Omega|^2)\, dxdt,$$

$$Y_2 := \int\limits_{r_- \leq |x| \leq r_+} |\Omega(x,0)|^2\, dx,$$

$$Z_2 := \int\limits_{-\frac{T_2}{4\mathcal{M}^{201}}}^{0} \int\limits_{10 r_- \leq |x| \leq \frac{r_+}{2}} (\mathcal{M}^{201} T_2^{-1} |\Omega|^2 + |\nabla \Omega|^2)\, dxdt.$$

Notice that for $\mathcal{M}$ sufficiently large, we have that

$$B(0,40r_-)\backslash B(0,10r_-) \subset \{10r_- \leq |x| \leq \tfrac{r_+}{2}\}.$$

Next, we use this, the separation condition (195), and the fact that for $\mathcal{M}$ large enough, (196) implies that

$$e^{e^{\mathcal{M}}} \geq 20r_- \geq 10R_2 \geq 20R_0 = 20K^\sharp T_2^{\frac{1}{2}} \geq \mathcal{M}^{100}\Big(\frac{T_2}{8\mathcal{M}^{201}}\Big)^{\frac{1}{2}} = \mathcal{M}^{100}\Big(\frac{T_1}{2}\Big)^{\frac{1}{2}}.$$

This gives that we can apply the concentration result of Step 1, taking there $T_1 = \frac{T_2}{4\mathcal{M}^{201}} = \frac{S_2}{4}$ and $R_* = 20r_-$. By (187), we have that

$$Z_2 \gtrsim \mathcal{M}^{201}\left(\frac{T_2}{4\mathcal{M}^{201}}\right)^{\frac{1}{2}} e^{-\frac{C\mathcal{M}^{348}(R_2)^2}{T_2}} T_2^{-1} \gtrsim T_2^{-\frac{1}{2}} e^{-\frac{C\mathcal{M}^{348}(R_2)^2}{T_2}}. \tag{202}$$

Therefore, one of the following two lower bounds holds:

$$T_2^{-\frac{1}{2}} \exp\Big(\frac{C\mathcal{M}^{1100}(R_2)^2}{T_2}\Big) \lesssim X_2, \tag{203}$$

$$T_2^{-\frac{1}{2}} \exp(-\exp(\mathcal{M}^{1123})) \lesssim e^{-\frac{C\mathcal{M}^{2200}(R_2)^2}{T_2}} T_2^{-\frac{1}{2}} \lesssim Y_2, \tag{204}$$

where we used the upper bound (197) for (204). The bound (204) can be used directly in Step 4 below. On the contrary, if (203) holds, more work needs to be done to transfer the lower bound on the enstrophy at time 0. This is the objective of Step 3 below.

Step 3: A final application of quantitative unique continuation. Assume that the bound (203) holds. The same reasoning as in [37] and the subsequent paper [7] (involving the pigeonhole principle three times) gives the following. There exists

$$8R_2 \leq R_3 \leq \tfrac{1}{8}\mathcal{M}^{1100} R_2, \tag{205}$$

$$\frac{1}{2}\exp\Big(-\frac{8R_3^2}{T_2}\Big) T_2 \leq -t_3 \leq \frac{T_2}{\mathcal{M}^{201}}, \tag{206}$$

and $x_3 \in B(0,R_3)\backslash B(0,\frac{R_3}{2})$ such that

$$T_2^{-\frac{1}{2}} \exp\Big(-\frac{18R_3^2}{T_2}\Big) \lesssim \int\limits_{2t_3}^{t_3} \int\limits_{B(x_3,(-t_3)^{\frac{1}{2}})} (T_2^{-1}|\Omega|^2 + |\nabla\Omega|^2)\, dxdt. \tag{207}$$

We apply now the Carleman inequality corresponding to quantitative unique continuation [7, Proposition B.2]. We apply this to the function $w: \overline{B(0,r)}\times[0,-20000t_3]\to\mathbb{R}^3$ defined for all $(x,t)\in\overline{B(0,r)}\times[0,-20000t_3]$ by

$$w(x,t)=\Omega(x+x_3,-t)$$

with

(208)
$$S=S_3:=-20000t_3,\quad r=r_3:=1000R_3\Big(-\frac{t_3}{T_2}\Big)^{\frac12},\quad \overline{s}_3=\underline{s}_3=-t_3,\quad \text{and}\quad C_{Carl}=1.$$

This gives

$$Z_3\lesssim e^{\frac{r_3^2}{500t_3}}X_3+(-t_3)^{\frac32}e^{-\frac{Cr_3^2}{t_3}}Y_3, \tag{209}$$

where

$$X_3:=\int_{-S_3}^{0}\int_{B(x_3,r_3)}(S_3^{-1}|\Omega|^2+|\nabla\Omega|^2)\,dxdt,\quad Y_3:=\int_{B(x_3,r_3)}|\Omega(x,0)|^2(-t_3)^{-\frac32}e^{\frac{|x-x_3|^2}{4t_3}}\,dx,$$

$$Z_3:=\int_{2t_3}^{t_3}\int_{B(x_3,\frac{r_3}{2})}(S_3^{-1}|\Omega|^2+|\nabla\Omega|^2)e^{\frac{|x-x_3|^2}{4t}}\,dxdt.$$

Using (196)–(197) and (205)–(206), we have that

$$\begin{aligned}B(x_3,(-t_3)^{\frac12})\subset B(x_3,\tfrac{r_3}{2})\subset B(x_3,r_3)\subset B\Big(x_3,\frac{|x_3|}{2}\Big)\\ \subset\{\tfrac{R_3}{4}<|y|<\tfrac32R_3\}\subset\{2R_2<|y|<\tfrac{3}{16}\mathcal{M}^{1100}R_2\}.\end{aligned} \tag{210}$$

Using this, (207), and $T_2^{-1}\le S_3^{-1}$, we have

$$T_2^{-\frac12}\exp\Big(-\frac{18R_3^2}{T_2}\Big)\lesssim\int_{2t_3}^{t_3}\int_{B(x_3,(-t_3)^{\frac12})}(T_2^{-1}|\Omega|^2+|\nabla\Omega|^2)e^{\frac{|x-x_3|^2}{4t}}\,dxdt\le Z_3. \tag{211}$$

Hence, we have

$$T_2^{-\frac12}\exp\Big(-\frac{18R_3^2}{T_2}\Big)\le C_{univ}e^{\frac{r_3^2}{500t_3}}X_3+C_{univ}(-t_3)^{\frac32}e^{-\frac{Cr_3^2}{t_3}}Y_3, \tag{212}$$

where C_{univ} is a positive universal constant. Using the bounds (199) along with (206) gives

$$\begin{aligned}C_{univ}e^{\frac{r_3^2}{500t_3}}X_3\lesssim S_3^{-2}r_3^3e^{\frac{r_3^2}{500t_3}}\lesssim(-t_3)^{-\frac12}e^{\frac{r_3^2}{1000t_3}}\lesssim T_2^{-\frac12}e^{-\frac{996R_3^2}{T_2}}\\ \le C'_{univ}T_2^{-\frac12}e^{-\frac{18R_3^2}{T_2}}e^{-978\cdot256(K^\sharp)^2}.\end{aligned} \tag{213}$$

As in [7], we fix $K^\sharp$ to be a sufficiently large universal constant such that

$$C'_{univ}e^{-978\cdot256(K^\sharp)^2}\le\frac12, \tag{214}$$

where $C'_{univ} \in (0,\infty)$ is the universal constant appearing in (213). Therefore, the term $C_{univ}e^{\frac{r_3^2}{500t_3}}X_3$ can be absorbed into the left-hand side of (212). From this and (212), we obtain

$$T_2^{-\frac{1}{2}}\exp\Big(-C\frac{R_3^2}{T_2}\Big) \lesssim \int\limits_{B(x_3,r_3)} |\Omega(x,0)|^2\,dx.$$

Using (196), the upper bound

$$R_3 \le \tfrac{1}{8}\mathcal{M}^{1100}R_2 \le \tfrac{1}{4}\mathcal{M}^{1100}\exp(100\mathcal{M}^{1122})R_0,$$

and (210), it follows that

$$T_2^{-\frac{1}{2}}\exp(-\exp(\mathcal{M}^{1123})) \lesssim \int\limits_{B\big(0,\frac{3}{16}\mathcal{M}^{1100}R_2\big)\setminus B(0,2R_2)} |\Omega(x,0)|^2\,dx. \tag{215}$$

Step 4: Conclusion: Summing the scales and lower bound for the global L^3 norm. Next, we use (215) and argue in a similar way to [7].[9] In particular, the pigeonhole principle, the quantitative estimate (199), integration by parts, and Hölder's inequality gives that

$$\int\limits_{B\big(0,\exp(\mathcal{M}^{1123})T_2^{\frac{1}{2}}\big)\setminus B\big(0,T_2^{\frac{1}{2}}\big)} |V(x,0)|^3\,dx \ge \exp\big(-\exp(\mathcal{M}^{1125})\big) \tag{216}$$

for all $8\mathcal{M}^{401}(-t'_0) \le T_2 \le 1$. We note that this holds for $T_2 = 8\mathcal{M}^{401}(-t'_0)$ using that V is smooth.

We then sum (216) over k scales starting with $T_2 = 8\mathcal{M}^{401}(-t'_0)$, where

$$k := \left\lfloor \tfrac{1}{2}\mathcal{M}^{-1123}\log\left(\frac{\mathcal{M}^{-401}}{8(-t'_0)}\right) + 1 \right\rfloor. \tag{217}$$

This gives that

$$\begin{aligned}&\exp\big(-\exp(\mathcal{M}^{1125})\big)\tfrac{1}{2}\mathcal{M}^{-1123}\log\left(\frac{\mathcal{M}^{-401}}{8(-t'_0)}\right)\\ &\le \int\limits_{B(0,\exp(\mathcal{M}^{1123}))\setminus B(0,(8\mathcal{M}^{401}(-t'_0))^{\frac{1}{2}})} |V(x,0)|^3\,dx \le \mathcal{M}^3.\end{aligned}$$

Rearranging this gives the lower bound (184) as required.

7. Proof of Theorem 2.

Without loss of generality, assume that $0 < \delta < \min(\mathrm{dist}(x_0,\partial B(4)),(\frac{1}{2}T_*)^{\frac{1}{2}})$. Assume that the hypothesis of Theorem 2 holds with the contrapositive of (10). Then there exists $N \in (0,\infty)$ such that

$$\sup_{0<s<t}\|v(\cdot,s)\|_{L^3(B(x_0,\delta))} \le N\Big(\log\log\log\Big(\frac{1}{(T_*-t)^{\frac{1}{4}}}\Big)\Big)^{\frac{1}{1129}} \quad \forall t\in(0,T_*). \tag{218}$$

Next, we fix $\varepsilon_0 > 0$ such that

[9] These arguments are in turn based on Tao's arguments in [37].

$$\frac{N}{\left(\log\log\log\left((\varepsilon_0)^{-\frac{1}{4}}\right)\right)^{\frac{1}{1128}-\frac{1}{1129}}} \leq 1 \quad \text{and} \tag{219}$$

$$\left(\log\log\log\left((\varepsilon_0)^{-\frac{1}{4}}\right)\right)^{\frac{1}{1128}} \geq \max\left\{\mathcal{M}_0, 16\delta^{-2}\|p\|^{\frac{3}{2}}_{L^{\frac{3}{2}}_{x,t}(B(0,4)\times(0,T_*))}\right\}. \tag{220}$$

Here, $\mathcal{M}_0$ is as in Theorem 1. Then (218)–(219) gives that

$$\sup_{0<s<t}\|v(\cdot,s)\|_{L^3(B(x_0,\delta))} \leq \left(\log\log\log\left(\frac{1}{(T_*-t)^{\frac{1}{4}}}\right)\right)^{\frac{1}{1128}} \quad \forall t \in (T_*-\varepsilon_0, T_*). \tag{221}$$

From now on, we take any $t \in (\max(\frac{1}{2}T_*, T_*-\varepsilon_0), T_*)$. Next, we rescale

$$(V_{x_0,\delta}(y,s), P_{x_0,\delta}(y,s)) := (\tfrac{\delta}{4}v(\tfrac{\delta}{4}x+x_0, \tfrac{\delta^2}{16}s+t), \tfrac{\delta^2}{16}p(\tfrac{\delta}{4}x+x_0, \tfrac{\delta^2}{16}s+t)).$$

Then $(V_{x_0,\delta}, P_{x_0,\delta})$ is a smooth solution to the Navier–Stokes equations in $B(0,4)\times[-16,0]$. By (221), we have

$$\|V_{x_0,\delta}\|_{L^\infty_t L^3_x(B(0,4)\times(-16,0))} \leq \mathcal{M} := \left(\log\log\log\left(\frac{1}{(T_*-t)^{\frac{1}{4}}}\right)\right)^{\frac{1}{1128}}. \tag{222}$$

By (220), we have that

$$\max\left\{\mathcal{M}_0, \|P_{x_0,\delta}\|^{\frac{3}{2}}_{L^{\frac{3}{2}}_{x,t}(Q(0,4))}\right\} \leq \mathcal{M}. \tag{223}$$

From (222)–(223), we see that we can apply Theorem 1. This gives

$$\tfrac{\delta}{4}\|v(\cdot,t)\|_{L^\infty(B(x_0,\frac{\delta}{8}))} = \|V_{x_0,\delta}(\cdot,0)\|_{L^\infty(B(0,\frac{1}{2}))} \leq \exp(\exp(\exp(\mathcal{M}^{1128})) = \frac{1}{(T_*-t)^{\frac{1}{4}}}.$$

So (218) implies that

$$\|v(\cdot,t)\|_{L^\infty(B(x_0,\frac{\delta}{8}))} \leq \frac{4}{\delta(T_*-t)^{\frac{1}{4}}} \quad \forall t \in (\max(\tfrac{1}{2}T_*, T_*-\varepsilon_0), T_*). \tag{224}$$

This implies that

$$v \in L^2_t L^\infty_x(B(x_0, \tfrac{\delta}{8}) \times (\max(\tfrac{1}{2}T_*, T_*-\varepsilon_0), T_*)). \tag{225}$$

By [35, Theorem 7], this implies that

$$v \in L^\infty_{x,t}(B(x_0,r)\times(T_*-r^2, T_*))$$

for all sufficiently small $r>0$. This violates the hypothesis of Theorem 2. Hence, the contrapositive (218) cannot hold, and we get the desired conclusion. ∎

Acknowledgement. The author thanks Dallas Albritton and Christophe Prange for helpful discussions. The author also wishes to thank Armen Shirikyan, Jin Tan, and Nikolay Tzvetkov, whose questions during a talk of the author prompted me to write down these results. The author thanks the anonymous referees for their valuable comments and suggestions.

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