Unified Fair Allocation of Goods and Chores via Copies

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We consider fair allocation of indivisible items in a model with goods, chores, and copies, as a unified framework for studying: (1) the existence of EFX and other solution concepts for goods with copies; (2) the existence of EFX and other solution concepts for chores. We establish a tight relation between these issues via two conceptual contributions: First, a refinement of envy-based fairness notions that we term envy without commons (denoted EFX\(\text{WC}\) when applied to EFX). Second, a formal duality theorem relating the existence of a host of (refined) fair allocation concepts for copies to their existence for chores. We demonstrate the usefulness of our duality result by using it to characterize the existence of EFX for chores through the dual environment, as well as to prove EFX existence in the special case of leveled preferences over the chores. We further study the hierarchy among envy-freeness notions without commons and their \(\alpha\)-MMS guarantees, showing for example that any EFX\(\text{WC}\) allocation guarantees at least \(\frac{4}{11}\)\(\alpha\)-MMS for goods with copies.

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1 INTRODUCTION

This work studies the relation between the existence of a fair allocation of indivisible items among agents – for various notions of fairness – in two different models that at first blush do not seem tightly related. The first model has indivisible goods with multiple copies of each good. The second model has indivisible chores (or “bads”), which are items with negative values.

Goods with copies. While the seminal work of Budish [18] on allocation of indivisible items includes copies as a key ingredient (motivated by important applications like the allocation of university courses to students), most subsequent works focus on a single copy of each item [see, e.g., 19]. A notable exception is [30], where they consider goods with multiple copies, where each agent may also receive more than one copy of a certain type of good. We consider a model where each agent may receive at most one copy of each good; we term such allocations “exclusive” (they are termed “valid” in [36], or “at most one” allocations in [35]). Our setting captures applications like course allocation, where a student cannot be allocated multiple seats in the same course. Another example is digital goods with a license quota (e.g., a university with X software licenses that allocates at most one license per student). Interestingly, even if a “fair” allocation (for an appropriate definition of fairness) is guaranteed to hold when each good has a single copy, this may significantly change when multiple copies are available. Considering fair allocation of goods with copies thus raises many open questions.

Chores. Fair allocation of chores is a topic of much recent research [e.g., 34]. The literature usually treats chores separately from goods [e.g., 4, 22, 28], and our current understanding of chores is...
lacking. For example, while it is known that an EFX allocation\(^1\) always exists for any number of goods (without copies) and three agents [22], almost nothing is known regarding EFX allocations of chores, and this is a major open problem. While this difference is a knowledge gap, other gaps provably hold. E.g., for goods, every allocation with MNW (maximum Nash welfare) satisfies the “envy bounded by a single item” (EF1) fairness notion as well as Pareto efficiency [19]. In contrast, for chores, there is no single valued welfare function whose maximization implies both fairness and efficiency [17].\(^2\)

**Relating the models.** Given the above, one may conclude that studying existence of solution concepts for fair allocation of goods with copies and of chores must be achieved by two disparate lines of research. However, a main contribution of this paper is to formalize a tight relation among the models, via a duality theorem that we develop for a large class of fairness notions. The connection between goods and chores has been informally alluded to in the past by Bogomolnaia et al. [17], who give a nice intuitive description: “Say that we must allocate 5 hours of a painful job […] Working 2 hours [on the job] is the same as being exempt [from the job] for 3 hours”. Very recently, Kulkarni et al. [36] employed this connection in their proofs. Our goal in this paper is to develop a formal treatment of this duality in the context of fair allocation existence, in order to enable a comprehensive exploration of the fundamental connection between fair allocations of goods, chores, and their copies.

### 1.1 Our Results

We now describe our duality results via a main motivating application.

**Main application.** Consider the well-studied fairness notion of EFX, in which each agent \(i\) prefers her own bundle over the bundle of any other agent \(j\), if one arbitrary good is removed from \(j\)'s bundle (in the related EF1 notion, the most valued good is removed from \(j\)’s bundle). As standard in the literature, we assume additive values/costs, and study exclusive allocations. In light of the recent progress on existence of EFX for goods, culminating with the result of [22] for three agents, a natural question is whether EFX existence persists in our first model of interest – goods with copies. The result in [23] finds that EFX exists for a mix of goods and chores with general but identical valuations (using the leximin allocation). We show (in Section 3) that an EFX allocation does not always exist for goods with multiple copies, and that this negative result holds even in a simple setting with any number \(n \geq 3\) of agents and identical values (see Example 1). This demonstrates how adding copies can completely change the fairness landscape.

To circumvent the underlying reason for inexistence with copies, we introduce a new fairness notion that we term “EFX without commons”, or “EFX\(_{WC}\)” for short. EFX\(_{WC}\) requires each agent \(i\) to prefer her own bundle over the bundle of any other agent \(j\), if an arbitrary good which is not also in \(i\)'s bundle is removed from \(j\)’s bundle. Intuitively, an agent compares herself to her peers while putting aside the intersection of their bundles, i.e., goods common to both. Where does EFX\(_{WC}\) fit in with other solution concepts? In the special case of a single copy per good, EFX\(_{WC}\) coincides with EFX. With multiple copies, EFX\(_{WC}\) is a strictly weaker fairness requirement than EFX (i.e., EFX\(_{WC}\) is implied by EFX but not vice versa). On the other hand, EFX\(_{WC}\) is a strictly stronger fairness requirement than EF1. This immediately raises the question of whether an EFX\(_{WC}\) allocation (unlike EFX) is guaranteed to exist for goods with copies.

At this point, the second model we study – fair allocation of chores – becomes relevant. We show that in any fair allocation setting where \(n\) is the number of agents, an EFX\(_{WC}\) allocation for goods with \(n-1\) copies per good exists if and only if an EFX allocation for chores exists when the

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\(^1\)In an EFX (resp. EF1) allocation, the envy among any two agents can be eliminated by removing any (resp. some) good from the envied bundle – see Section 2 for formal fairness definitions.

\(^2\)This phenomenon persists in the divisible case, where there is a polytime algorithm to find a competitive division for goods, but the same problem is PPAD-hard for chores [21].

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valuations are treated as costs (see Corollary 1). Thus, we have established a new characterization for the important open problem of existence of EFX for chores. This characterization “justifies” our new EFXWC notion, since its existence determines that of the well-studied EFX notion for chores.

**Duality theorems.** The characterization result for EFX with chores is obtained as a corollary of our duality “meta-theorem” (Theorem 2 in Section 4), connecting goods with copies on the one hand, and chores with copies on the other. This meta-theorem shows, for example, that any allocation of goods with copies is EFXWC if and only if its dual allocation for chores with copies is EFXWC. Since the dual allocation for goods with \(n-1\) copies is an allocation with one copy per chore, the characterization result follows. The meta-theorem is formulated in a general way that fits other envy-based solution concepts as well, and a similar duality meta-theorem (Theorem 3) applies to share-based solution concepts.\(^3\) The duality meta-theorems are in fact flexible enough to apply to settings with a mixture of goods and chores (see Remark 1).

**Fairness notions “without commons”.** In Section 5 we further explore “without commons” fairness, since its connection to EFX for chores seems to indicate it is the “right” notion to study for copies, and since disregarding common items is a natural way to limit envy among agents. We define similar notions to EFXWC for other solution concepts beyond EFX, in particular EF1WC and EFLWC.\(^4\) We study how these new concepts expand the hierarchy of envy-based fairness notions. Expansion of the hierarchy is important for two reasons: First, intermediate “without commons” fairness notions can serve as a technical stepping stone for making progress on the elusive existence of more standard notions. Second, as we show, “without commons” envy-freeness has good share-based fairness guarantees,\(^5\) especially in comparison to standard envy-freeness concepts like EF1. We thus believe such concepts to be of independent interest. Our results in this section are summarized in Figure 1.

Returning to our main application, while the general question of EFXWC existence for goods with copies is as hard as EFX for chores and thus beyond our current scope, Section 6 provides two important existence results for general goods with copies: Section 6.1 generalizes known results for goods and chores EF1 existence into EF1WC existence for identically ordinal preferences for general chores with copies. Section 6.2 provides a novel EFXWC existence result for the special case of leveled preferences [8, 40], where larger bundles are always preferred to smaller ones. This immediately implies the existence of EFX for chores with leveled preferences, which was not previously known, demonstrating the usefulness of our duality results.

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\(^3\)In share-based fairness, an agent compares her allocation to her “fair share” of the whole rather than to her peers’; see Section 2.

\(^4\)EFL (the basis for EFLWC) is due to [9].

\(^5\)As measured by their approximation of MMS, the maximin share of every agent [18].
1.2 Additional Related Literature

Solution concepts. An EF1 allocation is guaranteed to exist for goods without copies [39], for goods with copies [15], and for chores [3, 14]. Envy-freeness up to any item (EFX) existence is still a wide open problem for \( n \geq 4 \) agents and unrestricted additive valuations.\(^6\) An MMS allocation does not always exist, as was observed by Kurokawa et al. [37], but approximations of it may be guaranteed, prompting the notion of \( \alpha \)-MMS. Their work guarantees existence of \( \frac{3}{4} \)-MMS, and further works guarantee \( \frac{3}{4} \)-MMS or more for goods, depending on the number of agents \([28, 29]\). Bei et al. [12] extend this discussion to a mix of divisible and indivisible goods. For chores, Aziz et al. [5] show that MMS does not always exist for chores, and Huang and Lu [34] show the existence of a \( \frac{3}{5} \)-MMS approximation. The envy-free concept of “EFL”, which we find useful as an intermediary to argue about EFX, was introduced by Barman et al. [9], who show its existence for additive goods. For goods, envy-freeness notions can be sorted into a hierarchy (namely, EFX \( \implies \) EFL \( \implies \) EF1), with the notions separated by their \( \alpha \)-MMS guarantees, as shown by Amanatidis et al. [2]. We are able to establish a similar hierarchy for our more general goods with copies model.

Restricted allocations. Kroer and Peysakhovich [35] consider optimization of MNW by exclusive (“at most one”) allocations, showing these achieve an approximate CEEI in large enough instances. Biswas and Barman [15] consider a model where each agent has cardinality constraints over the amount of a same-typed good she can be allocated. Our model is a special case and thus their general results hold, namely existence of EF1 allocations and of \( \frac{1}{3} \)-MMS allocations. EFX and EFL allocations may not exist for certain instances in their model as well as in ours. We refine these notions by the introduction of “without commons” which enables our duality results, the hierarchy of concepts we describe, and our existence results. Other models with matroid constraints over allocations are presented in [16, 25].

Relation between goods and chores. Mixed instances of goods and chores are considered by Aziz et al. [3]. Ternary preferences, where items have some fixed positive value, fixed negative value, or zero value, are considered in various works [1, 6, 13]. As we show, a mixed model of goods and chores is a special case of our goods with copies model. Kulkarni et al. [36] note that a chore can be reinterpreted as \( n - 1 \) goods, and use this as a technical tool in their PTAS for computing the MMS guarantee in mixed manna settings. They do not explore further implications of this duality. Freeman et al. [27] study the notion of equitability for chores, and shows that while for goods an EQX and Pareto optimal allocation always exists, the same does not hold for chores. They show that using a different notion of DEQX, a DEQX and Pareto optimal chores allocation always exists.

2 PRELIMINARIES

In this section we present our model, provide definitions for leading measures of fair allocation of goods, and explain how they generalize to chores.

Our model. Let \( N \) be a set of \( n \) agents and let \( M \) be the set of items to allocate to the agents. Let \( \mathcal{T} \) be a set of item types to allocate to the agents (we slightly abuse notation and use \( \mathcal{T} \) to also denote an item set with a single copy per item). The item set \( M \) is a multiset of the type set \( \mathcal{T} \), and includes \( k_t \leq n \) copies of each item type \( t \in \mathcal{T} \).

Every agent \( i \in N \) has a valuation function \( v_i : 2^M \rightarrow \mathbb{R} \) mapping bundles of items to their values. An item \( t \in M \) is termed a good if \( v_i(t) \geq 0 \) for each agent \( i \) and \( > 0 \) for at least one agent, and a chore if \( v_i(t) \leq 0 \) for each agent \( i \). We assume that each item is either a good or a chore, i.e., we do

\(^6\)For goods without copies it is furthermore guaranteed that there exists an EF1 allocation that satisfies the efficiency condition of Pareto optimality (PO) [11, 19]. EF1+PO is also guaranteed for bi-valued chores [26]. EFX+PO is guaranteed for goods without copies with lexicographic preferences [32], and for a mix of goods and chores with seperable lexicographic preferences [31].
not allow an item to be a good for one agent and a chore for another. We use \( t \) when referring to a general item, \( g \) when referring to a good, and \( c \) when referring to a chore. Throughout we assume additive valuations, i.e., for every bundle of items \( S \subseteq \mathcal{M} \), \( v(S) = \sum_{t \in S} v(t) \). Note that additive valuations are monotone, i.e., for every two bundles \( S_1 \subseteq S_2 \subseteq \mathcal{M} \), we have \( v(S_1) \leq v(S_2) \) for goods and \( v(S_1) \geq v(S_2) \) for chores.

An allocation \( A = (A_1, \ldots, A_n) \) is a partition of all items among the agents (no item can be left unallocated). We focus on allocations that satisfy the following constraint:

**Definition 1.** An allocation \( A \) is **exclusive** if no two copies of the same item are allocated to the same agent.

(That is, by “allocation” we always refer to an exclusive allocation.) We denote the set of all (exclusive) allocations of items \( \mathcal{M} \) by \( X(M) \).

**Definition 2.** Consider exclusive allocations \( A, A' \in X(M) \). Then \( A' \) Pareto dominates \( A \) if for every agent \( i \), \( v_i(A'_i) \geq v_i(A_i) \) and \( v_j(A'_j) > v_j(A_j) \) for at least one agent \( j \). Allocation \( A \) is **Pareto optimal** if there is no \( A' \) that Pareto dominates it.

**Fairness notions for goods and chores.** The next definition summarizes many prominent fairness notions studied in previous literature, as defined for goods.

**Definition 3.** An allocation \( A \in X(M) \) of goods is:

- **\( \alpha \)-EFX** if \( \forall i, j \in \mathcal{N}, \forall g \in A_j : v_i(A_i) \geq \alpha v_i(A_j \setminus \{g\}) \);
- **\( \alpha \)-EF1** if \( \forall i, j \in \mathcal{N}, \exists g \in A_j : v_i(A_i) \geq \alpha v_i(A_j \setminus \{g\}) \);
- **\( \alpha \)-EFL** if \( \forall i, j \in \mathcal{N}, \text{either (1) } |A_j| \leq 1, \text{or (2) } \exists g \in A_j : v_i(A_i) \geq \alpha \max\{v_i(A_i \setminus \{g\}), v_j(g)\} \).

In words, EFL strengthens the EF1 requirement by not allowing to remove from \( A_j \) a highly valuable good, unless \( A_j \) is a singleton;

- **PROP (proportional)** if \( \forall i \in \mathcal{N} : v_i(A_i) \geq \frac{1}{n} v_i(M) \);
- **\( \alpha \)-MMS (maximin share)** for some \( 0 < \alpha \leq 1 \) if \( \forall i \in \mathcal{N} : v_i(A_i) \geq \alpha \max_{A \in X(M)} \min_{j \in \mathcal{N}} v_j(A_j) \).

1-MMS is termed MMS, 1-EFX is termed EFX, and so on. In Section 7.2.3 we show an extension to weighted envy notions.

We can distinguish between two classes of fairness notions appearing in Definition 3: **share-based** and **envy-based**. Interestingly, this classification determines how the fairness notions are generalized to chores.

First, PROP and MMS are share-based, meaning that an agent’s allocation is measured against what is deemed her fair share:

**Definition 4.** Given a valuation \( v \), the number of agents \( n \) and an item set \( \mathcal{M} \), a **share** function \( s \) outputs a real value, i.e., \( s(v, \mathcal{M}, n) \in \mathbb{R} \). An exclusive allocation \( A \) is called **s-share fair** if we have \( v_i(A_i) \geq s(v_i, \mathcal{M}, n), \forall i \in \mathcal{N} \).

For example, for PROP, \( s(v_i, \mathcal{M}, n) = \frac{1}{n} v_i(M) \). The definitions of PROP and MMS above hold for chores too, and in fact for any mixture of goods and chores.

The notions of EFX, EF1 and EFL are envy-based, meaning that an agent’s allocation is measured against those of her peers. The definition of envy-based notions for chores is different than their definition for goods in that, when an agent \( i \) compares her bundle to the bundle of another agent \( j \), she removes a single chore from her own bundle (while a good is removed from the other agent’s

\(^7\)In case \( A_j \) is empty, the good exists “in an empty sense” and the condition is satisfied.
bundle). For example, an exclusive allocation $A$ for chores is $\alpha$-EFX if for every agent pair $i, j \in N$ and chore $c \in A_i$:

$$\alpha w_i (A_i \setminus \{c\}) \geq v_i (A_j).$$

(1)

To more generally phrase the connection between envy-based fairness for goods and for chores we need the following definition.

**Definition 5** (Comparison criterion for goods and chores).

- Given a valuation function $v$ and two bundles $B_l$ and $B_U$, a comparison criterion $f$ outputs $F$ for “fair” and $\neg F$ for “unfair”, i.e., $f(v, B_l, B_U) \in \{F, \neg F\}$.
- Given a comparison criterion $f$, the complementary comparison criterion $f^c$ satisfies $f^c(v, B_l, B_U) = f(-v, B_l, B_U)$.

For example, let $f$ be the comparison criterion corresponding to $\alpha$-EFX for goods (see Definition 3), then $f^c$ coincides with the comparison criterion of $\alpha$-EFX for chores in Eq. (1). One can similarly define $E_F$ and $E_F$ for chores.

**Remark 1.** We define envy-based fair allocation solution concepts for the case where all items are goods and for the case where all items are chores. Our definition – and in fact all results in this paper – can be extended to a setting of additive valuations where some items are goods while others are chores, using the envy-based fair allocation solution concepts of Aziz et al. [3]. See more details in Section 7.2.1.

### 3 EFX FOR GOODS WITH COPIES

We begin by showing that EFX may not exist with copies.

**Example 1.** Consider $n \geq 3$ agents, $n + 1$ goods, and $\frac{n}{2} < k < n$ copies of each good.\(^8\) All agents share the same additive valuation function $v$, and for every item $g_w \in \mathcal{M}$ of type $w$:

$$v(g_w) = \begin{cases} w & w < n + 1; \\
\frac{n^2}{\alpha} & w = n + 1.
\end{cases}$$

**Proposition 1.** For goods with copies and any $\alpha > 0$, an $\alpha$-EFX allocation may not exist even for 3 agents. The same holds for $\alpha$-EFL.

**Proof.** Consider Example 1. We show the non-existence of $\alpha$-EFL allocations. Since every $\alpha$-EFL allocation is also an $\alpha$-EFL (a straight-forward generalization of the result of [9]), this shows no $\alpha$-EFX allocations exist for these settings. Fix an agent $l$ who does not receive a copy of item $g_{n+1}$. Since there are $k$ copies of item $g_1$ and $k$ copies of item $g_{n+1}$ where $2k > n$, there exists an agent $\ell$ which receives both item $g_1$ and item $g_{n+1}$. Agent $l$ EFL-envies agent $\ell$ since the first condition of EFL is not satisfied ($|A_\ell| \geq 2$), and for any $g \in A_h$ with $v_l (g) \leq v_l (A_l)$,

$$v_l (A_l) \leq \sum_{w \leq n} v_l (g_w) < \alpha \cdot v_l (g_{n+1}) \leq \alpha \cdot v_l (A_h \setminus \{g\}).$$

There is also a simple example with two agents and two good types $g_1, g_2$, where $g_1$ has a single copy and $g_2$ has two copies. The agents have identical valuation $v(g_1) > v(g_2)$. However, we choose to give an example with $k < n$, since it can be argued that instances with $k = n$ should be immediately reduced to simpler instances without the good type $t$.\(^8\)

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violating the second condition of EFL.

To handle the non-existence problem we introduce the new fairness notion of EFXWC – a generalization of EFX to a model with copies. When comparing two bundles, EFXWC puts aside all items common to both bundles, and then compares the remaining items in an EFX way:

**Definition 7 (α-EFXWC for goods).** An exclusive allocation $A$ of goods is $\alpha$-EFXWC if for every two agents $i, j \in \mathcal{N}$ and item $g \in A_i \setminus A_j$ it holds that $v_i(A_i \setminus A_j) \geq \alpha v_i((A_j \setminus A_i) \setminus \{g\})$.

We refer to 1-EFXWC as EFXWC.

**Remark 2.** We remark that:

- EFXWC is an envy-based notion and therefore its definition extends to chores using Definition 6.
- A model with no copies is a special case of our model, and when there is one copy of each good in our model then EFXWC is equivalent to EFX.
- EFX implies EFXWC: By additivity of the values, the inequality in Def. 7 is equivalent to the EFX inequality $v_i(A_i) \geq v_i((A_j \setminus A_i) \setminus \{g\})$, but EFX allows $g$ to be any item while EFXWC restricts its choice.

A main technical justification of EFXWC is its ability to give a “dual” view of goods and chores, which also yields a new characterization of existence of standard EFX allocations for chores. This is summarized by the following result (it is a corollary of Theorem 2 stated in the next section).

**Definition 8.** An allocation $A \in \mathcal{X}(\mathcal{M})$ of chores is $\alpha$-EFX if

$$\forall i, j \in \mathcal{N}, \forall g \in A_i : \alpha v_i(A_i \setminus \{g\}) \geq v_i(A_j \setminus \{g\}).$$

**Corollary 1 (Characterization of α-EFX existence for chores).** For any $\alpha > 0$, an $\alpha$-EFX allocation for chores exists in the standard setting without copies iff a $\alpha$-EFXWC allocation for goods exists in our setting with $k = n − 1$ copies of each good.

The following example illustrates the above.

**Example 2 (Special case of Example 1).** There are $n = 3$ agents and $4$ goods with $n − 1 = 2$ copies each. All agents have valuation $v$ as defined in Example 1 (the values of items $t_1, t_2, t_3, t_4$ are 1, 2, 3, 9 respectively).

In Example 2, the allocation $A_1 = \{t_1, t_2, t_4\}, A_2 = \{t_3, t_4\}$ and $A_3 = \{t_1, t_2, t_3\}$ is EFXWC. Indeed, agent 3 EFX-envies the others (and moreover EFL-envies them), but after putting aside common items, only item $t_4$ is left for the other agents, implying EFXWC. We now show that the “dual” allocation is EFX for chores. The dual allocation is given by $A_i^\circ = \mathcal{T} \setminus A_i$ for every $i$ (see formal definition in Section 4). I.e., $A_1^\circ = \{t_3\}, A_2^\circ = \{t_1, t_2\}$ and $A_3^\circ = \{t_4\}$. When the goods are treated as chores by simply taking the negation ($-v$) of the valuation (such that the values of $t_1, t_2, t_3, t_4$ are $-1, -2, -3, -9$ respectively), it is hard not to check that the dual allocation is EFX for these chores. Indeed, agent 3 envies the others, but removing any chore from agent 3’s single-chore bundle alleviates the envy.

A result in [42] shows that with $n = 3$ agents, a $\frac{1}{3}$-EFX approximation always exists for chores. By Corollary 1, for $n = 3$ agents and goods with two copies each, a $\frac{1}{3}$-EFXWC approximation always exists. This is in contrast with the result of Example 1, which shows that a $\frac{1}{3}$-EFX approximation

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9The definition we give uses $\alpha$ in a different way than the one common in the literature, which is $\forall i, j \in \mathcal{N}, \forall g \in A_i : v_i(A_i \setminus \{g\}) \geq \alpha v_i(A_j \setminus \{g\})$. In the common definition, goods approximation are with $0 < \alpha \leq 1$, and chores approximations are with $1 \leq \alpha$. However, for our duality framework, it is convenient to use the same $\alpha$ without the need to invert when taking the dual.
does not always exists (and in fact, for any $\alpha$-EFX). In Section 6.2 we show another setting (namely, leveled preferences) where EFXWC existence is guaranteed but $\alpha$-EFX is not, for any $\alpha > 0$.

4 FAIRNESS DUALITY FOR COPIES AND CHORES

In this section we show that allocations of goods and of chores are “dual” in a formal sense, and more importantly that fairness notions translate between dual allocations.

Definition 9 (Duality). The dual of a tuple $(A, v, M)$ is:

- The dual allocation $A^\circ$ is $A_i^\circ = T \setminus A_i$.
- The dual valuation is $v^\circ = -v$. Thus goods become chores and vice versa.
- The dual item set $M^\circ$ contains $n - k_i$ copies of every $t \in T$, where $k_i$ is the number of copies of $t$ in $M$.

Proposition 2 (Properties of dual transformations).

1. If $A$ is an exclusive allocation of item set $M$ then $A^\circ$ is an exclusive allocation of item set $M^\circ$.
2. The dual of the dual is the original: $A^{\circ\circ} = A$, $v^{\circ\circ} = v$ and $M^{\circ\circ} = M$.
3. The dual operation is a one-to-one mapping.

Proof. (1) $A^\circ$ is an allocation since each item $t$ is allocated $k_i$ times in $A$, so it does not appear in the bundles of $n - k_i$ agents. Every such agent is allocated a copy of $t$ in $A^\circ$, and other agents are not. $A^\circ$ is exclusive since by construction each bundle $A_i^\circ$ is contained in $T$ and has at most one copy of each item.

(2) $A_i^{\circ\circ} = T \setminus A_i^\circ = T \setminus (T \setminus A_i) = A_i$, $v^{\circ\circ} = -v^\circ = v$, and $M^{\circ\circ}$ contains $n - (n - k_i) = k_i$ items for all $t \in T$.

(3) Assume that $A_1, A_2, v_1, v_2, M_1, M_2$ satisfy $A_1^\circ = A_2^\circ$, $v_1^\circ = v_2^\circ$, $M_1^\circ = M_2^\circ$, then we have $A_1 = A_1^{\circ\circ} = A_2^{\circ\circ} = A_2$. The rest follows similarly.

Theorem 1. An exclusive allocation $A$ is Pareto optimal iff the dual allocation $A^\circ$ is Pareto optimal.

Proof. First we prove a property of the dual: If $v_i(B_i) > v_i(A_i)$ then $v_i^\circ(B_i^\circ) > v_i^\circ(A_i^\circ)$. From $v_i(B_i) > v_i(A_i)$, we have $v_i(T \setminus B_i) < v_i(T \setminus A_i)$, i.e. $v_i(B_i^\circ) < v_i(A_i^\circ)$. Because $v_i^\circ = -v_i$, we have $v_i^\circ(B_i^\circ) > v_i^\circ(A_i^\circ)$.

Suppose that allocation $B$ Pareto dominates allocation $A$. We prove that allocation $B^\circ$ Pareto dominates allocation $A^\circ$. By the property we just proved, we have $v_i^\circ(B_i) \geq v_i^\circ(A_i)$ for all $i \in N$ and there is a $j$ such that $v_j^\circ(B_j^\circ) > v_j^\circ(A_j^\circ)$. This directly implies the statement, as there is a Pareto dominating allocation over $A$ iff there is such an allocation for the dual.

4.1 A Meta-theorem for Envy-based Notions

The idea of $\alpha$-EFX$_{WC}$ generalizes to other envy-based notions:

Definition 10 (Comparison without commons). Given a comparison criterion $f$, the comparison criterion $f_{WC}$ satisfies $f_{WC}(v, B_1, B_2) = f(v, B_1 \setminus B_2, B_2 \setminus B_1)$, for all $v, B_1, B_2$.

As additional examples to $\alpha$-EFX$_{WC}$, consider $\alpha$-EF1$_{WC}$ and $\alpha$-EFL$_{WC}$. Note that by Definition 5, for any comparison criterion $f$ we have $(f_{WC})^c = (f^c)_{WC} \equiv f_{WC}^c$. We now show our first main theorem – that envy-based fairness holds under a duality transformation from goods to chores.

Theorem 2. Given a comparison criterion $f$, an exclusive allocation $A$ is $f_{WC}$-fair with respect to $v$ iff its dual $A^\circ$ is $f_{WC}^c$-fair with respect to $v^\circ$.
Theorem 2. For MMS, an exclusive allocation \( A \) is EFX if and only if \( A^\circ \) is s-share fair for \( \mathcal{M}^\circ, v^\circ \).

**Proof.** For MMS, we have
\[
MMS(v_i, \mathcal{M}, n) = \max_{A \in X(\mathcal{M})} \min_{j \in N} v_i(A_j) = \max_{A \in X(\mathcal{M})} \min_{j \in N} (v_i(T) - v_i(A_j^\circ))
\]
\[
= \max_{A^\circ \in X(\mathcal{M}^\circ)} \min_{j \in N} v_i^\circ(A_j^\circ) + v_i(T)
\]
\[
= v_i(T) + \max_{A^\circ \in X(\mathcal{M}^\circ)} \min_{j \in N} v_i^\circ(A_j) = v_i(T) + MMS(v_i^\circ, \mathcal{M}^\circ, n)
\]  

Equation (2)

Now consider some exclusive allocation \( A \) that is MMS fair for \( \mathcal{M}, v_i \). That is, for all \( i \in N \),
\[
v_i(A_i) \geq MMS(v_i, \mathcal{M}, n) \overset{\text{Eq. (2)}}{=} v_i(T) + MMS(v_i^\circ, \mathcal{M}^\circ, n).
\]

Equation (3)

Then,
\[
v_i^\circ(A_i^\circ) = -v_i(T \setminus A_i) = v_i(A_i) - v_i(T)
\]
\[
\overset{\text{Eq. (5)}}{\geq} v_i(T) + MMS(v_i^\circ, \mathcal{M}^\circ, n) - v_i(T) = MMS(v_i^\circ, \mathcal{M}^\circ, n).
\]

The converse is proved similarly.

\( \square \)

The important property of MMS that preserves it across the duality is that the change in valuation propagates through the share notion. We now develop a general discussion of this property through the notion of linear shares. The idea of a linear share is as follows. Focus on one agent and consider all values from her perspective. For any allocation, our dual transformation shifts the value of every bundle in her valuation by a constant \( d \) (in particular, \( d = v(T) \)). It is natural to require that her fair share for this allocation also shifts by the same constant \( d \). A linear share-based notion is one for which this property holds. The following definition formalizes this intuition, while generalizing the dual transformation to appropriate one-to-one mappings.
Definition 11 (Linear mapping). Given two type sets \( T \) and \( T' \), a one-to-one linear mapping \( f : T \rightarrow T' \) w.r.t. valuations \( v, v' \) and a constant \( d \) is a mapping such that there is a constant \( d \equiv df \) so that \( v(B) = v'(f(B)) + d \) for any subset \( B \subseteq T \).

Proposition 3. The dual as we define in Definition 9 is a linear mapping.

Proof. The dual as defined in Definition 9 with respect to item sets can be naturally extended to type sets (since two items of a certain type always map to items of some matching certain type). We prove that for every agent \( i \in \mathcal{N} \), the type sets \( T \) and \( T^o \) are linearly related w.r.t. valuations \( v_i, v_i^o \) and \( d = v_i(T) \). Given a bundle \( B_i \subseteq T \), the dual transformation is a one-to-one mapping that yields the allocation \( B_i^o \subseteq T \), and satisfies the condition in Definition 13:

\[
\begin{align*}
  v_i(B_i) - v_i^o(B_i^o) &= v_i(B_i) - (v_i(B_i^o)) \\
  &= v_i(B_i) + v_i(T \setminus B_i) = v_i(T).
\end{align*}
\]

A linear mapping \( f \) automatically defines a mapping over allocations. For an allocation \( B \in \mathcal{X}(M) \) with \( n \) bundles, we have \( B' = (f(B_1), \ldots, f(B_n)) \). The mapping then satisfies \( \forall i \in [n], v_i(B_i) = v_i(B'_i) + d \).

Definition 12 (Linear related). Given two types sets \( T \) and \( T' \) and corresponding items sets \( M \) and \( M' \), if there is a linear mapping \( f : T \rightarrow T' \) w.r.t. valuations \( v, v' \) such that in the extension to allocations, every exclusive allocation \( B \in \mathcal{X}(M) \) maps to an exclusive allocation \( B' \in \mathcal{X}(M') \), we say that the item sets are linearly related w.r.t. valuations \( v, v' \) and the constant \( d \).

We saw in Proposition 2 that the dual transformation maps an exclusive allocation to an exclusive allocation.

Definition 13 (Linear shares). A share \( s \) is linear if for any item sets \( M \) and \( M' \) that are linearly related w.r.t. \( v, v' \) and \( d \), we have \( s_i(M, n) = s_i'(M', n) + d \).

An example of linearly related item sets and valuations is \( M, M', v, v' \) where \( M' \) is equal to \( M \) with an additional \( n \) copies of a new item \( t \), and \( v' \) is equal to \( v \) with an additional value for \( t \). Thus if \( s \) is a linear share then \( s_i(M, n) = s_i'(M', n) - v(t) \).

Theorem 3. For any linear share \( s \), an exclusive allocation \( A \) is \( s \)-share fair for \( M, v \) iff \( A^o \) is \( s \)-share fair for \( M^o, v^o \).

Proof. We have shown that for every agent \( i \in \mathcal{N} \), item sets \( M \) and \( M^o \) are linearly related w.r.t. valuations \( v_i, v_i^o \) and \( d = v_i(T) \). Suppose that \( A \) is \( s \)-share fair for \( M, v_i \). By definition, we have \( v_i(A_i) \geq s(v_i, M, n) \) for all \( i \in \mathcal{N} \). Since the share \( s \) is linear, \( s(v_i, M, n) = s(v_i^o, M^o, n) + v_i(T) \). Therefore, \( v_i^o(A_i^o) \geq s(v_i^o, M^o, n) \forall i \in \mathcal{N} \), and \( A^o \) is \( s \)-share fair for \( M^o, v^o \). The converse is proved similarly.

4.2.1 Examples and non-examples of linear shares.

Proposition 4. MMS and PROP are linear.

Proof. Suppose that item sets \( M \) and \( M' \) are linearly related with \( v, v' \) and \( d \). We have a one to one mapping from exclusive allocations \( A \in \mathcal{X}(M) \) to exclusive allocations \( A' \in \mathcal{X}(M') \) such that \( v(A_i) = v'(A'_i) + d \) for all \( i \in \mathcal{N} \) and all allocations.

For MMS, we have

\[
\max_{A \in \mathcal{X}(M)} \min_{j \in \mathcal{N}} v_i(A_j) = \max_{A' \in \mathcal{X}(M')} \min_{j \in \mathcal{N}} v'_i(A_j) + d
\]

\[
= d + \max_{A' \in \mathcal{X}(M')} \min_{j \in \mathcal{N}} v'_i(A_j)
\]
We first analyze the connections between the new envy-based fairness notions and the more standard \( \alpha \)-EFX. Again by the \( \alpha \)-EFX condition, we show this is also the case for the new notion of \( \alpha \)-MPMMS. An exclusive allocation \( A \) is PROPc-fair for goods for \( M, v \) iff \( \alpha^c \) is PROPc-fair for chores for \( M^c, v^c \). The same statement holds for PROPx as well.

Interestingly, more nuanced share-based notions do not always satisfy linearity, thus portraying the possible limits of the duality between fair allocations of copies and chores. For example, \( \alpha \)-MMS is not linear, since a shift \( d \) in the valuation translates to a shift \( ad \) in the share. In Appendix C, we show this is also the case for the new notion of truncated proportional share (TPS), which is defined for goods in [7]. The notion is an extension of PROP and it has been shown that \( \text{PROP} \geq \text{TPS} \geq \text{MMS} \).

5 EXPANDING THE ENVY-BASED HIERARCHY

We first analyze the connections between the new envy-based fairness notions and the more standard ones, as summarized in Figure 1. Given the envy-based duality between goods with copies and chores with copies, it suffice to discuss goods with copies. In Section 5.1 we show that this hierarchical structure implies strictly different approximation guarantees to MMS.

Proposition 6. \( \alpha \)-EFX \( \implies \alpha \)-EFL WC \( \implies \alpha \)-EF1 WC.

Proof. \((\alpha \)-EFX \( \implies \alpha \)-EFL WC\) Fix an \( \alpha \)-EFX WC allocation \( A \) and \( i, j \in N \). We wish to show \( j \) does not \( \alpha \)-EFL WC-envy \( i \). If \( |A_i \setminus A_j| \leq 1 \), the first \( \alpha \)-EFL WC condition holds. Thus assume that \( |A_i \setminus A_j| \geq 2 \). Suppose that there is a good \( g \in A_i \setminus A_j \) s.t. \( \alpha_i(g) > \alpha_i(A_j \setminus A_i) \). Since there are at least 2 goods, there must be another good \( g^* \in (A_i \setminus A_j) \setminus g \). For \( g^* \), we have

\[
\alpha_i((A_i \setminus A_j) \setminus g^*) \geq \alpha_i((A_i \setminus A_j) \setminus g)) \leq \alpha_i(A_j \setminus A_i),
\]

violating the \( \alpha \)-EFX WC condition. We conclude that all goods \( g \in A_i \setminus A_j \) satisfy \( \alpha_i(g) \leq \alpha_i(A_j \setminus A_i) \). Again by the \( \alpha \)-EFX WC condition, \( \alpha_i((A_i \setminus A_j) \setminus \{g\})) \leq \alpha_i(A_j \setminus A_i) \). We can therefore choose an arbitrary good and the second EFL WC condition holds.

\((\alpha \)-EFL WC \( \implies \alpha \)-EF1 WC\) The only difference in definitions occurs when the second EFL WC condition is invoked. Then, there is an extra constraint over the choice of good \( g \) for EFL WC condition that does not appear in the EF1 WC definition. Therefore, the EFL WC definition is stricter than the EF1 WC, and the implication follows.

Example 3. \((EFL WC \nRightarrow EFX WC)\) Consider two agents and five goods with a single copy each, and identical valuations \( a = b = 1, c = d = 1 + \varepsilon, e = \varepsilon = 0.01 \). The allocation \( \{c, d, e\}, \{a, b\} \) is EFL WC but not EFX WC.

Proposition 7. \( EFX WC \nRightarrow EFL\), \( EFL \nRightarrow EF1 WC \).
## 4.5 MMS Approximations

It is a common practice in the literature to study the MMS approximation guarantees that various envy-based notions provide [2, 10, 12, 37]. In this section we show that our new envy-based fairness notions for multiple copies provide similar guarantees to their classic counterparts, which further demonstrates their usefulness.

We show upper bounds on the MMS approximation guarantees of the three “without commons” notions introduced above for the case of goods with copies. Since our duality theorems do not hold for $\alpha$-MMS, the results in this subsection cannot be directly transformed to chores. We also establish a lower bound for the approximation guarantee of EFL$_{WC}$, thus separating EFX$_{WC}$ and EFL$_{WC}$ from EF1$_{WC}$ by a factor that grows to infinity with $n$. It is interesting to note that the bounds we show for goods with copies are strictly lower than known bounds for goods without copies: for EFX$_{WC}$ we give an upper bound of $0.4$ while without copies a lower bound of $0.4$ is known [2]; for EFL$_{WC}$ we give an upper bound of $0.5$ while without copies a lower bound of $0.5$ is known [9].

### 5.1.1 EF1$_{WC}$

Amanatidis et al. [2] show that EF1 allocations do not guarantee an approximation strictly larger than $\frac{1}{n}$ to MMS for goods without copies. Upper bounds on $\alpha$ for goods without copies immediately apply to goods with copies, since the former is a special case of the latter. For completeness, we include an example showing that EF1$_{WC}$ cannot guarantee strictly more than $\frac{1}{n}$-MMS for goods with copies.

**Example 6 (EF1$_{WC}$ cannot guarantee strictly more than $\frac{1}{n}$-MMS for goods with copies.).** Consider an example with $n$ agents, all goods have one copy each, and identical valuations. Then EF1$_{WC}$ cannot guarantee a $\frac{1}{n}$-approximation to MMS for goods with copies.
We divide the agents into three disjoint sets: $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. As $\mathcal{A}$ is a disjoint set, we do not consider copies.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
Goods & $x$ & $x'$ & $x''$ & $y$ & $y'$ & $y_a$ & $y_b$ & $y_c$ \\
\hline
Copies & 6 & 7 & 3 & 3 & 3 & 3 & 1 & 1 \\
\hline
$\nu$ & 2.5 & I & I & I & $\frac{1}{2}$ & $\frac{1}{2}$ & $\frac{1}{2}$ & $\frac{1}{2}$ \\
\hline
\end{tabular}
\caption{EFX$_{WC}$ 0.4-MMS upper bound.}
\end{table}

valuations. Let $L_1, ..., L_n$ be goods, all with value 1, and let $H_1, ..., H_{n-1}$ be additional goods, all with value $n$. Then

$$\mathcal{A} = \{(H_\theta, L_\theta), \{L_n\}\}_{1 \leq \theta \leq n-1}$$

is an EFX$_{WC}$ allocation with $\nu(A_n) = \nu(L_n) = 1$, while an MMS of $n$ is guaranteed for agent $n$ by

$$\mathcal{A}' = \{(H_\theta), \{L_1, ..., L_n\}\}_{1 \leq \theta \leq n-1}$$

5.1.2 EFX$_{WC}$. We give an upper bound of 0.4 and a lower bound of $\frac{4}{11} \approx 0.36$.

**Example 7** (There is an EFX$_{WC}$ allocation which is at most 0.4-MMS for goods with copies). Consider 13 agents and 9 goods as given in Table 2, where $\nu$ is the valuation of agent 13, and all other agents value all goods as 1. The following allocation is EFX$_{WC}$:

$$\mathcal{A} = \{(H, x), (x', x''), (y, y', y_\theta), (x)\}_{\times 6 \times 3 \times 3 \times \theta \in \{a, b, c\}}$$

In the following allocation all agents have a value of exactly 2.5 in terms of $\nu$:

$$\mathcal{A}' = \{(H), (x, x', y), (x, x'', y'), (x, y_a, y_b, y_c)\}_{\times 6 \times 3 \times 3 \times 3}$$

We establish the following $\frac{4}{11}$-MMS lower bound by appropriately generalizing a proof of [2] to the case of goods with copies. We intentionally keep similar notations to allow easy comparison.

**Theorem 5.** An EFX$_{WC}$ allocation is at least $\frac{4}{11}$-MMS for goods with copies.

**Proof.** Suppose that allocation $\mathcal{A}$ is an EFX$_{WC}$ allocation. Let us take the perspective of agent $\alpha$. We divide the agents into three disjoint sets: $L_1 = \{i \in \mathcal{N} \mid |A_i \setminus A_\alpha| \leq 1\}$, $L_2 = \{i \in \mathcal{N} \mid |A_i \setminus A_\alpha| = 2\}$, $L_3 = \{i \in \mathcal{N} \mid |A_i \setminus A_\alpha| \geq 3\}$. Define the set of goods $\forall \theta \in \{1, 2, 3\}, S_\theta = \cup_{i \in L_\theta} (A_i \setminus A_\alpha)$.

**Claim 1.** EFX$_{WC}$ implies:

1. For any good $g \in S_2$, $\nu_\alpha (g) \leq \nu_\alpha (A_\alpha)$.
2. For any agent $i \in L_3$, $\nu_\alpha (A_i \setminus A_\alpha) \leq \frac{3}{2} \cdot \nu_\alpha (A_\alpha)$.

**Proof.** For the first inequality, let $i$ be an agent in the set $L_2$. By the condition of EFX$_{WC}$,

$$\forall g \in A_i \setminus A_\alpha, \nu_\alpha ((A_i \setminus A_\alpha) \setminus g) \leq \nu_\alpha (A_\alpha \setminus A_i) \leq \nu_\alpha (A_\alpha).$$

As $|A_i \setminus A_\alpha| = 2$, $\nu_\alpha (g) = \nu_\alpha ((A_i \setminus A_\alpha) \setminus g') \leq \nu_\alpha (A_\alpha)$, where $g'$ is another good in the set $A_i \setminus A_\alpha$. 

ACM Trans. Econ. Comput., Vol. 0, No. 0, Article 0. Publication date: 0.
For the second inequality, \( \min_{g \in A_i \setminus A_\alpha} v_\alpha (g) \leq \frac{1}{2} v_\alpha (A_\alpha) \), since otherwise we have for any good \( g' \in A_i \setminus A_\alpha, \)
\[
v_\alpha ((A_i \setminus A_\alpha) \setminus \{g'\}) = \sum_{g \in (A_i \setminus A_\alpha) \setminus \{g'\}} v_\alpha (g) \\
\geq \sum_{g \in (A_i \setminus A_\alpha) \setminus \{g'\}} \min_{g'' \in (A_i \setminus A_\alpha) \setminus \{g''\}} v_\alpha (g'') \\
\geq 2 \min_{g'' \in (A_i \setminus A_\alpha)} v_\alpha (g'') > 2 \cdot \frac{1}{2} v_\alpha (A_\alpha) = v_\alpha (A_\alpha),
\]
contradicting EFX\text{WC}. For \( g = \arg \min_{g' \in A_i \setminus A_\alpha} v_\alpha (g') \):
\[
v_\alpha (A_i \setminus A_\alpha) = v_\alpha ((A_i \setminus A_\alpha) \setminus \{g\}) + v_\alpha (g) \leq v_\alpha (A_\alpha) + \min_{g \in A_i \setminus A_\alpha} v_\alpha (g) \leq 2 \cdot \frac{3}{2} \cdot v_\alpha (A_\alpha).
\]

\[\square\]

Suppose that allocation \( A^* \) is a maximin share allocation for agent \( \alpha \). Let the allocation \( A' = \{A_i \in A^* \mid |A_i \cap S_1| = 0 \text{ and } |A_i \cap S_2| \leq 1\} \). We remove any bundle contains at least one good in \( S_1 \) or at least two goods in \( S_2 \) from the allocation \( A^* \). Notice that allocation \( A' \) cannot be empty. As \( \alpha \in L_1 \), the number of bundles is at least \( 1 + |S_1| + \frac{|S_2|}{2} \). And we remove at most \( |S_1| + \frac{|S_2|}{2} \) bundles. Let \( n' = |A'| \).

Next we prove that there is a bundle \( A'_j \) in the allocation \( A' \) such that \( v_\alpha (A'_j) \leq 11 \cdot v_\alpha (A_\alpha) \). Let \( y \) be the size of \( |L_3| \) and \( x \) be the number of goods in set \( S_2 \) appearing in the allocation \( A' \).

**Claim 2.** We have the following quantity relationships: (1) \( n' \geq x \), (2) \( n' \geq \frac{x}{2} + y \).

**Proof.** The first inequality holds as each bundle in \( A' \) has at most one good from \( S_2 \). For the second inequality, we have
\[
n = |L_1| + |L_2| + |L_3| \geq |S_1| + \frac{|S_2|}{2} + y.
\]
The number of removed bundles is at most \( |S_1| + \frac{|S_2| - x}{2} \). Therefore, \( n' \geq n - |S_1| - \frac{|S_2| - x}{2} \geq \frac{x}{2} + y \). \[\square\]

By Claim 1, the total sum of goods from sets \( S_2 \) and \( S_3 \) in allocation \( A' \) for agent \( \alpha \) is upper bounded by
\[
x \cdot \max_{g \in S_2} v_\alpha (g) + y \cdot \max_{i \in L_3} v_\alpha (A_i \setminus A_\alpha) \leq x \cdot v_\alpha (A_\alpha) + y \cdot \frac{3}{2} v_\alpha (A_\alpha).
\]

By Claim 2, the average valuation is bounded by
\[
\frac{x + \frac{3}{2} y}{n'} \cdot v_\alpha (A_\alpha) \leq \frac{x + \frac{3}{2} \cdot (n' - \frac{x}{2})}{n'} \cdot v_\alpha (A_\alpha) \leq \frac{1}{4} \cdot \frac{x + \frac{3}{2}}{n'} \cdot v_\alpha (A_\alpha) \leq \frac{7}{4} \cdot v_\alpha (A_\alpha).
\]
Therefore, there is an agent $j$ such that $v_a \left( A_j' \setminus A_a \right) \leq \frac{7}{4} \cdot v_a \left( A_a \right)$. We have

$$v_a \left( A_j' \right) = v_a \left( A_j' \setminus A_a \right) + v_a \left( A_j' \cap A_a \right)$$

$$\leq \frac{7}{4} \cdot v_a \left( A_a \right) + v_a \left( A_a \right) = \frac{11}{4} \cdot v_a \left( A_a \right).$$

MMS is $\min_{i \in N} v_a \left( A_i' \right) \leq v_a \left( A_j' \right) \leq \frac{11}{4} \cdot v_a \left( A_a \right)$. □

### 5.1.3 EFL$_{WC}$

We give an upper and a lower bound of one-third.

**Example 8 (There is an EFL$_{WC}$ Allocation with at Most $\frac{1}{3}$-MMS for Goods with Copies).** Consider $2\ell + 1$ agents and goods as given in Table 3, where $v$ is the valuation of agent $2\ell + 1$ and all other agents value all goods as 1. Then,

$$A = \{ \{H, x\}, \{x', y_0, z_0\}, \{x\} \}$$

$\times \ell$ \quad \begin{align*} \forall 1 \leq \theta \leq \ell \end{align*}

is EFL$_{WC}$, but $v(A_{2\ell+1}) = 1$, while a MMS of at least $3 - \frac{7}{\ell}$ is guaranteed by

$$A' = \{ \{H\}, \{x, x', y_0\}, \{x, z_1, ..., z_{\ell}\} \}.$$  

$\times \ell$ \quad \begin{align*} \forall 1 \leq \theta \leq \ell \end{align*}

**Theorem 6.** An EFL$_{WC}$ allocation is at least $\frac{1}{3}$-MMS for goods with copies.

**Proof.** Suppose that $A$ is an EFL$_{WC}$ allocation, and consider the perspective of an agent $i^*$. Among the remaining $n - 1$ agents, let $L_1$ be the set of agents satisfying the first EFL$_{WC}$ condition, and let $L_2$ be the remaining agents (that must thus satisfy the second condition). Denote $\ell = \left|L_1\right|, n - \ell - 1 = \left|L_2\right|$. Define $\forall \theta \in \{1, 2\}, S_\theta = \bigcup_{i \in L_\theta} (A_i \setminus A_{i^*})$ (we allow multiple copies in the same set). Notice that for any $j \in L_2$ we have by the second EFL$_{WC}$ condition that there is such good $g \in A_j$ with

$$\max \{ v_{i^*} \left( A_j \setminus (A_{i^*} \cup \{g\}) \right), v_{i^*} (g) \} \leq v_{i^*} (A_{i^*} \setminus A_j),$$

and so

$$v_{i^*} (A_j \setminus A_{i^*}) \leq v_{i^*} (A_j \setminus (A_{i^*} \cup \{g\})) + v_{i^*} (g)$$

$$\leq 2v_{i^*} (A_{i^*} \setminus A_j)$$

$$\leq 2v_{i^*} (A_{i^*}),$$

which implies

$$v_{i^*} (S_2) \leq \sum_{j \in L_2} v_{i^*} (A_j \setminus A_{i^*}) \leq 2(n - \ell - 1)v_{i^*} (A_{i^*}).$$

All goods that are not in $S_1, S_2$ must have a copy in $A_{i^*}$. Denote all the remaining goods’ copies $R$, then we overall have $M = S_1 \cup S_2 \cup R$.

In any allocation $A'$, there are at most $\ell$ agents with goods from $S_1$. That is since the first EFL$_{WC}$ condition for an agent $j$ requires $|A_j \setminus A_{i^*}| = 1$, and so we have $|S_1| = \sum_{j \in L_1} |A_j \setminus A_{i^*}| = \ell$. Let

<table>
<thead>
<tr>
<th>Goods</th>
<th>$H$</th>
<th>$x$</th>
<th>$x'$</th>
<th>$\forall 1 \leq \ell, y_1$</th>
<th>$\forall 1 \leq \ell, z_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copies</td>
<td>$\ell$</td>
<td>$\ell + 1$</td>
<td>$\ell$</td>
<td>$I$</td>
<td>$I$</td>
</tr>
<tr>
<td>$v$</td>
<td>$3$</td>
<td>$I$</td>
<td>$I$</td>
<td>$1 - \frac{7}{\ell}$</td>
<td>$\frac{7}{\ell}$</td>
</tr>
</tbody>
</table>

Table 3. EFL$_{WC}$ $\frac{1}{3}$-MMS upper bound
then be the set of at least \( n - \ell \) agents with no good from \( S_1 \), and let \( \{S_j^I\}_{j \in L'}, \{R_j^I\}_{j \in L'} \) be the allocations of \( S_2, R \) goods to these agents under \( A' \). Since

\[
\sum_{j \in L'} v_{i^*}(S_j^I) \leq v_{i^*}(S_2),
\]

there exists some \( j \in L' \) with

\[
v_{i^*}(S_j^I) \leq \frac{2(n - \ell - 1)}{|L'|} v_{i^*}(A_{i^*}) \leq \frac{2(n - \ell - 1)}{n - \ell} v_{i^*}(A_{i^*}) < 2 v_{i^*}(A_{i^*}).
\]

Since \( A' \) is an exclusive allocation, and since \( R \) includes only goods with a copy in \( A_{i^*} \), agent \( j \) satisfies \( v_{i^*}(R_j^I) \leq v_{i^*}(A_{i^*}) \), and overall \( v_{i^*}(A_j) = v_{i^*}(S_j^I) + v_{i^*}(S_j^I) + v_{i^*}(R_j^I) \leq 3 v_{i^*}(A_{i^*}) \). Since such an agent exists for any exclusive allocation, it exists for the MMS allocation, and so the minimal bundle in terms of \( v_{i^*} \) in that allocation is bounded by \( 3 v_{i^*}(A_{i^*}) \). This shows the \( \frac{1}{3} \)-MMS guarantee. \( \square \)

6 EXISTENCE RESULTS

In this section we show two existence results: The existence of EF1\(_{WC}\) for identical ordinal preferences, and the existence of EFX\(_{WC}\) for leveled preferences. For maximin share approximation, a formal reduction shows that identical ordinal preferences are the “hardest” case \([4, 34, 38]\). This is also intuitive, as in these cases the items are most contested. However, we complement the algorithm we provide in this section with an example that shows it to no longer work with instances that do not have identical ordinal preferences, due to problems that arise from the requirement of outputting an exclusive allocation. Leveled preferences are studied, e.g., in \([8, 40]\). The \( n = 3 \) example of \([37]\) that shows an instance that does not guarantee the maximin share, has both identical ordinal and leveled preferences.

6.1 EF1\(_{WC}\) for identical ordinal preferences

**Definition 14.** Valuations \( v_1, \ldots, v_n \) over a set of chores \( M \) have identical ordinal preferences, if for any two agents \( i, j \), and some chores \( c, c' \in M \), \( v_i(c) \geq v_i(c') \) if and only if \( v_j(c) \geq v_j(c') \).

A similar definition can be adapted to goods and to a mix of goods and chores.

We now show a round-robin algorithm for chores with copies. The algorithm can also apply to a setting of goods with copies, by first using the duality to consider it as an instance of chores with copies. As we saw in Theorem 2, the duality preserves EF1\(_{WC}\) allocations, and it is straightforward that it also preserves ordinal preferences.
**Algorithm 1**: Round-robin algorithm for chores with copies to find an EF1\textsubscript{WC} allocation for identical ordinal, additive preferences

**Input**: Agents \( N \), chores with copies \( M \), and non-positive, identical ordinal, additive valuations \( v_1, \ldots, v_n \)

**Output**: An EF1\textsubscript{WC} allocation \( A \)

1. Initialize an empty allocation \( A \)
2. \( S = M \);
3. while \( S \neq \emptyset \) do
   4. for \( i = 1 \) to \( n \) do
      5. Let \( C \subseteq S \) be all the chores \( c \in S \) that satisfy that \( \{ A_{i-1}, A_i \cup \{ c \} \} \) is exclusive.
      6. Let \( c' \) be the least costly chore \( c \in C \) (the one with maximal \( v_i(c) \)), if \( C \) is not empty.
      7. Let \( s_{c'} = \{ c' \} \) or \( \emptyset \) if \( C \) is empty.
      8. \( A_i = A_i \cup s_{c'} \);
      9. \( S = S \setminus s_{c'} \);
   10. end
11. end
12. return \( A \);

**Theorem 7.** With identical ordinal, additive valuations, Algorithm 1 outputs an EF1\textsubscript{WC} allocation.

**Proof.** First, we note that with identical ordinal, additive valuations, the algorithm indeed outputs an allocation. During the run of the algorithm with identical ordinal preferences, a chore \( c \) with lower preference is never selected before all copies of chores with higher preferences are selected. That is, if this happened and \( c \) was picked by some agent \( j \), it must mean that for any \( c' \) with \( v_j(c') > v_j(c) \), either all copies of \( c' \) were already selected, or \( j \) has a copy of them. If \( j \) selected a copy of \( c' \) it must be at least round before, and by the induction assumption, at this point all copies of higher preferences chores were already selected. Thus, during the \( n - 1 \) selections by other agents between the step where agent \( j \) selected \( c' \) and where it selected \( c \), all copies of \( c' \) are selected by the agents. Now assume towards contradiction the algorithm does not output an allocation. This only happens if \( S \) is never empty, which requires that at least once, \( s_{c'} \) is the empty set at line 7. If this were to happen, it is because no chore in \( S \) satisfies the constraint that by adding a copy of it to agent \( i \), the allocation remains exclusive. Then there is some such agent \( i \) so that \( S \) is not empty, but all available chores already have a copy in \( A_i \). However, this is never the case. There can not be multiple such chore types, as the lower preference among them being previously selected would imply that all the higher preference copies must have been selected (and thus not available). If there is exactly one such chore type, it must have been selected at least one round earlier by agent \( i \) and within this round all copies would have been selected by other agents, leading to a contradiction.

We now show that the allocation \( A \) that is returned is EF1\textsubscript{WC}. Consider some two agents \( i, j \):

- First, consider \( i < j \). We have \( |A_i \setminus A_j| \leq |A_j \setminus A_i| + 1 \). Let \( c = \arg \min_{\tilde{c} \in A_i \setminus A_j} \{ v_i(\tilde{c}) \} \) be the maximal cost chore in the set. Then note that \( |(A_i \setminus A_j) \setminus \{ c \}| \leq |A_j \setminus A_i| \) and if we order the elements by their order of being chosen, this yields an injection, where each element in \((A_i \setminus A_j) \setminus \{ c \}\) is a higher preference than the corresponding element in \( A_j \setminus A_i \).

- Consider \( i > j \). We have \( |A_i \setminus A_j| \leq |A_j \setminus A_i| \). Let
  \[
  c = \arg \min_{\tilde{c} \in A_i \setminus A_j} \{ v_i(\tilde{c}) \},
  c' = \arg \max_{\tilde{c} \in A_i \setminus A_j} \{ v_i(\tilde{c}) \},
  \]
  Then note that \( |(A_i \setminus A_j) \setminus \{ c \}| \leq |(A_j \setminus A_i) \setminus \{ c' \}| \) and if we order the elements by their order of being chosen, this yields an injection, where each element in \((A_i \setminus A_j) \setminus \{ c \}\) is a higher preference than the corresponding element in \((A_j \setminus A_i) \setminus \{ c' \}\).
preference than the corresponding element in \((A_j \setminus A_i) \setminus \{c’\}\). Since \(c’\) itself has non-positive value, this yields the \(\text{EF1}_{\text{WC}}\) condition.

\[ \square \]

We show how the algorithm may fail to produce an \(\text{EF1}_{\text{WC}}\) allocation once the preferences are not identically ordinal. Consider the following example:

**Example 9.** Consider three agents and 10 chores \(c_1, \ldots, c_{10}\), each with two copies. Agents 1 and 2 have \(v_1 = v_2\) and \(v_1(c_i) = -(C + 11 - i), v_2(c_i) = -(C + i)\) for some constant \(C > 0\). After the first five rounds of the while loop, \(A_1 = A_2 = \{c_{10}, c_9, c_8, c_7, c_6, c_5, c_4, c_3, c_2, c_1\}\). However, in the sixth round when computing \(c’\) for agent 3, all available chore copies fail to produce an exclusive allocation when added to \(A_3\), and so we continue to assign the remaining chore copies to agents 1 and 2 to arrive at the final allocation \(A_1 = \{c_{10}, c_9, c_8, c_7, c_6, c_5, c_4, c_3, c_2, c_1\}\), \(A_2 = \{c_{10}, c_9, c_8, c_7, c_6, c_5, c_4, c_3\}\), \(A_3 = \{c_1, c_2, c_3, c_4, c_5\}\). We have that \(v_1(A_1 \setminus \{c_1\}) = -C - 8 + \sum_{i=3}^{10} -(C + 11 - i) = -7C - 29\), while \(v_1(A_3) = \sum_{i=1}^{5} -(C + 11 - i) = -5C - 40\). Setting \(C > 5.5\), we get a violation of the \(\text{EF1}\) (and thus \(\text{EF1}_{\text{WC}}\)) condition.

### 6.2 Existence of \(\text{EFX}_{\text{WC}}\) for leveled preferences

**Definition 15.** A valuation \(v\) is a leveled preference for goods if for any two bundles, \(|B_1| > |B_2|\) \(\implies v(B_1) > v(B_2)\).

In particular, additive leveled preferences capture a notion of approximately cardinaly identical valuations (which we call \(\epsilon\)-balanced). This complements our discussion of identical ordinal valuations.

**Definition 16.** A valuation \(v\) is an \(\epsilon\)-balanced valuation for goods if for any two items \(i, j \in M\), \(v(i) < (1 + \epsilon)v(j)\).

**Lemma 1.** With \(m \geq 2\) agents, a \(\frac{1}{m-1}\)-balanced valuation is an additive leveled preference.

**Proof.** Consider any two bundles \(B_1, B_2\) with \(m \geq |B_1| > |B_2|\). Let \(B'_t\) be the \(t\) element (by some indexing) of bundle \(B'_t\). Then,

\[
v(B_2) = \sum_{t=1}^{|B_2|} v(B'_t) \leq |B_2| \max_{1 \leq t \leq |B_2|} v(B'_t) < (1 + \frac{1}{m-1}) |B_2| \min_{1 \leq t \leq |B_1|} v(B'_t)
\]

\[
\leq |B_1| \min_{1 \leq t \leq |B_1|} v(B'_t) = \sum_{t=1}^{|B_1|} v(B'_t) = v(B_1)
\]

\[ \square \]

An \(\alpha\)-\(\text{EFX}\) allocation does not always exist for leveled preferences over goods with copies:

**Example 10.** Consider \(n = 3\) (three agents) with identical additive valuations, and two goods \(\{a, b\}\), each with two copies, with \(v(a) = 1, v(b) = 1 + \frac{1}{a}\). For any allocation, there must be an agent \(i\) with both goods, and an agent \(j\) without the good \(b\). Agent \(j\) \(\alpha\)-\(\text{EFX}\) envies \(i\).

We prove that an \(\text{EFX}_{\text{WC}}\) allocation always exists for goods with copies with leveled preferences. By our duality framework (Theorem 2), this existence result holds for chores and in fact for mixed goods and chores (see the discussion in Section 7.2.1).

**Theorem 8.** There is an algorithm that always finds an \(\text{EFX}_{\text{WC}}\) exclusive allocation for goods with copies in the case of additive leveled preferences. Its runtime is \(O(n|T|^2)\).
We discuss challenges and generalizations of our approach.

To the best of our knowledge, we provide the first formal duality relationship between goods with valuation, and both agents get a bundle with the same cardinality as before, thus maintaining the sets of lower-level and upper-level bundle agents unchanged. We repeat the process.

Let agent $i$ get the bundle $(A_i \setminus \{g'\}) \cup \{g_{\max}\}$, and let agent $j$ get the bundle $(A_j \setminus \{g_{\max}\}) \cup \{g_{\min}\}$. After this operation, the allocation remains exclusive. Agent $i$ gets a strictly improved bundle by its valuation, and both agents get a bundle with the same cardinality as before, thus maintaining the sets of lower-level and upper-level bundle agents unchanged. We repeat the process.

We construct the potential function to show the number of steps is bounded and polynomial. For any good $g \in T$, let $\omega_i(g)$ be its ordinal position according to agent $i$’s preference over the goods, e.g., for the minimal good $g \in T$ by $i$ valuation we have $\omega_i(g) = 1$, and for the maximal good $g'$ we have $\omega_i(g') = |T|$. We consider the potential function

$$\psi(A) = \sum_{i \in N} \sum_{g \in A_i} \omega_i(g).$$

Notice that at each step this potential function strictly increases as we replace some good with a strictly preferred good for some lower level agent. Also note that $0 \leq \min_{A \in X(M)} \psi(A) \leq \max_{A \in X(M)} \psi(A) \leq n|T|^2$, and the function always returns an integer value. Thus the maximal number of substitution steps is in $O(n|T|^2)$.

The result of Theorem 8 implies that unlike Example 10, which is for general goods with copies, for goods with one copy each, there always exists an EFX allocation (since with one copy EFX and EFXWC are the same).

**Corollary 3.** There is an algorithm that finds an EFX allocation for goods with one copy and additive leveled preferences, with runtime $O(n|M|^2)$.

### 7 Discussion

To the best of our knowledge, we provide the first formal duality relationship between goods with copies and chores that establishes the equivalence of both settings for a broad class of fairness notions. We discuss challenges and generalizations of our approach.

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### 7.1 Challenges

Many existence results of envy-based fairness notions for goods without copies rely on the primitive of envy-cycle canceling, first shown by [39]. We thus note an important technical difference in proving existence for $f_{\text{WC}}$ notions. For goods with copies, it is sometimes impossible to cancel an envy-cycle without breaking the fairness notion. This was first pointed out in [14]. We give below another such example that has two additional properties: First, the allocation is EFX$_{\text{WC}}$ before cancelling the envy-cycle, but not even $\alpha$-EF1$_{\text{WC}}$ (for any choice of $\alpha$) after the cancellation. Second, in our example the choice of which envy-cycle to cancel is immaterial to the difficulty arising, as there is only one envy cycle.

**Example 11.** There are 3 agents and 7 good types. The valuation and the number of copies are listed in Table 4. Note that these valuations have identical ordinal preferences. Consider the allocation $A_1 = \{b, c, d, e, f\}$, $A_2 = \{H, a\}$, $A_3 = \{H, c, d, e\}$. It is an EFX$_{\text{WC}}$ allocation. We have that agent 1 envies agent 2 and vice versa. Let us reallocate the bundles to resolve this envy-cycle. Even though everyone gets a better bundle, it is not $\alpha$-EF1$_{\text{WC}}$, as agent 1 then gets the bundle $\{H, a\}$ but $\alpha$-EF1$_{\text{WC}}$-envies the bundle $\{H, c, d, e\}$.

Another intriguing property of goods with copies is that the MNW allocation is not even necessarily EF1$_{\text{WC}}$ (unlike the case of goods), as the following example shows.

**Definition 17.** The Nash welfare of an allocation is the product $NW(A) = \prod_{i \in N} v_i(A_i)$. The maximum Nash welfare allocation is $\text{MNW} = \arg \max_A NW(A)$.

**Example 12.** Consider Table 5 with $\epsilon = 10^{-6}$. The MNW allocation is the exclusive allocation $\{(a, b, c), \{a\}, \{d\}\}$. In this allocation agent 2 EF1$_{\text{WC}}$-envies agent 1.

### 7.2 Generalizations

#### 7.2.1 Mixed Goods and Chores

Our duality results can be extended to mixed goods and chores. To formally present this result, we need a whole set of new notation. For simplicity, we only give a high level idea here. Instead of defining the dual of a set of items, we define the dual of each item separately. A chore is the dual of a good.

The dual construction of one good can be described as follows. Suppose that we have a good $g$. We can construct a chore (or equivalently a dual good) $g^\circ$ such that $v_i'(g^\circ) = -v_i(g)$ for each agent.

---

**Table 4.** Envy-cycle cancellation failure for goods with copies.

<table>
<thead>
<tr>
<th>Goods</th>
<th>H</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copies</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>$v_1$</td>
<td>$3.2 - \alpha/2$</td>
<td>$\alpha$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>$v_2$</td>
<td>2</td>
<td>1.5</td>
<td>1</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
</tr>
<tr>
<td>$v_3$</td>
<td>2</td>
<td>1</td>
<td>0.1</td>
<td>0.5</td>
<td>0.5</td>
<td>0.1</td>
<td>0.1</td>
</tr>
</tbody>
</table>

**Table 5.** An MNW allocation that is not EF1$_{\text{WC}}$.

<table>
<thead>
<tr>
<th>Goods</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copies</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$v_1$</td>
<td>1</td>
<td>1</td>
<td>$\epsilon$</td>
<td></td>
</tr>
<tr>
<td>$v_2$</td>
<td>1</td>
<td>$\epsilon$</td>
<td>$\epsilon$</td>
<td>$\epsilon$</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$\epsilon$</td>
<td>$\epsilon$</td>
<td>$\epsilon$</td>
<td>1</td>
</tr>
</tbody>
</table>
Consider some share-based fairness notion. For the difference between the valuations, we have the following two properties: 1) The relative difference between two bundles does not change. 2) The absolute difference between two valuations is a constant. The first point is that \( \gamma_i(A_i) - \gamma_i(A_j) = \gamma'_i(A'_i) - \gamma'_i(A'_j) \). The second point is that \( \gamma_i(A_i) - \gamma'_i(A'_i) = \gamma_i(g) \). These two points will imply that the duality of linear share (Theorem 3) could be extended for this kind of duality.

The dual construction from a chore to a good can be obtained similarly. If we can make dual of pure goods (or pure chores) by making dual chore by chore (or good by good).

### 7.2.2 Beyond Additive Valuations

Some of our results can be extended beyond additive valuations. In particular, Theorem 2 holds for general valuations, and Proposition 6 (in particular, Eq. 4) holds for all monotone valuations, as indicated in the proof. Other results do not generalize. For example, Theorem 6 can not be extended, even to monotone and submodular valuations:

**Definition 18.** A valuation \( v \) of goods is submodular if \( \forall S_1, S_2 \subseteq M, v(S_1) + v(S_2) \geq v(S_1 \cup S_2) + v(S_1 \cap S_2) \).

**Example 13 (For monotone and submodular valuations, there is an EFL\(_{WC}\) allocation with at most \( \frac{1}{n} \)-MMS for goods with copies).** Consider \( n \) agents and goods \( \{y_j^i\}_{1 \leq i \leq n} \) each with one copy, where \( v \) is the valuation function of all agents:

\[
\gamma(S) = |\{i \mid \exists j, y_j^i \in S\}|
\]

In words, there are \( n^2 \) goods, partitioned into \( n \) groups, where goods in the same group are substitutes, and the valuation of a bundle of goods equals to the number of good groups present in the bundle. Then,

\[
A = \{\{y_1^0, \ldots, y_n^0\}, \ldots, \{y_1^n, \ldots, y_n^n\}\}
\]

is EFL\(_{WC}\) (in fact it is envy-free), but \( \gamma(A_i) = 1 \) for each agent \( i \), while a MMS of \( n \) is guaranteed by

\[
A' = \{\{y_1^{\theta}, \ldots, y_1^{(\theta+1) \mod n}, \ldots, y_{n}^{(\theta+n-1) \mod n}\} \}_{1 \leq \theta \leq n}
\]

### 7.2.3 Weighted Envy Notions

Definition 3 is general in the sense that it allows approximate envy notions. However, it can be further generalized to accommodate envy notions with subjective weights. Let \( W \) be a \( n \times n \) matrix of positive weights \( \{w^i_j\}_{1 \leq i \leq n} \). Define:

**Definition 3 (redefined and generalized).** An allocation \( A \in X(M) \) of goods is:

- **W-EFX** if \( \forall i, j \in N, \forall g \in A_j : \frac{\gamma_i(A_i \setminus (g))}{w^i_j} \geq \frac{\gamma_i(A_i)}{w^i_j} \);
- **W-EF1** if \( \forall i, j \in N, \exists g \in A_j : \frac{\gamma_i(A_i \setminus (g))}{w^i_j} \geq \frac{\gamma_i(A_i)}{w^i_j} \);

Notice that this definition generalizes \( \alpha \)-approximate notions: Being \( \alpha \)-EFX is the same as being W-EFX with the matrix \( w^i_j = \begin{cases} 1 & i = j \\ \frac{\alpha}{\alpha} & i \neq j \end{cases} \). It also generalizes the weighted notion of [20]: For some
weight vector \( w = \{w_i\}_{1 \leq j \leq n} \), being \( \text{WEF} \) with weights \( w \) is the same as being \( \text{W-EF} \) with the matrix \( \forall i, w_j = w_i \). The notion of \( \text{W-EF} \) (or alternatively \( \text{W-EF} \)) also allows for subjective weights, i.e., where every agent views the agents’ weights differently. Notice that approximate EFX with \( \alpha < 1 \) is a form of subjective weight EFX. We next define \( \text{W-EF} \) and \( \text{W-EF} \) for chores:

**Definition 8** (redefined and generalized). An allocation \( A \in X(M) \) of chores is:

- \( \text{W-EF} \) if \( \forall i, j \in N, \forall c \in A_j : w_i^j v_i (A_i \setminus \{c\}) \geq w_j^i v_i (A_j) \);
- \( \text{W-EF} \) if \( \forall i, j \in N, \exists c \in A_j : w_i^j v_i (A_i \setminus \{c\}) \geq w_j^i v_i (A_j) \);

Definition 10 and Theorem 2 can be used in an immediate and straight-forward way to define the notions of \( \text{W-EF} \) and \( \text{W-EF} \) for chores, and show that they are dual notions.

8 TECHNICAL SUMMARY AND FUTURE WORK

We recap our main results in this work. In Definition 1 we define exclusive allocations for goods (or chores) with copies. We see in Proposition 1 and Example 1 that EFX does not exist, even approximately and for \( n = 3 \) agents, for goods with copies. This is despite results that guarantee EFX existence for goods and approximate EFX existence for chores in the 3 agents setting [22, 42]. We define the more refined notion of EFX (Definition 7), that when considering the envy between two agents, only considers copies of goods that are unique to each agent. We see that this notion is much more natural to the goods with copies setting (Section 4, and in particular Theorem 2) in that it is preserved across the duality transformation (Definition 9). This immediately guarantees \( \frac{1}{2} - \text{EF} \) in the settings of Example 2 (which is a special case of Example 1, see Corollary 1). We show that many share notions common in the literature such as MMS, PROP and their various refinements are dual (Theorem 3), while other share notions such as TPS and \( \alpha \)-MMS are not. In Section 5, we establish a hierarchy of fairness notions summarized in Figure 1. Notably (but quite naturally), EFX is strictly stronger than EFX, while EF is strictly weaker than EF. We also find that similarly to goods without copies, the envy notions guarantee share notions approximations, albeit with adjusted constants. In Section 6, we develop two existence results for our new envy notions: We show that for identically ordinal additive preferences, an adaptation of the round-robin algorithm for chores with copies yields an EF allocation for chores with copies (and this naturally extends, by duality, to goods with copies). We show that for additive leveled preferences, an efficient swapping algorithm yields an EFX allocation for goods with copies. In Section 7, we show that achieving further existence results is complicated by the fact that useful and common primitives such as envy-cycle canceling or the max-Nash-welfare allocation do not guarantee even EF allocation in the case of goods with copies. An interesting avenue to consider is how to generalize the results of [9] to goods with copies, in the hope that it can yield EF existence results, maybe even together with Pareto optimality (the round-robin method which we utilize in Theorem 7 does not guarantee this).

For future work, we believe it is of interest to further investigate the duality phenomenon studied in this paper in the context of other fair division settings, e.g., divisible items. The main open challenge for the new fairness notions that we propose (such as EFX, EF) is to characterize their general existence for goods with copies. For this purpose, it is interesting to contrast our existence results with [33], which shows an example where an EFX allocation does not exist for a lexicographic mix of goods and chores. It is also worth investigating a more flexible model where the number of copies of each good is only loosely set, e.g., constrained between a minimal and maximal value. This is natural to the settings that motivate our work (such as student assignment to classes or licensing) and may permit further existence results.
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11The COMSOC version is accessible at https://drive.google.com/open?id=17b5r_kXT8I CIzvKkU7-0Vh_WRTAlosk.


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This immediately implies that $\mathcal{GMM}_S(v)$ one mapping from exclusive allocations $A$. Card-wise and groupwise MMS are linear shares.

\[ |A| \in X(M) \text{ with } \mu_k^i(A) \geq \mu_k^i(\bigcup_{j \in J} A_j) \]

PMMS and GMMS are linear.

Proof. Suppose that item sets $M$ and $M'$ are linearly related with $v, v'$ and $d$. We have a one to one mapping from exclusive allocations $A \in X(M)$ to exclusive allocations $A' \in X(M')$ such that $v(A_i) = v'(A'_i) + d$ for all $i \in N$ and all allocations. We prove that $\mathcal{GMM}_k^S$ is linear for any $k, i$. This immediately implies that $\mathcal{GMM}_k^S, \mathcal{GMMS}_i^k$ are linear and thus the statement of the proposition.

For $\mathcal{GMM}_k^S$, we have

\[ \max_{J \subseteq N_i \mid |J| = k,i \in J} \mu_k^i \left( \bigcup_{j \in J} A_j \right) = \max_{J \subseteq N_i \mid |J| = k,i \in J} \max_{B \in \mathcal{X}(\bigcup_{j \in J} A_j)} \min_{J \in [k]} v_i(B_j) = \]

\[ d + \max_{J \subseteq N_i \mid |J| = k,i \in J} \max_{B \in \mathcal{X}(\bigcup_{j \in J} A_j)} \min_{J \in [k]} v_i'(B'_j) = \]

where by $\mu_k^i$ we mean $\mu_k^i$ where the valuation function $v_i$ is replaced with $v'_i$. \hfill $\square$

B AN EXTENSION OF THEOREM 3 TO PROPc AND PROPx

Some relaxations to the notion of proportional share (PROP) were introduced in [24, 41].
Theorem 4. An exclusive allocation \( A \) is \( \text{PROP}_c \)-fair for goods for \( M, v \) iff \( A^o \) is \( \text{PROP}_c \)-fair for chores for \( M^o, v^o \). The same statement holds for \( \text{PROP}_x \) as well.

\[
\text{PROP}_c \text{-fair for chores:} \quad \exists Y \subseteq T \setminus A_i \text{ s.t. } |Y| \leq c, v(A_i \cup Y) \geq \text{PROP}(v_i, M, n); \\
\text{PROP}_x \text{-fair for goods:} \quad \forall Y \subseteq T \setminus A_i \text{ s.t. } |Y| = 1, v(A_i \cup Y) \geq \text{PROP}(v_i, M, n).
\]

We can define similarly for chores by replacing each \( T \setminus A_i \) with \( A_i \), and \( A_i \cup Y \) with \( A_i \setminus Y \).

Notice that this does not exactly match our general definition of \( s \)-share-fair in Definition 4, and moreover, requires different definitions for the goods and chores notions. However, with the appropriate adjustments, the proof of Theorem 3 can be extended to this case. We show this for \( \text{PROP}_c \), and \( \text{PROP}_x \) is done similarly.

Definition 20. An allocation \( A \) is \( \text{PROP}_c \)-fair for goods if for every agent \( i \)
\[
\exists Y \subseteq T \setminus A_i \text{ s.t. } |Y| \leq c, v(A_i \cup Y) \geq \text{PROP}(v_i, M, n);
\]
and is \( \text{PROP}_x \)-fair for goods if for every agent \( i \)
\[
\forall Y \subseteq T \setminus A_i \text{ s.t. } |Y| = 1, v(A_i \cup Y) \geq \text{PROP}(v_i, M, n).
\]

\( \text{PROP}_c \)-fair for chores: \quad \exists Y \subseteq T \setminus A_i \text{ s.t. } |Y| \leq c, v(A_i \cup Y) \geq \text{PROP}(v_i, M, n); \\
\text{PROP}_x \)-fair for goods: \quad \forall Y \subseteq T \setminus A_i \text{ s.t. } |Y| = 1, v(A_i \cup Y) \geq \text{PROP}(v_i, M, n).

We wish to further show that this does not just violate the linearity but also the duality. For this, we first have to define TPS for chores.

Definition 21. (Truncated Proportional Share for goods) Given an instance and an agent \( i \), the truncated proportional share is
\[
\text{TPS}(v_i, M, n) = \max_z \left\{ z \left| \frac{1}{n} \sum_{g \in M} \min\{v_i(g), z\} = z \right. \right\}.
\]

The following example shows that TPS is not a linear share:

**Example 14.** Suppose that there are 3 agents and 3 goods. Let us focus on one agent’s valuation: \( 2, 3, 5 \). The TPS would be 2 at this point. If we add 3 copies of a good with valuation 3, then \( \text{PROP} \) would be \( \frac{12}{3} > 5 \). After adding this good’s copies, the TPS becomes \( \frac{12}{3} \), which increases more than 3.

We wish to further show that this does not just violate the linearity but also the duality. For this, we first have to define TPS for chores.

**Definition 22.** (Truncated Proportional Share for chores) Given an instance and an agent \( i \), the truncated proportional share is
\[
\text{TPS}(v_i, M, n) = \min \left\{ \frac{1}{n} \cdot v(M), \min_{i \in M} \{v(i)\} \right\}.
\]
To the best of our knowledge, TPS was not defined for chores previously, so this definition requires some justification. First, it preserves the intuition of TPS for goods, where since large items prevent an equal distribution of goods, they are truncated for a more realistic estimation of the possible fair share. In the chores case, large items must be assigned, and so a fair share must consider receiving them as a singleton bundle. The definition also preserves the important property that $TPS \leq PROP$. Yet, it does not satisfy duality, as we see in the following example:

**Example 15.** Consider $n = 4$ agents, 6 goods with 1 copy each, and valuations 1, 3, 4, 6, 7, 19 respectively. It has $PROP = 10$, $TPS = 7$ as $\frac{1}{n} \sum_{g \in M} \min \{v(g), 7\} = \frac{1}{4} (1 + 3 + 4 + 6 + 7 + 7) = 7$, but for any $z > 7$, $\frac{1}{n} \sum_{g \in M} \min \{v(g), z\} = \frac{1}{4} (1 + 3 + 4 + 6 + 7 + z) = 7 + \frac{z-7}{4} \neq 7 + z - 7 = z$. The TPS value is implemented by allocation $A = \{\{1, 6\}, \{3, 4\}, \{7\}, \{19\}\}$. The dual chores instance has 6 chores with three copies each and valuations $-1, -3, -4, -6, -7, -19$ respectively. For this instance, $PROP = TPS = -30$ by direct calculation. But the dual allocation $A^c$ has a bundle with value $-33$, which is less than the TPS value.