SUBCOMPLEXES ON FILTERED MANIFOLDS

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Abstract. In this paper, we present a general construction of complexes on filtered manifolds. In the particular case of regular subRiemannian manifolds, we show that this yields the so-called Rumin complex when the manifold is also equipped with a compatible Riemannian metric. Another instance is set on a nilpotent Lie group equipped with a left invariant filtration; our scheme then computes the Lie algebra cohomology.

Contents

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1. Introduction

1.1. **Motivation.** The de Rham complex and its cohomology have inspired many mathematical ideas in the twentieth century, including Hodge theory and the Atiyah-Singer index theorem on Riemannian manifolds. An increasing number of schemes constructing other complexes and calculating their cohomologies have appeared in recent years in several geometric settings [CCY16, Tri54, DH22, Cas11, ČH23, LT23], especially in subRiemannian geometry.

In [FT23a], we obtained such a scheme on homogeneous nilpotent Lie groups by adapting ideas from parabolic geometry [ČSS01, CD01, DH22]. In the present paper, we push these ideas further to obtain two complexes on any regular subRiemannian manifold and on any filtered manifold equipped with a global coframe.

The importance of constructing complexes in these settings comes from the wide range of applications that ensue. These include:

- large scale geometry (especially the open conjecture classifying nilpotent Lie groups by quasi-isometries [PR18, PT19, BFP22]),
- analytic torsion [RS12, Kit20, Hal22] and currents [Can21, Vit22, JP12] for subRiemannian geometry, and
- K and index theories for subelliptic operators and C^*-algebras associated with subRiemannian structures [vE10, BvE14].

1.2. **Sketch of the construction of the complexes.** Let us briefly sketch our construction. The various claims will be proved in Sections 6 and 7. We consider two settings:

Setting (1): M is a filtered manifold equipped with a global coframe, and

Setting (2): M is a regular subRiemannian manifold equipped with a compatible Riemannian metric (in the sense of Definition 7.2).

The following differentials are naturally constructed over the filtered manifold M:

- the osculating Chevalley-Eilenberg differential $d_{\mathfrak{g}M}$ on the space $G\Omega^\bullet(M)$ of osculating forms over M, defined above each point $x \in M$ as the Chevalley-Eilenberg differential of the osculating group G_xM (see Section 5.2),
- the base differential $d_0 = \Phi^{-1} \circ d_{\mathfrak{g}M} \circ \Phi$ acting on $\Omega^\bullet(M)$ where $\Phi : \Omega^\bullet(M) \to G\Omega^\bullet(M)$ is the natural isomorphism identifying forms and osculating forms in our settings (see Sections 4.4 and 4.5.2 for Setting (1) and (2) respectively).

In both settings, d_0^t makes sense unambiguously.

We can now start our construction of the subcomplex from the de Rham differential d acting on the space $\Omega^\bullet(M)$ of forms over M. One could then define the operators P and for Π_0 on $\Omega^\bullet(M)$, where P is the projection onto $\ker d_0^t \cap \ker d_0^t d$ along $\operatorname{Im} d_0^t + \operatorname{Im} dd_0^t$, and Π_0 is the projection

$$\text{onto} \quad E_0 := \ker d_0 \cap \ker d_0^t \quad \text{along} \quad F_0 := \operatorname{Im} d_0 + \operatorname{Im} d_0^t \ .$$

We instead opt for the equivalent definitions of Π_0 as the orthogonal projection onto the kernel of the algebraic operator $\Box_0 := d_0 d_0^t + d_0^t d_0$, and of P as the generalised kernel projection of the differential operator $\Box := dd_0^t + d_0^t d$, i.e. $P = \frac{1}{2\pi i} \oint_{|z|=\varepsilon} (z - \Box)^{-1} dz$ for $\varepsilon > 0$ small enough. The map Π_0 is algebraic, while P is a differential operator. Once we define

$$L := P\Pi_0 + (\mathrm{I} - P)(\mathrm{I} - \Pi_0),$$

an invertible differential operator with L^{-1} also differential, we have the decomposition

$$L^{-1} dL = \Pi_0 L^{-1} dL \Pi_0 + (\mathrm{I} - \Pi_0) L^{-1} dL (\mathrm{I} - \Pi_0) =: D + C,$$

yielding two complexes $(E_0^\bullet, D)$ and $(F_0^\bullet, C)$ whose cohomologies are described as follows:

- The map C is conjugated to d_0 on F_0 via the invertible map g:

  $$Cg = gd_0, \quad \text{where} \quad g : F_0 \to F_0, \quad g := Cd_0^t \Box_0^{-1} + d_0^t \Box_0^{-1} d_0 \quad \text{on} \quad F_0.$$

  Since $(F_0^\bullet, d_0)$ is acyclic (i.e. has trivial cohomology), so is $(F_0^\bullet, C)$.
- The differential operator $\Pi_0 L^{-1}$ is a chain map between $(\Omega^\bullet(M), d)$ and $(E_0^\bullet, D)$, i.e. $(\Pi_0 L^{-1})d = D(\Pi_0 L^{-1})$. Since

  $$\mathrm{I} - L(\Pi_0 L^{-1}) = dh + hd\,, \quad \text{where} \quad h := L\, g d_0^t \Box_0^{-1}\, g^{-1} (\mathrm{I} - \Pi_0)\, L^{-1},$$

  this chain map is homotopically invertible. Consequently, the cohomology of $(E_0^\bullet, D)$ is linearly isomorphic to the de Rham cohomology of the manifold M.

We can modify the construction above by applying the same steps, but substituting the initial differential operator d with the algebraic part $\tilde{d}$ of the de Rham differential instead (see Section 3.1). This yields two complexes $(\tilde{E}_0^\bullet, \tilde{D})$ and $(\tilde{F}_0^\bullet, \tilde{C})$, with $(\tilde{F}_0^\bullet, \tilde{C})$ acyclic and $(\tilde{E}_0^\bullet, \tilde{D})$ linearly isomorphic to $(\Omega^\bullet(M), \tilde{d})$.

1.3. **Main results.** The main result of this paper is to present and justify all the steps and claims of the construction above (see Section 6). We also obtain explicit formulae for the new complexes $(E_0^\bullet, D)$ and $(\tilde{E}_0^\bullet, \tilde{D})$, showing that $E_0^\bullet = \tilde{E}_0^\bullet$ and that $D = \Pi_0 dP\Pi_0$ and $\tilde{D} = \Pi_0 d\tilde{P}\Pi_0$ (Section 7.4). Moreover, we recognise $(E_0^\bullet, D)$ as the subcomplex constructed by Rumin in [Rum90, Rum94, Rum99] on a regular subRiemannian manifold M equipped with a Riemannian structure. We also discuss the need for an extra hypothesis of compatibility between the Riemannian and subRiemannian structures (see Section 1.4 below).

At the end of the paper, we give an explicit construction for the operators D and $\tilde{D}$ on a 3D contact manifold. In a forthcoming paper, we will consider examples of Setting (1) given by nilpotent Lie groups equipped with left-invariant filtrations.

Our choice for Settings (1) and (2) here is motivated by recent advances in subRiemannian geometry, some listed in Section 1.1. We believe that the construction presented in Section 1.2 (and the slightly more general construction in Section 6) is flexible and that it will be adaptable to other settings in the future. An interesting generalisation would be to singular subRiemannian manifolds (e.g. Martinet distrubtions) and to the quasi-subRiemannian setting, for instance the Grushin plane.

1.4. **Compatibility.** Part of the motivation for the present work is to clarify the constructions in Rumin's initial papers [Rum90, Rum94, Rum99]. Indeed, we feel that the original definitions of both the base differential d_0 and its formal transpose d_0^t (or equivalently its partial inverse d_0^{-1}) might leave room for misunderstandings and inaccuracies. For example, in [Rum99], the need for a Riemannian metric when constructing the operator d_0^t is only hinted at. However, as discussed in Subsection 7.2.2, it is important to pay particular attention to the choice of the Riemannian metric. Indeed, the differential d_0 is defined on $\Omega^\bullet(M)$ via an identification with its osculating counterpart $d_{\mathfrak{g}M}$. However, this identification between forms and osculating forms may not be unique in general. More problematically, an arbitrary choice of Riemannian metric may not allow for a canonical identification of d_0^t with $d_{\mathfrak{g}M}^t$ - a step which seems essential in Rumin's construction.

For this identification to hold, we have determined an additional requirement: the Riemannian metric needs to be compatible with the subRiemannian structure in the sense explained in Subsection 7.2.2. The reason is that, in this setting, two metrics appear naturally on the osculating Lie algebra bundle:

- one induced by the subRiemannian structure (see Section 7.2.1), and
- the other induced by the filtration of the tangent space together with the Riemannian metric.

Our notion of compatibility mentioned above means that these two metrics on the osculating Lie algebra bundle coincide. This includes but goes beyond the Riemannian metric coinciding with the subRiemannian metric on the generating distribution. In this paper, we go through many elementary or well-known preliminaries to show how this compatibility condition emerges, yielding Setting (2) above.

We leave open the questions of the relation between two Rumin complexes on the same filtered manifold M but equipped with two different global coframes in Setting (1), or two different compatible Riemannian metrics in Setting (2). This is related to understanding the behaviour of the Rumin complex under changes of variables on filtered manifolds or pullback by Pansu-differentiable maps on Carnot groups [FT12, KMX28].

1.5. **Organisation of the paper.** In order to present the hypotheses of our setting and to remove any ambiguity on the objects we consider, the paper starts with well-known preliminaries on filtrations over vector spaces and on manifolds (Sections 2 and 3 respectively). Building on this, we present further preliminaries on how a filtration of the tangent bundle leads naturally to a filtration of the exterior algebra and a notion of weight on the corresponding graded space (see Section 4). When the manifold is filtered (i.e. when the tangent bundle is filtered and the commutator brackets of vector fields respect this filtration), then we can define the osculating Chevalley-Eilenberg differential $d_{\mathfrak{g}M}$ and other osculating objects (see Section 5). In Section 6, we present the general scheme for our construction. We then apply it to Settings (1) and (2) in Section 7, and to the particular case of 3D contact manifolds in Section 8.

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2. Preliminaries on filtrations of vector spaces

In this section, we briefly recall the construction of graded objects associated with a filtration of vector spaces. The notions presented below extend naturally to smooth vector bundles and their smooth sections, as well as to modules.

In this paper, all the vector spaces are taken over the field of real numbers.

2.1. **Filtered vector spaces.** A *filtered vector space* W is a vector space W with a filtration by vector subspaces:

$$\{0\} = W_0 \subseteq W_1 \subseteq \ldots \subseteq W_s = W. \tag{2.1}$$

In general, a filtration need not be finite; however, in this paper, all the filtrations considered will be finite as above.

A *graded vector space* G is a vector space G equipped with a gradation by vector subspaces, i.e. a direct sum decomposition by vector subspaces $G_j \subset G$:

$$G = \oplus_{j=1}^{s} G_j\,. \tag{2.2}$$

If G is graded, there is a natural *filtration associated with its gradation* (2.2), given by $\{0\} = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_s = G$, where $W_j := \oplus_{k\leq j} G_k$, $j = 1, \ldots, s$.

2.1.1. *The vector space* $\mathrm{gr}(W)$. Given a filtered vector space W with the filtration (2.1), its *associated graded space* is the vector space

$$\mathrm{gr}(W) := \oplus_{j=1}^{s} W_j / W_{j-1} \,. \tag{2.3}$$

Many of the summands above may be trivial. To avoid dealing with zero quotient spaces, it is enough to consider an appropriate choice of indices $w_0, w_1, \ldots, w_{s_0} \in \mathbb{N}$ for which the summands in (2.3) are not trivial, or equivalently for which the inclusions in (2.1) are strict:

$$\{0\} = W_{w_0} \subsetneq W_{w_1} \subsetneq \ldots \subsetneq W_{w_{s_0}} = W \ , \text{ and } \mathrm{gr}\,(W) = \oplus_{j=1}^{s_0} W_{w_j}/W_{w_{j-1}}. \tag{2.4}$$

2.1.2. *The linear map* $\mathrm{gr}(T)$. Consider a linear map $T : W \to W'$ between two vector spaces, each equipped with a filtration $\{0\} = W_0 \subseteq W_1 \subseteq \ldots \subseteq W_s = W$ and $\{0\} = W'_0 \subseteq W'_1 \subseteq \ldots \subseteq W'_{s'} = W'$. We may assume $s = s'$.

The map T is said to *respect the filtrations* when $T(W_j) \subseteq W'_j$ for every $j = 0, \ldots, s$. In this case, one can define the linear map $\mathrm{gr}(T) : \mathrm{gr}\,(W) \to \mathrm{gr}\,(W')$ as

$$\mathrm{gr}(T)\,(v \bmod W_j) := (Tv) \bmod W'_j \ \ , \ v \in W_{j+1} \ , \ j = 0, \ldots, s-1 \,.$$

2.1.3. *The basis* $\langle e \rangle$. If W is finite dimensional, then $\mathrm{gr}(W)$ has the same dimension as W. Given $n := \dim W = \dim \mathrm{gr}(W)$ and $n_j = \dim W_{w_j}/W_{w_{j-1}}$, we say that a basis $e = (e_1, \ldots, e_n)$ of W is *adapted to the filtration* (2.4), when $(e_1, \ldots, e_{n_1})$ is a basis of W_{w_1}, $(e_1, \ldots, e_{n_2+n_1})$ is a basis of W_{w_2}, and so on. For such a basis, we have that, for each $j = 0, \ldots, s_0 - 1$,

$$\langle e_i \rangle = e_i \bmod W_{w_j}, \quad i = d_j + 1, \ldots, d_{j+1} \ , \text{ with } d_j = n_0 + n_1 + \cdots + n_j \,,$$

gives a basis $\langle e \rangle := (\langle e_1 \rangle, \ldots, \langle e_n \rangle)$ of $\mathrm{gr}(W)$.

In particular, we get that for each $j = 0, \ldots, s_0 - 1$, the n_j-tuple $(\langle e_{d_j+1} \rangle, \ldots, \langle e_{d_{j+1}} \rangle)$ is a basis of $W_{w_{j+1}}/W_{w_j}$. In this sense, the basis $\langle e \rangle$ is graded and is said to be the graded basis associated with e.

2.1.4. *Matrix representation of linear maps.* In this paper, we will often use matrix representations of certain maps. To fix some notation, given a morphism $\varphi : \mathcal{V}_1 \to \mathcal{V}_2$ between two finite dimensional vector spaces, once we fix β_1 and β_2 two bases of $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively, we denote by

$$\mathrm{Mat}_{\beta_1}^{\beta_2}\,\varphi$$

the matrix representing φ in the bases β_1 and β_2.

Let us take e and e' two bases adapted to the filtrations (2.4) of two finite dimensional filtered vector spaces W and W' respectively. Assuming $s_0 = s'_0$, $n_j = \dim W_{w_j}/W_{w_{j-1}}$

and $n'_j = \dim W'_{w_j}/W'_{w_{j-1}}$, if a linear map $T\colon W \to W'$ respects the filtrations, the matrix representation of T in the bases e, e' is block-upper-triangular, i.e.

$$M := \mathrm{Mat}_e^{e'} T = \begin{pmatrix} M_1 & * & * \\ 0 & \ddots & * \\ 0 & & M_s \end{pmatrix},$$

with $M_j \in \mathbb{R}^{n'_j} \times \mathbb{R}^{n_j}$, with $j = 1, \dots, s_0$. Moreover, the matrix representing $\mathrm{gr}(T)$ in $\langle e\rangle, \langle e'\rangle$ is block-diagonal, i.e.

$$\mathrm{gr}(M) := \mathrm{Mat}_{\langle e\rangle}^{\langle e'\rangle} \mathrm{gr}(T) = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & & M_s \end{pmatrix}.$$

2.1.5. *Subfiltration.* We say that a vector subspace $\tilde{W} \subseteq W$ admits a subfiltration when $\tilde{W}_{w_j} := \tilde{W} \cap W_{w_j}$, $j = 1, \dots, s_0$ yields a filtration of $\tilde{W}$. In this case, the injection map $\tilde{W} \hookrightarrow W$ respects the filtrations and $\mathrm{gr}\,(\tilde{W})$ is a subspace of $\mathrm{gr}\,(W)$.

2.2. **Decreasing labelling and duality.** One can also consider a filtered vector space W with a decreasing filtration, that is a filtration with decreasing labelling:

$$W = V_0 \supseteq V_1 \supseteq \ldots \supseteq V_t = \{0\}\,. \tag{2.5}$$

It is not difficult to adapt all the constructions considered previously to a decreasing filtration. Indeed, it suffices to change the labels by setting $W_j := V_{t-j}$, so that, for instance, the associated graded space becomes $\mathrm{gr}\,(V) := \oplus_{j=0}^{t-1} V_j/V_{j+1}$. However, in this setting, the matrix representations will differ from those in Subsection 2.1.4, as the matrix representing a linear map that respects such decreasing filtrations will be block-lower-triangular.

We observe that if W has a decreasing filtration, then we obtain an increasing filtration of the dual space W^* of W by considering the filtration by the subspaces $V_j^\perp \subset W^*$, the annihilators of the subspaces $V_j \subset W$, that is

$$\{0\} = V_0^\perp \subseteq V_1^\perp \subseteq \ldots \subseteq V_t^\perp = W^*\ ,\ V_j^\perp = \{\ell \in W^* \mid \ell|_{V_j} \equiv 0\}\,,$$

also known as the dual filtration of (2.5).

Given two finite dimensional filtered vector spaces W and W', let us take e and e' two bases adapted to the decreasing filtrations $W = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_t = \{0\}$ and $W' = V'_0 \supseteq V'_1 \supseteq \cdots \supseteq V'_t = \{0\}$. Then for any linear map $T\colon W \to W'$ that respects these decreasing filtrations, the dual map

$$T^*\colon W'^* \to W^* \quad \text{where} \quad T^*(\ell) = \ell \circ T\ ,\ \forall\ \ell \in W'^*$$

respects the dual filtrations, and

$$\mathrm{Mat}_{e'^*}^{e^*} T^* = \left(\mathrm{Mat}_e^{e'} T\right)^t,$$

where e^* and e'^* denote the dual bases of e and e' respectively (one can easily show that if a basis e is adapted to a decreasing filtration (2.5), then e^* is adapted to its increasing dual filtration). In particular, this readily implies that the matrix representation of $T^*\colon W'^* \to W^*$ is block-upper-triangular, as pointed out in Subection 2.1.4.

Moreover, the vector spaces $\mathrm{gr}(V^*)$ and $(\mathrm{gr}\,(V))^*$ are canonically isomorphic.

2.3. **Graded vector spaces.** As explained earlier, given a graded vector space G, its gradation (2.2) naturally produces a filtration through $\oplus_{k\leq j} G_k$, while a filtration (2.1) of a vector space W yields a graded space $\mathrm{gr}\,(W)$ (2.3).

Definition 2.1. Let $G = \oplus_{j=1}^{s} G_j$ and $G' = \oplus_{j'=1}^{s'} G'_j$ be two graded vector spaces. We may assume $s = s'$. A linear map $T : G \to G'$ respects the gradations when $T(G_j) \subset G'_j$ for any $j = 1, \ldots, s$.

For instance, if T is a linear map between two filtered vector spaces that respects the filtrations, then $\mathrm{gr}\,(T)$ respects the gradations.

Consider a graded vector space G as above. Then its dual G^* is also graded with the dual gradation $G^* = \oplus_{j=1}^{s} G_j^{\perp}$. The k^{th} exterior algebra is also graded via the (possibly infinite) gradation

$$\wedge^k G = \oplus_{w=1}^{\infty} G^{k,w}\ , \text{ where } \ G^{k,w} := \sum_{j_1+\cdots+j_k=w} G_{j_1} \wedge \cdots \wedge G_{j_k}\ .$$

This leads to a (possibly infinite) gradation of the exterior algebra $\wedge^\bullet G$ which respects the wedge product:

$$\wedge^\bullet G = \oplus_{w=0}^{\infty} G^{\bullet,w}, \qquad G^{\bullet,w} = \oplus_{k=0}^{\infty} G^{k,w},$$

with $\wedge^0 G = G^{0,0} = \mathbb{R}$, and $G^{0,w} = \{0\}$ for $w > 0$. If G is finite dimensional, the gradations of $\wedge^k G$ and $\wedge^\bullet G$ are finite.

In the graded setting, there is a natural notion of weight: an element in G_j has weight j, and more generally, an element in $G^{\bullet,w}$ has weight w.

2.4. **Filtration on the exterior algebra.** A filtration (2.1) on a vector space W also induces a decreasing filtration on its k^{th} exterior algebra:

$$\wedge^k W = W^{k,\geq 0} \supseteq W^{k,\geq 1} \supseteq \ldots \supseteq W^{k,\geq k\cdot s+1} = \{0\}, \tag{2.6}$$

where

$$W^{k,\geq w} := \sum_{j_1+\cdots+j_k\geq w} W_{j_1} \wedge \cdots \wedge W_{j_k}\ .$$

The following lemma is easily checked.

Lemma 2.2. *Let W be a finite dimensional vector space equipped with a filtration* (2.1)*, and let $e = (e_1, \ldots, e_n)$ be a basis of W adapted to the filtration. Define the linear map*

$$\Psi : \wedge^k W \to \wedge^k \mathrm{gr}\,(W),$$

via

$$\Psi(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \langle e_{i_1} \rangle \wedge \cdots \wedge \langle e_{i_k} \rangle, \quad i_1 < \ldots < i_n.$$

Then Ψ is a linear isomorphism that respects the filtration (2.6) *of $\wedge^k W$ and the one induced by the gradation of $\wedge^k \mathrm{gr}\,(W)$. Moreover,* $\mathrm{gr}\,(\Psi)$ *provides an isomorphism between*

$$\mathrm{gr}(\wedge^k W) = \oplus_{w=0}^{k\cdot s} W^{k,\geq w}/W^{k,\geq w+1}$$

and $\wedge^k \mathrm{gr}\,(W)$ respecting their gradations.

Note that the isomorphism Ψ depends on the choice of the basis e.

2.5. **Euclidean filtered vector spaces.** In this Section, we consider a Euclidean space, that is, a finite dimensional vector space W equipped with a scalar product $(\cdot,\cdot)_W$.

Any subspace $W' \subset W$ is naturally Euclidean, its scalar product $(\cdot,\cdot)_{W'}$ being obtained by restricting $(\cdot,\cdot)_W$. Moreover, its orthogonal complement $W'^{(\perp,W)}$ in W is naturally identified with the quotient

$$W/W' \cong W'^{(\perp,W)}.$$

This quotient W/W' is also Euclidean if we equip it with the scalar product $(\cdot,\cdot)_{W/W'}$ corresponding to $(\cdot,\cdot)_{W'^{(\perp,W)}}$.

Beside being Euclidean, we further assume the vector space W also be filtered with filtration (2.4). The resulting graded space $\mathrm{gr}\, W$ inherits a natural scalar product $(\cdot,\cdot)_{\mathrm{gr}W}$ by imposing that the decomposition (2.3) is orthogonal, and that $(\cdot,\cdot)_{\mathrm{gr}W}$ restricted to each $W_{w_j}/W_{w_{j-1}}$ is given by $(\cdot,\cdot)_{W_{w_j}/W_{w_{j-1}}}$. Indeed, each $W_{w_j}/W_{w_{j-1}}$ is naturally isomorphic to the orthogonal complement $V_j := W_{n_{j-1}}^{(\perp,W_{n_j})}$ of $W_{w_{j-1}}$ in W_{w_j}. Therefore, by construction, the gradation by quotients on $\mathrm{gr}\,(W)$ is isomorphic to the following gradation of W:

$$W = V_1 \oplus^\perp V_2 \oplus^\perp \ldots \oplus^\perp V_{s_0},$$

which we will refer to as the *orthogonal gradation* or orthogonal graded decomposition of W (associated with the given scalar product and filtration).

We observe that when considering a basis $e = (e_1, \ldots, e_n)$ of W adapted to its filtration, after a Graham-Schmidt process, we may assume that $(e_1, \ldots, e_{n_1})$ is an orthonormal basis of $V_1 = W_1$, $(e_{n_1+1}, \ldots, e_{n_2+n_1})$ is an orthonormal basis of V_2, etc. In other words, after applying a Graham-Schmidt process, we can always obtain a basis e adapted to the given orthogonal gradation of W.

3. Preliminaries on manifolds

3.1. **The maps d and $\tilde{d}$ on a general manifold.** In this section, we recall briefly some well-known facts about the de Rham differential which are valid on any smooth manifold M (without any further assumptions). As is customary, d denotes the exterior differential on the space of smooth forms $\Omega^\bullet(M)$ on M. It is well known that the map $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ is a smooth differential map that satisfies the Leibniz property and $d^2 = 0$. The associated cochain complex $(\Omega^\bullet(M), d)$ is traditionally referred to as the de Rham complex.

The explicit formula for d is given for any $\omega \in \Omega^k(M)$ and $V_0, \ldots, V_k \in \Gamma(TM)$ by

$$d\omega(V_0, \ldots, V_k) = \sum_{i=0}^{k}(-1)^i V_i \left(\omega(V_0, \ldots, \hat{V}_i, \ldots, V_k)\right) + \sum_{0\leq i<j\leq k}(-1)^{i+j}\omega([V_i, V_j], V_0, \ldots, \hat{V}_i, \ldots, \hat{V}_j, \ldots, V_k),$$

the hats denoting omissions.

In this paper, the algebraic part of d is denoted by $\tilde{d}$. This is the map given by $\tilde{d} = 0$ on $\Omega^0(M)$, and for $k > 0$ and any $\omega \in \Omega^k(M)$, $V_0, \ldots, V_k \in \Gamma(TM)$, by

$$\tilde{d}\omega(V_0, \ldots, V_k) = \sum_{0\leq i<j\leq k}(-1)^{i+j}\omega([V_i, V_j], V_0, \ldots, \hat{V}_i, \ldots, \hat{V}_j, \ldots, V_k). \tag{3.1}$$

Lemma 3.1. *The map $\tilde{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ is smooth and algebraic. It satisfies the Leibniz property and $(\Omega^\bullet(M), \tilde{d})$ is a complex, i.e. $\tilde{d}^2 = 0$. The action of $\tilde{d}$ over $\Omega^1(M)$ is given by the explicit formula*

$$\tilde{d}\omega(V_0, V_1) = -\omega([V_0, V_1]) \ , \ \forall\, \omega \in \Omega^1(M),\ V_0, V_1 \in \Gamma(TM)\,.$$

Sketch of the proof. Since $\tilde{d} = d - (d - \tilde{d})$ and for any $\omega \in \Omega^k(M)$, $V_0, \ldots, V_k \in \Gamma(TM)$

$$(d - \tilde{d})\,\omega(V_0, \ldots, V_k) = \sum_{i=0}^{k}(-1)^i V_i \left(\omega(V_0, \ldots, \hat{V}_i, \ldots, V_k)\right)\ , \tag{3.2}$$

the Leibniz property of $\tilde{d}$ follows from the fact that both vector fields and d satisfy the Leibniz property.

The formula on 1-forms is easily computed from (3.1) by taking $k = 1$. This formula, together with the Jacobi identity for the commutator bracket, implies $\tilde{d}^2 = 0$ on $\Omega^1(M)$. Recursively and by the Leibniz rule, we get that $\tilde{d}^2 = 0$ on forms of any degree, since for any $\alpha \in \Omega^k(M)$ and any $\beta \in \Omega^\bullet(M)$

$$\begin{aligned}\tilde{d}^2(\alpha\wedge\beta) =& \tilde{d}\big(\tilde{d}\alpha\wedge\beta + (-1)^k\alpha\wedge\tilde{d}\beta\big)\\ =& (\tilde{d}^2\alpha)\wedge\beta + (-1)^{k+1}\tilde{d}\alpha\wedge\tilde{d}\beta + (-1)^k\tilde{d}\alpha\wedge\tilde{d}\beta + \alpha\wedge\tilde{d}^2\beta = 0\,.\end{aligned}$$

□

3.2. **Some natural concepts.** In this subsection, we recall some basic concepts in differential geometry.

3.2.1. *Frame and coframe.* Let E be a (real, finite dimensional, and smooth) vector bundle over a (smooth) manifold M with $\dim M = n$. A *frame* for E is a smooth section $(S_1, \ldots, S_n)$ of $E \times \ldots \times E = E^n$ such that $(S_1(x), \ldots, S_n(x))$ is a basis of E_x for every $x \in M$. A *coframe* for E is frame for the dual vector bundle E^*.

Throughout this paper, if the bundle E is not specified, we will take E to be the tangent bundle TM over the smooth manifold M and E^* to be the cotangent bundle. Frames for TM may not exist globally on the whole manifold M, but they can always be constructed locally, i.e. over an open neighbourhood of every point in M.

3.2.2. *Algebraic maps on bundles.* Let E and F be two smooth vector bundles over a manifold M. We say that a smooth map $T : \Gamma(E) \to \Gamma(F)$ is linear and algebraic when, for each $x \in M$, there exists a linear map $(T)_x : E_x \to F_x$ satisfying $T(\alpha)(x) = (T)_x(\alpha(x))$ for any $\alpha \in E$, $x \in M$. We will call $(T)_x$ the pointwise restriction of T above $x \in M$.

Remark 3.2. A (smooth) differential operator of order 0 is an algebraic linear map.

3.2.3. *Metric on bundles.* Let E be a vector bundle over a manifold M. By definition, a metric g on E is a family of scalar products $g_x := (\cdot, \cdot)_{E_x}$ on E_x depending smoothly on $x \in M$. The smooth dependence means that

$$g \in \Gamma(\mathrm{Sym}(E \otimes E)) .$$

When E is the tangent bundle TM of a smooth manifold M, the metric is said to be Riemannian and the couple (M, g) is referred to as a Riemannian manifold.

4. Filtration of tangent bundle

In this section, we consider an n-dimensional smooth manifold M whose tangent bundle TM is filtered by vector subbundles

$$M \times \{0\} = H^0 \subseteq H^1 \subseteq \ldots \subseteq H^s = TM . \tag{4.1}$$

Our aim is to extend the concepts of Section 2 to this setting. After an appropriate choice of indices like in (2.4), one can define the smooth vector bundle

$$\mathrm{gr}(TM) = \oplus_{i=1}^{s} H^i/H^{i-1} = \oplus_{j=1}^{s_0} H^{w_j}/H^{w_{j-1}} ,$$

with $n_j := \dim H^{w_j}/H^{w_{j-1}} > 0,\ j = 1, \ldots, s_0$, and $dim(M) = n = n_1 + n_2 + \cdots + n_{s_0}$.

4.1. **Adapted frames.**

The notion of adapted bases in the setting of filtered vector spaces naturally leads to the notion of adapted frames and coframes.

Definition 4.1. A frame $\mathbb{X} = (X_1, \ldots, X_n)$ for TM is said to be *adapted to the filtration* (4.1) when, for every $x \in M$, the basis $(X_1(x), \ldots, X_n(x))$ is adapted to the filtration $T_xM = H_x^{s_0} \supseteq \ldots \supseteq H_x^{w_1} \supseteq H_x^{w_0} = \{0\}$.

If $\mathbb{X} = (X_1, \ldots, X_n)$ is a frame for TM adapted to the filtration (4.1), then the associated graded basis on $\mathrm{gr}\,(T_xM)$

$$\langle \mathbb{X} \rangle_x = (\langle X_1 \rangle_x, \ldots, \langle X_n \rangle_x) ,\ x \in M ,$$

yields a frame $\langle \mathbb{X} \rangle$ for $\mathrm{gr}(TM)$ adapted to the gradation in the following sense.

Definition 4.2. A frame $\mathbb{Y} = (Y_1, \ldots, Y_n)$ for $\mathrm{gr}(TM)$ is *adapted to the gradation*, or *graded*, when $Y_1, \ldots, Y_{n_1}$ is a frame for H^{w_1}/H^{w_0}, $Y_{n_1+1}, \ldots, Y_{n_2+n_1}$ is a frame for H^{w_2}/H^{w_1}, and so on.

By duality, we obtain the following decreasing filtration of the cotangent bundle T^*M by vector bundles

$$T^*M = (H^{w_0})^\perp \supseteq (H^{w_1})^\perp \supseteq \ldots \supseteq (H^{w_{s_0}})^\perp = M \times \{0\}\,, \tag{4.2}$$

by considering the annihilators $(H^{w_j})^\perp$ of H^{w_j} for $j = 1, \ldots, s_0$, as follows.

Definition 4.3. A coframe Θ for TM is *adapted to the filtration* when, for every $x \in M$, the basis $\Theta(x) := (\theta^1(x), \ldots, \theta^n(x))$ is adapted to the filtration of $T_x^*M = (H_x^{w_0})^\perp \supseteq (H_x^{w_1})^\perp \supseteq \ldots \supseteq (H_x^{w_{s_0}})^\perp = \{0\}$.

This definition is equivalent to saying that, at each point $x \in M$, the final n_{s_0} covectors annihilate $H_x^{w_{s_0-1}}$. Furthermore, these final n_{s_0} covectors together with the preceding n_{s_0-1} covectors annihilate $H_x^{s_0-2}$, and so on. The associated frame $\langle\Theta\rangle = (\langle\theta^1\rangle, \ldots, \langle\theta^n\rangle)$ is a frame for $\mathrm{gr}(T^*M)$ that is adapted to the gradation in the following sense.

Definition 4.4. A frame $\Xi = (\xi^1, \ldots, \xi^n)$ for $\mathrm{gr}(T^*M)$ is said to be *adapted to the gradation* (or simply *graded*) when $\xi^n, \ldots, \xi^{n-n_{s_0}+1}$ is a frame for $(H^{w_{s_0-1}})^\perp/(H^{w_{s_0}})^\perp$, $\xi^{n-n_{s_0}}, \ldots, \xi^{n-n_{s_0-1}+1}$ is a frame for $(H^{w_{s_0-2}})^\perp/(H^{w_{s_0-1}})^\perp$, and so on.

One can readily check that if $\mathbb{X}$ is a frame adapted to a gradation for TM, then its dual Θ is an adapted coframe for TM. Conversely, if Θ is a coframe for T^*M adapted to a gradation of TM, then its dual $\Theta^* = \mathbb{X}$ is a graded frame for TM.

As mentioned in Subsection 3.2.1, frames for TM can always be constructed locally. Furthermore, through a pivoting process, one may easily construct a local frame for TM adapted to a given gradation. The inverse function theorem implies that, given a local frame $\mathbb{Y}$ of $\mathrm{gr}(TM)$ adapted to the gradation, we can then construct a local frame $\mathbb{X}$ of TM adapted to the filtration such that $\langle\mathbb{X}\rangle = \mathbb{Y}$.

4.2. **Weights for filtrations on tangent bundles.** Building on top of Section 4.1, we define below the natural notion of forms of weights at least w in this context.

4.2.1. *Weights of vector fields and 1-forms.* Given an increasing filtration (4.1) of TM, we denote by H^j the set of (based) vectors of weight at most j (shortened with $\leq j$). By duality, we say that the (based) covectors in $(H^{j-1})^\perp$ have weight at least j (shortened as $\geq j$). We extend this vocabulary to the space of smooth sections $\Gamma(TM)$ and $\Gamma(T^*M) = \Omega^1(M)$, that is vector fields and 1-forms respectively.

With our definition, a 1-form $\alpha \in \Omega^1(M)$ is of weight $\geq w$ when $\alpha(V) = 0$ for any $V \in \Gamma(H^{w-1})$. We denote the space of 1-forms of weight at least j by

$$\Omega^{1,\geq j}(M) := \Gamma((H^{j-1})^\perp), \qquad j = 1, 2, \ldots, s_0 + 1\,.$$

The filtration in (4.2) of the cotangent bundle then determines the following strict filtration

$$\Omega^1(M) = \Omega^{1,\geq w_1}(M) \supsetneq \Omega^{1,\geq w_2}(M) \supsetneq \ldots \supsetneq \Omega^{1,\geq w_{s_0+1}}(M) = \{0\}.$$

Example 4.5. If $\Theta := (\theta^1, \ldots, \theta^n)$ is a frame for T^*M adapted to the filtration (4.2), then all the 1-forms $\theta^1, \ldots, \theta^n$ are of weight $\geq w_1$. The 1-forms $\theta^{n_1+1}, \ldots, \theta^n$ are of weight $\geq w_2$, and more generally $\theta^{n_1+\cdots+n_{i-1}+1}, \ldots, \theta^n$ are of weight $\geq w_i$. By taking $n_0 = \dim H^{w_0} = 0$, we can state this as

$$\theta^j \in \Omega^{1,\geq w_i}(M), \quad j = n_0 + \cdots + n_{i-1} + 1, \ldots, n \;, \quad i = 1, \ldots, s_0 + 1.$$

4.2.2. *Weights of k-forms.* For $k = 0$, we set

$$C^\infty(M) = \Omega^0(M) := \Omega^{0,\geq 0}(M), \qquad \text{and} \qquad \Omega^{0,\geq 1}(M) := \{0\}.$$

For $k = 1, 2, \ldots$, we define the space of k-forms of weight at least w (shortened as $\geq w$) by

$$\Omega^{k,\geq w}(M) := \oplus_{i_1+\ldots+i_k \geq w} \Omega^{1,\geq i_1}(M) \wedge \ldots \wedge \Omega^{1,\geq i_k}(M).$$

In other words, a k-form is of weight at least w when it can be written locally as a linear combination of k-wedges of 1-form of weights at least $i_1, \ldots, i_k$ with $w \leq i_1 + \ldots + i_k$. From the definition, it follows that a k-form $\alpha \in \Omega^k(M)$ is of weight $\geq w$ when

$$\forall\; V_1 \in \Gamma(H^{i_1}), \ldots, V_k \in \Gamma(H^{i_k}) \qquad i_1 + \ldots + i_k < w \Longrightarrow \alpha(V_1, \ldots, V_k) = 0.$$

Example 4.6. Consider an adapted coframe Θ for TM. We will often use the shorthand

$$\Theta^{\wedge I} = \theta^{i_1} \wedge \ldots \wedge \theta^{i_k}$$

for the multi-index $I = (i_1, \ldots, i_k)$, always assuming $i_1 < \ldots < i_k$. Then $\Theta^{\wedge I}$ is of weight $\geq \upsilon_1 + \ldots + \upsilon_k$, where each θ^{i_j} is of weight $\geq \upsilon_j$.

Definition 4.7. The family $(\Theta^I)_I$ where I runs above all multi-indices of length k (taken with strictly increasing indices as in example 4.6 above) is a free basis of the $C^\infty(M)$-module $\Omega^\bullet(M)$ called the basis associated with the coframe Θ.

By construction, we have the inclusion $\Omega^{k,\geq w}(M) \wedge \Omega^{j,\geq w'}(M) \subseteq \Omega^{k+j,\geq w+w'}(M)$ for any $k, j \geq 0$. Moreover, given the increasing filtration (2.4) of TM, for any degree $k \geq 0$

$$\Omega^{k,\geq k\cdot w_{s_0}+1}(M) = \{0\} \;\text{ and }\; w \leq w' \implies \Omega^{k,\geq w}(M) \supseteq \Omega^{k,\geq w'}(M)\,.$$

In particular, the module $\Omega^k(M)$ over the ring $C^\infty(M)$ admits the filtration

$$\Omega^k(M) = \Omega^{k,\geq 0}(M) \supset \Omega^{k,\geq 1}(M) \supset \ldots \supset \Omega^{k,\geq w}(M) \supset \ldots \supset \Omega^{k,\geq k\cdot w_{s_0}+1}(M) = \{0\}. \tag{4.3}$$

The basis $(\Theta^I)_I$ introduced in Definition 4.7 is adapted to this filtration. We will discuss the gradation corresponding to such filtration over the space of smooth k-forms in Lemma 4.15.

4.2.3. *The bundle $\wedge^k \mathrm{gr}(T^*M)$ and its weights.* Given the filtration (4.1), by duality the smooth vector bundle

$$\mathrm{gr}(T^*M) = \oplus_{i=1}^{s}((H^{i-1})^\perp/(H^i)^\perp) = \oplus_{j=1}^{s_0}((H^{w_j-1})^\perp/(H^{w_j})^\perp)$$

is graded. We say that an element of $\mathrm{gr}(T^*M)$ is of weight j when it is in $(H^{j-1})^\perp/(H^j)^\perp$. We extend the same notion of weight to smooth sections of $\mathrm{gr}(T^*M)$.

Example 4.8. Let us consider the same adapted coframe Θ for TM as in Example 4.5. Then $\langle\theta^1\rangle, \ldots, \langle\theta^{n_1}\rangle$ have weight w_1, and for $i = 2, \ldots, s_0$, the sections $\langle\theta^{n_1+\cdots+n_{i-1}+1}\rangle, \ldots, \langle\theta^{n_1+\cdots+n_i}\rangle$ of $\mathrm{gr}(T^*M)$ have weight w_i. If we denote by υ_j the weight of $\langle\theta^j\rangle$, we have:

$$\upsilon_{n_1+\cdots+n_{i-1}+1} = \ldots = \upsilon_{n_1+\cdots+n_i} = w_i, \qquad i = 1, \ldots, s_0. \tag{4.4}$$

There is a natural identification between the following smooth vector bundles

$$\wedge^k \mathrm{gr}(T^*M) := \cup_{x\in M} \wedge^k \mathrm{gr}(T^*_xM) \cong \cup_{x\in M}(\wedge^k \mathrm{gr}(T_xM))^* =: \wedge^k \mathrm{gr}(TM)^*,$$

allowing us to consider directly $\wedge^k \mathrm{gr}(T^*M)$ in our discussion.

It is straightforward to check that $\wedge^k \mathrm{gr}(T^*M)$ is a module over $C^\infty(M)$ which, by Section 2.3, is naturally equipped with a gradation and a notion of weight. In other words,

$$\wedge^k \mathrm{gr}(T^*M) = \oplus_{w\in\mathbb{N}} \wedge^{k,w} \mathrm{gr}(T^*M),$$

where the elements of weight w are in the linear subbundle

$$\wedge^{k,w}\mathrm{gr}(T^*M) := \sum_{i_1+\ldots+i_k=w} ((H^{i_1-1})^\perp/(H^{i_1})^\perp) \wedge \ldots \wedge ((H^{i_k-1})^\perp/(H^{i_k})^\perp).$$

Once we extend this construction to the space of smooth sections of $\wedge^\bullet \mathrm{gr}\,(T^*M)$, we get the gradation

$$\Gamma(\wedge^\bullet \mathrm{gr}\,(T^*M)) = \oplus_{k,w\in\mathbb{N}_0} \Gamma(\wedge^{k,w}\mathrm{gr}\,(T^*M))\,, \tag{4.5}$$

where the k-forms of weight w are given by

$$\Gamma(\wedge^{k,w}\mathrm{gr}\,(T^*M)) = \sum_{i_1+\cdots+i_k=w} \Gamma((H^{i_1-1})^\perp)/\Gamma((H^{i_1})^\perp) \wedge \cdots \wedge \Gamma((H^{i_k-1})^\perp)/\Gamma((H^{i_k})^\perp)\,.$$

Example 4.9. We continue with the setting of Example 4.8 and examine the case of k-forms, $k > 1$. The sections $\langle\theta^{i_1}\rangle \wedge \ldots \wedge \langle\theta^{i_k}\rangle$ of $\wedge^k \mathrm{gr}(T^*M)$ are of weight

$$|(\upsilon_1, \ldots, \upsilon_k)| := \upsilon_{i_1} + \ldots + \upsilon_{i_k}.$$

We may use the shorthand notation $\langle\Theta\rangle^{\wedge I} = \langle\theta^{i_1}\rangle \wedge \ldots \wedge \langle\theta^{i_k}\rangle$ where I is the multi-index $I = (i_1, \ldots, i_k)$ with $i_1 < \ldots < i_k$.

We readily check the following properties:

Lemma 4.10. *The family $(\langle\Theta\rangle^I)_I$, where I runs above all multi-indices of length k and strictly increasing indices, is a basis of $\Gamma(\wedge^k \mathrm{gr}(T^*M))$ adapted to the gradation. Moreover, it is the basis associated to the basis $(\Theta^I)_I$ adapted to the filtration (4.3) of $\Omega^\bullet(M)$ in the sense described in Section 2.1.3.*

4.3. **The** gr **operation.**

Definition 4.11. A morphism $D : \Omega^\bullet(M) \to \Omega^\bullet(M)$ of $C^\infty(M)$-modules respects the filtration (4.3), when

$$\forall w \in \mathbb{N}_0 \qquad D\,\Omega^{\bullet,\geq w}(M) \subseteq \Omega^{\bullet,\geq w}(M).$$

The following property follows readily from Section 2:

Lemma 4.12. *Let $D : \Omega^\bullet(M) \to \Omega^\bullet(M)$ be a morphism of $C^\infty(M)$-modules that respects the filtration* (4.3)*. Then the map*

$$\mathrm{gr}(D) : \Gamma(\wedge^\bullet \mathrm{gr}(T^*M)) \to \Gamma(\wedge^\bullet \mathrm{gr}(T^*M)),$$

is a morphism of $C^\infty(M)$-modules that respects the gradation (4.5)*. If in addition, D is a differential operator, then $\mathrm{gr}(D)$ is also a differential operator of the same order or lower.*

As an operation, gr *is a module morphism from $\{D : \Omega^\bullet(M) \to \Omega^\bullet(M) :$ module morphism$\}$ to $\{T : \Gamma(\wedge^\bullet \mathrm{gr}(T^*M)) \to \Gamma(\wedge^\bullet \mathrm{gr}(T^*M)) :$ module morphism$\}$.*

By construction, $\mathrm{gr}(D) = 0$ if and only if D increases the weight in the following sense.

Definition 4.13. We say that a linear map $D : \Omega^\bullet(M) \to \Omega^\bullet(M)$ *increases the weights* of the filtration when

$$\forall w \in \mathbb{N}_0 \qquad D\,\Omega^{\bullet,\geq w}(M) \subseteq \Omega^{\bullet,\geq w+1}(M).$$

As there are only a finite number of weights, a linear map that increases the weights is necessarily nilpotent, that is

$$\mathrm{gr}\, D = 0 \Longrightarrow \quad \exists\, N_0 \in \mathbb{N} \ \text{ such that } D^{N_0} = 0. \tag{4.6}$$

This N_0 depends only on M and its filtration (4.1) (via the filtrations (4.3)), not on D. Throughout this paper, we will assume it to be a fixed integer.

Remark 4.14. Above each $x \in M$, we define $\mathrm{UT}(T_xM)$ as the set of invertible linear maps that preserve the filtration. Equivalently, this is the space of the maps whose matrix representations in a basis adapted to the filtration are block-upper-triangular. The resulting smooth subbundle $\mathrm{UT}(TM) = \cup_{x\in M}\mathrm{UT}(T_xM)$ of $\mathrm{GL}(TM)$ acts linearly on TM. It will also act linearly on $\mathrm{gr}(TM)$ via the operation gr, and on $\mathrm{gr}(TM)^* \cong \mathrm{gr}(T^*M)$ by duality. More generally, it will also act linearly on $\wedge^\bullet \mathrm{gr}(T^*M)$ in a way that respects the $\wedge$ product.

We denote by $\mathrm{diag}(\mathrm{gr}(T_xM))$ the set of linear maps that preserve the gradation. Equivalently, this is the space of linear maps whose matrix representations in a basis adapted to the gradation are block-diagonal. We observe that the resulting smooth subbundle $\mathrm{diag}(\mathrm{gr}(TM)) = \cup_{x\in M}\mathrm{diag}(\mathrm{gr}(T_xM))$ of $\mathrm{GL}(\mathrm{gr}(TM))$ coincides with $\mathrm{gr}(\mathrm{UT}(TM))$.

4.4. **Identifying $\Omega^\bullet(M)$ and $\Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$ via coframes.**

The existence of a coframe on M as in Examples 4.5,4.6, 4.8 and 4.9 is a strong hypothesis seldom satisfied globally. However, about each point, it is guaranteed on any open neighbourhood small enough, leading at least to a local analysis.

4.4.1. *Identification via free bases.* Roughly speaking, the existence of a coframe allows us to identify the free bases of $\Omega^\bullet(M)$ and $\Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$.

Lemma 4.15. *Let Θ be an adapted coframe for TM. Then the map*

$$\Phi : \Omega^\bullet(M) \to \Gamma(\wedge^\bullet \mathrm{gr}(T^*M)) \quad \textit{defined via} \quad \Phi(\Theta^I) = \langle\Theta\rangle^I$$

for any multi-index $I = (i_1, \ldots, i_k)$ with $1 \leq i_1 < \ldots < i_k \leq n$, is an isomorphism of $C^\infty(M)$-modules that respects the filtrations, even when restricted to $\Omega^k(M) \to \Gamma(\wedge^k \mathrm{gr}(T^*M))$ for *each* $k = 0, 1, \ldots, n$. *Moreover,* $\mathrm{gr}(\Phi)$ *is an isomorphism between the* $C^\infty(M)$*-modules* $\mathrm{gr}(\Omega^k(M))$ *and* $\Gamma(\wedge^k \mathrm{gr}(T^*M))$ *that respects the gradation.*

Proof. This follows directly from Lemma 2.2. □

Note that the map Φ is in fact algebraic, with

$$\Phi_x : \Gamma(\wedge^\bullet T_x^* M) \to \Gamma(\wedge^\bullet \mathrm{gr}(T^*M)) \quad \text{defined via} \quad \Phi_x(\Theta^I(x)) = \langle\Theta\rangle^I(x).$$

When restricted to $\Omega^1(M)$, the matrix representing the map Φ from Lemma 4.15 is just the identity matrix of $\mathbb{R}^n$:

$$\mathrm{Mat}_\Theta^{\langle\Theta\rangle}(\Phi : \Omega^1(M) \to \Gamma(\mathrm{gr}(T^*M))) = \mathrm{I}.$$

Moreover, for each $k = 2, \ldots, n$, using the notation $\Theta^k = (\Theta^I)_I$ and $\langle\Theta\rangle^k = (\langle\Theta\rangle^I)_I$ for the corresponding bases of $\Omega^k(M)$ and of $\Gamma(\wedge^k \mathrm{gr}(T^*M))$, the matrix representing $\Phi\colon \Omega^k(M) \to \Gamma(\wedge^k \mathrm{gr}(T^*M))$ is the identity matrix of $\mathbb{R}^{\dim \Omega^k(M)}$:

$$\mathrm{Mat}_{\Theta^k}^{\langle\Theta\rangle^k}(\Phi) = \mathrm{I}\,.$$

4.4.2. *Reversing* gr. Intertwining with Φ provides forms with a reverse operation to the operation gr described in Lemma 4.12.

Lemma 4.16. *We continue with the setting of Lemma 4.15.*

(1) *If* $T : \Gamma(\wedge^\bullet \mathrm{gr}(T^*M)) \to \Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$ *is a morphism of* $C^\infty(M)$*-modules, then*

$$T^\Phi := \Phi^{-1} \circ T \circ \Phi : \Omega^\bullet(M) \to \Omega^\bullet(M)$$

is a morphism of $C^\infty(M)$*-modules. If in addition* T *is a differential operator, then* T^Φ *is also a differential operator of the same order.*

(2) *If* $T : \Gamma(\wedge^\bullet \mathrm{gr}(T^*M)) \to \Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$ *is a morphism of* $C^\infty(M)$*-modules that respects the gradation, then* $T^\Phi : \Omega^\bullet(M) \to \Omega^\bullet(M)$ *respects the filtration and we have*

$$\mathrm{gr}(T^\Phi) = T.$$

(3) *If* $D : \Omega^\bullet(M) \to \Omega^\bullet(M)$ *respects the filtration, then* $D - (\mathrm{gr}(D))^\Phi$ *strictly increases weights, and so*

$$\mathrm{gr}(D) = \mathrm{gr}\left((\mathrm{gr}(D))^\Phi\right).$$

Proof. These properties are easily checked on Θ^I and $\langle\Theta\rangle^I$. □

4.4.3. *Scalar products on $\wedge^\bullet T_x^*M$ and of $\wedge^\bullet \mathrm{gr}(T_x^*M)$.* Here, we briefly sketch how the existence of a global coframe naturally yields scalar products both on $\Omega^\bullet(M)$ and $\Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$.

We continue with the setting of Lemma 4.15. For each $x \in M$, we denote by $\langle \cdot, \cdot \rangle_{T_x^*M}$ and $\langle \cdot, \cdot \rangle_{\mathrm{gr}(T_x^*M)}$ the scalar products on T_x^*M and $\mathrm{gr}(T_x^*M) \cong \mathrm{gr}(T_xM)^*$, so that $\theta^1(x), \ldots, \theta^n(x)$ and $\langle\theta^1\rangle(x), \ldots, \langle\theta^n\rangle(x)$ are orthogonal. We extend them to scalar products $\langle \cdot, \cdot \rangle_{\wedge^k T_x^*M}$ and $\langle \cdot, \cdot \rangle_{\wedge^k \mathrm{gr}(T_x^*M)}$ on $\wedge^k T_x^*M$ and $\wedge^k \mathrm{gr}(T_x^*M)$ for any $k = 2, \ldots, n$ via

$$\langle \alpha_1 \wedge \ldots \wedge \alpha_k, \beta_1 \wedge \ldots \wedge \beta_k \rangle_{\wedge^k T_x^*M} := \det \left(\langle \alpha_i, \beta_j \rangle_{T_x^*M} \right)_{1 \le i,j \le k}, \tag{4.7}$$

where $\alpha_1, \ldots, \alpha_k, \beta_1 \ldots, \beta_k \in T_x^*M$, and similarly for $\mathrm{gr}(\wedge^k T_x^*M) \cong \wedge^k \mathrm{gr}(T_x^*M)$. These extend naturally to scalar products on

$$\wedge^\bullet T_x^*M = \oplus^\perp \wedge^k T_x^*M \quad \text{and} \quad \wedge^\bullet \mathrm{gr}(T_x^*M) = \oplus^\perp \wedge^k \mathrm{gr}(T_x^*M)\,.$$

We will denote all of these scalar product as $\langle \cdot, \cdot \rangle$, their meaning being clear from the context. With the setting of Lemma 4.15, we check readily that Φ_x maps the scalar product of $\wedge^\bullet T_x^*M$ to the one of $\wedge^\bullet \mathrm{gr}(T_x^*M)$:

$$\Phi_x \colon \wedge^\bullet T_x^*M \to \wedge^\bullet \mathrm{gr}(T_x^*M)\,, \quad \langle \Phi_x(\alpha), \Phi_x(\beta) \rangle = \langle \alpha, \beta \rangle \ \forall\, \alpha, \beta \in \wedge^\bullet T_x^*M\,. \tag{4.8}$$

In our discussions, the bases $\Theta^k(x) = (\Theta^I)_I(x)$ and $\langle\Theta^k\rangle(x) = (\langle\Theta^I\rangle)_I(x)$ of $\wedge^k T_x^*M$ and of $\wedge^k \mathrm{gr}(T_x^*M)$ respectively are orthonormal. We will use the following notation to denote the elements of the bases in degree 0 and n:

$$1 = \Theta^\emptyset \text{ for } \wedge^0 T_x^*M \quad \text{and } \mathrm{vol}_\Theta(x) := \theta^1(x) \wedge \ldots \wedge \theta^n(x) \text{ for } \wedge^n T_x^*M$$

and

$$1 = \langle\Theta\rangle^\emptyset \text{ for } \wedge^0 \mathrm{gr}(T_x^*M) \text{ and } \langle \mathrm{vol}_\Theta \rangle(x) := \langle\theta^1\rangle(x) \wedge \ldots \wedge \langle\theta^n\rangle(x) \text{ for } \wedge^n \mathrm{gr}(T_x^*M)\,.$$

4.4.4. *Hodge-star, scalar product on $\Omega_c^\bullet(M)$ and $\Gamma_c(\wedge^\bullet \mathrm{gr}(T^*M))$, and transpose.* The Hodge star operator on $\wedge^\bullet T_x^*M$ or $\wedge^\bullet \mathrm{gr}(T_x^*M)$ is the map defined via

$$\star : \wedge^k T_x^*M \xrightarrow{\cong} \wedge^{n-k} T_x^*M\,,\ \alpha \wedge \star\beta := \langle \alpha, \beta \rangle \mathrm{vol}_\Theta(x) \ \forall\, \alpha, \beta \in \wedge^k T_x^*M,\ k = 1, \ldots, n\,,$$

and similarly on $\wedge^\bullet \mathrm{gr}(T_x^*M)$. As shown in (4.8), Φ_x maps the scalar product of $\wedge^\bullet T_x^*M$ to the one of $\wedge^\bullet \mathrm{gr}(T_x^*M)$, and so it maps the Hodge star operator of $\wedge^\bullet T_x^*M$ to the one of $\wedge^\bullet \mathrm{gr}(T_x^*M)$:

$$\forall \alpha, \beta \in \wedge^\bullet T_x^*M \qquad \star\, \Phi_x(\alpha) = \Phi_x(\star\alpha). \tag{4.9}$$

The $\star$ operator naturally extends to an algebraic operator on $\Omega^\bullet(M)$ and on $\Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$ respectively. Moreover, from the definition of $\star$, we have that

$$\alpha \wedge \star\beta = \beta \wedge \star\alpha \quad \text{on } \Omega^\bullet(M) \text{ and } \Gamma(\wedge^\bullet \mathrm{gr}(T^*M)),$$

and

$$\star 1 = \mathrm{vol}_\Theta,\ \star\mathrm{vol}_\Theta = 1, \quad \text{and} \quad \star\, 1 = \langle \mathrm{vol}_\Theta \rangle,\ \star\langle \mathrm{vol}_\Theta \rangle = 1.$$

More generally, the Hodge star operators on Θ^I and $\langle\Theta\rangle^I$ with multi-index $I = (i_1, \ldots, i_k)$ are given by

$$\star(\Theta^I) = (-1)^{\sigma(I)} \Theta^{\bar{I}}, \qquad \star(\langle\Theta\rangle^I) = (-1)^{\sigma(I)} \langle\Theta\rangle^{\bar{I}}$$

where $\bar{I} = (\bar{i}_1, \ldots, \bar{i}_{n-k})$ with $\bar{i}_1 < \bar{i}_2 < \ldots < \bar{i}_{n-k}$ is the multi-index complementing I, and $(-1)^{\sigma(I)}$ is the sign of the permutation $\sigma(I) = i_1 \cdots i_k \,\bar{i}_1 \cdots \bar{i}_{n-k}$. This completely characterises the Hodge star operators on both $\Omega^\bullet(M)$ and on $\Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$.

The previous properties imply readily that for any α, β in either $\Omega^k(M)$ or $\Gamma(\wedge^k \mathrm{gr}(T^*M))$, we have

$$\star \star \beta = (-1)^{k(n-k)}\beta \qquad \text{and} \qquad \langle \star\alpha, \star\beta\rangle = \langle \alpha, \beta\rangle\,.$$

We can now define the so-called L^2-inner products on the subspaces $\Omega^\bullet_c(M)$ and $\Gamma_c(\wedge^\bullet \mathrm{gr}(T^*M))$ of forms with compact support in $\Omega^\bullet(M)$ and $\Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$ respectively via

$$\langle \alpha, \beta\rangle_{L^2} := \int_M \alpha \wedge \star\beta = \int_M \langle \alpha, \beta\rangle \,\mathrm{vol}\;,\ \forall\, \alpha, \beta \in \Omega^\bullet_c(M) \text{ or } \Gamma_c(\wedge^\bullet \mathrm{gr}(T^*M))\,.$$

We have $\Phi(\Omega^\bullet_c(M)) = \Gamma_c(\wedge^\bullet \mathrm{gr}(T^*M))$ and

$$\forall \alpha, \beta \in \Omega^\bullet_c(M) \qquad \langle \alpha, \beta\rangle_{L^2} = \langle \Phi(\alpha), \Phi(\beta)\rangle_{L^2}. \tag{4.10}$$

4.4.5. *Properties of the transpose for the L^2-inner product.* The above definitions of the Hodge operator and of the scalar product show that if D is a differential operator on either $\Omega^\bullet(M)$ or $\Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$, then so are $\star D$ and $D\star$, as well as the formal transpose D^t defined as $\langle D\alpha, \beta\rangle_{L^2} = \langle \alpha, D^t\beta\rangle_{L^2}$ for any $\alpha \in \Omega^k_c(M)$ or $\Gamma_c(\wedge^k \mathrm{gr}(T^*M))$ and any $\beta \in \Omega^{k+1}_c(M)$ or $\Gamma_c(\wedge^{k+1} \mathrm{gr}(T^*M))$.

In the particular case of the de Rham differential d on forms, we observe that the transpose of $d^{(k)} = d : \Omega^k(M) \to \Omega^{k+1}(M)$ is a differential operator that satisfies

$$d^{(k,t)} = (-1)^{kn+1} \star d^{(n-k-1)}\star \colon \Omega^{k+1}(M) \to \Omega^k(M)\,.$$

We can also consider the transpose of the algebraic linear map $\tilde{d}^{(k)} = \tilde{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ of Subsection 3.1. $\tilde{d}^t$ is then an algebraic linear map satisfying

$$\tilde{d}^{(k,t)} = (-1)^{kn+1} \star \tilde{d}^{(n-k-1)}\star \colon \Omega^{k+1}(M) \to \Omega^k(M)\,.$$

As shown in Examples 4.17 and 4.18 below, the transpose D^t of a differential operator $D\colon \Omega^\bullet(M) \to \Omega^\bullet(M)$ that respects the filtration (4.3) does not necessarily respect the filtration. However, this will always be the case when the operator is of the form $\mathrm{gr}(D)^\Phi$.

Example 4.17. We consider the 3-dimensional Heisenberg group $\mathbb{H}$, and the canonical basis X, Y, T of its Lie algebra $\mathfrak{h}$. The only non-trivial bracket is $[X, Y] = T$. Identifying $\mathfrak{h}$ with the space of left-invariant vector fields on $\mathbb{H}$, we obtain a filtration of the tangent bundle of $\mathbb{H}$ in subbundles:

$$\{0\} = H^0 \subset H^1 := \mathrm{span}_{\mathbb{R}}\{X, Y\} \subset H^2 := \mathrm{span}_{\mathbb{R}}\{X, Y, T\} = T\mathbb{H}^1\,.$$

The dual of the left-invariant frame $\mathbb{X} := (X, Y, T)$ of $T\mathbb{H}$ yields the global co-frame $\Theta := (X^*, Y^*, T^*)$ (as well as the natural linear isomorphism between $\Omega^\bullet(\mathbb{H})$ and $C^\infty(\mathbb{H}) \otimes \wedge^\bullet \mathfrak{h}^*$).

Let us now consider the de Rham differential $d\colon \Omega^\bullet(\mathbb{H}) \to \Omega^\bullet(\mathbb{H})$ acting on the 1-form $f\,T^* \in \Omega^{1,\geq 2}(\mathbb{H})$ with $f \in C^\infty(\mathbb{H})$:

$$d(fT^*) = df \wedge T^* + f\,dT^* = Xf\,(X^* \wedge T^*) + Yf\,(Y^* \wedge T^*) - f\,(X^* \wedge Y^*) \in \Omega^{2,\geq 2}(\mathbb{H})\,.$$

Instead we consider the formal transpose $d^t = (-1)^{k\cdot 3+1} \star d\star \colon \Omega^{k+1}(\mathbb{H}) \to \Omega^k(\mathbb{H})$ acting on the 2-forms $g\,(X^* \wedge Y^*) \;\in\; \Omega^{2,\geq 2}(\mathbb{H})$ with $g \in C^\infty(\mathbb{H})$:

$$\begin{aligned} d^t(g\,(X^* \wedge Y^*)) &= \star\, d(g\,T^*) = \star(-g\,(X^* \wedge Y^*) + Xg\,(X^* \wedge T^*) + Y g\,(Y^* \wedge T^*)) \\ &= -\,g\,T^* - Xg\,Y^* + Y g\,X^* \in \Omega^{1,\geq 1}(\mathbb{H}) \end{aligned}$$

and so $d^t(\Omega^{2,\geq 2}(\mathbb{H})) \subset \Omega^{1,\geq 1}(\mathbb{H})$, but not $\Omega^{1,\geq 2}(\mathbb{H})$.

Applying similar computations, one can easily check that the transpose of the operator $\mathrm{gr}(d)^\Phi$ instead respects the filtration.

A situation similar to Example 4.17 also holds for the algebraic part of the de Rham differential $\tilde{d}\colon \Omega^\bullet(\mathbb{G}) \to \Omega^\bullet(\mathbb{G})$, when considering a nilpotent Lie group $\mathbb{G}$ with a filtration on its Lie algebra $\mathfrak{g}$ that is not coming from a homogeneous structure as in the following example:

Example 4.18. Let us consider the 4-dimensional Engel group $\mathbb{G}$, that is, the connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{g} = \mathrm{span}_\mathbb{R}\{X_1, X_2, X_3, X_4\}$ and $[X_1, X_i] = X_{i+1}$ for $i = 2, 3$. We identify $\mathfrak{g}$ with the space of left-invariant vector fields, and we consider the following left-invariant filtration of $T\mathbb{G}$

$$\{0\} = H^0 \subset H^1 = \mathrm{span}_\mathbb{R}\{X_1, X_2\} \subset H^2 = \mathrm{span}_\mathbb{R}\{X_1, X_2, X_3, X_4\} = T\mathbb{G}\,.$$

The coframe of the canonical frame $\mathbb{X} = (X_1, , X_2, X_3, X_4)$ is denoted by $\Theta = (\theta^1, \theta^2, \theta^3, \theta^4)$ Then, for an arbitrary form $f_1\,\theta^3 + f_2\,\theta^4$ in $\Omega^{1,\geq 2}(\mathbb{G})$, with $f_1, f_2 \in C^\infty(\mathbb{G})$, we have

$$\tilde{d}(f_1\,\theta^3 + f_2\,\theta^4) = -\,f_1\,(\theta^1 \wedge \theta^2) - f_2\,(\theta^1 \wedge \theta^3) \;\in \Omega^{2,\geq 2}(\mathbb{G})\,.$$

However, given a form $g\,(\theta^1 \wedge \theta^3) \;\in \Omega^{2,\geq 3}(\mathbb{G})$ with $g \in C^\infty(\mathbb{G})$, we obtain $\tilde{d}^t(g\,(\theta^1 \wedge \theta^3)) = -g\,\theta^4 \in \Omega^{1,\geq 2}(\mathbb{G})$.

Lemma 4.19. *We continue in the setting of Lemma 4.16, where $T : \Gamma(\wedge^\bullet \mathrm{gr}(T^*M)) \to \Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$ is a morphism of $C^\infty(M)$-modules. We consider the maps $T^\Phi = \Phi^{-1} \circ T \circ \Phi$ and $(\star T \star)^\Phi = \star T^\Phi \star$ acting on $\Omega^\bullet(M)$.*

Let us further assume that T respects the gradation.

1. *The $C^\infty(M)$-module morphism $\star T \star\colon \Gamma(\wedge^\bullet \mathrm{gr}(T^*M)) \to \Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$ also respects the gradation, and $(\star T \star)^\Phi = \star T^\Phi \star$. Consequently, the $C^\infty(M)$-module morphism $D := T^\Phi : \Omega^\bullet(M) \to \Omega^\bullet(M)$ respects the filtration and we have $\mathrm{gr}(\star D \star) = \star \mathrm{gr}(D) \star$.*
2. *If in addition, T is a differential operator, then T^t is also a differential operator of the same order as T, it respects the gradation, and satisfies $(T^t)^\Phi = (T^\Phi)^t$. Consequently, the differential operators on smooth forms $D := T^\Phi$ and its transpose D^t respect the filtration, and $\mathrm{gr}(D^t) = (\mathrm{gr}(D))^t$.*

Proof. The equality $(\star T \star)^\Phi = \star T^\Phi \star$ follows from (4.9).

We observe that $\langle\Theta\rangle^I$ is of weight $|I|$ while $\star\langle\Theta\rangle^I = (-1)^{\sigma(I)} \langle\Theta\rangle^{\bar{I}}$ is of weight $|\bar{I}| = Q - |I|$ where $Q := \upsilon_1 + \ldots + \upsilon_n$ is the weight of the volume form.

Since T respects the gradation and the weight of $\langle\Theta\rangle^{\bar{I}}$ is $|\bar{I}|$, $T(\langle\Theta\rangle^{\bar{I}})$ has the same weight $|\bar{I}|$. Consequently, $T(\langle\Theta\rangle^{\bar{I}})$ is a $C^\infty(M)$-linear combination of basis elements $\langle\Theta\rangle^J$ where the multi-indices J have weight $|J| = |\bar{I}| = Q - |I|$. Finally, $\star T(\Theta^{\bar{I}})$ has weight $Q - |J| = |I|$. We have therefore shown that the morphism $\star T \star$ maps elements of weight $|I|$ to elements of weight $|I|$, and so $\star T \star$ respects the gradation. Lemma 4.16 implies the rest of Part (1).

Part (2) follows from the definition of D^t and its link with $\star$ covered in Subsection 4.4.4. $\square$

4.5. **Identifying $\Omega^\bullet(M)$ and $\Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$ via metrics.** Let M be a manifold whose tangent bundle admits a filtration (4.1) by subbundles and is equipped with a metric g_{TM}.

4.5.1. *The natural orthogonal gradation of TM.* Note that from Section 2.5, we have that $\mathrm{gr}(TM)$ inherits a metric $g_{\mathrm{gr}(TM)}$. The same section also implies that, at each point $x \in M$, there exists an open neighbourhood U of x and a frame $\mathbb{X}$ of TM on U that is g_{TM}-orthonormal and adapted to the natural orthogonal gradation

$$TM = \oplus_j^\perp G^j,$$

where G^j is the orthogonal complement of $H^{w_{j-1}}$ in H^{w_j} for each $j = 1, \ldots, s_0$.

4.5.2. *The map Φ.* Let $x \in M$. For any $j = 1, \ldots, s_0$, if ξ is a covector in G_x^j (i.e. a linear form on T_xM vanishing on every G_x^k, $k \neq j$), we denote by $\Phi_x(\xi)$ the corresponding covector in $H_x^{w_j}/H_x^{w_{j-1}}$ (i.e. the corresponding linear form on $\mathrm{gr}(T_x^*M)$ vanishing on $H_x^{w_k}/H_x^{w_{k-1}}$, $k \neq j$). This extends uniquely to a linear bijective map $\Phi_x : T_x^*M \to \mathrm{gr}(T_x^*M)$, and then to a $C^\infty(M)$-module isomorphism

$$\Phi : \Omega^\bullet(M) \to \Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$$

respecting the Leibniz property.

Lemma 4.20. *Let M be a manifold whose tangent bundle admits a filtration by subbundles (4.1) and is equipped with a metric g_{TM}. Consider the map Φ constructed above.*

(1) *Let $\mathbb{X}$ be a g_{TM}-orthonormal local frame of TM on an open subset $U \subset M$ that is adapted to the natural orthogonal gradation $TM = \oplus^\perp G^j$. Denoting by $\Theta^\bullet$ the associated g_{TM}-orthonormal frame of $\Omega^\bullet(U)$, the map*

$$\Phi_\Theta \colon \Omega^k(U) \to \Gamma(\wedge^k T^*U) \text{ given by } \Phi_\Theta(\Theta^I) = \langle\Theta\rangle^I$$

for any increasing multi-index I of length k as constructed in Lemma 4.15, coincides with the map Φ constructed in Subsection 4.5.2 above. In other words, we have that

$$\Phi_\Theta = \Phi \quad \text{on } \Omega^\bullet(U) \cong \Omega^\bullet(M)|_U.$$

(2) *The properties described in Lemmata 4.15 and 4.16 also hold for the map Φ of Subsection 4.5.2.*

(3) *For each $x \in M$, the scalar products g_{T_xM} and $g_{\mathrm{gr}(T_xM)}$ on T_xM and $\mathrm{gr}(T_xM)$ respectively induce scalar products on their duals T_x^*M and $\mathrm{gr}(T_xM)^* \cong \mathrm{gr}(T_x^*M)$, and then scalar products $g_{\wedge^\bullet T_x^*M}$ on $\wedge^\bullet T_x^*M$ and $g_{\wedge^\bullet \mathrm{gr}(T_x^*M)}$ on $\wedge^\bullet \mathrm{gr}(T_x^*M)$ via (4.7). The map Φ_x sends the scalar product of $\wedge^\bullet T_x^*M$ onto the one of $\wedge^\bullet \mathrm{gr}(T_x^*M)$:*

$$g_{\wedge^\bullet \mathrm{gr}(T_x^*M)}(\cdot,\cdot) := g_{\wedge^\bullet T_x^*M}(\Phi_x^{-1}(\cdot), \Phi_x^{-1}(\cdot)).$$

(4) *If T and D are algebraic maps acting linearly on $\wedge^\bullet \mathrm{gr}(T^*M)$ and $\Omega^\bullet(M)$ respectively, then we define their transpose T^t and D^t as the algebraic maps acting linearly on $\wedge^\bullet \mathrm{gr}(T^*M)$ and $\Omega^\bullet(M)$ by their pointwise transpose, that is, $(T^t)_x$ and $(D^t)_x$ are the transpose of T_x and D_x for the scalar products $g_{\wedge^\bullet \mathrm{gr}(T_x^*M)}$ and $g_{\wedge^\bullet T_x^*M}$ at each point $x \in M$.*

If T is an algebraic map acting linearly on $\wedge^\bullet \mathrm{gr}(T^*M)$ respecting the gradation, then so is T^t. Moreover, the map $D := \Phi^{-1} \circ T \circ \Phi$ and its transpose D^t are both algebraic map acting linearly on $\Omega^\bullet(M)$ respecting the gradation, and satisfy:

$$D^t = \Phi^{-1} \circ T^t \circ \Phi \quad \textit{and} \quad \mathrm{gr}(D^t) = (\mathrm{gr}(D))^t.$$

In other words, Part (4) above means that Lemma 4.19 holds for algebraic maps with Φ as in Subsection 4.5.2.

Proof. We check that, by definition, $\Phi_x(\theta^i(x)) = \langle \theta^i \rangle(x)$, for any $i = 1, \dots, n$, $\theta^i \in \Theta^1$ and $x \in U$, and this readily implies Part (1). Part (2) is an immediate corollary. Fixing a frame $\mathbb{X}$ as in Part (1), Parts (3) and (4) follow from (4.8) and Lemma 4.19 respectively with Φ_Θ. □

4.5.3. *Case of an oriented manifold.* Assume in addition that M is an oriented manifold. If $\Theta^\bullet$ is the g_{TM}-orthonormal frame of $\Omega^\bullet(U)$ as in Part (1) of Lemma 4.20 and it is positively oriented, then $\mathrm{vol}_{g_{TM}}(x) := \theta^1(x) \wedge \ldots \wedge \theta^n(x)$ and $\langle \mathrm{vol}_{g_{TM}} \rangle(x) := \langle \theta^1(x) \rangle \wedge \ldots \wedge \langle \theta^n(x) \rangle$ are elements of $\wedge^n T_x^*M$ and $\wedge^n \mathrm{gr}(T_x^*M)$ respectively that are defined independently of such $\mathbb{X}$. In fact, we recognise $\mathrm{vol}_{g_{TM}}$ as the Riemannian volume form. We have

$$\langle \mathrm{vol}_{g_{TM}} \rangle := \Phi(\mathrm{vol}_{g_{TM}}) \ \in \Gamma(\wedge^\bullet(\mathrm{gr}(T^*M)))\,.$$

For any positively oriented and g_{TM}-orthonormal frame $\mathbb{X}$ of TM as in Part (1) of Lemma 4.20, its coframe $\Theta^\bullet$, we have

$$\forall\, x \in U \qquad \mathrm{vol}_{g_{TM}}(x) = \mathrm{vol}_\Theta(x) \quad \text{and} \quad \langle \mathrm{vol}_{g_{TM}} \rangle(x) = \langle \mathrm{vol}_\Theta \rangle(x).$$

The presence of a metric on an oriented manifold M allows us to define the associated Hodge star operation on $\wedge^\bullet T_x^*M$ and on $\wedge^\bullet \mathrm{gr}(T_x^*M)$, just like we did in Section 4.4.4 where we were assuming the presence of a global frame for TM. Just like in (4.9), the Hodge-$\star$ operator satisfies:

$$\forall\, \alpha, \beta \in \wedge^\bullet T_x^*M \qquad \star\, \Phi_x(\alpha) = \Phi_x(\star\alpha).$$

Given a coframe $\Theta^\bullet$ on $U \subset M$ as in (1) of Lemma 4.20, when restricting scalar products and the Hodge-$\star$ operator to U, they coincide with the objects defined via $\Theta^\bullet$ in Section 4.4.4. Consequently, as described in Section 4.4.4, we can analogously define the L^2-inner products on $\Omega_c^\bullet(M)$ and $\Gamma_c(\wedge^\bullet \mathrm{gr}(T^*M))$, and the relations between the $\star$ and the transpose also hold on $\Omega^\bullet(M)$ and $\Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$. In particular, the properties shown in Lemma 4.19 hold for the map Φ of Subsection 4.5.2.

4.6. **Properties of** gr **for algebraic maps.** Here, we describe some properties of gr in relation with algebraic maps (for the latter concept see Section 3.2.2).

The following properties are easily checked.

Lemma 4.21. *Let M be a smooth manifold whose tangent bundle is filtered by vector subbundles as in* (4.1)*.*

1. *Any algebraic morphism of $C^\infty(M)$-module D acting either on $\Omega^\bullet(M)$ or $\Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$ is determined by its pointwise restriction D_x at each $x \in M$. Conversely, any smooth map on $\wedge^\bullet T^*M$ or $\wedge^\bullet \mathrm{gr}(T^*M)$ that is linear at each fibre yields an algebraic morphism of $C^\infty(M)$-modules.*
2. *If a morphism of $C^\infty(M)$-modules D acting on $\Omega^\bullet(M)$ is algebraic and respects the filtration, then $\mathrm{gr}(D)$ is a morphism of $C^\infty(M)$-module acting on $\Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$ which is also an algebraic map that respects the gradation. Moreover, if D is invertible, then $\mathrm{gr}(D)$ is also invertible, and D^{-1} and $\mathrm{gr}(D)^{-1}$ are algebraic.*

The isomorphism Φ from Lemmata 4.15 and 4.16 together with the concept of algebraic maps allow us to construct inverses in some cases, and this provides a partial converse to (2) in Lemma 4.21 as follows.

Proposition 4.22. *Let M be a smooth manifold whose tangent bundle is filtered by vector subbundles as in* (4.1)*. Assume that M is equipped with either a global coframe $\Theta^\bullet$ adapted to the filtration or with a metric g_{TM}. In both cases, we consider the corresponding map Φ defined in Lemma 4.15 or in Section 4.5.2 respectively.*

Let $D : \Omega^\bullet(M) \to \Omega^\bullet(M)$ be a morphism of $C^\infty(M)$-modules that respects the filtration and such that $\mathrm{gr}(D)$ is algebraic. Assume also that $\mathrm{gr}(D)$ is fiberwise invertible in the sense that its restriction $\mathrm{gr}(D)_x$ at each $x \in M$ is a linear invertible map on $\wedge^\bullet T^*_x M$. Then the map D is invertible and the map $D^{-1} : \Omega^\bullet(M) \to \Omega^\bullet(M)$ respects the filtration. Moreover, $\mathrm{gr}(D^{-1}) = \mathrm{gr}(D)^{-1}$, and if D is in addition a differential operator acting on $\Omega^\bullet(M)$, then so is its inverse.

Proof. Let Φ be as in Lemma 4.15. By (2) in Lemma 4.16, the map $T := (\mathrm{gr}(D))^\Phi : \Omega^\bullet(M) \to \Omega^\bullet(M)$ respects the filtration and $\mathrm{gr}(T)^{-1} = \mathrm{gr}(D)^{-1}$. Moreover, by (2) of Lemma 4.21, T is invertible and its inverse is $T^{-1} = (\mathrm{gr}(D)^{-1})^\Phi$.

By Lemma 4.12, we have

$$\mathrm{gr}(T^{-1}D) = \mathrm{gr}(T^{-1})\mathrm{gr}(D) = \mathrm{gr}(D)^{-1}\mathrm{gr}(D) = \mathrm{I}\ , \ \text{so } \mathrm{gr}(T^{-1}D - \mathrm{I}) = \mathrm{gr}(T^{-1}D) - \mathrm{I} = 0\ .$$

Hence $T^{-1}D - \mathrm{I} =: B$ is nilpotent with $B^{N_0} = 0$ by (4.6). Since $T^{-1}D = \mathrm{I} + B$ and $\mathrm{I} + B$ is invertible with $(\mathrm{I} + B)^{-1} = \sum_{j=0}^{N_0} (-1)^j B^j$, we have $(\mathrm{I} + B)^{-1} T^{-1} D = \mathrm{I}$. This provides a left inverse for D and the formula below with $B = B_L$. We can do the same on the right. □

Corollary 4.23. *We continue with the setting of Proposition 4.22 and its proof. The following formulae holds for $D^{-1} : \Omega^\bullet(M) \to \Omega^\bullet(M)$:*

$$D^{-1} = \sum_{j=0}^{N_0} (-1)^j B_L^j T^{-1} = T^{-1} \sum_{j=0}^{N_0} (-1)^j B_R^j$$

where $T = \mathrm{gr}(D)^{\Phi} = \Phi \mathrm{gr}(D)^{-1}\Phi^{-1}$, *and* $B_L = T^{-1}D - \mathrm{I}$ *while* $B_R = DT^{-1} - \mathrm{I}$.

5. Filtered manifolds and osculating objects

In this section, we discuss the setting of filtered manifolds and some complexes naturally associated with them. In particular, we will define the osculating differential $d_{\mathfrak{g}M}$ and study its relation to the de Rham complex $(\Omega^\bullet(M), d)$. We will also use the algebraic part $\tilde{d}$ of d. The main result of this section is in Proposition 5.3 stating that $\mathrm{gr}(d) = \mathrm{gr}(\tilde{d}) = d_{\mathfrak{g}M}$.

5.1. **Setting and definitions.** A *filtered manifold* is a smooth manifold M equipped with a filtration of the tangent bundle TM by vector subbundles

$$M \times \{0\} = H^0 \subseteq H^1 \subseteq \ldots \subseteq H^s = TM$$

satisfying

$$[\Gamma(H^i), \Gamma(H^j)] \subseteq \Gamma(H^{i+j})\,, \tag{5.1}$$

with the convention that $H^i = TM$ when $i > s$.

Example 5.1. Any contact manifold, or more generally any regular subRiemannian manifold, is a filtered manifold. A related class of examples is given by nilpotent Lie groups equipped with a left-invariant filtration [LDT22].

For each $x \in M$, the quotient $\mathrm{gr}(T_xM) = \oplus_{i=1}^{s_0} (H^i_x/H^{i-1}_x)$ is naturally equipped with a Lie bracket $[\cdot,\cdot]_{\mathfrak{g}_xM}$, since

$$\forall\, f, g \in C^\infty(M),\ X \in \Gamma(H^i),\ Y \in \Gamma(H^j) \qquad [fX, gY] = fg[X,Y] + \Gamma(H^{i+j-1}).$$

When $\mathrm{gr}(T_xM)$ is equipped with this Lie bracket, we denote the resulting Lie algebra as

$$\mathfrak{g}_xM := (\mathrm{gr}(T_xM), [\cdot,\cdot]_{\mathfrak{g}_xM})\,.$$

It is naturally graded by

$$\mathfrak{g}_xM = \oplus_{i=1}^{s_0} \left(H^i_x/H^{i-1}_x\right),$$

and is therefore nilpotent. We denote by G_xM the corresponding connected simply connected nilpotent Lie group (sometimes called the nilpotentisation or the tangent cone [Mit85] of M at x). The unions

$$GM := \cup_{x\in M} G_xM \qquad \text{and} \qquad \mathfrak{g}M := \cup_{x \in M} \mathfrak{g}_xM,$$

are naturally equipped with a smooth bundle structure that are called [RS76, vEY17] the *bundles of osculating Lie groups and Lie algebras* over M.

We observe that the notions of weights defined in Section 4.2.3 and for the graded Lie algebra $\mathfrak{g}_xM$ coincide. We therefore obtain an analogous decomposition of $\wedge^\bullet \mathfrak{g}^*_xM$ in terms of weights and degrees:

$$\wedge^\bullet \mathfrak{g}^*_xM = \oplus_{k,w \in \mathbb{N}_0} \wedge^{k,w} \mathfrak{g}^*_xM\ , \quad \forall\, x \in M\ . \tag{5.2}$$

We introduce the following vocabulary.

Definition 5.2. The elements of $G\Omega^\bullet(M) := \Gamma(\wedge^\bullet \mathfrak{g}^* M)$ are called *osculating forms*. Moreover, for any $k, w \in \mathbb{N}_0$,

$$G\Omega^k(M) := \Gamma(\wedge^k \mathfrak{g}^* M) \qquad \text{and} \qquad G\Omega^{k,w}(M) := \Gamma(\wedge^{k,w} \mathfrak{g}^* M)$$

are called *osculating k-forms* (or osculating forms of degree k) and *osculating k-forms of weights w* respectively.

We say that a map $T\colon G\Omega^\bullet(M) \to G\Omega^\bullet(M)$ *respects the weights* of osculating forms when $T(G\Omega^{\bullet,w}(M)) \subseteq G\Omega^{\bullet,w}(M)$, and it *increases the weights* when $T(G\Omega^{\bullet,w}(M)) \subseteq \oplus_{w' \geq w} G\Omega^{\bullet,w'}(M)$.

5.2. **The osculating Chevalley-Eilenberg differential.** For each $x \in M$, $d_{\mathfrak{g}_x M}$ denotes the Chevalley-Eilenberg differential on the Lie group $G_x M$ viewed as the map

$$d_{\mathfrak{g}_x M} : \wedge^\bullet \mathfrak{g}_x^* M \to \wedge^\bullet \mathfrak{g}_x^* M$$

defined via $d_{\mathfrak{g}_x M}(\wedge^0 \mathfrak{g}_x^* M) = \{0\}$ for $k = 0$, for $k = 1$

$$\forall\, \alpha \in \wedge^1 \mathfrak{g}_x^* M\,,\ V_0, V_1 \in \mathfrak{g}_x M\ ,\quad d_{\mathfrak{g}_x M}\alpha(V_0, V_1) = -\alpha([V_0, V_1]_{\mathfrak{g}_x M}), \tag{5.3}$$

and more generally for $k > 0$, for any $\alpha \in \wedge^k \mathfrak{g}_x^* M$ and $V_0, \ldots V_k \in \mathfrak{g}_x M$,

$$d_{\mathfrak{g}_x M}\alpha(V_0, \ldots, V_k) = \sum_{0 \leq i < j \leq k} (-1)^{i+j} \alpha([V_i, V_j]_{\mathfrak{g}_x M}, V_0, \ldots, \hat{V}_i, \ldots, \hat{V}_j, \ldots, V_k). \tag{5.4}$$

It is well known [FT23a] that $d_{\mathfrak{g}_x M}$ is a linear map such that

$$d_{\mathfrak{g}_x M}^2 = 0\ ,\quad d_{\mathfrak{g}_x M}(\wedge^k \mathfrak{g}_x^* M) \subseteq \wedge^{k+1} \mathfrak{g}_x^* M,$$

and that it satisfies the Leibniz property on $\wedge^\bullet \mathfrak{g}_x^* M$, i.e.

$$\forall\, \alpha \in \wedge^k \mathfrak{g}_x^* M,\ \beta \in \wedge^\bullet \mathfrak{g}_x^* M\ ,\quad d_{\mathfrak{g}_x M}(\alpha \wedge \beta) = (d_{\mathfrak{g}_x M}\alpha) \wedge \beta + (-1)^k \alpha \wedge (d_{\mathfrak{g}_x M}\beta).$$

Moreover, $d_{\mathfrak{g}_x M}$ also preserves the weights of $\wedge^\bullet \mathfrak{g}_x^* M$:

$$d_{\mathfrak{g}_x M}(\wedge^{k,w} \mathfrak{g}_x^* M) \subseteq \wedge^{k+1,w} \mathfrak{g}_x^* M.$$

By construction, the algebraic map $d_{\mathfrak{g}M}$ defined by

$$d_{\mathfrak{g}M} : G\Omega^\bullet(M) \to G\Omega^\bullet(M) \qquad (d_{\mathfrak{g}M})_x = d_{\mathfrak{g}_x M},\ \forall\, x \in M,$$

is smooth. We call $(G\Omega^\bullet(M), d_{\mathfrak{g}M})$ the *osculating complex* or *osculating Chevalley-Eilenberg differential.*

5.3. **The osculating box operator.** Here we assume that the vector bundle $\wedge^\bullet \mathfrak{g}^* M$ is equipped with a metric $g_{\wedge^\bullet \mathfrak{g}^* M}$: each $\wedge^\bullet \mathfrak{g}_x^* M$ is equipped with a scalar product $g_{\wedge^\bullet \mathfrak{g}_x^* M}$ with a smooth dependence in $x \in M$. This scalar product allows us to consider, at each point $x \in M$, the transpose of the osculating differential $d_{\mathfrak{g}M}$, and then to define

$$\Box_{\mathfrak{g}M} := d_{\mathfrak{g}M} d_{\mathfrak{g}M}^t + d_{\mathfrak{g}M}^t d_{\mathfrak{g}M}\,.$$

We call this operator the *osculating box*. As $d_{\mathfrak{g}_M}$ is algebraic, so are $d_{\mathfrak{g}M}^t$ and $\Box_{\mathfrak{g}M}$.

Let us now assume that the decomposition (5.2) is orthogonal for the scalar product $g_{\wedge^\bullet \mathfrak{g}_x^* M}$ for each $x \in M$. This will arise naturally in Section 7.1, when the manifold is filtered and equipped with a global frame adapted to the filtration, and in Section 7.2 when M is a regular subRiemannian manifold.

With this assumption, we readily check that $d^t_{\mathfrak{g}M}$ preserves the weights, i.e. $d^t_{\mathfrak{g}_xM}(\wedge^{k,w}\mathfrak{g}^*_xM) \subseteq \wedge^{k-1,w}\mathfrak{g}^*_xM$, and $\Box_{\mathfrak{g}M}$ respects the degree and weights of osculating forms:

$$\Box_{\mathfrak{g}_xM}(\wedge^{k,w}\mathfrak{g}^*_xM) \subseteq \wedge^{k,w}\mathfrak{g}^*_xM.$$

Moreover, given the scalar product $g_{\wedge^\bullet\mathfrak{g}^*_xM}$, the linear map

$$\Box_{\mathfrak{g}_xM} := (\Box_{\mathfrak{g}M})_x = d_{\mathfrak{g}_xM}d^t_{\mathfrak{g}_xM} + d^t_{\mathfrak{g}_xM}d_{\mathfrak{g}_xM} \colon\ \wedge^\bullet\ \mathfrak{g}^*_xM \to \wedge^\bullet\mathfrak{g}^*_xM$$

is symmetric. We denote by $\Pi_{\mathfrak{g}_xM}$ the spectral orthogonal projection onto its kernel

$$E_{\mathfrak{g}_xM} := \ker(\Box_{\mathfrak{g}_xM}) = \ker d_{\mathfrak{g}_xM} \cap \ker d^t_{\mathfrak{g}_xM} = \operatorname{Im}\Pi_{\mathfrak{g}_xM}.$$

Note that the image of $\Box_{\mathfrak{g}_xM}$ is

$$F_{\mathfrak{g}_xM} := \operatorname{Im}(\Box_{\mathfrak{g}_xM}) = \operatorname{Im} d_{\mathfrak{g}_xM} + \operatorname{Im} d^t_{\mathfrak{g}_xM} = \ker\Pi_{\mathfrak{g}_xM} = E^\perp_{\mathfrak{g}_xM}.$$

By [FT23a, Lemma 2.1], the complex $(F^\bullet_{\mathfrak{g}M}, d_{\mathfrak{g}M})$ is acyclic, i.e. $d_{\mathfrak{g}M}(F_{\mathfrak{g}M}) \subseteq F_{\mathfrak{g}M}$, and the resulting cohomology is trivial:

$$d_{\mathfrak{g}_xM}(F_{\mathfrak{g}_xM}) = \ker(d_{\mathfrak{g}_xM} : F_{\mathfrak{g}_xM} \to F_{\mathfrak{g}_xM}) \subseteq \wedge^\bullet\mathfrak{g}^*_xM.$$

For $|z| = \varepsilon$ small enough, $z - \Box_{\mathfrak{g}_xM}$ is invertible, and by the Cauchy residue formula we have

$$\Pi_{\mathfrak{g}_xM} = \frac{1}{2\pi i}\oint_{|z|=\varepsilon}(z - \Box_{\mathfrak{g}_xM})^{-1}dz,$$

where the contour integration is over a circle about 0 of radius $\varepsilon > 0$ small enough. This defines a smooth algebraic map $\Pi_{\mathfrak{g}M}$ acting on $G\Omega^\bullet(M)$ that respects degrees and weights.

Proceeding as in [FT23a], at each $x \in M$, a partial inverse $d^{-1}_{\mathfrak{g}_xM}$ of $d_{\mathfrak{g}_xM}$ can be defined as

$$d^{-1}_{\mathfrak{g}_xM} := (d_{\mathfrak{g}_xM})^{-1}\mathrm{pr}_{\operatorname{Im} d_{\mathfrak{g}_xM}}\ ,$$

where we write pr_S to denote the orthogonal projection onto any S closed subspace of $\wedge^\bullet\mathfrak{g}^*_xM$.

Note that in our construction in Section 6, we will not use $d^{-1}_{\mathfrak{g}_xM}$. However, it is used in Rumin's construction recalled below and in Section 7 where we show that the two constructions coincide.

The resulting map $d^{-1}_{\mathfrak{g}M}$ on $G\Omega(M)$ is a smooth map that respects the filtration, and such that

$$(d^{-1}_{\mathfrak{g}M})^2 = 0 \ \text{ and } \ d^{-1}_{\mathfrak{g}_xM}(\wedge^{k+1,w}\mathfrak{g}^*_xM) \subseteq \wedge^{k,w}\mathfrak{g}^*_xM\ ,\ \forall\, k, w \in \mathbb{N}_0\,.$$

By [FT23a, Section 2], the kernel and image of $d^{-1}_{\mathfrak{g}M}$ coincide with the one of $d^t_{\mathfrak{g}M}$:

$$\ker d^t_{\mathfrak{g}_xM} = \ker d^{-1}_{\mathfrak{g}_xM} \quad \text{and} \quad \operatorname{Im} d^t_{\mathfrak{g}_xM} = \operatorname{Im} d^{-1}_{\mathfrak{g}_xM},$$

so

$$E_{\mathfrak{g}_xM} = \ker d_{\mathfrak{g}_xM} \cap \ker d^{-1}_{\mathfrak{g}_xM} \quad \text{and} \quad F_{\mathfrak{g}_xM} = \operatorname{Im} d_{\mathfrak{g}_xM} + \operatorname{Im} d^{-1}_{\mathfrak{g}_xM}.$$

By [FT23a, Proposition 2.2],

$$\Pi_{\mathfrak{g}M} = \mathrm{I} - d^{-1}_{\mathfrak{g}M}d_{\mathfrak{g}M} - d_{\mathfrak{g}M}d^{-1}_{\mathfrak{g}M}$$

and we have

$$d_{\mathfrak{g}M}^{-1}\Pi_{\mathfrak{g}M}=\Pi_{\mathfrak{g}M}d_{\mathfrak{g}M}^{-1}=0\ ,\quad d_{\mathfrak{g}M}\Pi_{\mathfrak{g}M}=\Pi_{\mathfrak{g}M}d_{\mathfrak{g}M}=0\,.$$

5.4. **The maps d and $\tilde d$ on a filtered manifold.** Let us show that $\mathrm{gr}(d)$ and $\mathrm{gr}(\tilde d)$ are well-defined, crucial operators, and that both $(\Gamma(\wedge^\bullet \mathrm{gr}(T^*M)),\mathrm{gr}(d))$ and $(\Gamma(\wedge^\bullet \mathrm{gr}(T^*M)),\mathrm{gr}(\tilde d))$ coincide with the osculating complex.

Proposition 5.3. *The maps d and $\tilde d$ respect the filtration with*

$$(d-\tilde d)\Omega^{k,\geq w}(M)\subseteq \Omega^{k,\geq w+1}(M),$$

for any degree $k\in\mathbb{N}_0$ and weight $w\in\mathbb{N}_0$. Moreover, the maps $\mathrm{gr}(d)$ and $\mathrm{gr}(\tilde d)$ acting on $G\Omega^\bullet(M)\cong\Gamma(\wedge^\bullet\mathrm{gr}(T^*M))$ coincide with $d_{\mathfrak{g}M}$:

$$\mathrm{gr}(d)=\mathrm{gr}(\tilde d)=d_{\mathfrak{g}M}\,.$$

Proof. From the definitions of d and $\tilde d$, both differentials as maps $\Omega^1(M)\to\Omega^2(M)$ respect the corresponding filtrations. As they obey the Leibniz rule, both d and $\tilde d$ respect the filtration on $\Omega^\bullet(M)$. Moreover, $\mathrm{gr}(d)$ and $\mathrm{gr}(\tilde d)$ satisfy the Leibniz property on $G\Omega^\bullet(M)\cong\Gamma(\wedge^\bullet\mathrm{gr}(T^*M))$.

Let $\omega\in\Omega^{k,\geq w}(M)$ and for each $j=0,\dots,k$ take $V_j\in H^{w_{i(j)}}$ with $w_{i(0)}+\dots+w_{i(k)}\leq w$. Since $w_{i(0)}+\dots+\hat w_{i(l)}+\dots+w_{i(k)}<w$, we have that $\omega(V_0,\dots,\hat V_l,\dots,V_k)=0$, and so $(d-\tilde d)\omega(V_0,\dots,V_k)=0$ by (3.2).

This shows $(d-\tilde d)\Omega^{k,\geq w}(M)\subseteq\Omega^{k,\geq w+1}(M)$, and implies $\mathrm{gr}(d)=\mathrm{gr}(\tilde d)$. We are left to prove that $\mathrm{gr}(\tilde d)=d_{\mathfrak{g}M}$.

If $\omega\in\Omega^1(M)$ and $V_{i_1}\in\Gamma(H^{w_{i(1)}}),V_{i_2}\in\Gamma(H^{w_{i(2)}})$, then by Lemma 3.1 and (5.1).

$$\begin{aligned}\tilde d\omega(V_1 \bmod H^{w_{i(1)}-1},V_2 \bmod H^{w_{i(2)}-1}) &= -\omega([V_1 \bmod H^{w_{i(1)}-1},V_2 \bmod H^{w_{i(2)}-1}])\\ &= -\omega([V_1,V_2] \bmod H^{w_{i(1)}+w_{i(2)}-1})\,.\end{aligned}$$

Hence, we have that for any $\omega\in(H^{w_j})^\perp$, $j=0,1,\dots,s_0-1$

$$\mathrm{gr}(\tilde d)(\omega \bmod (H^{w_{j+1}})^\perp)=-(\omega \bmod (H^{w_{j+1}})^\perp)([\cdot,\cdot]_{\mathfrak{g}M}).$$

We recognise $d_{\mathfrak{g}M}(\omega \bmod (H^{w_{j+1}})^\perp)$ from (5.3). Hence the maps $d_{\mathfrak{g}_M}$ and $\mathrm{gr}(\tilde d)$ coincide on $G\Omega^1(M)$. Since they both vanish on $G\Omega^0(M)$ and satisfy the Leibniz property, they coincide in fact on the whole $G\Omega^\bullet(M)$. □

6. General construction of subcomplexes on a filtered manifolds

Here, we present the general scheme to construct subcomplexes on filtered manifolds. It relies on the notions of a base differential and codifferential that we introduce in Section 6.1 (this terminology is borrowed from [GT53]). In Section 6.2, we construct certain operators $\Box,P,L$ which allow us to define the subcomplexes $(F_0^\bullet,C)$ and $(E_0^\bullet,D)$, with the latter computing the same cohomology as de Rham's (see Section 6.3). In Section 6.4, we also present an analogous construction, obtaining the operators $\tilde\Box,\tilde P,\tilde L$ and then the subcomplexes $(\tilde F_0^\bullet,\tilde C)$ and $(\tilde E_0^\bullet,\tilde D)$, with the latter computing the same cohomology as the complex $(\Omega^\bullet(M),\tilde d)$.

6.1. **Base differential and codifferential d_0 and δ_0.**

Definition 6.1. Let d_0 and δ_0 be two differential algebraic complexes on a filtered manifold M. We say that (d_0, δ_0) is a *pair of base differential and codifferential* (for a given manifold M equipped with a filtration (4.1)) when the following properties are satisfied:

- d_0 increases the degree by one while δ_0 decreases the degree by one, and both respect the filtration of $\Omega^\bullet(M)$:
$$d_0(\Omega^{k,\geq w}(M)) \subset \Omega^{k+1,\geq w}(M) \quad \text{and} \quad \delta_0(\Omega^{k+1,\geq w}(M)) \subset \Omega^{k,\geq w}(M),$$
- d_0 and δ_0 are disjoint (in the sense of [Kos61]), that is, for any $\alpha \in \Omega^\bullet(M)$
$$\delta_0 d_0 \alpha = 0 \Longrightarrow d_0\alpha = 0 \;\text{ and }\; d_0\delta_0\alpha = 0 \Longrightarrow \delta_0\alpha = 0\,,$$
- $\mathrm{gr}(d_0) = d_{\mathfrak{g}M}$ and $\mathrm{gr}(\delta_0) = d^t_{\mathfrak{g}M}$, the transpose being defined using a metric $g_{\wedge^\bullet\mathfrak{g}^*M}$ on the osculating bundle $\wedge^\bullet\mathfrak{g}^*M$ such that at any point $x \in M$, the decomposition (5.2) of $\wedge^\bullet\mathfrak{g}^*_xM$ by weights and degrees is $g_{\wedge^\bullet\mathfrak{g}^*_xM}$-orthogonal.

Given a pair (d_0, δ_0) of base differential and codifferential on a filtered manifold M, we define the associated *base box* via
$$\Box_0 := d_0\,\delta_0 + \delta_0\,d_0.$$

For any $x \in M$, we define the operator
$$\Pi_{0,x} := \frac{1}{2\pi i}\oint_{|z|=\epsilon}(z - \Box_{0,x})^{-1}dz,$$
for $\varepsilon > 0$ small enough.

Proposition 6.2. (1) *The operators $\Box_0$ and Π_0 are smooth and algebraic on $\Omega^\bullet(M)$. They respect its filtration and keep the degrees of the forms constant:*
$$\Box_0(\Omega^{k,\geq w}(M)) \subset \Omega^{k,\geq w}(M) \quad \textit{and} \quad \Pi_0(\Omega^{k,\geq w}(M)) \subset \Omega^{k,\geq w}(M).$$
They also satisfy
$$\mathrm{gr}(\Box_0) = \Box_{\mathfrak{g}M} \qquad \textit{and} \qquad \mathrm{gr}(\Pi_0) = \Pi_{\mathfrak{g}M}.$$
The algebraic subbundles
$$E_0 := \ker d_0 \cap \ker \delta_0 \quad \textit{and} \quad F_0 := \mathrm{Im}\, d_0 + \mathrm{Im}\,\delta_0,$$
admit subfiltrations and we have:
$$\mathrm{gr}(E_0) = E_{\mathfrak{g}M} \qquad \textit{and} \qquad \mathrm{gr}(F_0) = F_{\mathfrak{g}M}.$$

(2) *We have*
$$\Omega^\bullet(M) = E_0^\bullet \oplus F_0^\bullet \qquad \textit{with} \qquad E_0 = \ker \Box_0 \qquad \textit{and} \qquad F_0 = \mathrm{Im}\,\Box_0.$$
Moreover, Π_0 is the projection onto E_0 along F_0.

Proof. Part (1) is satisfied by construction. Disjointedness implies readily that $E_0 \cap F_0 = \{0\}$. Since $E_{\mathfrak{g}M} \oplus F_{\mathfrak{g}M} = \wedge^\bullet \mathfrak{g}^* M$, Part (1) implies that $\dim E_0 + \dim F_0 = \dim E_{\mathfrak{g}M} + \dim F_{\mathfrak{g}M} = \dim \Omega^\bullet(M)$. Hence $E_0 \oplus F_0 = \Omega^\bullet(M)$.

Disjointedness also implies $E_0 = \ker \Box_0$. Clearly, we also have $\operatorname{Im} \Box_0 \subset F_0$. From the Cauchy residue formula, proceeding as in the proof of [FT23a, Proposition 4.1], we check that $\Pi_0^2 = \Pi_0$. In other words, Π_0 is a projection of $\Omega^\bullet(M)$. We also check using the Cauchy residue that $\Pi_0 = \mathrm{I}$ on $\ker \Box_0$ while $\Pi_0 \Box_0 = 0$ so $\Pi_0 = 0$ on $\operatorname{Im} \Box_0$. We obtain Part (2) from $\Omega^\bullet(M) = E_0 \oplus F_0$. ☐

We call Π_0 the *base kernel projection*.

6.2. **The operators $\Box$, P and L.** We define the differential operator acting on $\Omega^\bullet(M)$:

$$\Box := d\,\delta_0 + \delta_0\, d.$$

Remark 6.3. The de Rham differential d will be replaced with its algebraic part $\tilde{d}$ in Section 6.4. The new construction will closely follow the steps below, as the only property that it relies on is that $\operatorname{gr}(d) = d_{\mathfrak{g}M}$.

Proposition 6.4. *The operator $\Box$ commutes with d and δ_0. It is a differential operator that preserves the degrees and the weights of forms:*

$$\Box(\Omega^{k,\geq w}(M)) \subseteq \Omega^{k,\geq w}(M).$$

We have

$$\operatorname{gr}(\Box) = \Box_{\mathfrak{g}M}.$$

Proof. Since d and δ_0 are complexes, they commute with $\Box$. Moreover, d and d_0 increase the degrees of forms by one, while δ_0 decreases the degrees by one. Hence, $\Box$ and $\Box_0$ preserve the degrees of forms. As d, d_0 and δ_0 respect the filtration, so do $\Box$ and $\Box_0$. We conclude with

$$\operatorname{gr}(\Box) = \operatorname{gr}(d)\operatorname{gr}(\delta_0) + \operatorname{gr}(\delta_0)\operatorname{gr}(d) = d_{\mathfrak{g}M} d^t_{\mathfrak{g}M} + d^t_{\mathfrak{g}M} d_{\mathfrak{g}M} = \Box_{\mathfrak{g}M}.$$

☐

Applying Proposition 6.4 together with Proposition 4.22 to $z - \Box$, the map

$$P := \frac{1}{2\pi i} \oint_{|z|=\varepsilon} (z - \Box)^{-1} dz,$$

is well-defined for $|z| = \varepsilon$ small enough.

Proposition 6.5. (1) *The map P is a differential operator acting on $\Omega^\bullet(M)$ that respects the filtration and the degree of the forms with $\operatorname{gr}(P) = \Pi_{\mathfrak{g}M}$. It commutes with $\Box$, d and δ_0.*

(2) *We have $\Box^{N_0} P = 0$ while $\Box^{N_0}$ acts on $\operatorname{Im} P$ where it is invertible. Moreover, P is the projection onto*

$$E := \ker \delta_0 \cap \ker(\delta_0 d) = \operatorname{Im} P = \ker \Box^{N_0},$$

along

$$F := \operatorname{Im}\delta_0 + \operatorname{Im} d\delta_0 = \ker P = \operatorname{Im}\Box^{N_0}.$$

These modules satisfy $\operatorname{gr}(E) = E_0$ *and* $\operatorname{gr}(F) = F_0$.

Proof. Part (1) follows from Proposition 6.4 and applying gr. From the Cauchy residue formula, proceeding as in the proof of [FT23a, Proposition 4.1], we check that $P^2 = P$. In other words, $P\colon \Omega^\bullet(M) \to \Omega^\bullet(M)$ is a projection.

For Part (2), $\operatorname{gr}(\Box P) = 0$ by the Cauchy formula and the properties of gr. Therefore $(\Box P)^{N_0} = 0$, but $(\Box P)^{N_0} = \Box^{N_0} P$ since P is a projection commuting with $\Box$. Therefore $\operatorname{Im} P \subseteq \ker \Box^{N_0}$.

Naturally, $\Box$ acts on $\operatorname{Im}\Box^{N_0}$ which is a submodule of $\Omega^\bullet(M)$ that inherits a filtration. Moreover, we check

$$\operatorname{gr}(\operatorname{Im}\Box^{N_0}) = \operatorname{Im}\operatorname{gr}(\Box^{N_0}) = \operatorname{Im}\Box_{\mathfrak{g}M} = \ker \Pi_{\mathfrak{g}M},$$

since $\Box_{\mathfrak{g}M}$ is symmetric and the orthogonal projection onto its kernel is $\Pi_{\mathfrak{g}M}$, and

$$\operatorname{gr}|_{\operatorname{Im}\Box^{N_0}}(\Box : \operatorname{Im}\Box^{N_0} \to \operatorname{Im}\Box^{N_0}) = \Box_{\mathfrak{g}M} : \operatorname{Im}\Pi_{\mathfrak{g}M} \to \operatorname{Im}\Pi_{\mathfrak{g}M}.$$

Adapting the proof of Proposition 4.22 to a submodule, we obtain that $\Box|_{\operatorname{Im}\Box^{N_0}}$ is invertible on $\operatorname{Im}\Box^{N_0}$. This implies that $\ker\Box^{N_0} \cap \operatorname{Im}\Box^{N_0} = \{0\}$ and $\ker\Box^{N_0} \oplus \operatorname{Im}\Box^{N_0} = \Omega^\bullet(M)$. We then readily check that P is the projection onto $\ker\Box^{N_0}$ along $\operatorname{Im}\Box^{N_0}$.

As $\Pi_{\mathfrak{g}M} d^t_{\mathfrak{g}M} = 0$, we have $\Pi_0\delta_0 = 0$. Hence, since P commutes with δ_0, we have

$$(P - \Pi_0)\delta_0 = P\delta_0 = \delta_0 P\,,$$

and so recursively, for any $k \geq 1$,

$$(P - \Pi_0)^k\delta_0 = P^k\delta_0 = \delta_0 P^k\,.$$

For $k \geq N_0$, $(P - \Pi_0)^k = 0$ since $\operatorname{gr}(P) = \Pi_{\mathfrak{g}M} = \operatorname{gr}(\Pi_0)$. As P is a projection, we have obtained $0 = P\delta_0 = \delta_0 P$. Moreover, since P commutes with d, we also have $Pd\delta_0 = dP\delta_0 = 0$, which implies the inclusion $\ker P \supset \operatorname{Im}\delta_0 + \operatorname{Im} d\delta_0 =: F$.

Since $d^2 = 0$ and $(\delta_0)^2 = 0$, we compute easily

$$\Box^{N_0} = (d\delta_0)^{N_0} + (\delta_0 d)^{N_0},$$

As P is the projection onto $\ker\Box^{N_0}$ along $\operatorname{Im}\Box^{N_0}$, we have

$$\operatorname{Im} P = \ker\Box^{N_0} \supset \ker\delta_0 \cap \ker(\delta_0 d) =: E,$$
$$\ker P = \operatorname{Im}\Box^{N_0} \subset \operatorname{Im}\delta_0 + \operatorname{Im}(d\delta_0) =: F\,.$$

These last inclusions then imply that $\ker P = F$, but also $\operatorname{Im} P = E$. ∎

Note that in our construction below, we do not need the characterisations of $\operatorname{Im} P$ and $\ker P$ as $E = \ker\delta_0 \cap \ker(\delta_0 d)$ and $F = \operatorname{Im}\delta_0 + \operatorname{Im} d\delta_0$. We will only need it when proving that this construction coincides with Rumin's.

We follow the same strategy as in the case of homogeneous groups [FT23a, Section 4.1]:

Proposition 6.6. *The differential operator L acting on $\Omega^\bullet(M)$ and defined as*

$$L := P\Pi_0 + (\mathrm{I} - P)(\mathrm{I} - \Pi_0),$$

preserves the degrees of the forms and respects the filtration with $\mathrm{gr}(L) = \mathrm{I}$. It is invertible and its inverse is a differential operator acting on $\Omega^\bullet(M)$. We have

$$PL = P\Pi_0 = L\Pi_0$$

and this implies

$$P = L\Pi_0 L^{-1} \qquad \text{and} \qquad (L^{-1}\Box L)\Pi_0 = \Pi_0(L^{-1}\Box L).$$

Consequently,

$$F = \ker P = L(\ker \Pi_0)L^{-1} = LF_0L^{-1}, \qquad E = \mathrm{Im}\, P = L(\mathrm{Im}\, \Pi_0)L^{-1} = LE_0L^{-1}.$$

Proof. By Lemma 4.12, we have

$$\mathrm{gr}(L) = \mathrm{gr}(P)\ \mathrm{gr}(\Pi_0) + \mathrm{gr}(\mathrm{I} - P)\ \mathrm{gr}(\mathrm{I} - \Pi_0) = \Pi_{\mathfrak{g}M}^2 + (\mathrm{I} - \Pi_{\mathfrak{g}M})^2 = \mathrm{I}.$$

We conclude with Proposition 4.22 and straightforward computations. □

6.3. **The complexes $(E_0^\bullet, D)$ and $(F_0^\bullet, C)$.** By Propositions 6.5 and 6.6, the de Rham differential d commutes with P, so $L^{-1}dL$ commutes with $L^{-1}PL = \Pi_0$. Hence, we can decompose the differential operator $L^{-1}dL$ as

$$L^{-1}dL \ = \ L^{-1}dL\Pi_0 \ + \ L^{-1}dL(\mathrm{I} - \Pi_0) \ = \ D \ + \ C\,,$$

where D and C are the differential operators

$$\begin{aligned} D &:= L^{-1}dL\Pi_0 \ = \ \Pi_0 L^{-1}dL\Pi_0, \\ C &:= L^{-1}dL(\mathrm{I} - \Pi_0) \ = \ (\mathrm{I} - \Pi_0)L^{-1}dL(\mathrm{I} - \Pi_0). \end{aligned}$$

Since $(L^{-1}dL)^2 = L^{-1}d^2L = 0$ and $DC = 0 = CD$, it follows that $D^2 = C^2 = 0$. Hence, the chain complex $L^{-1}dL$ decomposes into the direct sum of the two chain complexes: D acting on $E_0 = \mathrm{Im}\, \Pi_0$, and C acting on $F_0 = \ker \Pi_0$.

Proposition 6.7. *The maps C and d_0 are conjugated on F_0:*

$$Cg = gd_0 \quad \text{on } F_0\,,$$

where $g : F_0 \to F_0$ is the invertible operator defined as

$$g := C\delta_0\Box_0^{-1} + \delta_0\Box_0^{-1}d_0 \colon F_0 \to F_0\,.$$

Hence, the complexes $(F_0^\bullet, C)$ and $(F_0^\bullet, d_0)$ have the same cohomology.

Proof. Note that $\mathrm{Im}\, d_0 \subseteq \mathrm{Im}\, \Box_0 = F_0$ and that $\Box_0$ is invertible on $\mathrm{Im}\, \Box_0 = F_0$, so g is well-defined on F_0.

Since $\mathrm{gr}(L) = \mathrm{I}$, $\mathrm{gr}(d) = d_{\mathfrak{g}_M}$ and $\mathrm{gr}(\Pi_0) = \Pi_{\mathfrak{g}M}$, we have

$$\mathrm{gr}(C) = \mathrm{gr}(L^{-1}dL(\mathrm{I} - \Pi_0)) = d_{\mathfrak{g}_M}(\mathrm{I} - \Pi_{\mathfrak{g}M}),$$

and on $F_{\mathfrak{g}M}$

$$\begin{aligned}\mathrm{gr}|_{F_0}(g) &= \mathrm{gr}(C)\, d^t_{\mathfrak{g}_M} \Box^{-1}_{\mathfrak{g}M} + d^t_{\mathfrak{g}_M} \Box^{-1}_{\mathfrak{g}M} d_{\mathfrak{g}_M} \\ &= d_{\mathfrak{g}_M} d^t_{\mathfrak{g}_M} \Box^{-1}_{\mathfrak{g}M} + d^t_{\mathfrak{g}_M} \Box^{-1}_{\mathfrak{g}M} d_{\mathfrak{g}_M} \\ &= (d_{\mathfrak{g}_M} d^t_{\mathfrak{g}_M} + d^t_{\mathfrak{g}_M} d_{\mathfrak{g}_M}) \Box^{-1}_{\mathfrak{g}M} = \mathrm{I}_{F_{\mathfrak{g}M}}\,.\end{aligned}$$

Applying the proof of Proposition 4.22 to the submodule F_0, g is invertible on $\mathrm{Im}\, P_0 = F_0$.

Since $C^2 = 0$ and $d_0^2 = 0$, it is a straightfoward to check that $Cg = gd_0$ holds on F_0. □

Proposition 6.8. *The differential operator $\Pi_0 L^{-1}$ is a chain map between $(\Omega^\bullet(M), d)$ and (E_0, D), that is $(\Pi_0 L^{-1})d = D(\Pi_0 L^{-1})$. This chain map is homotopically invertible, with homotopic inverse given by L since we have*

$$(\Pi_0 L^{-1})L = \Pi_0, \quad \textit{and} \quad \mathrm{I} - L(\Pi_0 L^{-1}) = dh + hd\,,$$

where h is the differential operator acting on $\Omega^\bullet(M)$ and defined as

$$h := L\, g\delta_0 \Box_0^{-1}\, g^{-1}(\mathrm{I} - \Pi_0)\, L^{-1},$$

with g as in Proposition 6.7.

Proof. By construction, we have $d = L(D + C)L^{-1}$, with $D = \Pi_0 D \Pi_0$ and $C = (\mathrm{I} - \Pi_0)C(\mathrm{I} - \Pi_0)$, so $\Pi_0 L^{-1} d = \Pi_0 D \Pi_0 L^{-1} = D \Pi_0 L^{-1}$.

It remains to prove the properties regarding h. We first point out that h makes sense because $\mathrm{Im}\, \delta_0 \subset F_0$ and g acts on $F_0 = \mathrm{Im}\,(\mathrm{I} - \Pi_0)$ in an invertible way. The definitions of h and C together with Proposition 6.7 then yield:

$$\begin{aligned}L^{-1}(dh + hd)L &= L^{-1} dL\, g\delta_0 \Box_0^{-1}\, g^{-1}(\mathrm{I} - \Pi_0) + g\delta_0 \Box_0^{-1}\, g^{-1}(\mathrm{I} - \Pi_0)\, L^{-1} dL \\ &= C\, g\delta_0 \Box_0^{-1}\, g^{-1}(\mathrm{I} - \Pi_0) + g\delta_0 \Box_0^{-1}\, g^{-1} C \\ &= g d_0 \delta_0 \Box_0^{-1}\, g^{-1}(\mathrm{I} - \Pi_0) + g\delta_0 \Box_0^{-1}\, d_0 g^{-1}(\mathrm{I} - \Pi_0) \\ &= g\left(d_0 \delta_0 \Box_0^{-1} + \delta_0 \Box_0^{-1} d_0\right) g^{-1}(\mathrm{I} - \Pi_0) = \mathrm{I} - \Pi_0.\end{aligned}$$

The conclusion follows. □

Corollary 6.9. *The cohomology of $(E_0^\bullet, D)$ is linearly isomorphic to the de Rham cohomology of the manifold M.*

6.4. **The complexes $(E_0^\bullet, \tilde{D})$ and $(F_0^\bullet, \tilde{C})$.** In this section, we consider the complexes obtained by considering $\tilde{d}$ instead of d in the construction above (see Remark 6.3). The proofs are omitted, as the arguments are essentially the same as above. We start by defining the algebraic $\tilde{\Box}$ operator acting on $\Omega^\bullet(M)$:

$$\tilde{\Box} := \tilde{d}\,\delta_0 + \delta_0\,\tilde{d}.$$

It commutes with $\tilde{d}$ and δ_0 and preserves the degrees and the weights of forms:

$$\tilde{\Box}(\Omega^{k,\geq w}(M)) \subseteq \Omega^{k,\geq w}(M).$$

We have

$$\mathrm{gr}(\tilde{\Box}) = \Box_{\mathfrak{g}M}.$$

Applying Proposition 4.22 to $z - \tilde{\Box}$ locally at $x \in M$, together with Proposition 6.4, the map

$$\tilde{P} := \frac{1}{2\pi i} \oint_{|z|=\varepsilon} (z - \tilde{\Box})^{-1} dz,$$

is well-defined for $|z| = \varepsilon$ small enough. $\tilde{P}$ is then an algebraic operator acting on $\Omega^\bullet(M)$ that respects the filtration and the degree of forms, and $\mathrm{gr}(\tilde{P}) = \Pi_{\mathfrak{g}M}$. It commutes with $\tilde{\Box}$, $\tilde{d}$ and δ_0. We have $\tilde{\Box}^{N_0}\tilde{P} = 0$, while $\tilde{\Box}^{N_0}$ acts on $\mathrm{Im}\,\tilde{P}$ where it is invertible. Moreover, $\tilde{P}$ is the projection onto

$$\tilde{E} := \ker \delta_0 \cap \ker(\delta_0 \tilde{d}) = \mathrm{Im}\,\tilde{P} = \ker \tilde{\Box}^{N_0},$$

along

$$\tilde{F} := \mathrm{Im}\,\delta_0 + \mathrm{Im}\,\tilde{d}\delta_0 = \ker \tilde{P} = \mathrm{Im}\,\tilde{\Box}^{N_0}.$$

These modules satisfy $\mathrm{gr}(\tilde{E}) = E_0$ and $\mathrm{gr}(\tilde{F}) = F_0$.

The algebraic operator $\tilde{L}$ acting on $\Omega^\bullet(M)$ and defined as

$$\tilde{L} := \tilde{P}\Pi_0 + (\mathrm{I} - \tilde{P})(\mathrm{I} - \Pi_0),$$

preserves the degree of forms, respects the filtration, and $\mathrm{gr}(\tilde{L}) = \mathrm{I}$. It is invertible and its inverse is an algebraic operator acting on $\Omega^\bullet(M)$. We have

$$\tilde{P}\tilde{L} = \tilde{P}\Pi_0 = \tilde{L}\Pi_0, \qquad \tilde{P} = \tilde{L}\Pi_0 L^{-1} \qquad \text{and} \qquad (\tilde{L}^{-1}\tilde{\Box}\tilde{L})\Pi_0 = \Pi_0(\tilde{L}^{-1}\tilde{\Box}\tilde{L}).$$

Consequently,

$$\tilde{F} = \ker \tilde{P} = \tilde{L}(\ker \Pi_0)\tilde{L}^{-1} = \tilde{L}F_0\tilde{L}^{-1}, \qquad \tilde{E} = \mathrm{Im}\,\tilde{P} = \tilde{L}(\mathrm{Im}\,\Pi_0)\tilde{L}^{-1} = \tilde{L}E_0\tilde{L}^{-1}.$$

We define the two differentials $\tilde{D}$ on $E_0 = \mathrm{Im}\,\Pi_0$, and $\tilde{C}$ on $F_0 = \ker \Pi_0$ via

$$\tilde{L}^{-1}\tilde{d}\tilde{L} = \tilde{L}^{-1}\tilde{d}\tilde{L}\Pi_0 + \tilde{L}^{-1}\tilde{d}\tilde{L}(\mathrm{I} - \Pi_0) =: \tilde{D} + \tilde{C}$$

The maps $\tilde{C}$ and d_0 are conjugated on F_0:

$$\tilde{C}\tilde{g} = \tilde{g}d_0 \quad \text{on } F_0\,,$$

where $\tilde{g} : F_0 \to F_0$ is the invertible operator defined as

$$\tilde{g} := \tilde{C}\delta_0\Box_0^{-1} + \delta_0\Box_0^{-1}d_0 \colon F_0 \to F_0\,.$$

The complexes $(F_0^\bullet, \tilde{C})$ and $(F_0^\bullet, d_0)$ have the same cohomology.

The differential operator $\Pi_0\tilde{L}^{-1}$ is a chain map between $(\Omega^\bullet(M), \tilde{d})$ and $(E_0, \tilde{D})$, that is $(\Pi_0\tilde{L}^{-1})\tilde{d} = \tilde{D}(\Pi_0\tilde{L}^{-1})$. This chain map is homotopically invertible, with homotopic inverse given by $\tilde{L}$ since we have

$$(\Pi_0\tilde{L}^{-1})\tilde{L} = \Pi_0, \quad \text{and} \quad \mathrm{I} - \tilde{L}(\Pi_0\tilde{L}^{-1}) = \tilde{d}\tilde{h} + \tilde{h}\tilde{d}\,,$$

where $\tilde{h}$ is the algebraic operator acting on $\Omega^\bullet(M)$ and defined as

$$\tilde{h} := \tilde{L}\,\tilde{g}\delta_0\Box_0^{-1}\,\tilde{g}^{-1}(\mathrm{I} - \Pi_0)\,\tilde{L}^{-1}.$$

Consequently, the cohomology of $(E_0^\bullet, \tilde{D})$ is linearly isomorphic to the cohomology of $\tilde{d}$ of the manifold M.

7. The two settings and the main results

In this section, we apply the general construction described in Section 6 to the two following settings:

(1) the case of a global coframe, and
(2) the case of a regular subRiemannian manifold.

In each case, we construct a pair of base differential an codifferential. We obtain further properties for the resulting subcomplexes. Furthermore, this construction in these settings turns out to be an alternative to the one we are presenting below, which follows arguments by Rumin.

7.1. **The case of a global coframe.** Here, we assume that M is equipped with a (global) coframe $\Theta^\bullet$ adapted to the filtration of T^*M. As mentioned above, this is a strong hypothesis. It is satisfied in the case of left-invariant filtrations on nilpotent Lie groups, but rarely in other more general settings. However, studying this case is important because its results may be applied locally to any filtered manifold.

The existence of a global coframe $\Theta^\bullet$ adapted to the filtration (4.1) naturally leads to the following pair of base differential and codifferential:

$$d_0 := (d_{\mathfrak{g}M})^\Phi = \Phi^{-1} \circ d_{\mathfrak{g}M} \circ \Phi, \quad \text{and} \quad \delta_0 := (d^t_{\mathfrak{g}M})^\Phi = \Phi^{-1} \circ d^t_{\mathfrak{g}M} \circ \Phi,$$

where the map $\Phi = \Phi_\Theta \colon \Omega^\bullet(M) \to \Gamma(\wedge^\bullet \mathrm{gr}(T^*M)) \cong G\Omega^\bullet(M)$ is defined in Lemma 4.15 using the frame $\Theta^\bullet$, and the transpose $d^t_{\mathfrak{g}M}$ is taken with respect to the scalar product on $\wedge^\bullet \mathfrak{g}^*_x M = \wedge^\bullet \mathrm{gr}(T^*_x M)$, $x \in M$, defined in Section 4.4.3. One can readily check that this (d_0, δ_0) is a pair of base differential and codifferential in the sense of Definition 6.1.

Remark 7.1. When expressed in terms of the bases $\Theta^\bullet$ and $\langle\Theta\rangle^\bullet$ respectively, the maps $d_0 \colon \Omega^\bullet(M) \to \Omega^\bullet(M)$ and $d_{\mathfrak{g}M} \colon G\Omega^\bullet(M) \to G\Omega^\bullet(M)$ are realised by same matrices, as

$$\mathrm{Mat}^{\Theta^\bullet}_{\Theta^\bullet}(d_0) = (\mathrm{Mat}^{\langle\Theta\rangle^\bullet}_{\langle\Theta\rangle^\bullet} d_{\mathfrak{g}M}) \ \text{ and } \ \mathrm{Mat}^{\Theta^\bullet}_{\Theta^\bullet}(d^t_0) = (\mathrm{Mat}^{\langle\Theta\rangle^\bullet}_{\langle\Theta\rangle^\bullet}(d_{\mathfrak{g}M}))^t,$$

where we are using the same notation as in Sections 2.1.4 and 4.4.

Moreover, if M is a graded Lie group and $\Theta^\bullet$ is a basis adapted to the gradation, then $\Phi \colon \Omega^\bullet(M) \to G\Omega^\bullet(M)$ is the identity map, and the Chevalley-Eilenberg differential $\tilde{d} \colon \Omega^\bullet(M) \to \Omega^\bullet(M)$ coincides with the algebraic differential $d_0 = (d_{\mathfrak{g}M})^\Phi$, see also [FT23a].

7.2. **Case of a regular subRiemannian manifolds.** This section is devoted to the case of regular subRiemannian manifolds, which are a particular case of filtered manifolds (see the definition below). We can therefore apply our construction on open subsets equipped with a coframe adapted to the filtration to define subcomplexes locally. However, if carried out in a naive way, such a construction may lead to the creation of local objects that are not well-defined, in the sense that they may not coincide when constructed on intersecting open subsets.

In this section, we show that if we equip the filtered manifold with a Riemannian metric that is compatible with its subRiemannian structure (as explained in Section 7.2.2 below), then our construction is globally well-defined. Moreover, it coincides with the local construction from Section 7.1, once we choose a suitable coframe $\Theta^\bullet$.

7.2.1. *The osculating metric of a regular subRiemannian manifold.* Recall that a subRiemannian manifold is a smooth manifold M equipped with a bracket generating distribution $\mathcal{D} \subset TM$ and with a metric $g_{\mathcal{D}}$ on $\mathcal{D}$. Let $\Gamma^1 = \Gamma(\mathcal{D})$ be the set of smooth sections of $\mathcal{D}$, and $\Gamma^i := [\Gamma^1, \Gamma^{i-1}] + \Gamma^{i-1}$, for $i > 1$. As $\mathcal{D}$ is bracket generating, there exists i such that $\Gamma^i = TM$, and we denote by s the smallest such integer i. If, for every $i = 1, \ldots, s$, there exists a subbundle $H^i \subset TM$ for which $\Gamma^i = \Gamma(H^i)$ is the set of smooth sections on H^i, then M is said to have an *regular* subRiemannian structure. Clearly, the H^is provide the structure of filtered manifold.

Consider a regular subRiemannian manifold M as above. In this case, the osculating Lie algebras are stratified

$$\mathfrak{g}_x M = \oplus_{i=1}^s \mathfrak{g}_{i,x}, \qquad \mathfrak{g}_{i,x} = H_x^i / H_x^{i+1} \ , \ \forall\, x \in M \,,$$

and the subspaces $\mathfrak{g}_{i,x} \subset \mathfrak{g}_x M$ are given by imposing $\mathfrak{g}_1 := \mathcal{D}$, and

$$\mathfrak{g}_{2,x} = [\mathfrak{g}_{1,x}, \mathfrak{g}_{1,x}]_{\mathfrak{g}_x M} \ , \ \mathfrak{g}_{3,x} = [\mathfrak{g}_{1,x}, \mathfrak{g}_{2,x}]_{\mathfrak{g}_x M} \ , \ldots , \ \mathfrak{g}_{i,x} = [\mathfrak{g}_{1,x}, [\mathfrak{g}_{i-1,x}]_{\mathfrak{g}_x M} \ , \ldots .$$

The metric $g_{\mathcal{D}}$ on $\mathcal{D}$ naturally induces a scalar product on each osculating Lie algebra fibre $\mathfrak{g}_x M$ [Mon02, p. 188]. Let us briefly recall its construction.

At every point $x \in M$, the map

$$(\mathfrak{g}_{1,x})^j = \mathfrak{g}_{1,x} \times \ldots \times \mathfrak{g}_{1,x} \to \mathfrak{g}_{j,x} \ , \ (V_1, \ldots, V_j) \longmapsto [V_1, [V_2, [\cdots, V_j]_{\mathfrak{g}_x M}]_{\mathfrak{g}_x M} \cdots]_{\mathfrak{g}_x M} \ ,$$

is j-linear and surjective. Hence, the metric $g_{\mathcal{D}}$ on $\mathfrak{g}_1 = H^1 = \mathcal{D}$ induces a scalar product on $(\mathfrak{g}_{1,x})^j$ and then on $\mathfrak{g}_{j,x}$ (which is the image of the above map, and therefore inherits the scalar product from the orthogonal complement of the kernel). Constructing this for $j = 2, \ldots, s$, we obtain a scalar product $g_{\mathfrak{g}_x M}$ on $\mathfrak{g}_x M$. By construction, the dependence in x is smooth, and $(g_{\mathfrak{g}M})(x) := g_{\mathfrak{g}_x M}$, $x \in M$, defines an element $g_{\mathfrak{g}M} \in \Gamma(\mathrm{Sym}(\mathfrak{g}M \otimes \mathfrak{g}M))$. Therefore, $g_{\mathfrak{g}M}$ is a metric on the osculating bundle $\mathfrak{g}M$ of Lie algebras. We call $g_{\mathfrak{g}M}$ the *osculating metric* induced by $g_{\mathcal{D}}$.

7.2.2. *Compatible Riemannian metrics.* We now assume that, in addition to being a regular subRiemannian manifold, M is also equipped with a Riemannian metric, that is, with a metric g_{TM} on its tangent bundle TM. As already seen in Sections 2.5 and 4.5.1, g_{TM} will also induce a metric $g_{\mathrm{gr}(TM)}$ on $\mathrm{gr}(TM) \cong \mathfrak{g}M$. The two metrics $g_{\mathfrak{g}M}$ and $g_{\mathrm{gr}(TM)}$ will be different in general.

Definition 7.2. A Riemannian metric g_{TM} is *compatible with the subRiemannian structure* of a regular subRiemannian manifold M (or compatible for short) when the two metrics $g_{\mathrm{gr}(TM)}$ and $g_{\mathfrak{g}M}$ on the osculating Lie algebra bundle $\mathfrak{g}M \cong \mathrm{gr}(TM)$ coincide.

By construction, $g_{\mathfrak{g}M}$ coincides on $\mathcal{D} = H^1 = \mathfrak{g}_1$ with $g_{\mathcal{D}}$. Hence, a necessary condition for g_{TM} to be compatible is for g_{TM} to coincide with $g_{\mathcal{D}}$ on $\mathcal{D}$. However, it is not sufficient generally. For example, in the case of the 5-dimensional stratifiable Lie group $N_{5,3,2}$ in

[LDT22] (keeping the notation of [LDT22]), if we take $\Theta = \{\theta^1, \theta^2, \theta^3, \theta^4, \theta^5\}$ and $\hat{\Theta} = \{\theta^1, \theta^2, \theta^3, \theta^4, \theta^4 + \theta^5\}$ as global left-invariant coframes of $TN_{5,3,2}$, then the corresponding subcomplexes $(E_0^\bullet, D)$ and $(\hat{E}_0^\bullet, \hat{D})$ do not coincide [FT23b]. In other words, just extending the metric $g_{\mathcal{D}}$ from $\mathcal{D}$ to an arbitrary Riemannian metric on TM is not sufficient to ensure compatibility. However, compatible metrics can always be constructed.

Lemma 7.3. *Let M be a (second countable) regular subRiemannian manifold. We can always construct a compatible Riemannian metric.*

Proof. Consider a sequence of non-negative functions $\varphi_i \in C_c^\infty(M)$, $i \in \mathbb{N}$, such that the sum $\sum_{i\in\mathbb{N}} \varphi_i$ on M is locally finite and constantly equal to 1. We may assume that the support of each φ_i is small enough so that it is included in an open subset $U_i \subset M$ where a frame $\mathbb{X}_i$ of TU_i exists. After a Graham-Schmidt procedure, we may assume that each $\mathbb{X}_i$ is such that the corresponding frame $\langle \mathbb{X}_i \rangle$ of the vector bundle $\mathfrak{g}M|_{U_i}$ is $g_{\mathfrak{g}M}$-orthonormal. Denoting by g_i the corresponding metric on U_i such that $\mathbb{X}_i$ is orthonormal, one can easily check that $g = \sum_{i\in\mathbb{N}} \varphi_i g_i$ is a compatible Riemannian metric. □

7.2.3. *The base differential and codifferential associated with a compatible Riemannian metric.* Let M be a regular subRiemannian manifold equipped with a Riemannian metric g_{TM}. We consider the map $\Phi\colon \Omega^\bullet(M) \to G\Omega^\bullet(M) \cong \Gamma(\wedge^\bullet \mathrm{gr}(T^*M))$ defined via the natural orthogonal gradation of TM, as well as the scalar products on $\wedge^\bullet T_x^*M$ and $\wedge^\bullet \mathrm{gr}(T_x^*M)$ associated with g_{TM}, as presented in Section 4.5.2.

The compatibility implies that the scalar product on $\wedge^\bullet \mathrm{gr}(T_x^*M)$ (defined as in Section 4.5.2 using g_{TM}) coincides with the scalar product on $\wedge^\bullet \mathfrak{g}_x^*M$ induced by the osculating metric $g_{\mathfrak{g}M}$. Consequently, we can define without ambiguity the following differential and codifferential as

$$d_0 := (d_{\mathfrak{g}M})^\Phi = \Phi^{-1} \circ d_{\mathfrak{g}M} \circ \Phi, \quad \text{and} \quad \delta_0 := (d_{\mathfrak{g}M}^t)^\Phi = \Phi^{-1} \circ d_{\mathfrak{g}M}^t \circ \Phi,$$

and we check readily that they are a pair of base differential and codifferential in the sense of Definition 6.1. Moreover, by Lemma 4.20, d_0 and δ_0 can be equivalently constructed as in Section 7.1 on any open subset $U \subset M$ where a g_{TM}-orthonormal frame adapted to the natural orthogonal gradation exists.

7.3. **Main results.** Here, we present our results for the two settings above:

Setting (1): M is a filtered manifold equipped with a global coframe Θ, or

Setting (2): M is a regular subRiemannian manifold equipped with a compatible Riemannian metric.

Using the map Φ defined in Lemma 4.15 for Case (1) or in Section 4.5.2 for Case (2), we can consider the pair of base differential and codifferential:

$$d_0 := (d_{\mathfrak{g}M})^\Phi = \Phi^{-1} \circ d_{\mathfrak{g}M} \circ \Phi, \quad \text{and} \quad \delta_0 := (d_{\mathfrak{g}M}^t)^\Phi = \Phi^{-1} \circ d_{\mathfrak{g}M}^t \circ \Phi.$$

Note that here, using the L^2-inner product on $\Omega_c^\bullet(M)$, we have

$$\delta_0 = d_0^t.$$

Moreover, the associated base box is given by

$$\Box_0 = (\Box_{\mathfrak{g}M})^{\Phi} = \Phi^{-1} \circ \Box_{\mathfrak{g}M} \circ \Phi = d_0\, d_0^t + d_0^t\, d_0,$$

with kernel projector

$$\Pi_0 = (\Pi_{\mathfrak{g}M})^{\Phi} = \Phi^{-1} \circ \Pi_{\mathfrak{g}M} \circ \Phi = \frac{1}{2\pi i} \oint_{|z|=\epsilon} (z - \Box_0)^{-1} dz.$$

The map Π_0 is the projection onto

$$E_0 := \ker(\Box_0) = \Phi^{-1} \circ E_{\mathfrak{g}M} \circ \Phi = \ker d_0 \cap \ker d_0^t = \operatorname{Im} \Pi_0,$$

along

$$F_0 := \operatorname{Im}(\Box_0) = \Phi^{-1} \circ F_{\mathfrak{g}M} \circ \Phi = \operatorname{Im} d_0 + \operatorname{Im} d_0^t = \ker \Pi_0.$$

Theorem 7.4. *We continue within either Setting (1) or Setting (2).*

(1) *By applying the construction in Section 6, we obtain two pairs of subcomplexes: $(E_0^\bullet, D)$ and $(E_0^\bullet, \tilde{D})$ whose cohomologies are linearly isomorphic to $(\Omega^\bullet(M), d)$ and $(\Omega^\bullet(M), \tilde{d})$ respectively, and the subcomplexes $(F_0^\bullet, C)$ and $(F_0^\bullet, \tilde{C})$ which are acyclic.*

(2) *Consider Setting (2) of a regular subRiemannian manifold equipped with a compatible metric g_{TM}, and an open subset $U \subset M$ where a g_{TM}-orthonormal frame $\mathbb{X}$ adapted to the gradation exists. Then U equipped with the coframe $\Theta^\bullet$, dual of $\mathbb{X}$, satisfies Setting (1), and the local subcomplexes $(E_0^\bullet, D)_U$ and $(F_0^\bullet, C)_U$, as well as $(E_0^\bullet, \tilde{D})_U$ and $(F_0^\bullet, \tilde{C})_U$ constructed on U following the scheme of Section 6, coincide with $(E_0^\bullet, D)$, $(F_0^\bullet, C)$, $(E_0^\bullet, \tilde{D})$, and $(F_0^\bullet, \tilde{C})$ respectively restricted to U.*

(3) *Writing $\tilde{D}^{(k)}$ for the restriction of $\tilde{D}$ to $\Omega^k(M)$, we have*

$$\tilde{D}^{(k,t)} = (-1)^{kn+1} \star\, \tilde{D}^{(n-k-1)} \star \colon \Omega^{k+1}(M) \to \Omega^k(M)\,.$$

We have an analogous formula for D^t, the formal transpose of D either

- *in Setting (1), or*
- *in Setting (2), with the additional hypothesis that M is oriented.*

Proof. The properties of $(F_{\mathfrak{g}M}^\bullet, d_{\mathfrak{g}M})$ imply that the complex $(F_0^\bullet, d_0)$ is acyclic. Part (1) follows from the results of Sections 6.3 and 6.4. Part (2) is a consequence of Lemma 4.20, see Section 7.2.3.

Let us prove Part (3) for D in Setting (1). We may write

$$d_0^{(k)} : \Omega^k(M) \to \Omega^{k+1}(M)\ ,\ d_0^{(k-1,t)} : \Omega^k(M) \to \Omega^{k-1}(M)\ ,$$
$$\Box_0^{(k)} : \Omega^k(M) \to \Omega^k(M)\ ,\ \Pi_0^{(k)} : \Omega^k(M) \to \Omega^k(M)\ ,$$

for their restrictions to the space of k-forms of the operators d_0, d_0^t, $\Box_0$, and Π_0 respectively, and $d_{\mathfrak{g}M}^{(k,t)} : G\Omega^{k+1}(M) \to G\Omega^k(M)$ for the restriction of $d_{\mathfrak{g}M}$ to $G\Omega^{k+1}(M)$. By [FT23a, Section 2.1.6], we have

$$d_{\mathfrak{g}M}^{(k,t)} = (-1)^{kn+1} \star d_{\mathfrak{g}M}^{(n-k-1)} \star \colon G\Omega^{k+1}(M) \to G\Omega^k(M)\,,$$

and so, by Lemma 4.19,

$$d_0^{(k,t)} = (-1)^{kn+1} \star d_0^{(n-k-1)} \star \colon \Omega^{k+1}(M) \to \Omega^k(M)\,.$$

Writing $\square^{(k)}$, $P^{(k)}$ and $L^{(k)}$ for the restriction of $\square, P, L$ to $\Omega^k(M)$, and proceeding as in the proof of [FT23a, Lemma 4.3], we obtain

$$\star\square^{(k)}\star = (-1)^{k(n-k)}\square \quad , \quad \star P^{(k)}\star = (-1)^{k(n-k)}P^{(n-k,t)} \ ,$$

and

$$\star L^{(k)}\star = (-1)^{k(n-k)}L_1^{(n-k)} \ , \quad \text{where} \quad L_1 := P^t\Pi_0 + (\mathrm{I}-P^t)(\mathrm{I}-\Pi_0)\,.$$

The linear map $L_1^{(k)} : \Omega^k(M) \to \Omega^k(M)$ is an invertible differential operator, whose inverse is also a differential operator. We can now conclude Part (3) for D in Setting (1) as in the proof of [FT23a, Lemma 4.7].

The adaptations of the argument for $\tilde{D}$ in Setting (1) or (2) are straightforward. For D in Setting (2), the existence of a global Hodge-$\star$ operation and a transpose for the L^2-inner product requires the manifold to be oriented, see Section 4.5.3. With the addition of this hypothesis, the adaptation is again straightforward. $\square$

7.4. **Equivalent constructions for the subcomplexes $(E_0^\bullet, D)$ and $(E_0^\bullet, \tilde{D})$.** Here, we present an interpretation of Rumin's construction [Rum99] of the complex that bears his name, extended to Settings (1) and (2). In our presentation, we highlight the difference between objects living on the manifold and their osculating counterparts.

We first define the map

$$d_0^{-1} := \Phi^{-1} \circ d_{\mathfrak{g}M}^{-1} \circ \Phi \colon \Omega^\bullet(M) \to \Omega^\bullet(M)\,, \tag{7.1}$$

which is defined via the partial inverse $d_{\mathfrak{g}M}^{-1}$ of the osculating differential $d_{\mathfrak{g}M}$ (see Section 5.3). From the properties of $d_{\mathfrak{g}M}^{-1}$, we see that $(d_0^{-1})^2 = 0$, it respects the filtration and decreases the degree by one:

$$\forall\, k, w \in \mathbb{N}_0 \qquad d_0^{-1}(\Omega^{k+1,\geq w}(M)) \subseteq \Omega^{k,\geq w}(M)\,.$$

We have

$$\ker d_0^t = \ker d_0^{-1} \quad \text{and} \quad \mathrm{Im}\, d_0^t = \mathrm{Im}\, d_0^{-1},$$

so

$$E_0 = \ker d_0 \cap \ker d_0^{-1} \quad \text{and} \quad F_0 = \mathrm{Im}\, d_0 + \mathrm{Im}\, d_0^{-1}.$$

Moreover,

$$\Pi_0 = \mathrm{I} - d_0^{-1}d_0 - d_0 d_0^{-1}, \tag{7.2}$$

and we have

$$d_0^{-1}\Pi_0 = \Pi_0 d_0^{-1} = 0\ , \quad d_0\Pi_0 = \Pi_0 d_0 = 0\,.$$

Lemma 7.5. *Let us consider the differential operators acting on $\Omega^\bullet(M)$ defined by:*

$$b := d_0^{-1}d_0 - d_0^{-1}d = -d_0^{-1}(d-d_0) \quad \textit{and} \quad b_1 := d_0 d_0^{-1} - d d_0^{-1} = -(d-d_0)d_0^{-1}.$$

(1) *The maps b and b_1 are nilpotent. Consequently, $\mathrm{I}-b$ and $\mathrm{I}-b_1$ are invertible, and $(\mathrm{I}-b)^{-1}$ and $(\mathrm{I}-b_1)^{-1}$ are well-defined differential operators acting on $\Omega^\bullet(M)$. We have:*

$$b d_0^{-1} = d_0^{-1}b_1, \quad \textit{and} \quad (\mathrm{I}-b)^{-1}d_0^{-1} = d_0^{-1}(\mathrm{I}-b_1)^{-1}.$$

(2) *The differential operator* $\Pi\colon \Omega^\bullet(M) \to \Omega^\bullet(M)$ *defined as*

$$\Pi := (\mathrm{I}-b)^{-1}d_0^{-1}d + d(\mathrm{I}-b)^{-1}d_0^{-1} = d_0^{-1}(\mathrm{I}-b_1)^{-1}d + dd_0^{-1}(\mathrm{I}-b_1)^{-1}\,,$$

is the projection onto

$$F := \operatorname{Im} d_0^{-1} + \operatorname{Im} dd_0^{-1} = \operatorname{Im} d_0^t + \operatorname{Im} dd_0^t\,,$$

along

$$E := \ker d_0^{-1} \cap \ker(d_0^{-1}d) = \ker d_0^t \cap \ker(d_0^t d)\,.$$

Proof. By Proposition 5.3 and Lemma 4.16, d and d_0 increase the degree of forms by one, respect the filtration, and $\mathrm{gr}(d) = d_{\mathfrak{g}M} = \mathrm{gr}(d_0)$. As d_0^{-1} decreases the degree of forms by one, b respects the filtration, and $\mathrm{gr}(b) = -\mathrm{gr}(d_0^{-1})\,\mathrm{gr}(d-d_0) = 0$. Consequently, it is nilpotent with $b^{N_0} = 0$. The proof then follows as in [FT23a, Lemma 3.9]. □

Corollary 7.6. *The projections* Π *and* P *constructed in Lemma 7.5 and Proposition 6.5 respectively coincide.*

Following Rumin's notation, the two projections onto F and E along E and F are denoted respectively as

$$\Pi_F := \Pi \qquad \text{and} \qquad \Pi_E := \mathrm{I} - \Pi\,.$$

We then obtain Rumin's construction and its equivalence with the one presented in this paper.

Theorem 7.7. *We consider either Setting (1) or Setting (2).*

(1) *(M. Rumin) The de Rham complex* $(\Omega^\bullet(M), d)$ *splits into two subcomplexes* $(E^\bullet, d)$ *and* $(F^\bullet, d)$*. Moreover, the differential operator defined as*

$$d_c := \Pi_0 d\, \Pi_E\, \Pi_0\colon \Omega^\bullet(M) \to \Omega^\bullet(M)\ ,$$

satisfies

$$d_c^2 = 0 \quad , \quad d_c(\Omega^k(M)) \subset \Omega^{k+1}(M)\,.$$

Moreover, we have

$$\Pi_E = \Pi_E\Pi_0\Pi_E \quad \textit{and} \quad \Pi_0\Pi_E\Pi_0 = \Pi_0.$$

and so $E_0 = \operatorname{Im}\Pi_0 = \Pi_0 E$*, and the complex* $(E^\bullet, d)$ *is conjugated to* $(E_0^\bullet, d_c)$ *via* Π_0*.*

(2) *The maps* d_c *and* D *coincide, i.e.* $d_c = D$*.*

Proof. Adapting the proof of [FT23a, Lemma 3.10], it is easy to check that the operators d_0^{-1}, Π_E and Π_F satisfy the following properties:

1. $d_0^{-1}\Pi_E = \Pi_E d_0^{-1} = 0$;
2. $d\Pi_F = \Pi_F d$, and $d\Pi_E = \Pi_E d$;
3. $\Pi_E(\mathrm{I}-\Pi_0)\Pi_E = 0$;
4. on $\Omega^k(M)$, we have $\Pi_F^t = (-1)^{k(n-k)} \star \Pi_F \star$ and $\Pi_E^t = (-1)^{k(n-k)} \star \Pi_E \star$.

These readily imply Part (1) and Part (2), after adapting the proofs of Theorems 3.11 and 4.9 in [FT23a] respectively. □

This equivalent construction also works if we replace d with $\tilde{d}$, yielding a complex $(E_0^\bullet, \tilde{d}_c)$ which coincides with $\tilde{D}$.

8. Case of a three-dimensional contact manifold (locally)

On a three-dimensional contact manifold, the existence of Darboux coordinates implies that the manifold is filtered and can be locally described as follows. The manifold M is equipped with a frame X, Y, T satisfying

$$\begin{aligned}
[X, Y] &= c_0 T + c_1 X + c_2 Y, \\
[X, T] &= c_3 X + c_4 Y, \\
[Y, T] &= c_5 X + c_6 Y,
\end{aligned}$$

with $c_j \in C^\infty(M)$, $j = 0, \dots, 6$, with c_0 nowhere vanishing.

The associated free basis $\Theta^\bullet$ of $\Omega^\bullet(M)$ is given by

$$\begin{aligned}
k &= 0 \qquad 1(\geq 0), \\
k &= 1 \qquad X^*(\geq 1),\ Y^*(\geq 1),\ T^*(\geq 2), \\
k &= 2 \qquad X^* \wedge Y^*(\geq 2),\ X^* \wedge T^*(\geq 3),\ Y^* \wedge T^*(\geq 3), \\
k &= 3 \qquad \mathrm{vol} = X^* \wedge Y^* \wedge T^*(\geq 4).
\end{aligned}$$

The numbers in parenthesis refer to the $\geq$ weight of the form.

8.1. Description of $\tilde{d}, d, d_{\mathfrak{g}M}, d_0, d_0^t$.

8.1.1. *Description of* $\tilde{d}$. For $k = 0, 3$, $\tilde{d}^{(k)} = 0$. For $k = 1, 2$, let us describe $\tilde{d}$ in matrix form with respect to the canonical basis described above.

$$\mathrm{Mat}(\tilde{d}^{(1)}) = \begin{pmatrix} -c_1 & -c_2 & -c_0 \\ -c_3 & -c_4 & 0 \\ -c_5 & -c_6 & 0 \end{pmatrix}, \qquad \mathrm{Mat}(\tilde{d}^{(2)}) = \begin{pmatrix} c_3 + c_6 & -c_1 & -c_2 \end{pmatrix}.$$

8.1.2. *Description of* d. For $k = 3$, we have $d^{(3)} = 0$, while for $k = 0$, we have $df(V) = Vf$, $f \in C^\infty(M)$. Hence,

$$\begin{aligned}
\mathrm{Mat}(d^{(0)}) &= \begin{pmatrix} X \\ Y \\ T \end{pmatrix} \\
\mathrm{Mat}(d^{(1)}) &= \begin{pmatrix} -Y & X & 0 \\ -T & 0 & X \\ 0 & -T & Y \end{pmatrix} + \mathrm{Mat}(\tilde{d}^{(1)}) \\
\mathrm{Mat}(d^{(2)}) &= \begin{pmatrix} T & -Y & X \end{pmatrix} + \mathrm{Mat}(\tilde{d}^{(2)})
\end{aligned}$$

8.1.3. *Description of $d_{\mathfrak{g}M}$.* The osculating group is isomorphic to the Heisenberg group at any point $p \in M$:

$$[\langle X\rangle_p, \langle Y\rangle_p]_{\mathfrak{g}_x M} = c_0(p)\langle T\rangle_p, \qquad [\langle X\rangle_p, \langle T\rangle_p]_{\mathfrak{g}_x M} = 0 = [\langle Y\rangle_p, \langle T\rangle_p]_{\mathfrak{g}_x M}.$$

The associated free basis of $G\Omega^\bullet(M)$ is

$$\begin{array}{ll} k=0 & 1(=0), \\ k=1 & \langle X^*\rangle(=1),\ \langle Y^*\rangle(=1),\ \langle T^*\rangle(=2), \\ k=2 & \langle X^*\rangle\wedge\langle Y^*\rangle(=2),\ \langle X^*\rangle\wedge\langle T^*\rangle(=3),\ \langle Y^*\rangle\wedge\langle T^*\rangle(=3), \\ k=3 & \langle X^*\rangle\wedge\langle Y^*\rangle\wedge\langle T^*\rangle(=4). \end{array}$$

The number in parenthesis refers to the weight of the osculating form.

For $k = 0, 2, 3$, $d^{(k)}_{\mathfrak{g}M} = 0$. For $k = 1$, let us describe $d_{\mathfrak{g}M}$ in matrix form with respect to the canonical basis described above.

$$\mathrm{Mat}(d^{(1)}_{\mathfrak{g}M}) = \begin{pmatrix} 0 & 0 & -c_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

8.1.4. *Description of d_0 and d_0^t.* For $k = 0, 2, 3$, $d^{(k)}_0 = 0$ while for $k = 1$, the description of $d^{(1)}_0$ in matrix form with respect to the canonical basis of $\Omega^\bullet(M)$ given above coincides with $\mathrm{Mat}(d^{(1)}_{\mathfrak{g}M})$. For $k = 0, 1, 3$, $d^{(k,t)}_0 = 0$. For $k = 2$, the matrix description of d^t_0 coincides with the transpose of $\mathrm{Mat}(d^{(1)}_{\mathfrak{g}M})$. Consequently,

$$\mathrm{Mat}(d^{(1)}_0) = \begin{pmatrix} 0 & 0 & -c_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{Mat}(d^{(1,t)}_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -c_0 & 0 & 0 \end{pmatrix}.$$

8.2. **Construction of D.**

8.2.1. *Description of $\Box_0$, P_0.* For $k = 0, 3$, $\Box^{(k)}_0 = 0$, while $\Box^{(1)}_0 = d^{(1,t)}_0 d^{(1)}_0$ and $\Box^{(2)}_0 = d^{(1)}_0 d^{(1,t)}_0$ are represented by the following diagonal matrices:

$$\begin{aligned} \mathrm{Mat}\big(\Box^{(1)}_0\big) &= \mathrm{Mat}\big(d^{(1,t)}_0\big)\mathrm{Mat}\big(d^{(1)}_0\big) = \mathrm{diag}\big(0, 0, c_0^2\big)\ , \\ \mathrm{Mat}\big(\Box^{(2)}_0\big) &= \mathrm{Mat}\big(d^{(1)}_0\big)\mathrm{Mat}\big(d^{(1,t)}_0\big) = \mathrm{diag}\big(c_0^2, 0, 0\big)\ . \end{aligned}$$

Consequently, $\Pi^{(k)}_0$ is the identity for $k = 0, 3$, while for $k = 1, 2$,

$$\mathrm{Mat}\big(\Pi^{(1)}_0\big) = \mathrm{diag}(1, 1, 0) \qquad \text{and} \qquad \mathrm{Mat}\big(\Pi^{(2)}_0\big) = \mathrm{diag}(0, 1, 1)\ .$$

Consequently, $E^{(0)}_0 = \Omega^0(M) = \Phi^{-1}\big(\langle 1\rangle\big)$ and $E^{(3)}_0 = \Omega^3(M) = \Phi^{-1}\big(\langle X^* \wedge Y^* \wedge T^*\rangle\big)$ while

$$E^{(1)}_0 = \Phi^{-1}\big(\langle X^*\rangle \oplus \langle Y^*\rangle\big) \quad \text{and} \quad E^{(2)}_0 = \Phi^{-1}\big(\langle X^* \wedge T^*\rangle \oplus \langle Y^* \wedge T^*\rangle\big).$$

8.2.2. *Description of* $\square$, P, L. For $k=0,3$, we have $\square^{(k)}=0$, while for $k=1,2$

$$\mathrm{Mat}\big(\square^{(1)}\big)=\mathrm{Mat}\big(d_0^{(1,t)}\big)\mathrm{Mat}\big(d^{(1)}\big)=c_0\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ Y+c_1 & -X+c_2 & c_0\end{pmatrix},$$

$$\mathrm{Mat}\big(\square^{(2)}\big)=\mathrm{Mat}\big(d^{(1)}\big)\mathrm{Mat}\big(d_0^{(1,t)}\big)=-c_0\begin{pmatrix}-c_0 & 0 & 0\\ X & 0 & 0\\ Y & 0 & 0\end{pmatrix}.$$

Consequently, $P^{(k)}$ is the identity on $\Omega^k(M)$ for $k=0,3$. Since $\big((z-\square_0)(\square_0-\square)\big)^j=0$ for $j>1$, the formulae for Cauchy residues and the von Neumann series simplify into

$$P=\Pi_0+\Pi_0(\square_0-\square)c_0^{-2}\mathrm{pr}_{c_0^2}+c_0^{-2}\mathrm{pr}_{c_0^2}(\square_0-\square)\Pi_0\ ,$$

where $\mathrm{pr}_{c_0^2}$ denotes the projection onto the c_0^2-eigenspace of $\square_0$. Hence, for $k=1,2$,

$$\mathrm{Mat}\big(P^{(1)}\big)=\mathrm{diag}(1,1,0)-c_0^{-1}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ Y+c_1 & -X+c_2 & 0\end{pmatrix},$$

$$\mathrm{Mat}\big(P^{(2)}\big)=\mathrm{diag}(0,1,1)+c_0^{-1}\begin{pmatrix}0 & 0 & 0\\ X & 0 & 0\\ Y & 0 & 0\end{pmatrix}.$$

We now describe

$$L=P\Pi_0+(\mathrm{I}-P)(\mathrm{I}-\Pi_0)=\mathrm{I}+(P-\Pi_0)(-\mathrm{I}+2\Pi_0).$$

For $k=0,3$, $L^{(k)}$ is the identity on $\Omega^k(M)$ while for $k=1,2$,

$$\mathrm{Mat}\big(L^{(1)}\big)=\mathrm{I}_3-c_0^{-1}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ Y+c_1 & -X+c_2 & 0\end{pmatrix}$$

$$\mathrm{Mat}\big(L^{(2)}\big)=\mathrm{I}_3-c_0^{-1}\begin{pmatrix}0 & 0 & 0\\ X & 0 & 0\\ Y & 0 & 0\end{pmatrix},$$

where $\mathrm{I}_3=\mathrm{diag}(1,1,1)$ denotes the 3-by-3 identity matrix.

8.2.3. *Computation of* D. We now describe

$$D^{(k)}=\big(L^{(k+1)}\big)^{-1}d^{(k)}L^{(k)}\Pi_0^{(k)},$$

for $k=0,1,2$.

$$\mathrm{Mat}\big(D^{(0)}\big)=\begin{pmatrix}X\\ Y\\ T+c_0^{-1}((Y+c_1)X+(-X+c_2)Y)\end{pmatrix}=\begin{pmatrix}X\\ Y\\ 0\end{pmatrix},$$

$$\mathrm{Mat}(D^{(1)}) = \begin{pmatrix} 0 & 0 & 0 \\ -T - c_3 - c_0^{-1}X(Y+c_1) & -c_4 + c_0^{-1}X(X-c_2) & 0 \\ -c_5 - c_0^{-1}Y(Y+c_1) & -T - c_6 + Yc_0^{-1}(X-c_2) & 0 \end{pmatrix},$$

$$\mathrm{Mat}(D^{(2)}) = \begin{pmatrix} 0 & -Y-c_1 & X-c_2 \end{pmatrix}.$$

The zeros in the matrices above illustrate D acting on E_0 while being trivial on F_0.

8.3. **Construction of $\tilde{D}$.**

8.3.1. *Description of* $\tilde{\Box}$, $\tilde{P}$, $\tilde{L}$. For $k = 0, 3$, we have $\tilde{\Box}^{(k)} = 0$, while for $k = 1, 2$

$$\mathrm{Mat}\left(\tilde{\Box}^{(1)}\right) = \mathrm{Mat}\left(d_0^{(1,t)}\right)\mathrm{Mat}\left(\tilde{d}^{(1)}\right) = c_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_1 & c_2 & c_0 \end{pmatrix},$$

$$\mathrm{Mat}\left(\tilde{\Box}^{(2)}\right) = \mathrm{Mat}\left(\tilde{d}^{(1)}\right)\mathrm{Mat}\left(d_0^{(1,t)}\right) = \mathrm{diag}\left(c_0^2, 0, 0\right).$$

Consequently, $\tilde{P}^{(k)}$ is the identity on $\Omega^k(M)$ for $k = 0, 3$. Since $\left((z - \Box_0)(\Box_0 - \tilde{\Box})\right)^j = 0$ for $j > 1$, the formulae for Cauchy residues and the von Neumann series simplify into

$$\tilde{P} = \Pi_0 + \Pi_0(\Box_0 - \tilde{\Box})c_0^{-2}\mathrm{pr}_{c_0^2} + c_0^{-2}\mathrm{pr}_{c_0^2}(\Box_0 - \tilde{\Box})\Pi_0 ,$$

where $\mathrm{pr}_{c_0^2}$ denotes the projection onto the c_0^2-eigenspace of $\Box_0$. Hence, for $k = 1, 2$,

$$\mathrm{Mat}\left(\tilde{P}^{(1)}\right) = \mathrm{diag}(1,1,0) - c_0^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_1 & c_2 & 0 \end{pmatrix}$$

$$\mathrm{Mat}\left(\tilde{P}^{(2)}\right) = \mathrm{diag}(0,1,1) .$$

We now describe

$$\tilde{L} = \tilde{P}\Pi_0 + (\mathrm{I} - \tilde{P})(\mathrm{I} - \Pi_0) = \mathrm{I} + (\tilde{P} - \Pi_0)(-\mathrm{I} + 2\Pi_0) .$$

For $k = 0, 3$, $\tilde{L}^{(k)}$ is the identity on $\Omega^k(M)$ while for $k = 1, 2$,

$$\mathrm{Mat}\left(\tilde{L}^{(1)}\right) = \mathrm{I}_3 - c_0^{-1} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_1 & c_2 & 0 \end{pmatrix}, \quad \mathrm{Mat}\left(\tilde{L}^{(2)}\right) = \mathrm{I}_3 ,$$

where $\mathrm{I}_3 = \mathrm{diag}(1,1,1)$ denotes the 3-by-3 identity matrix.

8.3.2. *Computation of* $\tilde{D}$. We now describe

$$\tilde{D}^{(k)} = \left(L^{(k+1)}\right)^{-1}\tilde{d}^{(k)}\tilde{L}^{(k)}\Pi_0^{(k)}$$

for $k = 0, 1, 2$.

$$\mathrm{Mat}\left(\tilde{D}^{(0)}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathrm{Mat}\left(\tilde{D}^{(1)}\right) = \begin{pmatrix} 0 & 0 & 0 \\ -c_3 & -c_4 & 0 \\ -c_5 & -c_6 & 0 \end{pmatrix},$$

$$\mathrm{Mat}\left(\tilde{D}^{(2)}\right) = \begin{pmatrix} 0 & -c_1 & -c_2 \end{pmatrix}.$$

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