A SUB-RIEMANNIAN MAXIMUM MODULUS THEOREM

FEDERICO BUSEGHIN, NICOLÒ FORCILLO, AND NICOLA GAROFALO

ABSTRACT. In this note we prove a sub-Riemannian maximum modulus theorem in a Carnot group. Using a nontrivial counterexample, we also show that such result is best possible, in the sense that in its statement one cannot replace the right-invariant horizontal gradient with the left-invariant one.

1. Introduction and statement of the result

The maximum modulus theorem for the standard Laplacian in $\mathbb{R}^n$ can be formulated as follows. Consider the unit ball $B = \{x \in \mathbb{R}^n \mid |x| < 1\}$. Suppose that $f \in C^2(B)$ solves $\Delta f = c$ in B, where $c \in \mathbb{R}$, and that furthermore $|\nabla f| \leq 1$ in B. If $|\nabla f(0)| = 1$, then $f(x) = f(0) + \langle \omega, x\rangle$, for some $\omega \in \mathbb{S}^{n-1}$ (in fact, $\omega = \nabla f(0)$). One way of proving this result is the following. Let $u(x) = |\nabla f(x)|^2$. Then we have in B,

$$\Delta u = 2||\nabla^2 f||^2 + 2\langle \nabla f, \nabla(\Delta f)\rangle \geq 0, \tag{1.1}$$

where we have denoted by $\nabla^2 f$ the Hessian of f, and by $||\nabla^2 f||^2$ the square of its Hilbert-Schmidt norm. Since by the hypothesis, we have $0 \leq u \leq 1$ in B, and $u(0) = 1$, by the strong maximum principle it follows that $u \equiv 1$ in B. This implies $\Delta u \equiv 0$ in B, and since $\langle \nabla f, \nabla(\Delta f)\rangle \equiv 0$, it follows from (1.1) that

$$0 = \Delta u = ||\nabla^2 f||^2 \geq 0,$$

which implies $||\nabla^2 f||^2 \equiv 0$. Taylor's formula thus gives the desired conclusion $f(x) = f(0) + \langle \nabla f(0), x\rangle$ in B.

In this note we are concerned with a sub-Riemannian version of this maximum modulus theorem (for the birth of sub-Riemannian geometry the reader should see [5]). This question has an interest in non-holonomic free boundary problems, see [7] and [15]. Specifically, let $\mathbb{G}$ be a stratified nilpotent Lie group, aka Carnot group (see [12] and also [1]), and consider the left- and right-invariant vector fields associated with a orthonormal basis $\{e_1, ..., e_m\}$ of the bracket generating (horizontal) layer $\mathfrak{g}_1$ of the Lie algebra $\mathfrak{g}$, see Section 2. Given a function $f \in C^1(\mathbb{G})$,

2020 Mathematical subject classification: 35R03, 35H10, 31C05

Key words and phrases. Bochner formulas. Right-invariant vector fields. Maximum modulus theorem.

The third author has been supported in part by a Progetto SID (Investimento Strategico di Dipartimento): "Aspects of nonlocal operators via fine properties of heat kernels", University of Padova, 2022. He has also been partially supported by a Visiting Professorship at the Arizona State University.

respectively denote by

$$|\nabla_H f|^2 = \sum_{i=1}^m (X_i f)^2, \qquad |\tilde{\nabla}_H f|^2 = \sum_{i=1}^m (\tilde{X}_i f)^2, \tag{1.2}$$

the left- and right-invariant *carré du champ* of f relative to $\{X_1, ..., X_m\}$. It is clear that, if we indicate with $e \in \mathbb{G}$ the group identity, since $X_i(e) = \tilde{X}_i(e)$ for $i = 1, ..., m$, we have $|\nabla_H f(e)|^2 = |\tilde{\nabla}_H f(e)|^2$. But the two objects in (1.2) are substantially different, except in the trivial situation in which the function f depends exclusively on the horizontal variables, see for instance (2.6) below. Consider now the left-invariant horizontal Laplacian associated with the basis $\{e_1, ..., e_m\}$, defined by

$$\Delta_H = \sum_{i=1}^m X_i^2. \tag{1.3}$$

Thanks to the result in [19] this operator is hypoelliptic (but not real-analytic hypoelliptic, in general). Given a non-isotropic gauge ρ, centred at the group identity $e \in \mathbb{G}$, for every $r > 0$ we indicate $B_r = \{p \in \mathbb{G} \mid \rho(p) < r\}$. Also, a point $p \in \mathbb{G}$ will be routinely identified with its logarithmic coordinates in the Lie algebra $\mathfrak{g}$. We will thus write $p \cong (z, \sigma)$, where $z = (z_1, ..., z_m)$ will denote the variable in the horizontal, bracket generating layer of $\mathfrak{g}$, and σ will indicate the bulk variable in the remaining layers, see Section 2. We note that, thanks to the stratification of the Lie algebra $\mathfrak{g}$, a linear function $f : \mathbb{G} \to \mathbb{R}$ cannot possibly depend on the variable σ, and must therefore be of the form $f(p) = \alpha + \langle \omega, z\rangle$, for some $\alpha \in \mathbb{R}$ and $\omega \in \mathbb{R}^m$. We prove the following result.

Theorem 1.1. *Let $f \in C^2(B_r)$ for a certain $r > 0$, and suppose that for some $c \in \mathbb{R}$ we have $\Delta_H f = c$ in B_r and*

$$|\tilde{\nabla}_H f|^2 \le 1. \tag{1.4}$$

If $|\tilde{\nabla}_H f(e)|^2 = 1$, then f depends only on the horizontal variables z, and in fact

$$f(p) = f(e) + \langle \nabla_z f(e), z\rangle.$$

We emphasise that Theorem 1.1 has a best possible character, in the sense that it is false if in its statement we replace the condition (1.4) with the seemingly "more natural" left-invariant one

$$|\nabla_H f|^2 \le 1. \tag{1.5}$$

We prove in fact the following nontrivial phenomenon.

Proposition 1.2. *In the Heisenberg group $\mathbb{H}^n$ there exist non linear solutions of the equation $\Delta_H f = 0$ in a ball B_r (for some $r > 0$), such that $|\nabla_H f(e)|^2 = 1$, and for which moreover (1.5) hold.*

Theorem 1.1 and Proposition 1.2 underline a striking difference between Riemannian and sub-Riemannian geometry. We recall that the celebrated identity of Bochner states that on a Riemannian manifold M one has for $f \in C^3(M)$

$$\Delta(|\nabla f|^2) = 2||\nabla^2 f||^2 + 2\langle\nabla(\Delta f), \nabla f\rangle + 2\,\mathrm{Ric}(\nabla f, \nabla f), \tag{1.6}$$

where $\mathrm{Ric}(\cdot,\cdot)$ indicates the Ricci tensor on M, see e.g. [6, Sec. 4.3 on p.18]. This implies in particular that if $\Delta f = c$ for some $c \in \mathbb{R}$, and $\mathrm{Ric}(\cdot,\cdot) \geq 0$, then

$$\Delta(|\nabla f|^2) \geq 2||\nabla^2 f||^2. \tag{1.7}$$

In sub-Riemannian geometry the fundamental subharmonicity property (1.7) fails miserably, see [15], and also (3.19) in Section 3 below. To remedy this negative situation one must bring the right-invariant vector fields $\tilde{X}_i$ to center stage. We mention that in harmonic analysis and partial differential equations the idea of using right-invariant vector fields in a left-invariant problem has been fruitfully explored in the works [21], [2], [7], [13], [14], [20], [4].

Concerning the results in this note we mention that, similarly to the Euclidean maximum modulus theorem, the proof of Theorem 1.1 is based on a Bochner type identity for the right-invariant *carré du champ* $|\tilde{\nabla}_H f|^2$, see Proposition 3.1 below, and on Bony's strong maximum principle. The stratification of the Lie algebra ultimately plays again a role in concluding that f must be linear. To prove Proposition 1.2, instead, we construct a nontrivial explicit example of a harmonic function in the Heisenberg group $\mathbb{H}^1$ for which the maximum modulus theorem fails for the left-invariant *carré du champ*, see (3.7) below.

This note contains three sections. Besides the present one, in Section 2 we collect some background material that is needed in Section 3, where we prove Theorem 1.1 and Proposition 1.2.

2. Background material

In this section we collect some background material that is needed in the rest of the paper. A Carnot group of step $k \geq 1$ is a simply-connected real Lie group $(\mathbb{G}, \circ)$ whose Lie algebra $\mathfrak{g}$ is stratified and k-nilpotent. This means that there exist vector spaces $\mathfrak{g}_1, ..., \mathfrak{g}_k$ such that:

(i) $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$;
(ii) $[\mathfrak{g}_1, \mathfrak{g}_j] = \mathfrak{g}_{j+1}$, $j = 1, ..., k-1$, $\quad [\mathfrak{g}_1, \mathfrak{g}_k] = \{0\}$.

We assume that $\mathfrak{g}$ is endowed with a scalar product $\langle\cdot,\cdot\rangle$ with respect to which the layers $\mathfrak{g}_j'$s, $j = 1, ..., r$, are mutually orthogonal. We let $m_j = \dim\, \mathfrak{g}_j$, $j = 1, ..., k$, and denote by $N = m_1 + ... + m_k$ the topological dimension of $\mathbb{G}$. From the assumption (ii) on the Lie algebra it is clear that any basis of the first layer $\mathfrak{g}_1$ bracket generates the whole of $\mathfrak{g}$. Because of such special role $\mathfrak{g}_1$ is usually called the horizontal layer of the stratification. For ease of notation we henceforth write $m = m_1$.

Given $\xi, \eta \in \mathfrak{g}$, the Baker-Campbell-Hausdorff formula reads

$$\exp(\xi) \circ \exp(\eta) = \exp\left(\xi + \eta + \frac{1}{2}[\xi, \eta] + \frac{1}{12}\big\{[\xi, [\xi, \eta]] - [\eta, [\xi, \eta]]\big\} + ...\right), \tag{2.1}$$

where the dots indicate commutators of order four and higher, see [22, Sec. 2.15]. Since by (ii) above all commutators of order k and higher are trivial, in every Carnot group the series in the right-hand side of (2.1) is finite. Notice that (2.1) assignes a group law in $\mathbb{G}$, which is noncommutative when $k > 1$. If $p = \exp \xi, p' = \exp \xi'$, the group law $p \circ p'$ in $\mathbb{G}$ is obtained from (2.1) by the algebraic commutation relations between the elements of its Lie algebra.

Carnot groups are naturally equipped with a one-parameter family of automorphisms $\{\delta_\lambda\}_{\lambda > 0}$ which are called the group dilations. One first defines a family of non-isotropic dilations Δ_λ :

$\mathfrak{g} \to \mathfrak{g}$ in the Lie algebra by assigning the formal degree j to the j-th layer $\mathfrak{g}_j$ in the stratification of $\mathfrak{g}$. This means that if $\xi = \xi_1 + ... + \xi_k \in \mathfrak{g}$, with $\xi_j \in \mathfrak{g}_j$, $j = 1, ..., k$, one lets

$$\Delta_\lambda \xi = \Delta_\lambda(\xi_1 + \cdots + \xi_k) = \lambda\xi_1 + \cdots + \lambda^k \xi_k. \tag{2.2}$$

One then uses the exponential mapping to push forward (2.2) to the group $\mathbb{G}$, i.e., we define a one-parameter family of group automorphisms $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ by the equation

$$\delta_\lambda(p) = \exp \circ \Delta_\lambda \circ \exp^{-1}(p), \qquad p \in \mathbb{G}. \tag{2.3}$$

We will denote by ρ a fixed homogeneous norm in $\mathbb{G}$, see p. 8 in [12], and set

$$B_r = \{p \in \mathbb{G} \mid \rho(p) < r\}. \tag{2.4}$$

Notice that, since $\rho(\delta_\lambda(p)) = \lambda\rho(p)$, we infer that for every $p \in B_r$ one has $\delta_\lambda(p) \in B_r$ for every $0 \le \lambda \le 1$. This means that B_r is starlike with respect to the group identity $e \in \mathbb{G}$, see [16]. This simple, yet basic property, will be used in the proof of Theorem 1.1 below.

Given a Carnot group $(\mathbb{G}, \circ)$, we denote the left-translation operator by $L_p(p') = p \circ p'$ and with dL_p its differential. The right-translation will be denoted by $R_p(p') = p' \circ p$, and its differential by dR_p. If we fix an orthonormal basis $\{e_1, ..., e_m\}$ of the horizontal layer $\mathfrak{g}_1$, then we can define respectively left- and right-invariant vector fields by the formulas

$$X_i(p) = dL_p(e_i), \qquad \tilde{X}_i(p) = dR_p(e_i). \tag{2.5}$$

As we have mentioned, the two objects $|\tilde{\nabla}_H f|^2$ and $|\nabla_H f|^2$ differ substantially. For instance, in the special case in which $\mathbb{G}$ is a group of step $k = 2$, with group constants b^ℓ_{ij}, and (logarithmic) coordinates $p = (z_1, ..., z_m, \sigma_1, ..., \sigma_{m_2})$, one has

$$X_i = \partial_{z_i} + \frac{1}{2}\sum_{\ell=1}^{m_2}\sum_{r=1}^{m} b^\ell_{ri} z_r \partial_{\sigma_\ell}, \qquad \tilde{X}_j = \partial_{z_j} - \frac{1}{2}\sum_{\ell'=1}^{m_2}\sum_{s=1}^{m} b^{\ell'}_{sj} z_s \partial_{\sigma_{\ell'}}.$$

This gives

$$|\nabla_H f|^2 - |\tilde{\nabla}_H f|^2 = 2\sum_{\ell=1}^{m_2}\left(\sum_{1\le i<j\le m} b^\ell_{ij}\left(z_i\partial_{z_j} f - z_j \partial_{z_i} f\right)\right)\partial_{\sigma_\ell} f, \tag{2.6}$$

see [14, Lemma 2.3]. Since $b^\ell_{ji} = -b^\ell_{ij}$, a direct computation shows, for $i, j = 1, ..., m$,

$$[X_i, \tilde{X}_j] = -\frac{1}{2}\sum_{\ell=1}^{m_2}\sum_{r=1}^{m} b^\ell_{ij}\partial_{\sigma_\ell} - \frac{1}{2}\sum_{\ell=1}^{m_2}\sum_{r=1}^{m} b^\ell_{ji}\partial_{\sigma_\ell} = 0.$$

More in general, see [7, Lemma 5.1], if in an arbitrary Carnot group $\mathbb{G}$ for any $\zeta \in \mathfrak{g}$ we respectively indicate with Z, and $\tilde{Z}$ the left- and right-invariant vector fields on $\mathbb{G}$ defined by the Lie formulas

$$Zf(p) = \frac{d}{dt} f(p \circ \exp(t\zeta))\big|_{t=0}, \qquad \tilde{Z}f(p) = \frac{d}{dt} f(\exp(t\zeta) \circ p)\big|_{t=0}. \tag{2.7}$$

For any $\eta, \zeta \in \mathfrak{g}$, for the corresponding vector fields on $\mathbb{G}$ we have the following basic commutation identities

$$[Y, \tilde{Z}] = [\tilde{Y}, Z] = 0. \tag{2.8}$$

Such identities can be easily verified using (2.7) and the Baker-Campbell-Hausdorff formula. From (2.8) we have in particular $[X_i, \tilde{X}_j] = 0$, for $i, j = 1, ..., m$. This fact will be used in the proof of Proposition 3.1 below.

Denote now by $\mathscr{Z}$ the infinitesimal generator of the dilations $\{\delta_\lambda\}_{\lambda>0}$. We recall that this vector field acts on a function $f : \mathbb{G} \to \mathbb{R}$ according to the Lie formula

$$\mathscr{Z} f(p) = \lim_{\lambda\to 1} \frac{f\big(\delta_\lambda(p)\big) - f(p)}{\lambda - 1}, \qquad p \in \mathbb{G}. \tag{2.9}$$

Using (2.3) and (2.9), it is easy to recognise that, if we identify $p = \exp((z, \sigma)) \in \mathbb{G}$ with its logarithmic coordinates $(z_1, ..., z_m, \sigma_{2,1}, ..., \sigma_{2,m_2}, ..., \sigma_{k,1}, ..., \sigma_{k,m_k})$, then the vector field $\mathscr{Z}$ takes the form

$$\mathscr{Z} = \sum_{j=1}^{m} z_j \frac{\partial}{\partial z_j} + 2 \sum_{s=1}^{m_2} \sigma_{2,s} \frac{\partial}{\partial \sigma_{2,s}} + ... + k \sum_{\ell=1}^{m_k} \sigma_{k,\ell} \frac{\partial}{\partial \sigma_{k,\ell}}. \tag{2.10}$$

From the representation (2.10) and the Baker-Campbell-Hausdorff formula one can prove the following result. For every $j = 1, ..., k$ denote by $\{e_{j,1}, ..., e_{j,m_j}\}$ an orthonormal basis of the layer $\mathfrak{g}_j$ of the Lie algebra, keeping in mind that when $j = 1$ we write $\{e_1, ..., e_m\}$. As in (2.5), define left-invariant vector fields in $\mathbb{G}$ by setting

$$X_{j,\ell}(p) = dL_p(e_{j,\ell}), \qquad j = 1, ..., k,\ \ \ell = 1, ..., m_j.$$

Again, we denote by $\{X_1, ..., X_m\}$ the family $\{X_{1,1}, ..., X_{1,m_1}\}$.

Proposition 2.1. *There exist polynomials in $\mathbb{G}$, $Q_{j,\ell}$, $j = 1, ..., k$, $\ell = 1, ..., m_j$, where for each $j = 1, ..., k$ the function $Q_{j,\ell}$ is homogeneous of degree j, i.e., $\delta_\lambda Q_{j,\ell} = \lambda^j Q_{j,\ell}$, such that*

$$\mathscr{Z} = \sum_{\ell=1}^{m} Q_{1,\ell} X_\ell + \sum_{\ell=1}^{m_2} Q_{2,\ell} X_{2,\ell} + ... + \sum_{\ell=1}^{m_k} Q_{k,\ell} X_{k,\ell}.$$

In the logarithmic coordinates (z, σ) one has $Q_{1,\ell}(p) = z_\ell$, $\ell = 1, ..., m$.

In particular, Proposition 2.1 shows that $\mathscr{Z}$ involves differentiation not just in the horizontal directions, but in any layer of the stratification of $\mathfrak{g}$. As a consequence, the vector field $\mathscr{Z}$ is not horizontal.

3. Proof of Theorem 1.1 and Proposition 1.2

In this section we prove Theorem 1.1 and Proposition 1.2. We begin by recalling the following result from [15], see also [14].

Proposition 3.1 (Right Bochner type identity)**.** *Let $\mathbb{G}$ be a Carnot group, $f \in C^3(\mathbb{G})$, then one has*

$$\Delta_H(|\tilde{\nabla}_H f|^2) = 2\langle \tilde{\nabla}_H f, \tilde{\nabla}_H(\Delta_H f)\rangle + 2 \sum_{i=1}^{m} |\tilde{\nabla}_H(X_i f)|^2. \tag{3.1}$$

If in particular $\Delta_H f = c$ *for some* $c \in \mathbb{R}$*, then we have*

$$\Delta_H(|\tilde{\nabla}_H f|^2) = 2\sum_{i=1}^m |\tilde{\nabla}_H (X_i f)|^2 \geq 0. \tag{3.2}$$

Proof. The proof is a straightforward calculation that uses the commutation identities $[X_i, \tilde{X}_j] = 0$, $i, j = 1, ..., m$, see (2.8) above. One finds

$$\begin{aligned}\Delta_H(|\tilde{\nabla}_H f|^2) &= 2\sum_{i,j=1}^m \tilde{X}_j(X_i X_i f)\tilde{X}_j f + 2\sum_{i,j=1}^m \tilde{X}_j(X_i f)\tilde{X}_j(X_i f)\\ &= 2\sum_{j=1}^m \tilde{X}_j(\Delta_H f)\tilde{X}_j f + 2\sum_{i=1}^m |\tilde{\nabla}_H(X_i f)|^2,\end{aligned}$$

from which (3.1) immediately follows.

□

We can now present the proof of the maximum modulus theorem.

Proof of Theorem 1.1. Suppose $\Delta_H f = c$ in B_r. By the hypoellipticity of Δ_H, we know that $f \in C^\infty(B_r)$. Consider the function $u = |\tilde{\nabla}_H f|^2$. In view of (3.2) in Proposition 3.1 such function satisfies $\Delta_H u \geq 0$. From (1.4) we have $0 \leq u \leq 1$ in B_r. Furthermore, the assumption $|\tilde{\nabla}_H f(e)|^2 = 1$ implies that $u(e) = 1$. By the connectedness of B_r and Bony's strong maximum principle [3], we conclude that $u \equiv 1$ in B_r. In particular, $\Delta_H u = 0$ in B_r. With this information in hands, we return to (3.2) and infer that in B_r we must have

$$0 = \Delta_H u = \sum_{i=1}^m |\tilde{\nabla}_H(X_i f)|^2 \ \geq 0, \tag{3.3}$$

which gives, in particular, $\sum_{i=1}^m |\tilde{\nabla}_H(X_i f)|^2 \equiv 0$. Now (3.3) implies $\tilde{\nabla}_H(X_i f) \equiv 0$, for every $i = 1, ..., m$. Since also the right-derivatives $\tilde{X}_j$, $j = 1, ..., m$, satisfy the bracket generating assumption (see [12]), we infer that for every fixed $j = 1, ..., k$, and $\ell = 1, ..., m_j$, we have for all first derivatives of $X_i f$, $i \in \{1, ..., m\}$,

$$\partial_{\sigma_{j,\ell}}(X_i f) \equiv 0 \quad \text{ in } B_r.$$

From the connectedness of B_r we thus conclude that there exist constants $c_1, ..., c_m \in \mathbb{R}$ such that we have in B_r

$$X_i f \equiv c_i, \quad \text{for every} \ \ i = 1, ..., m. \tag{3.4}$$

We claim that (3.4) implies the desired conclusion

$$f(p) = f(e) + \langle \nabla_z f(e), z\rangle,$$

for every $p \in B_r$. This can be seen as follows. In view of the bracket generating property of the vector fields X_i, we infer from (3.4) that for every $i = 2, ..., k$ and $\ell = 1, ..., m_i$, we have

$$X_{i,\ell} f \equiv 0.$$

Inserting this information in the expression of $\mathscr{L}f$ given by Proposition 2.1, we conclude that for every $p \in B_r$ one has

$$\mathscr{L}f(p) = \sum_{\ell=1}^{m} Q_{1,\ell} X_\ell f = \langle z, c\rangle,$$

where $c = (c_1, ..., c_m)$ is defined by (3.4). This gives $\mathscr{L}f(\delta_\lambda p) = \lambda\langle z, c\rangle$. But we have observed in Section 2 that for every $r > 0$ the set B_r is starlike with respect to the identity e. This means that for every $p \in B_r$ the trajectory of dilations $\delta_\lambda(p) \in B_r$ for every $0 \le \lambda \le 1$. The fundamental theorem of calculus thus gives

$$f(p) - f(e) = \int_0^1 \frac{d}{d\lambda} f(\delta_\lambda p) d\lambda = \int_0^1 \frac{1}{\lambda} \mathscr{L}f(\delta_\lambda p) d\lambda = \langle c, z\rangle = \langle \nabla_z f(e), z\rangle, \tag{3.5}$$

which gives the desired conclusion.

□

Finally, we provide the

Proof of Proposition 1.2. The Heisenberg group $\mathbb{H}^n$ is perhaps the most important example of a Lie group of step $k = 2$. Consider the first group $\mathbb{H}^1$ endowed with the left-invariant generators of the Lie algebra given by

$$X_1 = \partial_x - \frac{y}{2}\,\partial_\sigma, \qquad X_2 = \partial_y + \frac{x}{2}\,\partial_\sigma. \tag{3.6}$$

If we let $T = \partial_\sigma$, then the only nontrivial commutator is $[X_1, X_2] = T$. In $\mathbb{H}^1$ we now consider the function

$$f(x, y, \sigma) = x + 6y\sigma - x^3 - 21\left(x\sigma^2 - \frac{1}{8}xy^4 + \frac{1}{3}y^3\sigma - \frac{1}{40}x^5\right). \tag{3.7}$$

For the reader's understanding, we mention that $f = P_1 + P_3 - 21\ P_5$, where $P_1(x, y, \sigma) = x$, $P_3(x, y, \sigma) = 6y\sigma - x^3$, and $P_5(x, y, \sigma) = x\sigma^2 - \frac{1}{8}xy^4 + \frac{1}{3}y^3\sigma - \frac{1}{40}x^5$, are three harmonic polynomials of homogeneous degree $1, 3$ and 5. This means that they solve in $\mathbb{H}^1$ the equation $\Delta_H P_i = 0$, $i = 1, 3, 5$, and that $P_i(\delta_\lambda p) = \lambda^i P_i(p)$, where $\delta_\lambda p = (\lambda x, \lambda y, \lambda^2\sigma)$ are the group non-isotropic dilations, see (2.2) and (2.3) above (the class of solid harmonics in $\mathbb{H}^1$ was introduced in [17], see also [18] for multi-dimensional generalisations). From (3.7) and (3.6), simple computations give

$$\begin{cases} X_1 f = 1 - 3|z|^2 - 21\sigma^2 + \frac{21}{8}x^4 + \frac{49}{8}y^4 + 21xy\sigma, \\ \\ X_2 f = 6\sigma + 3xy - 21\sigma|z|^2 + 7xy^3. \end{cases} \tag{3.8}$$

Furthermore, we have

$$X_1^2 f = -6x + \frac{21}{2}x^3 + 42y\sigma - \frac{21}{2}xy^2, \qquad X_2^2 f = 6x - \frac{21}{2}x^3 - 42y\sigma + \frac{21}{2}xy^2. \tag{3.9}$$

In particular $\Delta_H f = 0$ in $\mathbb{H}^1$ (this conclusion could also be derived from the fact that f is the sum of three harmonic polynomials, except that we have not verified explicitly that $\Delta_H P_i = 0$).

In order to simplify the understanding of $|\nabla_H f|^2$, keeping in mind that $z = (x, y)$, we express (3.8) in the following way

$$X_1 f = 1 - 3|z|^2 - 21\sigma^2 + g(x, y, \sigma), \qquad X_2 f = 6\sigma + h(x, y, \sigma), \tag{3.10}$$

where we have respectively denoted

$$g(x, y, \sigma) = \frac{21}{8}x^4 + \frac{49}{8}y^4 + 21xy\sigma, \qquad h(x, y, \sigma) = 3xy - 21\sigma|z|^2 + 7xy^3. \tag{3.11}$$

Since both g and h vanish at the origin, it is clear from that

$$|\nabla_H f(0, 0, 0)|^2 = X_1 f(0, 0, 0)^2 + X_2 f(0, 0, 0)^2 = 1. \tag{3.12}$$

The reader should notice that

$$g(x, y, \sigma) = O((|z|^2 + \sigma^2)^{3/2}), \qquad h(x, y, \sigma) = O(|z|^2 + \sigma^2). \tag{3.13}$$

Using (3.10) and (3.13) it is now not difficult to verify that

$$|\nabla_H f(x, y, \sigma)|^2 = 1 - 6(|z|^2 + \sigma^2) + k(x, y, \sigma), \tag{3.14}$$

where

$$\begin{aligned} k(x, y, \sigma) = {} & 2\ g(x, y, \sigma) + g(x, y, \sigma)^2 + h(x, y, \sigma)^2 + 12\ \sigma h(x, y, \sigma) - 6\ |z|^2 g(x, y, \sigma) \\ & - 42\ \sigma^2 g(x, y, \sigma) + 126\ |z|^2\sigma^2 + 9\ |z|^4 + 441\ \sigma^4. \end{aligned} \tag{3.15}$$

It is clear that $k(0, 0, 0) = 0$ and a moment's thought reveals that near $(0, 0, 0)$ we have

$$k(x, y, \sigma) = O((|z|^2 + \sigma^2)^{3/2}). \tag{3.16}$$

From (3.14) and (3.16) it is therefore clear that there exists $\rho > 0$ such that for every $0 < |z|^2 + \sigma^2 < \rho^2$ we have

$$|\nabla_H f(x, y, \sigma)|^2 < 1, \tag{3.17}$$

therefore $|\nabla_H f|^2$ has a strict maximum in $e = (0, 0, 0)$ in the Euclidean ball $B_\rho^{(e)} = \{(x, y, \sigma) \in \mathbb{H}^1 \mid |z|^2 + \sigma^2 < \rho^2\}$, and yet f is not linear. This proves that Theorem 1.1 cannot possibly hold if we replace (1.4) with (1.5), thus establishing the proposition.

□

Some final comments are in order. In view of (3.12), (3.17) and of Bony's strong maximum principle, it is clear that it is not possible for the harmonic function f in (3.7) to satisfy

$$\Delta_H(|\nabla_H f|^2) \geq 0$$

in a connected open set contained in $B_\rho^{(e)}$. In fact, we are next going to show that the opposite inequality does hold in a full neighbourhood of the origin. From (3.14) we see that

$$\begin{aligned} \Delta_H(|\nabla_H f|^2) &= -6\Delta_H(|z|^2 + \sigma^2) + \Delta_H k(x, y, \sigma) = -24 - 12|\nabla_H \sigma|^2 + \Delta_H k(x, y, \sigma) \\ &= -24 - 3|z|^2 + \Delta_H k(x, y, \sigma). \end{aligned} \tag{3.18}$$

From (3.11) and (3.15) we observe that $k(x, y, \sigma)$ is a sum of polynomials having homogeneous degree ≥ 4 with respect to the non-isotropic group dilations. Therefore, $-3|z|^2 + \Delta_H k(x, y, \sigma)$ is the sum of polynomials with homogeneous degree ≥ 2. Consequently, by (3.18) we must have

$$\Delta_H(|\nabla_H f|^2) \leq 0, \tag{3.19}$$

in a sufficiently small neighbourhood of the origin. For further instances of this typically sub-Riemannian negative phenomenon, and some basic implications of it, the reader should see [15].

References

1. A. Bellaïche, *The tangent space in sub-Riemannian geometry*, in *Sub-Riemannian geometry*, 1-78, Progr. Math., 144, Birkhäuser, Basel, 1996.
2. I. Birindelli & E. Lanconelli, *A negative answer to a one-dimensional symmetry problem in the Heisenberg group.* Calc. Var. Partial Differential Equations **18** (2003), no. 4, 357-372.
3. J.-M. Bony. *Principe du maximum, inégalite de Harnack et unicité du probléme de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier **19** (1969), 277-304.
4. L. Capogna, G. Citti & M. Manfredini, *Regularity of mean curvature flow of graphs on Lie groups free up to step* 2. Nonlinear Anal. **126** (2015), 437-450.
5. E. Cartan, *Sur la représentation géométrique des systèmes matériels non holonomes*, Proc. Internat. Congress Math., vol.4, Bologna, 1928, 253-261.
6. B. Chow, P. Lu & L. Ni, *Hamilton's Ricci flow.* Graduate Studies in Mathematics, 77. American Mathematical Society, Providence, RI; Science Press Beijing, New York, 2006. xxxvi+608 pp.
7. D. Danielli, N. Garofalo & A. Petrosyan, *The sub-elliptic obstacle problem: $C^{1,\alpha}$ regularity of the free boundary in Carnot groups of step two.* Adv. Math. **211** (2007), no. 2, 485-516.
8. F. Ferrari & N. Forcillo, *A new glance to the Alt-Caffarelli-Friedman monotonicity formula.* Math. Eng. **2** (2020), no. 4, 657-679.
9. F. Ferrari & N. Forcillo, *Some remarks about the existence of an Alt-Caffarelli-Friedman monotonicity formula in the Heisenberg group*, ArXiv:2001.04393
10. F. Ferrari & N. Forcillo, *A counterexample to the monotone increasing behavior of an Alt-Caffarelli-Friedman formula in the Heisenberg group*, ArXiv: 2203.06232
11. G. B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups.* Ark. Mat. 13 (1975), no. 2, 161-207.
12. G. Folland & E. M. Stein, *Hardy spaces on homogeneous Carnot groups*, Mathematical Notes, 28, Princeton University Press, Princeton, N.J., 1982.
13. N. Garofalo, *Geometric second derivative estimates in Carnot groups and convexity.* Manuscripta Math. **126** (2008), no. 3, 353-373.
14. N. Garofalo, *Gradient bounds for the horizontal horizontal p-Laplacian on a Carnot group and some applications.* Manuscripta Math. **130** (2009), no. 3, 375-385.
15. N. Garofalo, *A note on monotonicity and Bochner formulas in Carnot groups*, Proceedings of the Royal Society of Edinburgh. Section A: Mathematics. Published online by Cambridge University Press: 05 September 2022.
16. F. Dragoni, N. Garofalo & P. Salani, *Starshapedness for fully non-linear equations in Carnot groups.* J. Lond. Math. Soc. (2) **99** (2019), no. 3, 901-918.
17. P. C. Greiner, *Spherical harmonics on the Heisenberg group.* Canad. Math. Bull. **23** (1980), no. 4, 383-396.
18. P. C. Greiner & T. H. Koornwinder, *Variations on the Heisenberg spherical harmonics.* Stichting Mathematisch Centrum, Afdeling Zuivere Wiskunde, preprint no.186 (1983).
19. L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math. **119** (1967), 147-171.
20. V. Martino & A. Montanari, *Lipschitz continuous viscosity solutions for a class of fully nonlinear equations on Lie groups.* J. Geom. Anal. **24** (2014), no. 1, 169-189.
21. D. Müller, M. M. Peloso & F. Ricci, *On the solvability of homogeneous left-invariant differential operators on the Heisenberg group.* J. Funct. Anal. **148** (1997), no. 2, 368-383.

22. V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1974.

Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, United Kingdom
Email address: fb588@bath.ac.uk

Department of Mathematics, Michigan State University, East Lansing, MI 48824
Email address: forcill1@msu.edu

Dipartimento d'Ingegneria Civile e Ambientale (DICEA), Università di Padova, Via Marzolo, 9 - 35131 Padova, Italy
Email address: nicola.garofalo@unipd.it