INFINITE-TIME BLOWING-UP SOLUTIONS TO SMALL PERTURBATIONS OF THE YAMABE FLOW

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Abstract. In this paper, we examine a PDE aspect of the Yamabe flow as an energy-critical parabolic equation of the fast-diffusion type. It is well-known that under the validity of the positive mass theorem, the Yamabe flow on a smooth closed Riemannian manifold M exists for all time t and uniformly converges to a solution to the Yamabe problem on M as $t \to \infty$. We show that such results no longer hold if some arbitrarily small and smooth perturbation is imposed on it, by constructing solutions to the perturbed flow that blow up at multiple points on M in the infinite time. We also examine the stability of the blow-up phenomena under a negativity condition on the Ricci curvature at blow-up points.

1. Introduction

1.1. **History.** Let (M, g_0) be a smooth closed Riemannian manifold of dimension $N \ge 3$. The Yamabe flow on (M, g_0) is defined as

$$\begin{cases} \dfrac{\partial}{\partial t} g(t) = -\left(S\left[g(t)\right] - \overline{S}\left[g(t)\right]\right) g(t) \quad \text{for } t \in (0,T), \\ g(0) = g_0 \end{cases} \tag{1.1}$$

where T is the maximal time, $S\left[g(t)\right]$ is the scalar curvature of the metric $g(t)$, and $\overline{S}\left[g(t)\right]$ is the volume-averaged value of $S\left[g(t)\right]$ over M. If we set $g(t) = u(t)^{\frac{4}{N-2}} g_0$ where $u(t)$ is a positive function on M for each $t \in (0,T)$, then (1.1) is reduced to an energy-critical parabolic equation of the fast-diffusion type

$$pu^{p-1}u_t = \frac{N+2}{4}\left(\kappa_N \Delta_{g_0} u - S\left[g_0\right]u + \overline{S}\left[g(t)\right]u^p\right) \quad \text{on } M \times (0,T) \tag{1.2}$$

where $\kappa_N := \frac{4(N-1)}{N-2}$ and $p := \frac{N+2}{N-2}$.

Let E_{g_0} be the energy corresponding to equation (1.2),

$$E_{g_0}(u) := \frac{\int_M \left(\kappa_N |\nabla_{g_0} u|_{g_0}^2 + S\left[g_0\right]u^2\right) dv_{g_0}}{\left(\int_M |u|^{p+1} dv_{g_0}\right)^{\frac{2}{p+1}}}$$

where dv_{g_0} is the volume form on (M, g_0), and $Y(M, g_0)$ is the Yamabe constant on (M, g_0) defined by

$$Y(M, g_0) = \inf\left\{E_{g_0}(u) : u \in C^\infty(M) \setminus \{0\}\right\}.$$

If $Y(M, g_0) \le 0$, a relatively simple argument based on the maximum principle shows that the flow (1.1) exists globally ($T = \infty$) and converges to a metric of constant scalar curvature as $t \to \infty$.

However, treating (1.1) in the case $Y(M, g_0) > 0$ is challenging, because one has to exclude the possibility of blow-up phenomena. Thanks to the series of works by Chow [12], Ye [52], Schwetlick and Struwe [46], and Brendle [6, 7], it is now known that (1.1) is always globally well-defined in time and uniformly converges to a metric of constant scalar curvature as $t \to \infty$, under the validity

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of the positive mass theorem; see also [9] in which Carlotto, Chodosh, and Rubinstein showed the existence of a Yamabe flow on a certain manifold converging at a polynomial rate.

One of the main ingredients of the proof by Brendle in [6, 7] is to construct a suitable family $\{\bar{u}_{z,\epsilon} \in C^\infty(M) : z \in M, \epsilon > 0 \text{ small}\}$ of test functions satisfying

$$\begin{cases} \sup\{E_{g_0}(\bar{u}_{z,\epsilon}) : z \in M, \epsilon > 0 \text{ small}\} \le Y(\mathbb{S}^N, g_{\mathbb{S}^N}), \\ \lim_{\epsilon \to 0} \epsilon^{\frac{N-2}{2}} \bar{u}_{z,\epsilon}(\exp_z(\epsilon x)) = W_{1,0}(x) \quad \text{for } z \in M,\ x \in T_zM \end{cases} \tag{1.3}$$

where $(\mathbb{S}^N, g_{\mathbb{S}^N})$ is the N-dimensional unit-sphere in $\mathbb{R}^{N+1}$ with the canonical metric $g_{\mathbb{S}^N}$, exp is the exponential map, and $W_{1,0}$ is the standard bubble in $\mathbb{R}^N$, that is,

$$W_{1,0}(x) = \frac{\alpha_N}{(1+|x|^2)^{\frac{N-2}{2}}} \quad \text{for } x \in \mathbb{R}^N, \quad \alpha_N := (N(N-2))^{\frac{N-2}{4}}. \tag{1.4}$$

Given a C^2-function h on M, let $E_{g_0,h}$ be a perturbed energy of E_{g_0} given as

$$E_{g_0,h}(u) = \frac{\int_M \left[\kappa_N |\nabla_{g_0} u|_{g_0}^2 + (S[g_0] + h)u^2\right] dv_{g_0}}{\left(\int_M |u|^{p+1} dv_{g_0}\right)^{\frac{2}{p+1}}}.$$

If $N \ge 5$ and $h(z) > 0$ at some $z \in M$, then no test function $\bar{u}_{z,\epsilon}$ satisfying (1.3) is available. For instance, if $N \ge 7$, M is non-locally conformally flat, and $\bar{u}_{z,\epsilon}$ is a 'bubble-like' function satisfying the second condition in (1.3), then

$$E_{g_0,h}(\bar{u}_{z,\epsilon}) \simeq Y(\mathbb{S}^N, g_{\mathbb{S}^N}) + \tilde{c}_1 h(z)\epsilon^2 - \tilde{c}_2 \|\text{Weyl}[g_0](z)\|_{g_0}^2 \epsilon^4$$

as $\epsilon \to 0$, so

$$E_{g_0,h}(\bar{u}_{z,\epsilon}) > Y(\mathbb{S}^N, g_{\mathbb{S}^N}) \quad \text{provided } \epsilon > 0 \text{ small enough.}$$

Here, $\tilde{c}_1, \tilde{c}_2 > 0$, Weyl$[g_0]$ is the Weyl tensor on (M, g_0), and $\|\cdot\|_{g_0}$ is the tensor norm in the metric g_0.

As a means of understanding the PDE aspect of the Yamabe flow, one can examine the behavior of a solution to the evolution equation corresponding to the energy $E_{g_0,h}$. Setting

$$L_{g_0,h} u = \kappa_N \Delta_{g_0} - (S[g_0] + h) \tag{1.5}$$

so that $L_{g_0} := L_{g_0,0}$ is the conformal Laplacian on (M, g_0), we consider

$$\begin{cases} pu^{p-1} u_t = \dfrac{N+2}{4}(L_{g_0,h} u + \kappa_N u^p) & \text{on } M \times (0,\infty), \quad p = \frac{N+2}{N-2}, \\ u > 0 & \text{on } M \times (0,\infty), \\ u(\cdot, 0) = u_0 > 0 & \text{on } M.^1 \end{cases} \tag{1.6}$$

The main result of this paper is to show that (1.6) may exhibit infinite-time blow-up phenomena, which are in particular driven by the bubbles (also known as the Talenti-Aubin instantons)

$$W_{\mu,\xi}(x) = \mu^{-\frac{N-2}{2}} W_{1,0}(\mu^{-1}(x-\xi)), \quad x \in \mathbb{R}^N,\ \mu > 0,\ \xi \in \mathbb{R}^N; \tag{1.7}$$

cf. (1.4). It is a classical result that the solution set of the Yamabe problem in $\mathbb{R}^N$

$$-\Delta W = W^p, \quad W > 0 \quad \text{in } \mathbb{R}^N \tag{1.8}$$

is precisely $\{W_{\mu,\xi} : \mu > 0, \xi \in \mathbb{R}^N\}$, which corresponds to a family of standard metrics on N-dimensional spheres in $\mathbb{R}^{N+1}$ via the stereographic projection.

[1] The average $\overline{S}[g(t)]$ of the scalar curvature $S[g(t)]$ in (1.1) or (1.2) uniformly converges to a constant $\kappa_N > 0$ as $t \to \infty$ provided $Y(M, g_0) > 0$. In this viewpoint, $S[g(t)]$ was replaced with κ_N in (1.6).

1.2. **Main theorems.** We now list the main theorems of this paper and some remarks on them.

The following theorem precisely describes the infinite-time blow-up phenomena of equation (1.6).

Theorem 1.1. *Let (M, g_0) be a smooth closed Riemannian manifold of dimension $N \geq 5$ such that*

$$Y_h(M, g_0) := \inf\{E_{g_0,h}(u) : u \in C^\infty(M) \setminus \{0\}\} > 0. \tag{1.9}$$

Assume that h is a C^2-function on M such that $\max_M h > 0$. Given a point $z_0 \in M$ such that $h(z_0) > 0$, there exists an initial datum $u_{z_0,0}$ such that (1.6) has a solution u_{z_0} blowing-up at z_0 as $t \to \infty$. More precisely, there is a constant $C > 0$ depending only on (M, g_0), N, h, and z_0 such that

$$\begin{cases} u_{z_0}(z,t) = \left(1 + O\left(|\exp_{z_0}^{-1}(z)|^2\right)\right) W_{\mu(t),\xi(t)}\left(\exp_{z_0}^{-1}(z)\right) & \text{if } d_{g_0}(z, z_0) \leq \delta_0, \\ u_{z_0}(z,t) \leq C\mu^{\frac{N-2}{2}}(t) G(z, z_0) & \text{if } d_{g_0}(z, z_0) > \delta_0 \end{cases} \tag{1.10}$$

for $t > 0$ large enough, where

- $\exp$ *is the exponential map on (M, g_0), and $d_{g_0}(z, z_0)$ is the geodesic distance between z and z_0;*
- *$\delta_0 > 0$ is a sufficiently small number;*
- *$\mu : [0,\infty) \to (0,\infty)$ and $\xi : [0,\infty) \to \mathbb{R}^N$ are parameters such that $\mu(t) \simeq t^{-\frac{1}{2}}$ and $|\xi(t)| \lesssim t^{-1+\varepsilon}$ for all $t > 0$ large and some $\varepsilon \in (0,1)$ small;*
- *G is the Green's function of the perturbed conformal Laplacian $L_{g_0,h}$ defined in (2.2).*

In fact, (1.6) also possesses solutions which blow up at multiple points as $t \to \infty$.

Theorem 1.2. *Let (M, g_0) be a smooth closed Riemannian manifold of dimension $N \geq 5$ such that $Y_h(M, g_0) > 0$. Assume that h is a C^2-function on M such that $\max_M h > 0$. Given any $k \in \mathbb{N}$, choose a k-tuple $\mathbf{z}_0 := \{z_0^{(1)}, \ldots, z_0^{(k)}\}$ of distinct points on M such that $h(z_0^{(l)}) > 0$ for $l = 1, \ldots, k$. Then there exists an initial datum $u_{\mathbf{z}_0,0}$ such that (1.6) has a solution $u_{\mathbf{z}_0}$ blowing-up at each point $z_0^{(1)}, \ldots, z_0^{(k)}$ as $t \to \infty$. More precisely, there is a constant $C > 0$ depending only on (M, g_0), N, h, k, and $\mathbf{z}_0$ such that*

$$\begin{cases} u_{\mathbf{z}_0}(z,t) = \left(1 + O\left(\left|\exp_{z_0^{(l)}}^{-1}(z)\right|^2\right)\right) W_{\mu^{(l)}(t),\xi^{(l)}(t)}\left(\exp_{z_0^{(l)}}^{-1}(z)\right) & \text{if } d_{g_0}\left(z, z_0^{(l)}\right) \leq \delta_0,\ l = 1, \ldots, k, \\ u_{\mathbf{z}_0}(z,t) \leq C\mu^{\frac{N-2}{2}}(t) \sum_{l=1}^{k} G\left(z, z_0^{(l)}\right) & \text{otherwise} \end{cases}$$

for $t > 0$ large enough, where $\mu^{(l)} : [0,\infty) \to (0,\infty)$ and $\xi^{(l)} : [0,\infty) \to \mathbb{R}^N$ are parameters such that $\mu^{(l)}(t) \simeq t^{-\frac{1}{2}}$ and $|\xi^{(l)}(t)| \lesssim t^{-1+\varepsilon}$ for all $t > 0$ large and some $\varepsilon \in (0,1)$ small.

Regarding the stability of the solution found above, we have the following result.

Corollary 1.3. *In the setting of Theorem 1.2, we further assume that the largest eigenvalue of the Ricci curvature tensor at $z_0^{(l)}$ is less than or equal to $-\frac{6}{N-4} h(z_0^{(l)})$ for each $l = 1, \ldots, k$. Let $\sigma \in (0,1)$. There is a k-codimensional manifold $\mathcal{M}_{\mathbf{z}_0}$ in $C^{2,\sigma}(M)$ containing $u_{\mathbf{z}_0}$ such that if $u_0 \in \mathcal{M}_{\mathbf{z}_0}$ is sufficiently close to $u_{\mathbf{z}_0,0}$, then (1.6) has a solution u blowing-up at each point $z_0^{(1)}, \ldots, z_0^{(k)}$ as $t \to \infty$.*

In the above corollary, the technical condition on the Ricci curvature comes from the ODE system (5.43) that each $\xi^{(l)}$ solves.

Remark 1.4. Theorems 1.1 and 1.2 show that the Yamabe flow is an equation at the borderline guaranteeing the global existence and uniform boundness of solutions. A couple of remarks regarding them are in order.

(1) If $Y(M,g_0) > 0$ and h is a function such that $\|h\|_{L^{N/2}(M)} < Y(M,g_0)$, then a simple application of Hölder's inequality yields that $Y_h(M,g_0) > 0$. Therefore, if (M,g_0) is a smooth closed Riemannian manifold of dimension $N \geq 5$ such that $Y(M,g_0) > 0$, (1.6) exhibits the infinite-time blow-up phenomena provided $h \in C^2(M)$ is sufficiently small and $\max_M h > 0$.

(2) Our results can be regarded as parabolic analogues of the theorems of Micheletti, Pistoia, and Vétois [38], and of Esposito, Pistoia, and Vétois [22] which assert the existence of blow-up solutions to slightly perturbed (elliptic) Yamabe problems; see also [43, 39, 50, 11, 40]. Their results are related to $C^2(M)$-compactness property of the solution set of the Yamabe problem on (M,g_0); refer to [31, 8] and references therein.

In the elliptic case, the blow-up points must be a critical point of some function involving the function h and a geometric quantity on (M,g_0); either the Weyl curvature or the $O(1)$-order term of the Green's function of the conformal Laplacian on (M,g_0). In our evolutionary setting, we only require that h is positive at each blow-up point.

(3) Suppose that (1.9) holds and $\min_M h < 0$. We expect that given any initial condition $u_0 > 0$ on M, a solution to (1.6) exists and converges to a stationary solution as $t \to \infty$. For a corresponding Dirichlet problem on a smooth bounded domain in $\mathbb{R}^N$, such expectation was rigorously verified by Jin and Xiong [29].

On the other hand, our approach suggests that if the domain of (1.6) is replaced with $M \times (-\infty, 0)$ and a k-tuple of points $\mathbf{z}_0 = \{z_0^{(1)}, \dots, z_0^{(k)}\}$ on M such that $h(z_0^{(l)}) < 0$ for $l = 1, \dots, k$ is given, then there exists an ancient solution which blows up at each point $z_0^{(1)}, \dots, z_0^{(k)}$ as $t \to -\infty$.

(4) The blow-up rates of the multiple blowing-up solution constructed in Theorem 1.2 are all comparable to $t^{-\frac{1}{2}}$, while their blow-up points are distinct.

On the other hand, Daskalopoulos, del Pino, and Sesum [15] constructed ancient solutions to the Yamabe flow on $\mathbb{S}^N$ which have the bubble-tower structure. These solutions look like superpositions of k-bubbles, where the blow-up rates $\{\mu^{(1)}(t), \dots, \mu^{(k)}(t)\}$ satisfy $|\log(\mu^{(l_1)}(t)/\mu^{(l_2)}(t))| \to \infty$ as $t \to -\infty$ for $1 \leq l_1 \neq l_2 \leq k$, and the blow-up points converge to a point on $\mathbb{S}^N$. For the energy-critical nonlinear heat (or Fujita) equation in $\mathbb{R}^N$, del Pino, Musso, and Wei [19] and Sun, Wei, and Zhang [49] built solutions which exhibit the bubble-tower phenomena as $t \to \infty$ and $t \to -\infty$, respectively. We believe that similar solutions exist within our framework.

To the best of our knowledge, there is no result on the existence of a multiple blowing-up solution in the parabolic setting where the blow-up rates are all comparable and the blow-up points collapse into a single point (known as a clustering solution). Our model (1.6) may provide such an example under a suitable geometric condition.

(5) The formal analysis for $N = 4$ suggests that there exists a solution to (1.6) that blows-up in infinite time, with a blow-up rate of $\mu(t) \simeq t^{-\frac{1}{2}}(1 + O(\frac{\log\log t}{|\log t|}))$ as $t \to \infty$. However, further work is necessary to establish the existence of such a solution rigorously.

For $N = 3$, Del Pino, Musso, and Wei [17] constructed infinite-time blowing-up solutions such that the dynamics of the scaling parameters are governed by fractional-order ODEs. In contrast, we expect that (1.6) with $N = 3$ does not have infinite-time blow-up solution, provided $\|h\|_{L^\infty(M)}$ is small enough. This is due to the positive mass of the Green's function for the conformal Laplacian L_{g_0} on (M,g_0). A similar observation was made by Li and Zhu [35] for the small linear perturbations of the 3-dimensional (elliptic) Yamabe problem, whose solution set is L^∞-bounded for the same reason.

We plan to investigate the above cases in a forthcoming work.

There have been extensive studies on blow-up solutions to various time-dependent energy-critical equations; e.g. the critical nonlinear heat equations [23, 47, 13, 14, 17, 24, 18, 19, 49], the harmonic map heat flows and the nematic liquid crystal flows [10, 2, 51, 42, 16, 33], the critical wave equations

[30, 32, 25, 20, 21, 26, 27], the wave maps equations and the Yang-Mills equations [41, 44], the critical Schrödinger equations and map equations [36, 37], and so on. Our contribution towards this direction is to build solutions to energy-critical degenerate parabolic equations on general Riemannian manifolds, blowing up at an arbitrary number of points.

Furthermore, several results on the optimal regularity and sharp extinction rates for fast diffusion equations in smooth bounded domains in $\mathbb{R}^N$ were proved recently; refer to [28, 29, 5, 4] among others. In their proofs, Aronson-Bénilan type inequalities (namely, bounds for $u^{-1}|u_t|$; see [1]) appear as one of the key tools. In our analysis of (1.6), such an inequality is derived in a very specific manner; see (4.4) below and its proof.

1.3. **Novelty of the proof.** In order to establish Theorems 1.1 and 1.2 as well as Corollary 1.3, we will apply the modulation argument combined with the inner-outer gluing procedure. Our presentation is influenced primarily by the paper [14] that studied infinite-time multiple bubbling solutions to the critical nonlinear heat equations in smooth bounded domains in $\mathbb{R}^N$. However, because of the delicate structure of (1.6) resulting from the conformal property of the manifold M, we need a variety of new ideas and techniques. In this subsection, we briefly explain them.

(1) Construction of approximate solutions: It is one of the tricky parts of the proof, since approximate solutions must be sufficiently regular and very close to true solutions at every point on M regardless of its distance from the blow-up points.

At points away from the blow-up points, the correct shape of the approximate solution turns out to be the Green's function G of the perturbed conformal Laplacian $L_{g_0,h}$ multiplied by the $\frac{N-2}{2}$-th power of the dilation parameter (up to a constant multiple).

At points near a blow-up point, say $z_0 \in M$, the approximate solution looks like the bubbles (1.7) at the main order, but we have to know more detailed information to reduce its error sufficiently. To this aim, we deform it by multiplying it by a constant multiple of $d_{g_0}(\cdot, z_0)^{N-2}G(\cdot, z_0)$ and then attaching a small correction term Ψ that solves the linear equation (2.30). The existence of the correction term is tied to the solvability of the ODE (2.31) that the leading order term of the dilation parameter satisfies. Also, for each fixed t, we require that the correction term $\Psi(y,t)$ decays faster than $|y|^{4-N}$ as $|y| \to \infty$, which is achieved thanks to a remarkable cancellation among the terms constituting the inhomogeneous term of (2.30).

In the middle region, we control the approximate solution so that it is regular enough and bounded by a constant multiple of the $\frac{N-2}{2}$-th power of the dilation parameter.

The choice of the approximate solutions was inspired by Schoen's solution [45] for the Yamabe problem and Brendle's solution [6] for the Yamabe flow.

(2) Degeneracy of the equation: In [15], Daskalopoulos, del Pino, and Sesum constructed type II ancient compact solutions to the Yamabe flow on $\mathbb{S}^N$. In [48], Sire, Wei, and Zheng built finite-time extinguishing solutions to fast diffusion equations with the critical exponent in smooth bounded domains in $\mathbb{R}^N$. From the PDE point of view, our results are closely related to these works.

To handle the degeneracy of the equation (u^{p-1} in (1.6) for our setting), the authors in [48] lifted the spatial domain to a subset of $\mathbb{S}^N$ via the stereographic projection so that the equation becomes uniformly parabolic. This method works for our inner problem.[2] However, we need a new idea to treat the outer problem because its spatial domain is a general Riemannian manifold M, which cannot be embedded into $\mathbb{S}^N$ in general. We will overcome this technical difficulty with the help of a maximum principle adapted to our setting; see Lemma 4.5.

(3) Choice of the norms: A delicate issue throughout the proof is to choose appropriate norms to work with.

[2]The proof of a Liouville-type theorem for (4.22)–(4.23), which appears during the analysis of the outer problem, also uses this method. See the proof of Lemma 4.3.

As a preliminary step to solving the outer problem (3.2) and obtaining a priori estimates of the solution, we examine the associated inhomogeneous problem (4.1). By employing the weighted integral norms defined in Definition 1.7, we first show the global existence of a weak solution. To depict the shape of the solution more precisely, we also use norms that control the pointwise behavior of functions. A weighted L^∞ setting works well for the standard nonlinear heat equation as demonstrated in [14], but it does not for (1.6) because of its degeneracy u^{p-1}. To handle the terms caused by the degeneracy such as $(\psi_{\mu,\xi})_t$ in (3.4) in a pointwise sense, we will devise various Hölder-type norms on $M \times [t_0, \infty)$; refer to Definition 1.8. In defining them, we must reflect that the scaling properties in the spatial variable and in the time variable are different from each other, which makes the analysis considerably complicated.

An analogous explanation also applies to the inner problem (3.6).

1.4. **Structure of the paper.** From Section 2 to 5, we concern the proof of Theorem 1.1: In Section 2, we construct approximate solutions which behave as in (1.10). In Section 3, we decompose equation (1.6) into the outer problem (3.2) and the inner problem (3.6). In Section 4, we prove the unique solvability of the outer problem and derive a priori estimates of the solution. In Section 5, we determine the dilation and translation parameters by analyzing the ODE system that they solve, and find the unique solution to the inner problem as well as its a priori estimates. All the information obtained will allow us to complete the proof of the theorem.

Finally, necessary modifications to esbalish Theorem 1.2 and the proof of Corollary 1.3 is presented in Section 6.

1.5. **Notations.** We collect some notations used throughout the paper.

- The Einstein convention is used throughout the paper. Unless otherwise stated, the indices i, j, q, r, and s take values from 1 to the dimension N of the underlying manifold M, and l and m range over values from 1 to the number k of blowing-up points of solutions.

- For a tensor field T on (M, g), a notation such as $T_{;a}$ stands for a covariant derivative of T.

- $\mathbb{S}^N$ is the standard unit sphere in $\mathbb{R}^{N+1}$, $g_{\mathbb{S}^N}$ is its canonical metric, and $\left|\mathbb{S}^N\right|$ is its surface measure.

- ∇_g, Δ_g, $\langle \cdot, \cdot \rangle_g$, and $|\cdot|_g$ are the gradient, the Laplace-Beltrami operator, the inner product, and the norm on (M, g), respectively. On $(\mathbb{S}^N, g_{\mathbb{S}^N})$, we write $\nabla_{\mathbb{S}^N} = \nabla_{g_{\mathbb{S}^N}}$, etc. In the Euclidean space $\mathbb{R}^N$, we write $\nabla = \nabla_x$, $\Delta = \Delta_x$, etc, where the subscript x denotes the variable in $\mathbb{R}^N$.

- $i(M, g_0)$ is the injectivity radius of (M, g_0).

- For a surface integral, dS denotes the volume form on the domain of integration.

- Given a number $\delta > 0$ and a metric g on M, let $B^N(x, \delta) = \{y \in \mathbb{R}^N : |y - x| < \delta\}$ for $x \in \mathbb{R}^N$ and $B_g(z, \delta) = \{\xi \in M : d_g(z, \xi) < \delta\}$ for $z \in M$.

- For a function f and $\ell \in \mathbb{N}$, we set $\partial_i f = \frac{\partial f}{\partial x_i}$ for $i = 1, \ldots, N$, $\dot{f} = f_t = \partial_t f = \frac{\partial f}{\partial t}$, and

$$f_{[\ell]}(x) = f(x) - \sum_{|\gamma|=0}^{\ell-1} \frac{x^\gamma}{\gamma!} \partial_\gamma f(0) = O(|x|^\ell) \tag{1.11}$$

for $|x|$ small, where γ is a multi-index.

- For $\ell \in \mathbb{N} \cup \{0\}$, $\sigma \in (0, 1)$, $T > 0$, and a set Ω, we write $C^{\ell+\sigma}(\Omega) = C^{\ell,\sigma}(\Omega)$ and $C^{2\ell+\sigma, \ell+\sigma/2}(\Omega \times [0, T])$ to refer the Hölder space and the parabolic Hölder space, respectively.

- $\mathrm{supp}(f)$ is the support of a function f.

- Let $\eta \in C^\infty(\mathbb{R})$ be a function such that

$$\begin{cases} \eta(r) \ge 0,\ \eta'(r) \le 0 \text{ for } r \in \mathbb{R}, \\ \eta(r) = 1 \text{ for } r \in (-\infty, 1] \text{ and } 0 \text{ for } r \in [2, \infty), \end{cases} \tag{1.12}$$

and $\eta_\delta(r) = \eta(\delta^{-1} r)$ for $r \in \mathbb{R}$ and $\delta > 0$. By abuse of notation, we often write $\eta_\delta(x) = \eta_\delta(|x|)$ for $x \in \mathbb{R}^N$.

- $C, \zeta > 0$ are universal constants that may vary from line to line.

1.6. **Norms.** We introduce all the norms that will be used throughout the paper. Let $\delta_0 > 0$ be the small number in the statement of Theorem 1.1, for which we impose that $4\delta_0 < i(M, g_0)$. Let also $t_0 > 0$ be a large number, and $\mu = \mu(t) > 0$ and $\xi = \xi(t) \in \mathbb{R}^N$ be small functions on $[t_0, \infty)$ tending to 0 as $t \to \infty$. The functions $\mu_0 = \mu_0(t)$ and $u_{\mu,\xi} = u^{(2)}_{\mu,\xi}(z,t)$ are defined in (2.5) and (2.39) below, respectively.

Definition 1.5 (Local Hölder semi-norms)**.** Assume that $\sigma \in (0,1)$. For a function $\lambda : [t_0, \infty) \to \mathbb{R}$, we set

$$[\lambda]_{C_t^{\sigma/2}}(t) = \sup \left\{ \frac{|\lambda(t_1) - \lambda(t_2)|}{|t_1 - t_2|^{\sigma/2}} : t_1, t_2 \in (\max\{t_0, t-1\}, t),\ t_1 \ne t_2 \right\}.$$

For a function $\psi : M \times [t_0, \infty) \to \mathbb{R}$, we set

$$[\psi]_{C_z^\sigma}(z,t) = \sup \left\{ \frac{|\psi(z_1,t) - \psi(z_2,t)|}{d_{g_0}(z_1,z_2)^\sigma} : z_1, z_2 \in B_{g_0}(z, r_0(z,t)),\ z_1 \ne z_2 \right\}$$

and

$$[\psi]_{C_t^{\sigma/2}}(z,t) = \sup \left\{ \frac{|\psi(z,t_1) - \psi(z,t_2)|}{|t_1 - t_2|^{\sigma/2}} : t_1, t_2 \in (\max\{t_0, t-1\}, t),\ t_1 \ne t_2 \right\}$$

where

$$r_0(z,t) := \begin{cases} \dfrac{\mu_0(t)}{2} & \text{for } z \in B_{g_0}(z_0, \mu_0(t)), \\ \dfrac{d_{g_0}(z,z_0)}{2} & \text{for } z \in B_{g_0}(z_0, \delta_0) \setminus B_{g_0}(z_0, \mu_0(t)), \\ \dfrac{\delta_0}{2} & \text{for } z \in M \setminus B_{g_0}(z_0, \delta_0), \end{cases}$$

and $t \in [t_0, \infty)$.

Let Ω be a smooth domain in $\mathbb{R}^N$. For a function $\psi : \Omega \times [t_0, \infty) \to \mathbb{R}$, we set

$$[\psi]_{C_\Omega^\sigma}(y,t) = \sup \left\{ \frac{|\psi(y_1,t) - \psi(y_2,t)|}{|y_1 - y_2|^\sigma} : y_1, y_2 \in B^N(y, r_1(y)) \cap \Omega,\ y_1 \ne y_2 \right\}$$

where

$$r_1(y) := \begin{cases} \dfrac{1}{2} & \text{in } \Omega \cap \{|y| < 1\}, \\ \dfrac{|y|}{2} & \text{in } \Omega \cap \{|y| \ge 1\}. \end{cases}$$

Definition 1.6 (A Hölder norm for functions on $[t_0, \infty)$)**.** For numbers $\nu > 0$ and $\sigma \in (0,1)$ and functions $\lambda : [t_0, \infty) \to \mathbb{R}$ and $\xi = (\xi_1, \ldots, \xi_N) : [t_0, \infty) \to \mathbb{R}^N$, we define

$$\begin{cases} \|\lambda\|_{\nu;\sigma} = \sup\limits_{t \in [t_0,\infty)} \mu_0^{-\nu}(t) \left[|\lambda(t)| + [\lambda]_{C_t^{\sigma/2}}(t) \right], \\ \|\xi\|_{\nu;\sigma} = \sup\limits_{t \in [t_0,\infty)} \mu_0^{-\nu}(t) \displaystyle\sum_{i=1}^N \left[|\xi_i(t)| + [\xi_i]_{C_t^{\sigma/2}}(t) \right]. \end{cases} \tag{1.13}$$

Definition 1.7 (Weighted integral norms for functions on M or $M\times[t_0,\infty)$)**.** Let $M_\tau = M\times[\tau,\tau+1]$ for $\tau \ge t_0$.

- Local in time weighted L^2, H^1 and H^2 norms: We set

$$\|\psi_0\|_{\mathcal{H}^1} = \left[\int_M (|\nabla_{g_0}\psi_0|_{g_0}^2 + \psi_0^2) dv_{g_0}\right]^{\frac{1}{2}}$$

for a function ψ_0 on M, and

$$\begin{aligned}
\|\psi(\cdot,\tau)\|_{L^2(M)} &= \left[\int_M \left(|\psi|^2 u_{\mu,\xi}^{p-1}\right)(z,\tau) dv_{g_0}\right]^{\frac{1}{2}}, \\
\|\psi\|_{L^2(M_\tau)} &= \left[\iint_{M_\tau} \left(|\psi|^2 u_{\mu,\xi}^{p-1}\right)(z,t) dv_{g_0} dt\right]^{\frac{1}{2}}, \\
\|\psi\|_{H^1(M_\tau)} &= \left\|u_{\mu,\xi}^{-\frac{p-1}{2}} \nabla_{g_0}\psi\right\|_{L^2(M_\tau)} + \|\psi\|_{L^2(M_\tau)}, \\
\|\psi\|_{H^2(M_\tau)} &= \|\psi_t\|_{L^2(M_\tau)} + \left\|u_{\mu,\xi}^{1-p} L_{g_0,h}\psi\right\|_{L^2(M_\tau)} + \|\psi\|_{H^1(M_\tau)}
\end{aligned}$$

for a function ψ on $M\times[t_0,\infty)$.

- Global in time weighted L^2, H^1 and H^2 norms: Given two positive numbers t_0 and s_0 such that $t_0 < s_0 - 1$, we set

$$\begin{aligned}
\|\psi\|_{L^2_{t_0,s_0}} &= \sup_{\tau\in[t_0,s_0-1]} \|\psi\|_{L^2(M_\tau)}, \\
\|\psi\|_{H^1_{t_0,s_0}} &= \sup_{\tau\in[t_0,s_0-1]} \|\psi\|_{H^1(M_\tau)}, \\
\|\psi\|_{H^2_{t_0,s_0}} &= \sup_{\tau\in[t_0,s_0-1]} \|\psi\|_{H^2(M_\tau)}.
\end{aligned}$$

In the case that $s_0=\infty$, we omit the subscript s_0 so that $\|\psi\|_{L^2_{t_0,\infty}} = \|\psi\|_{L^2_{t_0}}$, etc.

Observe that $\|\psi\|_{H^2(M_\tau)}$ is a norm if $Y_h(M,g_0)>0$ holds; see (1.9).

Definition 1.8 (Weighted Hölder norms for functions on $M\times[t_0,\infty)$)**.** Given $\alpha\in\mathbb{R}$ and $\gamma\ge 0$, we define

$$w_{\alpha,\gamma}(z,t) = \mu^\alpha(t)\cdot \max\left\{\frac{\eta_{\delta_0}(d_{g_0}(z,z_0))}{\mu^{\alpha+\gamma}(t) + |\exp_{z_0}^{-1}(z)-\xi(t)|^{\alpha+\gamma}},\ 2^{3(\alpha+\gamma)}\delta_0^{-(\alpha+\gamma)}\right\} \tag{1.14}$$

on $M\times[t_0,\infty)$. Here, η_{δ_0} is the cut-off function defined after (1.12).

- The weighted L^∞ norms: Given $\alpha>0$ and $\rho\ge -\frac{N-2}{2}$, we define

$$\|f\|_{*,\alpha,\rho} = \left\|\left(\mu_0^\rho w_{\alpha,2}\right)^{-1} u_{\mu,\xi}^{p-1} f\right\|_{L^\infty(M\times[t_0,\infty))}, \quad \|\psi_0\|_{**,\alpha} = \left\|\left(\mu_0^\rho(t_0) w_{\alpha,0}\right)^{-1}\psi_0\right\|_{L^\infty(M)},$$

and

$$\|\psi\|_{*',\alpha,\rho} = \left\|\left(\mu_0^\rho w_{\alpha,0}\right)^{-1}\psi\right\|_{L^\infty(M\times[t_0,\infty))}.$$

- The weighted Hölder norms: Given $\alpha>0$, $\rho\ge -\frac{N-2}{2}$, and $\sigma\in(0,1)$, we define

$$\|f\|_{*,\alpha,\rho;\sigma} = \left\|\left(\mu_0^\rho w_{\alpha,2}\right)^{-1}\left|u_{\mu,\xi}^{p-1}f\right| + \left(\mu_0^\rho w_{\alpha,2+\sigma}\right)^{-1}\left[u_{\mu,\xi}^{p-1}f\right]_{C_z^\sigma} + \left(\mu_0^\rho w_{\alpha-\sigma,2}\right)^{-1}\left[u_{\mu,\xi}^{p-1}f\right]_{C_t^{\sigma/2}}\right\|_{L^\infty(M\times[t_0,\infty))}, \tag{1.15}$$

$$\|\psi_0\|_{**,\alpha;\sigma} = \sum_{\ell=0}^{2}\left\|\left(\mu_0^\rho(t_0)w_{\alpha,\ell}\right)^{-1}\left|\nabla_{g_0}^\ell\psi_0\right| + \left(\mu_0^\rho(t_0)w_{\alpha,\ell+\sigma}\right)^{-1}\left[\nabla_{g_0}^\ell\psi_0\right]_{C_z^\sigma}\right\|_{L^\infty(M)}, \tag{1.16}$$

and

$$
\begin{aligned}
&\|\psi\|_{*',\alpha,\rho;\sigma} \\
&= \sum_{\ell=0}^{2} \left\| (\mu_0^\rho w_{\alpha,\ell})^{-1} \left|\nabla_{g_0}^\ell \psi\right| + (\mu_0^\rho w_{\alpha,\ell+\sigma})^{-1} \left[\nabla_{g_0}^\ell \psi\right]_{C_z^\sigma} + (\mu_0^\rho w_{\alpha-\sigma,\ell})^{-1} \left[\nabla_{g_0}^\ell \psi\right]_{C_t^{\sigma/2}} \right\|_{L^\infty(M\times[t_0,\infty))} \qquad (1.17)\\
&+ \left\| (\mu_0^\rho w_{\alpha-2,0})^{-1} |\psi_t| + (\mu_0^\rho w_{\alpha-2,\sigma})^{-1} [\psi_t]_{C_z^\sigma} + (\mu_0^\rho w_{\alpha-2-\sigma,0})^{-1} [\psi_t]_{C_t^{\sigma/2}} \right\|_{L^\infty(M\times[t_0,\infty))}.
\end{aligned}
$$

Definition 1.9 (Weighted Hölder norms for functions on $\Omega\times[t_0,\infty)$)**.** Let Ω be a smooth domain in $\mathbb{R}^N$, $a\in(0,N-2)$, $b>0$, and $s_0>\frac{3t_0}{2}\gg 1$.

- The weighted L^∞ norms: We set

$$
\|\mathcal{H}\|_{\sharp,a+2,b;t_0,s_0} = \left\| \mu_0^{-b} \left(1+|\bar{y}|^{a+2}\right) \left(W_{1,0}^{p-1}\mathcal{H}\right)(\bar{y},t) \right\|_{L^\infty(\mathbb{R}^N\times[t_0,s_0))}
$$

and

$$
\|\psi\|_{\sharp',a,b;t_0,s_0(\Omega)} = \left\| \mu_0^{-b} \left(1+|\bar{y}|^{a}\right) \psi(\bar{y},t) \right\|_{L^\infty(\Omega\times[t_0,\infty))}.
$$

In the case that $s_0=\infty$, we write $\|\mathcal{H}\|_{\sharp,a+2,b} = \|\mathcal{H}\|_{\sharp,a+2,b;t_0,s_0}$ and $\|\psi\|_{\sharp',a,b(\Omega)} = \|\psi\|_{\sharp',a,b;t_0,s_0(\Omega)}$.

- The weighted Hölder norms: Given $\sigma\in(0,1)$, we set

$$
\begin{aligned}
\|\mathcal{H}\|_{\sharp,a+2,b;\sigma} = \Big\| \mu_0^{-b} \big\{ &\left(1+|\bar{y}|^{a+2}\right) \left|W_{1,0}^{p-1}\mathcal{H}\right|(\bar{y},t) \\
&+ \left(1+|\bar{y}|^{a+2+\sigma}\right) \left[W_{1,0}^{p-1}\mathcal{H}\right]_{C^\sigma_{\mathbb{R}^N}}(\bar{y},t) + \left(1+|\bar{y}|^{a+2-\sigma}\right) \left[W_{1,0}^{p-1}\mathcal{H}\right]_{C_t^{\sigma/2}}(\bar{y},t) \big\} \Big\|_{L^\infty(\mathbb{R}^N\times[t_0,\infty))} \qquad (1.18)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\psi\|_{\sharp',a,b;\sigma(\Omega)} \\
&= \sum_{\ell=0}^{2} \Big\| \mu_0^{-b} \Big\{ \left(1+|\bar{y}|^{a+\ell}\right) \left|\nabla_{\bar{y}}^\ell \psi(\bar{y},t)\right| + \left(1+|\bar{y}|^{a+\ell+\sigma}\right) \left[\nabla_{\bar{y}}^\ell \psi\right]_{C^\sigma_\Omega}(\bar{y},t) \qquad (1.19)\\
&\qquad + \left(1+|\bar{y}|^{a+\ell-\sigma}\right) \left[\nabla_{\bar{y}}^\ell \psi\right]_{C_t^{\sigma/2}}(\bar{y},t) \Big\} \Big\|_{L^\infty(\Omega\times[t_0,\infty))} \\
&+ \Big\| \mu_0^{-b} \Big\{ \left(1+|\bar{y}|^{a-2}\right) |\psi_t(\bar{y},t)| + \left(1+|\bar{y}|^{a-2+\sigma}\right) [\psi_t]_{C^\sigma_\Omega}(\bar{y},t) + \left(1+|\bar{y}|^{a-2-\sigma}\right) [\psi_t]_{C_t^{\sigma/2}}(\bar{y},t) \Big\} \Big\|_{L^\infty(\Omega\times[t_0,\infty))}.
\end{aligned}
$$

2. Construction of approximate solutions

From Section 2 to Section 5, we will concern the proof of Theorem 1.1, which asserts the existence of solutions to (1.6) blowing-up at a single point $z_0\in M$ as $t\to\infty$. During the proof, we will consider the equation

$$
\begin{cases}
pu^{p-1}u_t = \dfrac{N+2}{4}(L_{g_0,h}u + \kappa_N u^p) & \text{on } M\times(t_0,\infty),\\
u>0 & \text{on } M\times(t_0,\infty),\\
u(\cdot,t_0) = u_0 > 0 & \text{on } M
\end{cases} \qquad (2.1)
$$

for $t_0>0$ large enough. Clearly, a time translation $u(\cdot-t_0)$ of a solution u to (2.1) solves (1.6).

In this section, we construct approximate solutions through two stages.

In Subsection 2.1, we define the first approximate solution which resembles to a bubble near z_0 and to the Green's function G of the perturbed conformal Laplacian $L_{g_0,h}$ away from z_0.

In Subsection 2.2, we refine the first approximate solutions in a small neighborhood of z_0, by adding solutions of certain linearized equations at the approximate solutions.

2.1. **First approximate solutions.** Throughout the paper, we always assume that $N \ge 5$.

Let G be the Green's function of $L_{g_0,h}$, i.e.,

$$-L_{g_0,h}G(z,z_0) = \delta_{z_0}(z) \quad \text{on } M \tag{2.2}$$

where δ_{z_0} is the Dirac measure supported at $z_0 \in M$.

Given a pair of parameters $(\mu,\xi) = (\mu(t),\xi(t))$, we are going to define the first approximate solution $u^{(1)}_{\mu,\xi}$ to (1.6).

Expression of $\mu(t)$: Let

$$Z_{N+1}(y) = y \cdot \nabla W_{1,0}(y) + \frac{N-2}{2}W_{1,0}(y) = \alpha_N \left(\frac{N-2}{2}\right) \frac{1-|y|^2}{(1+|y|^2)^{\frac{N}{2}}} \quad \text{for } y \in \mathbb{R}^N \tag{2.3}$$

and positive numbers

$$c_1 = p\int_{\mathbb{R}^N} W_{1,0}^{p-1}Z_{N+1}^2 \quad \text{and} \quad c_2 = -\int_{\mathbb{R}^N} W_{1,0}Z_{N+1}. \tag{2.4}$$

Reminding the hypothesis that $h(z_0) > 0$, we set

$$d_0 = \frac{1}{\sqrt{h(z_0)}} \quad \text{and} \quad \mu_0(t) = \sqrt{\frac{2c_1}{(N+2)c_2 t}} \quad \text{for } t \in [t_0,\infty) \tag{2.5}$$

where $t_0 > 0$ is a sufficiently large number. Then we assume that $\mu(t)$ has the form

$$\mu(t) = d_0\mu_0(t) + \lambda(t) =: \bar{\mu}(t) + \lambda(t) \quad \text{for } t \in [t_0,\infty) \tag{2.6}$$

where $\lambda(t)$ is a higher-order term.

Expression of $\lambda(t)$ and $\xi(t)$: The parameters $\lambda(t) \in \mathbb{R}$ as well as $\xi(t) = (\xi_1,\dots,\xi_N)(t) \in \mathbb{R}^N$ will be determined in Subsection 5.2. Until then, we assume that

$$\|\lambda\|_{\nu_1;\sigma_0} + \|\dot{\lambda}\|_{\nu_1+2;\sigma_0} + \|\xi\|_{\nu_2;\sigma_0} + \|\dot{\xi}\|_{\nu_2+2;\sigma_0} \le C \tag{2.7}$$

where $\nu_1 = \nu_2 = 2 - \varepsilon_0$ for some small numbers $\varepsilon_0 \in (0,1)$ and $\sigma_0 \in (0,1)$; refer to (1.13) for the definition of the norms.

Expression of approximate solutions: From now on, we will often identify points $z \in B_{g_0}(z_0, 2\delta_0)$ and $x = \exp_{z_0}^{-1}(z) \in B^N(0,2\delta_0)$ via g_0-normal coordinates centered at z_0. Given a pair (μ,ξ) satisfying (2.5)–(2.7), we define

$$u^{(1)}_{\mu,\xi}(z,t) = \gamma_N^{-1}\kappa_N G(z,z_0) v^{(1)}_{\mu,\xi}(z,t) \quad \text{on } M \times [t_0,\infty) \tag{2.8}$$

where

$$v^{(1)}_{\mu,\xi}(z,t) = \begin{cases} |x|^{N-2}\mu^{-\frac{N-2}{2}}W_{1,0}(\mu^{-1}(x-\xi)) & \text{if } |x| = d_{g_0}(z,z_0) \le \delta_0, \\ \alpha_N \mu^{\frac{N-2}{2}} \left[1 + \eta_{\delta_0}(x)\left\{\frac{|x|^{N-2}}{(\mu^2+|x-\xi|^2)^{(N-2)/2}} - 1\right\}\right] & \text{if } \delta_0 < |x| = d_{g_0}(z,z_0) \le 2\delta_0, \\ \alpha_N \mu^{\frac{N-2}{2}} & \text{if } d_{g_0}(z,z_0) > 2\delta_0. \end{cases} \tag{2.9}$$

Here, κ_N, $\alpha_N > 0$ are the numbers appearing in (1.2) and (1.4), and $\gamma_N := [(N-2)|\mathbb{S}^{N-1}|]^{-1}$.

Lemma 2.1. *The function $u^{(1)}_{\mu,\xi}$ is of class $C^{2+\sigma_0,1+\sigma_0/2}(M \times [t_0,\infty))$. Moreover, it is positive on $M \times [t_0,\infty)$ provided $t_0 > 0$ large enough.*

Proof. Since $Y_h(M,g_0) > 0$, the principal $L^2(M)$-eigenvalue of the operator $L_{g_0,h}$ is positive, and so $L_{g_0,h}$ satisfies the maximum principle on M. Therefore, the Green's function G of $L_{g_0,h}$ is uniquely determined and positive on M. From this fact and (2.5)–(2.7), we deduce the assertion. □

We remark that the definition of $u^{(1)}_{\mu,\xi}$ is motivated by Schoen [45].

By slightly modifying the proof of [34, Lemma 6.4] and taking δ_0 smaller if needed, we see

$$\gamma_N^{-1}\kappa_N|x|^{N-2}G(z,z_0) = 1 + \sum_{\ell=2}^{N-3} P_\ell(x) + c|x|^{N-2}\log|x| + O\left(|x|^{N-2}\right)$$

for $z = \exp_{g_0}(x) \in B_{g_0}(z_0,\delta_0)$ where P_ℓ is a homogeneous polynomial of degree ℓ, and c is a number that is zero for N odd.

Lemma 2.2. *It holds that*

$$P_2(x) = \hat{c}_1 R_{ij}(z_0)x_ix_j + \hat{c}_2 S(z_0)|x|^2 + \hat{c}_3 h(z_0)|x|^2 \tag{2.10}$$

where

$$\hat{c}_1 := \frac{1}{12}, \quad \hat{c}_2 := -\frac{1}{24(N-1)}, \quad \hat{c}_3 := -\frac{N-2}{8(N-1)(N-4)}, \tag{2.11}$$

$R_{ij}(z_0)$ is a component of the Ricci curvature tensor at z_0 on (M,g_0), and $S(z_0)$ is the scalar curvature at z_0 on (M,g_0).

Proof. We continue to identify two points $z \in B_{g_0}(z_0,2\delta_0)$ and $x = \exp_{z_0}^{-1}(z) \in B^N(0,2\delta_0)$. It is well-known that

$$\begin{cases} \sqrt{|g_0|}(x) = 1 - \dfrac{1}{6}R_{ij}(z_0)x_ix_j + \sqrt{|g_0|}_{[3]}(x), \\ g_0^{ij}(x) = \delta_{ij} + \dfrac{1}{3}R_{iqjr}(z_0)x_qx_r + (g_0^{ij})_{[3]}(x) \end{cases} \tag{2.12}$$

where

- $R_{iqjr}(z_0)$ is a component of the Riemannian curvature tensor at z_0 on (M,g_0);
- $\sqrt{|g_0|}_{[3]}$ and $(g_0^{ij})_{[3]}$ are the remainder terms in the Taylor expansions of $\sqrt{|g_0|}$ and g_0^{ij}, respectively, defined by (1.11).

We set

$$P(x) = \gamma_N^{-1}\kappa_N|x|^{N-2}G(x,z_0) - 1 \quad \text{for } x \in B^N(0,2\delta_0) \tag{2.13}$$

and write $P_2(x) = c_{ij}x_ix_j$ for some $c_{ij} \in \mathbb{R}$ so that

$$P(x) = c_{ij}x_ix_j + P_{[3]}(x) = c_{ij}x_ix_j + O(|x|^3). \tag{2.14}$$

Since $\gamma_N\Delta|x|^{2-N} = L_{g_0,h}G(z,z_0) = \delta_{z_0}(z)$, it holds that

$$|x|^2\Delta P - 2(N-2)x_i\partial_iP + |x|^N\left(\Delta_{g_0} - \Delta\right)\left\{(1+P)|x|^{2-N}\right\} - \kappa_N^{-1}|x|^2(S+h)(1+P) = 0. \tag{2.15}$$

Putting (2.12) and (2.14) into (2.15), we obtain

$$-4(N-2)c_{ij}x_ix_j + 2\sum_{i=1}^N c_{ii}|x|^2 + \frac{N-2}{3}R_{ij}(z_0)x_ix_j - \kappa_N^{-1}(S(z_0)+h(z_0))|x|^2 = O(|x|^3).$$

Solving this identity, we deduce (2.10). □

Define the error $\mathcal{S}(u)$ of a positive function u on $M \times [t_0,\infty)$ as

$$\mathcal{S}(u) = -pu^{p-1}u_t + \frac{N+2}{4}\left(L_{g_0,h}u + \kappa_N u^p\right)$$

where $p = \frac{N+2}{N-2}$. In the next lemmas, we compute the error of the first approximate solution $u^{(1)}_{\mu,\xi}$.

Lemma 2.3. *For $z = \exp_{z_0}(x) = \exp_{z_0}(\mu y + \xi) \in B_{g_0}(z_0, \delta_0)$ and $t \in [t_0, \infty)$,*

$$\mu^{\frac{N+2}{2}} \mathcal{S}\left(u^{(1)}_{\mu,\xi}\right)(y,t) = \mathcal{E}_0[\mu](y,t) + \mathcal{E}_1[\mu,\xi](y,t). \tag{2.16}$$

Here,

$$\mathcal{E}_0[\mu](y,t) := \mu^{-1}\dot{\mu}\left(pW_{1,0}^{p-1} Z_{N+1}\right)(y) + \mu^2 \mathcal{F}_0(y) \tag{2.17}$$

and

$$\begin{aligned}\mathcal{E}_1[\mu,\xi](y,t) := \mu^{-1}\dot{\xi}\cdot\left(pW_{1,0}^{p-1}\nabla W_{1,0}\right)(y) + \mathcal{F}_1[\mu,\xi](y,t)& \\ + \mu^3 a^{\{1\}} + \mu^{\nu_2+2} a^{\{2\}} + \mu\dot{\mu} a^{\{0\}} + \mu\dot{\xi}\cdot \mathbf{a}^{\{-1\}}&\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}\mathcal{F}_0(y) := \frac{(N+2)\kappa_N}{4} &\left[\left\{\left(2\hat{c}_1 + 2N\hat{c}_2 - \kappa_N^{-1}\right) S(z_0) + \left(2N\hat{c}_3 - \kappa_N^{-1}\right) h(z_0)\right\} W_{1,0}(y)\right. \\ &+ 4\left\{\hat{c}_2 S(z_0) + \hat{c}_3 h(z_0)\right\} y_j \partial_j W_{1,0}(y) \\ &\left.+ \frac{4}{N-2}\left\{\hat{c}_1 R_{ij}(z_0) y_i y_j + \hat{c}_2 S(z_0)|y|^2 + \hat{c}_3 h(z_0)|y|^2\right\} W_{1,0}^p(y)\right],\end{aligned} \tag{2.19}$$

$$\begin{aligned}\mathcal{F}_1[\mu,\xi](y,t) := (N+2)\kappa_N \mu &\left[\left\{\hat{c}_2 S(z_0) + \hat{c}_3 h(z_0)\right\} \xi_j \partial_j W_{1,0}(y)\right. \\ &\left.+ \frac{2}{N-2}\left\{\hat{c}_1 R_{ij}(z_0) y_i \xi_j + \hat{c}_2 S(z_0) y_i \xi_i + \hat{c}_3 h(z_0) y_i \xi_i\right\} W_{1,0}^p(y)\right],\end{aligned} \tag{2.20}$$

and $\hat{c}_1$, $\hat{c}_2$, and $\hat{c}_3$ are constants defined in (2.11)*. Furthermore, $a^{\{\gamma\}} \in \mathbb{R}$ and $\mathbf{a}^{\{\gamma\}} \in \mathbb{R}^N$ are C^1 functions of $(y, \mu, \mu^{-1}\xi)$ such that*

$$\sum_{\ell=0}^{1}\left(1 + |y|^{N-\gamma+\ell}\right)\left(\left|\nabla_y^\ell a^{\{\gamma\}}\right| + \left|\nabla_y^\ell \mathbf{a}^{\{\gamma\}}\right|\right) + \mu_0^{-2}\left(1 + |y|^{N-\gamma}\right)\left(\left|\partial_t a^{\{\gamma\}}\right| + \left|\partial_t \mathbf{a}^{\{\gamma\}}\right|\right) \tag{2.21}$$

is uniformly bounded.

Proof. We recall the functions P, $W_{\mu,\xi}$, and P_2 given in (2.13), (1.7), and (2.10), respectively.

Let us write $u^{(1)}_{\mu,\xi} = (1+P)W_{\mu,\xi}$ on $B_{g_0}(z_0,\delta_0) \times [t_0,\infty)$. Then

$$\begin{aligned}\mathcal{S}\left(u^{(1)}_{\mu,\xi}\right) = -p(1+P)^p \partial_t W_{\mu,\xi} + \frac{N+2}{4}&\left[\kappa_N \Delta_{g_0}\left((1+P)W_{\mu,\xi}\right)\right. \\ &\left.+\kappa_N (1+P)^p W_{\mu,\xi}^p - (S+h)(1+P)W_{\mu,\xi}\right]\end{aligned} \tag{2.22}$$

in the (x,t)-variable. Also, thanks to (2.6)–(2.7), we see

$$\begin{aligned}&-p(1+P)^p \partial_t W_{\mu,\xi} \\ &\quad= p\mu^{-\frac{N+4}{2}} W_{1,0}^{p-1}(y)\left[\dot{\mu} Z_{N+1}(y) + \dot{\xi}\cdot\nabla W_{1,0}(y)\right] + \mu^{-\frac{N}{2}}\dot{\mu} a^{\{0\}} + \mu^{-\frac{N}{2}}\dot{\xi}\cdot\mathbf{a}^{\{-1\}}.\end{aligned} \tag{2.23}$$

Setting

$$\begin{aligned}\mathcal{R} = &\left[\kappa_N \Delta_{g_0}\left((1+P)W_{\mu,\xi}\right) + \kappa_N (1+P)^p W_{\mu,\xi}^p - (S+h)(1+P)W_{\mu,\xi}\right] \\ &- \left[\kappa_N (\Delta P_2) W_{\mu,\xi} + 2\kappa_N \nabla P_2 \cdot \nabla W_{\mu,\xi} - (S+h)(z_0) W_{\mu,\xi}\right. \\ &\left.\qquad -\frac{\kappa_N}{3} R_{ij}(z_0) x_i \partial_j W_{\mu,\xi} + \kappa_N (p-1) P_2 W_{\mu,\xi}^p\right],\end{aligned}$$

we claim that

$$\mu^{\frac{N+2}{2}} \mathcal{R}(y,t) = \mu^3 a^{\{1\}} + \mu^{\nu_2+2} a^{\{2\}}. \tag{2.24}$$

By (1.8), it holds that

$$\begin{aligned}\mathcal{R} &= \kappa_N(\Delta_{g_0}P - \Delta P_2)W_{\mu,\xi} + 2\kappa_N\left(\langle\nabla_{g_0}P, \nabla_{g_0}W_{\mu,\xi}\rangle_{g_0} - \nabla P_2\cdot\nabla W_{\mu,\xi}\right)\\ &- [(S+h)(1+P)-(S+h)(z_0)]\,W_{\mu,\xi} + \kappa_N\left[(1+P)(\Delta_{g_0}-\Delta)W_{\mu,\xi} + \frac{1}{3}R_{ij}(z_0)x_i\partial_jW_{\mu,\xi}\right]\\ &+ \kappa_N\left[(1+P)^p-(1+P)-(p-1)P_2\right]W_{\mu,\xi}^p. \end{aligned} \tag{2.25}$$

From this, we readily deduce that (2.24) holds for $|y|\le 1$. Suppose that $|y|\ge 1$. Putting (1.4), (1.7), and the estimate

$$[(1+P)^p-(1+P)-(p-1)P_2]\,W_{\mu,\xi}^p = \mu^{-\frac{N-4}{2}}a^{\{1\}}$$

into (2.25) yields

$$\begin{aligned}\left(\alpha_N\mu^{\frac{N-2}{2}}\right)^{-1}\mathcal{R} &= \kappa_N(\Delta_{g_0}P-\Delta P_2)|x|^{2-N} + 2\kappa_N\left(\langle\nabla_{g_0}P,\nabla_{g_0}|x|^{2-N}\rangle_{g_0} - \nabla P_2\cdot\nabla|x|^{2-N}\right)\\ &- [(S+h)(1+P)-(S+h)(z_0)]\,|x|^{2-N}\\ &+ \kappa_N\left[(1+P)(\Delta_{g_0}-\Delta)|x|^{2-N} + \frac{1}{3}R_{ij}(z_0)x_i\partial_j|x|^{2-N}\right]\\ &+ \mu^{-(N-3)}a^{\{1\}} + \mu^{-(N-2)+\nu_2}a^{\{2\}}.\end{aligned}$$

Then (2.15) and (2.10)–(2.11) imply

$$\begin{aligned}\mu^{\frac{N+2}{2}}\mathcal{R}(y,t) &= -\mu^N\alpha_N\left[\kappa_N\Delta P_2|x|^{2-N} + 2\kappa_N\nabla P_2\cdot\nabla|x|^{2-N} - (S+h)(z_0)|x|^{2-N}\right.\\ &\qquad\left.+\frac{(N-2)\kappa_N}{3}R_{ij}(z_0)x_ix_j|x|^{2-N}\right] + \mu^3a^{\{1\}} + \mu^{\nu_2+2}a^{\{2\}}\\ &= -\mu^N\alpha_N\kappa_N\underbrace{\left[2\hat{c}_1+2N\hat{c}_2-4(N-2)\hat{c}_2-\kappa_N^{-1}\right]}_{=0}S(z_0)|x|^{2-N}\\ &\quad-\mu^N\alpha_N\kappa_N\underbrace{\left[2N\hat{c}_3-4(N-2)\hat{c}_3-\kappa_N^{-1}\right]}_{=0}h(z_0)|x|^{2-N} + \mu^3a^{\{1\}} + \mu^{\nu_2+2}a^{\{2\}}\\ &= \mu^3a^{\{1\}} + \mu^{\nu_2+2}a^{\{2\}}.\end{aligned}$$

We note that the term involving $R_{ij}(z_0)x_ix_j|x|^{-N}$ vanished here, because its coefficient is a multiple of $4\hat{c}_1 - \frac{1}{3} = 0$. Therefore, the assertion (2.24) is true.

Now, by applying (2.10) once more, we derive

$$(\Delta P_2)W_{\mu,\xi} = \mu^{-\frac{N-2}{2}}\left[(2\hat{c}_1+2N\hat{c}_2)\,S(z_0) + 2N\hat{c}_3h(z_0)\right]W_{1,0}(y)$$

and

$$\nabla P_2\cdot\nabla W_{\mu,\xi} = 2\mu^{-\frac{N}{2}}\left[\hat{c}_1R_{ij}(z_0)(\mu y_i+\xi_i) + (\hat{c}_2S(z_0)+\hat{c}_3h(z_0))(\mu y_j+\xi_j)\right]\partial_jW_{1,0}(y).$$

Accordingly,

$$\begin{aligned}&\Delta_{g_0}\left((1+P)W_{\mu,\xi}\right) + (1+P)^pW_{\mu,\xi}^p - \kappa_N^{-1}(S+h)(1+P)W_{\mu,\xi}\\ &= (\Delta P_2)W_{\mu,\xi} + 2\nabla P_2\cdot\nabla W_{\mu,\xi} - \kappa_N^{-1}(S+h)(z_0)W_{\mu,\xi} - \frac{1}{3}R_{ij}(z_0)x_i\partial_jW_{\mu,\xi} + (p-1)P_2W_{\mu,\xi}^p\\ &\quad+\mu^{-\frac{N-4}{2}}a^{\{1\}} + \mu^{\nu_2-\frac{N-2}{2}}a^{\{2\}}\\ &= \mu^{-\frac{N-2}{2}}\left[\left(2\hat{c}_1+2N\hat{c}_2-\kappa_N^{-1}\right)S(z_0) + \left(2N\hat{c}_3-\kappa_N^{-1}\right)h(z_0)\right]W_{1,0}(y)\\ &\quad+\mu^{-\frac{N}{2}}\left[4\left(\hat{c}_2S(z_0)+\hat{c}_3h(z_0)\right)(\mu y_j+\xi_j)\right]\partial_jW_{1,0}(y)\end{aligned} \tag{2.26}$$

$$+\mu^{-\frac{N+2}{2}}\left(\frac{4}{N-2}\right)\left\{\hat{c}_1R_{ij}(z_0)(\mu y_i+\xi_i)(\mu y_j+\xi_j)+(\hat{c}_2S(z_0)+\hat{c}_3h(z_0))\,|\mu y+\xi|^2\right\}W_{1,0}^p(y)$$
$$+\mu^{-\frac{N-4}{2}}a^{\{1\}}+\mu^{\nu_2-\frac{N-2}{2}}a^{\{2\}}.$$

Plugging (2.23) and (2.26) into (2.22), we establish the desired estimate (2.16). □

Lemma 2.4. *Let $\varepsilon_1\in(0,1)$ be a small number. It holds that*

$$\left[1-\eta_{\mu_0^{\varepsilon_1}}(d_{g_0}(z,z_0))\right]\mathcal{S}\left(u_{\mu,\xi}^{(1)}\right)(z,t)$$
$$=\mu^{\frac{N}{2}-\zeta_1\varepsilon_1}\dot{\mu}b_1(z,t)+\mu^{\frac{N+2}{2}-\zeta_1\varepsilon_1}b_2(z,t)+\mu^{\frac{N-2}{2}+\nu_2-\zeta_1\varepsilon_1}b_3(z,t) \quad (2.27)$$

for $(z,t)\in M\times[t_0,\infty)$. Here,
- *$\eta_{\mu_0^{\varepsilon_1}}=\eta_0(\mu_0^{-\varepsilon_1}\cdot)$ is the cut-off function introduced after (1.12);*
- *$\zeta_1>0$ is a number depending only on N, and $\nu_2=2-\varepsilon_0$;*
- *b_1, b_2, and b_3 are functions on $M\times[t_0,\infty)$ such that*

$$\sum_{\ell=0}^{1}\left\|\nabla_z^\ell b\right\|_{L^\infty(M\times[t_0,\infty))}+\sup_{(z,t)\in M\times[t_0,\infty)}\mu_0^{-2}(t)[b]_{C_t^{\sigma_0/2}}(z,t)\le C\delta_0^{-\zeta} \quad (2.28)$$

for $b=b_1$, b_2 and b_3, where $C,\zeta>0$ are constants depending only on (M,g_0), N, h, and z_0.

Proof. The proof is split into two steps.

Step 1. We assume that $d_{g_0}(z,z_0)>2\delta_0$. Because $L_{g_0,h}G(z,z_0)=0$, we simply obtain

$$\begin{aligned}\mathcal{S}\left(u_{\mu,\xi}^{(1)}\right)&=\mathcal{S}\left(\gamma_N^{-1}\kappa_N\alpha_NG(z,z_0)\mu^{\frac{N-2}{2}}\right)\\&=-\left(\frac{N+2}{2}\right)\left(\gamma_N^{-1}\kappa_N\alpha_NG(z,z_0)\right)^p\mu^{\frac{N}{2}}\dot{\mu}+\frac{(N+2)\kappa_N}{4}\left(\gamma_N^{-1}\kappa_N\alpha_NG(z,z_0)\right)^p\mu^{\frac{N+2}{2}}\\&=\mu^{\frac{N}{2}-\zeta_1\varepsilon_1}\dot{\mu}b_1+\mu^{\frac{N+2}{2}-\zeta_1\varepsilon_1}b_2.\end{aligned} \quad (2.29)$$

By taking $b_3=0$, we get (2.27) for this case.

Step 2. We assume that $\mu_0^{\varepsilon_1}\le d_{g_0}(z,z_0)\le 2\delta_0$. Let $\hat{b}_1,\hat{b}_2,\dots$ be functions satisfying (2.28) for $b=\hat{b}_1,\hat{b}_2,\dots$. Arguing as in (2.29), we find

$$\mathcal{S}\left(u_{\mu,\xi}^{(1)}\right)=\frac{(N+2)\kappa_N}{4\gamma_N}L_{g_0,h}\left(G(z,z_0)\,v_{\mu,\xi}^{(1)}\right)+\mu^{\frac{N}{2}-\zeta_1\varepsilon_1}\dot{\mu}\hat{b}_1+\mu^{\frac{N+2}{2}-\zeta_1\varepsilon_1}\hat{b}_2.$$

Also, applying $L_{g_0,h}G(z,z_0)=0$ once more, we see

$$L_{g_0,h}\left(G(z,z_0)\,v_{\mu,\xi}^{(1)}\right)=\kappa_N\left[2\left\langle\nabla_{g_0}G(z,z_0),\nabla_{g_0}v_{\mu,\xi}^{(1)}\right\rangle_{g_0}+G(z,z_0)\Delta_{g_0}v_{\mu,\xi}^{(1)}\right].$$

On the other hand, we infer from (2.9) and the mean value theorem that

$$\begin{aligned}\partial_iv_{\mu,\xi}^{(1)}(x,t)&=\alpha_N\mu^{\frac{N-2}{2}}\left[\frac{\partial_i\left(\eta_{\delta_0}(x)\right)}{(\mu^2+|x-\xi|^2)^{(N-2)/2}}\left\{|x|^{N-2}-(\mu^2+|x-\xi|^2)^{\frac{N-2}{2}}\right\}\right.\\&\qquad\left.+\frac{(N-2)\eta_{\delta_0}(x)}{(\mu^2+|x-\xi|^2)^{N/2}}\left\{x_i|x|^{N-4}\left(\mu^2-2x\cdot\xi+|\xi|^2\right)+|x|^{N-2}\xi\right\}\right]\\&=\mu^{\frac{N+2}{2}}\hat{b}_{3i}+\mu^{\frac{N-2}{2}+\nu_2}\hat{b}_{4i}\end{aligned}$$

and

$$\partial_{ij}v_{\mu,\xi}^{(1)}(x,t)=\mu^{\frac{N+2}{2}}\hat{b}_{3ij}+\mu^{\frac{N-2}{2}+\nu_2}\hat{b}_{4ij}.$$

Consequently, by setting

$$\begin{aligned} b_1 &= \left[1-\eta_{\mu_0^{\varepsilon_1}}(d_{g_0}(\cdot,z_0))\right]\hat{b}_1,\\ b_2 &= \left[1-\eta_{\mu_0^{\varepsilon_1}}(d_{g_0}(\cdot,z_0))\right]\left[\hat{b}_2+\kappa_N\left\{2g_0^{ij}\partial_i G(\cdot,z_0)+\left(\frac{\partial_i\sqrt{|g_0|}}{\sqrt{|g_0|}}g_0^{ij}+\partial_i g_0^{ij}\right)G(\cdot,z_0)\right\}\hat{b}_{3j}+\kappa_N g_0^{ij}G(\cdot,z_0)\hat{b}_{3ij}\right], \end{aligned}$$

and so on, we deduce (2.27). Note that $\text{supp}(b_3)\subset B_{g_0}(z_0,2\delta_0)\times[t_0,\infty)$. □

2.2. **Second approximation solutions.** In this subsection, we refine the approximate solutions to reduce their errors in the ball $B_{g_0}(z,z_0)$ significantly. To this end, we introduce a linear equation

$$\Delta\Psi(\cdot,t)+pW_{1,0}^{p-1}\Psi(\cdot,t)=-\frac{4}{(N+2)\kappa_N}\mathcal{E}_0[\bar{\mu}](\cdot,t)\quad\text{in } \mathbb{R}^N,\quad \Psi(\cdot,t)\in\dot{W}^{1,2}(\mathbb{R}^N) \tag{2.30}$$

for each $t\in[t_0,\infty)$, where $\bar{\mu}$ and $\mathcal{E}_0[\bar{\mu}]$ are given in (2.6) and (2.17), respectively.

By virtue of the definition of $\bar{\mu}$, we have

$$\bar{\mu}^{-1}\dot{\bar{\mu}}=-\frac{(N+2)c_2}{4c_1}h(z_0)\bar{\mu}^2\quad\text{on } [t_0,\infty). \tag{2.31}$$

Hence the function $\bar{\mu}^{-2}\mathcal{E}_0[\bar{\mu}]$ is independent of t, and a solution Ψ to (2.30) is decomposed into $\Psi(y,t)=\bar{\mu}^2(t)Q(y)$ on $\mathbb{R}^N\times[t_0,\infty)$ where Q satisfies

$$\Delta Q+pW_{1,0}^{p-1}Q=-\frac{4}{(N+2)\kappa_N}\left(\bar{\mu}^{-2}\mathcal{E}_0[\bar{\mu}]\right)(y)\quad\text{in } \mathbb{R}^N,\quad Q\in\dot{W}^{1,2}(\mathbb{R}^N). \tag{2.32}$$

From (2.17) and (2.19), we immediately see that $(\bar{\mu}^{-2}\mathcal{E}_0[\bar{\mu}])(y)=O(|y|^{2-N})$ for $|y|$ large. In fact, as we shall see in the next lemma, a notable cancellation among the terms of $\mathcal{F}_0$ occurs, so we actually have a better decay estimate for $\bar{\mu}^{-2}\mathcal{E}_0[\bar{\mu}]$.

Lemma 2.5. *There exists a constant $C>0$ depending only on (M,g_0), N, h, and z_0 such that*

$$\left|\left(\bar{\mu}^{-2}\mathcal{E}_0[\bar{\mu}]\right)(y)\right|\le\frac{C}{1+|y|^N}\quad\textit{for } (y,t)\in\mathbb{R}^N\times[t_0,\infty). \tag{2.33}$$

Proof. A direct computation shows that

$$\begin{aligned} &\frac{1}{\alpha_N\hat{c}_2}\left[\left(2\hat{c}_1+2N\hat{c}_2-\kappa_N^{-1}\right)W_{1,0}(y)+4\hat{c}_2y_j\partial_jW_{1,0}(y)\right]\\ &\qquad=\frac{1}{\alpha_N\hat{c}_3}\left[\left(2N\hat{c}_3-\kappa_N^{-1}\right)W_{1,0}(y)+4\hat{c}_3y_j\partial_jW_{1,0}(y)\right]=\frac{4(N-2)}{(1+|y|^2)^{\frac{N}{2}}}. \end{aligned}$$

In light of (2.17), (2.19), (2.31), and the above identity, we obtain (2.33). □

Lemma 2.6. *Equation* (2.32) *has a solution.*

Proof. It is well-known that the space of all bounded solutions to

$$L_0[\Psi]:=W_{1,0}^{1-p}\left(\Delta\Psi+pW_{1,0}^{p-1}\Psi\right)=0\quad\text{in } \mathbb{R}^N$$

is spanned by the function Z_{N+1} in (2.3) and

$$Z_i(y):=\partial_iW_{1,0}(y)=-(N-2)\alpha_N\frac{y_i}{(1+|y|^2)^{\frac{N}{2}}}\quad\text{for } y\in\mathbb{R}^N\text{ and } i=1,\dots,N. \tag{2.34}$$

This fact, (2.33), the dimensional assumption $N\ge5$, and the Fredholm alternative imply that (2.32) is solvable if and only if

$$\int_{\mathbb{R}^N}\left(\bar{\mu}^{-2}\mathcal{E}_0[\bar{\mu}]\right)(y)Z_n(y)dy=0 \tag{2.35}$$

for $n=1,\dots,N+1$.

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By parity, (2.35) holds automatically for $n = 1, \dots, N$. Besides, setting

$$c_3 = \int_{\mathbb{R}^N} (y \cdot \nabla W_{1,0}(y)) Z_{N+1}(y) dy \quad \text{and} \quad c_4 = \int_{\mathbb{R}^N} |y|^2 \left(W_{1,0}^p Z_{N+1} \right)(y) dy,$$

we deduce from (2.17), (2.19), (2.4), (2.11), and (2.31) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \mathcal{E}_0[\bar{\mu}](y,t) Z_{N+1}(y) dy \\
&= c_1 \bar{\mu}^{-1} \dot{\bar{\mu}} + \frac{(N+2)\kappa_N}{4} \underbrace{\left[-(2N\hat{c}_3 - \kappa_N^{-1}) c_2 + 4\hat{c}_3 c_3 + \frac{4}{N-2} \hat{c}_3 c_4 \right]}_{=\kappa_N^{-1} c_2} h(z_0) \bar{\mu}^2 \\
&\quad + \frac{(N+2)\kappa_N}{4} \underbrace{\left[-\left(2\hat{c}_1 + 2N\hat{c}_2 - \kappa_N^{-1} \right) c_2 + \left(\frac{4\hat{c}_1}{N} + 4\hat{c}_2 - \frac{1}{3N} \right) c_3 + \frac{4}{N-2} \left(\frac{\hat{c}_1}{N} + \hat{c}_2 \right) c_4 \right]}_{=0} S(z_0) \bar{\mu}^2 \\
&= c_1 \bar{\mu}^{-1} \dot{\bar{\mu}} + \frac{(N+2)c_2}{4} h(z_0) \bar{\mu}^2 = 0.
\end{aligned} \tag{2.36}$$

Thus (2.32) is solvable. □

Lemma 2.7. *There is a constant $C > 0$ depending on Q such that*

$$\left| \nabla_y^\ell Q(y) \right| \le \frac{C \log(2 + |y|)}{|y|^{N-2+\ell}} \quad \text{for } y \in \mathbb{R}^N \text{ and } \ell = 0, 1, 2, 3. \tag{2.37}$$

Proof. By employing the rescaling argument with the condition $Q \in \dot{W}^{1,2}(\mathbb{R}^N)$, we obtain

$$|Q(y)| \le \frac{C}{1 + |y|^{\frac{N-2}{2}}} \quad \text{for } y \in \mathbb{R}^N.$$

Then, having (2.33) in hand, we apply the comparison principle to (2.32) repeatedly, which produces

$$|Q(y)| \le \frac{C}{1 + |y|^{N-2-\varepsilon}} \quad \text{for } y \in \mathbb{R}^N \tag{2.38}$$

where $\varepsilon > 0$ is any small number. By (2.32), (2.33), and (2.38),

$$\Delta Q(y) = -\frac{4}{(N+2)\kappa_N} \left(\bar{\mu}^{-2} \mathcal{E}_0[\bar{\mu}] \right)(y) - p W_{1,0}^{p-1} Q = O\left(\frac{1}{|y|^N} \right) \quad \text{for } |y| \ge 1.$$

Consequently, from the Green's representation formula for Q, we get

$$|Q(y)| \le C \int_{\mathbb{R}^N} \frac{|\Delta Q(\tilde{y})|}{|y - \tilde{y}|^{N-2}} d\tilde{y} \le C \int_{\mathbb{R}^N} \frac{1}{|y - \tilde{y}|^{N-2}} \frac{d\tilde{y}}{1 + |\tilde{y}|^N} \le \frac{C \log(2 + |y|)}{|y|^{N-2}} \quad \text{for } |y| \ge 1$$

where we estimate the second integral by decomposing the domain $\mathbb{R}^N$ into

$$\overline{B^N(0, R_0)} \cup \overline{B^N(y, R_0)} \cup \left(\mathbb{R}^N \setminus \left(B^N(0, R_0) \cup B^N(y, R_0) \right) \right) \quad \text{for } R_0 := \frac{|y|}{2}.$$

Therefore, (2.37) for $\ell = 0$ holds. The gradient estimate for Q, that is, (2.37) for $\ell = 1, 2, 3$ follows from elliptic regularity. □

Let $\Psi_0 = \bar{\mu}^2 Q_0$ be the unique solution to (2.30) such that

$$\int_{\mathbb{R}^N} (Q_0 Z_n)(y) dy = 0 \quad \text{for } n = 1, \dots, N+1.$$

We now define the second (or, refined) approximate solution

$$u_{\mu,\xi}^{(2)}(z,t) = \gamma_N^{-1} \kappa_N G(z, z_0) v_{\mu,\xi}^{(2)}(z,t) \quad \text{on } M \times [t_0, \infty) \tag{2.39}$$

where

$$v_{\mu,\xi}^{(2)}(z,t)$$
$$= \begin{cases} |x|^{N-2}\mu^{-\frac{N-2}{2}}\left[W_{1,0}(y)+\Psi_0(y,t)\right] & \text{if } d_{g_0}(z,z_0)\le\delta_0,\\ \mu^{\frac{N-2}{2}}\Big[\alpha_N+\eta_{\delta_0}(x)\Big\{\alpha_N\left(\frac{|x|^{N-2}}{(\mu^2+|x-\xi|^2)^{(N-2)/2}}-1\right) \\ \qquad +\mu^{-(N-2)}|x|^{N-2}\Psi_0(y,t)\Big)\Big] & \text{if } \delta_0<d_{g_0}(z,z_0)\le 2\delta_0,\\ \mu^{\frac{N-2}{2}}\alpha_N & \text{if } d_{g_0}(z,z_0)>2\delta_0. \end{cases} \tag{2.40}$$

Here, $z=\exp_{z_0}(x)=\exp_{z_0}(\mu y+\xi)$ for $z\in B_{g_0}(z_0,2\delta_0)$. By taking t_0 large, we see from (2.37) that $u_{\mu,\xi}^{(2)}$ is of class $C^{2+\sigma_0,1+\sigma_0/2}(M\times[t_0,\infty))$ and positive on $M\times[t_0,\infty)$. The next lemmas measure its error.

Lemma 2.8. *It holds that*

$$\mu^{\frac{N+2}{2}}\mathcal{S}\left(u_{\mu,\xi}^{(2)}\right)(y,t)=\mathcal{E}_2[\mu,\xi](y,t) \tag{2.41}$$

for $y\in B^N(-\mu^{-1}\xi,\mu^{-1}\delta_0)$ *and* $t\in[t_0,\infty)$. *Here,*

$$\begin{aligned}\mathcal{E}_2[\mu,\xi](y,t):=\bar\mu^{-1}\left(\dot\lambda-\bar\mu^{-1}\dot{\bar\mu}\lambda\right)\left(pW_{1,0}^{p-1}Z_{N+1}\right)(y)+\mu^{-1}\dot\xi\cdot\left(pW_{1,0}^{p-1}\nabla W_{1,0}\right)(y)+2\bar\mu\lambda\mathcal{F}_0(y)\\ +\mathcal{F}_1[\mu,\xi](y,t)+\mu^3a^{\{1\}}+\mu^{2\nu_1}a^{\{2\}}+\mu\dot\mu a^{\{0\}}+\mu^{\nu_1-2}\dot\lambda\,a^{\{-2\}}+\mu\dot\xi\cdot\mathbf{a}^{\{-1\}}\end{aligned} \tag{2.42}$$

where

- $\mathcal{F}_0$, $\mathcal{F}_1$, *and* P *are the functions in* (2.19), (2.20), *and* (2.13), *respectively, and* ν_1 *and* ν_2 *are numbers in* (2.7);
- $a^{\{\gamma\}}\in\mathbb{R}$ *and* $\mathbf{a}^{\{\gamma\}}\in\mathbb{R}^N$ *are* C^1 *functions of* $(y,\mu,\mu^{-1}\xi,\mu^{-1}\lambda)$ *satisfying* (2.21).

Proof. We see from (2.8)–(2.9), (2.39)–(2.40), (2.16), and (2.30) that

$$\begin{aligned}\mu^{\frac{N+2}{2}}\mathcal{S}\left(u_{\mu,\xi}^{(2)}\right)&=\mu^{\frac{N+2}{2}}\mathcal{S}\left(u_{\mu,\xi}^{(1)}+(1+P)\mu^{-\frac{N-2}{2}}\Psi_0\right)\\ &=\frac{(N+2)\kappa_N}{4}\left(\Delta_{g_0(x)}\Psi_0+pW_{1,0}^{p-1}\Psi_0\right)+\mu^{\frac{N+2}{2}}\mathcal{S}\left(u_{\mu,\xi}^{(1)}\right)+\mathcal{A}(\Psi_0)\\ &=\frac{(N+2)\kappa_N}{4}\left(\Delta_{g_0(x)}-\Delta\right)\Psi_0+\left(\mathcal{E}_0[\mu]-\mathcal{E}_0[\bar\mu]\right)+\mathcal{E}_1[\mu,\xi]+\mathcal{A}(\Psi_0)\end{aligned} \tag{2.43}$$

in the (y,t)-variable. Here,

$$\begin{aligned}\mathcal{A}(\Psi_0):=&\frac{(N+2)\kappa_N}{4}\left[(W_{1,0}+\Psi_0)^p-W_{1,0}^p-pW_{1,0}^{p-1}\Psi_0\right]\\ &+p\mu^{-1}\dot\mu\left[(W_{1,0}+\Psi_0)^{p-1}\left\{y\cdot\nabla(W_{1,0}+\Psi_0)+\frac{N-2}{2}(W_{1,0}+\Psi_0)\right\}-W_{1,0}^{p-1}Z_{N+1}\right]\\ &+p\mu^{-1}\dot\xi\cdot\left[(W_{1,0}+\Psi_0)^{p-1}\nabla(W_{1,0}+\Psi_0)-W_{1,0}^{p-1}\nabla W_{1,0}\right]+\mu^4\log(2+|y|)a^{\{2\}}.\end{aligned}$$

By Taylor's theorem and (2.37),

$$\mathcal{A}(\Psi_0)=\log(2+|y|)\left[\mu^4a^{\{2\}}+\mu\dot\mu a^{\{-2\}}+\mu\dot\xi\cdot\mathbf{a}^{\{-3\}}\right]. \tag{2.44}$$

Also, in view of (2.7), (2.12), (2.37), it holds that

$$\left(\Delta_{g_0(x)}-\Delta\right)\Psi_0=\mu^4\log(2+|y|)a^{\{2\}} \tag{2.45}$$

and

$$\begin{aligned}\mathcal{E}_0[\mu]-\mathcal{E}_0[\bar\mu]=\bar\mu^{-1}\left(\dot\lambda-\bar\mu^{-1}\dot{\bar\mu}\lambda\right)\left(pW_{1,0}^{p-1}Z_{N+1}\right)(y)+2\bar\mu\lambda\mathcal{F}_0(y)\\ +\mu^{2\nu_1}a^{\{2\}}+\mu^{\nu_1-2}\dot\lambda\,a^{\{-2\}}.\end{aligned} \tag{2.46}$$

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Inserting (2.44)–(2.46) and (2.18) into (2.43), and then arranging the resulting terms, we deduce the desired equality (2.41). □

Lemma 2.9. *It holds that*

$$\left[1-\eta_{\mu_0^{\varepsilon_1}}(d_{g_0}(z,z_0))\right]\mathcal{S}\left(u^{(2)}_{\mu,\xi}\right)(z,t) = \mu^{\frac{N}{2}-\zeta_2\varepsilon_1}\dot{\mu}b_4(z,t)+\mu^{\frac{N+2}{2}-\zeta_2\varepsilon_1}b_5(z,t)+\mu^{\frac{N-2}{2}+\nu_2-\zeta_2\varepsilon_1}b_6(z,t) \quad (2.47)$$

for $(z,t)\in M\times[t_0,\infty)$. *Here,*

- $\varepsilon_1\in(0,1)$ *is the small number in Lemma 2.4, and* $\eta_{\mu_0^{\varepsilon_1}}$ *is the cut-off function introduced after* (1.12);
- $\zeta_2>0$ *is a number depending only on* N, *and* $\nu_2=2-\varepsilon_0$;
- b_4, b_5, *and* b_6 *are functions on* $M\times[t_0,\infty)$ *such that* (2.28) *holds for* $b=b_4$, b_5, *and* b_6.

Proof. If $d_{g_0}(z,z_0)>2\delta_0$, then by (2.8), (2.39), and (2.29), we have

$$\mathcal{S}\left(u^{(2)}_{\mu,\xi}\right)=\mathcal{S}\left(u^{(1)}_{\mu,\xi}\right)=\mu^{\frac{N}{2}-\zeta_1\varepsilon_1}\dot{\mu}b_1+\mu^{\frac{N+2}{2}-\zeta_1\varepsilon_1}b_2.$$

Suppose that $\mu_0^{\varepsilon_1}\le d_{g_0}(z,z_0)\le 2\delta_0$. Setting

$$\tilde{u}^{(2)}_{\mu,\xi}=\gamma_N^{-1}\kappa_N G(z,z_0)\cdot\mu^{-\frac{N-2}{2}}\bar{\mu}^2\eta_{\delta_0}(x)|x|^{N-2}Q_0(y),$$

we write

$$\mathcal{S}\left(u^{(2)}_{\mu,\xi}\right)=\mathcal{S}\left(u^{(1)}_{\mu,\xi}\right)+\frac{N+2}{4}L_{g_0,h}\tilde{u}^{(2)}_{\mu,\xi} - \left\{\partial_t\left(u^{(1)}_{\mu,\xi}+\tilde{u}^{(2)}_{\mu,\xi}\right)^p-\partial_t\left(u^{(1)}_{\mu,\xi}\right)^p\right\}+\frac{(N+2)\kappa_N}{4}\left\{\left(u^{(1)}_{\mu,\xi}+\tilde{u}^{(2)}_{\mu,\xi}\right)^p-\left(u^{(1)}_{\mu,\xi}\right)^p\right\}. \quad (2.48)$$

Let $\hat{b}_5$, $\hat{b}_6$, and $\hat{b}_7$ be functions satisfying (2.28). Applying (2.37), we argue as in Step 2 of the proof of Lemma 2.4. Then we obtain

$$L_{g_0,h}\tilde{u}^{(2)}_{\mu,\xi}=\mu^{\frac{N+2}{2}-\varepsilon_1\zeta_2}\hat{b}_5. \quad (2.49)$$

Also, we compute

$$\partial_t\left(u^{(1)}_{\mu,\xi}+\tilde{u}^{(2)}_{\mu,\xi}\right)^p-\partial_t\left(u^{(1)}_{\mu,\xi}\right)^p = p\left[\left\{\left(u^{(1)}_{\mu,\xi}+\tilde{u}^{(2)}_{\mu,\xi}\right)^{p-1}-\left(u^{(1)}_{\mu,\xi}\right)^{p-1}\right\}\partial_t u^{(1)}_{\mu,\xi}+\left(u^{(1)}_{\mu,\xi}+\tilde{u}^{(2)}_{\mu,\xi}\right)^{p-1}\partial_t\tilde{u}^{(2)}_{\mu,\xi}\right]=\mu^{\frac{N+4}{2}-\varepsilon_1\zeta_2}\dot{\mu}\hat{b}_6 \quad (2.50)$$

and

$$\left(u^{(1)}_{\mu,\xi}+\tilde{u}^{(2)}_{\mu,\xi}\right)^p-\left(u^{(1)}_{\mu,\xi}\right)^p=\mu^{\frac{N+6}{2}-\varepsilon_1\zeta_2}\hat{b}_7. \quad (2.51)$$

By substituting (2.49)–(2.51), and (2.27) into (2.48), and then defining b_4, b_5, and b_6 suitably, we obtain (2.47). □

3. Inner-outer gluing procedure

For the sake of brevity, we write $u_{\mu,\xi}=u^{(2)}_{\mu,\xi}$ in the sequel.

In the rest of the paper, we construct a remainder term $\psi_{\mu,\xi}$ such that $u=u_{\mu,\xi}+\psi_{\mu,\xi}$ is a solution to (2.1). To this end, we apply the inner-outer gluing procedure as in [14]. It amounts to decomposing $\psi_{\mu,\xi}$ into two parts

$$\psi_{\mu,\xi}=\psi^{\mathrm{out}}_{\mu,\xi}+\eta_{\mu_0^{\varepsilon_1}}(d_{g_0}(z,z_0))\,\hat{\psi}^{\mathrm{in}}_{\mu,\xi}\left(\exp_{z_0}^{-1}(z),t\right)\quad\text{on } M\times[t_0,\infty) \quad (3.1)$$

and determining $\psi^{\mathrm{out}}_{\mu,\xi}$ and $\hat{\psi}^{\mathrm{in}}_{\mu,\xi}$ by solving so-called outer and inner problems.

Outer problem: Let $\psi^{\mathrm{out}}_{\mu,\xi}$ be a function on $M\times[t_0,\infty)$ solving

$$pu_{\mu,\xi}^{p-1}\left(\psi_{\mu,\xi}^{\rm out}\right)_t = \frac{N+2}{4}L_{g_0,h}\psi_{\mu,\xi}^{\rm out} + \mathcal{W}_{\mu,\xi}\psi_{\mu,\xi}^{\rm out} + \left(1-\eta_{\mu_0^{\varepsilon_1}}\right)\mathcal{S}(u_{\mu,\xi}) + \mathcal{J}_1\left[\hat\psi_{\mu,\xi}^{\rm in}\right] + \mathcal{J}_2\left[\psi_{\mu,\xi}^{\rm out},\hat\psi_{\mu,\xi}^{\rm in}\right]. \quad (3.2)$$

Here, $L_{g_0,h}$ is the differential operator defined in (1.5),

$$\mathcal{W}_{\mu,\xi} := \frac{(N+2)\kappa_N p}{4}\left[u_{\mu,\xi}^{p-1} - \eta_{\mu_0^{\varepsilon_1}}\left(u_{\mu,\xi}^{(1)}\right)^{p-1}\right],$$

$$\begin{aligned}\mathcal{J}_1\left[\hat\psi_{\mu,\xi}^{\rm in}\right] :=&\ \frac{(N+2)\kappa_N p}{4}\left[u_{\mu,\xi}^{p-1} - \left(u_{\mu,\xi}^{(1)}\right)^{p-1}\right]\eta_{\mu_0^{\varepsilon_1}}\hat\psi_{\mu,\xi}^{\rm in} - p\left[u_{\mu,\xi}^{p-1} - \left(u_{\mu,\xi}^{(1)}\right)^{p-1}\right]\eta_{\mu_0^{\varepsilon_1}}\left(\hat\psi_{\mu,\xi}^{\rm in}\right)_t \\ &+ \frac{(N+2)\kappa_N}{4}\left[\left(\Delta_{g_0}\eta_{\mu_0^{\varepsilon_1}}\right)\hat\psi_{\mu,\xi}^{\rm in} + 2\left\langle\nabla_{g_0}\eta_{\mu_0^{\varepsilon_1}},\nabla_{g_0}\hat\psi_{\mu,\xi}^{\rm in}\right\rangle_{g_0}\right]\end{aligned} \quad (3.3)$$

and

$$\begin{aligned}\mathcal{J}_2\left[\psi_{\mu,\xi}^{\rm out},\hat\psi_{\mu,\xi}^{\rm in}\right] :=&\ \frac{(N+2)\kappa_N}{4}\left[(u_{\mu,\xi}+\psi_{\mu,\xi})^p - u_{\mu,\xi}^p - pu_{\mu,\xi}^{p-1}\psi_{\mu,\xi}\right] \\ &- p\left[(u_{\mu,\xi}+\psi_{\mu,\xi})^{p-1} - u_{\mu,\xi}^{p-1}\right]\left[\left(1-\eta_{\mu_0^{\varepsilon_1}/2}\right)(u_{\mu,\xi})_t + (\psi_{\mu,\xi})_t\right].\end{aligned} \quad (3.4)$$

Inner problem: As before, we identify points $z \in B_{g_0}(z_0,\delta_0)$ and $x \in B^N(0,\delta_0)$ via g_0-normal coordinates at $z_0 \in M$. Let $\hat\psi_{\mu,\xi}^{\rm in}$ be a function on $B^N(0,2\mu_0^{\varepsilon_1}) \times [t_0,\infty)$ satisfying

$$\begin{aligned}p\left(u_{\mu,\xi}^{(1)}\right)^{p-1}\left(\hat\psi_{\mu,\xi}^{\rm in}\right)_t =&\ \frac{N+2}{4}\left[L_{g_0,h}\hat\psi_{\mu,\xi}^{\rm in} + \kappa_N p\left(u_{\mu,\xi}^{(1)}\right)^{p-1}\left(\hat\psi_{\mu,\xi}^{\rm in} + \psi_{\mu,\xi}^{\rm out}\right)\right] \\ &+ \mu^{-\frac{N+2}{2}}\mathcal{E}_2[\mu,\xi](\mu^{-1}(x-\xi),t) \\ &- p\left[\left(u_{\mu,\xi} + \hat\psi_{\mu,\xi}^{\rm in} + \psi_{\mu,\xi}^{\rm out}\right)^{p-1} - u_{\mu,\xi}^{p-1}\right]\eta_{\mu_0^{\varepsilon_1}/2}\left(u_{\mu,\xi}\right)_t.\end{aligned}$$

We write $x = \bar\mu\bar y + \xi$ and define $\psi_{\mu,\xi}^{\rm in}$ by the relation

$$\begin{aligned}\hat\psi_{\mu,\xi}^{\rm in}(x,t) &= \gamma_N^{-1}\kappa_N|x|^{N-2}G(x,z_0)\cdot\bar\mu^{-\frac{N-2}{2}}\psi_{\mu,\xi}^{\rm in}\left(\bar\mu^{-1}(x-\xi),t\right) \\ &= (1+P(x))\bar\mu^{-\frac{N-2}{2}}\psi_{\mu,\xi}^{\rm in}(\bar y,t)\end{aligned} \quad (3.5)$$

for $(\bar y,t) \in B^N(-\bar\mu^{-1}\xi, 2\bar\mu^{-1}\mu_0^{\varepsilon_1})$ where P is the function in (2.13). Then $\psi_{\mu,\xi}^{\rm in}$ solves

$$\begin{aligned}pW_{1,0}^{p-1}\left(\psi_{\mu,\xi}^{\rm in}\right)_t =&\ \frac{(N+2)\kappa_N}{4}\left(\Delta\psi_{\mu,\xi}^{\rm in} + pW_{1,0}^{p-1}\psi_{\mu,\xi}^{\rm in}\right) \\ &+ \left(\frac{\bar\mu}{\mu}\right)^{\frac{N-2}{2}}(1+P)^{-p}(x)\mathcal{E}_2[\mu,\xi](y,t) + \mathcal{K}_1\left[\psi_{\mu,\xi}^{\rm in}\right] + \mathcal{K}_2\left[\psi_{\mu,\xi}^{\rm in},\psi_{\mu,\xi}^{\rm out}\right]\end{aligned} \quad (3.6)$$

in the $(\bar y,t)$-variable, where $y = \mu^{-1}\bar\mu\bar y$,

$$\begin{aligned}\mathcal{K}_1\left[\psi_{\mu,\xi}^{\rm in}\right] :=&\ \frac{(N+2)\kappa_N p}{4}\left[W_{1,0}^{p-1}(y) - W_{1,0}^{p-1}(\bar y)\right]\psi_{\mu,\xi}^{\rm in} - p\left[W_{1,0}^{p-1}(y) - W_{1,0}^{p-1}(\bar y)\right]\left(\psi_{\mu,\xi}^{\rm in}\right)_t \\ &+ pW_{1,0}^{p-1}(y)\left[\bar\mu^{-1}\dot{\bar\mu}\left(\bar y\cdot\nabla\psi_{\mu,\xi}^{\rm in} + \frac{N-2}{2}\psi_{\mu,\xi}^{\rm in}\right) + \bar\mu^{-1}\dot\xi\cdot\nabla\psi_{\mu,\xi}^{\rm in}\right] \\ &+ \frac{(N+2)\kappa_N}{4}\left[\left(\frac{\mu}{\bar\mu}\right)^2(1+P)^{1-p}(x)\Delta_{g_0(x)}\psi_{\mu,\xi}^{\rm in} - \Delta\psi_{\mu,\xi}^{\rm in}\right] - \frac{N+2}{4}\mu^2(1+P(x))(S+h)(x)\psi_{\mu,\xi}^{\rm in} \\ &+ \frac{(N+2)\kappa_N}{4}\left(\frac{\mu}{\bar\mu}\right)^2(1+P)^{-p}(x)\left[2\bar\mu g_0^{ij}(x)(\partial_{x_i}P)(x)\partial_j\psi_{\mu,\xi}^{\rm in} + \bar\mu^2(\Delta_{g_0}P)(x)\psi_{\mu,\xi}^{\rm in}\right]\end{aligned} \quad (3.7)$$

and

$$\begin{aligned}\mathcal{K}_2\left[\psi^{\rm in}_{\mu,\xi},\psi^{\rm out}_{\mu,\xi}\right]:=&\ \frac{(N+2)\kappa_N p}{4}\left(\frac{\bar\mu}{\mu}\right)^{\frac{N-2}{2}}(1+P)^{-1}(x)W_{1,0}^{p-1}(y)\mu^{\frac{N-2}{2}}\psi^{\rm out}_{\mu,\xi}(x,t)\\ &-p\left(\frac{\bar\mu}{\mu}\right)^{\frac{N-2}{2}}\left[\left\{\left(W_{1,0}+\bar\mu^2Q_0\right)(y)+\left(\frac{\mu}{\bar\mu}\right)^{\frac{N-2}{2}}\psi^{\rm in}_{\mu,\xi}+(1+P)^{-1}(x)\mu^{\frac{N-2}{2}}\psi^{\rm out}_{\mu,\xi}(x,t)\right\}^{p-1}\right.\\ &\left.-\left(W_{1,0}+\bar\mu^2Q_0\right)^{p-1}(y)\right]\eta_{\mu_0^{\varepsilon_1}/2}(x)\mu^{\frac{N-2}{2}}\left\{\mu^{-\frac{N-2}{2}}\left(W_{1,0}+\bar\mu^2Q_0\right)(y)\right\}_t.\end{aligned} \tag{3.8}$$

If $\psi^{\rm out}_{\mu,\xi}$ and $\psi^{\rm in}_{\mu,\xi}$ solve (3.2) and (3.6), respectively, then $u=u_{\mu,\xi}+\psi_{\mu,\xi}$ will satisfy (2.1) provided it is positive on $M\times[t_0,\infty)$. In Sections 4 and 5, we will look for solutions to (3.2) and (3.6) with appropriate choices of parameters (μ,ξ) and initial conditions u_0, and verify that u is indeed positive.

4. Outer problem

This section is devoted to the analysis of the outer problem (3.2).

In Subsections 4.1 and 4.2, we develop existence theory and a priori estimates on a solution to an associated inhomogeneous problem (4.1). The main technical point is to control the degenerate factor $u^{p-1}_{\mu,\xi}$.

In Subsections 4.3 and 4.4, we apply the results for (4.1) and the contraction mapping theorem to derive the unique solvability of (3.2) as well as several a priori estimates on the solution.

4.1. **Inhomogeneous problem associated to (3.2): Weighted H^2 estimate.** In this subsection, we will prove the unique existence of a solution to the inhomogeneous problem

$$\begin{cases} pu^{p-1}_{\mu,\xi}\psi_t=\dfrac{N+2}{4}L_{g_0,h}\psi+\mathcal{W}_{\mu,\xi}\psi+u^{p-1}_{\mu,\xi}f & \text{on } M\times(t_0,\infty),\\ \psi(\cdot,t_0)=\psi_0 & \text{on } M\end{cases} \tag{4.1}$$

in a weighted H^2 space by deriving a priori estimate for solutions. Here and after, we mean by a solution to (4.1) and other related equations a function that satisfies them in a weak sense.

We first establish the following version of a priori weighted H^2 estimate. Refer to Definition 1.7 for the definition of the norms.

Lemma 4.1. *Suppose that $Y_h(M,g_0)>0$, $t_0<s_0-1$, $\|f\|_{L^2_{t_0,s_0}}+\|\psi_0\|_{\mathcal{H}^1}<\infty$ and (2.5)–(2.7) hold. Then there is a constant $C>0$ depending only on (M,g_0), N, h, and z_0 such that every solution ψ to*

$$\begin{cases} pu^{p-1}_{\mu,\xi}\psi_t=\dfrac{N+2}{4}L_{g_0,h}\psi+\mathcal{W}_{\mu,\xi}\psi+u^{p-1}_{\mu,\xi}f & \text{on } M\times(t_0,s_0),\\ \psi(\cdot,t_0)=\psi_0 & \text{on } M\end{cases} \tag{4.2}$$

satisfies

$$\|\psi\|_{H^2_{t_0,s_0}}\le C\left(\|\psi\|_{L^2_{t_0,s_0}}+\|f\|_{L^2_{t_0,s_0}}+\|\psi_0\|_{\mathcal{H}^1}\right) \tag{4.3}$$

provided $t_0>0$ is large enough.

Proof. The proof is inspired by that of [15, Lemma 3.2]. We will divide it into four steps.

Throughout the proof, we assume that C depends only on (M,g_0), N, h, and z_0, and in particular, is independent of s_0.

Step 1. We claim that

$$u^{-1}_{\mu,\xi}|(u_{\mu,\xi})_t|\le C\mu_0^2. \tag{4.4}$$

If $z \in B_{g_0}(z_0, \delta_0)$, we infer from (2.39)–(2.40), (2.37), and (2.5)–(2.7) that

$$
\begin{aligned}
u_{\mu,\xi}^{-1}(u_{\mu,\xi})_t = \frac{N-2}{2}\mu^{-1}\dot{\mu} + \left[\frac{1}{(1+|y|^2)^{\frac{N-2}{2}}} + \bar{\mu}^2 Q_0(y)\right]^{-1} \\
\times \left[\frac{(N-2)\mu^{-1}(\dot{\xi}\cdot y - \dot{\mu})}{(1+|y|^2)^{\frac{N}{2}}} + O(\mu_0^4)Q_0(y) + O(\mu_0)\left(\dot{\mu}y - \dot{\xi}\right)\cdot \nabla Q_0(y)\right] \\
= O\left(\mu^{-1}\dot{\mu} + \frac{\mu^{-1}\dot{\xi}}{1+|y|}\right) = O(\mu_0^2)
\end{aligned}
$$

in the (y,t)-variable.

If $z \in B_{g_0}(z_0, 2\delta_0) \setminus \overline{B_{g_0}(z_0, \delta_0)}$, then

$$u_{\mu,\xi}^{-1}(u_{\mu,\xi})_t = \frac{N-2}{2}\mu^{-1}\dot{\mu} + O\left(\mu_0^2\left(\mu_0^2 \log \mu_0 + \delta_0^{-1}\mu_0^{\nu_2}\right)\right) = O(\mu_0^2).$$

If $z \in M \setminus \overline{B_{g_0}(z_0, 2\delta_0)}$, then

$$u_{\mu,\xi}^{-1}(u_{\mu,\xi})_t = \frac{N-2}{2}\mu^{-1}\dot{\mu} = O(\mu_0^2).$$

Consequently, the claim follows.

Step 2. Fixing any $t \in [t_0, s_0]$, we multiply (4.1) by ψ and integrate the resultant equality over M. Then we obtain

$$
\begin{aligned}
\frac{p}{2}\,\partial_t \int_M \psi^2 u_{\mu,\xi}^{p-1} dv_{g_0} - \frac{N+2}{4}\int_M \psi L_{g_0,h}\psi\, dv_{g_0} \\
= \int_M \mathcal{W}_{\mu,\xi}\psi^2 dv_{g_0} + \int_M f\psi u_{\mu,\xi}^{p-1} dv_{g_0} + \frac{p(p-1)}{2}\int_M \left[u_{\mu,\xi}^{-1}(u_{\mu,\xi})_t\right]\psi^2 u_{\mu,\xi}^{p-1} dv_{g_0}. \quad (4.5)
\end{aligned}
$$

On the other hand, Hölder's inequality yields

$$\int_M \psi^2 u_{\mu,\xi}^{p-1} dv_{g_0} \le C\left(\int_M u_{\mu,\xi}^{p+1} dv_{g_0}\right)^{\frac{p-1}{p+1}}\left(\int_M |\psi|^{p+1} dv_{g_0}\right)^{\frac{2}{p+1}} \le C\left(\int_M |\psi|^{p+1} dv_{g_0}\right)^{\frac{2}{p+1}}.$$

Hence, together with the condition $Y_h(M, g_0) > 0$ and the Sobolev inequality, we deduce

$$\int_M \psi^2 u_{\mu,\xi}^{p-1} dv_{g_0} \le C\int_M (|\nabla_{g_0}\psi|_{g_0}^2 + \psi^2)\, dv_{g_0} \le -C\int_M \psi L_{g_0,h}\psi\, dv_{g_0}. \quad (4.6)$$

Owing to (4.4), (4.6), the bound $|\mathcal{W}_{\mu,\xi}| \le C u_{\mu,\xi}^{p-1}$ on $M \times [t_0, s_0]$, and Hölder's inequality, we see from (4.5) that

$$
\begin{aligned}
&\partial_t \int_M \psi^2 u_{\mu,\xi}^{p-1} dv_{g_0} + \int_M (|\nabla_{g_0}\psi|_{g_0}^2 + \psi^2)\, dv_{g_0} \\
&\le C\left(\int_M \mathcal{W}_{\mu,\xi}\psi^2 dv_{g_0} + \int_M f\psi u_{\mu,\xi}^{p-1} dv_{g_0} + \mu_0^2(t)\int_M \psi^2 u_{\mu,\xi}^{p-1} dv_{g_0}\right) \quad (4.7)\\
&\le C\left(\|\psi(\cdot,t)\|_{L^2(M)}^2 + \|f(\cdot,t)\|_{L^2(M)}^2\right).
\end{aligned}
$$

Given any $\tau \in (t_0, s_0 - 1]$, we set a function $\chi(t) = t - \tau \in [0,1]$ for $t \in [\tau, \tau+1]$. For such t,

$$\partial_t\left(\chi(t)\int_M \psi^2 u_{\mu,\xi}^{p-1} dv_{g_0}\right) + \chi(t)\int_M (|\nabla_{g_0}\psi|_{g_0}^2 + \psi^2)\, dv_{g_0} \le C\left(\|\psi(\cdot,t)\|_{L^2(M)}^2 + \|f(\cdot,t)\|_{L^2(M)}^2\right)$$

by (4.7). Integrating it over $t \in [\tau, \tau+1]$, we arrive at

$$\int_M \left(\psi^2 u_{\mu,\xi}^{p-1}\right)(z,\tau+1)\,dv_{g_0} + \iint_{M_\tau} \chi(t)(|\nabla_{g_0}\psi|_{g_0}^2 + \psi^2)\,dv_{g_0}dt \\ \le C\left(\|\psi\|_{L^2(M_\tau)}^2 + \|f\|_{L^2(M_\tau)}^2\right). \quad (4.8)$$

From (4.7), we also derive

$$\int_M \left(\psi^2 u_{\mu,\xi}^{p-1}\right)(z,t_0+1)\,dv_{g_0} + \iint_{M_{t_0}} (|\nabla_{g_0}\psi|_{g_0}^2 + \psi^2)\,dv_{g_0}dt \\ \le C\left(\|\psi\|_{L^2(M_{t_0})}^2 + \|f\|_{L^2(M_{t_0})}^2 + \|\psi_0\|_{\mathcal{H}^1}^2\right). \quad (4.9)$$

To get this inequality, we use the initial datum ψ_0 in (4.1) instead of introducing the cut-off function $\chi(t)$, and (4.6).

Step 3. Fixing $t \in [t_0, s_0]$ again, we multiply (4.1) by ψ_t, integrate the result over M, and then apply Hölder's inequality and $|\partial_t \mathcal{W}_{\mu,\xi}| \le C\mu_0^2 u_{\mu,\xi}^{p-1}$ on $M \times [t_0, s_0]$. Then we observe

$$\partial_t\left[-\frac{N+2}{4}\int_M \psi L_{g_0,h}\psi\,dv_{g_0} - \int_M \mathcal{W}_{\mu,\xi}\psi^2 dv_{g_0}\right] + \|\psi_t\|_{L^2(M)}^2 \\ \le C\left(\int_M |\partial_t \mathcal{W}_{\mu,\xi}|\psi^2 dv_{g_0} + \|f\|_{L^2(M)}^2\right) \le C\left(\mu_0^2(t)\|\psi\|_{L^2(M)}^2 + \|f\|_{L^2(M)}^2\right).$$

Using $\chi^2(t)$ as a cut-off function and employing (4.8), we find

$$\iint_{M_\tau} \chi^2(t)\psi_t^2 u_{\mu,\xi}^{p-1} dv_{g_0}dt \le C\left[\int_M (\mathcal{W}_{\mu,\xi}\psi^2)(z,\tau+1)\,dv_{g_0} + \iint_{M_\tau}\chi(t)(|\nabla_{g_0}\psi|_{g_0}^2 + \psi^2)\,dv_{g_0}dt \right. \\ \left. + \mu_0^2(t)\|\psi\|_{L^2(M_\tau)}^2 + \|f\|_{L^2(M_\tau)}^2\right] \\ \le C\left(\|\psi\|_{L^2(M_\tau)}^2 + \|f\|_{L^2(M_\tau)}^2\right) \quad (4.10)$$

for all $\tau \in (t_0, s_0 - 1]$. Furthermore, we have

$$\|\psi_t\|_{L^2(M_{t_0})}^2 \le C\left(\|\psi\|_{L^2(M_{t_0})}^2 + \|f\|_{L^2(M_{t_0})}^2 + \|\psi_0\|_{\mathcal{H}^1}^2\right) \quad (4.11)$$

as in (4.9).

Step 4. Combining (4.8)–(4.11), using (4.6), and performing algebraic manipulations, we conclude that

$$\|\psi_t\|_{L^2_{t_0,s_0}} + \|\psi\|_{H^1_{t_0,s_0}} \le C\left(\|\psi\|_{L^2_{t_0,s_0}} + \|f\|_{L^2_{t_0,s_0}} + \|\psi_0\|_{\mathcal{H}^1}\right).$$

By (4.1) and the previous estimate,

$$\|u_{\mu,\xi}^{1-p}L_{g_0,h}\psi\|_{L^2_{t_0,s_0}} \le C\left(\|\psi_t\|_{L^2_{t_0,s_0}} + \|\psi\|_{L^2_{t_0,s_0}} + \|f\|_{L^2_{t_0,s_0}}\right) \\ \le C\left(\|\psi\|_{L^2_{t_0,s_0}} + \|f\|_{L^2_{t_0,s_0}} + \|\psi_0\|_{\mathcal{H}^1}\right).$$

Adding the above two estimates, we immediately obtain (4.3). □

Next, we improve Lemma 4.1 by dropping the norm of ψ in the right-hand side of (4.3). We need some preliminary definitions.

Definition 4.2. Let Π be the inverse of the stereographic projection

$$\Pi(y) = \left(\frac{2y}{1+|y|^2}, -\frac{1-|y|^2}{1+|y|^2}\right) \in \mathbb{S}^N_n := \mathbb{S}^N \setminus \{(0,\dots,0,1)\} \subset \mathbb{R}^{N+1}$$

for $y \in \mathbb{R}^N$, and $\Pi_* f : \mathbb{S}^N_n \to \mathbb{R}$ be a weighted push-forward of a function $f : \mathbb{R}^N \to \mathbb{R}$ given by

$$(\Pi_* f)(\tilde{y}) = \left(\frac{1+|y|^2}{2}\right)^{\frac{N-2}{2}} f(y) \quad \text{for } \tilde{y} := \Pi(y) \in \mathbb{S}^N_n.$$

Lemma 4.3. *Suppose that all the conditions of Lemma 4.1 hold, and $s_0 > \frac{3t_0}{2}$. Then there is a constant $C > 0$ depending only on (M, g_0), N, h, and z_0 such that every solution ψ to (4.2) satisfies*

$$\|\psi\|_{H^2_{t_0,s_0}} \le C \left(\|f\|_{L^2_{t_0,s_0}} + \|\psi_0\|_{\mathcal{H}^1}\right) \tag{4.12}$$

provided $t_0 > 0$ is large enough.

Proof. Thanks to (4.3), it suffices to show that

$$\|\psi\|_{L^2_{t_0,s_0}} \le C \left(\|f\|_{L^2_{t_0,s_0}} + \|\psi_0\|_{\mathcal{H}^1}\right). \tag{4.13}$$

Throughout the proof, we assume that $C > 0$ depends only on (M, g_0), N, h, and z_0.

Step 1. To establish (4.13), we argue by contradiction. Suppose that there are increasing sequences $\{t_\ell\}_{\ell\in\mathbb{N}}$, $\{s_\ell\}_{\ell\in\mathbb{N}}$ of positive numbers such that

$$s_\ell > \frac{3t_\ell}{2} \quad \text{and} \quad t_\ell,\, s_\ell \to \infty \quad \text{as } \ell \to \infty, \tag{4.14}$$

sequences $\{\mu_\ell\}_{\ell\in\mathbb{N}}$ and $\{\xi_\ell\}_{\ell\in\mathbb{N}}$ of parameters satisfying (2.5)–(2.7), and sequences $\{\psi_\ell\}_{\ell\in\mathbb{N}}$, $\{f_\ell\}_{\ell\in\mathbb{N}}$, $\{\psi_{0\ell}\}_{\ell\in\mathbb{N}}$ of functions which satisfy

$$\begin{cases} p u_{\mu_\ell,\xi_\ell}^{p-1} (\psi_\ell)_t = \dfrac{N+2}{4} L_{g_0,h}\psi_\ell + \mathcal{W}_{\mu_\ell,\xi_\ell}\psi_\ell + u_{\mu_\ell,\xi_\ell}^{p-1} f_\ell & \text{on } M \times (t_\ell, s_\ell), \\ \psi_\ell(\cdot, t_\ell) = \psi_{0\ell} & \text{on } M \end{cases} \tag{4.15}$$

and

$$\|\psi_\ell\|_{L^2_{t_\ell,s_\ell}} = 1, \quad \|f_\ell\|_{L^2_{t_\ell,s_\ell}} + \|\psi_{0\ell}\|_{\mathcal{H}^1} \to 0 \quad \text{as } \ell \to \infty. \tag{4.16}$$

There exists $\tau_\ell \in [t_\ell, s_\ell - 1]$ such that

$$\frac{1}{2} \le \iint_{M_{\tau_\ell}} \psi_\ell^2 u_{\mu_\ell,\xi_\ell}^{p-1} dv_{g_0} dt \le 1 \quad \text{and} \quad \|\psi_\ell\|_{H^1_{t_\ell,s_\ell}} \le C \tag{4.17}$$

where the latter inequality is valid by virtue of Lemma 4.1. By passing to a subsequence if necessary, we may assume that $\{\tau_\ell\}_{\ell\in\mathbb{N}}$ is increasing.

The main assertion of this step is

$$\liminf_{\ell\to\infty} (\tau_\ell - t_\ell) = \infty. \tag{4.18}$$

Let us prove it. We define

$$\Psi_\ell(\tau) := \int_M \left(\psi_\ell^2 u_{\mu_\ell,\xi_\ell}^{p-1}\right)(\cdot, \tau) dv_{g_0} \quad \text{for } \tau \in [t_\ell, s_\ell - 1].$$

If we integrate (4.7) over $s \in [t_\ell, \tau]$, we find

$$\Psi_\ell(\tau) \le C \left[\int_{t_\ell}^{\tau} \Psi_\ell(s) ds + (\tau - t_\ell + 1)\left(\|f_\ell\|^2_{L^2_{t_\ell,s_\ell}} + \|\psi_{0\ell}\|^2_{\mathcal{H}^1}\right)\right].$$

By Grönwall's inequality and (4.16), it follows that

$$\Psi_\ell(\tau) \le o(1) \cdot (\tau - t_\ell + 1) e^{C(\tau - t_\ell)}.$$

This and (4.17) yield the assertion (4.18).

Step 2. Set

$$\phi_\ell(z, t) = \psi_\ell(z, t + \tau_\ell), \quad u_\ell(z, t) = u_{\mu_\ell,\xi_\ell}(z, t + \tau_\ell),$$

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$$\mathcal{W}_\ell(z,t) = \mathcal{W}_{\mu_\ell,\xi_\ell}(z,t+\tau_\ell), \quad \tilde{f}_\ell(z,t) = f_\ell(z,t+\tau_\ell).$$

We infer from (4.15), (4.17) and (4.18) that

$$pu_\ell^{p-1}(\phi_\ell)_t = \frac{N+2}{4} L_{g_0,h}\phi_\ell + \mathcal{W}_\ell\,\phi_\ell + u_\ell^{p-1}\tilde{f}_\ell \quad \text{on } M\times(t_\ell-\tau_\ell,1)$$

where $t_\ell - \tau_\ell \to -\infty$ as $\ell\to\infty$, and

$$\frac{1}{2} \le \iint_{M_0} \phi_\ell^2 u_\ell^{p-1} dv_{g_0}dt \le 1 \quad \text{and} \quad \|\psi_\ell\|_{H^1_{t_\ell-\tau_\ell,1}} \le C. \tag{4.19}$$

Moreover, by (2.39), (4.17) and the Sobolev inequality, there exists a large number $R_1 > 0$ such that

$$\begin{aligned}
&\int_0^1 \int_{M\setminus B_{g_0}(z_0,R_1\tilde{\mu}_\ell)} \phi_\ell^2 u_\ell^{p-1} dv_{g_0}dt \\
&\le C\int_0^1 \left[\int_{M\setminus B_{g_0}(z_0,\delta_0)} \delta_0^{-2N}\mu_0^N(\tau_\ell)dv_{g_0} + \int_{\{R_1\le|y-\tilde{\xi}_\ell|\le\delta_0\tilde{\mu}_\ell^{-1}\}} W_{1,0}^{p+1}(y)dy\right]^{\frac{2}{N}} dt \\
&\le C\left(\delta_0^{-4}\mu_0^2(\tau_\ell) + R_1^{-2}\right) \le \frac{1}{4}
\end{aligned} \tag{4.20}$$

for all large $\ell\in\mathbb{N}$. Here, $\tilde{\mu}_\ell := \mu_\ell(\cdot+\tau_\ell)$ and $\tilde{\xi}_\ell := \xi_\ell(\cdot+\tau_\ell)$.

Step 3. According to (4.19) and (4.20),

$$\int_0^1 \int_{B_{g_0}(z_0,R_1\tilde{\mu}_\ell)} \phi_\ell^2 u_\ell^{p-1} dv_{g_0}dt \ge \frac{1}{4}. \tag{4.21}$$

Identifying $z\in B_{g_0}(z_0,\frac{\delta_0}{4})$ and $x = \tilde{\mu}_\ell y + \tilde{\xi}_\ell \in B^N(0,\frac{\delta_0}{4})$ as before, we define

$$\varphi_\ell(y,t) = \tilde{\mu}_\ell^{\frac{N-2}{2}}\phi_\ell(\tilde{\mu}_\ell y+\tilde{\xi}_\ell,t) \quad \text{for } (y,t)\in B_\ell\times(t_\ell-\tau_\ell,1)$$

where $B_\ell := B^N\left(-\tilde{\mu}_\ell^{-1}\tilde{\xi}_\ell, \frac{\delta_0}{4}\tilde{\mu}_\ell^{-1}\right)$. It solves

$$\begin{aligned}
&p(1+P)^{p-1}(x)\left(W_{1,0}+\tilde{\bar{\mu}}_\ell^2 Q_0\right)^{p-1}(\varphi_\ell)_t \\
&= \frac{N+2}{4}\left[\kappa_N\Delta_{g_0(x)}\varphi_\ell - \tilde{\mu}_\ell^2(S+h)\varphi_\ell\right] + \tilde{\mu}_\ell^2\overline{\mathcal{W}}_\ell\varphi_\ell + (1+P)^{p-1}(x)\left(W_{1,0}+\tilde{\bar{\mu}}_\ell^2 Q_0\right)^{p-1}\bar{f}_\ell \\
&\quad + p(1+P)^{p-1}(x)\left(W_{1,0}+\tilde{\bar{\mu}}_\ell^2 Q_0\right)^{p-1}\left[\tilde{\mu}_\ell^{-1}\dot{\tilde{\mu}}_\ell\left(y\cdot\nabla\varphi_\ell + \frac{N-2}{2}\varphi_\ell\right) + \tilde{\mu}_\ell^{-1}\dot{\tilde{\xi}}_\ell\cdot\nabla\varphi_\ell\right]
\end{aligned}$$

in $B_\ell\times(t_\ell-\tau_\ell,1)$, where P is the function in (2.13), $\tilde{\bar{\mu}}_\ell := \bar{\mu}_\ell(\cdot+\tau_\ell)$,

$$\overline{\mathcal{W}}_\ell(y,t) := \mathcal{W}_\ell(\tilde{\mu}_\ell y+\tilde{\xi}_\ell,t) \quad \text{and} \quad \bar{f}_\ell(y,t) := \tilde{f}_\ell(\tilde{\mu}_\ell y+\tilde{\xi}_\ell,t).$$

Also, in light of (4.19) and (4.21),

$$\int_0^1\int_{B^N(0,2R_1)} \varphi_\ell^2 W_{1,0}^{p-1}dydt \ge C \quad \text{and} \quad \sup_{\tau\in[t_\ell-\tau_\ell,0]}\int_\tau^{\tau+1}\int_{B_\ell}|\nabla\varphi_\ell|^2dydt \le C.$$

Hence, there exists a function φ_∞ in $\mathbb{R}^N\times(-\infty,1)$ such that for each $\tau\in[t_\ell-\tau_\ell,0]$,

$$\varphi_\ell \to \varphi_\infty \begin{cases} \text{weakly in } \dot{W}^{1,2}(\mathbb{R}^N\times(\tau,\tau+1)) \\ \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^N\times(\tau,\tau+1)) \\ \text{a.e. in } \mathbb{R}^N\times(\tau,\tau+1) \end{cases} \quad \text{as } \ell\to\infty,$$

up to a subsequence. In particular, exploiting (4.16) and (2.5)–(2.7), we see that

$$pW_{1,0}^{p-1}(\varphi_\infty)_t = \frac{(N+2)\kappa_N}{4}\Delta\varphi_\infty \quad \text{in } \mathbb{R}^N \times (-\infty, 1) \tag{4.22}$$

as well as

$$\int_0^1 \int_{B^N(0,2R_1)} \varphi_\infty^2 W_{1,0}^{p-1} dydt \ge C \quad \text{and} \quad \sup_{\tau\in(-\infty,0]} \int_\tau^{\tau+1} \int_{\mathbb{R}^N} |\nabla\varphi_\infty|^2 dydt \le C. \tag{4.23}$$

In the next step, we will deduce that

$$\varphi_\infty = 0 \quad \text{in } \mathbb{R}^N \times (-\infty, 1), \tag{4.24}$$

a contradiction to the first inequality in (4.23). Therefore, (4.13) is valid.

Step 4. Referring to Definition 4.2, we write $\bar\varphi_\infty = \Pi_*\varphi_\infty$ on $\mathbb{S}^N_n$. From (4.22), the second inequality in (4.23), and the conformal covariance of conformal Laplacians, we observe

$$(\bar\varphi_\infty)_t = \frac{\kappa_N}{N}\left[\Delta_{\mathbb{S}^N}\bar\varphi_\infty - \frac{N(N-2)}{4}\bar\varphi_\infty\right] \quad \text{in } \mathbb{S}^N_n \times (-\infty, 1) \tag{4.25}$$

and

$$\sup_{\tau\in(-\infty,0]} \int_\tau^{\tau+1} \int_{\mathbb{S}^N} \left(|\nabla_{\mathbb{S}^N}\bar\varphi_\infty|^2_{\mathbb{S}^N} + \bar\varphi_\infty^2\right) dS_{\tilde y}dt \le C. \tag{4.26}$$

Parabolic regularity theory and (4.26) imply that $(0,\ldots,0,1) \in \mathbb{S}^N$ is a removable singularity of $\bar\varphi_\infty$ and $\|\bar\varphi_\infty\|_{C^\infty(\mathbb{S}^N\times(-\infty,1))} \le C$.

Let us apply a well-known argument to prove that $\bar\varphi_\infty = 0$ on $\mathbb{S}^N \times (-\infty, 1)$. By using (4.25) and Bochner's formula, we obtain

$$\begin{aligned}\left(|\nabla\bar\varphi|^2\right)_t &= 2\langle\nabla\bar\varphi, \nabla\bar\varphi_t\rangle = \frac{2\kappa_N}{N}\left[\langle\nabla\bar\varphi, \nabla\Delta\bar\varphi\rangle - \frac{N(N-2)}{4}|\nabla\bar\varphi|^2\right] \\ &= \frac{\kappa_N}{N}\left[\Delta|\nabla\bar\varphi|^2 - 2\left|\nabla^2\bar\varphi\right|^2 - 2\mathrm{Ric}(\nabla\bar\varphi, \nabla\bar\varphi) - \frac{N(N-2)}{2}|\nabla\bar\varphi|^2\right] \le \frac{\kappa_N}{N}\Delta|\nabla\bar\varphi|^2\end{aligned}$$

where $\mathrm{Ric} = (N-1)g_{\mathbb{S}^N}$ is the Ricci curvature on $\mathbb{S}^N$, and we wrote $\bar\varphi = \bar\varphi_\infty$, $\nabla = \nabla_{\mathbb{S}^N}$, etc. Hence, for any $\tau_0 \in (-\infty, 1)$ fixed and $t > \tau_0$,

$$\begin{aligned}\left(\bar\varphi^2 + \frac{2\kappa_N}{N}(t-\tau_0)|\nabla\bar\varphi|^2\right)_t &\le \frac{2\kappa_N}{N}\left[\bar\varphi\left\{\Delta\bar\varphi - \frac{N(N-2)}{4}\bar\varphi\right\} + |\nabla\bar\varphi|^2 + \frac{\kappa_N}{N}(t-\tau_0)\Delta|\nabla\bar\varphi|^2\right] \\ &\le \frac{\kappa_N}{N}\Delta\left[\bar\varphi^2 + \frac{2\kappa_N}{N}(t-\tau_0)|\nabla\bar\varphi|^2\right].\end{aligned}$$

By the maximum principle,

$$\bar\varphi^2 + \frac{2\kappa_N}{N}(t-\tau_0)|\nabla\bar\varphi|^2 \le \max_{\tilde y\in\mathbb{S}^N} \bar\varphi^2(\tilde y, \tau_0) \le C.$$

Thus

$$|\nabla\bar\varphi(\tilde y, t)| \le \frac{C}{\sqrt{t-\tau_0}} \quad \text{for all } (\tilde y, t) \in \mathbb{S}^N \times [\tau_0, 1).$$

Taking $\tau_0 \to -\infty$ shows that $\bar\varphi$ depends only on t. In view of (4.25),

$$\bar\varphi(t) = \bar c_1 e^{-\frac{\kappa_N(N-2)}{4}t} + \bar c_2 \quad \text{in } (-\infty, 1) \quad \text{for some } \bar c_1, \bar c_2 \in \mathbb{R}.$$

Since it must be bounded in $(-\infty, 1)$, we have that $\bar c_1 = 0$. From (4.25) again, we conclude that $\bar c_2 = 0$.

Now, (4.24) follows at once. □

Using a priori estimate (4.12), we deduce the unique existence of a solution to (4.1).

Corollary 4.4. *Suppose that $Y_h(M,g_0)>0$, $\|f\|_{L^2_{t_0}}+\|\psi_0\|_{\mathcal{H}^1}<\infty$ and (2.5)–(2.7) hold. Then there exists a unique solution ψ to (4.1) satisfying*

$$\|\psi\|_{H^2_{t_0}} \le C\left(\|f\|_{L^2_{t_0}}+\|\psi_0\|_{\mathcal{H}^1}\right). \tag{4.27}$$

Proof. Pick an increasing sequence $\{s_\ell\}_{\ell\in\mathbb{N}}$ such that $\frac{3t_0}{2}<s_\ell\to\infty$ as $\ell\to\infty$, and let ψ_ℓ be a unique solution to a uniformly parabolic equation (4.2) with $s_0=s_\ell$. Then a priori estimate (4.12) with $s_0=s_\ell$ is true for ψ_ℓ, and the constant $C>0$ in (4.12) is independent of $\ell\in\mathbb{N}$. Thus, passing to a subsequence, ψ_ℓ converges weakly to a function ψ_∞ in a Banach space equipped with the norm $\|\cdot\|_{H^2_{t_0,t}}$ for each $t>\frac{3t_0}{2}$. As a result, ψ_∞ is the only function satisfying (4.1) and (4.27). Setting $\psi=\psi_\infty$, we conclude the proof. □

4.2. **Inhomogeneous problem associated to** (3.2)**: Pointwise estimate.** We next derive a pointwise estimate for a solution to the inhomogeneous problem (4.1). A main tool is a version of the maximum principle presented in the following lemma.

Lemma 4.5. *Suppose that ψ satisfies*

$$\begin{cases} pu_{\mu,\xi}^{p-1}\psi_t \ge \dfrac{N+2}{4}L_{g_0,h}\psi+\mathcal{W}_{\mu,\xi}\psi & \text{on } M\times(t_0,\infty),\\ \psi(\cdot,t_0)\ge 0 & \text{on } M\end{cases}$$

in a weak sense. Then $\psi\ge 0$ on $M\times[t_0,\infty)$.

Proof. Fix $s_0>t_0$ and let $M(s_0)=M\times(t_0,s_0)$. We set

$$\phi(z,t)=e^{-C_0t}\psi(z,t)\quad \text{for } (z,t)\in M\times[t_0,\infty)$$

where

$$C_0:=p^{-1}\left(\min_{M(s_0)}u_{\mu,\xi}^{p-1}\right)^{-1}\left[\frac{N+2}{4}\max_M(|S|+|h|)+\max_{M(s_0)}|\mathcal{W}_{\mu,\xi}|+1\right].$$

Then

$$\begin{aligned}0\ge&-\frac{N+2}{4}\left[\kappa_N\iint_{M(s_0)}\langle\nabla_{g_0}\phi,\nabla_{g_0}\omega\rangle_{g_0}+\iint_{M(s_0)}(S+h)\phi\omega\right]+\iint_{M(s_0)}\mathcal{W}_{\mu,\xi}\phi\omega\\ &+p(p-1)\iint_{M(s_0)}u_{\mu,\xi}^{-1}(u_{\mu,\xi})_t\phi\omega u_{\mu,\xi}^{p-1}-C_0p\iint_{M(s_0)}\phi\omega u_{\mu,\xi}^{p-1}+p\iint_{M(s_0)}\phi\omega_tu_{\mu,\xi}^{p-1}\\ &-p\int_M(\phi\omega u_{\mu,\xi}^{p-1})(\cdot,s_0)+p\int_M(\phi\omega u_{\mu,\xi}^{p-1})(\cdot,t_0)\end{aligned} \tag{4.28}$$

for any nonnegative regular function ω on $M(s_0)$, and

$$f_0:=\frac{N+2}{4}(S+h)-\mathcal{W}_{\mu,\xi}+p(p-1)u_{\mu,\xi}^{-1}(u_{\mu,\xi})_t\cdot u_{\mu,\xi}^{p-1}+C_0pu_{\mu,\xi}^{p-1}>0\quad\text{on } M(s_0)$$

by (4.4).

Fix a function $\omega_0\in C^\infty(M)$ such that $\omega_0\ge0$ on M. By standard parabolic theory, the backward uniformly parabolic equation

$$\begin{cases} pu_{\mu,\xi}^{p-1}\omega_t+\dfrac{N+2}{4}\kappa_N\Delta_{g_0}\omega-f_0\omega=0 & \text{on } M(s_0),\\ \omega(\cdot,s_0)=\omega_0 & \text{on } M\end{cases}$$

has a unique solution $\omega\ge0$ on $M(s_0)$. Taking it as a test function for (4.28) and employing $\phi(\cdot,t_0)\ge0$ on M, we find

$$0\ge -p\int_M(\phi u_{\mu,\xi}^{p-1})(\cdot,s_0)\omega_0dv_{g_0}.$$

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Since ω_0 and s_0 were chosen arbitrarily, we must have that $\phi(\cdot,t_0) \ge 0$, namely, $\psi(\cdot,t_0) \ge 0$ on $M \times [t_0,\infty)$. □

In the rest of this subsection, we will deduce a priori pointwise estimates for (4.1) by means of barrier and rescaling arguments. For the definition of the associated norms, refer to Definition 1.8.

Lemma 4.6. *Suppose that $Y_h(M,g_0) > 0$, $\|f\|_{L^2_{t_0}} + \|\psi_0\|_{\mathcal{H}^1} < \infty$, and (2.5)–(2.7) hold so that equation (4.1) admits the unique solution ψ; refer to Corollary 4.4. Assume further that $\|f\|_{*,\alpha,\rho} + \|\psi_0\|_{**,\alpha} < \infty$ for some $\alpha \in (0, N-2)$ and $\rho \ge -\frac{N-2}{2}$. Then there exist constants $C > 0$ large and $\delta_1 > 0$ small depending only on (M,g_0), N, h, z_0, α, and ρ such that*

$$|\psi(z,t)| \le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho}\mu_0^\rho(t) + \|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\right] w_{\alpha,0}(z,t) \tag{4.29}$$

for $(z,t) \in M \times [t_0,\infty)$.

Proof. Throughout the proof, we assume that $C > 0$ depends only on (M,g_0), N, h, z_0, α, and ρ. Moreover, we write $\psi = \psi[f,\psi_0]$, emphasizing the dependence of ψ on f and ψ_0. Let $\psi_1 = \psi[0,\psi_0]$ and $\psi_2 = \psi[f,0]$ so that $\psi = \psi_1 + \psi_2$.

STEP 1: A BOUND FOR ψ_1. First of all, we assume that $d_{g_0}(z,z_0) < \frac{\delta_0}{4}$ so that $u(x,t) = (1 + P(x))\mu^{-\frac{N-2}{2}}[W_{1,0}(y) + \bar\mu^2 Q_0(y)]$ for $x = \exp_{z_0}^{-1}(z) = \mu y + \xi \in B^N(0,\frac{\delta_0}{4})$ and $t \in [t_0,\infty)$. Given a small number $\delta_1 > 0$, we set

$$\tilde\psi_{11}(z,t) = \|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)} \times \begin{cases} 2-|y|^2 & \text{for } |y| \le 1, \\ |y|^{-\alpha} & \text{for } |y| > 1. \end{cases}$$

If $|y| \le 1$, then

$$\begin{aligned} pu_{\mu,\xi}^{p-1}(\tilde\psi_{11})_t &= p\mu^{-2}W_{1,0}^{p-1}(y)\left(1+O(\mu_0^2)\right)\|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\left[-\delta_1(2-|y|^2)+O(\mu_0^2)\right] \\ &\ge -4p\alpha_N^{p-1}\delta_1\|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\mu^{-2}, \\ -L_{g_0,h}\tilde\psi_{11} &= \|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}L_{g_0,h}|y|^2 \ge \frac{3N}{2}\|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\mu^{-2}, \end{aligned}$$

and

$$\left|\mathcal{W}_{\mu,\xi}\tilde\psi_{11}\right| = o(\mu_0^{-2})\|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}.$$

Thus we obtain

$$pu_{\mu,\xi}^{p-1}(\tilde\psi_{11})_t - \frac{N+2}{4}L_{g_0,h}\tilde\psi_{11} - \mathcal{W}_{\mu,\xi}\tilde\psi_{11} \ge \frac{N(N+2)}{4}\|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\mu^{-2} > 0 \tag{4.30}$$

for sufficiently small $\delta_1 > 0$.

If $|y| > 1$, then

$$\begin{aligned} pu_{\mu,\xi}^{p-1}(\tilde\psi_{11})_t &= p\mu^{-2}W_{1,0}^{p-1}(y)\left(1+O(\delta_0^2)\right)\|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\left[-\delta_1+O(\mu_0^2)\right]|y|^{-\alpha} \\ &\ge -2p\alpha_N^{p-1}\delta_1\|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\mu^{-2}|y|^{-(\alpha+4)} \end{aligned}$$

and

$$-\frac{N+2}{4}L_{g_0,h}\tilde\psi_{11} - \mathcal{W}_{\mu,\xi}\tilde\psi_{11} \ge \left[C + O\left(\mu_0^{\frac{8}{N-2}}\log\mu_0\right)\right]\|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\mu^{-2}|y|^{-(\alpha+2)}$$

where $C > 0$ is a constant depending only on N and α. Therefore,

$$pu_{\mu,\xi}^{p-1}(\tilde\psi_{11})_t - \frac{N+2}{4}L_{g_0,h}\tilde\psi_{11} - \mathcal{W}_{\mu,\xi}\tilde\psi_{11} \ge C\|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\mu^{-2}|y|^{-(\alpha+2)} > 0 \tag{4.31}$$

for small δ_0, $\delta_1 > 0$. On the other hand, there exists a large constant $c > 0$ depending only on (M,g_0), N, and α such that

$$(c\tilde\psi_{11} \pm \psi_1)(z,t_0) = (c\tilde\psi_{11})(z,t_0) \pm \psi_0(z) \ge 0 \tag{4.32}$$

for $z \in B_{g_0}(z_0, \frac{\delta_0}{4})$.

Next, we handle the case that $d_{g_0}(z, z_0) > \frac{\delta_0}{6}$. Let v be a solution to $-L_{g_0,h} v = 1$ on M. Owing to the condition $Y_h(M, g_0) > 0$, the strong maximum principle holds for the elliptic operator $L_{g_0,h}$. Hence, such v exists uniquely in $C^2(M)$ and is positive on M. Let us set

$$\tilde{\psi}_{12}(z,t) = 2^{3\alpha} \left(\min_{z \in M} v(z) \right)^{-1} \delta_0^{-\alpha} \|\psi_0\|_{**,\alpha} \mu_0^{\rho}(t_0) e^{-\delta_1(t-t_0)} \mu^{\alpha}(t) v(z) \quad \text{on } M \times [t_0, \infty).$$

A simple calculation shows that if $\frac{\delta_0}{6} < d_{g_0}(z, z_0) < \frac{\delta_0}{4}$ and $t \in [t_0, \infty)$, then $\tilde{\psi}_{12}(z,t) > \tilde{\psi}_{11}(z,t)$. Besides,

$$p u_{\mu,\xi}^{p-1} (\tilde{\psi}_{12})_t \ge -C\delta_1 \delta_0^{-(\alpha+4)} \|\psi_0\|_{**,\alpha} \mu_0^{\rho}(t_0) e^{-\delta_1(t-t_0)} \mu^{\alpha+2}$$

and

$$-\frac{N+2}{4} L_{g_0,h} \tilde{\psi}_{12} - \mathcal{W}_{\mu,\xi} \tilde{\psi}_{12} = \left[C + O\left(\delta_0^{-4} \mu_0^2\right) \right] \delta_0^{-\alpha} \|\psi_0\|_{**,\alpha} \mu_0^{\rho}(t_0) e^{-\delta_1(t-t_0)} \mu^{\alpha}.$$

Consequently, we have

$$p u_{\mu,\xi}^{p-1} (\tilde{\psi}_{12})_t - \frac{N+2}{4} L_{g_0,h} \tilde{\psi}_{12} - \mathcal{W}_{\mu,\xi} \tilde{\psi}_{12} \ge C\delta_0^{-\alpha} \|\psi_0\|_{**,\alpha} \mu_0^{\rho}(t_0) e^{-\delta_1(t-t_0)} \mu^{\alpha} > 0 \tag{4.33}$$

taking $\delta_1 > 0$ smaller if needed. In addition, it holds that

$$(c\tilde{\psi}_{12} \pm \psi_1)(z, t_0) \ge 0 \quad \text{on } M \tag{4.34}$$

for some large $c > 0$.

Now, letting $\tilde{\psi}_{11}(z,t) = 0$ for $z \in M \setminus B_{g_0}(z, \frac{\delta_0}{4})$ and $t \in [t_0, \infty)$, we define a function

$$\tilde{\psi} = \max \left\{ \tilde{\psi}_{11}, \tilde{\psi}_{12} \right\} \quad \text{on } M \times [t_0, \infty).$$

By applying Lemma 4.5 for functions $c\tilde{\psi} \pm \psi_1$ and using (4.30)–(4.34), we deduce

$$|\psi_1(z,t)| \le C \|\psi_0\|_{**,\alpha} \mu_0^{\rho}(t_0) e^{-\delta_1(t-t_0)} w_{\alpha,0}(z,t) \quad \text{on } M \times [t_0, \infty). \tag{4.35}$$

Step 2: A bound for ψ_2. For $z \in B_{g_0}(z_0, \frac{\delta_0}{4})$, we redefine

$$\tilde{\psi}_{21}(z,t) = \delta_0^{-2} \|f\|_{*,\alpha,\rho} \mu^{\rho}(t) \times \begin{cases} 2 - |y|^2 & \text{for } |y| \le 1, \\ |y|^{-\alpha} & \text{for } |y| > 1 \end{cases}$$

where $x = \exp^{-1}(z) = \mu y + \xi \in B^N(0, \frac{\delta_0}{4})$ and $t \in [t_0, \infty)$.

To treat the case that $d_{g_0}(z, z_0) > \frac{\delta_0}{6}$, we also set

$$\tilde{\psi}_{22}(z,t) = 2^{3\alpha} \left(\min_{z \in M} v(z) \right)^{-1} \delta_0^{-(\alpha+2)} \|f\|_{*,\alpha,\rho} \mu^{\alpha+\rho}(t) v(z) \quad \text{on } M \times [t_0, \infty).$$

Then, arguing as in Step 1, we arrive at

$$|\psi_2(z,t)| \le C\delta_0^{-2} \|f\|_{*,\alpha,\rho} \mu_0^{\rho}(t) w_{\alpha,0}(z,t) \quad \text{on } M \times [t_0, \infty). \tag{4.36}$$

Summing (4.35) and (4.36) up, we get (4.29). □

Corollary 4.7. *Suppose that all the conditions of Lemma 4.6 hold, and $\|f\|_{*,\alpha,\rho;\sigma_0} + \|\psi_0\|_{**,\alpha;\sigma_0} < \infty$ where $\sigma_0 \in (0,1)$ is the number appearing in (2.7). There exist constants $C > 0$ large and $\delta_1 > 0$ small depending only on (M, g_0), N, h, z_0, α, ρ, and σ_0 such that*

$$\begin{cases} \left|\nabla_{g_0}^{\ell} \psi(z,t)\right| \le C \left[\delta_0^{-2} \|f\|_{*,\alpha,\rho;\sigma_0} \mu_0^{\rho}(t) + \|\psi_0\|_{**,\alpha;\sigma_0} \mu_0^{\rho}(t_0) e^{-\delta_1(t-t_0)}\right] w_{\alpha,\ell}(z,t), \\ \left[\nabla_{g_0}^{\ell} \psi\right]_{C_z^{\sigma_0}} (z,t) \le C \left[\delta_0^{-2} \|f\|_{*,\alpha,\rho;\sigma_0} \mu_0^{\rho}(t) + \|\psi_0\|_{**,\alpha;\sigma_0} \mu_0^{\rho}(t_0) e^{-\delta_1(t-t_0)}\right] w_{\alpha,\ell+\sigma_0}(z,t), \\ \left[\nabla_{g_0}^{\ell} \psi\right]_{C_t^{\sigma_0/2}} (z,t) \le C \left[\delta_0^{-2} \|f\|_{*,\alpha,\rho;\sigma_0} \mu_0^{\rho}(t) + \|\psi_0\|_{**,\alpha;\sigma_0} \mu_0^{\rho}(t_0) e^{-\delta_1(t-t_0)}\right] w_{\alpha-\sigma_0,\ell}(z,t), \end{cases} \tag{4.37}$$

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and

$$\begin{cases} |\psi_t(z,t)| \le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^\rho(t) + \|\psi_0\|_{**,\alpha;\sigma_0}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\right] w_{\alpha-2,0}(z,t), \\ [\psi_t]_{C_z^{\sigma_0}}(z,t) \le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^\rho(t) + \|\psi_0\|_{**,\alpha;\sigma_0}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\right] w_{\alpha-2,\sigma_0}(z,t), \\ [\psi_t]_{C_t^{\sigma_0/2}}(z,t) \le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^\rho(t) + \|\psi_0\|_{**,\alpha;\sigma_0}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\right] w_{\alpha-2-\sigma_0,0}(z,t) \end{cases} \tag{4.38}$$

for $(z,t) \in M \times [t_0,\infty)$ *and* $\ell \in \{0,1,2\}$. *In particular, we have that*

$$\|\psi\|_{*',\alpha,\rho;\sigma_0} \le C_1\delta_0^{-2}\left(\|f\|_{*,\alpha,\rho;\sigma_0} + \|\psi_0\|_{**,\alpha;\sigma_0}\right) \tag{4.39}$$

where $C_1 > 0$ *is a constant depending only on* (M,g_0), N, h, z_0, α, ρ, *and* σ_0.

Proof. The proof is decomposed into two steps.

STEP 1. We examine the case that $d_{g_0}(z,z_0) \le \frac{\delta_0}{12}$.

We set

$$\phi(y,t) = \psi(z,t) \quad \text{for } y \in B^N\left(-\xi, \frac{\delta_0}{4\mu}\right) \text{ and } t \in [t_0,\infty).$$

It solves

$$\begin{aligned} \phi_t = \frac{N-2}{4}\mu^{-2}u_{\mu,\xi}^{1-p}(x,t)\Delta_{g_0(x)}\phi + \mu^{-1}\left(\dot{\mu}y + \dot{\xi}\right)\cdot\nabla\phi \\ + p^{-1}u_{\mu,\xi}^{1-p}(x,t)\left[\frac{N+2}{4}(S+h)(x) + \mathcal{W}_{\mu,\xi}(x,t)\right]\phi + p^{-1}f(x,t). \end{aligned} \tag{4.40}$$

For a number $t > t_0 + 2$ and a set $\Omega \subset \mathbb{R}^{N+1}$, we define

$$\Omega_{11} = \{|y| \le 6\} \times [t-1,t], \quad \Omega_{12} = \{|y| \le 8\} \times [t-2,t],$$

and a standard parabolic Hölder norm

$$\|\phi\|_{C^{\sigma_0,\sigma_0/2}(\Omega)} = \|\phi\|_{L^\infty(\Omega)} + \sup_{(y_1,t_1)\neq(y_2,t_2)\in\Omega} \frac{|\phi(y_1,t_1) - \phi(y_2,t_2)|}{|y_1-y_2|^{\sigma_0} + |t_1-t_2|^{\sigma_0/2}}.$$

A straightforward computation shows that the $C^{\sigma_0,\sigma_0/2}(\Omega_{12})$-norms of the coefficients in (4.40) are uniformly bounded in t. Therefore, the interior parabolic Schauder estimate is applicable, which gives

$$\sum_{\ell=0}^{2}\left\|\nabla^\ell\phi\right\|_{C^{\sigma_0,\sigma_0/2}(\Omega_{11})} + \|\phi_t\|_{C^{\sigma_0,\sigma_0/2}(\Omega_{11})} \le C\left(\|\phi\|_{L^\infty(\Omega_{12})} + \|f(x,t)\|_{C^{\sigma_0,\sigma_0/2}(\Omega_{12})}\right). \tag{4.41}$$

Given any (y_1,t_1), $(y_2,t_2) \in \Omega_{12}$, we have

$$\frac{|f(x_1,t_1) - f(x_2,t_2)|}{|y_1-y_2|^{\sigma_0} + |t_1-t_2|^{\sigma_0/2}} \le (1+O(\delta_0^2))\mu^{\sigma_0}(t_1)\frac{|f(x_1,t_1) - f(x_{12},t_1)|}{d_{g_0}(x_1,x_{12})^{\sigma_0}} + \frac{|f(x_{12},t_1) - f(x_2,t_2)|}{|t_1-t_2|^{\sigma_0/2}}$$

where

$$x_1 := \mu(t_1)y_1 + \xi(t_1), \quad x_2 := \mu(t_2)y_2 + \xi(t_2) \quad \text{and} \quad x_{12} := \mu(t_1)y_2 + \xi(t_1).$$

Moreover, if $|y_1 - y_2| < \delta$ for a number $\delta > 0$ small enough, then $d_{g_0}(x_1,x_2) < \mu_0(t_1)$ and so

$$\begin{aligned} |f(x_1,t_1) - f(x_{12},t_1)| &\le u_{\mu,\xi}^{1-p}(x_1,t_1)\left[u_{\mu,\xi}^{p-1}f\right]_{C_z^{\sigma_0}}(x_1,t_1)\, d_{g_0}(x_1,x_{12})^{\sigma_0} \\ &\quad + \sup_{\theta\in(0,1)}\left|\nabla_x(u_{\mu,\xi}^{1-p})(x_1+\theta(x_{12}-x_1),t_1)\right|\left|(u_{\mu,\xi}^{p-1}f)(x_{12},t_1)\right| \\ &\le C\left[\mu^{-\sigma_0}(t_1)\|f\|_{*,\alpha,\rho;\sigma_0} + d_{g_0}(x_1,x_{12})^{1-\sigma_0}\|f\|_{*,\alpha,\rho}\right]\mu_0^\rho(t_1)d_{g_0}(x_1,x_{12})^{\sigma_0}. \end{aligned}$$

If $|t_1 - t_2| < 1$, then

$$|x_{12} - x_2| \le C\mu_0^3(t_1)(|y_2| + o(1))|t_1-t_2| \le C\mu_0^3(t_1)|t_1-t_2|,$$

which implies

$$\begin{aligned}|f(x_{12},t_1)-f(x_2,t_2)| &\le |f(x_{12},t_1)-f(x_2,t_1)|+|f(x_2,t_1)-f(x_2,t_2)|\\ &\le C\left[\mu_0^{2\sigma_0}(t_1)|t_1-t_2|^{\sigma_0/2}\|f\|_{*,\alpha,\rho;\sigma_0}+\|f\|_{*,\alpha,\rho}\right]\mu_0^{\rho}(t_1)|t_1-t_2|^{\sigma_0/2}\\ &\le C\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^{\rho}(t_1)|t_1-t_2|^{\sigma_0/2}.\end{aligned}$$

It follows that

$$\|f(x,t)\|_{C^{\sigma_0,\sigma_0/2}(\Omega_{12})}\le C\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^{\rho}(t). \tag{4.42}$$

Substituting (4.29) and (4.42) into (4.41) gives us that

$$\begin{cases}\left|\nabla^{\ell}\psi(x,t)\right|=\mu^{-\ell}\left|\nabla^{\ell}\phi(y,t)\right|\le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^{\rho}+\|\psi_0\|_{**,\alpha}\mu_0^{\rho}(t_0)e^{-\delta_1(t-t_0)}\right]\mu_0^{-\ell},\\ \mu_0^{\sigma_0}\left[\nabla^{\ell}\psi\right]_{C_z^{\sigma_0}}(x,t)+\left[\nabla^{\ell}\psi\right]_{C_t^{\sigma_0/2}}(x,t)\\ \qquad\qquad\le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^{\rho}+\|\psi_0\|_{**,\alpha}\mu_0^{\rho}(t_0)e^{-\delta_1(t-t_0)}\right]\mu_0^{-\ell}\end{cases} \tag{4.43}$$

and

$$\begin{cases}|\psi_t(x,t)|\le|\phi_t(y,t)|+\mu^{-1}\left(\dot\mu|y|+\dot\xi\right)|\nabla\psi(x,t)|\\ \qquad\le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^{\rho}+\|\psi_0\|_{**,\alpha}\mu_0^{\rho}(t_0)e^{-\delta_1(t-t_0)}\right],\\ \mu_0^{\sigma_0}[\psi_t]_{C_z^{\sigma_0}}(x,t)+[\psi_t]_{C_t^{\sigma_0/2}}(x,t)\le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^{\rho}+\|\psi_0\|_{**,\alpha}\mu_0^{\rho}(t_0)e^{-\delta_1(t-t_0)}\right]\end{cases} \tag{4.44}$$

for $\ell\in\{0,1,2\}$, $|x-\xi|<4\mu$, and $t>t_0+2$.

Let n_0 be a natural number such that $2^{n_0}\le\frac{\delta_0}{24}\mu^{-1}(t)<2^{n_0+1}$. In the rest of this step, we will estimate ϕ on each set

$$\Omega_{n1}:=\left\{2^n<|y|<2^{n+1}\right\}\times\left[t-2^{-2n},t\right]\quad\text{for } n=1,\dots,n_0.$$

With this aim, we introduce

$$\widetilde\Omega_{n1}:=\{1<|Y|<2\}\times[-1,0],\quad \widetilde\Omega_{n2}:=\left\{\frac12<|Y|<\frac52\right\}\times[-2,0]$$

and

$$\varphi(Y,\tau)=\phi\left(2^nY,t+2^{-2n}\tau\right)\quad\text{for } (Y,\tau)\in\widetilde\Omega_{n2}.$$

By (4.40), it is a solution to

$$\begin{aligned}\varphi_\tau=&\ \frac{N-2}{4}2^{-4n}\mu^{-2}u_{\mu,\xi}^{1-p}\left(x,t+2^{-2n}\tau\right)\Delta_{g_0(x)}\varphi+2^{-2n}\mu^{-1}\left(\dot\mu y+\dot\xi\right)\cdot\nabla\varphi\\ &+p^{-1}2^{-2n}u_{\mu,\xi}^{1-p}\left(x,t+2^{-2n}\tau\right)\left[\frac{N+2}{4}(S+h)(x)+\mathcal{W}_{\mu,\xi}\left(x,t+2^{-2n}\tau\right)\right]\varphi\\ &+2^{-2n}p^{-1}f\left(x,t+2^{-2n}\tau\right)\end{aligned}$$

in $\widetilde\Omega_{n2}$. A direct computation gives that the above equation is uniformly parabolic and all the $C^{\sigma_0,\sigma_0/2}(\widetilde\Omega_{n2})$-norms of the coefficients in it are uniformly bounded in n and t. Hence

$$\begin{aligned}\sum_{\ell=0}^{2}\left\|\nabla^{\ell}\varphi\right\|_{C^{\sigma_0,\sigma_0/2}(\widetilde\Omega_{n1})}+\|\varphi_\tau\|_{C^{\sigma_0,\sigma_0/2}(\widetilde\Omega_{n1})}\\ \le C\left(\|\varphi\|_{L^\infty(\widetilde\Omega_{n2})}+\left\|2^{-2n}f\left(x,t+2^{-2n}\tau\right)\right\|_{C^{\sigma_0,\sigma_0/2}(\widetilde\Omega_{n2})}\right),\end{aligned} \tag{4.45}$$

which can be interpreted as

$$\begin{cases} \left|\nabla^\ell \psi(x,t)\right| = (\mu 2^n)^{-\ell} \left|\nabla^\ell \varphi(Y,0)\right| \\ \qquad\qquad \le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^\rho + \|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\right]\dfrac{\mu_0^{-\ell}}{|y|^{\alpha+\ell}}, \\ (\mu_0|y|)^{\sigma_0}\left[\nabla^\ell\psi\right]_{C_z^{\sigma_0}}(x,t) + |y|^{-\sigma_0}\left[\nabla^\ell\psi\right]_{C_t^{\sigma_0}}(x,t) \\ \qquad\qquad \le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^\rho + \|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\right]\dfrac{\mu_0^{-\ell}}{|y|^{\alpha+\ell}} \end{cases} \tag{4.46}$$

and

$$\begin{cases} |\psi_t(x,t)| \le C\mu^2|\nabla\varphi(Y,\tau)| + 2^{2n}|\varphi_\tau(Y,\tau)| \\ \qquad\qquad \le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^\rho + \|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\right]\dfrac{1}{|y|^{\alpha-2}}, \\ (\mu_0|y|)^{\sigma_0}[\psi_t]_{C_z^{\sigma_0}}(x,t) + |y|^{-\sigma_0}[\psi_t]_{C_t^{\sigma_0/2}}(x,t) \\ \qquad\qquad \le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^\rho + \|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\right]\dfrac{1}{|y|^{\alpha-2}} \end{cases} \tag{4.47}$$

for $\ell \in \{0,1,2\}$, $2\mu < |x-\xi| < \frac{\delta_0}{12}$, and $t \ge t_0+2$. To get the estimates for the $C_t^{\sigma_0}$-norms in (4.46)–(4.47), we need (4.45) with t replaced by any number in the interval $[t-1+2^{2n}, t]$.

To treat the case that $t \in [t_0, t_0+2]$, we repeat the above argument with the interior Schauder estimate for Cauchy problems. For example, instead of (4.41), we use

$$\begin{aligned} &\sum_{\ell=0}^{2}\left\|\nabla^\ell\phi\right\|_{C^{\sigma_0,\sigma_0/2}(\Omega'_{11})} + \|\phi_t\|_{C^{\sigma_0,\sigma_0/2}(\Omega'_{11})} \\ &\qquad\qquad \le C\left(\|\phi\|_{L^\infty(\Omega'_{12})} + \|f(x,t)\|_{C^{\sigma_0,\sigma_0/2}(\Omega'_{12})} + \sum_{\ell=0}^{2}\left\|\nabla^\ell\phi_0\right\|_{C^{\sigma_0}(\{|y|\le 8\})}\right) \end{aligned}$$

where $\phi_0(y) := \psi_0(x)$ for $y \in B^N\left(-\xi, \frac{\delta_0}{4\mu}\right)$,

$$\Omega'_{11} := \{|y| \le 6\} \times [t_0, t_0+2] \quad \text{and} \quad \Omega'_{12} := \{|y| \le 8\} \times [t_0, t_0+2].$$

Then, the corresponding estimate to (4.43) is

$$\left|\nabla^\ell\psi(x,t)\right| + \mu_0^{\sigma_0}\left[\nabla^\ell\psi\right]_{C_z^{\sigma_0}}(x,t) + \left[\nabla^\ell\psi\right]_{C_t^{\sigma_0/2}}(x,t) \le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^\rho + \|\psi_0\|_{**,\alpha;\sigma_0}\mu_0^\rho(t_0)\right]\mu_0^{-\ell},$$

and we can deduce an analogous estimate to (4.44), (4.46), and (4.47).

Step 2. We examine the case that $d_{g_0}(z, z_0) \ge \frac{\delta_0}{12}$.

Define a function $\tilde\psi$ on $M \times [t_0, \infty)$ by

$$\psi(z,t) = \tilde\psi(z, s(t)) \quad \text{where } s(t) := \delta_0^4 t^2.$$

It solves

$$\tilde\psi_s = \frac{N-2}{8\delta_0^4 t}u_{\mu,\xi}^{1-p}\Delta_{g_0}\tilde\psi + \frac{1}{2p\,\delta_0^4 t}u_{\mu,\xi}^{1-p}\left[\frac{N+2}{4}(S+h) + \mathcal{W}_{\mu,\xi}\right]\tilde\psi + \frac{1}{2p\,\delta_0^4 t}f(z,t).$$

If $d_{g_0}(z,z_0) \ge \frac{\delta_0}{24}$, a coefficient function $(\delta_0^4 t)^{-1}u_{\mu,\xi}^{1-p}$ is bounded from above, bounded away from 0 (because $(\delta_0^4 t)^{-1}u_{\mu,\xi}^{1-p} \simeq t^{-1}\mu_0^{-2} \simeq 1$), and uniformly Lipschitz. Therefore, we conclude from the parabolic Schauder estimate (for Cauchy problems) that

$$\begin{cases} \left|\nabla^\ell \psi(z,t)\right| = \left|\nabla^\ell \tilde{\psi}(z,s)\right| \le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^\rho + \|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\right]\delta_0^{-(\alpha+\ell)}\mu_0^\alpha, \\ \delta_0^{\sigma_0}\left[\nabla^\ell\psi\right]_{C_z^{\sigma_0}}(x,t) + (\delta_0^{-1}\mu_0)^{\sigma_0}\left[\nabla^\ell\psi\right]_{C_t^{\sigma_0}}(x,t) \\ \qquad \le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^\rho + \|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\right]\delta_0^{-(\alpha+\ell)}\mu_0^\alpha \end{cases} \tag{4.48}$$

for $\ell \in \{0,1,2\}$ and

$$\begin{cases} |\psi_t(z,t)| \le 2\delta_0^4 t|\tilde{\psi}_s(z,s)| \le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^\rho + \|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\right]\delta_0^{2-\alpha}\mu_0^{\alpha-2} \\ \delta_0^{\sigma_0}[\psi_t]_{C_z^{\sigma_0}}(x,t) + (\delta_0^{-1}\mu_0)^{\sigma_0}[\psi_t]_{C_t^{\sigma_0}}(x,t) \\ \qquad \le C\left[\delta_0^{-2}\|f\|_{*,\alpha,\rho;\sigma_0}\mu_0^\rho + \|\psi_0\|_{**,\alpha}\mu_0^\rho(t_0)e^{-\delta_1(t-t_0)}\right]\delta_0^{2-\alpha}\mu_0^{\alpha-2} \end{cases} \tag{4.49}$$

provided $d_{g_0}(z,z_0) \ge \frac{\delta_0}{12}$.

Combining (4.43)–(4.49), we find the desired gradient estimates (4.37) and (4.38). □

4.3. **Unique solvability of** (3.2) **and a priori estimate of the solution.** Denote $\mathcal{B}_{\bar\mu,\xi} := B^N(-\bar\mu^{-1}\xi, 2\bar\mu^{-1}\mu_0^{\varepsilon_1})$. Let $\hat\psi^{\rm in}$ be a function on $B^N(-0, 2\mu_0^{\varepsilon_1}) \times [t_0,\infty)$ and $\psi^{\rm in}$ be a function on $\mathcal{B}_{\bar\mu,\xi} \times [t_0,\infty)$ such that

$$\hat\psi^{\rm in}(x,t) = (1+P(x))\bar\mu^{-\frac{N-2}{2}}\psi^{\rm in}(\bar y,t)\,; \tag{4.50}$$

cf. (3.5). We remark that $\psi^{\rm in}$ needs not to be a solution to (3.6) at this moment. Letting also ψ_0 be a function on M, we assume that

$$\left\|\psi^{\rm in}\right\|_{\sharp',a,b;\sigma_0(\mathcal{B}_{\bar\mu,\xi})} \le C_{21} \quad \text{and} \quad \|\psi_0\|_{**,\alpha;\sigma_0} \le C_{22}\mu_0^{\delta_2}(t_0) \tag{4.51}$$

for some C_{21}, $C_{22} > 0$, $a \in (\sigma_0, N-2)$, $b \simeq 3$, $\alpha \in (0,a)$ to be determined later, and an arbitrarily chosen $\delta_2 > 0$; the $\sharp'$- and $**$-norms are given in (1.19) and (1.16), respectively, and $\sigma_0 \in (0,1)$ is the number appearing in (2.7). By employing the existence theory and a priori estimates for the inhomogeneous problem (4.1), we shall prove the unique solvability of the outer problem (3.2), i.e.,

$$\begin{cases} pu_{\mu,\xi}^{p-1}\left(\psi^{\rm out}\right)_t = \dfrac{N+2}{4}L_{g_0,h}\psi^{\rm out} + \mathcal{W}_{\mu,\xi}\psi^{\rm out} + u_{\mu,\xi}^{p-1}f_{\mu,\xi}^{\rm out}\left[\psi^{\rm out},\hat\psi^{\rm in}\right] & \text{on } M\times(t_0,\infty), \\ \psi^{\rm out}(\cdot,t_0) = \psi_0 & \text{on } M \end{cases} \tag{4.52}$$

where

$$f_{\mu,\xi}^{\rm out}\left[\psi^{\rm out},\hat\psi^{\rm in}\right] := u_{\mu,\xi}^{1-p}\left[\left(1-\eta_{\mu_0^{\varepsilon_1}}\right)\mathcal{S}(u_{\mu,\xi}) + \mathcal{J}_1\left[\hat\psi^{\rm in}\right] + \mathcal{J}_2\left[\psi^{\rm out},\hat\psi^{\rm in}\right]\right]. \tag{4.53}$$

Proposition 4.8. *Assume that* (4.51) *and* (2.5)–(2.7) *hold, and* $t_0 > 0$ *is sufficiently large. Suppose also that* $\alpha \in (0,a)$ *and* $\beta \in (0,b]$ *satisfy* $\alpha+\beta = N-2$, *and set* $\rho = -\frac{N-2}{2}+\beta = \frac{N-2}{2}-\alpha$. *Choose any*

$$\delta_3 \in (0, \min\{2,\ \beta,\ \beta(p-1),\ a+b-(N-2),\ (p-1)(a+b-(N-2))\}]\,. \tag{4.54}$$

Then (4.52) *possesses a unique solution* $\psi^{\rm out} = \psi^{\rm out}[\lambda,\xi,\dot\lambda,\dot\xi,\hat\psi^{\rm in},\psi_0]$ *such that*

$$\left\|\psi^{\rm out}\right\|_{*',\alpha,\rho;\sigma_0} \le C\delta_0^{-2}\mu_0^{\delta_4}(t_0) \tag{4.55}$$

where $C > 0$ *is a constant depending only on* (M,g_0), N, h, z_0, α, *and* σ_0,

$$\delta_4 \in (0, \min\{\nu_2 - \zeta_2\varepsilon_1,\ 2(p-1), \delta_3\}) \cap (0, \min\{(1-\varepsilon_1)(a-\alpha), \delta_2\}], \tag{4.56}$$

and the $*'$*-norm is defined in* (1.17).

The proof of Proposition 4.8 consists of several lemmas. Let $\mathbf{1}_{N=5}$ be a quantity equal to 1 if $N=5$ and 0 if $N \ge 6$.

Lemma 4.9. *We have*

$$\left\| u_{\mu,\xi}^{1-p} \left(1 - \eta_{\mu_0^{\varepsilon_1}}\right) \mathcal{S}(u_{\mu,\xi}) \right\|_{*,\alpha,\rho;\sigma_0} \le C_{31} \delta_0^{-\zeta_{31}} \mu_0^{\nu_2 - \zeta_2 \varepsilon_1}(t_0) \tag{4.57}$$

where

- C_{31}, $\zeta_{31} > 0$ *are constants depending only on* (M, g_0), N, h, z_0, α, *and* σ_0*;*
- $\varepsilon_1 \in (0,1)$ *is the small number in Lemma 2.4,* $\zeta_2 > 0$ *is the number depending only on* N *in Lemma 2.9, and* $\nu_2 = 2 - \varepsilon_0$ *is the number in* (2.7)*;*
- *the* $*$*-norm is defined in* (1.15)*.*

Proof. Write $\widetilde{\mathcal{S}} = \left(1 - \eta_{\mu_0^{\varepsilon_1}}\right) \mathcal{S}(u_{\mu,\xi})$. Applying (2.47) and (2.5)–(2.7), we observe

$$\sum_{\ell=0}^{1} \left|\nabla_z^\ell \widetilde{\mathcal{S}}(z,t)\right| + \left[\left(1 - \eta_{\mu_0^{\varepsilon_1}}\right) \widetilde{\mathcal{S}}\right]_{C_t^{\sigma_0/2}}(z,t) \le C\delta_0^{-\zeta} \left(\mu_0^{\frac{N+2}{2} - \zeta_2\varepsilon_1} + \mu_0^{\frac{N-2}{2} + \nu_2 - \zeta_2\varepsilon_1}\right).$$

Hence

$$\left[(\mu_0^\rho w_{\alpha,2})^{-1} \left|\widetilde{\mathcal{S}}(z,t)\right| + (\mu_0^\rho w_{\alpha,2+\sigma_0})^{-1} \left[\widetilde{\mathcal{S}}(z,t)\right]_{C_z^{\sigma_0}} + (\mu_0^\rho w_{\alpha-\sigma_0,2})^{-1} \left[\widetilde{\mathcal{S}}(z,t)\right]_{C_t^{\sigma_0/2}}\right](z,t)$$
$$\le C\delta_0^{-\zeta} \mu_0^{-\alpha-\rho} \mu_0^{\frac{N-2}{2} + \nu_2 - \zeta_2\varepsilon_1} = C\delta_0^{-\zeta} \mu_0^{\nu_2 - \zeta_2\varepsilon_1},$$

which reads (4.57). □

Lemma 4.10. *We have*

$$\left\| u_{\mu,\xi}^{1-p} \mathcal{J}_1 \left[\psi^{\text{out}}, \hat{\psi}^{\text{in}}\right] \right\|_{*,\alpha,\rho;\sigma_0} \le C_{32} \mu_0^{\delta_4}(t_0) \left\|\psi^{\text{in}}\right\|_{\sharp',a,b;\sigma_0(\mathcal{B}_{\bar{\mu},\xi})} \tag{4.58}$$

where $C_{32} > 0$ *depends only on* (M, g_0), N, h, z_0, α, *and* σ_0, *and* $\delta_4 > 0$ *is a number satisfying* (4.56).

Proof. It suffices to consider only points in the set $\{x \in \mathbb{R}^N : |x| \le 2\mu_0^{\varepsilon_1}\} \supset \text{supp}(\eta_{\mu_0^{\varepsilon_1}})$.

In view of (2.8)–(2.9), (2.39)–(2.40), (2.13), and (2.37), we have

$$\begin{aligned}\left|u_{\mu,\xi}^{p-1} - \left(u_{\mu,\xi}^{(1)}\right)^{p-1}\right|(x,t) &= (1 + P(x))^{p-1} \mu^{-2} \left|(W_{1,0} + \Psi_0)^{p-1} - W_{1,0}^{p-1}\right|(y,t) \\ &\le C\mu_0^{-2} \left[\mathbf{1}_{N=5}\, \bar{\mu}^2 \left(W_{1,0}^{p-2} Q_0\right)(y) + \left(\bar{\mu}^2 Q_0(y)\right)^{p-1}\right] \\ &\le C \left[\mathbf{1}_{N=5} \log(2 + |y|) + \mu_0^{-2+\frac{8}{N-2}} \log(2 + |y|)^{p-1}\right] \frac{1}{1 + |y|^4}.\end{aligned}$$

Besides, by (4.50),

$$\left|\hat{\psi}^{\text{in}}(x,t)\right| \le C\bar{\mu}^{-\frac{N-2}{2}} \left|\psi^{\text{in}}(\bar{y},t)\right| \le C\mu_0^{-\frac{N-2}{2}} \left\|\psi^{\text{in}}\right\|_{\sharp',a,b(\mathcal{B}_{\bar{\mu},\xi})} \frac{\mu_0^b}{1 + |\bar{y}|^a} \tag{4.59}$$

and

$$\left|\hat{\psi}_t^{\text{in}}(x,t)\right| \le C\mu_0^{-\frac{N-2}{2}} \left\|\psi^{\text{in}}\right\|_{\sharp',a,b(\mathcal{B}_{\bar{\mu},\xi})} \frac{\mu_0^b}{1 + |\bar{y}|^{a-2}}$$

where $x = \mu y + \xi = \bar{\mu}\bar{y} + \xi$. Accordingly,

$$\begin{aligned}&\left\| p u_{\mu,\xi}^{1-p} \eta_{\mu_0^{\varepsilon_1}} \left[u_{\mu,\xi}^{p-1} - \left(u_{\mu,\xi}^{(1)}\right)^{p-1}\right] \left[\frac{(N+2)\kappa_N}{4} \left|\hat{\psi}^{\text{in}}\right| + \left|\hat{\psi}_t^{\text{in}}\right|\right] \right\|_{*,\alpha,\rho} \\ &\le C \left[\mathbf{1}_{N=5}\, \mu_0^2 |\log\mu_0| + (\mu_0^2 |\log\mu_0|)^{p-1}\right] \left\|\psi^{\text{in}}\right\|_{\sharp',a,b(\mathcal{B}_{\bar{\mu},\xi})} \sup_{(\bar{y},t) \in \mathcal{B}_{\bar{\mu},\xi} \times [t_0,\infty)} \frac{\mu_0^{b-\beta}}{1 + |\bar{y}|^{a-\alpha}} \qquad (4.60) \\ &\le C \left[\mathbf{1}_{N=5}\, \mu_0^2 |\log\mu_0| + (\mu_0^2 |\log\mu_0|)^{p-1}\right](t_0) \left\|\psi^{\text{in}}\right\|_{\sharp',a,b(\mathcal{B}_{\bar{\mu},\xi})}\end{aligned}$$

where the second inequality comes from the assumption that $\alpha \in (0,a)$ and $\beta \in (0,b]$.

On the other hand, we see

$$\begin{aligned}
&\left|\left(\Delta_{g_0}\eta_{\mu_0^{\varepsilon_1}}\right)\hat{\psi}^{\text{in}}\right|(x,t) + \left[\left|\nabla_{g_0}\eta_{\mu_0^{\varepsilon_1}}\right|\left|\nabla_{g_0}\hat{\psi}^{\text{in}}\right|\right](x,t) \\
&\le C\mu_0^{-\frac{N-2}{2}}\left\|\psi^{\text{in}}\right\|_{\sharp',a,b(\mathcal{B}_{\bar{\mu},\xi})}\left(\mu_0^{-2\varepsilon_1}\frac{\mu_0^b}{1+|\bar{y}|^a} + \mu_0^{-\varepsilon_1}\frac{\mu_0^b}{1+|\bar{y}|^{a+1}}\right) \\
&\le C\mu_0^{-\frac{N+2}{2}}\left\|\psi^{\text{in}}\right\|_{\sharp',a,b(\mathcal{B}_{\bar{\mu},\xi})}\frac{\mu_0^b}{1+|\bar{y}|^{a+2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\|u_{\mu,\xi}^{1-p}\cdot\frac{(N+2)\kappa_N}{4}\left[\left(\Delta_{g_0}\eta_{\mu_0^{\varepsilon_1}}\right)\hat{\psi}^{\text{in}} + 2\left\langle\nabla_{g_0}\eta_{\mu_0^{\varepsilon_1}},\nabla_{g_0}\hat{\psi}^{\text{in}}\right\rangle_{g_0}\right]\right\|_{*,\alpha,\rho} \\
&\le C\left\|\psi^{\text{in}}\right\|_{\sharp',a,b(\mathcal{B}_{\bar{\mu},\xi})}\sup_{\left\{(\bar{y},t):\bar{\mu}^{-1}\mu_0^{\varepsilon_1}\le|\bar{y}+\bar{\mu}^{-1}\xi|\le 2\bar{\mu}^{-1}\mu_0^{\varepsilon_1},\, t\ge t_0\right\}}\frac{\mu_0^{b-\beta}}{1+|\bar{y}|^{a-\alpha}} \qquad (4.61)\\
&\le C\mu_0^{(1-\varepsilon_1)(a-\alpha)}(t_0)\left\|\psi^{\text{in}}\right\|_{\sharp',a,b(\mathcal{B}_{\bar{\mu},\xi})}.
\end{aligned}$$

Combining (4.60) and (4.61) gives

$$\left\|u_{\mu,\xi}^{1-p}\mathcal{J}_1\left[\psi^{\text{out}},\hat{\psi}^{\text{in}}\right]\right\|_{*,\alpha,\rho} \le C\mu_0^{\delta_4}(t_0)\left\|\psi^{\text{in}}\right\|_{\sharp',a,b(\mathcal{B}_{\bar{\mu},\xi})}.$$

Further computations yield a weighted Hölder bound of $\mathcal{J}_1$ and so (4.58). We omit the details. □

Lemma 4.11. *We have*

$$\begin{aligned}
\left\|u_{\mu,\xi}^{1-p}\mathcal{J}_2\left[\psi^{\text{out}},\hat{\psi}^{\text{in}}\right]\right\|_{*,\alpha,\rho;\sigma_0} \le C_{33}\delta_0^{-\zeta_{33}}\Bigg[&\mathbf{1}_{N=5}\Bigg\{\left(\left\|\psi^{\text{out}}\right\|_{*',\alpha,\rho;\sigma_0} + \left\|\psi^{\text{in}}\right\|_{\sharp',a,b;\sigma_0(\mathcal{B}_{\bar{\mu},\xi})}\right)\mu_0^{4-3\varepsilon_1}(t_0) \\
&+\left(\left\|\psi^{\text{out}}\right\|_{*',\alpha,\rho;\sigma_0}^2 + \left\|\psi^{\text{in}}\right\|_{\sharp',a,b;\sigma_0(\mathcal{B}_{\bar{\mu},\xi})}^2\mu_0^{\delta_3}(t_0)\right)\Bigg\} \\
&+\Bigg\{\left(\left\|\psi^{\text{out}}\right\|_{*',\alpha,\rho;\sigma_0}^{p-1} + \left\|\psi^{\text{in}}\right\|_{\sharp',a,b;\sigma_0(\mathcal{B}_{\bar{\mu},\xi})}^{p-1}\right)\mu_0^{4-(N-2)\varepsilon_1}(t_0) \qquad (4.62)\\
&+\left(\left\|\psi^{\text{out}}\right\|_{*',\alpha,\rho;\sigma_0}^{p} + \left\|\psi^{\text{in}}\right\|_{\sharp',a,b;\sigma_0(\mathcal{B}_{\bar{\mu},\xi})}^{p}\mu_0^{\delta_3}(t_0)\right)\Bigg\}\Bigg]
\end{aligned}$$

where $C_{33}, \zeta_{33} > 0$ depends only on (M, g_0), N, h, z_0, α, and σ_0, and $\delta_3 > 0$ is the number satisfying (4.54)*.*

Proof. By (3.4), (3.1), and (4.4),

$$\mathcal{J}_2\left[\psi^{\text{out}},\hat{\psi}^{\text{in}}\right] = A_1 + A_2 \quad \text{on } M\times[t_0,\infty)$$

where

$$|A_1| \le C\left[\mathbf{1}_{N=5}u_{\mu,\xi}^{p-2}\left\{\left|\psi^{\text{out}}\right|^2 + \eta_{\mu_0^{\varepsilon_1}}^2\left|\hat{\psi}^{\text{in}}\right|^2\right\} + \left\{\left|\psi^{\text{out}}\right|^p + \eta_{\mu_0^{\varepsilon_1}}^p\left|\hat{\psi}^{\text{in}}\right|^p\right\}\right]$$

and

$$\begin{aligned}
|A_2| \le C\Bigg[&\mathbf{1}_{N=5}u_{\mu,\xi}^{p-2}\left\{\left|\psi^{\text{out}}\right| + \eta_{\mu_0^{\varepsilon_1}}\left|\hat{\psi}^{\text{in}}\right|\right\} + \left\{\left|\psi^{\text{out}}\right|^{p-1} + \eta_{\mu_0^{\varepsilon_1}}^{p-1}\left|\hat{\psi}^{\text{in}}\right|^{p-1}\right\}\Bigg] \\
&\times\left[\mu_0^{\frac{N+2}{2}-(N-2)\varepsilon_1} + \left|\psi_t^{\text{out}}\right| + \eta_{\mu_0^{\varepsilon_1}}\left|\hat{\psi}_t^{\text{in}}\right|\right].
\end{aligned}$$

Let us estimate the term A_1. We have

$$\left(\mu_0^\rho w_{\alpha,2}\right)^{-1}\left[\mathbf{1}_{N=5}u_{\mu,\xi}^{p-2}\left|\psi^{\text{out}}\right|^2 + \left|\psi^{\text{out}}\right|^p\right]$$

$$\begin{aligned}&\le C\left[\mathbf{1}_{N=5}\left\|\psi^{\rm out}\right\|_{*',\alpha,\rho}^2 \mu_0^{\rho} u_{\mu,\xi}^{p-2}\left(w_{\alpha,2}^{-1} w_{\alpha,0}^2\right)+\left\|\psi^{\rm out}\right\|_{*',\alpha,\rho}^p \mu_0^{\rho(p-1)}\left(w_{\alpha,2}^{-1} w_{\alpha,0}^p\right)\right] \\ &\le C\delta_0^{-\zeta}\left[\mathbf{1}_{N=5}\left\|\psi^{\rm out}\right\|_{*',\alpha,\rho}^2\left(\mu_0^{\beta}+\mu_0^2\right)+\left\|\psi^{\rm out}\right\|_{*',\alpha,\rho}^p\left(\mu_0^{\beta(p-1)}+\mu_0^2\right)\right]\end{aligned}$$

on $M\times[t_0,\infty)$. In addition, for $x=\mu y+\xi\in\operatorname{supp}(\eta_{\mu_0^{\varepsilon_1}})$, it holds that

$$\begin{aligned}&(\mu_0^{\rho} w_{\alpha,2})^{-1}\left[\mathbf{1}_{N=5} u_{\mu,\xi}^{p-2}\eta_{\mu_0^{\varepsilon_1}}^2\left|\hat\psi^{\rm in}\right|^2+\eta_{\mu_0^{\varepsilon_1}}^p\left|\hat\psi^{\rm in}\right|^p\right](x,t) \\ &\le C\left[\mathbf{1}_{N=5}\left\|\psi^{\rm in}\right\|_{\sharp',a,b(\mathcal{B}_{\bar\mu,\xi})}^2 \mu_0^{2b+\alpha-3}\left(1+|y|^{\alpha+1-2a}\right)+\left\|\psi^{\rm in}\right\|_{\sharp',a,b(\mathcal{B}_{\bar\mu,\xi})}^p \mu_0^{bp-\beta}\left(1+|y|^{\alpha+2-ap}\right)\right] \\ &\le C\left[\mathbf{1}_{N=5}\left\|\psi^{\rm in}\right\|_{\sharp',a,b(\mathcal{B}_{\bar\mu,\xi})}^2\left(\mu_0^{2b-\beta}+\mu_0^{2(a+b-2)}\right)+\left\|\psi^{\rm in}\right\|_{\sharp',a,b(\mathcal{B}_{\bar\mu,\xi})}^p\left(\mu_0^{bp-\beta}+\mu_0^{(a+b)p-N}\right)\right]\end{aligned}$$

where we applied (4.59) for the first inequality.

Arguing as above, we can also estimate

$$\begin{aligned}&(\mu_0^{\rho} w_{\alpha,2})^{-1}|A_2| \\ &\le C\delta_0^{-\zeta}\left[\mathbf{1}_{N=5}\left\{\left(\|\psi^{\rm out}\|_{*',\alpha,\rho}+\|\psi^{\rm in}\|_{\sharp',a,b(\mathcal{B}_{\bar\mu,\xi})}\right)\mu_0^{4-3\varepsilon_1}+\left(\|\psi^{\rm out}\|_{*',\alpha,\rho}^2+\|\psi^{\rm in}\|_{\sharp',a,b(\mathcal{B}_{\bar\mu,\xi})}^2\mu_0^{\delta_3}(t_0)\right)\right\}\right. \\ &\quad\left.+\left\{\left(\|\psi^{\rm out}\|_{*',\alpha,\rho}^{p-1}+\|\psi^{\rm in}\|_{\sharp',a,b(\mathcal{B}_{\bar\mu,\xi})}^{p-1}\right)\mu_0^{4-(N-2)\varepsilon_1}+\left(\|\psi^{\rm out}\|_{*',\alpha,\rho}^{p}+\|\psi^{\rm in}\|_{\sharp',a,b(\mathcal{B}_{\bar\mu,\xi})}^{p}\mu_0^{\delta_3}(t_0)\right)\right\}\right]\end{aligned}$$

on $M\times[t_0,\infty)$. To deduce the inequality, we repeatedly use the relations $\alpha+\beta=N-2$, $\alpha\in(0,a)$, and $\beta\in(0,b]$.

Combining all the computations together, we establish a weighted L^∞-estimate for $\mathcal{J}_2$, that is, (4.62) where the weighted Hölder norms are replaced with the associated weighted L^∞-norms.

Further computations yield a weighted Hölder bound of $\mathcal{J}_2$ and so (4.62). We omit the details. □

Completion of the proof of Proposition 4.8. For the parameters (μ,ξ) satisfying (2.5)–(2.7) and the function ψ_0 satisfying (4.51), let $\mathcal{T}_{\mu,\xi}^{\rm out}[f,\psi_0]$ be a unique solution to (4.1) obtained in Corollary 4.4. A function $\psi^{\rm out}$ is a solution to (4.52) if and only if it satisfies

$$\psi^{\rm out}=\mathcal{T}_{\mu,\xi}^{\rm out}\left[f_{\mu,\xi}^{\rm out}\left[\psi^{\rm out},\hat\psi^{\rm in}\right],\psi_0\right] \tag{4.63}$$

where $f_{\mu,\xi}^{\rm out}$ is the map given in (4.53). We shall prove the existence of a fixed point of the operator $\mathcal{T}_{\mu,\xi}^{\rm out}[f_{\mu,\xi}^{\rm out}[\cdot,\hat\psi^{\rm in}],\psi_0]$ by applying the contraction mapping theorem on the set

$$\mathcal{D}^{\rm out}:=\left\{\psi^{\rm out}:\|\psi^{\rm out}\|_{*',\alpha,\rho;\sigma_0}\le 2C_1(C_{21}C_{32}+C_{22})\delta_0^{-2}\mu_0^{\delta_4}(t_0)\right\} \tag{4.64}$$

where C_1, C_{21}, C_{22}, C_{32}, $C_{33}>0$ are the numbers appearing in (4.39), (4.51), (4.58), and (4.62).

By employing (4.39), (3.3), (3.4), (4.57), (4.58), (4.62), (4.51), and (4.56), and taking a larger $t_0>0$ if required, we obtain

$$\begin{aligned}&\left\|\mathcal{T}_{\mu,\xi}^{\rm out}\left[f_{\mu,\xi}^{\rm out}\left[\psi^{\rm out},\hat\psi^{\rm in}\right],\psi_0\right]\right\|_{*',\alpha,\rho;\sigma_0} \\ &\le C_1\delta_0^{-2}\left(\left\|f_{\mu,\xi}^{\rm out}\left[\psi^{\rm out},\hat\psi^{\rm in}\right]\right\|_{*,\alpha,\rho;\sigma_0}+\|\psi_0\|_{**,\alpha;\sigma_0}\right) \\ &\le C_1\delta_0^{-2}\left[C_{31}\delta_0^{-\zeta_{31}}\mu_0^{\nu_2-\zeta_2\varepsilon_1}(t_0)+C_{21}C_{32}\mu_0^{\delta_4}(t_0)+C_{22}\mu_0^{\delta_2}(t_0)\right. \\ &\qquad+C_{33}\delta_0^{-\zeta_{33}}\left\{\mathbf{1}_{N=5}\left(2C_{21}\mu_0^{4-3\varepsilon_1}(t_0)+\left\|\psi^{\rm out}\right\|_{*',\alpha,\rho;\sigma_0}^2+C_{21}^2\mu_0^{\delta_3}(t_0)\right)\right. \\ &\qquad\qquad\left.\left.+\left(2C_{21}^{p-1}\mu_0^{4-(N-2)\varepsilon_1}(t_0)+\left\|\psi^{\rm out}\right\|_{*',\alpha,\rho;\sigma_0}^p+C_{21}^p\mu_0^{\delta_3}(t_0)\right)\right\}\right] \\ &\le 2C_1(C_{21}C_{32}+C_{22})\delta_0^{-2}\mu_0^{\delta_4}(t_0)\end{aligned}$$

for any $\psi \in \mathcal{D}^{\text{out}}$. Furthermore,

$$\begin{aligned}
&\left\|\mathcal{T}^{\text{out}}_{\mu,\xi}\left[f^{\text{out}}_{\mu,\xi}\left[\psi^{\text{out}}_1, \hat{\psi}^{\,\text{in}}\right], \psi_0\right] - \mathcal{T}^{\text{out}}_{\mu,\xi}\left[f^{\text{out}}_{\mu,\xi}\left[\psi^{\text{out}}_2, \hat{\psi}^{\,\text{in}}\right], \psi_0\right]\right\|_{*',\alpha,\rho;\sigma_0} \\
&\le C_1\delta_0^{-2}\left\|\mathcal{J}_2\left[\psi^{\text{out}}_1, \hat{\psi}^{\,\text{in}}\right] - \mathcal{J}_2\left[\psi^{\text{out}}_2, \hat{\psi}^{\,\text{in}}\right]\right\|_{*,\alpha,\rho;\sigma_0} \\
&\le C\delta_0^{-2}\left[\left\|\left(u_{\mu,\xi} + \psi^{\text{out}}_1\right)^p_+ - \left(u_{\mu,\xi} + \psi^{\text{out}}_2\right)^p_+ - pu^{p-1}_{\mu,\xi}\left(\psi^{\text{out}}_1 - \psi^{\text{out}}_2\right)\right\|_{*,\alpha,\rho;\sigma_0}\right. \\
&\quad + \left\|\left\{\left(u_{\mu,\xi} + \psi^{\text{out}}_1\right)^{p-1}_+ - \left(u_{\mu,\xi} + \psi^{\text{out}}_2\right)^{p-1}_+\right\}\left\{\left(1 - \eta_{\mu_0^{\varepsilon_1}}\right)(u_{\mu,\xi})_t + \left(\psi^{\text{out}}_1\right)_t\right\}\right\|_{*,\alpha,\rho;\sigma_0} \\
&\quad \left. + \left\|\left\{\left(u_{\mu,\xi} + \psi^{\text{out}}_2\right)^{p-1}_+ - (u_{\mu,\xi})^{p-1}_+\right\}\left(\psi^{\text{out}}_1 - \psi^{\text{out}}_2\right)_t\right\|_{*,\alpha,\rho;\sigma_0}\right] \\
&\le C\delta_0^{-\zeta}\left[\mathbf{1}_{N=5}\mu_0^{\delta_4}(t_0) + \mu_0^{\delta_4(p-1)}(t_0)\right]\left\|\psi^{\text{out}}_1 - \psi^{\text{out}}_2\right\|_{*',\alpha,\rho;\sigma_0}
\end{aligned} \tag{4.65}$$

for any $\psi_1, \psi_2 \in \mathcal{D}^{\text{out}}$. Therefore, the operator $\mathcal{T}^{\text{out}}_{\mu,\xi}[f^{\text{out}}_{\mu,\xi}[\cdot, \hat{\psi}^{\,\text{in}}], \psi_0]$ is a contraction map on $\mathcal{D}^{\text{out}}$, and it has a fixed point in $\mathcal{D}^{\text{out}}$, namely, a point satisfying (4.63). Finally, (4.55) follows directly from (4.64). □

4.4. **Derivatives of the solution to** (3.2) **with respect to parameters.** In the following proposition, we provide quantitative estimates on the derivatives of ψ^{out} with respect to its parameters.

Proposition 4.12. *Suppose that all the conditions of Proposition 4.8 hold. There exists a constant C, $\zeta > 0$ depending only on (M, g_0), N, h, z_0, α, and σ_0 such that*

$$\begin{cases}
\left\|\partial_\lambda\psi^{\text{out}}\left[\bar{\lambda}\right]\right\|_{*',\alpha,\rho;\sigma_0} \le C\delta_0^{-\zeta}\mu_0^{\delta_4}(t_0)\mu_0^{\nu_1-1}\left\|\bar{\lambda}\right\|_{\nu_1}, \\
\left\|\partial_\xi\psi^{\text{out}}\left[\bar{\xi}\right]\right\|_{*',\alpha,\rho;\sigma_0} \le C\delta_0^{-\zeta}\mu_0^{\delta_4}(t_0)\mu_0^{\nu_2-1}\left\|\bar{\xi}\right\|_{\nu_2}, \\
\left\|\partial_{\dot{\lambda}}\psi^{\text{out}}[\dot{\bar{\lambda}}]\right\|_{*',\alpha,\rho;\sigma_0} \le C\delta_0^{-\zeta}\mu_0^{\nu_1+3-\zeta\varepsilon_1}\left\|\dot{\bar{\lambda}}\right\|_{\nu_1+2}, \\
\left\|\partial_{\dot{\xi}}\psi^{\text{out}}[\dot{\bar{\xi}}]\right\|_{*',\alpha,\rho;\sigma_0} \le C\delta_0^{-\zeta}\mu_0^{\nu_2+4-\zeta\varepsilon_1}\left\|\dot{\bar{\xi}}\right\|_{\nu_2+2}, \\
\left\|\partial_{\hat{\psi}^{\,\text{in}}}\psi^{\text{out}}\left[\bar{\psi}\right]\right\|_{*',\alpha,\rho;\sigma_0} \le C\delta_0^{-\zeta}\mu_0^{\delta_4}(t_0)\left\|\mu_0^{-\frac{N-2}{2}}\bar{\psi}\right\|_{\sharp',a,b;\sigma_0}
\end{cases} \tag{4.66}$$

and

$$\left\|\psi^{\text{out}}[\psi_{01}] - \psi^{\text{out}}[\psi_{02}]\right\|_{*',\alpha,\rho;\sigma_0} \le C\delta_0^{-2}\left\|\psi_{01} - \psi_{02}\right\|_{**,\alpha;\sigma_0} \tag{4.67}$$

Here, we write

$$\partial_\lambda\psi^{\text{out}}[\bar{\lambda}] := \partial_s\psi^{\text{out}}\left[\lambda + s\bar{\lambda}, \xi, \dot{\lambda}, \dot{\xi}, \hat{\psi}^{\,\text{in}}, \psi_0\right]\Big|_{s=0}, \quad \psi^{\text{out}}[\psi_0] := \psi^{\text{out}}[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \hat{\psi}^{\,\text{in}}, \psi_0],$$

and so on. Also, the ν- and $\sharp'$-norms were defined in (1.13) *and* (1.19)*, respectively.*

Proof. The implicit function theorem implies that the functions $\partial_\lambda\psi^{\text{out}}[\bar{\lambda}], \ldots, \partial_{\hat{\psi}^{\,\text{in}}}\psi^{\text{out}}\left[\bar{\psi}\right]$ are well-defined in a neighborhood of given data $(\lambda, \xi, \dot{\lambda}, \dot{\xi}, \hat{\psi}^{\,\text{in}}, \psi_0)$. Examining the equations that the functions solve, we can deduce (4.66). We skip the details, because the necessary computations are similar to those in the proof of Proposition 4.8; refer to [14, Proposition 4.2] where an analogous result was obtained for critical nonlinear heat equation.

Furthermore, by (4.39) and (4.65),

$$\begin{aligned}
&\left\|\psi^{\text{out}}[\psi_{01}] - \psi^{\text{out}}[\psi_{02}]\right\|_{*',\alpha,\rho;\sigma_0} \\
&\le C\delta_0^{-2}\left(\left\|\mathcal{J}_2\left[\psi^{\text{out}}[\psi_{01}], \hat{\psi}^{\text{in}}\right] - \mathcal{J}_2\left[\psi^{\text{out}}[\psi_{02}], \hat{\psi}^{\text{in}}\right]\right\|_{*,\alpha,\rho;\sigma_0} + \left\|\psi_{01} - \psi_{02}\right\|_{**,\alpha;\sigma_0}\right) \\
&\le C\delta_0^{-2}\left(o(1)\left\|\psi^{\text{out}}[\psi_{01}] - \psi^{\text{out}}[\psi_{02}]\right\|_{*',\alpha,\rho;\sigma_0} + \left\|\psi_{01} - \psi_{02}\right\|_{**,\alpha;\sigma_0}\right).
\end{aligned}$$

Here, the term $o(1)$ tends to 0 as $t_0 \to \infty$. Hence, by enlarging t_0 if necessary, we deduce (4.67). □

5. Inner problem and the completion of the proof of Theorem 1.1

This section is devoted to the analysis of the inner problem (3.6), which will allow us to complete the proof of the main theorem.

In Subsection 5.1, we establish the unique solvability and a priori estimate of the non-uniform inhomogeneous parabolic equation

$$\begin{cases} pW_{1,0}^{p-1}\psi_t = \dfrac{(N+2)\kappa_N}{4}\left(\Delta\psi + pW_{1,0}^{p-1}\psi\right) + W_{1,0}^{p-1}\mathcal{H} & \text{in } \mathbb{R}^N \times (t_0,\infty), \\ \psi(\cdot,t_0) = e_0(t_0)Z_0 & \text{in } \mathbb{R}^N \end{cases} \tag{5.1}$$

under the assumption that

$$\int_{\mathbb{R}^N} \mathcal{H}(\bar{y},t)\left(W_{1,0}^{p-1}Z_n\right)(\bar{y})d\bar{y} = 0 \quad \text{for } t \in [t_0,\infty) \text{ and } n = 1,\dots,N+1. \tag{5.2}$$

Here,

- $e_0(t_0)$ is a real number to be determined by $\mathcal{H}$;
- $Z_1,\dots,Z_{N+1}$ are the functions in (2.3) and (2.34);
- Z_0 is the unique positive radial function in $\mathbb{R}^N$ such that

$$\begin{cases} L_0[Z_0] = W_{1,0}^{1-p}\left(\Delta Z_0 + pW_{1,0}^{p-1}Z_0\right) = \mathfrak{m}_0 Z_0 \quad \text{in } \mathbb{R}^N, \\ \mathfrak{m}_0 > 0, \ \displaystyle\int_{\mathbb{R}^N} W_{1,0}^{p-1}Z_0^2 = 1. \end{cases}$$

It is clear that $\mathfrak{m}_0 = p-1$ and Z_0 is a constant multiple of $W_{1,0}$; refer to [3, Appendix].

Given parameters (λ,ξ), and functions ψ^{in} and ψ_0 satisfying (2.5)–(2.7) and (4.51), respectively, let $\psi^{\text{out}} = \psi^{\text{out}}[\lambda,\xi,\dot{\lambda},\dot{\xi},\hat{\psi}^{\text{in}},\psi_0]$ be the unique solution to (4.52) found in Proposition 4.8. In Subsection 5.2, we reduce

$$\int_{\mathbb{R}^N} \eta_{2\mu_0^{\varepsilon_1}}(x)\left[\left(\frac{\bar{\mu}}{\mu}\right)^{\frac{N-2}{2}}(1+P)^{-p}(x)\mathcal{E}_2[\mu,\xi](y,t) + \mathcal{K}_1\left[\psi^{\text{in}}\right] + \mathcal{K}_2\left[\psi^{\text{in}},\psi^{\text{out}}\right]\right]Z_n(\bar{y})d\bar{y} = 0 \tag{5.3}$$

for $n = 1,\dots,N+1$ and $t \in [t_0,\infty)$ into a system of nonlinear ODEs (5.30) and (5.39) of (λ,ξ) and solve it; refer to (1.12), (2.42), (3.1), (3.5), (3.7), (3.8), and Subsection 2.1 for the definition of the notations.

In Subsection 5.3, we solve (3.6) by applying (5.3), the unique solvability of (5.1) under (5.2), and the contraction mapping theorem.

In Subsection 5.4, we complete the proof of Theorem 1.1.

5.1. Linear theory.

In this subsection, we examine (5.1) provided $a \in (\sigma_0, N-2)$ and $b \simeq 3$.

Proposition 5.1. *Suppose that $\|\mathcal{H}\|_{\sharp,a+2,b;\sigma_0} < \infty$ and (5.2) holds. Then one can find $\psi = \psi[\mathcal{H}]$ and $e_0 = e_0[\mathcal{H}] \in \mathbb{R}$ satisfying (5.1). These ψ and e_0 are linear in $\mathcal{H}$, and*

$$\|\psi\|_{\sharp',a,b;\sigma_0(\mathbb{R}^N)} + \|e_0\|_{b;\sigma_0} \le C_4\|\mathcal{H}\|_{\sharp,a+2,b;\sigma_0} \tag{5.4}$$

for a constant $C_4 > 0$ depending only on N, a, b and σ_0. Here, the norms of ψ, e_0, and $\mathcal{H}$ are given in Definitions 1.9 and 1.6.

As an intermediate step to proving the proposition, we will analyze a uniformly parabolic equation on $\mathbb{S}^N$

$$\begin{cases} p\,\varphi_t = \dfrac{\kappa_N p}{N}\left(\Delta_{\mathbb{S}^N}\varphi + N\varphi\right) + \widetilde{\mathcal{H}} - \tilde{c}_0(t) & \text{on } \mathbb{S}^N \times (t_0,s_0), \\ \varphi(\tilde{y},t_0) = 0 & \text{on } \mathbb{S}^N \end{cases} \tag{5.5}$$

for $s_0 > \frac{3t_0}{2}$ large enough. Here, $\widetilde{\mathcal{H}}$ is a function satisfying

$$\left\|\widetilde{\mathcal{H}}\right\|_{\widetilde{\sharp},a,b;t_0,s_0} := \left\|\mu_0^{-b}[d_{\mathbb{S}^N}(\tilde{y})]^{N-a}\widetilde{\mathcal{H}}(\tilde{y},t)\right\|_{L^\infty(\mathbb{S}^N\times[t_0,s_0))} < \infty,$$

where $d_{\mathbb{S}^N}(\tilde{y}) := \arccos(\tilde{y}_{N+1})$ is the geodesic distance on $\mathbb{S}^N$ between $\tilde{y}_{\text{north}} := (0,\dots,0,1)$ and $\tilde{y}$, and

$$\int_{\mathbb{S}^N} \widetilde{\mathcal{H}}(\tilde{y},t)\tilde{y}_n dS_{\tilde{y}} = 0 \quad \text{for } t \in [t_0,s_0) \text{ and } n = 1,\dots,N+1. \tag{5.6}$$

Also, $\tilde{c}_0$ is defined as

$$\tilde{c}_0(t) = \frac{1}{|\mathbb{S}^N|}\int_{\mathbb{S}^N} \widetilde{\mathcal{H}}(\tilde{y},t)dS_{\tilde{y}} \quad \text{for } t \in [t_0,s_0). \tag{5.7}$$

By integrating the equation in (5.5) over $\mathbb{S}^N$, and applying (5.7) and the initial condition on φ, we find

$$\int_{\mathbb{S}^N} \varphi(\tilde{y},t)dS_{\tilde{y}} = 0 \quad \text{for } t \in [t_0,s_0). \tag{5.8}$$

Moreover, testing $\tilde{y}_n$ on (5.5) for $n = 1,\dots,N+1$, and exploiting (5.6) and the fact that N is the second eigenvalue of $-\Delta_{\mathbb{S}^N}$ with eigenfunctions $\tilde{y}_1,\dots,\tilde{y}_{N+1}$, we obtain

$$\int_{\mathbb{S}^N} \varphi(\tilde{y},t)\tilde{y}_n dS_{\tilde{y}} = 0 \quad \text{for } t \in [t_0,s_0) \text{ and } n = 1,\dots,N+1. \tag{5.9}$$

Lemma 5.2. *Assume that* $\left\|\widetilde{\mathcal{H}}\right\|_{\widetilde{\sharp},a,b;t_0,s_0} < \infty$ *for* $s_0 > \frac{3t_0}{2}$ *large, and* (5.6) *holds. If* φ *is a solution to* (5.5) *such that* φ, $D_{\tilde{y}}\varphi$, $D^2_{\tilde{y}}\varphi$, $\varphi_t \in L^2(\mathbb{S}^N\times(t_0,s_0))$, *then there exists a constant* $C > 0$ *depending only on* N, a *and* b *such that*

$$\|\varphi\|_{\widetilde{\sharp},a+2,b;t_0,s_0} + \sup_{t\in[t_0,s_0)} \mu_0^{-b}(t)|\tilde{c}_0(t)| \le C\left\|\widetilde{\mathcal{H}}\right\|_{\widetilde{\sharp},a,b;t_0,s_0}. \tag{5.10}$$

Proof. To deduce the lemma, we will argue as in the proof of [48, Lemma 4.1]; see Remark 1.4 (3). The proof is divided into two steps.

Throughout the proof, C denotes a universal constant independent of s_0.

STEP 1. We insist that

$$\sup_{t\in[t_0,s_0)} \mu_0^{-b}(t)|\tilde{c}_0(t)| \le C\left\|\widetilde{\mathcal{H}}\right\|_{\widetilde{\sharp},a,b;t_0,s_0} \quad \text{and} \quad \|\varphi\|_{\widetilde{\sharp},a+2,b;t_0,s_0} < \infty. \tag{5.11}$$

The first inequality in (5.11) is obvious. Let us consider the second one. If $\delta > 0$ is any small number, parabolic regularity theory ensures the existence of $K_1 > 0$ depending on φ and $\widetilde{\mathcal{H}}$ such that

$$\|\varphi\|_{C^0\left((\mathbb{S}^N\setminus B_{\mathbb{S}^N}(\tilde{y}_{\text{north}},\delta))\times[t_0,s_0)\right)} \le K_1.$$

Also, the geodesic distance $d_{\mathbb{S}^N}(\tilde{y}) = \arccos(\tilde{y}_{N+1})$ on $\mathbb{S}^N$ between $\tilde{y}_{\text{north}}$ and $\tilde{y}$ satisfies

$$\Delta_{\mathbb{S}^N}[d_{\mathbb{S}^N}(\tilde{y})]^{a-(N-2)} = ((N-2)-a)\left(-a+O(\delta^2)\right)[d_{\mathbb{S}^N}(\tilde{y})]^{a-N} \quad \text{in } B_{\mathbb{S}^N}(\tilde{y}_{\text{north}},\delta).$$

Therefore, if we set

$$v(\tilde{y},t) = K_2\left\|\widetilde{\mathcal{H}}\right\|_{\widetilde{\sharp},a,b;t_0,s_0}\mu_0^b(t)[d_{\mathbb{S}^N}(\tilde{y})]^{a-(N-2)} \quad \text{in } B_{\mathbb{S}^N}(\tilde{y}_{\text{north}},\delta)\times(t_0,s_0),$$

then

$$\begin{cases} (v\pm\varphi)_t \ge \dfrac{\kappa_N}{N}\left[\Delta_{\mathbb{S}^N}(v\pm\varphi) + N(v\pm\varphi)\right] & \text{in } B_{\mathbb{S}^N}(\tilde{y}_{\text{north}},\delta)\times(t_0,s_0),\\ (v\pm\varphi)(\tilde{y},t_0) = v(\tilde{y},t_0) \ge 0 & \text{in } B_{\mathbb{S}^N}(\tilde{y}_{\text{north}},\delta),\\ v\pm\varphi \ge 0 & \text{on } \partial B_{\mathbb{S}^N}(\tilde{y}_{\text{north}},\delta)\times(t_0,s_0) \end{cases}$$

provided $K_2 \gg K_1$. The parabolic maximum principle implies the second inequality in (5.11).

STEP 2. We assert that

$$\|\varphi\|_{\widetilde{\sharp},a+2,b;t_0,s_0} \le C \left\|\widetilde{\mathcal{H}}\right\|_{\widetilde{\sharp},a,b;t_0,s_0}. \tag{5.12}$$

To the contrary, suppose that there exist increasing sequences $\{t_\ell\}_{\ell\in\mathbb{N}}$, $\{s_\ell\}_{\ell\in\mathbb{N}}$ of positive numbers for which (4.14) holds, and sequences $\{\varphi_\ell\}_{\ell\in\mathbb{N}}$, $\{\widetilde{\mathcal{H}}_\ell\}_{\ell\in\mathbb{N}}$ of functions which satisfy

$$\begin{cases} p\,(\varphi_\ell)_t = \dfrac{\kappa_N p}{N}\left(\Delta_{\mathbb{S}^N}\varphi_\ell + N\varphi_\ell\right) + \widetilde{\mathcal{H}}_\ell - \tilde{c}_\ell(t) & \text{on } \mathbb{S}^N \times (t_\ell, s_\ell), \\ \varphi_\ell(\tilde{y}, t_\ell) = 0 & \text{on } \mathbb{S}^N, \end{cases} \tag{5.13}$$

$$\int_{\mathbb{S}^N} \widetilde{\mathcal{H}}_\ell(\tilde{y},t)\tilde{y}_n dS_{\tilde{y}} = 0 \quad \text{for } t \in [t_\ell, s_\ell) \text{ and } n = 1,\dots,N+1, \tag{5.14}$$

and

$$\|\varphi_\ell\|_{\widetilde{\sharp},a+2,b;t_\ell,s_\ell} = 1, \quad \left\|\widetilde{\mathcal{H}}_\ell\right\|_{\widetilde{\sharp},a,b;t_\ell,s_\ell} \to 0 \quad \text{as } \ell \to \infty. \tag{5.15}$$

Here, $\tilde{c}_\ell$ is defined as

$$\tilde{c}_\ell(t) := \frac{1}{|\mathbb{S}^N|}\int_{\mathbb{S}^N} \widetilde{\mathcal{H}}_\ell(\tilde{y},t)dS_{\tilde{y}} \quad \text{for } t \in [t_\ell, s_\ell). \tag{5.16}$$

By (5.8), (5.9) and (5.15), we have

$$\int_{\mathbb{S}^N} \varphi_\ell(\tilde{y},t)dS_{\tilde{y}} = \int_{\mathbb{S}^N} \varphi_\ell(\tilde{y},t)\tilde{y}_n dS_{\tilde{y}} = 0 \quad \text{for } t \in [t_\ell, s_\ell) \text{ and } n = 1,\dots,N+1, \tag{5.17}$$

and there is a sequence $\{(\tilde{y}_\ell, \tau_\ell)\}_{\ell\in\mathbb{N}}$ such that $\{\tau_\ell\}_{\ell\in\mathbb{N}}$ is increasing and

$$\begin{cases} \tilde{y}_\ell = (\tilde{y}_{\ell 1}, \dots, \tilde{y}_{\ell(N+1)}) \in \mathbb{S}^N, \quad \lim\limits_{\ell\to\infty} \tilde{y}_\ell = \tilde{y}_\infty \in \mathbb{S}^N, \quad \tau_\ell \in (t_\ell, s_\ell), \\ \dfrac{1}{2} \le \mu_0^{-b}(\tau_\ell)[d_{\mathbb{S}^N}(\tilde{y}_\ell)]^{N-2-a}|\varphi_\ell(\tilde{y}_\ell, \tau_\ell)| \le 1 \quad \text{for all } \ell \in \mathbb{N}. \end{cases} \tag{5.18}$$

Let us check that $\tilde{y}_\infty = \tilde{y}_{\text{north}}$. If not, we would have a small number $\delta > 0$ such that $d_{\mathbb{S}^N}(\tilde{y}_\ell) \ge \delta$ for all large $\ell \in \mathbb{N}$. Let $\chi : \mathbb{S}^N \to [0,1]$ be a smooth function such that $\chi(\tilde{y}) = 1$ if $d_{\mathbb{S}^N}(\tilde{y}) \ge \frac{\delta}{2}$, $\chi(\tilde{y}) = 0$ if $d_{\mathbb{S}^N}(\tilde{y}) \le \frac{\delta}{4}$, and $|\nabla\chi| \le \frac{4}{\delta}$ on $\mathbb{S}^N$. We test (5.13) against $\chi^2\varphi_\ell$ for each $\tau \in (t_\ell, s_\ell)$, integrate the result over (t_ℓ, τ), and apply (5.11), the initial condition on φ_ℓ, (5.15), and Grönwall's inequality. Then we obtain

$$\int_{\left\{d_{\mathbb{S}^N}(\tilde{y}) \ge \frac{\delta}{2}\right\}} \varphi_\ell^2(\tilde{y},\tau)dS_{\tilde{y}} \le o(1)\left[\int_{t_\ell}^{\tau} \mu_0^{2b}(t)dt\right] e^{C(\tau - t_\ell)} \quad \text{for } \tau \in [t_\ell, s_\ell).$$

Combining this with parabolic estimates yields that $\liminf_{\ell\to\infty}(\tau_\ell - t_\ell) = \infty$, as otherwise

$$\begin{aligned} &\mu_0^{-b}(\tau_\ell)\|\varphi_\ell\|_{C^0(\{d_{\mathbb{S}^N}(\tilde{y})\ge\delta\}\times[\tau_\ell-1,\tau_\ell])} \\ &\le C\mu_0^{-b}(\tau_\ell)\left[\|\varphi_\ell\|_{L^2(\{d_{\mathbb{S}^N}(\tilde{y})\ge\frac{\delta}{2}\}\times[\tau_\ell-2,\tau_\ell])} + \left\|\widetilde{\mathcal{H}}_\ell\right\|_{L^\infty(\{d_{\mathbb{S}^N}(\tilde{y})\ge\frac{\delta}{2}\}\times[\tau_\ell-2,\tau_\ell])} + \|\tilde{c}_\ell\|_{L^\infty([\tau_\ell-2,\tau_\ell])}\right] \\ &\le o(1)\left[\left\{\int_{\tau_\ell-2}^{\tau_\ell}\int_{t_\ell}^{\tau}\left(\frac{\tau_\ell}{t}\right)^b e^{C(\tau-t_\ell)}dtd\tau\right\}^{\frac{1}{2}} + 1\right] = o(1), \end{aligned}$$

contradicting (5.18). It follows that the limit equation of the sequence $\{\mu_0^{-b}(t+\tau_\ell)\,\varphi_\ell(\tilde{y}, t+\tau_\ell)\}_{\ell\in\mathbb{N}}$ is

$$\begin{cases} (\bar{\varphi}_\infty)_t = \dfrac{\kappa_N}{N}\left(\Delta_{\mathbb{S}^N}\bar{\varphi}_\infty + N\bar{\varphi}_\infty\right) & \text{on } \mathbb{S}^N \times (-\infty, 0), \\ |\bar{\varphi}_\infty(\tilde{y},t)| \le [d_{\mathbb{S}^N}(\tilde{y})]^{a-(N-2)} & \text{for } (\tilde{y},t) \in \mathbb{S}^N \times (-\infty,0), \end{cases} \tag{5.19}$$

and it holds that $\bar{\varphi}_\infty(\tilde{y}_\infty, 0) \ne 0$ and

$$\int_{\mathbb{S}^N} \bar{\varphi}_\infty(\tilde{y},t)dS_{\tilde{y}} = \int_{\mathbb{S}^N} \bar{\varphi}_\infty(\tilde{y},t)\tilde{y}_n dS_{\tilde{y}} = 0 \quad \text{for } t \in (-\infty, 0) \text{ and } n = 1,\dots,N+1. \tag{5.20}$$

By the growth condition on $\bar{\varphi}_\infty$ in (5.19) and parabolic regularity, $\bar{\varphi}_\infty$ is in fact smooth on $\mathbb{S}^N \times (-\infty, 0)$ and $\|\bar{\varphi}_\infty\|_{C^\infty(\mathbb{S}^N \times (-\infty,0))} \le C$. Also, (5.20) gives

$$\mathfrak{B}(\bar{\varphi}_\infty(\cdot,t)) \ge 0 \quad \text{for } t \in (-\infty, 0) \quad \text{where } \mathfrak{B}(\bar{\varphi}) := \int_{\mathbb{S}^N} \left(|\nabla_{\mathbb{S}^N} \bar{\varphi}|^2 - N\bar{\varphi}^2\right) dS_{\tilde{y}}.$$

Testing $(\bar{\varphi}_\infty)_t$ on (5.19) and the equation of $(\bar{\varphi}_\infty)_t$, respectively, we get

$$\int_{-t}^0 \int_{\mathbb{S}^N} (\bar{\varphi}_\infty)_t^2 = \frac{\kappa_N}{2N} \left[\mathfrak{B}(\bar{\varphi}_\infty(\cdot,-t)) - \mathfrak{B}(\bar{\varphi}_\infty(\cdot,0))\right] \le C\|\bar{\varphi}_\infty\|_{C^1((\mathbb{S}^N \times (-\infty,0)))} \le C$$

and

$$\partial_t \int_{\mathbb{S}^N} (\bar{\varphi}_\infty)_t^2 = -\mathfrak{B}(\bar{\varphi}_\infty(\cdot,t)) \le 0$$

for all $t \in (-\infty, 0]$. Hence $(\bar{\varphi}_\infty)_t = 0$ on $\mathbb{S}^N \times (-\infty, 0]$. By plugging it into (5.19) and employing (5.20) again, we conclude that $\bar{\varphi}_\infty = 0$ on $\mathbb{S}^N \times (-\infty, 0]$. This is a contradiction, and we must have that $\tilde{y}_\infty = \tilde{y}_{\text{north}}$. Especially, $\nu_\ell := d_{\mathbb{S}^N}(\tilde{y}_\ell) \to 0$ and $(\tilde{y}_{\ell 1}, \ldots, \tilde{y}_{\ell N}) \to 0 \in \mathbb{R}^N$ as $\ell \to \infty$.

Now, we regard φ_ℓ as the function in $B^N(0, \frac{1}{2}) \times (t_\ell, s_\ell)$, abusing the notation

$$\varphi_\ell(y,t) = \varphi_\ell\left(\left(y, \sqrt{1-|y|^2}\right), t\right) \quad \begin{cases} \text{for } (y,t) \in B^N\left(0, \frac{1}{2}\right) \times (t_\ell, s_\ell) \\ \text{so that } \left(\left(y, \sqrt{1-|y|^2}\right), t\right) \in \mathbb{S}^N \times (t_\ell, s_\ell), \end{cases}$$

and define

$$\tilde{\varphi}_\ell(Y,\tau) = \mu_0^{-b}(\tau_\ell)\nu_\ell^{N-2-a} \varphi_\ell\left(\tilde{y}_{\ell 1} + \nu_\ell Y_1, \ldots, \tilde{y}_{\ell N} + \nu_\ell Y_N, \tau_\ell + \nu_\ell^2 \tau\right)$$
$$\text{for } Y = (Y_1, \ldots, Y_N) \in B^N\left(0, \frac{1}{4\nu_\ell}\right) \text{ and } \tau \in \left(\frac{t_\ell - \tau_\ell}{\nu_\ell^2}, 0\right].$$

As $\ell \to \infty$, $\tilde{\varphi}_\ell$ tends to a function $\tilde{\varphi}_\infty$ in $\mathbb{R}^N \times (-\infty, 0]$ which satsifies

$$\begin{cases} (\tilde{\varphi}_\infty)_\tau = \dfrac{\kappa_N}{N} \Delta \tilde{\varphi}_\infty & \text{in } \mathbb{R}^N \times (-\infty, 0), \\ |\tilde{\varphi}_\infty(Y,\tau)| \le |Y - Y_0|^{a-(N-2)} & \text{for } (Y,\tau) \in \mathbb{R}^N \times (-\infty, 0] \text{ and some } Y_0 \in \mathbb{S}^{N-1}, \end{cases} \tag{5.21}$$

and $\frac{1}{2} \le |\tilde{\varphi}_\infty(0,0)| \le 1$. However, a standard comparison argument shows that the only solution to (5.21) is the trivial one. This is a contradiction, and (5.12) must hold.

Inequality (5.10) is an immediate consequence of (5.11) and (5.12). □

Corollary 5.3. *Assume that* $\left\|\widetilde{\mathcal{H}}\right\|_{\widetilde{\sharp},a,b} := \left\|\widetilde{\mathcal{H}}\right\|_{\widetilde{\sharp},a,b;t_0,\infty} < \infty$, *and* (5.6) *and* (5.7) *hold for* $s_0 = \infty$. *Then one can find a solution* φ *to the equation*

$$\begin{cases} p\,\varphi_t = \dfrac{\kappa_N p}{N} \left(\Delta_{\mathbb{S}^N} \varphi + N\varphi\right) + \widetilde{\mathcal{H}} - \tilde{c}_0(t) & \text{on } \mathbb{S}^N \times (t_0, \infty), \\ \varphi(\tilde{y}, t_0) = 0 & \text{on } \mathbb{S}^N, \end{cases} \tag{5.22}$$

which automatically satisfies (5.8) *and* (5.9) *for* $s_0 = \infty$. *Also,* $\varphi = \varphi[\widetilde{\mathcal{H}}]$ *and* $\tilde{c}_0 = \tilde{c}_0[\widetilde{\mathcal{H}}]$ *are linear in* $\widetilde{\mathcal{H}}$, *and*

$$\|\varphi\|_{\widetilde{\sharp},a+2,b} + \sup_{t \in [t_0,\infty)} \mu_0^{-b}(t)|\tilde{c}_0(t)| \le C \left\|\widetilde{\mathcal{H}}\right\|_{\widetilde{\sharp},a,b} \tag{5.23}$$

for a constant $C > 0$ *depending only on* N, a *and* b.

Proof. Given an increasing sequence $\{s_\ell\}_{\ell \in \mathbb{N}}$ such that $\frac{3t_0}{2} < s_\ell \to \infty$ as $\ell \to \infty$, let us consider the problem

$$\begin{cases} p\,(\varphi_\ell)_t = \dfrac{\kappa_N p}{N} \left(\Delta_{\mathbb{S}^N} \varphi_\ell + N\varphi_\ell\right) + \widetilde{\mathcal{H}}_\ell - \tilde{c}_\ell(t) & \text{on } \mathbb{S}^N \times (t_0, s_\ell), \\ \varphi_\ell(\tilde{y}, t_0) = 0 & \text{on } \mathbb{S}^N. \end{cases} \tag{5.24}$$

Here,

$$\widetilde{\mathcal{H}}_\ell(\tilde{y},t) := \min\left\{\widetilde{\mathcal{H}}(\tilde{y},t),\ell\right\} - \sum_{n=1}^{N+1} \tilde{d}_{n\ell}(t)\tilde{y}_n \in L^\infty(\mathbb{S}^N \times (t_0,s_\ell)),$$

$$\tilde{d}_{n\ell}(t) := \frac{N+1}{|\mathbb{S}^N|}\int_{\mathbb{S}^N} \min\left\{\widetilde{\mathcal{H}}(\tilde{y},t),\ell\right\}\tilde{y}_n dS_{\tilde{y}} \quad \text{for } t \in (t_0,s_\ell) \text{ and } n = 1,\dots,N+1,$$

and $\tilde{c}_\ell$ is the quantity defined by (5.16) in which t_ℓ is replaced with t_0. Clearly, (5.14) holds provided $t_\ell = t_0$, and $\tilde{d}_{n\ell}(t) \to 0$ and $\widetilde{\mathcal{H}}_\ell \to \widetilde{\mathcal{H}}$ a.e. as $\ell \to \infty$.

By applying the Galerkin method, we can construct a solution φ_ℓ to (5.24) such that φ_ℓ, $D_{\tilde{y}}\varphi_\ell$, $D^2_{\tilde{y}}\varphi_\ell$, $(\varphi_\ell)_t \in L^2(\mathbb{S}^N \times (t_0,s_\ell))$. Lemma 5.2 tells us that such φ_ℓ is unique and

$$\|\varphi_\ell\|_{\tilde{\sharp},a+2,b;t_0,s_\ell} + \sup_{t\in[t_0,s_\ell)} \mu_0^{-b}(t)|\tilde{c}_\ell(t)| \le C\left\|\widetilde{\mathcal{H}}_\ell\right\|_{\tilde{\sharp},a,b;t_0,s_\ell} \le C\left\|\widetilde{\mathcal{H}}\right\|_{\tilde{\sharp},a,b} \tag{5.25}$$

for a constant $C > 0$ independent of $\ell \in \mathbb{N}$. By (5.14), we also have (5.17) provided $t_\ell = t_0$.

Fix $\varepsilon \in (0, \frac{a}{N-a})$. Given any $s > t_0$, parabolic regularity and (5.25) yield that

$$\left\||\varphi_\ell| + |D_{\tilde{y}}\varphi_\ell| + |D^2_{\tilde{y}}\varphi_\ell| + |(\varphi_\ell)_t|\right\|_{L^{1+\varepsilon}(\mathbb{S}^N\times(t_0,s))} \le C$$

for some $C > 0$ independent of $\ell \in \mathbb{N}$. Therefore, we have a function φ_∞ on $\mathbb{S}^N \times (t_0,\infty)$ such that

$$\varphi_\ell \to \varphi_\infty \begin{cases} \text{weakly in } W^{1,1+\varepsilon}_{\text{loc}}(\mathbb{S}^N \times (t_0,\infty)), \\ \text{strongly in } L^{1+\varepsilon}_{\text{loc}}(\mathbb{S}^N \times (t_0,\infty)), \\ \text{a.e. on } \mathbb{S}^N \times (t_0,\infty) \end{cases} \qquad \text{as } \ell \to \infty,$$

along a subsequence. In particular, φ_∞ satisfies (5.22) with $\tilde{c}_0(t) = \lim_{\ell\to\infty} \tilde{c}_\ell(t)$. Setting $\varphi = \varphi_\infty$, we also infer from (5.17) and (5.25) that (5.8)–(5.9) for $s_0 = \infty$ and (5.23) are valid.

The linear dependence of φ and $\tilde{c}_0$ in $\widetilde{\mathcal{H}}$ is obvious. This completes the proof. □

Corollary 5.4. *Assume that $\|\mathcal{H}\|_{\sharp,a+2,b} < \infty$ and (5.2) holds. Then one can find a solution ϕ to the equation*

$$\begin{cases} pW_{1,0}^{p-1}\phi_t = \dfrac{(N+2)\kappa_N}{4}\left(\Delta\phi + pW_{1,0}^{p-1}\phi\right) \\ \qquad\qquad\qquad + W_{1,0}^{p-1}\mathcal{H} - c_0(t)W_{1,0}^{p-1}Z_0 & \text{in } \mathbb{R}^N \times (t_0,\infty), \\ \phi(\bar{y},t_0) = 0 & \text{in } \mathbb{R}^N \end{cases} \tag{5.26}$$

for a suitable c_0, which also satisfies

$$\int_{\mathbb{R}^N} \phi(\bar{y},t)\left(W_{1,0}^{p-1}Z_n\right)(\bar{y})d\bar{y} = 0 \quad \text{for } t \in [t_0,\infty) \text{ and } n = 0,\dots,N+1. \tag{5.27}$$

Also, $\phi = \phi[\mathcal{H}]$ and $c_0 = c_0[\mathcal{H}]$ are linear in $\mathcal{H}$, and

$$\|\phi\|_{\sharp',a,b(\mathbb{R}^N)} + \sup_{t\in[t_0,\infty)} \mu_0^{-b}(t)|c_0(t)| \le C\|\mathcal{H}\|_{\sharp,a+2,b} \tag{5.28}$$

for a constant $C > 0$ depending only on N, a and b.

Proof. Let $\Pi : \mathbb{R}^N \to \mathbb{S}^N_n$ and $\Pi_* f : \mathbb{S}^N_n \to \mathbb{R}$ be the inverse of the stereographic projection and the weighted push-forward of $f : \mathbb{R}^N \to \mathbb{R}$ in Definition 4.2, respectively.

We set $\widetilde{\mathcal{H}} = \Pi_*\mathcal{H}$ for each fixed $t \in [t_0,\infty)$. If $d_{\mathbb{S}^N}(\tilde{y})$ is the geodesic distance on $\mathbb{S}^N$ between $\tilde{y}_{\text{north}} = (0,\dots,0,1)$ and $\tilde{y} = \Pi(\bar{y})$ for $\bar{y} \in \mathbb{R}^N$, then

$$d_{\mathbb{S}^N}(\tilde{y}) = \arccos\left(\frac{|\bar{y}|^2-1}{|\bar{y}|^2+1}\right) = \frac{2}{|\bar{y}|}\left[1 + O\left(\frac{1}{|\bar{y}|^2}\right)\right] \quad \text{for } |\bar{y}| \text{ large.}$$

Hence, $\|\mathcal{H}\|_{\sharp,a+2,b} < \infty$ is reduced to $\left\|\widetilde{\mathcal{H}}\right\|_{\widetilde{\sharp},a,b} < \infty$. Besides, each $\Pi_* Z_\ell$ being a constant multiple of $\tilde{y}_\ell$, (5.2) is transformed into (5.6) with $s_0 = \infty$.

Given the solution $(\varphi, \tilde{c}_0)$ to (5.22) deduced from Corollary 5.3, we set (ϕ, c_0) by

$$\varphi = \Pi_* \phi \quad \text{and} \quad \tilde{c}_0(t) := c_0(t)(\Pi_* Z_0) \quad \text{for } t \in [t_0, \infty),$$

noting that $\Pi_* Z_0$ is a positive constant. By virtue of conformality of the map Π, it satisfies (5.26)–(5.28). □

Corollary 5.5. *Suppose that all the assumptions on Proposition 5.1 hold. Let ϕ be a solution to* (5.26) *satisfying* (5.28). *Then there exists a constant $C > 0$ depending only on N, a, b and σ_0 such that*

$$\|\phi\|_{\sharp',a,b;\sigma_0(\mathbb{R}^N)} + \|c_0\|_{b;\sigma_0} \le C\|\mathcal{H}\|_{\sharp,a+2,b;\sigma_0}. \tag{5.29}$$

Proof. From the relation

$$c_0(t) = \int_{\mathbb{R}^N} W_{1,0}^{p-1} Z_0 \mathcal{H}(\bar{y}, t) d\bar{y} \quad \text{for } t \in [t_0, \infty),$$

it is easy to see that $[c_0]_{C_t^{\sigma_0/2}}$ is controlled by $\|\mathcal{H}\|_{\sharp,a+2,b;\sigma_0}$ provided $a > \sigma_0$. Using this fact and (5.28), one can estimate $\|\phi\|_{\sharp',a,b;\sigma_0(\mathbb{R}^N)}$ as in Step 1 in the proof of Corollary 4.7. We skip the details. □

Proof of Proposition 5.1. Let ϕ be the solution to (5.26) satisfying (5.27) and (5.29) found in Corollary 5.4. Also, we set $\psi = \phi + e_0(t)Z_0$ where

$$e_0(t) := -\frac{1}{p}\int_t^\infty \exp\left(\kappa_N(t-\tau)\right) c_0(\tau) d\tau \quad \text{for } t \in [t_0, \infty).$$

Then

$$\dot{e}_0 - \kappa_N e_0 = p^{-1} c_0 \quad \text{on } [t_0, \infty),$$

so ψ satisfies (5.1). Since the function c_0 is linear in $\mathcal{H}$, so is e_0. Moreover, by (5.29),

$$\|e_0\|_{b;\sigma_0} \le C \sup_{t\in[t_0,\infty)} \mu_0^{-b}(t)|c_0(t)| \le C\|\mathcal{H}\|_{\sharp,a+2,b}.$$

This together with the condition $a < N-2$ yields

$$\|\psi\|_{\sharp',a,b;\sigma_0(\mathbb{R}^N)} \le \|\phi\|_{\sharp',a,b;\sigma_0(\mathbb{R}^N)} + \|e_0 Z_0\|_{\sharp',a,b;\sigma_0(\mathbb{R}^N)} \le C\|\mathcal{H}\|_{\sharp,a+2,b;\sigma_0}$$

as desired. □

5.2. **Choice of parameters.** As mentioned at the beginning of this section, we reduce (5.3) into a system of nonlinear ODEs of $(\lambda, \xi_1, \dots, \xi_N)$. Recall ν_1 and ν_2 in (2.7), δ_4 in (4.56), and a, b, α, β, ρ, σ_0, and $\psi^{\text{out}} = \psi^{\text{out}}[\lambda, \xi, \dot{\lambda}, \dot{\xi}, \hat{\psi}^{\text{in}}, \psi_0]$ in Proposition 4.8.

Lemma 5.6. *Assume that* (4.51) *and* (2.5)–(2.7) *hold. For $n = 1, \dots, N$,* (5.3) *is equivalent to*

$$\begin{pmatrix} \dot{\xi}_1 \\ \vdots \\ \dot{\xi}_n \end{pmatrix} - \frac{N-4}{6h(z_0)t}\begin{pmatrix} R_{11}(z_0) & \cdots & R_{1N}(z_0) \\ \vdots & \ddots & \vdots \\ R_{N1}(z_0) & \cdots & R_{NN}(z_0) \end{pmatrix}\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix} = \mu_0^4 \begin{pmatrix} \Theta_{11} \\ \vdots \\ \Theta_{N1} \end{pmatrix} + \mu_0^{b+\nu_1}\begin{pmatrix} \Theta_{12} \\ \vdots \\ \Theta_{N2} \end{pmatrix} + \mu_0^{\delta_4}(t_0)\mu_0^{\beta+1}\begin{pmatrix} \Theta_{13} \\ \vdots \\ \Theta_{N3} \end{pmatrix} \tag{5.30}$$

on $[t_0, \infty)$. Here, $\Theta_n = \Theta_{n1}, \Theta_{n2}, \Theta_{n3}$ is a function on $[t_0, \infty)$ such that

- $\|\Theta_n\|_{0;\sigma_0} \le C_5 \delta_0^{-\zeta_5}$ *for some C_5, $\zeta_5 > 0$ where the norm is defined in* (1.13);

- *It depends on the parameters* λ, $\dot{\lambda}$, ξ, $\dot{\xi}$ *and the functions* ψ^{in}, ψ_0. *Besides, it satisfies*

$$\begin{cases} \|\Theta_n[\lambda_1]-\Theta_n[\lambda_2]\|_{0;\sigma_0} \le C\delta_0^{-\zeta}\|\lambda_1-\lambda_2\|_{\nu_1;\sigma_0}, \\ \|\Theta_n[\xi_1]-\Theta_n[\xi_2]\|_{0;\sigma_0} \le C\delta_0^{-\zeta}\|\xi_1-\xi_2\|_{\nu_2;\sigma_0}, \\ \left\|\Theta_n[\dot{\lambda}_1]-\Theta_n[\dot{\lambda}_2]\right\|_{0;\sigma_0} \le C\delta_0^{-\zeta}\|\dot{\lambda}_1-\dot{\lambda}_2\|_{\nu_1+2;\sigma_0}, \\ \left\|\Theta_n[\dot{\xi}_1]-\Theta_n[\dot{\xi}_2]\right\|_{0;\sigma_0} \le C\delta_0^{-\zeta}\|\dot{\xi}_1-\dot{\xi}_2\|_{\nu_2+2;\sigma_0}, \\ \left\|\Theta_n[\psi_1^{\text{in}}]-\Theta_n[\psi_2^{\text{in}}]\right\|_{0;\sigma_0} \le C\delta_0^{-\zeta}\left\|\psi_1^{\text{in}}-\psi_2^{\text{in}}\right\|_{\sharp',a,b;\sigma_0(\mathcal{B}_{\bar{\mu},\xi})} \end{cases} \tag{5.31}$$

and

$$\|\Theta_n[\psi_{01}]-\Theta_n[\psi_{02}]\|_{0;\sigma_0} \le C\delta_0^{-\zeta}\mu_0^{-\delta_4}(t_0)\,\|\psi_{01}-\psi_{02}\|_{**,\alpha;\sigma_0} \tag{5.32}$$

for $C, \zeta > 0$ *large depending only on* (M, g_0), N, h, z_0, a, b, α, *and* σ_0.

Proof. Throughout the proof, we will use the notation Θ_n to refer functions which behave as in the statement of the lemma. They may vary from line to line and even in the same line.

Fixing $n = 1, \dots, N$, we will examine

$$\begin{aligned} B_{n1} &:= \left(\frac{\bar{\mu}}{\mu}\right)^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \eta_{2\mu_0^{\varepsilon_1}}(x)(1+P)^{-p}(x)\mathcal{E}_2[\mu,\xi](y,t)Z_n(\bar{y})d\bar{y}, \\ B_{n2} &:= \int_{\mathbb{R}^N} \eta_{2\mu_0^{\varepsilon_1}}(x)\mathcal{K}_1\left[\psi^{\text{in}}\right]Z_n(\bar{y})d\bar{y}, \quad B_{n3} := \int_{\mathbb{R}^N} \eta_{2\mu_0^{\varepsilon_1}}(x)\mathcal{K}_2\left[\psi^{\text{in}},\psi^{\text{out}}\right]Z_n(\bar{y})d\bar{y} \end{aligned} \tag{5.33}$$

for $t \in [t_0, \infty)$, respectively. Here, $x = \mu y + \xi = \bar{\mu}\bar{y} + \xi$.

Estimate on B_{n1}: To estimate B_{n1}, we treat the quantity $\mathcal{E}_2[\mu,\xi](y,t)$ in (2.42) term by term.

First, we have

$$\begin{aligned} &\bar{\mu}^{-1}\left(\dot{\lambda}-\bar{\mu}^{-1}\dot{\bar{\mu}}\lambda\right)\left(\frac{\bar{\mu}}{\mu}\right)^{\frac{N-2}{2}} \int_{\mathbb{R}^N} \eta_{2\mu_0^{\varepsilon_1}}(x)(1+P)^{-p}(x)\left(pW_{1,0}^{p-1}Z_{N+1}\right)(y)Z_n(\bar{y})d\bar{y} \\ &= \mu_0^{\nu_1+1}\Theta_n\left(1+\mu_0^{\nu_1-1}\Theta_n\right)\left[\int_{\mathbb{R}^N}\left(\eta_{2\mu_0^{\varepsilon_1}}(x)-1\right)\left(pW_{1,0}^{p-1}Z_{N+1}Z_n\right)(\bar{y})d\bar{y}\right. \\ &\qquad + \int_{\mathbb{R}^N} \eta_{2\mu_0^{\varepsilon_1}}(x)\left\{\left(pW_{1,0}^{p-1}Z_{N+1}\right)(y)-\left(pW_{1,0}^{p-1}Z_{N+1}\right)(\bar{y})\right\}Z_n(\bar{y})d\bar{y} \\ &\qquad \left. + \int_{\mathbb{R}^N} \eta_{2\mu_0^{\varepsilon_1}}(x)\left\{(1+P)^{-p}(x)-1\right\}\left(pW_{1,0}^{p-1}Z_{N+1}\right)(y)Z_n(\bar{y})d\bar{y}\right] \\ &= \mu_0^{\nu_1+1}\Theta_n\left(\mu_0^{(N+1)(1-\varepsilon_1)}\Theta_n+\mu_0^{\nu_1-1}\Theta_n+\mu_0^2\Theta_n\right) = \mu_0^{2\nu_1}\Theta_n, \end{aligned}$$

and similarly,

$$\left(\frac{\bar{\mu}}{\mu}\right)^{\frac{N-2}{2}}\mu^{-1}\dot{\xi}\cdot\int_{\mathbb{R}^N}\eta_{2\mu_0^{\varepsilon_1}}(x)(1+P)^{-p}(x)\left(pW_{1,0}^{p-1}\nabla W_{1,0}\right)(y)Z_n(\bar{y})d\bar{y} = c_5\bar{\mu}^{-1}\dot{\xi}_n+\mu_0^{\nu_1+\nu_2}\Theta_n$$

where

$$c_5 := p\int_{\mathbb{R}^N} W_{1,0}^{p-1}Z_1^2 > 0. \tag{5.34}$$

Second, we infer from (2.19) and the parity that

$$2\bar{\mu}\lambda\left(\frac{\bar{\mu}}{\mu}\right)^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\eta_{2\mu_0^{\varepsilon_1}}(x)(1+P)^{-p}(x)\mathcal{F}_0(y)Z_n(\bar{y})d\bar{y} = 2\bar{\mu}\lambda\int_{\mathbb{R}^N}(\mathcal{F}_0Z_n)(\bar{y})d\bar{y}+\mu_0^{2\nu_1}\Theta_n = \mu_0^{2\nu_1}\Theta_n.$$

Third, (2.20) and the identity

$$\frac{2}{N-2}\int_{\mathbb{R}^N}(y\cdot\nabla W_{1,0}(y))W_{1,0}^p(y)dy = -\int_{\mathbb{R}^N}|\nabla W_{1,0}|^2$$

imply

$$\left(\frac{\bar\mu}{\mu}\right)^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\eta_{2\mu_0^{\varepsilon_1}}(x)(1+P)^{-p}(x)\mathcal{F}_1[\mu,\xi](y)Z_n(\bar y)d\bar y$$
$$=\int_{\mathbb{R}^N}(\mathcal{F}_1[\bar\mu,\xi]Z_n)(\bar y)d\bar y+\mu_0^{\nu_1+\nu_2}\Theta_n=-\frac{(N+2)\kappa_N}{12N}\bar\mu\xi_jR_{jn}(z_0)\int_{\mathbb{R}^N}|\nabla W_{1,0}|^2+\mu_0^{\nu_1+\nu_2}\Theta_n.$$

Lastly,

$$\left(\frac{\bar\mu}{\mu}\right)^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\eta_{2\mu_0^{\varepsilon_1}}(x)(1+P)^{-p}(x)\Big[\mu^3a^{\{1\}}+\mu^{2\nu_1}a^{\{2\}}$$
$$+\mu\dot\mu a^{\{0\}}+\mu^{\nu_1-2}\dot\lambda\, a^{\{-2\}}+\mu\dot\xi\cdot\mathbf{a}^{\{-1\}}\Big]Z_n(\bar y)d\bar y=\mu_0^3\Theta_n.$$

From the above calculations, (2.5), and (5.34), we conclude that

$$B_{n1}=c_5\bar\mu^{-1}\left[\dot\xi_n-\frac{N-4}{6h(z_0)t}R_{jn}(z_0)\xi_j\right]+\mu_0^3\Theta_n. \tag{5.35}$$

Estimate on B_{n2}: To estimate B_{n2}, we handle the quantity $\mathcal{K}_1[\psi^{\rm in}]$ in (3.7) term by term.

As an illustration, we consider the integrals involving the first and third terms of $\mathcal{K}_1[\psi^{\rm in}]$: The mean value theorem, (1.19), (4.51), and the assumption $a>\sigma_0$ lead to

$$\int_{\mathbb{R}^N}\eta_{2\mu_0^{\varepsilon_1}}(x)\left[W_{1,0}^{p-1}(\bar y)-W_{1,0}^{p-1}(y)\right]\left(\psi^{\rm in}\right)_tZ_n(\bar y)d\bar y=\mu_0^{b+\nu_1-1}\Theta_n$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}\eta_{2\mu_0^{\varepsilon_1}}(x)\left[\left(\frac{\mu}{\bar\mu}\right)^2(1+P)^{1-p}(x)(\Delta_{g_0(x)}\psi^{\rm in})(\bar y,t)-\Delta\psi^{\rm in}(\bar y,t)\right]Z_n(\bar y)d\bar y\\
&=\left(\frac{\mu}{\bar\mu}\right)^2\int_{\mathbb{R}^N}\eta_{2\mu_0^{\varepsilon_1}}(x)\left[(1+P)^{1-p}(x)-1\right]\left(\Delta_{g_0(x)}\psi^{\rm in}\right)(\bar y,t)Z_n(\bar y)d\bar y\\
&+\left[\left(\frac{\mu}{\bar\mu}\right)^2-1\right]\int_{\mathbb{R}^N}\eta_{2\mu_0^{\varepsilon_1}}(x)\left(\Delta_{g_0(x)}\psi^{\rm in}\right)(\bar y,t)Z_n(\bar y)d\bar y\\
&+\int_{\mathbb{R}^N}\eta_{2\mu_0^{\varepsilon_1}}(x)\left[\left(\Delta_{g_0(\bar\mu\cdot+\xi)}-\Delta\right)\psi^{\rm in}\right](\bar y,t)Z_n(\bar y)d\bar y\\
&=\mu_0^{b+1}\Theta_n+\mu_0^{b+\nu_1-1}\Theta_n+\left(\mu_0^{a+b+1-\sigma_0}\Theta_n+\mu_0^{b+2}\Theta_n\right)=\mu_0^{b+\nu_1-1}\Theta_n.\end{aligned} \tag{5.36}$$

In a similar manner, one can compute the integrals involving the remaining terms of $\mathcal{K}_1[\psi^{\rm in}]$, deriving

$$B_{n2}=\mu_0^{b+\nu_1-1}\Theta_n. \tag{5.37}$$

Estimate on B_{n3}: We consider two terms of $\mathcal{K}_2[\psi^{\rm in},\psi^{\rm out}]$ in (3.8) separately.

First, by (1.17) and (4.55),

$$\left(\frac{\bar\mu}{\mu}\right)^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\eta_{2\mu_0^{\varepsilon_1}}(x)(1+P)^{-1}(x)W_{1,0}^{p-1}(y)\mu^{\frac{N-2}{2}}\psi^{\rm out}(x,t)Z_n(\bar y)d\bar y=\mu_0^{\delta_4}(t_0)\mu_0^{\beta}\Theta_n.$$

Second, the mean value theorem yields

$$\left(\frac{\bar\mu}{\mu}\right)^{\frac{N-2}{2}}\int_{\mathbb{R}^N}\eta_{2\mu_0^{\varepsilon_1}}(x)\left[\left\{\left(W_{1,0}+\bar\mu^2Q_0\right)(y)+\left(\frac{\mu}{\bar\mu}\right)^{\frac{N-2}{2}}\psi^{\rm in}+(1+P)^{-1}(x)\mu^{\frac{N-2}{2}}\psi^{\rm out}(x,t)\right\}^{p-1}\right.$$
$$\left.-\left(W_{1,0}+\bar\mu^2Q_0\right)^{p-1}(y)\right]\eta_{\mu_0^{\varepsilon_1}/2}\,\mu^{\frac{N-2}{2}}\left\{\mu^{-\frac{N-2}{2}}\left(W_{1,0}+\bar\mu^2Q_0\right)(y)\right\}_tZ_n(\bar y)d\bar y$$

$= \mu_0^{b+2}\Theta_n + \mu_0^{\delta_4}(t_0)\mu_0^{\beta+2}\Theta_n.$

Consequently,

$$B_{n3} = \mu_0^{b+2}\Theta_n + \mu_0^{\delta_4}(t_0)\mu_0^{\beta}\Theta_n. \tag{5.38}$$

Notice that (5.3) is equivalent to the equation $B_{n1}+B_{n2}+B_{n3}=0$. By combining (5.35), (5.37), and (5.38), we establish (5.30).

A closer look at the above computations with (4.66)–(4.67) gives (5.31)–(5.32). The details are omitted. □

Lemma 5.7. *Assume that* (4.51) *and* (2.5)–(2.7) *hold. For $n = N+1$,* (5.3) *is equivalent to*

$$\dot\lambda(t) + \frac{3}{2t}\lambda(t) = \mu_0^4\Theta_{(N+1)1} + \mu_0^{b+\nu_1}\Theta_{(N+1)2} + \mu_0^{a+b+1-\sigma_0}\Theta_{(N+1)3} + \mu_0^{\delta_4}(t_0)\mu_0^{\beta+1}\Theta_{(N+1)4} \tag{5.39}$$

on $[t_0,\infty)$. Here, $\Theta_{N+1} = \Theta_{(N+1)1},\dots,\Theta_{(N+1)4}$ is a function on $[t_0,\infty)$ that behaves as the function Θ_n described in the statement of Lemma 5.6.

Proof. The proof is analogous to that of the previous lemma. Let $B_{(N+1)1}$, $B_{(N+1)2}$, and $B_{(N+1)3}$ be the quantities obtained by taking $n = N+1$ in (5.33). It suffices to estimate each of them.

Estimate on $B_{(N+1)1}$: In view of (2.4) and (2.5)–(2.6) (see also (2.31)), we know

$$\frac{N+2}{2}\bar\mu^2 h(z_0)c_2 - \bar\mu^{-1}\dot{\bar\mu}c_1 = \frac{3}{2t}c_1.$$

By (2.42), (2.36), and the above identity, the dominating term of $B_{(N+1)1}$ turns out to be

$$\int_{\mathbb{R}^N}\left[\bar\mu^{-1}\left(\dot\lambda - \bar\mu^{-1}\dot{\bar\mu}\lambda\right)pW_{1,0}^{p-1}(\bar y) + 2\bar\mu\lambda\mathcal{F}_0(\bar y)\right]Z_{N+1}(\bar y)d\bar y$$
$$= \bar\mu^{-1}\left(\dot\lambda - \bar\mu^{-1}\dot{\bar\mu}\lambda\right)c_1 + \frac{N+2}{2}\bar\mu\lambda h(z_0)c_2 = \bar\mu^{-1}c_1\left(\dot\lambda + \frac{3}{2t}\lambda\right).$$

Treating the other parts of $B_{(N+1)1}$, we obtain

$$B_{(N+1)1} = \bar\mu^{-1}c_1\left(\dot\lambda + \frac{3}{2t}\lambda\right) + \mu_0^3\Theta_{N+1}.$$

Estimate on $B_{(N+1)2}$ and $B_{(N+1)3}$: Arguing as in the derivation of (5.37) and (5.38), we deduce

$$B_{(N+1)2} = \mu_0^{b+\nu_1-1}\Theta_{N+1} + \mu_0^{a+b-\sigma_0}\Theta_{N+1}$$

and

$$B_{(N+1)3} = \mu_0^{b+2}\Theta_{N+1} + \mu_0^{\delta_4}(t_0)\mu_0^{\beta}\Theta_{N+1}.$$

Observe that the above estimate on $B_{(N+1)2}$ has one more term than that on B_{n2} in (5.37). It is because the integrand of $B_{(N+1)2}$ decays slower than that of B_{n2} as $|\bar y|\to\infty$; compare (2.3) and (2.34). □

We next solve the system (5.30) and (5.39) of nonlinear ODEs, thereby determining the parameters (λ,ξ).

Proposition 5.8. *Suppose that* (4.51) *holds and $t_0 > 0$ is large enough. Given $\nu_1 = \nu_2 = 2-\varepsilon_0$ in* (2.7)*, δ_4 in* (4.56)*, and a, b, α, β, σ_0, and ψ^{out} in Proposition 4.8, we further assume that $\delta_4 \le \varepsilon_0$, $a \in [\sigma_0+\varepsilon_0, N-2)$, and $b = \beta = 3-\varepsilon_0$. Then there exists a solution $(\lambda[\hat\psi^{\text{in}},\psi_0], \xi[\hat\psi^{\text{in}},\psi_0])$ to the system* (5.30) *and* (5.39) *of ODEs which satisfies*

$$\|\lambda\|_{\nu_1;\sigma_0} + \|\dot\lambda\|_{\nu_1+2;\sigma_0} + \|\xi\|_{\nu_2;\sigma_0} + \|\dot\xi\|_{\nu_2+2;\sigma_0} \le C\delta^{-\zeta}\mu_0^{\delta_4}(t_0) \tag{5.40}$$

where C, $\zeta > 0$ are constants depending only on (M, g_0), N, h, z_0, a, b, α, σ_0, and ε_0; compare with (2.7). *Additionally,*

$$\left\|\lambda[\hat{\psi}_1^{\rm in}] - \lambda[\hat{\psi}_2^{\rm in}]\right\|_{\nu_1} + \left\|\xi[\hat{\psi}_1^{\rm in}] - \xi[\hat{\psi}_2^{\rm in}]\right\|_{\nu_2} \le C\delta_0^{-\zeta}\mu_0^{\delta_4}(t_0)\left\|\hat{\psi}_1^{\rm in} - \hat{\psi}_2^{\rm in}\right\|_{\sharp',a,b;\sigma_0(\mathcal{B}_{\bar{\mu},\xi})} \tag{5.41}$$

and

$$\|\lambda[\psi_{01}] - \lambda[\psi_{02}]\|_{\nu_1} + \|\xi[\psi_{01}] - \xi[\psi_{02}]\|_{\nu_2} \le C\delta_0^{-\zeta}\|\psi_{01} - \psi_{02}\|_{**,\alpha;\sigma_0}. \tag{5.42}$$

Proof. The proof is decomposed into three steps.

Step 1. Let $\mathcal{M}$ be an $N \times N$ symmetric matrix and $\bar{\mathfrak{h}}(t) := (\mathfrak{h}_1(t), \ldots, \mathfrak{h}_N(t))$ a function on $[t_0, \infty)$ such that $\|\bar{\mathfrak{h}}\|_{\nu;\sigma} < \infty$ for some numbers $\nu > 2$ and $\sigma \in (0, 1)$. We assert that there exists a solution $\xi(t) = (\xi_1(t), \ldots, \xi_N(t))$ to the ODE system

$$\dot{\xi}(t) + \frac{1}{t}\mathcal{M}\xi(t) = \bar{\mathfrak{h}}(t) \quad \text{on } [t_0, \infty), \quad \xi(t) \to 0 \quad \text{as } t \to \infty \quad \text{and} \quad \|\dot{\xi}\|_{\nu-\epsilon;\sigma} \le C\|\bar{\mathfrak{h}}\|_{\nu;\sigma} \tag{5.43}$$

where ξ and $\bar{\mathfrak{h}}$ are regarded as column vectors, $\epsilon > 0$ is any small number, and $C > 0$ is a constant depending only on N, $\mathcal{M}$, ν, ϵ, and σ.

Indeed, since $\mathcal{M}$ is symmetric, there exist an orthogonal matrix $\mathcal{Q}$ and a diagonal matrix $\mathcal{D} = \mathrm{diag}(\varsigma_1, \ldots, \varsigma_N)$ such that $\mathcal{M} = \mathcal{Q}^T\mathcal{D}\mathcal{Q}$. If we set $\xi(t)$ by

$$\xi(t) = \mathcal{Q}^T\tilde{\xi}(t) \quad \text{where} \quad \tilde{\xi}_i(t) := \begin{cases} t^{-\varsigma_i}\displaystyle\int_{t_0}^t s^{\varsigma_i}\left(\mathcal{Q}\bar{\mathfrak{h}}\right)_i(s)ds & \text{if } \varsigma_i \ge \frac{\nu}{2} - 1, \\ -t^{-\varsigma_i}\displaystyle\int_t^\infty s^{\varsigma_i}\left(\mathcal{Q}\bar{\mathfrak{h}}\right)_i(s)ds & \text{if } \varsigma_i < \frac{\nu}{2} - 1, \end{cases} \tag{5.44}$$

then it satisfies (5.43). Note that one can select $\epsilon = 0$ unless $\varsigma_i = \frac{\nu}{2} - 1$ for some $i = 1, \ldots, N$.

Step 2. Let $\mathfrak{h}(t) = (\mathfrak{h}_1(t), \ldots, \mathfrak{h}_{N+1}(t))$ be a function on $[t_0, \infty)$ and

$$\mathcal{M} = -\frac{N-4}{6h(z_0)}\begin{pmatrix} R_{11}(z_0) & \cdots & R_{1N}(z_0) \\ \vdots & \ddots & \vdots \\ R_{N1}(z_0) & \cdots & R_{NN}(z_0) \end{pmatrix}. \tag{5.45}$$

By adjusting ε_0 suitably, we may assume that $\varsigma_i \ne \frac{\nu_1}{2} = \frac{\nu_2}{2}$ for all $i = 1, \ldots, N$, and so we may take $\epsilon = 0$. Let ξ be the solution to (5.43) chosen in the previous step,

$$\lambda(t) = t^{-\frac{3}{2}}\int_{t_0}^t s^{\frac{3}{2}}\mathfrak{h}_{N+1}(s)ds \quad \text{so that } \dot{\lambda}(t) + \frac{3}{2t}\lambda(t) = \mathfrak{h}_{N+1}(t) \quad \text{on } [t_0, \infty), \tag{5.46}$$

and

$$\mathcal{T}^{\rm par}[\mathfrak{h}] = (\Xi, \Lambda) := (\dot{\xi}, \dot{\lambda}). \tag{5.47}$$

By (5.43) and (5.46),

$$\begin{aligned} \|\mathcal{T}^{\rm par}[\mathfrak{h}]\|_{\nu_2+2,\,\nu_1+2;\sigma_0} &= \|(\Xi, \Lambda)\|_{\nu_2+2,\,\nu_1+2;\sigma_0} := \|\Xi\|_{\nu_2+2;\sigma_0} + \|\Lambda\|_{\nu_1+2;\sigma_0} \\ &\le C_6(\|(\mathfrak{h}_1, \ldots, \mathfrak{h}_n)\|_{\nu_2+2;\sigma_0} + \|\mathfrak{h}_{N+1}\|_{\nu_1+2;\sigma_0}) = C_6\|\mathfrak{h}\|_{\nu_2+2,\,\nu_1+2;\sigma_0} \end{aligned} \tag{5.48}$$

for some $C_6 > 0$.

Step 3. Parameters (λ, ξ) solve the system (5.30) and (5.39) of nonlinear ODEs if and only if

$$(\Xi, \Lambda) = \mathcal{T}^{\rm par}\left[f^{\rm par}\left[\Xi, \Lambda, \hat{\psi}^{\rm in}, \psi_0\right]\right] \tag{5.49}$$

where

$$f_n^{\rm par}\left[\Xi, \Lambda, \hat{\psi}^{\rm in}, \psi_0\right] := \begin{cases} \mu_0^4\Theta_{n1} + \mu_0^{b+\nu_1}\Theta_{n2} + \mu_0^{\delta_4}(t_0)\mu_0^{\beta+1}\Theta_{n3} & \text{for } n = 1, \ldots, N, \\ \mu_0^4\Theta_{n1} + \mu_0^{b+\nu_1}\Theta_{n2} + \mu_0^{a+b+1-\sigma_0}\Theta_{n3} + \mu_0^{\delta_4}(t_0)\mu_0^{\beta+1}\Theta_{n4} & \text{for } n = N+1 \end{cases}$$

and

$$f^{\rm par}\left[\Xi,\Lambda,\hat{\psi}^{\rm in},\psi_0\right] := \left(f_1^{\rm par}\left[\Xi,\Lambda,\hat{\psi}^{\rm in},\psi_0\right],\dots,f_{N+1}^{\rm par}\left[\Xi,\Lambda,\hat{\psi}^{\rm in},\psi_0\right]\right)$$

on $[t_0,\infty)$.

We claim that there exists a point (Ξ,Λ) on the set

$$\mathcal{D}^{\rm par} := \left\{\mathfrak{h} : \|\mathfrak{h}\|_{\nu_2+2,\,\nu_1+2;\sigma_0} \le 4\delta_0^{-\zeta_5}\mu_0^{\delta_4}(t_0)C_5C_6\right\}$$

for which (5.49) holds, where C_5, $\zeta_5>0$ are the numbers in the statement of Lemma 5.6.

To see this, we infer from (5.48) and the conditions $a\ge\sigma_0+\varepsilon_0$ and $b+1=\beta+1=\nu_2+2$ that

$$\begin{aligned}\left\|\mathcal{T}^{\rm par}\left[f^{\rm par}\left[\Xi,\Lambda,\hat{\psi}^{\rm in},\psi_0\right]\right]\right\|_{\nu_2+2,\,\nu_1+2;\sigma_0} &\le C_6\left\|f^{\rm par}\left[\Xi,\Lambda,\hat{\psi}^{\rm in},\psi_0\right]\right\|_{\nu_2+2,\,\nu_1+2;\sigma_0}\\ &\le 4\delta_0^{-\zeta_5}\mu_0^{\delta_4}(t_0)C_5C_6.\end{aligned}$$

In addition, (5.31) says

$$\begin{aligned}&\left\|\mathcal{T}^{\rm par}\left[f^{\rm par}\left[\Xi_1,\Lambda_1,\hat{\psi}^{\rm in},\psi_0\right]\right]-\mathcal{T}^{\rm par}\left[f^{\rm par}\left[\Xi_2,\Lambda_2,\hat{\psi}^{\rm in},\psi_0\right]\right]\right\|_{\nu_2+2,\,\nu_1+2;\sigma_0}\\ &\le C\left\|f^{\rm par}\left[\Xi_1,\Lambda_1,\hat{\psi}^{\rm in},\psi_0\right]-f^{\rm par}\left[\Xi_2,\Lambda_2,\hat{\psi}^{\rm in},\psi_0\right]\right\|_{\nu_2+2,\,\nu_1+2;\sigma_0}\\ &\le C\delta_0^{-\zeta}\mu_0^{\delta_4}(t_0)\|(\Xi_1,\Lambda_1)-(\Xi_2,\Lambda_2)\|_{\nu_2+2,\,\nu_1+2;\sigma_0} \le \frac{1}{2}\|(\Xi_1,\Lambda_1)-(\Xi_2,\Lambda_2)\|_{\nu_2+2,\,\nu_1+2;\sigma_0}\end{aligned}$$

provided $t_0>0$ large enough. By the contraction mapping theorem, the assertion holds.

Checking (5.40) for the associated parameters (λ,ξ) is a simple task. In addition, a closer look at the above computations with (5.31)–(5.32) gives (5.41)–(5.42). The proof is finished. □

5.3. **Unique solvability of (3.6).** We are in position to solve the inner problem (3.6). We define

$$\begin{aligned}&f_{\mu,\xi}^{\rm in}\left[\psi^{\rm in},\psi^{\rm out}\right]\\ &= \eta_{2\mu_0^{\varepsilon_1}}(x)W_{1,0}^{1-p}\left[\left(\frac{\bar{\mu}}{\mu}\right)^{\frac{N-2}{2}}(1+P)^{-p}(x)\mathcal{E}_2[\mu,\xi](y,t)+\mathcal{K}_1\left[\psi^{\rm in}\right]+\mathcal{K}_2\left[\psi^{\rm in},\psi^{\rm out}\right]\right]; \quad (5.50)\end{aligned}$$

cf. (5.3). If a function $\psi^{\rm in}$ satisfies

$$\begin{cases}pW_{1,0}^{p-1}\left(\psi^{\rm in}\right)_t = \dfrac{(N+2)\kappa_N}{4}\left(\Delta\psi^{\rm in}+pW_{1,0}^{p-1}\psi^{\rm in}\right)+W_{1,0}^{p-1}f_{\mu,\xi}^{\rm in}\left[\psi^{\rm in},\psi^{\rm out}\right] & \text{in } \mathbb{R}^N\times(t_0,\infty),\\ \psi^{\rm in}(\cdot,t_0)=e_0(t_0)Z_0 & \text{in } \mathbb{R}^N\end{cases} \quad (5.51)$$

for some $e_0(t_0)\in\mathbb{R}$, then it solves (3.6) in $B^N(0,\mathcal{B}_{\bar{\mu},\xi})\times(t_0,\infty)$ where $\mathcal{B}_{\bar{\mu},\xi}=B^N(-\bar{\mu}^{-1}\xi,2\bar{\mu}^{-1}\mu_0^{\varepsilon_1})$. Furthermore, estimate (5.52) given below clearly implies the first inequality of (4.51).

Proposition 5.9. *Suppose that the second inequality in* (4.51) *holds and $t_0>0$ is large enough. Given numbers a, b, α, β, and σ_0 in Proposition 5.8, we further assume that $a\in[\sigma_0+\varepsilon_0,\min\{N-4,\alpha+2\}]$. Then one can find $\psi^{\rm in}=\psi^{\rm in}[\psi_0]$ and $e_0=e_0[\psi_0]$ satisfying* (5.51),

$$\left\|\psi^{\rm in}\right\|_{\sharp',a,b;\sigma_0(\mathbb{R}^N)}+\mu_0^{-b}(t_0)|e_0(t_0)|\le C, \quad (5.52)$$

and

$$\begin{aligned}&\left\|\psi^{\rm in}[\psi_{01}]-\psi^{\rm in}[\psi_{02}]\right\|_{\sharp',a,b;\sigma_0\left(\mathcal{B}_{\bar{\mu},\xi}\right)}+\mu_0^{-b}(t_0)|e_0[\psi_{01}](t_0)-e_0[\psi_{02}](t_0)|\\ &\le C\delta_0^{-\zeta}\|\psi_{01}-\psi_{02}\|_{**,\alpha;\sigma_0} \quad (5.53)\end{aligned}$$

for a constant C, $\zeta>0$ depending only on (M,g_0), N, h, z_0, a, b, α, and σ_0. Here, (λ,ξ) and $\psi^{\rm out}$ are the ones determined in Propositions 5.8 and 4.8, respectively.

As in the proof of Proposition 4.8, we start by estimating $f^{\text{in}}_{\mu,\xi}\left[\psi^{\text{in}},\psi^{\text{out}}\right]$ in the $\sharp$-norm, defined in (1.18).

Lemma 5.10. *We have*

$$\left\|\eta_{2\mu_0^{\varepsilon_1}}(x)W_{1,0}^{1-p}\left(\frac{\bar{\mu}}{\mu}\right)^{\frac{N-2}{2}}(1+P)^{-p}(x)\mathcal{E}_2[\mu,\xi](y,t)\right\|_{\sharp,a+2,b;\sigma_0}\le C_{71} \tag{5.54}$$

where $C_{71}>0$ is a constant depending only on (M,g_0), N, h, z_0, α, and σ_0.

Proof. It follows from (2.42) that the dominating term of $\mathcal{E}_2[\mu,\xi]$ is a constant multiple of $\bar{\mu}\lambda W_{1,0}(y)$. For this term, we have

$$\begin{aligned}
&\mu_0^{-b}\left(\frac{\bar{\mu}}{\mu}\right)^{\frac{N-2}{2}}\bar{\mu}\lambda\Big[\left(1+|\bar{y}|^{a+2}\right)\left|\eta_{2\mu_0^{\varepsilon_1}}(x)(1+P)^{-p}(x)W_{1,0}(y)\right|\\
&\qquad+\left(1+|\bar{y}|^{a+2+\sigma_0}\right)\left[\eta_{2\mu_0^{\varepsilon_1}}(\bar{\mu}\cdot+\xi)(1+P)^{-p}(\bar{\mu}\cdot+\xi)W_{1,0}\left(\frac{\bar{\mu}}{\mu}\cdot\right)\right]_{C^{\sigma_0}_{\mathbb{R}^N}}(\bar{y},t)\Big]\\
&\le C\frac{\mu_0^{\nu_1+1-b}}{1+|\bar{y}|^{N-4-a}}\le C
\end{aligned}$$

and

$$\begin{aligned}
&\mu_0^{-b}\left(1+|\bar{y}|^{a+2-\sigma_0}\right)\left[\bar{\mu}\lambda\left(\frac{\bar{\mu}}{\mu}\right)^{\frac{N-2}{2}}\eta_{2\mu_0^{\varepsilon_1}}(\bar{\mu}\cdot+\xi)(1+P)^{-p}(\bar{\mu}\cdot+\xi)W_{1,0}\left(\frac{\bar{\mu}}{\mu}\cdot\right)\right]_{C_t^{\sigma_0/2}}(\bar{y},t)\\
&\qquad\le C\frac{\mu_0^{\nu_1+3-b}}{1+|\bar{y}|^{N-4-a+\sigma_0}}\le C\mu_0^2(t_0)
\end{aligned}$$

where we employed the condition $a\in(0,N-4]$; see Definition 1.5 for the definition of the local Hölder semi-norms.

Handling the other terms of $\mathcal{E}_2[\mu,\xi]$ in an analogous way, we establish (5.54). □

Lemma 5.11. *We have*

$$\left\|\eta_{2\mu_0^{\varepsilon_1}}(x)W_{1,0}^{1-p}\mathcal{K}_1\left[\psi^{\text{in}}\right]\right\|_{\sharp,a+2,b;\sigma_0}\le C_{72}\mu_0^{2\varepsilon_1}(t_0)\left\|\psi^{\text{in}}\right\|_{\sharp',a,b;\sigma_0(\mathcal{B}_{\bar{\mu},\xi})} \tag{5.55}$$

where $C_{72}>0$ is a constant depending only on (M,g_0), N, h, z_0, α, and σ_0.

Proof. Arguing as in the proof of Lemma 4.10, we can estimate the first three terms of $\mathcal{K}_1[\psi^{\text{in}}]$ in (3.7).

Let us consider the fourth term. From the inequalities

$$\begin{aligned}
&\mu_0^{-b}\left(1+|\bar{y}|^{a+2}\right)\eta_{2\mu_0^{\varepsilon_1}}(x)\left|\left(\frac{\mu}{\bar{\mu}}\right)^2(1+P)^{1-p}(x)(\Delta_{g_0(x)}\psi^{\text{in}})(\bar{y},t)-\Delta\psi^{\text{in}}(\bar{y},t)\right|\\
&\le C\mu_0^{-b}\left(1+|\bar{y}|^{a+2}\right)\left[\left(\mu_0^2|\bar{y}|^2+\mu_0^{\nu_1-1}\right)\left|\nabla^2\psi^{\text{in}}(\bar{y},t)\right|+\mu_0^2|\bar{y}||\nabla\psi^{\text{in}}(\bar{y},t)|\right]\\
&\le C\mu_0^{2\varepsilon_1}(t_0)\left\|\psi^{\text{in}}\right\|_{\sharp',a,b(\mathcal{B}_{\bar{\mu},\xi})},
\end{aligned}$$

we obtain its weighted L^∞-bound; cf. (5.36). By further inspection, we deduce a weighted Hölder estimate.

Handling the remaining terms of $\mathcal{K}_1[\psi^{\text{in}}]$ in an analogous way, we establish (5.55). □

Lemma 5.12. *We have*

$$\left\|\eta_{2\mu_0^{\varepsilon_1}}(x)W_{1,0}^{1-p}\mathcal{K}_2\left[\psi^{\text{in}},\psi^{\text{out}}\right]\right\|_{\sharp,a+2,b;\sigma_0}\le C_{73}\left(\mu_0^2(t_0)\left\|\psi^{\text{in}}\right\|_{\sharp',a,b;\sigma_0(\mathcal{B}_{\bar{\mu},\xi})}+\left\|\psi^{\text{out}}\right\|_{*',\alpha,\rho;\sigma_0}\right) \tag{5.56}$$

where $C_{73} > 0$ is a constant depending only on (M, g_0), N, h, z_0, α, and σ_0.

Proof. Let us consider the first term of $\mathcal{K}_2[\psi^{\text{in}}]$. Applying the condition $a \in (0, \alpha+2]$, we deduce

$$\begin{aligned}
&\mu_0^{-b}\left(1+|\bar{y}|^{a+2}\right)\eta_{2\mu_0^{\varepsilon_1}}(x)\left|\left(\frac{\bar{\mu}}{\mu}\right)^{\frac{N-2}{2}}(1+P)^{-1}(x)W_{1,0}^{p-1}(y)\mu^{\frac{N-2}{2}}\psi_{\mu,\xi}^{\text{out}}(x,t)\right| \\
&\le C\left\|\psi^{\text{out}}\right\|_{*',\alpha,\rho;\sigma_0}\sup_{(\bar{y},t)\in\mathcal{B}_{\bar{\mu},\xi}\times[t_0,\infty)}\frac{\mu_0^{\beta-b}}{1+|\bar{y}|^{\alpha-a+2}} \le C\left\|\psi^{\text{out}}\right\|_{*',\alpha,\rho;\sigma_0},
\end{aligned}$$

from which we obtain a weighted L^∞-bound. By further inspection, we deduce a weighted Hölder estimate.

Employing the mean value theorem, we can handle the second term of $\mathcal{K}_1[\psi^{\text{in}}]$. Inequality (5.56) then readily follows. □

Completion of the proof of Proposition 5.9. Let $\mathcal{T}^{\text{in}}[\mathcal{H}] := \psi[\mathcal{H}]$ and $e_0[\mathcal{H}]$ be the solution to (5.1) found in Proposition 5.1. A function ψ^{in} solves (5.51) if it satisfies

$$\psi^{\text{in}} = \mathcal{T}^{\text{in}}\left[f_{\mu,\xi}^{\text{in}}\left[\psi^{\text{in}}, \psi^{\text{out}}\left[\psi^{\text{in}};\psi_0\right]\right]\right]$$

where $f_{\mu,\xi}^{\text{in}}$ is the map defined in (5.50), and the notation $\psi^{\text{out}}[\psi^{\text{in}};\psi_0]$ emphasizes the dependence of ψ^{out} on ψ^{in} and ψ_0.

We claim that the operator $\mathcal{T}^{\text{in}} \circ f_{\mu,\xi}^{\text{in}}$ has a fixed point on the set

$$\mathcal{D}^{\text{in}} := \left\{\psi^{\text{in}} : \left\|\psi^{\text{in}}\right\|_{\sharp',a,b;\sigma_0(\mathbb{R}^N)} \le 2C_4C_{71}\right\}$$

where C_4 and C_{71} are the numbers appearing in (5.4) and (5.54).

By using (5.4), (5.50), (5.54), (5.55), (5.56), and (4.55), and taking a large $t_0 > 0$ if needed, we get

$$\begin{aligned}
&\left\|\mathcal{T}^{\text{in}}\left[f_{\mu,\xi}^{\text{in}}\left[\psi^{\text{in}},\psi^{\text{out}}\left[\psi^{\text{in}};\psi_0\right]\right]\right]\right\|_{\sharp',a,b;\sigma_0(\mathbb{R}^N)} \\
&\le C_4\left\|f_{\mu,\xi}^{\text{in}}\left[\psi^{\text{in}},\psi^{\text{out}}\left[\psi^{\text{in}};\psi_0\right]\right]\right\|_{\sharp,a+2,b;\sigma_0} \\
&\le C_4\left[C_{71}+\left(C_{72}\mu_0^{2\varepsilon_1}(t_0)+C_{73}\mu_0^2(t_0)\right)\left\|\psi^{\text{in}}\right\|_{\sharp',a,b;\sigma_0\left(\mathcal{B}_{\bar{\mu},\xi}\right)}+C_{73}\left\|\psi^{\text{out}}\right\|_{*',\alpha,\rho;\sigma_0}\right] \le 2C_4C_{71}.
\end{aligned} \tag{5.57}$$

Moreover, by the linearity of $\mathcal{T}^{\text{in}}$, (5.4), (4.66), and (5.41) (see also the derivation of (5.55)),

$$\begin{aligned}
&\left\|\mathcal{T}^{\text{in}}\left[f_{(\mu,\xi)[\psi_1^{\text{in}}]}^{\text{in}}\left[\psi_1^{\text{in}},\psi^{\text{out}}\left[\psi_1^{\text{in}};\psi_0\right]\right]\right]-\mathcal{T}^{\text{in}}\left[f_{(\mu,\xi)[\psi_2^{\text{in}}]}^{\text{in}}\left[\psi_2^{\text{in}},\psi^{\text{out}}\left[\psi_2^{\text{in}};\psi_0\right]\right]\right]\right\|_{\sharp',a,b;\sigma_0(\mathbb{R}^N)} \\
&\le C\left\|f_{(\mu,\xi)[\psi_1^{\text{in}}]}^{\text{in}}\left[\psi_1^{\text{in}},\psi^{\text{out}}\left[\psi_1^{\text{in}};\psi_0\right]\right]-f_{(\mu,\xi)[\psi_2^{\text{in}}]}^{\text{in}}\left[\psi_2^{\text{in}},\psi^{\text{out}}\left[\psi_2^{\text{in}};\psi_0\right]\right]\right\|_{\sharp,a+2,b;\sigma_0} \\
&\le C\delta_0^{-\zeta}\mu_0^{\min\{\delta_4,2\varepsilon_1\}}(t_0)\left\|\psi_1^{\text{in}}-\psi_2^{\text{in}}\right\|_{\sharp',a,b;\sigma_0\left(\mathcal{B}_{\bar{\mu},\xi}\right)} \le \frac{1}{2}\left\|\psi_1^{\text{in}}-\psi_2^{\text{in}}\right\|_{\sharp',a,b;\sigma_0\left(\mathcal{B}_{\bar{\mu},\xi}\right)}
\end{aligned}$$

where the notation $(\mu,\xi) = (\mu,\xi)[\psi^{\text{in}}]$ stresses the dependence of (μ,ξ) in ψ^{in}. By the contraction mapping theorem, the assertion holds.

We now have a unique solution ψ^{in} to (5.51) with the desired $\sharp'$-norm bound given in (5.52). Also, the bound on $e_0(t_0)$ in (5.52) immediately follows from (5.4) and (5.57). A further inspection based on (4.67) and (5.42) gives (5.53), concluding the proof. □

5.4. **Completion of the proof of Theorem 1.1.** Let z_0 be a point on M such that $h(z_0) > 0$. In Subsection 2.1, we selected sufficiently small numbers ε_0, ε_1, $\sigma_0 \in (0,1)$ and set $\nu_1 = \nu_2 = 2-\varepsilon_0$. Take

$$a = N-4, \quad b = 3-\varepsilon_0, \quad \alpha = N-5+\varepsilon_0, \quad \beta = 3-\varepsilon_0, \quad \delta_2 = \varepsilon_0,$$

and any $\delta_4 \in (0, \varepsilon_0]$ satisfying (4.56). Choose also $\psi_0 = 0$ on M. From the discussion in Section 3 and Propositions 4.8, 5.8, and 5.9, we find a solution u_{z_0} to (1.6) of the form $u_{z_0} = u_{\mu,\xi} + \psi_{\mu,\xi}$ on $M \times [t_0, \infty)$ where $u_{\mu,\xi} = u^{(2)}_{\mu,\xi}$ and $\psi_{\mu,\xi}$ are given in (2.39) and (3.1), respectively. Estimates (4.55), (2.7), and (5.52) (or (4.51)) imply that $u_{z_0} > 0$ on $M \times [t_0, \infty)$ and (1.10) holds. Consequently, the proof of Theorem 1.1 is completed.

Note that we have a freedom to choose the initial value $\psi_0 = \psi^{\text{out}}(\cdot, t_0)$ on the outer problem (3.2) or (4.52), provided the second inequality of (4.51) holds. In Subsection 6.2, we will further analyze this observation to establish the k-codimensional stability stated in Corollary 1.3.

6. Proof of Theorem 1.2 and Corollary 1.3

Throughout the section, $k \in \mathbb{N}$ is fixed and l can take any integer between 1 and k.

6.1. **Proof of Theorem 1.2.** In this subsection, we provide the outline of the proof of Theorem 1.2, pointing out the changes needed with respect to the one bubble case.

By choosing the number $\delta_0 > 0$ sufficiently small, we may assume that $\mathbf{z}_0 = (z_0^{(1)}, \dots, z_0^{(k)})$ is an element of the configuration set

$$\mathcal{C} := \left\{ \left(z_0^{(1)}, \dots, z_0^{(k)}\right) \in M^k : d_{g_0}\left(z_0^{(l)}, z_0^{(m)}\right) \geq c\delta_0 \text{ for } 1 \leq l \neq m \leq k \right\}$$

where $c > 0$ is a number determined by (M, g_0), N, h, and k. Given $l = 1, \dots, k$, we write

$$d^{(l)} := \frac{1}{\sqrt{h\left(z_0^{(l)}\right)}} \quad \text{and} \quad \mu^{(l)}(t) = d^{(l)}\mu_0(t) + \lambda^{(l)}(t) =: \bar{\mu}^{(l)}(t) + \lambda^{(l)}(t) \quad \text{for } t \in [t_0, \infty) \tag{6.1}$$

where μ_0 is the function in (2.5), and $\lambda^{(l)}$ is a higher-order term. We assume that (2.7) holds for $(\lambda, \xi) = (\lambda^{(l)}, \xi^{(l)})$.

By substituting $z_0^{(l)}$ and $\bar{\mu}^{(l)}$ for z_0 and $\bar{\mu}$, respectively, we define the analogue $P^{(l)}$ of P in (2.14), and the analogue $\Psi_0^{(l)}$ of Ψ_0 in the paragraph after Lemma 2.7. In (2.39)–(2.40), we put z_0, (μ, ξ) and Ψ_0 in place of $z_0^{(l)}$, $(\mu^{(l)}, \xi^{(l)})$, respectively, to define $u^{(2)}_{\mu^{(l)},\xi^{(l)}}$ and $v^{(2)}_{\mu^{(l)},\xi^{(l)}}$. Then we set the refined approximate solution

$$u_{\boldsymbol{\mu},\boldsymbol{\xi}}(z,t) = u^{(2)}_{\boldsymbol{\mu},\boldsymbol{\xi}}(z,t) = \sum_{l=1}^{k} u^{(2)}_{\mu^{(l)},\xi^{(l)}}(z,t) \quad \text{on } M \times [t_0, \infty) \tag{6.2}$$

where $\boldsymbol{\mu} := (\mu^{(1)}, \dots, \mu^{(k)}) \in (0,\infty)^k$ and $\boldsymbol{\xi} := (\xi^{(1)}, \dots, \xi^{(k)}) \in (\mathbb{R}^N)^k$. It holds that

$$\mu^{\frac{N+2}{2}} \mathcal{S}\left(u^{(2)}_{\boldsymbol{\mu},\boldsymbol{\xi}}\right)(y,t) = \mathcal{E}_2\left[\mu^{(l)}, \xi^{(l)}\right](y,t)$$

for $z = \exp_{z_0^{(l)}}(x) = \exp_{z_0^{(l)}}(\mu^{(l)}y + \xi^{(l)}) \in B_{g_0}(z_0^{(l)}, \delta_0)$ and $t \in [t_0, \infty)$, where $\mathcal{E}_2$ is the function in (2.42). Indeed, the interaction between two different bubbles is estimated by

$$\mathbf{1}_{N=5}\mu_0^3 a^{\{1\}} + \mu_0^{N-2} a^{\{N-4\}} \left(\lesssim \mathbf{1}_{N=5}\mu_0^3 a^{\{1\}} + \mu_0^4 a^{\{2\}} \right),$$

which is smaller than the main order terms of $\mathcal{E}_2$. An analogous formula to (2.47) is also true.

The norms in Subsection 1.6 must be adjusted accordingly. For instance, every $u_{\mu,\xi}$ in the norms has to be replaced with $u_{\boldsymbol{\mu},\boldsymbol{\xi}}$, and the weight $w_{\alpha,\gamma}$ in (1.14) needs to be redefined as

$$w_{\alpha,\gamma}(z,t) = \max\left\{\sum_{l=1}^{k} \eta_{\delta_0}\left(\left|x^{(l)}\right|\right) \frac{\mu_0^{-\gamma}}{1+\left|\left(\mu^{(l)}\right)^{-1}\left(x^{(l)}-\xi^{(l)}\right)\right|^{\alpha+\gamma}},\right. \left. 2^{3(\alpha+\gamma)}\min\left\{h\left(z_0^{(1)}\right),\dots,h\left(z_0^{(k)}\right)\right\}^{-\frac{\alpha+\gamma}{2}} \delta_0^{-(\alpha+\gamma)}\mu_0^{\alpha}\right\}$$

where $x^{(l)} := \exp^{-1}_{z_0^{(l)}}(z)$.

Let $\psi_{\boldsymbol{\mu},\boldsymbol{\xi}}$ stand for the remainder term such that $u = u_{\boldsymbol{\mu},\boldsymbol{\xi}} + \psi_{\boldsymbol{\mu},\boldsymbol{\xi}}$ is a solution to (2.1). We assume that it can be written as

$$\psi_{\boldsymbol{\mu},\boldsymbol{\xi}} = \psi^{\text{out}}_{\boldsymbol{\mu},\boldsymbol{\xi}} + \psi^{\text{in}}_{\boldsymbol{\mu},\boldsymbol{\xi}} \quad \text{on } M\times[t_0,\infty) \tag{6.3}$$

where $\psi^{\text{in}}_{\boldsymbol{\mu},\boldsymbol{\xi}}$ has the form

$$\psi^{\text{in}}_{\boldsymbol{\mu},\boldsymbol{\xi}}(z,t) = \sum_{l=1}^{k} \eta_{\mu_0^{\varepsilon_1}}\left(\left|x^{(l)}\right|\right)\left(1+P^{(l)}\left(x^{(l)}\right)\right)\left(\bar\mu^{(l)}\right)^{-\frac{N-2}{2}}\left(\psi^{\text{in}}_{\boldsymbol{\mu},\boldsymbol{\xi}}\right)^{(l)}\left(\bar y^{(l)},t\right) \tag{6.4}$$

for $(z,t)\in M\times[t_0,\infty)$ and $x^{(l)} = \bar\mu^{(l)}\bar y^{(l)} + \xi^{(l)}$. The inner-outer gluing procedure consists of finding a solution $\psi^{\text{out}}_{\boldsymbol{\mu},\boldsymbol{\xi}}$ to the outer problem on $M\times[t_0,\infty)$ and a solution $(\psi^{\text{in}}_{\boldsymbol{\mu},\boldsymbol{\xi}})^{(l)}$ of an inner problem on $B^N(-(\bar\mu^{(l)})^{-1}\xi^{(l)}, 2(\bar\mu^{(l)})^{-1}\mu_0^{\varepsilon_1})\times[t_0,\infty)$ for $l=1,\dots,k$.

Minor modifications of the arguments in Sections 4 and 5 yield a priori estimates for inhomogeneous equations associated to the outer and inner problems. For example, the only change required in the proof of Lemma 4.3 is to replace (4.21) with

$$\int_0^1 \int_{B_{g_0}\left(z_0^{(m)}, R_1\mu_\ell^{(m)}(\cdot+\tau_\ell)\right)} \phi_\ell^2 u_\ell^{p-1} dv_{g_0} dt \ge \frac{1}{4k}$$

for some $m\in\{1,\dots,k\}$. The existence of such m is guaranteed by the pigeonhole principle. To prove Lemma 4.6, we divide the manifold M into $k+1$ subsets $B_{g_0}(z_0^{(1)}, \frac{\delta_0}{4}),\dots,B_{g_0}(z_0^{(k)},\frac{\delta_0}{4})$, and $M\setminus\bigcup_{l=1}^k B_{g_0}(z_0^{(l)},\frac{\delta_0}{6})$, and then examine the behavior of ψ on each subset.

By employing the a priori estimates for inhomogeneous equations and adopting the arguments in Sections 4 and 5 once again, we establish the existence of $\psi^{\text{out}}_{\boldsymbol{\mu},\boldsymbol{\xi}}$, $(\lambda^{(l)},\xi^{(l)})$, and $(\psi^{\text{in}}_{\boldsymbol{\mu},\boldsymbol{\xi}})^{(l)}$ for $l=1,\dots,k$ as well as estimates on their respective norms. Combining this information leads us to complete the proof of Theorem 1.2.

6.2. **Proof of Corollary 1.3.** By modifying the argument in the proof of [14, Corollary 1.1] suitably, one can prove the corollary. Here we give a sketch the proof.

As in the previous subsection, a quantity with the superscript (l) indicates that it is related to the l-th blow-up point $z_0^{(l)}$.

Fix $l=1,\dots,k$ and let $\mathcal{M}$ be the matrix in (5.45) with $z_0 = z_0^{(l)}$. If $\varsigma_1,\dots,\varsigma_N$ are the eigenvalues of $\mathcal{M}$, then the assumption on the Ricci curvature implies that

$$\min\{\varsigma_1,\dots,\varsigma_N\} \ge 1 > 1 - \frac{\varepsilon_0}{2} = \frac{\nu_2+2}{2} - 1. \tag{6.5}$$

Set $(\overline{f^{\text{par}}})^{(l)} = ((f_1^{\text{par}})^{(l)},\dots,(f_N^{\text{par}})^{(l)})$ and an orthogonal matrix $\mathcal{Q}$ such that $\mathcal{M} = \mathcal{Q}^T\mathcal{D}\mathcal{Q}$ with $\mathcal{D} = \text{diag}(\varsigma_1,\dots,\varsigma_N)$. In light of (6.5), (5.44), (5.46), (5.47), and (5.49), we can select the parameters

$(\lambda^{(l)}, \xi^{(l)})$ by

$$\lambda^{(l)}(t) = t^{-\frac{3}{2}} \int_{t_0}^t s^{\frac{3}{2}} \left(f_{N+1}^{\text{par}}\right)^{(l)}(s)ds$$

and

$$\xi^{(l)}(t) = \mathcal{Q}^T \tilde{\xi}^{(l)}(t) \quad \text{where} \quad \tilde{\xi}_i^{(l)}(t) = t^{-\varsigma_i} \int_{t_0}^t s^{\varsigma_i} \left[\mathcal{Q}\left(\overline{f^{\text{par}}}\right)^{(l)}\right]_i (s)ds \quad \text{for } i = 1, \ldots, N. \tag{6.6}$$

They satisfy (2.7), and

$$\mu^{(l)}(t_0) = d^{(l)}\mu_0(t_0), \quad \xi^{(l)}(t_0) = 0 \quad \text{and} \quad \xi^{(l)}(t) \to 0 \quad \text{as } t \to \infty \tag{6.7}$$

by (6.1).[3]

Let ψ_0 be the initial condition for the outer problem, and $e_0^{(l)}[\psi_0]Z_0$ the initial condition for the inner problem of $(\psi^{\text{in}}_{\boldsymbol{\mu},\boldsymbol{\xi}})^{(l)}$. Recalling (6.2) and (6.3)–(6.4), we choose the initial datum for (2.1) of the form

$$\begin{aligned} u(z,t_0) = u_{\boldsymbol{\mu},\boldsymbol{\xi}}(z,t_0) + \sum_{l=1}^k \eta_{\mu_0^{\varepsilon_1}(t_0)}\left(\left|x^{(l)}\right|\right)\left(1 + P^{(l)}\left(x^{(l)}\right)\right)\left(\bar{\mu}^{(l)}(t_0)\right)^{-\frac{N-2}{2}} e_0^{(l)}[0](t_0)Z_0(z) \\ + F[\psi_0](z) \quad \text{for } z \in M \end{aligned}$$

where

$$\begin{aligned} F[\psi_0](z) &:= \psi_0(z) \\ &+ \sum_{l=1}^k \eta_{\mu_0^{\varepsilon_1}(t_0)}\left(\left|x^{(l)}\right|\right)\left(1 + P^{(l)}\left(x^{(l)}\right)\right)\left(\bar{\mu}^{(l)}(t_0)\right)^{-\frac{N-2}{2}} \left[e_0^{(l)}[\psi_0](t_0) - e_0^{(l)}[0](t_0)\right] Z_0(z). \end{aligned}$$

According to (6.7), the solution $u(z,t)$ to (2.1) blows up precisely at $z_0^{(1)}, \ldots, z_0^{(k)}$ on M. Besides, F is a C^1-function on $C^{2,\sigma_0}(M)$ (see (5.53)), $F[0] = 0$, and $DF[\Psi_0] = \text{Id}$ on the subspace $W := \cap_{l=1}^k \ker(D_{\psi_0} e_0^{(l)}[0])$ of $C^{2,\sigma_0}(M)$. Hence the inverse function theorem says that there is a manifold of codimension $\text{codim}(W) \le k$ in a neighborhood of $0 \in C^{2,\sigma_0}(M)$ such that each element is expressed as $F[\psi_0]$ for some $\psi_0 \in C^{2,\sigma_0}(M)$ near 0. Calling such a manifold $\mathcal{M}_{\mathbf{z}_0}$ and taking $\sigma = \sigma_0$, we conclude the proof.

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