A New Transform Method I: Domain Dependent Fundamental Solutions and Integral Representations.

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Abstract

A new method for solving boundary value problems (BVPs) for linear and certain non-linear PDEs was introduced by one of the authors in the late 90's [Fokas, 1997]. For *linear* PDEs this method constructs novel integral representations which are formulated in the Fourier (transform) space. In this paper we present a simplified way of obtaining these representations for elliptic PDEs, namely we introduce an algorithm for constructing particular, domain dependent, integral representations of the associated fundamental solutions, which are then substituted into Green's integral representations. Furthermore we extend this new method from BVPs in polygons to BVPs in polar co-ordinates. In the sequel to this paper [Spence and Fokas, 2009], these results are used to solve particular BVPs, which elucidate the fact that this method has substantial advantages over the classical transform method.

1 Introduction.

1.1 Background.

In 1747 d'Alembert derived the wave equation, which was the first PDE in the history of mathematics. Soon after, d'Alembert and Euler discovered a general method for constructing large classes of solutions, namely the method of *separation of variables*. Daniel Bernoulli, in his attempt to solve the wave equation, introduced the infinite sine series and Euler discovered the standard formula for the coefficients of a Fourier series. Fourier, in his attempts to understand heat diffusion, inaugurated in 1807 the era of linearization that dominated mathematical physics for the first half of the nineteenth century. In 1828 Green introduced the powerful approach of integral representations that can be obtained via Green's functions (or more precise via the fundamental solutions). Separation of variables led to the *spectral analysis* of ordinary differential operators and to the solution of PDEs via a *transform pair*. The prototypical such pair is the Fourier transform (FT), variations include the sine, cosine, Laplace, and Mellin transforms, as well as their discrete analogues.

In 1967 a new method, called the Inverse Scattering Transform (IST) method, was introduced for solving the initial value problem of certain non-linear evolution PDEs called "integrable". These equations include the celebrated Korteweg-de Vries and nonlinear Schrödinger equations. The distinguishing feature of these equations is the existence of a Lax pair, namely each of these equations can be written as the compatibility condition of two *linear eigenvalue equations* forming the associated Lax pair. Gelfand and one of the authors have emphasized [Fokas and Gelfand, 1994] that *the IST is based on a deeper type of separability.* Indeed, the spectral analysis of the t-independent part of the Lax pair yields an appropriate *nonlinear* FT pair. It should be noted that after constructing this nonlinear transform, the t-dependent part of the Lax pair yields the time evolution of the nonlinear Fourier data; in this sense, inspite of the fact that the IST is applicable to nonlinear PDEs, *IST still follows the logic of separation of variables.*

A unified approach for solving both linear and integrable non-linear PDEs in two dimensions, motivated by the IST method, was introduced by one of the authors in 1997 [Fokas, 1997]. This method goes *beyond* separation of variables. Indeed, this method is based on the *simultaneous spectral analysis* of both parts of the Lax pair and thus it corresponds to the *synthesis* as opposed to separation of variables.

For integrable non-linear PDEs, the simultaneous spectral analysis of the associated Lax pair is the only known method for constructing the novel integral representations of the method of [Fokas, 1997]. However, for linear PDEs, in addition to this method there exist several other approaches. Among these approaches it appears that for *evolution* PDEs the simplest approach is to use the Fourier transform and contour deformations [Fokas, 2002], whereas for *elliptic* PDEs the simplest approach is to employ Green's integral representations [Fokas and Zyskin, 2002]. This paper presents a simplification of the latter approach and also extends the method to PDEs in polar, as opposed to Cartesian, co-ordinates.

In this paper we consider the second order linear elliptic PDE

$$\Delta u(\boldsymbol{x}) + \lambda u(\boldsymbol{x}) = -f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{1.1}$$

where λ is a real constant, $f(\boldsymbol{x})$ a given function, and Ω is some 2 dimensional domain with a piecewise smooth boundary. For $\lambda = 0$ this is Poisson's equation, $\lambda > 0$ the Helmholtz equation, and $\lambda < 0$ the Modified Helmholtz equation.

For a boundary value problem (BVP) to be well-posed, certain *boundary conditions* must be prescribed; the ones of most physical importance are:

- Dirichlet: $u(\boldsymbol{x}) = \text{known}$, $\boldsymbol{x} \in \partial\Omega$
- Neumann: $\frac{\partial u}{\partial n}(\boldsymbol{x}) = \text{known}$, $\boldsymbol{x} \in \partial\Omega$
- Robin: $\frac{\partial u}{\partial n}(\boldsymbol{x}) + \alpha u(\boldsymbol{x}) = \text{known}$, $\alpha = \text{constant}, \quad \boldsymbol{x} \in \partial\Omega$,

where $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$, where $\mathbf{n}$ is the unit outward-pointing normal to Ω. More complicated boundary conditions can involve derivatives at angles to the boundary. One can also prescribe mixed boundary conditions, such as Dirichlet on part of the domain, and Neumann on another part.

1.2 Green's integral representation.

Green's theorem gives the following integral representation of the solution of (1.1)

$$u(\boldsymbol{x}) = \int_{\partial\Omega} \left(E(\boldsymbol{\xi}, \boldsymbol{x}) \frac{\partial u}{\partial n}(\boldsymbol{\xi}) - u(\boldsymbol{\xi}) \frac{\partial E}{\partial n}(\boldsymbol{\xi}, \boldsymbol{x}) \right) dS(\boldsymbol{\xi}) + \int_{\Omega} f(\boldsymbol{\xi}) E(\boldsymbol{\xi}, \boldsymbol{x}) dV(\boldsymbol{\xi}), \quad \boldsymbol{x} \in \Omega, \tag{1.2}$$

where E is the fundamental solution (sometimes called the free space Green's function) satisfying

$$\left(\Delta_{\boldsymbol{\xi}} + \lambda \right) E(\boldsymbol{\xi}, \boldsymbol{x}) = -\delta(\boldsymbol{\xi} - \boldsymbol{x}), \quad \boldsymbol{\xi} \in \Omega, \tag{1.3}$$

where δ denotes the Dirac delta function, and dS, dV denote the surface and volume elements. For the different signs of λ, E is given by the expressions below.

- Laplace/Poisson ($\lambda = 0$): $E = -\frac{1}{2\pi} \log |\boldsymbol{\xi} - \boldsymbol{x}|$.
- Helmholtz ($\lambda > 0$): $E = \frac{i}{4} H_0^{(1)}(\sqrt{\lambda} |\boldsymbol{\xi} - \boldsymbol{x}|)$ (with assumed time-dependence $e^{-i\omega t}$ this corresponds to outgoing waves).
- Modified Helmholtz ($\lambda < 0$): $E = \frac{1}{2\pi} K_0(\sqrt{-\lambda} |\boldsymbol{\xi} - \boldsymbol{x}|)$,

where $H_0^{(1)}$ is a Hankel function and K_0 a modified Bessel function.

We note that a drawback for both Helmholtz and modified Helmholtz in 2-d is that the fundamental solution is given in terms of a special function.

1.3 The method of [Fokas, 1997].

The method of [Fokas, 1997] has two basic ingredients:

1. The integral representation.
2. The global relation.

The integral representation (IR). This is the analogue of Green's IR in the transform space. Indeed, the solution u is given as an integral involving *transforms* of the boundary values, whereas Green's IR expresses u as an integral involving directly the boundary values. In this paper the IR is obtained by first constructing particular integral representations depending on the domain of the fundamental solution E ("domain-dependent fundamental solutions") and then substituting these representations into Green's integral formula of the solution and interchanging the orders of integration.

The global relation (GR). The global relation is Green's divergence form of the equation integrated over the domain, where one employs a one-parameter family of solutions of the adjoint equation instead of the fundamental solution. Indeed

$$0 = \int_{\partial\Omega} \left(v(\boldsymbol{\xi}) \frac{\partial u}{\partial n}(\boldsymbol{\xi}) - u(\boldsymbol{\xi}) \frac{\partial v}{\partial n}(\boldsymbol{\xi}) \right) dS(\boldsymbol{\xi}) + \int_{\Omega} f(\boldsymbol{\xi}) v(\boldsymbol{\xi}) dV(\boldsymbol{\xi}), \tag{1.4}$$

where v is any solution of the adjoint of (1.1):

$$\Delta v(\boldsymbol{x}) + \lambda v(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \Omega; \tag{1.5}$$

(equation (1.5) is (1.3) without the delta function on the right hand side). Separation of variables gives a one-parameter family of solutions of the adjoint equation depending on the parameter $k \in \mathbb{C}$ (the separation constant). For example, for the Poisson equation, separation of variables in Cartesian co-ordinates yields four solutions of the adjoint equation, $v = e^{\pm ikx \pm ky}$, $k \in \mathbb{C}$. All equations (1.4) obtained from these different choices of v shall be refered to as "the global relation". This relation was called the global relation as it contains global, as opposed to local, information about the boundary values.

Just like Green's IR, the new IR contains contributions from both known and unknown boundary values. However, it turns out that the GR involves *precisely* the transforms of the boundary values appearing in the IR. The main idea of the method of [Fokas, 1997] is that, for certain boundary value problems, one can **use the information given in the GR about the transforms of the unknown boundary values to eliminate these unknowns from the IR.**

This method yields the solution as an integral in the complex k plane involving transforms of the known boundary values. This novel solution formula has two significant features:

1. The integrals can be deformed to involve exponentially decaying integrands.

2. The expression is uniformly convergent at the boundary of the domain.

These features give rise to both analytical and numerical advantages in comparison with the classical methods. In particular, for linear evolution PDEs, the effective numerical evaluation of the solution is given in [Flyer and Fokas, 2008].

1.4 Plan of the paper.

The novel integral representations for the interior of a convex polygon and for certain domains in polar co-ordinates are obtained in sections §2 and §3 respectively. This is achieved by first constructing appropriate "domain dependent fundamental solutions" and then substituting these into Green's integral representation. These results are discussed further in §4.

In the sequel to this paper [Spence and Fokas, 2009], the global relations for these domains are constructed, and then the integral representations and global relations are used to solve certain boundary value problems for Helmholtz in a wedge and for Poisson in a circular wedge.

2 Integral representations for polygons.

Notations. We identify $\mathbb{R}^2$ with $\mathbb{C}$. Let $z = x + iy$ be the physical variable, and $z' = \xi + i\eta$ the "dummy variable" of integration. We will consider u both as a function of (x, y), and as a function of $(z, \bar{z})$ interchangeably. Using the chain rule, (1.1) becomes

$$\frac{\partial^2}{\partial z \partial \bar{z}} u + \frac{\lambda}{4} = -\frac{f}{4}.$$

For the Helmholtz equation $\lambda = 4\beta^2$, $\beta \in \mathbb{R}^+$, and for the Modified Helmholtz equation $\lambda = -4\beta^2$, $\beta \in \mathbb{R}^+$. The subscripts T and N will denote tangential and normal respectively.

2.1 Co-ordinate system dependent fundamental solutions.

In Cartesian co-ordinates, (1.3) is

$$E_{\xi\xi} + E_{\eta\eta} + \lambda E = -\delta(\xi - x)\delta(\eta - y). \tag{2.1}$$

The differential operator

$$-\frac{d^2}{d\xi^2} u = \lambda u$$

on $(-\infty, \infty)$ possesses the completeness relation

$$\delta(\xi - x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik_1(x-\xi)} dk_1, \tag{2.2}$$

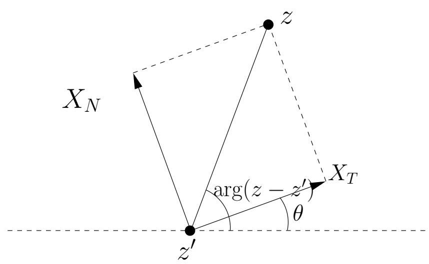


Figure 1: The points z, z' and the local co-ordinate system (X_T, X_N)

i.e. the Fourier transform [Stakgold, 1967]. Similarly, the completeness relation for the η co-ordinate is

$$\delta(\eta - y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik_2(y-\eta)}dk_2. \tag{2.3}$$

These two completeness relations give rise to the following integral representation of E:

$$E(\xi,\eta,x,y) = \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} dk_1 dk_2 \frac{e^{ik_1(x-\xi)+ik_2(y-\eta)}}{k_1^2+k_2^2-\lambda}, \tag{2.4}$$

where the contours of integration must be suitably deformed to avoid the poles of the integrand (the contours of (2.2) and (2.3) should be deformed *before* using them to obtain (2.4)).

Proposition 2.1 ***(Integral representation of E for modified Helmholtz).*** *For the modified Helmholtz equation, the fundamental solution E is given by*

$$E(z',z) = \frac{1}{4\pi}\int_0^\infty \frac{dk}{k}\exp\left(i\beta\left(k(z-z')e^{-i\theta} - \frac{1}{k}\overline{(z-z')}e^{i\theta}\right)\right), \quad \theta \le \arg(z-z') \le \theta+\pi. \tag{2.5}$$

(Note that if $\arg(z-z') = \theta$ or $\theta+\pi$ then the integral in (2.5) is not absolutely convergent.)

Proof . Starting with (2.4) where $\lambda = -4\beta^2$, define X_T, X_N by

$$z - z' = (x-\xi) + i(y-\eta) = (X_T + iX_N)e^{i\theta}, \tag{2.6}$$

so that, if $\theta \le \arg(z-z') \le \theta+\pi$, then $X_N = \mathrm{Im}\big((z-z')e^{-i\theta}\big) \ge 0$, see Figure 1.

In a similar way, rotate the (k_1,k_2) plane by letting

$$k_1 + ik_2 = (k_T + ik_N)e^{i\theta},$$

so that

$$\begin{aligned} k_1(x-\xi) + k_2(y-\eta) &= k_T X_T + k_N X_N, \\ {k_1}^2 + {k_2}^2 &= {k_T}^2 + {k_N}^2 \end{aligned} \tag{2.7}$$

and (2.4) becomes

$$E(\xi,\eta,x,y)=\frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2}dk_T dk_N\frac{e^{ik_TX_T+ik_NX_N}}{k_T^2+k_N^2+4\beta^2}. \tag{2.8}$$

Now perform the k_N integral by closing the contour in the upper half plane (as $X_N\geq 0$) to obtain

$$E(\xi,\eta,x,y)=\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{e^{ik_TX_T-\sqrt{k_T^2+4\beta^2}X_N}}{\sqrt{k_T^2+4\beta^2}}dk_T. \tag{2.9}$$

The substitution $k_T=\beta(l-1/l)$ transforms (2.9) to

$$E(\xi,\eta,x,y)=\frac{1}{4\pi}\int_0^{\infty}\frac{dl}{l}\exp\left(i\beta\left(l-\frac{1}{l}\right)X_T-\beta\left(l+\frac{1}{l}\right)X_N\right),$$

which becomes (2.5) after using (2.6) and relabelling l as k.

The right hand side of (2.5) defines a function of $z-z'$ in a half plane. To show that this expression defines a function of $z-z'$ in the whole plane (except zero) we must show that the right hand sides of (2.5) for different values of θ are equal for $z-z'$ in the common domain of definition. This is achieved by rotating the contour, using Cauchy's theorem and the analyticity of the integrand in k. $\square$

The main differences between Helmholtz and Modified Helmholtz are the following:

1. There are *two* fundamental solutions.
2. The contours of integrations for both the fundamental solutions contain circular arcs as well as rays in the complex plane.

These differences follow from the fact that for $\lambda>0$ the integral representation of E (2.4) has poles on the contour thus it is *not* well-defined. There are two different choices of contour which resolve this ambiguity, and these two choices yield two fundamental solutions.

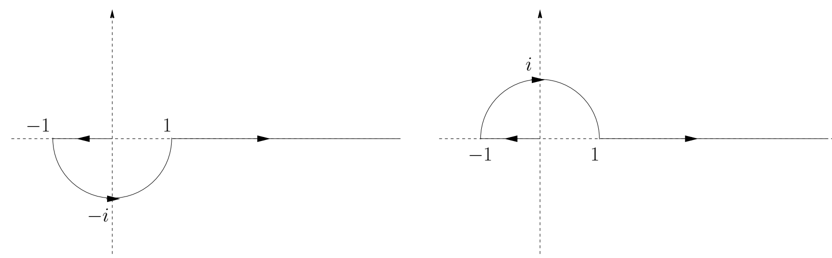
Proposition 2.2 ***(Integral representation of E_{out} and E_{in} for Helmholtz).*** *For the Helmholtz equation, the two fundamental solutions are given by*

$$E_{out}(z',z)=\frac{i}{4}H_0^{(1)}(2\beta|z-z'|)=\frac{1}{4\pi}\int_{L_{out}}\frac{dk}{k}\exp\left(i\beta\left(k(z-z')e^{-i\theta}+\frac{1}{k}(\overline{z-z'})e^{i\theta}\right)\right), \tag{2.10a}$$

$$E_{in}(z',z)=-\frac{i}{4}H_0^{(2)}(2\beta|z-z'|)=\frac{1}{4\pi}\int_{L_{in}}\frac{dk}{k}\exp\left(i\beta\left(k(z-z')e^{-i\theta}+\frac{1}{k}(\overline{z-z'})e^{i\theta}\right)\right), \tag{2.10b}$$

$$\theta\leq\arg(z-z')\leq\theta+\pi,$$

where the contours L_{out} and L_{in} are shown in figure 2.1. (Note that if $\arg(z-z')=\theta$ *or* $\theta+\pi$ *then the integrals in* (2.10) *are not absolutely convergent.)*



(a) The contour L_{out} of integration in the k plane for $H_0^{(1)}$, i.e. outgoing.

(b) The contour L_{in} of integration in the k plane for $H_0^{(2)}$, i.e. incoming.

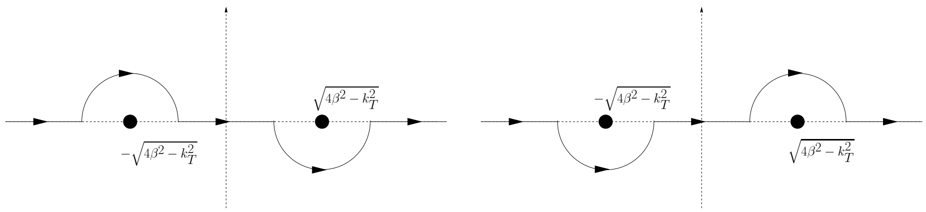
Figure 2: The contours L_{out} and L_{in}.

Proof . For Helmholtz the fundamental solution is given by (2.4) with $\lambda = 4\beta^2$, $\beta \in \mathbb{R}$. As before, first perform the k_N integral. For $|k_T| \geq 2\beta$ the poles are on the imaginary axis (like for Modified Helmholtz) at $k_N = \pm i\sqrt{k_T^2 - 4\beta^2}$. For $|k_T| \leq 2\beta$ the poles are on the real axis, at $k_N = \pm\sqrt{4\beta^2 - k_T^2}$, and the k_N integral is not well defined unless the path of integration around the poles is specified. The two paths around the poles yielding non-zero contributions are given in figure 2.1. The two choices differ in their asymptotic behaviour at infinity, which correspond to outgoing or incoming waves respectively (with the assumed time dependence $e^{-i\omega t}$). The asymptotics can be determined by applying the method of steepest descent to (2.10). Perform the k_N integration to obtain

$$E(\xi,\eta,x,y) = \frac{1}{4\pi}\int_{|k_T|>2\beta} \frac{e^{ik_T X_T - \sqrt{k_T^2-4\beta^2}|X_N|}}{\sqrt{k_T^2-4\beta^2}} dk_T \pm \frac{i}{4\pi}\int_{-2\beta}^{2\beta} \frac{e^{ik_T X_T \pm i\sqrt{4\beta^2-k_T^2}|X_N|}}{\sqrt{4\beta^2-k_T^2}} dk_T \tag{2.11}$$

with the top sign corresponding to $H_0^{(1)}$ and the bottom sign to $H_0^{(2)}$. As with modified Helmholtz, a change of variables can be used to eliminate the square roots. In this case this change of variables is $k_T = \beta\,(l + 1/l)$. This is motivated by the fact that if $l = e^{i\phi}$ then $k_T = 2\beta\cos\phi$, and hence $\sqrt{4\beta^2 - k_T^2} = \pm 2\beta\sin\phi = \mp i\beta(l - 1/l)$. Due to the square roots there are several choices for the range of integration in l, but all these choices lead to equivalent answers. In order to get the same integrand for both integrals in (2.11) we choose $l \in (1,\infty)$ and $l \in (0,-1)$ in the first integral, $\phi \in (-\pi, 0)$ in the second for outgoing, and $\phi \in (0,\pi)$ in the second for incoming. A few lines of computation as well as relabelling l as k yields (2.10). The proof that (2.10) defines a function of $z - z'$ in the whole plane follows by contour deformation in a similar way to that for Modified Helmholtz. $\square$

For Poisson, this algorithm of constructing representations of the fundamental solution results in a representation that is *formal*, namely it involves divergent integrals. This is a consequence of the fact that the fundamental solution for Poisson, $-\frac{1}{2\pi}\log|\boldsymbol{\xi} - \boldsymbol{x}|$, does not decay at infinity, and so its Fourier transform



(a) $H_0^{(1)}$ i.e. Outgoing (b) $H_0^{(2)}$ i.e. Incoming

Figure 3: The contours for the k_N integration in the fundamental solution to obtain $H_0^{(1)}$ and $H_0^{(2)}$

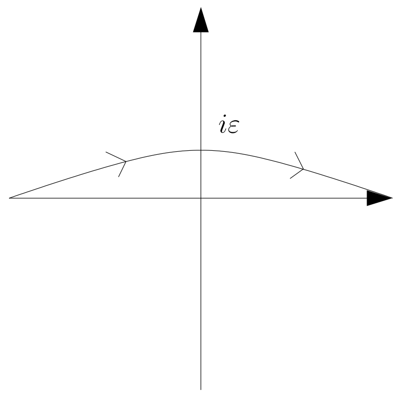


Figure 4: The deformed contour of integration in the k_T plane for Poisson.

is not well defined in a classical sense. Nevertheless, when the representation of the fundamental solution is substituted into Green's IR, the resulting IR for u *is* well defined.

Proposition 2.3 ***(Integral representation of*** E ***for Poisson).*** *For Poisson's equation, a formal representation of the fundamental solution is given by*

$$E(z', z) = \lim_{\varepsilon \to 0} \frac{1}{4\pi} \left(\int_{i\varepsilon}^{\infty} \frac{dk}{k} \exp\left(ike^{-i\theta}(z - z')\right) + \int_{-i\varepsilon}^{\infty} \frac{dk}{k} \exp\left(-ike^{i\theta}\overline{(z - z')}\right) \right), \tag{2.12}$$

$$\theta \leq \arg(z - z') \leq \theta + \pi.$$

(Note that if $\arg(z - z') = \theta$ *or* $\theta + \pi$ *then the integrals in* (2.12) *are not absolutely convergent.)*

Proof . Start with (2.4) with $\lambda = 0$ and perform the same change of variables, (2.6) and (2.7), as for modified Helmholtz to obtain (2.8) with $\beta = 0$. Deform the k_T contour above or below the singularity at 0 so it passes through $i\varepsilon$, see figure 2.1. Perform the k_N integral by closing the contour in the upper half plane to enclose the pole at $k_N = ik_T$ for $\Re k_T > 0$ and $k_N = -ik_T$ for $\Re k_T < 0$, to obtain

$$E(\xi,\eta,x,y) = \frac{1}{4\pi}\left(\int_{i\varepsilon}^{\infty} dk_T\, \frac{e^{ik_TX_T-k_TX_N}}{k_T} + \int_{-\infty}^{i\varepsilon} dk_T\, \frac{e^{ik_TX_T+k_TX_N}}{-k_T}\right). \tag{2.13}$$

Now let $k_T \mapsto -k_T$ in the second integral and use (2.6) to obtain the right hand side of (2.12) without the $\varepsilon \to 0$ limit (after relabelling k_T as k).

The rotation of contours that shows (2.5) and (2.10) are well defined functions of $z - z'$ in the whole plane requires that the contours start from zero and hence fails for (2.13). Taking the $\varepsilon \to 0$ limit allows this argument to proceed for (2.13), however the integrals do not converge in this limit. □ □

Remark 2.4 ***(Derivation via one transform)*** *The IRs of E have been obtained by taking two transforms to obtain* (2.4)*, then computing one integral. Alternatively, they can be obtained by taking one transform, then solving one ODE. For example, after rotating co-ordinates using* (2.6)*, equation* (2.1) *with $\lambda = -4\beta^2$ becomes*

$$E_{X_TX_T} + E_{X_NX_N} - 4\beta^2 E = -\delta(X_T)\delta(X_N).$$

Taking the Fourier transform in X_T yields

$$\widehat{E}_{X_NX_N} - (4\beta^2 + k_T^2)\widehat{E} = -e^{ik_TX_T}\delta(X_N),$$

which can be solved using a 1-d Green's function to give

$$\widehat{E} = \frac{e^{ik_TX_T - \sqrt{k_T^2+4\beta^2}|X_N|}}{2\sqrt{k_T^2+4\beta^2}}.$$

Then the inverse Fourier transform yields (2.9)*.*

2.2 Domain dependent fundamental solutions for a convex polygon.

Let $\Omega^{(i)}$ be the interior of a convex polygon in $\mathbb{R}^2$. Let $\partial\Omega$ denote the boundary of the polygon, oriented *anticlockwise*, where the vertices of the polygon $z_1, z_2, ..., z_n$ are labelled anticlockwise. Let S_j be the side (z_j, z_{j+1}) and let $\alpha_j = \arg(z_{j+1} - z_j)$ be the angle of S_j.

Proposition 2.5 . *Let $z' \in S_j$, $z \in \Omega^{(i)}$.*

- *For Modified Helmholtz,*

$$E(z',z) = \frac{1}{4\pi}\int_{l_j} \frac{dk}{k} \exp\left(i\beta\left(k(z-z') - \frac{\overline{(z-z')}}{k}\right)\right), \tag{2.14}$$

where

$$l_j = \{k \in \mathbb{C} : k = le^{-i\alpha_j}, l \in (0,\infty)\}. \tag{2.15}$$

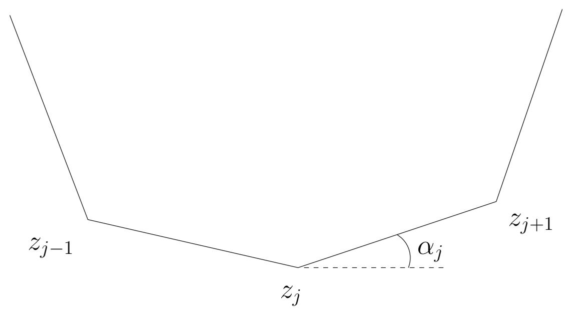


Figure 5: The convex polygon $\Omega^{(i)}$.

- *For Helmholtz,*

$$E(z',z) = \frac{1}{4\pi}\int_{L_{out\,j}} \frac{dk}{k}\exp\left(i\beta\left(k(z-z') + \frac{\overline{(z-z')}}{k}\right)\right), \tag{2.16}$$

where

$$L_{out\,j} = \{k \in \mathbb{C} : k = le^{-i\alpha_j}, l \in L_{out}\}. \tag{2.17}$$

- *For Poisson,*

$$E(z',z) = \lim_{\varepsilon\to 0}\frac{1}{4\pi}\left(\int_{l_{j\varepsilon}} \frac{dk}{k}\exp\left(ik(z-z')\right) + \int_{\overline{l_{j\varepsilon}}} \frac{dk}{k}\exp\left(-ik\overline{(z-z')}\right)\right), \tag{2.18}$$

where

$$l_{j\varepsilon} = \{k \in \mathbb{C} : k = le^{-i\alpha_j}, l \in (i\varepsilon,\infty)\}.$$

Proof . For $z' \in S_j$, $z \in \Omega^{(i)}$, convexity implies

$$\alpha_j < \arg(z-z') < \alpha_j + \pi,$$

so (2.5) holds with $\theta = \alpha_j$. Let $k \mapsto ke^{i\alpha_j}$ to get (2.14). Similarly for (2.18), (2.16). □

2.3 The integral representations for a convex polygon.

The integral representations are obtained by substituting the domain dependent fundamental solutions into Green's IR.

If u is a solution of (1.1) for $\Omega = \Omega^{(i)}$, then Green's IR is

$$u(z,\bar{z}) = \sum_{j=1}^{n}\int_{z_j}^{z_{j+1}} E\left(u_\xi d\eta - u_\eta d\xi\right) - u\Big(E_\xi d\eta - E_\eta d\xi\Big) + \int_\Omega f\,E\,d\xi d\eta, \tag{2.19a}$$

$$= -i\sum_{j=1}^{n}\int_{z_j}^{z_{j+1}} E\left(u_{z'}dz' - u_{\bar{z}'}d\bar{z}'\right) - u\Big(E_{z'}dz' - E_{\bar{z}'}d\bar{z}'\Big) + \int_\Omega f\,E\,d\xi d\eta, \quad z\in\Omega. \tag{2.19b}$$

If the polygon is *open*, then u must obey the following boundary conditions at infinity:

- $\lambda = 0$ (Poisson), $u \to 0$ as $r \to \infty$.
- $\lambda = -4\beta^2$ (Modified Helmholtz), $u \to 0$ as $r \to \infty$.
- $\lambda = 4\beta^2$ (Helmholtz),

$$\sqrt{r}\left(\frac{\partial u}{\partial r} - 2i\beta u\right) \to 0 \text{ as } r \to \infty, \tag{2.20}$$

where $r = \sqrt{x^2 + y^2}$.

2.4 Modified Helmholtz.

Proposition 2.6 ***(Integral representation of the solution of modified Helmholtz).*** *Let $\lambda = -4\beta^2$. Let u be a solution of* (1.1) *in the domain $\Omega^{(i)}$. Suppose that u has the integral representation* (2.19)*. Then u also has the* alternative *representation*

$$u(z,\bar{z}) = \frac{1}{4\pi i}\sum_{j=1}^{n}\int_{l_j}\frac{dk}{k}e^{i\beta\left(kz-\frac{\bar{z}}{k}\right)}\widehat{u}_j(k) + F(z,\bar{z}), \quad z \in \Omega, \tag{2.21}$$

where l_j, $j = 1, ..., n$, are rays in the complex k plane, oriented from 0 to ∞ and defined by (2.15)*. The transforms of the boundary values of u on the side j, denoted by $\{\widehat{u}_j(k)\}_1^n$, are given by*

$$\widehat{u}_j(k) = \int_{z_j}^{z_{j+1}} e^{-i\beta\left(kz'-\frac{\bar{z}'}{k}\right)}[(u_{z'} + i\beta ku)dz' - (u_{\bar{z}'} + \frac{\beta}{ik}u)d\bar{z}'], \quad j = 1, .., n, \quad z_{n+1} = z_1, \tag{2.22}$$

$$= \int_{z_j}^{z_{j+1}} e^{-i\beta\left(kz'+\frac{\bar{z}'}{k}\right)}\left[i\frac{\partial u}{\partial n}(z'(s)) + i\beta\left(k\frac{dz'}{ds} + \frac{1}{k}\frac{d\bar{z}'}{ds}\right)u(z'(s))\right]ds, \quad z' = z'(s) \in S_j,$$

where n is the outward *pointing normal to the polygon (in* both *interior and exterior cases). The forcing term is given by*

$$F(z,\bar{z}) = \iint_{\Omega} f(\xi,\eta)\, E(z',z)\, d\xi d\eta, \tag{2.23}$$

where E is the fundamental solution for Modified Helmholtz.

Proof of proposition 2.6. Substituting (2.14) into the first term on the right hand side of (2.19b) (the boundary integral) and interchanging the order of integration, this term becomes the first term on the right hand side of (2.21).

□

2.5 Helmholtz.

For Helmholtz there are two fundamental solutions. For Ω unbounded we can choose one of these solutions using the requirement that we require *outgoing* waves. However, when $\Omega = \Omega^{(i)}$ the domain is bounded and hence it is *not* possible to choose between the outgoing and incoming fundamental solutions. It turns out that substituting the integral representation of *either* of these fundamental solutions into Green's integral representation yields the *same* result for $\Omega = \Omega^{(i)}$.

Proposition 2.7 ***(Integral representation of the solution of Helmholtz).*** *Let $\lambda = 4\beta^2$. Let u be a solution of* (1.1) *in the domain $\Omega^{(i)}$. Suppose that u has the integral representation* (2.19). *Then u also has the* alternative *representation*

$$u(z,\bar{z}) = \frac{1}{4\pi i}\sum_{j=1}^{n}\int_{L_{out\,j}} \frac{dk}{k} e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)}\widehat{u}_j(k) + F(z,\bar{z}), \tag{2.24}$$

where $L_{out\,j}$, $j = 1, ..., n$, are rays in the complex k plane oriented from 0 to ∞ and defined by (2.17) *where the contour L_{out} is shown in Figure 2.1. The transforms of the boundary values of u on the side j, denoted by $\{\widehat{u}_j(k)\}_1^n$, are given by*

$$\widehat{u}_j(k) = \int_{z_j}^{z_{j+1}} e^{-i\beta\left(kz'+\frac{\bar{z}'}{k}\right)}[(u_{z'} + i\beta k u)dz' - (u_{\bar{z}'} - \frac{\beta}{ik}u)d\bar{z}'], \quad j = 1,..,n, \quad z_{n+1} = z_1, \tag{2.25}$$

$$= \int_{z_j}^{z_{j+1}} e^{-i\beta\left(kz'-\frac{\bar{z}'}{k}\right)}\left[i\frac{\partial u}{\partial n}(s) + i\beta\left(k\frac{dz'}{ds} - \frac{1}{k}\frac{d\bar{z}'}{ds}\right)u(s)\right]ds, \quad z' = z'(s) \in S_j,$$

where n is the outward *pointing normal to the polygon (in* both *interior and exterior cases). The forcing term is given by*

$$F(z,\bar{z}) = \iint_{\Omega} f(\xi,\eta)\, E(z',z)\, d\xi d\eta, \tag{2.26}$$

where E is the fundamental solution for Helmholtz.

Proof of proposition 2.7. This follows in *exactly* the same way as for modified Helmholtz, using of course (2.16) instead of (2.14). If E_{in} is used in (2.19) instead of E_{out} then we obtain (2.24) with L_{out} replaced by L_{in}. In order to show that (2.24) and the corresponding representation obtained by replacing L_{out} with L_{in} are equivalent, it is necessary to use the following *global relation* for Helmholtz,

$$\sum_{j=1}^{n}\widehat{u}_j(k) + i\widehat{f}(k) = 0, \quad k \in \mathbb{C}. \tag{2.27}$$

This equation is derived in [Spence and Fokas, 2009], see proposition 3.2. For simplicity consider $f = 0$ (the case of non-zero f is considered in [Spence, 2009]). The two representations differ by

$$\frac{1}{4\pi i}\oint_{\{|k|=1\}}\frac{dk}{k}e^{i\beta\left(kz+\frac{\bar{z}}{k}\right)}\left(\sum_{j=1}^{n}\widehat{u}_j(k)\right).$$

Indeed, adding the circle oriented anticlockwise to L_{out} transforms L_{out} to to L_{in}; similar considerations apply to the contours L_{outj} which are just L_{out} rotated by certain angles. Due to the global relation (2.27), the above integral vanishes. $\square$

2.6 Poisson.

The representation of the fundamental solution for Poisson (2.18) is formal, since the integrals do not converge in the limit of $\varepsilon \to 0$. However, after the domain-dependent fundamental solution (2.18) is substituted into Green's IR, the $\varepsilon \to 0$ limit *does* exist because the solution u satisfies the following consistency condition:

$$\int_{\partial\Omega}\frac{\partial u}{\partial n}ds := \int_{\partial\Omega}u_\xi d\eta - u_\eta d\xi = -\int_{\Omega}f d\xi d\eta. \tag{2.28}$$

This equation can be obtained by integrating (1.1) over Ω and applying Green's theorem (recall that $\partial\Omega$ is oriented anticlockwise). These conditions imply that $k = 0$ is a *removable* singularity.

Proposition 2.8 ***(Integral representation of the solution of Poisson's equation).*** *Let $\lambda = 0$. Let u be a solution of* (1.1) *in the domain $\Omega^{(i)}$. Suppose that u has the integral representations* (2.19)*. Then u also has the* alternative *representation*

$$u(z,\bar{z}) = \frac{1}{4\pi i}\sum_{j=1}^{n}\int_{l_j}\frac{dk}{k}e^{ikz}\widehat{u}_j(k) - \frac{1}{4\pi i}\sum_{j=1}^{n}\int_{\overline{l_j}(z)}\frac{dk}{k}e^{-ik\bar{z}}\widetilde{u}_j(k) + F(z,\bar{z}), \quad z \in \Omega. \tag{2.29}$$

The transforms of the boundary values of u on the side j, denoted by $\{\widehat{u}_j(k)\}_1^n$ and $\{\widetilde{u}_j(k)\}_1^n$, are defined by

$$\widehat{u}_j(k) = \int_{z_j}^{z_{j+1}} e^{-ikz'}\left[(u_{z'} + iku)\,dz' - u_{\bar{z}'}d\bar{z}'\right], \quad j = 1,..,n, \quad z_{n+1} = z_1, \tag{2.30}$$

$$= \int_{z_j}^{z_{j+1}} e^{-ikz'}\left[i\frac{\partial u}{\partial n}(s) + ik\frac{dz'}{ds}u(s)\right]ds, \quad z' = z'(s) \in S_j$$

and

$$\widetilde{u}_j(k) = \int_{z_j}^{z_{j+1}} e^{ik\bar{z}} \left[(u_{\bar{z}} - iku)d\bar{z} - u_z dz\right], \quad j = 1,..,n, \quad z_{n+1} = z_1, \tag{2.31}$$
$$= \int_{z_j}^{z_{j+1}} e^{ik\bar{z}'} \left[-i\frac{\partial u}{\partial n}(s) - ik\frac{d\bar{z}'}{ds}u(s)\right] ds, \quad z' = z'(s) \in S_j.$$

The contours l_j, $j = 1, ..., n$, are the same as in Proposition 2.6, (2.15) *and $\overline{l_j}$, $j = 1, ..., n$, are the complex conjugates of l_j, $j = 1, ..., n$, that is, the rays in the complex k-plane oriented toward infinity defined by*

$$\overline{l_j} = \{k \in \mathbb{C} : k = le^{i\alpha_j}, l \in (0, \infty)\}. \tag{2.32}$$

The forcing term is given by

$$F(z, \bar{z}) = \iint_{\Omega^{(i)}} f(\xi, \eta)\, E(z', z)\, d\xi d\eta, \tag{2.33}$$

where E is the fundamental solution for Poisson.

Remark 2.9 ***(Removable singularity at*** $k = 0$ ***and consistency conditions).*** *The point $k = 0$ is a* removable *singularity in* (2.29)*. Indeed the consistency condition* (2.28) *implies that*

$$-i\sum_{j=1}^{n} \widehat{u_j}(0) = i\sum_{j=1}^{n} \widetilde{u_j}(0) = -\iint_{\Omega^{(i)}} f(\xi, \eta)\, d\xi d\eta.$$

Proof of proposition 2.8. This is identical to the proof of prop. 2.6 using (2.18) instead of (2.14) and also letting $\varepsilon \to 0$ at the end. $\square$

Remark 2.10 ***(Non-convex polygons).*** *The co-ordinate system dependent fundamental solutions of §2.1 can also be used to obtain integral representations for non-convex polygons [Spence, 2009], [Charalambopoulos et al., 2009]. However in this case, the contours of integration depend on the position of z; namely they depend on the position of z relative to each side of the polygon.*

Remark 2.11 ***(Forcing term).*** *In order to obtain a spectral representation of the forcing term in Green's IR using* (2.5),(2.10)*, and* (2.12) *we require that for $z' \in \Omega$ and $z \in \Omega$, $\theta \leq \arg(z - z') \leq \theta + \pi$ for some θ. This is* impossible *unless we split the domain Ω into two regions by a line through z, where we are free to choose the angle of the split. This means that the transforms of the forcing term depend on z, which is inconvenient, but apparently unavoidable [Spence, 2009]. A spectral representation of the forcing term is useful for the numerical evaluation of the solution, since it is then possible to use contour deformations to obtain integrands which decay exponentially as $|k| \to \infty$.*

2.7 Relation to classical results.

The idea of obtaining integral representations of E is certainly not new. However, the representations presented here apparently contain certain novel features explained below.

The integral representations of the different fundamental solution, (2.5),(2.12), and (2.10) differ from the classical representations in two novel ways:

Rotation to half-planes. Starting with (2.4), one of the following operations is usually performed:

1. Rotate the k_1, k_2 co-ordinates so k_2 lies in direction of $(x-\xi, y-\eta)$; for Modified Helmholtz this yields

$$E(\xi,\eta,x,y) = \frac{1}{(2\pi)^2}\iint_{\mathbb{R}^2} dk_1 dk_2 \frac{e^{ik_2\sqrt{(x-\xi)^2+(y-\eta)^2}}}{k_1^2+k_2^2+4\beta^2}.$$

   Computing the k_2 integral, this becomes

$$E(\xi,\eta,x,y) = \frac{1}{4\pi}\int_{-\infty}^{\infty} dk_1 \frac{e^{-\sqrt{k_1^2+4\beta^2}\sqrt{(x-\xi)^2+(y-\eta)^2}}}{\sqrt{k_1^2+4\beta^2}}. \qquad (2.34)$$

   Such integral representations appear on p.56-57 and 279 of [Stakgold, 1967] vol. 2 and on p. 376 of [Abramowitz and Stegun, 1965] (equations 9.6.23, 9.6.24).

2. Compute the k_2 integral without first performing a rotation; for Modified Helmholtz this yields

$$E(\xi,\eta,x,y) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{ik_1(x-\xi)-\sqrt{k_1^2+4\beta^2}|y-\eta|}}{\sqrt{k_1^2+4\beta^2}} dk_1,$$

   which is (2.9) with $\theta = 0$ in (2.6). This appears on p. 298 of [Ablowitz and Fokas, 2003]; the analogue for Helmholtz, which is (2.11) with $\theta = 0$ in (2.6), appears on p. 278 of [Duffy, 2001] equation (5.1.22) (actually, equation (5.1.22) has a error in the sign of the square root in the exponent, but the equation (5.1.21), from which it is obtained, is correct.)

Change of variables to eliminate square roots. The transformations $k_T = \beta(l - 1/l)$ for Modified Helmholtz and $k_T = \beta(l + 1/l)$ for Helmholtz, transform $\sqrt{4\beta^2 + k_T^2}$ and $\sqrt{k_T^2 - 4\beta^2}$ into $\beta(l + 1/l)$ and $\beta(l - 1/l)$ respectively (modulo a sign depending on the range of l). These transformations eliminate the square roots at the cost of introducing a pole at $l = \infty$. These transformations have been used earlier but only in polar co-ordinates, in particular using them in the two analogues of (2.34) for Helmholtz yields the IRs for $H_0^{(1)}$ and $H_0^{(2)}$ involving the contours of integration L_{out} and L_{in} respectively, see [Watson, 1966] §6.21 p. 179. It is surprising that apparently these transformations have *not* been used before in the solution of Helmholtz in cartesian co-ordinates. Indeed, [Ockendon et al., 2003] §5.8.3 page 191 states that "in a half plane transform methods for Helmholtz equation ... are cursed by the presence of branch points in the transform plane".

It is important to note that it is not possible to obtain the integral representations of propositions 2.6-2.8 using the representations of E of the form (2.34), without first transforming (2.34) into the representations of propositions 2.1-2.3. This is because the integrand of (2.34) cannot be written as a function of (x, y) multiplied by a function of (ξ, η), and thus it is not possible to interchange the physical and spectral integrals when (2.34) is substituted into Green's integral representation.

3 Integral representations for polar co-ordinates.

Notations. Let (r, θ) be the physical variable, and (ρ, ϕ) be the "dummy variable" of integration.

We will consider only Poisson, $\lambda = 0$, and Helmholtz, $\lambda = \beta^2$ (note that this is different to $\lambda = 4\beta^2$ used for Helmholtz in §2).

3.1 Co-ordinate system dependent fundamental solutions.

In polar co-ordinates, (1.3) becomes

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial E}{\partial \rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 E}{\partial \phi^2}+\beta^2 E = -\frac{\delta(\rho-r)\delta(\phi-\theta)}{\rho}, \quad 0<r,\rho<\infty, \quad \theta,\phi,\ 2\pi \text{ periodic.} \tag{3.1}$$

For reasons which will be explained later (see remark 3.4 and §3.3), we consider the *non-periodic* fundamental solution defined by

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial E_s}{\partial \rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 E_s}{\partial \phi^2}+\beta^2 E_s = -\frac{\delta(\rho-r)\delta(\phi-\theta)}{\rho}, \quad 0<r,\rho<\infty, \quad -\infty<\theta,\phi<\infty. \tag{3.2}$$

Proposition 3.1 ***(Integral representation of E_s for Helmholtz).*** *For the Helmholtz equation, the outgoing non-periodic fundamental solution E_s can be expressed in terms of radial eigenfunctions (*the radial representation*) in the form*

$$E_s(\rho,\phi;r,\theta) = \lim_{\varepsilon\to 0}\frac{i}{4}\left(\int_0^{i\infty} dk\, e^{\varepsilon k^2} H_k^{(1)}(\beta\rho)J_k(\beta r)e^{ik|\theta-\phi|} + \int_0^{-i\infty} dk\, e^{\varepsilon k^2} H_k^{(1)}(\beta\rho)J_k(\beta r)e^{-ik|\theta-\phi|}\right). \tag{3.3}$$

*Alternatively, it can be expressed in terms of angular eigenfunctions (*the angular representation*) in the form*

$$E_s(\rho, \phi; r, \theta) = \frac{i}{4}\left(\int_0^\infty dk\, H_k^{(1)}(\beta r_>)J_k(\beta r_<)e^{ik(\theta-\phi)} + \int_0^\infty dk\, H_k^{(1)}(\beta r_>)J_k(\beta r_<)e^{-ik(\theta-\phi)}\right), \tag{3.4}$$

where $r_> = \max(r, \rho)$*,* $r_< = \min(r, \rho)$ *and* $-\infty < \theta, \phi < \infty$*.*

Proof . The differential operator

$$-\frac{d^2}{d\theta^2}u = \lambda u$$

on $(-\infty, \infty)$ possesses the completeness relation

$$\delta(\theta - \phi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik_2(\theta-\phi)}dk_2, \tag{3.5}$$

i.e. the Fourier transform([Stakgold, 1967], chapter 4). The completeness relation associated with the operator

$$-\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) - \beta^2\rho u = \lambda\frac{u}{\rho} \tag{3.6}$$

on $0 < \rho < \infty$, with the additional condition that the eigenfunctions satisfy the *outgoing* radiation condition, is

$$\rho\delta(r - \rho) = \lim_{\varepsilon\to 0}\frac{1}{2}\int_{-i\infty}^{i\infty} dk_1\, k_1\, e^{\varepsilon k_1^2}\, H_{k_1}^{(1)}(\beta r_1)J_{k_1}(\beta r_2), \tag{3.7}$$

where either $r_1 = r, r_2 = \rho$ or visa versa, see remark 3.2. The completeness relations (3.5) and (3.7) give rise to the following integral representation of E:

$$E_s(\rho, \phi; r, \theta) = \lim_{\varepsilon\to 0} -\frac{1}{4\pi}\int_{-i\infty}^{i\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \frac{e^{ik_2(\theta-\phi)}\, k_1\, e^{\varepsilon k^2}\, H_k^{(1)}(\beta r_1)J_k(\beta r_2)}{k_1^2 - k_2^2}. \tag{3.8}$$

Choose $r_1 = \rho$ and $r_2 = r$, so that E_s satisfies the outgoing radiation condition in ρ. (In fact, r and ρ can be interchanged in the right hand side of (3.3) using an identity involving integrals of Bessel functions, [Spence, 2009].) If $\theta > \phi$, close the k_2 integral in the upper half k_2 plane, enclosing the pole at $k_2 = k_1$ for $\Im k_1 \geq 0$ and at $k_2 = -k_1$ for $\Im k_1 \leq 0$. Relabel k_1 as k to obtain (3.3) for $\theta > \phi$. Similarly, if $\theta < \phi$, close in the lower half k_2 plane to obtain (3.3) for $\theta < \phi$.

Evaluating the k_1 integral requires knowledge of the asymptotics of the product of the Bessel functions as $k \to \infty$. For x, y fixed, the following formulae are

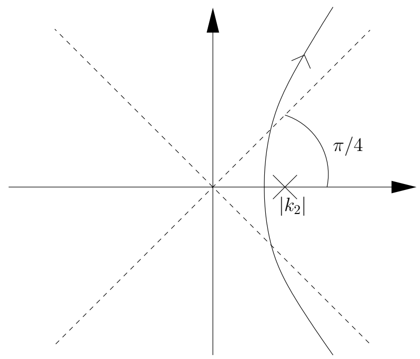


Figure 6: The deformed contour in the k_1 plane.

valid:

$$kJ_k(y)H_k^{(1)}(x) \sim -\frac{i}{\pi}\left(\frac{y}{x}\right)^k, \qquad |k| \to \infty,\ -\frac{\pi}{2} < \arg k < \frac{\pi}{2}, \tag{3.9}$$

$$kJ_k(y)H_k^{(1)}(x) \sim -2i\frac{\sin k\pi}{\pi}e^{-i\pi k}\left(\frac{-2k}{ex}\right)^{-k}\left(\frac{-2k}{ey}\right)^{-k}, \quad |k| \to \infty,\ \frac{\pi}{2} < \arg k < \frac{3\pi}{2}. \tag{3.10}$$

Hence the product $kJ_k(y)H_k^{(1)}(x)$ is bounded at infinity only when $\Re k > 0$ and $x > y$.

For $r_1 > r_2$ deform the k_1 contour to some contour in the right half plane, enclosing the pole at $k_1 = |k_2|$, with $\frac{\pi}{4} < \arg k_1 < \frac{\pi}{2}$ as $|k| \to \infty$ for $\Im k_1 > 0$ and $-\frac{\pi}{2} < \arg k_1 < -\frac{\pi}{4}$ as $|k| \to \infty$ for $\Im k_1 < 0$, see figure 6. Now the k_1 integral converges absolutely even with $\varepsilon = 0$, so by the dominated convergence theorem ε can be set to zero in the integrand. After calculating the residue, this results in

$$E_s(\rho, \phi; r, \theta) = \frac{i}{4}\int_{-\infty}^{\infty} dk_2\, H_{|k_2|}^{(1)}(\beta r_>)J_{|k_2|}(\beta r_<)e^{ik_2(\theta-\phi)}.$$

Relabel k_2 as k and let $k \mapsto -k$ for $k < 0$ to obtain (3.3). The proof that (3.3) and (3.4) define the same function follows by deforming the integrals on $(0, i\infty)$ and $(0, -i\infty)$ to $(0, \infty)$ using (3.9) and the fact, mentioned earlier, that r and ρ can be interchanged in the right hand side of (3.3). $\square$

Remark 3.2 ***(The Kontorovich-Lebedev transform)*** *D.S. Jones has proved rigorously that the following transform pair is valid:*

$$g(k) = \int_0^\infty dy f(y)H_k^{(j)}(y), \tag{3.11}$$

$$xf(x) = -(-1)^j \lim_{\varepsilon\to 0}\frac{1}{2}\int_{-i\infty}^{i\infty} dk\, e^{\varepsilon k^2} k\, J_k(x)g(k), \quad j = 1, 2, \tag{3.12}$$

where if the eigenfunctions satisfy the outgoing *radiation condition* $j = 1$ *and if the eigenfunctions satisfy the* incoming *radiation condition* $j = 2$*, [Jones, 1980]. Jones has shown that* (3.12) *without the regularising term* $\exp \varepsilon k^2$ *(which is obtained by the spectral analysis of* (3.6)*) diverges even for the function* $g(y) = e^{-ay}$, $\Re a > 0$*. Essentially the reason for this divergence is that the product of* H_k *and* J_k *is unbounded on* $i\mathbb{R}$*. However, most of the solutions to BVPs obtained by using* (3.11) *and* (3.12) *without the term* $\exp \varepsilon k^2$ *are correct because the contour is deformed (albeit illegally) and the resulting expression converges. The term* $\exp \varepsilon k^2$ *justifies rigorously the contour deformation, after which* ε *can be set to zero.*

For the Poisson equation, the algorithm results in formal representations of the non-periodic fundamental solutions, as in the case of Cartesian co-ordinates. Nevertheless, when these representations are substituted into Green's IR, the resulting representation for u is well-defined.

Proposition 3.3 ***(Integral representation of*** E_s ***for Poisson)*** *For the Poisson equation the non-periodic fundamental solution* E_s *can be expressed formally either in terms of radial eigenfunctions (*the radial representation*) in the form*

$$E_s(\rho, \phi; r, \theta) = \lim_{\varepsilon \to 0} \frac{1}{4\pi} \left(\int_{\varepsilon}^{i\infty} \frac{dk}{k} \left(\frac{\rho}{r}\right)^k e^{ik|\theta-\phi|} + \int_{\varepsilon}^{-i\infty} \frac{dk}{k} \left(\frac{\rho}{r}\right)^k e^{-ik|\theta-\phi|} \right), \tag{3.13}$$

*or in terms of angular eigenfunctions (*the angular representation*) in the form*

$$E_s(\rho, \phi; r, \theta) = \lim_{\varepsilon \to 0} \frac{1}{4\pi} \left(\int_{i\varepsilon}^{\infty} \frac{dk}{k} \left(\frac{r_>}{r_<}\right)^{-k} e^{ik(\theta-\phi)} + \int_{-i\varepsilon}^{\infty} \frac{dk}{k} \left(\frac{r_>}{r_<}\right)^{-k} e^{-ik(\theta-\phi)} \right), \tag{3.14}$$

where $r_> = \max(r, \rho)$*,* $r_< = \min(r, \rho)$*,* $-\infty < \theta, \phi < \infty$*.*

Proof. The operator in θ is the same as for Helmholtz, and so the appropriate completeness relation is (3.5). The differential operator

$$-\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) = \lambda \frac{u}{\rho}$$

on $0 < \rho < \infty$, possesses the completeness relation

$$\rho\, \delta(r - \rho) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left(\frac{\rho}{r}\right)^{k_1} dk_1, \tag{3.15}$$

i.e. the Mellin transform([Stakgold, 1967], chapter 4, p.308). After rewriting (3.2) in the form

$$\rho \frac{\partial}{\partial \rho}\left(\rho \frac{\partial E_s}{\partial \rho}\right) + \frac{\partial^2 E_s}{\partial \phi^2} = -\rho\, \delta(r - \rho)\delta(\theta - \phi),$$

the above two completeness relations give rise to the following integral representation of E:

$$E_s(\rho, \phi; r, \theta) = -\frac{1}{4\pi^2 i} \int_{-i\infty}^{i\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \frac{e^{ik_2(\theta-\phi)}}{k_1^2 - k_2^2} \left(\frac{\rho}{r}\right)^{k_1}. \tag{3.16}$$

If $\theta > \phi$, deform the k_1 integral off $i\mathbb{R}$ near the origin so that it passes through $\varepsilon \neq 0$. Then close the k_2 integral in the upper half k_2 plane, enclosing the pole at $k_2 = k_1$ for $\Im k_1 \geq 0$ and at $k_2 = -k_1$ for $\Im k_1 \leq 0$. Relabel k_1 as k to obtain (3.13), without the limit, for $\theta > \phi$. Similarly, if $\theta < \phi$, close in the lower half k_2 plane to obtain (3.13), without the limit, for $\theta < \phi$.

If $\rho > r$, deform the k_2 integral off $\mathbb{R}$ near the origin so that it passes through $i\varepsilon$, $\varepsilon \neq 0$. Then close the k_1 integral in the left half k_1 plane enclosing the pole at $k_1 = -k_2$ for $\Re k_2 \geq 0$ and at $k_1 = k_2$ for $\Re k_2 \leq 0$. Let $k_2 \mapsto -k_2$ for $\Re k_1 \leq 0$ and relabel k_2 as k to obtain (3.14) for $r > \rho$. Similarly, if $\rho < r$, close in the right half k_1 plane to obtain (3.14) for $\rho < r$ without the limit. The proof that (3.13) and (3.14) define the same function follows by contour deformation in a similar manner to the analogous proof for Helmholtz, except that the differences become zero in the limit $\varepsilon \to 0$.

$\square$

Remark 3.4 ***(Periodicity).*** *For some domains in polar co-ordinates, periodicity in the angle co-ordinate is natural. The operator*

$$-\frac{d^2}{d\theta^2}u = \lambda u$$

on $(0, 2\pi)$ *with* θ *periodic, possesses the completeness relation*

$$\delta(\theta - \phi) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in(\theta-\phi)}.$$

The analogue of (3.8) *is now*

$$E(\rho, \phi; r, \theta) = \lim_{\varepsilon \to 0} -\frac{1}{4\pi} \int_{-i\infty}^{i\infty} dk_1 \sum_{n=-\infty}^{\infty} \frac{e^{in(\theta-\phi)}\, e^{\varepsilon k_1^2}\, k_1\, H_{k_1}^{(1)}(\beta r_1) J_{k_1}(\beta r_2)}{k_1^2 - n^2}. \tag{3.17}$$

Computing the k_1 *integral yields the angular expansion*

$$E(\rho, \phi; r, \theta) = \frac{i}{4} \sum_{n=-\infty}^{\infty} H_n^{(1)}(\beta r_>) J_n(\beta r_<) e^{in(\theta-\phi)}. \tag{3.18}$$

Equation (3.18) *is the analogue of* (3.4). *However, the summation in* (3.17) cannot *be computed and therefore there does* not *exist an analogue of* (3.3). *This is consistent with the following construction: if one attempts to obtain the "radial"* *representation for Helmholtz by taking the Kontorovich-Lebedev transform in ρ of* (3.1) *(see remark 2.4), one finds*

$$\frac{d^2\widetilde{E}}{d\phi^2}(k,\phi) + k^2\widetilde{E}(k,\phi) = -\delta(\phi-\theta) r H_k^{(1)}(\beta r).$$

For solutions periodic in ϕ, $k \in \mathbb{Z}$, and under this restriction the Kontorovich-Lebedev transform cannot *be inverted. Similar considerations apply for Poisson.*

The fact that there do not exist radial representations under periodicity implies that there do not exist IRs in the transform space for any domain other than the interior and exterior of the circle. The so-called "hybrid method" of [Dassios and Fokas, 2008] for solving boundary value problems in the interior and exterior of the sphere, uses the analogue of (3.18) *in 3-d to obtain an IR in the transform space (the construction of the GR and the solution of boundary value problems in 3-d follow steps similar to the method in 2-d described in this paper).*

Remark 3.5 (Derivation of both co-ordinate dependent fundamental solutions using only one completeness relation). *In the proof of proposition 3.1 it was shown that* (3.3) *and* (3.4) *can be obtained from each other by contour deformation. Combining this with remark 2.4 shows that both* (3.3) *and* (3.4) *can actually be obtained using either one of the completeness relations* (3.5) *or* (3.11)*, although this approach is less algorithmic than the approach in the proofs of proposition 3.1.*

3.2 Novel integral representations for certain domains in polar co-ordinates.

In the sequel [Spence and Fokas, 2009], the co-ordinate dependent fundamental solutions of propositions 3.1 and 3.3 are used to construct novel integral representations for two examples in polar co-ordinates: the Helmholtz equation in the wedge

$$D_1 = \{a < r < \infty, \quad 0 < \theta < \alpha\}, \tag{3.19}$$

and the Poisson equation in the circular wedge,

$$D_2 = \{a < r < \infty, \quad -\alpha < \theta < \alpha\}. \tag{3.20}$$

After constructing the associated global relations, certain BVPs are solved using the steps outlined in the introduction.

3.3 Relation to classical results.

The novel idea of section §3.1 is to **consider the non-periodic fundamental solution, E_s, instead of E** (recall that there does *not* exist a radial representation under periodicity). The non-periodic fundamental solution was first introduced by Sommerfeld in 1896 when he extended the method of images to solve the problem of diffraction by a half-line, that is Helmholtz in the domain

$$\Omega = \{\mathbb{R}^2 \backslash \{x > 0\}\} = \{0 < r < \infty, \, 0 < \theta < 2\pi\}.$$

Sommerfeld required 4π *periodicity* under which the problem is solvable by images. Similarly, letting $-\infty < \phi, \theta < \infty$ (i.e. θ is *not* periodic) converts a wedge of arbitrary angle (less than 2π) into an infinite strip, which can then be solved using an infinite number of images. Sommerfeld expressed both his 4π periodic and non-periodic fundamental solutions (called "Riemann-surface" or "branched" solutions) as integrals of exponentials over the so-called Sommerfeld contours, [Sommerfeld, 1964] p.249. The expansion of the the non-periodic fundamental solution in angular eigenfunctions (3.4) is presented in [Stakgold, 1967] vol. 2 page 270. However, perhaps due to the unfamiliarity with the Kontorovich-Lebedev transform, the radial representation (3.3) does not appear to be known.

The periodic angular representation (3.18) and its analogue in 3-d is well known, e.g. [Morse and Feshbach, 1953] vol 1 p.827; these authors are aware of the non-availability of a radial representation (vol. 1 p.829).

4 Conclusions.

Integral representations of the solution of the *homogeneous* version of (1.1) in a convex polygon were given in [Fokas, 2001], using the simultaneous spectral analysis of the associated Lax pair (or equivalently the spectral analysis of an equivalent differential form). In addition, a representation for the derivative, u_z, of the solution of the Laplace equation was also obtained. In [Fokas and Zyskin, 2002] these representations were re-derived by substituting (2.4) into Green's integral representations and performing one k integration; the analogous method for deriving the integral representations for evolution PDEs was first presented in [Bressloff, 1997].

Boundary value problems for the *inhomogeneous* equation $u_{\bar{z}} = f$ were solved in [Fokas and Pinotsis, 2006] by first subtracting off a particular solution and then solving the homogeneous equation (in general this leads to a more complicated expression for the solution).
The main achievements of this paper are the following:

- Integral representations are presented for the *inhomogeneous* equation (1.1).
- The integral representations contain only the Dirichlet and Neumann boundary values.
- An integral representation for the solution u of Poisson has been obtained (instead of an integral representation for the derivative u_z).

- The method has been extended from polygonal domains to domains in polar co-ordinates.

In addition, the connection with Green is now better understood and the domain-dependent fundamental solutions provide a simpler method than [Fokas and Zyskin, 2002] or [Fokas, 2001] for obtaining the novel integral representations.

In the conclusions of the sequel, [Spence and Fokas, 2009], a direct comparison is made between this method and the classical transform method, and the method is also placed in context with respect to other approaches, such as the Sommerfeld-Malyuzhinets technique.

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