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A New Transform Method II: the Global Relation, and Boundary Value Problems in Polar Co-ordinates.

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Abstract

A new method for solving boundary value problems (BVPs) for linear and certain non-linear PDEs was introduced by one of the authors in the late 90's [Fokas, 1997]. For *linear* PDEs this method constructs novel integral representations which are formulated in the Fourier (transform) space. In a previous paper [Spence and Fokas, 2009], a simplified way of obtaining these representations was presented. In the current paper, first, the second ingredient of the new method, namely the derivation of the so-called “global relation” – an equation involving transforms of the boundary values – is presented. Then, using the global relation as well as the integral representation derived in the previous paper, certain BVPs in polar co-ordinates are solved. These BVPs elucidate the fact that this method has substantial advantages over the classical transform method.

1 Introduction.

1.1 Background.

In this paper we consider the second order linear elliptic PDE

$$\Delta u(\mathbf{x}) + \lambda u(\mathbf{x}) = -f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \quad (1.1)$$

where λ is a real constant, $f(\mathbf{x})$ a given function, and Ω is some 2 dimensional domain with a piecewise smooth boundary. For $\lambda = 0$ this is Poisson's equation, $\lambda > 0$ the Helmholtz equation, and $\lambda < 0$ the Modified Helmholtz equation.

For a boundary value problem (BVP) to be well-posed, certain *boundary conditions* must be prescribed; the ones of most physical importance are:

- Dirichlet: $u(\mathbf{x}) = \text{known}, \mathbf{x} \in \partial\Omega$
- Neumann: $\frac{\partial u}{\partial n}(\mathbf{x}) = \text{known}, \mathbf{x} \in \partial\Omega$
- Robin: $\frac{\partial u}{\partial n}(\mathbf{x}) + \alpha u(\mathbf{x}) = \text{known}, \alpha = \text{constant}, \mathbf{x} \in \partial\Omega,$

where $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$, where \mathbf{n} is the unit outward-pointing normal to Ω . More complicated boundary conditions can involve derivatives at angles to the boundary. One can also prescribe mixed boundary conditions, such as Dirichlet on part of the domain, and Neumann on another part.

Green's theorem gives the following integral representation of the solution of (1.1)

$$u(\mathbf{x}) = \int_{\partial\Omega} \left(E(\boldsymbol{\xi}, \mathbf{x}) \frac{\partial u}{\partial n}(\boldsymbol{\xi}) - u(\boldsymbol{\xi}) \frac{\partial E}{\partial n}(\boldsymbol{\xi}, \mathbf{x}) \right) dS(\boldsymbol{\xi}) + \int_{\Omega} f(\boldsymbol{\xi}) E(\boldsymbol{\xi}, \mathbf{x}) dV(\boldsymbol{\xi}), \quad \mathbf{x} \in \Omega, \quad (1.2)$$

where E is the fundamental solution (sometimes called the free space Green's function) satisfying

$$\left(\Delta_{\boldsymbol{\xi}} + \lambda \right) E(\boldsymbol{\xi}, \mathbf{x}) = -\delta(\boldsymbol{\xi} - \mathbf{x}), \quad \boldsymbol{\xi} \in \Omega. \quad (1.3)$$

For the different signs of λ , E is given by the expressions below.

- Laplace/Poisson ($\lambda = 0$): $E = -\frac{1}{2\pi} \log |\boldsymbol{\xi} - \mathbf{x}|$.
- Helmholtz ($\lambda > 0$): $E = \frac{i}{4} H_0^{(1)}(\sqrt{\lambda} |\boldsymbol{\xi} - \mathbf{x}|)$ (with assumed time-dependence $e^{-i\omega t}$ this corresponds to outgoing waves).
- Modified Helmholtz ($\lambda < 0$): $E = \frac{1}{2\pi} K_0(\sqrt{-\lambda} |\boldsymbol{\xi} - \mathbf{x}|)$,

where $H_0^{(1)}$ is a Hankel function and K_0 a modified Bessel function.

We note that a drawback for both Helmholtz and modified Helmholtz in 2-d is that the fundamental solution is given in terms of a special function.

Starting with a given boundary value problem in a separable domain, i.e. a domain of the form $\Omega = \{a_1 \leq \xi_1 \leq b_1\} \times \{a_2 \leq \xi_2 \leq b_2\}$ where ξ_j are the co-ordinates under which the differential operator is separable, the method of **separation of variables** consists of the following steps:

1. Separate the PDE into two ODEs.
2. Concentrate on one of these ODEs and derive the associated completeness relation (i.e. transform pair) depending on the boundary conditions. This is achieved by finding the 1-dimensional Green's function, $g(x, \xi; \nu)$, with eigenvalue ν and integrating g over a large circle in the complex ν plane:

$$\delta(x - \xi) = - \lim_{R \rightarrow \infty} \frac{1}{2\pi i} \oint_{|\nu|=R} g(x, \xi; \nu) d\nu. \quad (1.4)$$

3. Apply this transform to the PDE and use integration by parts to derive the ODE associated with this transform.
4. Solve this ODE using an appropriate 1-dimensional Green's function, or variation of parameters.

The solution of the boundary value problem is given as a superposition of eigenfunctions of the ODE considered in step 2 (either an integral or a sum depending on whether the ODE has a continuous or discrete spectrum).

For each boundary value problem there exist two different representations of the solution depending on which ODE was considered in step 2. To show that these two representations are equivalent requires two steps:

1. Go into the complex plane, either by deforming contours (if the solution is given as an integral), or by converting the series solution into an integral using the identity

$$\sum_{n=-\infty}^{\infty} f(n) = \int_C \frac{f(k)}{1 - e^{-2\pi ik}} dk, \quad (1.5)$$

where C is a contour which encloses the real k axis (in the positive sense) but no singularities of $f(k)$. This latter procedure is known as **the Watson transformation** [Watson, 1918].

2. Deform contours to enclose the singularities of $f(k)$ and evaluate the integral as residues/branch cut integrals.

Completeness: the rigorous proof that the transform derived in step 2 is complete can usually be achieved in two ways:

- (a) A direct proof of (1.4) for a given Green's function via integration in the complex plane [Titchmarsh, 1962].
- (b) If the inverse of the differential operator is a compact self-adjoint operator on a Hilbert space, then the spectral theorem implies that the eigenfunctions are complete (see e.g. [Evans, 1998] §6.5.1, [Stakgold, 1967] vol. 1 §3.3, §4.2).

The main limitations of this method for solving boundary value problems are the following:

- It fails for BVPs with *non-separable* boundary conditions (for example, those which include a derivative at an angle to the boundary).
- The appropriate transform depends on the boundary conditions and so the process must be repeated for different boundary conditions.

- The solution is not uniformly convergent on the whole boundary of the domain (since it is given as a superposition of eigenfunctions of one of the ODEs).
- A priori, it is not clear which of the two representations is better for practical purposes. For computations, an integral representation is generally superior to an infinite series. In the cases where only an infinite series is available, several techniques have been used in order to improve the convergence of this series, see e.g. [Duffy, 2001] §5.8.
- For the Helmholtz equation, the PDE, and hence at least one of the separated ODEs, involves a radiation condition and so is not self-adjoint. Thus, completeness of the transform associated with the ODE with the radiation condition is not guaranteed. The direct method (a) mentioned earlier can still be used in some cases [Cohen, 1964].

The implementation of these steps is classical and can be found in many books, for example [Stakgold, 1967] volume 1 chapter 4, [Morse and Feshbach, 1953] §5.1, [Friedman, 1956] chapters 4 and 5, [Keener, 1995] chapters 7 and 8, and [Ockendon et al., 2003] §4.4, 5.7, 5.8. A excellent overview, including historical remarks, is given in [Keller, 1979].

1.2 The method of [Fokas, 1997].

The method of [Fokas, 1997] has two basic ingredients:

1. The integral representation.
2. The global relation.

The integral representation (IR). This is the analogue of Green’s IR in the transform space. Indeed, the solution u is given as an integral involving *transforms* of the boundary values, whereas Green’s IR expresses u as an integral involving directly the boundary values. The IR can be obtained by first constructing particular integral representations depending on the domain of the fundamental solution E (“domain-dependent fundamental solutions”) and then substituting these representations into Green’s integral formula of the solution and interchanging the orders of integration [Spence and Fokas, 2009].

The global relation (GR). The global relation is Green’s divergence form of the equation integrated over the domain, where one employs the solution of the adjoint equation instead of the fundamental solution. Indeed

$$0 = \int_{\partial\Omega} \left(v(\boldsymbol{\xi}) \frac{\partial u}{\partial n}(\boldsymbol{\xi}) - u(\boldsymbol{\xi}) \frac{\partial v}{\partial n}(\boldsymbol{\xi}) \right) dS(\boldsymbol{\xi}) + \int_{\Omega} f(\boldsymbol{\xi}) v(\boldsymbol{\xi}) dV(\boldsymbol{\xi}), \quad (1.6)$$

where v is any solution of the adjoint of (1.1):

$$\Delta v(\mathbf{x}) + \lambda v(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega; \quad (1.7)$$

(equation (1.7) is (1.3) without the delta function on the right hand side). Separation of variables gives a one-parameter family of solutions of the adjoint equation depending on the parameter $k \in \mathbb{C}$ (the separation constant). For example, for the Poisson equation, separation of variables in Cartesian co-ordinates yields four solutions of the adjoint equation, $v = e^{\pm ikx \pm ky}$, $k \in \mathbb{C}$. All equations (1.6) obtained from these different choices of v shall be referred to as “the global relation”. This relation was called the global relation as it contains global, as opposed to local, information about the boundary values.

Just like Green’s IR, the new IR contains contributions from both known and unknown boundary values. However, it turns out that the GR involves *precisely* the transforms of the boundary values appearing in the IR. The main idea of the method of [Fokas, 1997] is that, for certain boundary value problems, one can **use the information given in the GR about the transforms of the unknown boundary values to eliminate these unknowns from the IR.** This can be achieved in three steps:

1. **Use the GR to express the transforms of the unknown boundary values appearing in the IR in terms of the smallest possible subset of the other functions appearing in the GR** (if there exist different possibilities, use the one which yields the smallest number of unknowns in each equation).
2. **Identify the domains in the complex k plane where the integrands are bounded and also identify the location of poles.** At this stage some unknowns can be eliminated directly using analyticity and employing Cauchy’s theorem.
3. **Deform contours and use the GR again so that the contribution from the unknown boundary values vanish by analyticity** (employing Cauchy’s theorem).

For certain domains it is possible to solve the given BVP by using only a subset of the above three steps. For example, Helmholtz in a wedge, which is solved in §4.1, only requires step 1, whereas Poisson in a circular wedge, which is solved in §4.2, requires steps 1 and 2. On the other hand, Helmholtz in the exterior of the circle, which actually played a prominent role in the development of classical transforms (including the discovery of the Watson transformation) [Keller, 1979], requires all three steps [Spence, 2009].

This method yields the solution as an integral in the complex k plane involving transforms of the known boundary values. This novel solution formula has two significant features:

1. The integrals can be deformed to involve exponentially decaying integrands.
2. The expression is uniformly convergent at the boundary of the domain.

These features give rise to both analytical and numerical advantages in comparison with classical methods. In particular, for linear evolution PDEs, the effective numerical evaluation of the solution is given in [Flyer and Fokas, 2008].

1.3 Plan of the paper.

In the previous paper [Spence and Fokas, 2009], the appropriate domain-dependent fundamental solutions were constructed both for polygons and for domains in polar co-ordinates. In §2 these representations are used to construct novel integral representations for two particular domains in polar co-ordinates. In §3 the global relations for these domains are presented. In §4 the usefulness of the results of §2 and §3 is illustrated by solving certain boundary value problems for Helmholtz in a wedge and for Poisson in a circular wedge. These results are discussed further in §5.

2 Integral representations for polar co-ordinates.

Notations. Let (r, θ) be the physical variable, and (ρ, ϕ) be the “dummy variable” of integration.

We will consider only Poisson, $\lambda = 0$, and Helmholtz, $\lambda = \beta^2$.

The following two propositions were proved in [Spence and Fokas, 2009]:

Proposition 2.1 (*Integral representation of E_s for Helmholtz*). *For the Helmholtz equation, the outgoing non-periodic fundamental solution E_s can be expressed in terms of radial eigenfunctions (the radial representation) in the form*

$$E_s(\rho, \phi; r, \theta) = \lim_{\varepsilon \rightarrow 0} \frac{i}{4} \left(\int_0^{i\infty} dk e^{\varepsilon k^2} H_k^{(1)}(\beta\rho) J_k(\beta r) e^{ik|\theta-\phi|} + \int_0^{-i\infty} dk e^{\varepsilon k^2} H_k^{(1)}(\beta\rho) J_k(\beta r) e^{-ik|\theta-\phi|} \right). \quad (2.1)$$

Alternatively, it can be expressed in terms of angular eigenfunctions (the angular representation) in the form

$$E_s(\rho, \phi; r, \theta) = \frac{i}{4} \left(\int_0^\infty dk H_k^{(1)}(\beta r_>) J_k(\beta r_<) e^{ik(\theta-\phi)} + \int_0^\infty dk H_k^{(1)}(\beta r_>) J_k(\beta r_<) e^{-ik(\theta-\phi)} \right), \quad (2.2)$$

where $r_> = \max(r, \rho)$, $r_< = \min(r, \rho)$ and $-\infty < \theta, \phi < \infty$.

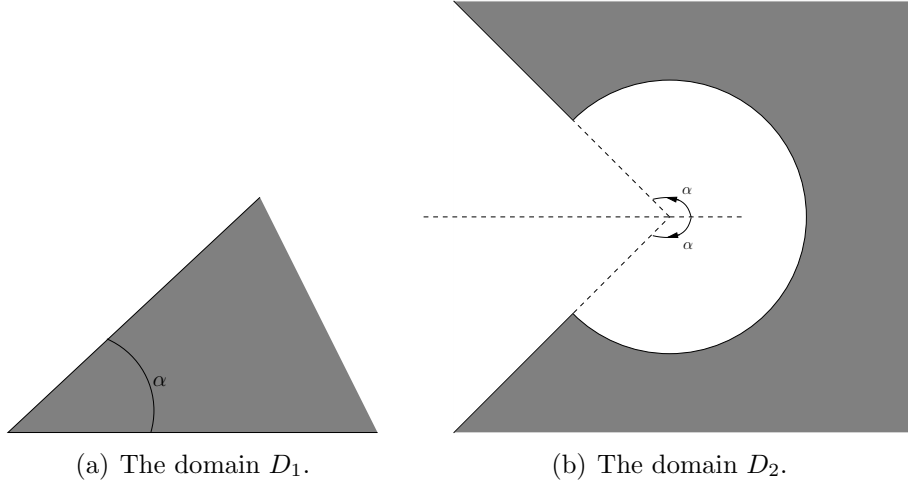


Figure 1: The domains D_1 and D_2 .

Proposition 2.2 (*Integral representation of E_s for Poisson*) For the Poisson equation the non-periodic fundamental solution E_s can be expressed formally either in terms of radial eigenfunctions (the radial representation) in the form

$$E_s(\rho, \phi; r, \theta) = \lim_{\varepsilon \rightarrow 0} \frac{1}{4\pi} \left(\int_{\varepsilon}^{i\infty} \frac{dk}{k} \left(\frac{\rho}{r}\right)^k e^{ik|\theta-\phi|} + \int_{\varepsilon}^{-i\infty} \frac{dk}{k} \left(\frac{\rho}{r}\right)^k e^{-ik|\theta-\phi|} \right), \quad (2.3)$$

or in terms of angular eigenfunctions (the angular representation) in the form

$$E_s(\rho, \phi; r, \theta) = \lim_{\varepsilon \rightarrow 0} \frac{1}{4\pi} \left(\int_{i\varepsilon}^{\infty} \frac{dk}{k} \left(\frac{r_{>}}{r_{<}}\right)^{-k} e^{ik(\theta-\phi)} + \int_{-i\varepsilon}^{\infty} \frac{dk}{k} \left(\frac{r_{>}}{r_{<}}\right)^{-k} e^{-ik(\theta-\phi)} \right), \quad (2.4)$$

where $r_{>} = \max(r, \rho)$, $r_{<} = \min(r, \rho)$, $-\infty < \theta, \phi < \infty$.

2.1 Novel integral representations for certain domains in polar co-ordinates.

In order to illustrate the new integral representations we consider two examples in polar co-ordinates: the Helmholtz equation in the wedge

$$D_1 = \{a < r < \infty, \quad 0 < \theta < \alpha\}, \quad (2.5)$$

see figure 1(a), and the Poisson equation in the circular wedge,

$$D_2 = \{a < r < \infty, \quad -\alpha < \theta < \alpha\}, \quad (2.6)$$

see figure 1(b),

The associated novel integral representations can be obtained by substituting the representations of the fundamental solutions of propositions 2.2 and 2.1 into Green's integral representation in polar co-ordinates, i.e. in the equation

$$u(r, \theta) = \int_{\partial D} \left(u \frac{\partial E_s}{\partial \phi} - E_s \frac{\partial u}{\partial \phi} \right) \frac{d\rho}{\rho} + \rho \left(E_s \frac{\partial u}{\partial \rho} - u \frac{\partial E_s}{\partial \rho} \right) d\phi + \iint_D f(\rho, \phi) E_s(\rho, \phi; r, \theta) \rho d\rho d\phi. \quad (2.7)$$

In addition, u satisfies the following boundary conditions at infinity:

$$\lambda = 0 \quad (\text{Poisson}), \quad u \rightarrow 0 \text{ as } r \rightarrow \infty. \quad (2.8)$$

$$\lambda = \beta^2 \quad (\text{Helmholtz}), \quad \sqrt{r} \left(\frac{\partial u}{\partial r} - i\beta u \right) \rightarrow 0 \text{ as } r \rightarrow \infty. \quad (2.9)$$

Furthermore, f satisfies appropriate conditions such that the integral $\int_D f(\rho, \phi) \rho d\rho d\phi$ is well defined.

Proposition 2.3 (Helmholtz in D_1). *Let u be a solution of (1.1) with $\lambda = \beta^2$ in the domain D_1 defined by (2.5), which satisfies the radiation condition at infinity (2.9). Then u is given by*

$$\begin{aligned} u(r, \theta) = \lim_{\varepsilon \rightarrow 0} \frac{i}{4} & \left(\int_0^{i\infty} dk e^{\varepsilon k^2} J_k(\beta r) e^{ik\theta} [-ikD_0(k) - N_0(k)] \right. \\ & + \int_0^{-i\infty} dk e^{\varepsilon k^2} J_k(\beta r) e^{-ik\theta} [ikD_0(k) - N_0(k)] \\ & - \int_0^{i\infty} dk e^{\varepsilon k^2} J_k(\beta r) e^{-ik\theta} e^{ik\alpha} [ikD_\alpha(k) - N_\alpha(k)] \\ & \left. - \int_0^{-i\infty} dk e^{\varepsilon k^2} J_k(\beta r) e^{ik\theta} e^{-ik\alpha} [-ikD_\alpha(k) - N_\alpha(k)] \right) \\ & + \iint_\Omega d\rho d\phi \rho f(\rho, \phi) E_s(\rho, \phi; r, \theta), \end{aligned} \quad (2.10)$$

where

$$D_\chi(k) = \int_0^\infty \frac{d\rho}{\rho} H_k^{(1)}(\beta\rho) u(\rho, \chi), \quad N_\chi(k) = \int_0^\infty \frac{d\rho}{\rho} H_k^{(1)}(\beta\rho) u_\theta(\rho, \chi), \quad \chi = 0 \text{ or } \alpha, \quad (2.11)$$

with E_s given by either (2.1) or (2.2).

Proof. On the boundary $\{\phi = \alpha, \infty > \rho > a\}$, use the radial representations of E_s (2.1) with $\theta < \phi$. On the boundary $\{\phi = 0, 0 < \rho < \infty\}$, use the radial representations of E_s (2.1) with $\theta > \phi$. \square

Proposition 2.4 (Poisson in D_2). *Let u be a solution of (1.1) with $\lambda = 0$ in the domain D_2 defined by (2.6), which satisfies the boundary condition (2.8) at infinity. Then u is given by*

$$\begin{aligned}
4\pi u(r, \theta) &= \int_0^{i\infty} \frac{dk}{k} \left(\frac{r}{a}\right)^{-k} (e^{ik\theta} e^{ik\alpha} [-ikD_{-\alpha}(k) - N_{-\alpha}(k)] - e^{-ik\theta} e^{ik\alpha} [ikD_{\alpha}(k) - N_{\alpha}(k)]) \\
&+ \int_0^{-i\infty} \frac{dk}{k} \left(\frac{r}{a}\right)^{-k} (e^{-ik\theta} e^{-ik\alpha} [ikD_{-\alpha}(k) - N_{-\alpha}(k)] - e^{ik\theta} e^{-ik\alpha} [-ikD_{\alpha}(k) - N_{\alpha}(k)]) \\
&+ \int_0^{\infty} \frac{dk}{k} \left(\frac{r}{a}\right)^{-k} (e^{ik\theta} [-aN(-ik) + kD(-ik)] + e^{-ik\theta} [-aN(ik) + kD(ik)]) \\
&+ 4\pi \iint_{\Omega} d\rho d\phi \rho f(\rho, \phi) E_s(\rho, \phi; r, \theta), \tag{2.12}
\end{aligned}$$

where the functions $D_{\chi}(k), N_{\chi}(k), \chi = 0, \alpha, D(\pm ik), N(\pm ik)$ are defined as follows:

$$D_{\chi}(k) = \int_a^{\infty} \frac{d\rho}{\rho} \left(\frac{\rho}{a}\right)^k u(\rho, \chi), \quad N_{\chi}(k) = \int_a^{\infty} \frac{d\rho}{\rho} \left(\frac{\rho}{a}\right)^k u_{\theta}(\rho, \chi), \quad \chi = 0 \text{ or } \alpha, \tag{2.13}$$

and

$$D(\pm ik) = \int_{-\alpha}^{\alpha} d\phi e^{\pm ik\phi} u(a, \phi), \quad N(\pm ik) = \int_{-\alpha}^{\alpha} d\phi e^{\pm ik\phi} u_r(a, \phi), \tag{2.14}$$

with E_s given by either (2.3) or (2.4) with $\varepsilon = 0$.

Proof. On the boundary $\{\phi = \alpha, \infty > \rho > a\}$, use the radial representations of E_s (2.3) with $\theta < \phi$. On the boundary $\{\rho = a, \alpha > \phi > -\alpha\}$ use the angular representations of E_s (2.4) with $r > \rho$. On the boundary $\{\phi = 0, 0 < \rho < \infty\}$, use the radial representations of E_s (2.3) with $\theta > \phi$. After letting $\varepsilon \rightarrow 0$, the singularity at $k = 0$ on the contours of integration is *removable*: the consistency condition

$$\int_{\partial D} r \frac{\partial u}{\partial r} d\theta - \frac{1}{r} \frac{\partial u}{\partial \theta} dr + \iint_D f(r, \theta) r dr d\theta = 0,$$

yields the following relation for the boundary values:

$$-N_0(0) + N_{\alpha}(0) - aN(0) = - \iint_D d\rho d\phi \rho f(\rho, \phi).$$

This equation is the GR for this problem evaluated at $k = 0$, see (3.18). \square

3 The global relation.

3.1 Polygons.

Let $\Omega^{(i)}$ be the interior of a convex polygon in \mathbb{R}^2 . Let $\partial\Omega$ denote the boundary of the polygon, oriented *anticlockwise*, where the vertices of the polygon z_1, z_2, \dots, z_n

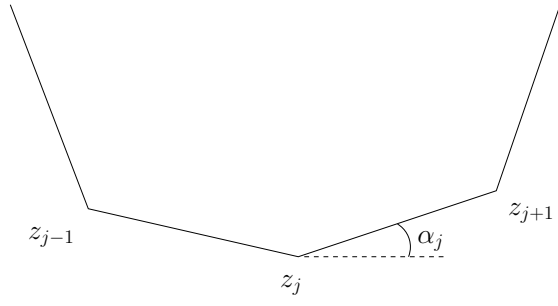


Figure 2: The convex polygon $\Omega^{(i)}$.

are labelled anticlockwise. Let S_j be the side (z_j, z_{j+1}) and let $\alpha_j = \arg(z_{j+1} - z_j)$ be the angle of S_j . For the Helmholtz equation $\lambda = 4\beta^2$, $\beta \in \mathfrak{R}^+$, and for the Modified Helmholtz equation $\lambda = -4\beta^2$, $\beta \in \mathfrak{R}^+$.

Let u be a solution of (1.1) for $\Omega = \Omega^{(i)}$. In Cartesian co-ordinates the GR (1.6) is

$$\int_{\partial D} u (v_\eta d\xi - v_\xi d\eta) - v (u_\eta d\xi - u_\xi d\eta) = - \int_D f v d\xi d\eta, \quad (3.1)$$

where v is any solution of the adjoint equation (1.7). Equivalently, (3.1) written in complex co-ordinates becomes

$$\int_{\partial D} u (v_{z'} dz' - v_{\bar{z}'} d\bar{z}') - v (u_{z'} dz' - u_{\bar{z}'} d\bar{z}') = i \int_D f v d\xi d\eta. \quad (3.2)$$

Lemma 3.1 (Adjoint solutions). *The following are particular solutions of the adjoint equation (1.7):*

$$\lambda = 0, \quad v = \text{either } e^{-ikz} \text{ or } e^{ik\bar{z}}, \quad (3.3a)$$

$$\lambda = -4\beta^2, \quad v = e^{-i\beta\left(kz' - \frac{\bar{z}'}{k}\right)}, \quad (3.3b)$$

$$\lambda = 4\beta^2, \quad v = e^{-i\beta\left(kz' + \frac{\bar{z}'}{k}\right)}. \quad (3.3c)$$

Proof . Separation of variables yields that the exponential function $v = e^{m_1\xi + m_2\eta}$ satisfies (1.7) iff

$$m_1^2 + m_2^2 + \lambda = 0. \quad (3.4)$$

For $\lambda = 0$ a natural way to parametrize this 1-parameter family of solutions is $m_1 = \pm ik$, $m_2 = \pm k$, which leads to the solutions

$$e^{-ik\xi + k\eta}, \quad e^{ik\xi + k\eta},$$

which are (3.3a). Two more solutions are obtained by letting $k \rightarrow -k$.

For $\lambda = -4\beta^2$, a natural parametrization of (3.4) is $m_1 = \pm 2\beta \sin \phi$ and $m_2 = \pm 2\beta \cos \phi$; which letting $k = e^{i\phi}$ yields

$$m_1 = \mp i\beta \left(k - \frac{1}{k} \right), \quad m_2 = \pm \beta \left(k + \frac{1}{k} \right).$$

This parametrisation leads to four particular solutions namely (3.3b), as well as to those solutions obtained from (3.3b) using the transformations $k \rightarrow -k$ and $k \rightarrow 1/k$.

In a similar way, for $\lambda = 4\beta^2$ a natural parametrization of (3.4) is $m_1 = \pm 2i\beta \sin \phi$ and $m_2 = \pm 2i\beta \cos \phi$, which leads to the particular solution (3.3c), as well as to those solutions obtained from (3.3)c using the transformations $k \rightarrow -k$ and $k \rightarrow 1/k$. \square

Substituting these adjoint solutions into (3.2) immediately yields the following GRs.

Proposition 3.2 (Global relation for $\Omega^{(i)}$) *Let u be a solution of (1.1) in the domain $\Omega^{(i)}$. Then the following relations, called global relations, are valid.*

Modified Helmholtz ($\lambda = -4\beta^2$, $\beta \in \mathbb{R}$):

$$\sum_{j=1}^n \widehat{u}_j(k) + i\widehat{f}(k) = 0, \quad k \in \mathbb{C}, \quad (3.5)$$

where $\{\widehat{u}_j(k)\}_1^n$ are defined by

$$\widehat{u}_j(k) = \int_{z_j}^{z_{j+1}} e^{-i\beta(kz' - \frac{z'}{k})} [(u_{z'} + i\beta k u) dz' - (u_{\bar{z}'} + \frac{\beta}{ik} u) d\bar{z}'], \quad j = 1, \dots, n, \quad z_{n+1} = z_1,$$

and $\widehat{f}(k)$ is defined by

$$\widehat{f}(k) = \iint_{\Omega^{(i)}} e^{-i\beta(kz' - \frac{z'}{k})} f(\xi, \eta) d\xi d\eta. \quad (3.6)$$

Helmholtz ($\lambda = 4\beta^2$, $\beta \in \mathbb{R}$):

$$\sum_{j=1}^n \widehat{u}_j(k) + i\widehat{f}(k) = 0, \quad k \in \mathbb{C}, \quad (3.7)$$

where $\{\widehat{u}_j(k)\}_1^n$ are defined by

$$\begin{aligned} \widehat{u}_j(k) &= \int_{z_j}^{z_{j+1}} e^{-i\beta(kz' + \frac{z'}{k})} [(u_{z'} + i\beta k u) dz' - (u_{\bar{z}'} - \frac{\beta}{ik} u) d\bar{z}'], \quad j = 1, \dots, n, \quad z_{n+1} = z_1, \\ &= \int_{z_j}^{z_{j+1}} e^{-i\beta(kz' - \frac{z'}{k})} \left[i \frac{\partial u}{\partial n}(s) + i\beta \left(k \frac{dz'}{ds} - \frac{1}{k} \frac{d\bar{z}'}{ds} \right) u(s) \right] ds, \quad z' = z'(s) \in S_j, \end{aligned}$$

and $\widehat{f}(k)$ is defined by

$$\widehat{f}(k) = \iint_{\Omega^{(i)}} e^{-i\beta(kz' + \frac{z'}{k})} f(\xi, \eta) d\xi d\eta, \quad (3.8)$$

Poisson ($\lambda = 0$):

$$\sum_{j=1}^n \widehat{u}_j(k) + i\widehat{f}(k) = 0, \quad (3.9a)$$

$$\sum_{j=1}^n \widetilde{u}_j(k) - i\widetilde{f}(k) = 0, \quad k \in \mathbb{C}, \quad (3.9b)$$

where $\{\widehat{u}_j(k)\}_1^n$, $\{\widetilde{u}_j(k)\}_1^n$, $\widehat{f}(k)$ are defined by

$$\begin{aligned} \widehat{u}_j(k) &= \int_{z_j}^{z_{j+1}} e^{-ikz'} [(u_{z'} + iku) dz' - u_{\bar{z}'} d\bar{z}'], \quad j = 1, \dots, n, \quad z_{n+1} = z_1, \quad (3.10) \\ &= \int_{z_j}^{z_{j+1}} e^{-ikz'} \left[i \frac{\partial u}{\partial n}(s) + ik \frac{dz'}{ds} u(s) \right] ds, \quad z' = z'(s) \in S_j \end{aligned}$$

and

$$\begin{aligned} \widetilde{u}_j(k) &= \int_{z_j}^{z_{j+1}} e^{ik\bar{z}'} [(u_{\bar{z}'} - iku) d\bar{z}' - u_z dz'], \quad j = 1, \dots, n, \quad z_{n+1} = z_1, \quad (3.11) \\ &= \int_{z_j}^{z_{j+1}} e^{ik\bar{z}'} \left[-i \frac{\partial u}{\partial n}(s) - ik \frac{d\bar{z}'}{ds} u(s) \right] ds, \quad z' = z'(s) \in S_j. \end{aligned}$$

and $\widehat{f}(k)$, $\widetilde{f}(k)$ are defined by

$$\widehat{f}(k) = \iint_{\Omega^{(i)}} e^{-ikz'} f(\xi, \eta) d\xi d\eta, \quad \widetilde{f}(k) = \iint_{\Omega^{(i)}} e^{ik\bar{z}'} f(\xi, \eta) d\xi d\eta. \quad (3.12)$$

Remark 3.3 (*The existence of two global relations for Poisson and one for modified Helmholtz and Helmholtz*). In the particular adjoint solution (3.3b) for Modified Helmholtz, let $k \rightarrow k/\beta$ to obtain

$$v = e^{-ikz' + i\frac{\beta^2 \bar{z}'}{k}}. \quad (3.13)$$

For $\beta = 0$ this reduces to the expression for v of the first global relation for Poisson. Letting $k \rightarrow \beta^2/k$ in (3.13), and then letting $\beta = 0$ yields $e^{ik\bar{z}}$, which is the expression for v of the second global relation for Poisson. Thus the two adjoint solutions for Poisson can be obtained from the single adjoint solution for Modified Helmholtz (3.13).

Remark 3.4 (*Unbounded domains*). If $\Omega^{(i)}$ is unbounded then k must be restricted so that the integral on the boundary at infinity is zero, see [Spence, 2009] or [Fokas and Spence, 2010].

3.2 Polar co-ordinates

Let u be a solution of (1.1). If Ω is unbounded assume u satisfies the boundary conditions (2.8) and (2.9) at infinity. In polar co-ordinates the GR (1.6) is

$$\int_{\partial D} r \left(v \frac{\partial u}{\partial r} - u \frac{\partial v}{\partial r} \right) d\theta + \frac{1}{r} \left(u \frac{\partial v}{\partial \theta} - v \frac{\partial u}{\partial \theta} \right) dr + \iint_D f(r, \theta) v(r, \theta; r, \theta) r dr d\theta = 0, \quad (3.14)$$

where v is any solution of the adjoint equation (1.7) which, if Ω is unbounded, satisfies the same boundary conditions at infinity as u (this means that the integral at infinity is zero).

Lemma 3.5 (Adjoint solutions) *There are four particular solutions of the adjoint equation which can be obtained by separation of variables in polar co-ordinates; two of these solutions are given by*

$$\beta = 0, \quad v = e^{\pm ik\theta} r^k, \quad (3.15a)$$

$$\beta \neq 0, \quad v = e^{\pm ik\theta} B_k(\beta r), \quad (3.15b)$$

where $B_k(r)$ denotes a solution of the Bessel equation of order k ; two more solutions can be obtained by $k \mapsto -k$.

Remark 3.6 (The restrictions on k in the adjoint solutions). *Depending on whether the domain contains the origin or is unbounded, some of the particular solutions for v are disallowed. Indeed, first consider the Poisson equation. At infinity, assume $u(r, \theta) = \mathcal{O}(r^{-\varepsilon})$ as $r \rightarrow \infty$ where $\varepsilon > 0$; actually $u(r, \theta) = \mathcal{O}(r^{-\pi/\gamma})$ as $r \rightarrow \infty$, where γ is the angle of the wedge the domain makes at infinity [Jones, 1986]. Then,*

$$\int_{C_R} r \left(v \frac{\partial u}{\partial r} - u \frac{\partial v}{\partial r} \right) d\theta + \frac{1}{r} \left(u \frac{\partial v}{\partial \theta} - v \frac{\partial u}{\partial \theta} \right) dr \rightarrow 0 \text{ as } R \rightarrow \infty \iff \Re k < \varepsilon,$$

with v given by (3.15a). At the origin, assume $u(r, \theta) = \mathcal{O}(r^\varepsilon)$ as $r \rightarrow 0$ where $\varepsilon > 0$; actually $u(r, \theta) = \mathcal{O}(r^{\pi/\gamma})$ as $r \rightarrow 0$, where γ is the angle of the wedge the domain makes at 0 [Jones, 1986]. Then, for the integral in the global relation to exist, we require $\Re k > -\varepsilon$.

For Helmholtz, $\int_{C_R} \rightarrow 0$ as $R \rightarrow \infty$ iff u and v satisfy the radiation condition (2.9). Therefore if the domain is unbounded, v must be either $e^{\pm ik\theta} H_k^{(1)}(\beta r)$, or a similar expression with $k \mapsto -k$. At the origin, assume $u(r, \theta) = \mathcal{O}(r^\varepsilon)$ as $r \rightarrow 0$ where $\varepsilon > 0$; actually $u(r, \theta) = \mathcal{O}(r^{\pi/\gamma})$ as $r \rightarrow 0$ where γ is the wedge angle [Jones, 1986]. Then, for the integral to exist, $B_k(\beta r)$ must be bounded as $r \rightarrow 0$. Recall that $J_k(\beta r)$ is bounded as $r \rightarrow 0$ for $\Re k \geq 0$ and $H_k^{(1)}(\beta r)$ is bounded as $r \rightarrow 0$ for $\Re k = 0$.

Proposition 3.7 (*Global relation for Helmholtz in the domain D_1 defined by (2.5)*). Let u be the solution of (1.1) with $\lambda = \beta^2$ in the domain D_1 defined by (2.5), see figure 1(a), and let u satisfy (2.9). Then

$$\pm ikD_0(k) - N_0(k) - e^{\pm ik\alpha} [\pm ikD_\alpha(k) - N_\alpha(k)] = -F(k, \pm ik), \quad \Re k = 0, \quad (3.16)$$

where $D_\chi(k), N_\chi(k)$, $\chi = 0$ or α , are defined by (2.11) and $F(k, \pm ik)$ are defined by

$$F(k, \pm ik) = \int_0^\infty d\rho \int_0^\alpha d\phi \rho f(\rho, \phi) H_k^{(1)}(\beta\rho) e^{\pm ik\phi}. \quad (3.17)$$

Proof. Substitute (3.15b) with $B_k = H_k^{(1)}$ into (3.14) (according to remark 3.6, v must satisfy the radiation condition). Parametrise the sides of D_1 by $\{\phi = \alpha, \infty > \rho > a\}$ and $\{\phi = 0, 0 < \rho < \infty\}$. The region of validity of k is specified by remark 3.6. \square

Proposition 3.8 (*Global relation for Poisson in the domain D_2 defined by (2.6)*). Let u be the solution of (1.1) with $\lambda = 0$ in the domain D_2 defined by (2.6), see figure 1(b), and let u satisfy (2.8). Then

$$\begin{aligned} e^{\mp ik\alpha} [\pm ikD_{-\alpha}(k) - N_{-\alpha}(k)] - e^{\pm ik\alpha} [\pm ikD_\alpha(k) - N_\alpha(k)] - [aN(\pm ik) - kD(\pm ik)] \\ = -F(k, \pm ik), \quad \Re k < \frac{\pi}{\alpha}, \end{aligned} \quad (3.18)$$

where $D(\pm ik), N(\pm ik)$ are defined by (2.14), $D_\chi(k), N_\chi(k)$, $\chi = 0$ or α , are defined by (2.13) and $F(k, \pm ik)$ are defined by

$$F(k, \pm ik) = \int_0^\infty d\rho \int_{-\alpha}^\alpha d\phi \rho f(\rho, \phi) \left(\frac{\rho}{a}\right)^k e^{\pm ik\phi}. \quad (3.19)$$

Proof. Substitute (3.15a) into (3.14) and parametrise the sides of D_2 by $\{\phi = \alpha, \infty > \rho > a\}$, $\{\rho = a, \alpha > \phi > -\alpha\}$, and $\{\phi = -\alpha, 0 < \rho < \infty\}$. The region of validity in k is specified by remark 3.6. \square

4 The solution of certain BVPs in polar coordinates.

In this section we solve the Dirichlet problem for Helmholtz in the wedge D_1 (2.5) and the Neumann problem for Poisson in the circular wedge D_2 (2.6). This is achieved by employing the integral representations of propositions 2.3 and 2.4, as well as the global relations of propositions 3.7 and 3.8 respectively, and then following the steps outlined in the introduction.

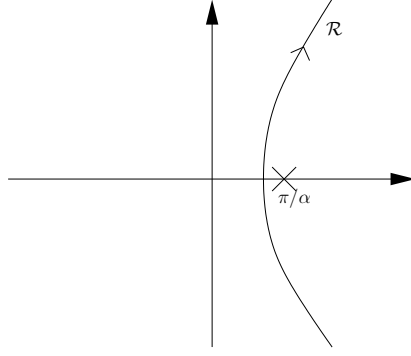


Figure 3: The contour \mathcal{R} in the k plane.

4.1 The Helmholtz equation in a wedge

Proposition 4.1 (The Dirichlet problem). *Let $u(a, \theta)$ satisfy (1.1) with $\lambda = \beta^2$ in the domain D_1 defined by (2.5), with the Dirichlet boundary conditions*

$$u(r, 0) = d_0(r), \quad 0 < r < \infty, \quad (4.1a)$$

$$u(r, \alpha) = d_\alpha(r), \quad 0 < r < \infty, \quad (4.1b)$$

where $d_0(r), d_\alpha(r), f(r, \theta)$ are given. Then u is given by

$$\begin{aligned} u(r, \theta) = & \frac{1}{2} \int_{\mathcal{R}} \frac{dk k}{\sin k\alpha} \left(\sin k\theta (D_\alpha^{(R)}(k, r) + \tilde{D}_\alpha^{(L)}(k, r)) + \sin k(\alpha - \theta) (D_0^{(R)}(k, r) + \tilde{D}_0^{(L)}(k, r)) \right) \\ & - \frac{i}{4} \lim_{\varepsilon \rightarrow 0} \int_0^{i\infty} \frac{dk}{\sin k\alpha} e^{\varepsilon k^2} J_k(\beta r) (\sin k(\alpha - \theta) F(k, ik) + e^{ik\alpha} \sin k\theta F(k, -ik)) \\ & - \frac{i}{4} \lim_{\varepsilon \rightarrow 0} \int_0^{-i\infty} \frac{dk}{\sin k\alpha} e^{\varepsilon k^2} J_k(\beta r) (\sin k(\alpha - \theta) F(k, -ik) + e^{-ik\alpha} \sin k\theta F(k, ik)) \\ & + \iint_{\Omega} d\rho d\phi \rho f(\rho, \phi) E_s(\rho, \phi; r, \theta), \end{aligned} \quad (4.2)$$

where the functions $D_\chi^{(R)}(k, r)$ and $\tilde{D}_\chi^{(L)}(k, r)$, $\chi = 0$ or α , are defined as follows:

$$D_\chi^{(R)}(k, r) = J_k(\beta r) \int_r^\infty H_k^{(1)}(\beta\rho) d_\chi(\rho) \frac{d\rho}{\rho}, \quad \chi = 0 \text{ or } \alpha, \quad 0 < r < \infty, \quad \Re k \geq 0, \quad (4.3)$$

$$\tilde{D}_\chi^{(L)}(k, r) = H_k^{(1)}(\beta r) \int_0^r J_k(\beta\rho) d_\chi(\rho) \frac{d\rho}{\rho}, \quad \chi = 0 \text{ or } \alpha, \quad 0 < r < \infty, \quad \Re k \geq 0,$$

and \mathcal{R} is a contour in the right half plane with $\arg k = \pm\varphi$ at infinity, $0 < \varphi < \pi/2$, and such that when $\Im k = 0$, $\Re k < \pi/\alpha$, see figure 3. The functions $F(k, \pm ik)$ are defined by (3.17), and the function E_s appearing in the last term of (4.2) is given by either (2.2) or (2.1).

Proof .

Step 1. The IR (2.10) contains the two unknown functions N_0, N_α . The GRs (3.16) are two equations containing these two unknown functions. Hence solving (3.16) for N_0, N_α we find

$$N_\alpha(k) = \frac{1}{2i \sin k\alpha} \left(-2ikD_0(k) + 2ik \cos k\alpha D_\alpha(k) + F(k, -ik) - F(k, ik) \right),$$

and

$$N_0(k) = \frac{1}{2i \sin k\alpha} \left(2ikD_\alpha(k) - 2ik \cos k\alpha D_0(k) - e^{-ik\alpha} F(k, -ik) + e^{ik\alpha} F(k, ik) \right).$$

Substituting these expressions into the IR yields the solution (4.2) but with the first term replaced by

$$\frac{1}{2} \lim_{\varepsilon \rightarrow 0} \int_{-i\infty}^{i\infty} dk k e^{\varepsilon k^2} J_k(\beta r) \left(\frac{D_\alpha(k) \sin k\theta + D_0(k) \sin k(\alpha - \theta)}{\sin k\alpha} \right). \quad (4.4)$$

We now seek to deform the contour from $i\mathbb{R}$ to a contour on which the integral converges absolutely so we can let $\varepsilon = 0$. The asymptotics

$$kJ_k(y)H_k^{(1)}(x) \sim -\frac{i}{\pi} \left(\frac{y}{x} \right)^k, \quad |k| \rightarrow \infty, \quad -\frac{\pi}{2} < \arg k < \frac{\pi}{2}, \quad (4.5)$$

$$kJ_k(y)H_k^{(1)}(x) \sim -2i \frac{\sin k\pi}{\pi} e^{-i\pi k} \left(\frac{-2k}{ex} \right)^{-k} \left(\frac{-2k}{ey} \right)^{-k}, \quad |k| \rightarrow \infty, \quad \frac{\pi}{2} < \arg k < \frac{3\pi}{2}. \quad (4.6)$$

imply that the product $J_k(\beta r)H_k^{(1)}(\beta\rho)$ is bounded at infinity for $\Re k > 0$ provided that $\rho > r$. Our plan is to split the relevant integrals in D_0 and D_α and to use the following identity in order to exchange the arguments of H_k and J_k :

$$\int_{-i\infty}^{i\infty} dk J_k(x)H_k^{(1)}(y)Q(k) = \int_{-i\infty}^{i\infty} dk J_k(y)H_k^{(1)}(x)Q(k), \quad (4.7)$$

where $Q(k) = -Q(-k)$. To prove this identity, expand H_k as a linear combination of J_k and J_{-k} , using the definition of H_k , and then let $k \mapsto -k$ in the term involving J_{-k} . Note that (4.7) also establishes reciprocity between r and ρ in the Kontorovich-Lebedev transform, (4.8) below.

The definitions of D_χ , (2.11), and of $D_\chi^{(R)}$, (4.3), imply the identity

$$J_k(\beta r)D_\chi(k) = D_\chi^{(L)}(k, r) + D_\chi^{(R)}(k, r), \quad \chi = 0 \text{ or } \alpha, \quad 0 < r < \infty, \quad \Re k = 0,$$

where $D_\chi^{(L)}$, $\chi = 0$ or α , are defined by

$$D_\chi^{(L)}(k, r) = J_k(\beta r) \int_0^r H_k^{(1)}(\beta\rho) d_\chi(\rho) \frac{d\rho}{\rho}, \quad \chi = 0 \text{ or } \alpha, \quad 0 < r < \infty, \quad \Re k = 0.$$

Using (4.7) in the terms involving $D_\chi^{(L)}(k, r)$, deforming the contour to from $i\mathbb{R}$ to \mathcal{R} , and setting $\varepsilon = 0$, proves that (4.4) is equal to the first term of (4.2). \square

Remark 4.2 (Recovery of classical representations). *The Dirichlet problem of Helmholtz in a wedge can be solved using either a sine-series in θ or the Kontorovich-Lebedev transform in r . For simplicity consider $f = 0$. To obtain the sine-series solution starting with (4.2), evaluate the integral as residues at the poles $n\pi/\alpha$, $n \in \mathbb{Z}^+$:*

$$u(r, \theta) = i \left(\frac{\pi}{\alpha} \right)^2 \sum_{n=1}^{\infty} n \sin \frac{n\pi\theta}{\alpha} \left\{ D_0^{(R)} \left(\frac{n\pi}{\alpha}, r \right) + \tilde{D}_0^{(L)} \left(\frac{n\pi}{\alpha}, r \right) \right. \\ \left. - (-1)^n \left[D_\alpha^{(R)} \left(\frac{n\pi}{\alpha}, r \right) + \tilde{D}_\alpha^{(L)} \left(\frac{n\pi}{\alpha}, r \right) \right] \right\}.$$

The solution obtained by the Kontorovich-Lebedev transform is the same as (4.2), [Jones, 1980] or [Jones, 1986] §9.19, page 587. Since there are no boundaries in r , the solution is given as an integral which is uniformly convergent at ∂D_1 , thus this is the best possible representation.

Remark 4.3 (Verifying the boundary conditions). *In order to verify the boundary conditions (4.1), evaluate (4.2) at $\theta = 0$ and $\theta = \alpha$. In this respect we note that by employing (4.7) and by following similar steps to those used earlier, the Kontorovich-Lebedev transform:*

$$g(k) = \int_0^\infty dy f(y) H_k^{(1)}(y), \quad (4.8a)$$

$$x f(x) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2} \int_{-i\infty}^{i\infty} dk e^{\varepsilon k^2} k J_k(x) g(k), \quad (4.8b)$$

(see remark 3.2 of [Spence and Fokas, 2009] or [Jones, 1980]) can be written as

$$f(r) = \frac{1}{2} \int_{\mathcal{R}} dk k \left(J_k(\beta r) \int_r^\infty \frac{d\rho}{\rho} H_k^{(1)}(\beta\rho) f(\rho) + H_k^{(1)}(\beta r) \int_0^r \frac{d\rho}{\rho} J_k(\beta\rho) f(\rho) \right).$$

Furthermore (2.1) implies

$$\iint_{\Omega} d\rho d\phi \rho f E_s(\theta = 0) = \lim_{\varepsilon \rightarrow 0} \frac{i}{4} \left(\int_0^{i\infty} dk e^{\varepsilon k^2} J_k(\beta r) F(k, ik) + \int_0^{-i\infty} dk e^{\varepsilon k^2} J_k(\beta r) F(k, -ik) \right)$$

and

$$\iint_{\Omega} d\rho d\phi \rho f E_s(\theta = \alpha) = \lim_{\varepsilon \rightarrow 0} \frac{i}{4} \left(\int_0^{i\infty} dk e^{\varepsilon k^2} J_k(\beta r) e^{ik\alpha} F(k, -ik) + \int_0^{-i\infty} dk e^{\varepsilon k^2} J_k(\beta r) e^{-ik\alpha} F(k, ik) \right).$$

4.2 The Poisson equation in a circular wedge.

Proposition 4.4 (The Neumann problem). *Let $u(a, \theta)$ satisfy (1.1) with $\lambda = 0$ in the domain D_2 defined by (2.6), with the Neumann boundary conditions*

$$u_\theta(r, -\alpha) = n_{-\alpha}(r), \quad a < r < \infty, \quad (4.9a)$$

$$u_r(a, \theta) = n(\theta), \quad -\alpha < \theta < \alpha, \quad (4.9b)$$

$$u_\theta(r, \alpha) = n_\alpha(r), \quad a < r < \infty, \quad (4.9c)$$

with the condition (2.8), and with the following consistency condition

$$\int_a^\infty \frac{d\rho}{\rho} n_\alpha(\rho) - \int_a^\infty \frac{d\rho}{\rho} n_{-\alpha}(\rho) - a \int_{-\alpha}^\alpha d\phi n(\phi) + \int_a^\infty d\rho \int_{-\alpha}^\alpha d\phi \rho f(\rho, \phi) = 0. \quad (4.10)$$

Then, u is given by

$$\begin{aligned} 4\pi u(r, \theta) = & \int_0^{i\infty} \frac{dk}{k} \left(\frac{r}{a}\right)^{-k} \left\{ e^{-ik\theta} e^{iak} \left[\frac{A(k)}{\Delta(k)} + N_\alpha(k) - N_\alpha(-k) \right] \right. \\ & \left. + e^{ik\theta} e^{iak} \left[\frac{B(k)}{\Delta(k)} - N_{-\alpha}(k) + N_{-\alpha}(-k) \right] \right\} \\ & + \int_0^{-i\infty} \frac{dk}{k} \left(\frac{r}{a}\right)^{-k} \left\{ e^{ik\theta} e^{-iak} \left[\frac{A(k)}{\Delta(k)} - N_\alpha(-k) + N_\alpha(k) \right] + e^{-ik\theta} e^{-iak} \left[\frac{B(k)}{\Delta(k)} + N_{-\alpha}(-k) - N_{-\alpha}(k) \right] \right\} \\ & + \int_0^\infty \frac{dk}{k} \left(\frac{r}{a}\right)^{-k} \left\{ 2e^{-ik\theta} [e^{iak} N_\alpha(-k) - e^{-iak} N_{-\alpha}(-k) - aN(ik) + F(-k, ik)] \right. \\ & \left. + 2e^{ik\theta} [e^{-iak} N_\alpha(-k) - e^{iak} N_{-\alpha}(-k) - aN(-ik) + F(-k, -ik)] - e^{-ik\theta} F(-k, ik) - e^{ik\theta} F(-k, -ik) \right\} \\ & + 4\pi \iint_\Omega d\rho d\phi \rho f(\rho, \phi) E_s(\rho, \phi; r, \theta) \end{aligned} \quad (4.11)$$

where $\Delta, A, B, N_{\pm\alpha}, N$ are defined as follows:

$$N_{\pm\alpha}(k) = \int_a^\infty \frac{d\rho}{\rho} \left(\frac{\rho}{a}\right)^k n_{\pm\alpha}(\rho), \quad N(\pm ik) = \int_{-\alpha}^\alpha d\phi e^{\pm ik\phi} n(\phi),$$

$$\Delta(k) = e^{2iak} - e^{-2iak},$$

$$\begin{aligned} A(k) = & -(e^{2iak} + e^{-2iak})[N_\alpha(k) + N_\alpha(-k)] + 2[N_{-\alpha}(k) + N_{-\alpha}(-k)] + 2a[e^{iak} N(ik) + e^{-iak} N(-ik)] \\ & - e^{iak}[F(k, ik) + F(-k, ik)] - e^{-iak}[F(-k, -ik) + F(k, -ik)], \end{aligned}$$

$$\begin{aligned} B(k) = & (e^{2iak} + e^{-2iak})[N_{-\alpha}(k) + N_{-\alpha}(-k)] - 2[N_\alpha(k) + N_\alpha(-k)] + 2a[e^{iak} N(-ik) + e^{-iak} N(ik)] \\ & - e^{-iak}[F(k, ik) + F(-k, ik)] - e^{iak}[F(-k, -ik) + F(k, -ik)]. \end{aligned}$$

The functions $F(k, \pm ik)$ are defined by (3.17), and E_s is given by either (2.4) or (2.3) with $\varepsilon = 0$.

Remark 4.5 (Consistency condition). The consistency condition (4.10) is

$$-N_{-\alpha}(0) + N_{\alpha}(0) - aN(0) + F(0,0) = 0.$$

Proof .

Step 1. The two GRs (3.18) can be rewritten as

$$ie^{ik\alpha}D_{\alpha}(k) - ie^{-ik\alpha}D_{-\alpha}(k) - D(ik) = I(k) \quad (4.12)$$

and

$$-ie^{-ik\alpha}D_{\alpha}(k) + ie^{ik\alpha}D_{-\alpha}(k) - D(-ik) = J(k), \quad \Re k < \frac{\pi}{2\alpha}, \quad (4.13)$$

where

$$I(k) = \frac{1}{k} (e^{ik\alpha}N_{\alpha}(k) - e^{-ik\alpha}N_{-\alpha}(k) - aN(ik) + F(k, ik)),$$

$$J(k) = \frac{1}{k} (e^{-ik\alpha}N_{\alpha}(k) - e^{ik\alpha}N_{-\alpha}(k) - aN(-ik) + F(k, -ik)).$$

Letting $k \mapsto -k$ in (4.12) and (4.13) yields

$$ie^{-ik\alpha}D_{\alpha}(-k) - ie^{ik\alpha}D_{-\alpha}(-k) - D(-ik) = I(-k) \quad (4.14)$$

and

$$-ie^{ik\alpha}D_{\alpha}(-k) + ie^{-ik\alpha}D_{-\alpha}(-k) - D(ik) = J(-k), \quad \Re k > -\frac{\pi}{2\alpha}. \quad (4.15)$$

The IR (2.12) contains the four unknown functions $D_{\pm\alpha}(k), D(\pm ik)$, which can be expressed in terms of the two functions, $D_{\pm\alpha}(-k)$ as follows:

$$iD_{-\alpha}(k) = -iD_{-\alpha}(-k) + \frac{1}{\Delta(k)} (e^{-ik\alpha}(I(k) - J(-k)) - e^{ik\alpha}(I(-k) - J(k))), \quad (4.16)$$

$$iD_{\alpha}(k) = -iD_{\alpha}(-k) + \frac{1}{\Delta(k)} (e^{ik\alpha}(I(k) - J(-k)) - e^{-ik\alpha}(I(-k) - J(k))), \quad (4.17)$$

$$\frac{\pi}{2\alpha} < \Re k < \frac{\pi}{2\alpha}.$$

Indeed, (4.14) and (4.15) imply

$$D(-ik) = ie^{-ik\alpha}D_{\alpha}(-k) - ie^{ik\alpha}D_{-\alpha}(-k) - I(-k), \quad (4.18)$$

$$D(ik) = -ie^{ik\alpha}D_{\alpha}(-k) + ie^{-ik\alpha}D_{-\alpha}(-k) - J(-k), \quad \Re k > -\frac{\pi}{2\alpha}. \quad (4.19)$$

Eliminating $D(ik)$ from (4.12) and (4.15), and $D(-ik)$ from (4.13) and (4.14) yields two equations for the combinations of the unknown functions $D_{-\alpha}(k) -$

$D_{-\alpha}(-k)$ and $D_{\alpha}(k) - D_{\alpha}(-k)$. Solving these two equations for these two combinations yields (4.16) and (4.17).

Substituting (4.16)-(4.19) into the IR, the terms involving the unknowns $D_{\pm\alpha}(-k)$ are proportional to the following expression:

$$\begin{aligned} & \int_0^{i\infty} dk \left(\frac{r}{a}\right)^{-k} \left(e^{ik\theta} e^{ik\alpha} D_{-\alpha}(-k) + e^{ik\alpha} e^{-ik\theta} D_{\alpha}(-k) \right) \\ & + \int_0^{-i\infty} dk \left(\frac{r}{a}\right)^{-k} \left(-e^{-ik\theta} e^{-ik\alpha} D_{-\alpha}(-k) - e^{-ik\alpha} e^{ik\theta} D_{\alpha}(-k) \right) \\ & + \int_0^{\infty} dk \left(\frac{r}{a}\right)^{-k} \left(e^{ik\theta} e^{-ik\alpha} D_{\alpha}(-k) - e^{ik\alpha} e^{ik\theta} D_{-\alpha}(-k) \right. \\ & \quad \left. - e^{-ik\theta} e^{ik\alpha} D_{-\alpha}(-k) + e^{-ik\alpha} e^{-ik\theta} D_{-\alpha}(-k) \right). \end{aligned}$$

This expression vanishes using Cauchy's theorem. Indeed $e^{ik(\theta+\alpha)}$ and $e^{ik(\alpha-\theta)}$ decay for $\Im k > 0$, $D_{\pm\alpha}(-k)$ are bounded at infinity for $\Re k \geq 0$ and are of $\mathcal{O}\left(\frac{1}{k}\right)$ as $k \rightarrow \infty$ for $\Re k \geq 0$ (using integration by parts), and $\left(\frac{r}{a}\right)^k$ is bounded for $\Re k > 0$. Also, Jordan's lemma, see e.g. [Ablowitz and Fokas, 2003] page 222) implies that the integral at infinity vanishes.

The remaining terms yield

$$\begin{aligned} 4\pi u(r, \theta) &= \int_0^{i\infty} \frac{dk}{k} \left(\frac{r}{a}\right)^{-k} \left\{ e^{-ik\theta} e^{iak} \left[\frac{A(k)}{\Delta(k)} + N_{\alpha}(k) \right] + e^{ik\theta} e^{iak} \left[\frac{B(k)}{\Delta(k)} - N_{-\alpha}(k) \right] \right\} \\ &+ \int_0^{-i\infty} \frac{dk}{k} \left(\frac{r}{a}\right)^{-k} \left\{ e^{ik\theta} e^{-iak} \left[\frac{A(k)}{\Delta(k)} + N_{\alpha}(k) \right] + e^{-ik\theta} e^{-iak} \left[\frac{B(k)}{\Delta(k)} - N_{-\alpha}(k) \right] \right\} \\ &+ \int_0^{\infty} \frac{dk}{k} \left(\frac{r}{a}\right)^{-k} \left\{ e^{-ik\theta} \left[e^{iak} N_{\alpha}(-k) - e^{-iak} N_{-\alpha}(-k) - 2aN(ik) + F(-k, ik) \right] \right. \\ &\quad \left. + e^{ik\theta} \left[e^{-iak} N_{\alpha}(-k) - e^{iak} N_{-\alpha}(-k) - 2aN(-ik) + F(-k, -ik) \right] \right\} \\ &+ 4\pi \iint_{\Omega} d\rho d\phi \rho f(\rho, \phi) E_s(\rho, \phi; r, \theta). \end{aligned} \tag{4.20}$$

These integrals contain poles on the contours at $k = 0$. However, the integrals taken together are well defined. In order for *each* integral to be well defined we will rewrite (4.20) using the following identity:

$$\begin{aligned} & - \left(\int_{\infty}^0 + \int_0^{-i\infty} \right) \frac{dk}{k} \left(\frac{r}{a}\right)^{-k} \left(e^{ik\theta} e^{ik\alpha} N_{-\alpha}(-k) - e^{-ik\theta} e^{ik\alpha} N_{\alpha}(-k) \right) \\ & \left(\int_{\infty}^0 + \int_0^{i\infty} \right) \frac{dk}{k} \left(\frac{r}{a}\right)^{-k} \left(e^{-ik\theta} e^{-ik\alpha} N_{-\alpha}(-k) - e^{ik\theta} e^{-ik\alpha} N_{\alpha}(-k) \right) = 0. \end{aligned} \tag{4.21}$$

Adding (4.21) to (4.20) yields (4.11). In order to establish (4.21), using analyticity considerations similar to those used above, we find

$$\begin{aligned}
& - \left(\int_{\infty}^{\varepsilon} + \int_{\varepsilon}^{-i\varepsilon} + \int_{-i\varepsilon}^{-i\infty} \right) \frac{dk}{k} \left(\frac{r}{a} \right)^{-k} \left(e^{ik\theta} e^{ik\alpha} N_{-\alpha}(-k) - e^{-ik\theta} e^{ik\alpha} N_{\alpha}(-k) \right) \\
& \left(\int_{\infty}^{\varepsilon} + \int_{\varepsilon}^{i\varepsilon} + \int_{i\varepsilon}^{i\infty} \right) \frac{dk}{k} \left(\frac{r}{a} \right)^{-k} \left(e^{-ik\theta} e^{-ik\alpha} N_{-\alpha}(-k) - e^{ik\theta} e^{-ik\alpha} N_{\alpha}(-k) \right) = 0.
\end{aligned} \tag{4.22}$$

Letting $\varepsilon \rightarrow 0$, the residue contributions from the pole at $k = 0$ cancel and (4.22) follows, where the integrals are understood in the principal value sense, i.e.

$$\left(\int_{-\infty}^0 + \int_0^{\infty} \right) \frac{dk}{k} f(k) = \int_0^{\infty} \frac{dk}{k} (f(k) - f(ik)).$$

Now the definitions of $A(k)$ and $B(k)$ imply

$$A(0) = -4S, \quad B(0) = -4S,$$

where

$$S = N_{\alpha}(0) - N_{-\alpha}(0) - aN(0) + F(0, 0) = 0.$$

Thus, $A(0) = B(0) = 0$. In addition, the functions $A(k)$ and $B(k)$ are even functions of k , thus $A'(0) = B'(0) = 0$, and these relations imply that each bracket appearing in the integrals of (4.11) vanishes at $k = 0$. For example, the first bracket evaluated at $k = 0$ yields

$$\frac{O(k^2)}{4i\alpha k} + N_{\alpha}(0) - N_{-\alpha}(0) = 0.$$

Thus $k = 0$ is a removable singularity in *each* term of the RHS of equation (4.11). \square

Remark 4.6 (Comparison with the classical representations). *The Neumann problem of Poisson in a circular wedge can be solved using either a cosine-series in θ or the Mellin sine transform in r :*

$$\begin{aligned}
\hat{g}(k) &= \int_0^{\infty} \frac{d\rho}{\rho} \sin \left(k \log \left(\frac{\rho}{a} \right) \right) g(\rho), \\
g(r) &= \frac{2}{\pi} \int_0^{\infty} dk \sin \left(k \log \left(\frac{r}{a} \right) \right) \hat{g}(k), \quad a < r < \infty,
\end{aligned}$$

[Stakgold, 1967] volume 1 p. 316. *The cosine-series solution is uniformly convergent at $r = a$ but not at $\theta = \pm\alpha$; the Mellin sine transform solution is uniformly convergent at $\theta = \pm\alpha$ but not at $r = a$. The solution (4.11) is the best possible representation: it is an integral which is uniformly convergent at $\theta = \pm\alpha$ and at $r = a$ (furthermore the integral can either be evaluated as residues or deformed to give the cosine-series solution and Mellin sine solution respectively).*

	The method of [Fokas, 1997]	Transforms
Mathematical input	<ul style="list-style-type: none"> • Completeness relation for each separated ODE in whole space (independent of the domain and boundary conditions) • Green's theorem 	<ul style="list-style-type: none"> • Completeness relation for one separated ODE dependent on domain and boundary conditions (bcs)
Implementation given a BVP	<ul style="list-style-type: none"> • Same steps independent of bcs • Algebraic manipulation • Unknowns vanish by Cauchy 	<ul style="list-style-type: none"> • Different transforms for different bcs • Integration by parts • Solve ODE using 1-d Green's function
Solution	<ul style="list-style-type: none"> • Uniformly convergent at $\partial\Omega$ • Given as an integral, deform contour so integrand decays exponentially 	<ul style="list-style-type: none"> • Not uniformly convergent at $\partial\Omega$ • Either an infinite sum or an integral depending on domain. If an integral, can deform contour so integrand decays exponentially
Boundary conditions	<ul style="list-style-type: none"> • Separable and some non-separable 	<ul style="list-style-type: none"> • Only separable
Domains	<ul style="list-style-type: none"> • Separable and some non-separable 	<ul style="list-style-type: none"> • Only separable

Table 1: Comparison of the method of [Fokas, 1997] and classical transforms in 2-D

5 Conclusions.

5.1 The method of [Fokas, 1997] versus the classical transform method.

A natural question is “how does the method of [Fokas, 1997] compare to the classical transform method?”. Table 5.1 compares the method of [Fokas, 1997] and the classical transform method, and shows that the method of [Fokas, 1997] requires less mathematical input, is simpler to implement, yields more useful solution formulae, and is more widely applicable than the classical transform method. In section 4 this method was applied to two boundary value problems in separable domains with separable boundary conditions. The method can also be used to solve boundary value problems in non-separable domains [Dassios and Fokas, 2005], [Fokas, 2008],[Spence, 2009], [Fokas and Kalimeris, 2009], [Kalimeris and Fokas, 2009], as well as in separable domains with non-separable boundary conditions [Fokas, 2008],[Spence, 2009].

5.2 How the method of [Fokas, 1997] is related to other approaches.

Shanin. The GR for Helmholtz in the interior of an equilateral triangle was first discovered by Shanin [Shanin, 1997] (later published as [Shanin, 2000]). In these

papers the eigenvalues and eigenfunctions of the Laplacian for impedance boundary conditions are found and the Dirichlet problem of the Helmholtz equation is solved in terms of an infinite series (an integral representation of the solution is also obtained by substituting an integral representation of the fundamental solution into Green’s integral representation).

The synthesis of “Fourier” with “Green” in the background of “Cauchy”. The method of [Fokas, 1997] combines the ideas of Green and the classical transform method by constructing the analogue of Green’s integral representation in the transform space. Furthermore, for the elimination of the unknown boundary values a crucial role is played by analyticity and Cauchy’s theorem. Several investigations have made some attempts in this direction, see for example [Croisille and Lebeau, 1999],[Gautesen, 2005],[Bernard, 2006], however it appears that this “synthesis” has not been achieved before.

Integral representations in the complex plane. Representations of solutions to ODEs as integrals in the complex plane were pioneered by Laplace following earlier investigations by Euler. In hindsight, the “moral” of the Watson transformation is that the best representation of the solution of a separable PDEs is an integral representation in the complex k plane (which can be deformed to either of the two representations obtained by transforms). This representation is precisely the one obtained by the new method. In this sense, it is surprising that no-one tried to find this representation directly until the emergence of the method of [Fokas, 1997].

The Sommerfeld-Malyuzhinets technique. The Sommerfeld-Malyuzhinets (S-M) technique is a method for solving Helmholtz in a wedge by assuming that the solution can be written as an integral in the complex plane over the so-called Sommerfeld contour, see e.g. [Osipov and Norris, 1999], [Babich et al., 2008]. It appears that for a wedge the method of [Fokas, 1997] is related to the S-M method: the change of variables $k = e^{i\phi}$, $z = re^{i\theta}$ in the exponential $\exp i\beta \left(kz + \frac{z}{k}\right)$ appearing in the integral representation of Helmholtz in a convex polygon (see proposition 2.7 of [Spence and Fokas, 2009]) transforms it to $\exp 2i\beta r \cos(\theta - \phi)$, which appears in the solutions given by the S-M method (recall that the frequency is 2β). Although a detailed investigation of this connection will be presented elsewhere, we can already make the following remarks: (a) The S-M technique is a method for solving only the Helmholtz equation in a particular domain whereas the method of [Fokas, 1997] is applicable to a wide variety of PDEs and domains. (b) The solution formula of S-M is an integral involving a contour in the complex plane which is independent of the wedge angle, whereas the solution given by the method of [Fokas, 1997] involves integrals over contours which depend on the domain. (c) The S-M technique requires the solution of a functional-difference equation as well as Cauchy’s theorem, whereas the method of [Fokas, 1997] requires only Cauchy’s theorem as well as the algebraic manipulation of the global relation.

5.3 Extension of the method to 3 dimensions.

The method of [Fokas, 1997] is applied to evolution PDEs in 2 space and 1 time dimensions in [Fokas, 2002] and [Kalimeris and Fokas, 2009]. Regarding elliptic PDEs we note the following: appropriate domain dependent fundamental solutions for separable domains in 3 dimensions can be constructed using the method of [Spence and Fokas, 2009] and are expressed as double integrals over two complex parameters. The construction of the global relation is similar to the construction presented here, but now the global relation contains two complex parameters. The solution of BVPs in 3 dimensions also follows the steps outlined in §1.2, but is more involved due to the fact that the solution, like the domain dependent fundamental solutions, is expressed as a double integral.

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References

- [Ablowitz and Fokas, 2003] Ablowitz, M. and Fokas, A. (2003). *Complex Variables: Introduction and Applications*. CUP, 2nd edition.
- [Babich et al., 2008] Babich, V., Lyalinov, M., and Grikurov, V. (2008). *Diffraction theory: The Sommerfeld-Malyuzhinets technique*. Alpha Science.
- [Bernard, 2006] Bernard, J. (2006). A spectral approach for scattering by impedance polygons. *The Quarterly Journal of Mechanics and Applied Mathematics*, 59(4):517.
- [Cohen, 1964] Cohen, D. (1964). Separation of Variables and Alternative Representations for Non-Selfadjoint Boundary Value Problems. *Communications on Pure and Applied Mathematics*, 17:1–22.
- [Croisille and Lebeau, 1999] Croisille, J. and Lebeau, G. (1999). *Diffraction by an immersed elastic wedge*. Lecture notes in mathematics volume 1723, Springer Verlag.
- [Dassios and Fokas, 2005] Dassios, G. and Fokas, A. (2005). The basic elliptic equations in an equilateral triangle. *Proc. R. Soc. Lond. A*, 461:2721–2748.
- [Duffy, 2001] Duffy, D. (2001). *Green's Functions With Applications*. CRC Press.
- [Evans, 1998] Evans, L. (1998). *Partial differential equations*. American Mathematical Society Providence, RI.
- [Flyer and Fokas, 2008] Flyer, N. and Fokas, A. (2008). A hybrid analytical–numerical method for solving evolution partial differential equations. I. The half-line. *Proceedings of the Royal Society A*, 464(2095):1823.
- [Fokas, 1997] Fokas, A. (1997). A unified transform method for solving linear and certain nonlinear PDEs. *Proc. R. Soc. Lond. A*, 453:1411–1443.
- [Fokas, 2002] Fokas, A. (2002). A new transform method for evolution partial differential equations. *IMA J. Appl. Math.*, 67(6):559–590.

- [Fokas, 2008] Fokas, A. (2008). *A Unified Approach to Boundary Value Problems*. CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM.
- [Fokas and Kalimeris, 2009] Fokas, A. and Kalimeris, K. (2009). Eigenvalues for the Laplace operator in the interior of an equilateral triangle. *preprint*.
- [Fokas and Spence, 2010] Fokas, A. and Spence, E. (2010). The Wiener-Hopf technique revisited. *in preparation*.
- [Friedman, 1956] Friedman, B. (1956). *Principles and Techniques of Applied Mathematics*. John Wiley & Sons Inc.
- [Gautesen, 2005] Gautesen, A. (2005). Diffraction of plane waves by a wedge with impedance boundary conditions. *Wave Motion*, 41(3):239–246.
- [Jones, 1980] Jones, D. (1980). The Kontorovich-Lebedev Transform. *IMA Journal of Applied Mathematics*, 26(2):133–141.
- [Jones, 1986] Jones, D. (1986). Acoustic and electromagnetic waves. *Oxford/New York, Clarendon Press/Oxford University Press, 1986, 764 p*.
- [Kalimeris and Fokas, 2009] Kalimeris, K. and Fokas, A. (2009). The heat equation in the interior of an equilateral triangle. *Studies In Applied Mathematics*., to appear.
- [Keener, 1995] Keener, J. (1995). *Principles of Applied Mathematics*. Perseus Books.
- [Keller, 1979] Keller, J. (1979). Progress and Prospects in the Theory of Linear Wave Propagation. *SIAM Review*, 21:229.
- [Morse and Feshbach, 1953] Morse, P. and Feshbach, H. (1953). *Methods of Theoretical Physics*. McGraw-Hill Science/Engineering/Math.
- [Ockendon et al., 2003] Ockendon, J., Howison, S., Lacey, A., and Movchan, A. (2003). *Applied Partial Differential Equations*. Oxford University Press, USA.
- [Osipov and Norris, 1999] Osipov, A. and Norris, A. (1999). The Malyuzhinets theory for scattering from wedge boundaries: a review. *Wave motion*, 29(4):313–340.
- [Shanin, 1997] Shanin, A. (1997). Excitation of wave field in a triangle with impedance boundary conditions. *Day of Diffraction*, 97:205–214.
- [Shanin, 2000] Shanin, A. (2000). Excitation of a wave field in a triangular domain with impedance boundary conditions. *Journal of Mathematical Sciences*, 102(4):4328–4338.
- [Spence, 2009] Spence, E. (2009). *Boundary value problems for linear elliptic PDEs*. PhD thesis, University of Cambridge.
- [Spence and Fokas, 2009] Spence, E. and Fokas, A. (2009). A new transform method I: Domain dependent fundamental solutions and integral representations. *preprint*.
- [Stakgold, 1967] Stakgold, I. (1967). *Boundary value problems of mathematical physics, Volumes I and II*. New York: The Macmillan Company; London: Collier-Macmillan Ltd.
- [Titchmarsh, 1962] Titchmarsh, E. (1962). *Eigenfunction Expansions Associated with Second-order Differential Equations*. Clarendon Press, 2nd edition.
- [Watson, 1918] Watson, G. (1918). The Diffraction of Electric Waves by the Earth. *Proceedings of the Royal Society of London. Series A*, 95(666):83–99.