# ISOTHERMIC SURFACES AND CONSERVATION LAWS

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*In memory of Joe Wolf*

ABSTRACT. For CMC surfaces in 3-dimensional space forms, we relate the moment class of Korevaar–Kusner–Solomon to a second cohomology class arising from the integrable systems theory of isothermic surfaces. In addition, we show that both classes have a variational origin as Noether currents.

## 1. Introduction

The purpose of this paper is to record and explore a puzzling coincidence that we observed during a MATRIX workshop. The setting is classical differential geometry of surfaces and the coincidence relates two Lie algebra valued cohomology classes associated to a constant mean curvature (CMC) surface in a 3-dimensional space form.

The first of these classes is the *moment class* of Kusner and his collaborators [22–24] of which the following is a paradigm: let $\Sigma \subset \mathbb{R}^3$ have constant mean curvature $H$, $Y$ be a Killing field on $\mathbb{R}^3$ and $\gamma$ a 1-cycle in $\Sigma$ bounding a 2-cycle $D$ in $\mathbb{R}^3$. Then

$$\int_\gamma (Y,\nu) - 2H\int_D (Y,N)$$

*depends only on the homology class of* $\gamma$. Here $\nu, N$ are appropriate unit normals to $\gamma$ in $\Sigma$ and $D$ in $\mathbb{R}^3$, respectively. Letting $Y$ vary in the Lie algebra $\mathfrak{g}_{\mathbb{R}^3}$ of Killing fields, we arrive at a cohomology class $\mu \in H^1(\Sigma,\mathbb{R}) \otimes \mathfrak{g}^*_{\mathbb{R}^3}$, the *moment class*. This has been used to demonstrate the necessity of the balancing conditions that Kapouleas [21] needed in his celebrated existence theorem for CMC surfaces. A similar analysis can be carried out for CMC immersed hypersurfaces in any Riemannian manifold $M^{k+1}$ under mild topological conditions to produce $\mu \in H^{k-1}(\Sigma,\mathbb{R})\otimes\mathfrak{g}^*_M$, with $\mathfrak{g}_M$ the Killing algebra of $M$.

Returning to the $\mathbb{R}^3$ case, Meeks–Pérez–Tinaglia [25] observe that the restriction $\mu_{|\mathbb{R}^3}$ to the constant vector fields has a de Rham representative given by the *flux form*:

$$\mathrm{d}x \times (N + Hx) \in \Omega^1_\Sigma \otimes \mathbb{R}^3$$

where $x\colon \Sigma \to \mathbb{R}^3$ is the inclusion and $N$ the normal to $x$. The starting point of the present paper is the observation that the flux form essentially appears as a component of another closed Lie algebra valued form, the *retraction form*, which is the corner-stone of the modern theory of isothermic surfaces.

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Recall that an immersed surface $x\colon \Sigma \to \mathbb{R}^3$ is *isothermic* if, away from umbilics, it locally admits conformal coordinates $u, v$ which are simultaneously curvature line coordinates. This amounts to the local existence of holomorphic quadratic differentials that commute with the second fundamental form $\mathrm{II}$ of $x$. We say that $x$ is globally isothermic [28] if there is a globally defined holomorphic quadratic differential $q \in H^0(\Sigma, K^2)$ which so commutes. CMC surfaces are globally isothermic since then $\mathrm{II}^{2,0}$ is holomorphic by the Codazzi equation.

Globally isothermic surfaces comprise an integrable system [12]. This can, in large part, be understood as a consequence of the existence of a pencil of flat connections with gauge potential $\eta$ which is a closed 1-form with values in the Lie algebra $\mathfrak{c}_{\mathbb{R}^3}$ of *conformal* vector fields on $\mathbb{R}^3$. This $\eta \in \Omega^1_\Sigma \otimes \mathfrak{c}_{\mathbb{R}^3}$, which is constructed from $x$ and $q$, is the retraction form. When $x$ is a CMC surface, the component of $\eta$ in $\mathfrak{so}(3) \cong \wedge^2\mathbb{R}^3$, the infinitesimal rotations about 0, is cohomologous to

$$\mathrm{d}x \wedge (N + Hx)$$

which coincides with the flux form after using the cross product to identify $\wedge^2\mathbb{R}^3$ with $\mathbb{R}^3$. The alert reader will spot that something unusual is going on since we are identifying the rotational part of $\eta$ with the (dual of) the translational part of $\mu$.

In our attempts to understand this, we arrived at the following context and results. Let $M$ be a 3-dimensional space form with $\mathfrak{g}_M$ the Lie algebra of Killing fields on $M$ and $\mathfrak{c}_M$ the Lie algebra of conformal vector fields so that $\mathfrak{g}_M \leq \mathfrak{c}_M$. Our first observation is:

*There is an isomorphism $S\colon \mathfrak{g}_M \cong \mathfrak{g}_M^*$ which is equivariant for the action of the isometry group of $M$.*

Now let $x\colon \Sigma \to M$ be a CMC, and therefore isothermic, surface with moment class $\mu \in H^1(\Sigma, \mathbb{R}) \otimes \mathfrak{g}_M^*$ and retraction class $[\eta] \in H^1(\Sigma, \mathbb{R}) \otimes \mathfrak{c}_M$. Then

1. $[\eta]$ *takes values in* $\mathfrak{g}_M$.
2. $S[\eta] = \mu$.

As corollaries of this analysis, we find that parallel pairs of CMC surfaces in space forms have the same moment class and we provide an explicit de Rham representative of the moment class which generalises the flux form.

Here is the plan of the paper. We describe the moment class and retraction form in Sections 2 and 3 respectively. In Section 4 we prove our main results. However, while we provide complete proofs, we lack a conceptual understanding of *why* our results are true. In Section 5, we attempt a partial resolution by showing that both moment and retraction classes have a variational origin as Noether currents. That said, a new idea is required to relate the variational problems involved.

Finally: it was an honour to participate in Joe Wolf's final workshop. The generosity and gentleness with which he shared his wisdom was a delight to witness.

## 2. A conservation law for CMC hypersurfaces

We begin by setting out an approach to the result of Korevaar–Kusner–Solomon [23].

Let $x\colon \Sigma^k \to M^n$ be an immersion of an oriented $k$-manifold into an oriented Riemannian $n$-manifold. We denote the metric on $M$ by $(\,,\,)$. Equip $\Sigma$ with the induced metric $\mathrm{I} = (\mathrm{d}x, \mathrm{d}x)$ and volume form $\mathrm{vol}_x \in \Omega^k_\Sigma$. With $N\Sigma$ the normal bundle of $x$, we have an orthogonal decomposition

$$x^*TM = \mathrm{d}x(T\Sigma) \oplus N\Sigma$$

with a corresponding decomposition of vector fields along $x$:

$$X \circ x = \mathrm{d}x(X^\top) + X^\perp,$$

for $X \in \Gamma\, TM$. Let $\mathrm{II} \in \Gamma\, S^2T^*\Sigma \otimes N\Sigma$ be the second fundamental form of $x$ and $\mathbf{H} = \frac{1}{k}\,\mathrm{trace}\,\mathrm{II} \in \Gamma\, N\Sigma$ the mean curvature vector.

With this in hand, we have

**Lemma 2.1.** *For $Y$ a Killing field on $M$,*

$$\mathrm{d}i_{Y^\top}\mathrm{vol}_x = k(Y \circ x, \mathbf{H})\mathrm{vol}_x. \tag{2.1}$$

*Proof.* Using the Cartan identity we have

$$\mathrm{d}i_{Y^\top}\mathrm{vol}_x = L_{Y^\top}\mathrm{vol}_x = (\mathrm{trace}\,\nabla^\Sigma Y^\top)\mathrm{vol}_x,$$

where $\nabla^\Sigma$ is the Levi-Civita connection on $\Sigma$. On the other hand, with $\nabla^M$ the (pullback of) the Levi-Civita connection on $M$ and $A$ the shape operator, we have

$$\nabla^\Sigma Y^\top = (\nabla^M Y)^\top + A^{Y^\perp}. \tag{2.2}$$

Now $Y$ is Killing so $(\nabla^M Y)^\top$ is skew and so trace-free while

$$\mathrm{trace}\, A^{Y^\perp} = (Y \circ x, \mathrm{trace}\,\mathrm{II}) = k(Y \circ x, \mathbf{H}).$$

□

When $x$ is a hypersurface so that $n = k+1$, we take $N \in \Gamma\, N\Sigma$ to be the unit normal for which $x^*(i_N\mathrm{vol}_M) = \mathrm{vol}_x$ and so define the mean curvature $H$ by $HN = \mathbf{H}$. Now (2.1) reads

$$\mathrm{d}i_{Y^\top}\mathrm{vol}_x = kH(Y \circ x, N)\mathrm{vol}_x = kHx^*(i_Y\mathrm{vol}_M). \tag{2.3}$$

Now suppose that the de Rham cohomology $H^k(M,\mathbb{R}) = H^{k-1}(M,\mathbb{R}) = 0$. Then, another application of the Cartan identity gives

$$\mathrm{d}i_Y\mathrm{vol}_M = L_Y\mathrm{vol}_M = 0,$$

since $Y$ is Killing. Our hypotheses on the cohomology of $M$ now guarantee the existence of $\alpha_Y \in \Omega^{k-1}_M$, unique up to the addition of an exact $(k-1)$-form, with

$$\mathrm{d}\alpha_Y = i_Y\mathrm{vol}_M.$$

In this case, (2.3) becomes

$$\mathrm{d}i_{Y^\top}\mathrm{vol}_x = kH\,\mathrm{d}(x^*\alpha_Y). \tag{2.4}$$

We therefore conclude:

**Proposition 2.2.** *Let $M^{k+1}$ be an oriented Riemannian manifold with $H^k(M,\mathbb{R}) = H^{k-1}(M,\mathbb{R}) = 0$, $Y$ a Killing field on $M$ and $x\colon \Sigma^k \to M^{k+1}$ be an immersed oriented hypersurface with* constant *mean curvature $H$. Then*

$$\mathrm{d}(i_{Y^\top}\mathrm{vol}_x - kHx^*\alpha_Y) = 0. \tag{2.5}$$

Denote by $\mu_x(Y) \in H^{k-1}(\Sigma,\mathbb{R})$ the cohomology class of $i_{Y^\top}\mathrm{vol}_x - kHx^*\alpha_Y$ and let $\mathfrak{g}_M$ denote the Lie algebra of Killing fields of $M$. Observe that the uniqueness of $\alpha_Y$ modulo exact forms ensures that $Y \mapsto \alpha_Y$ induces a linear map $\mathfrak{g}_M \to \Omega^{k-1}_M/\,\mathrm{d}\Omega^{k-2}_M$ so that $Y \mapsto \mu_x(Y)$ is also a linear map defining the *moment class*

$$\mu_x \in H^{k-1}(\Sigma,\mathbb{R}) \otimes \mathfrak{g}_M^*. \tag{2.6}$$

Let $G_M$ be the isometry group of $M$ and $g \in G_M$. For $p \in M$, we have

$$Y(gp) = \mathrm{d}_p g(\mathrm{Ad}(g)^{-1}Y)(p) \tag{2.7}$$

so that

$$i_{\mathrm{Ad}(g)^{-1}Y}\mathrm{vol}_M = g^*(i_Y \mathrm{vol}_M).$$

It follows at once that

$$\alpha_{\mathrm{Ad}(g)^{-1}Y} \equiv g^*\alpha_Y \mod \mathrm{d}\Omega^{k-2}_M.$$

Again from (2.7), we deduce that

$$i_{(\mathrm{Ad}(g)^{-1}Y)^\top}\mathrm{vol}_x = i_{Y^\top}\mathrm{vol}_{gx}$$

and so finally that

$$\mu_{gx}(Y) = \mu_x(\mathrm{Ad}(g)^{-1}Y).$$

Otherwise said:

**Proposition 2.3** (c.f. [24])**.** *The moment class is equivariant for the co-adjoint action of $G_M$ on $\mathfrak{g}^*_M$:*

$$\mu_{gx} = \mathrm{Ad}^*(g)\mu_x,$$

*for all $g \in G_M$.*

As an application of these ideas, we recover the following conservation law of Korevaar–Kusner–Solomon:

**Theorem 2.4** ([23, Remark 3.17; 24; 25, Theorem 3.4])**.** *In the situation of Proposition 2.2, further suppose that $\Sigma$ is the boundary of a $(k+1)$-manifold $\Omega$ and that $x$ extends to a local diffeomorphism $\hat{x}\colon \Omega \to M$.*

*Let $\gamma$ be a hypersurface in $\Sigma$ and $D$ a hypersurface in $\Omega$ with $\partial D = \gamma$. Then*

$$\int_\gamma (Y,\nu)\mathrm{vol}_{x_{|\gamma}} - kH\int_D (Y,N)\mathrm{vol}_{\hat{x}_{|D}}$$

*depends only on the homology class of $\gamma$. Here $\nu$ is the normal to $\gamma$ in $\Sigma$ and $N$ the normal to $\hat{x}_{|D}$.*

*Proof.* We recognise the integrands: $(Y,\nu)\mathrm{vol}_{x_{|\gamma}} = i_{Y^\top}\mathrm{vol}_x$ on $\gamma$ while $(Y,N)\mathrm{vol}_{\hat{x}_{|D}} = \hat{x}^*(i_Y\mathrm{vol}_M) = \mathrm{d}(\hat{x}^*\alpha_Y)$ on $D$. Thus, by Stokes' Theorem, the integral is

$$\int_\gamma i_{Y^\top}\mathrm{vol}_x - kH\int_D \mathrm{d}(\hat{x}^*\alpha_Y) = \int_\gamma (i_{Y^\top}\mathrm{vol}_x - kHx^*\alpha_Y)$$

which is simply the evaluation of $\mu_x$ on the homology class of $\gamma$. □

We would like to find an explicit closed $(k-1)$-form representing $\mu_x$, or, equivalently, an explicit representative of the coset of $\alpha_Y$ in $\Omega^{k-1}_M/\mathrm{d}\Omega^{k-2}_M$. When $M = \mathbb{R}^{k+1}$ this is possible. Indeed, it is an amusing exercise to check that

$$\alpha_Y \equiv \begin{cases} \frac{1}{k} i_X i_Y \mathrm{vol}_{\mathbb{R}^{k+1}} \mod \Omega^{k-2}_{\mathbb{R}^{k+1}}, & \text{for } Y \text{ an infinitesimal translation;} \\ \frac{1}{k+1} i_X i_Y \mathrm{vol}_{\mathbb{R}^{k+1}} \mod \Omega^{k-2}_{\mathbb{R}^{k+1}}, & \text{for } Y \text{ an infinitesimal rotation about } 0, \end{cases}$$

where $X$ is the radial vector field, $X(p) = p$, for $p \in \mathbb{R}^{k+1}$. However, even when $M = S^{k+1}$, this seems to be difficult.

That said, when $k = 2$ and $M$ is a space-form[1], we can find such a representative by recourse to the theory of isothermic surfaces, to which we now turn.

[1]That is, $M$ is simply connected with constant sectional curvatures.

## 3. Isothermic surfaces

From now on, we restrict attention to immersions $x\colon \Sigma^2 \to M^n$ of an oriented surface into an oriented Riemannian manifold.

**Definition** (Isothermic surface)**.** An immersion $x\colon \Sigma^2 \to M^n$ is *isothermic* if, away from umbilics, it locally admits conformal curvature line coordinates, thus conformal coordinates $u, v$ for I for which the coordinate vector fields $\partial/\partial u, \partial/\partial v$ are eigenvectors for all shape operators $A$ of $x$.

*Remarks.*

1. It follows that all shape operators of an isothermic $x$ commute.
2. Isothermicity is a condition on the trace-free part of the shape operators and so is preserved by conformal diffeomorphisms: if $x\colon \Sigma \to M$ is isothermic and $T\colon M \to \hat{M}$ is a conformal diffeomorphism then $T \circ x$ is also isothermic.

The conformal structure of the induced metric I makes $\Sigma$ into a Riemann surface, so that complex variable methods can be applied. This offers a different take on the isothermic condition: for $u, v$ conformal curvature line coordinates, $z := u + iv$ is a holomorphic coordinate and $q := \mathrm{d}z^2$ is a holomorphic quadratic differential which commutes with shape operators in the sense that

$$Q(A^\nu U, V) = Q(U, A^\nu V), \tag{3.1}$$

for all $U, V \in T\Sigma$ and $\nu \in N\Sigma$, where $Q = 2\operatorname{Re} q$. This prompts (c.f. Smyth [28]):

**Definition** (Globally isothermic surface)**.** An immersion $x\colon \Sigma^2 \to M^n$ is *globally isothermic* if there is a holomorphic quadratic differential $q \in H^0(K^2)$ for which (3.1) holds.

In this case, we say that $(x, q)$ is globally isothermic.

We remark that globally isothermic surfaces are isothermic: away from zeroes of $q$, we may locally find a holomorphic coordinate $z$ with $q = \mathrm{d}z^2$.

**Example.** Let $x\colon \Sigma \to M^3$ take values in a 3-dimensional space form. In this setting, it is well known [20, p. 137] that $H$ is constant if and only if the *Hopf differential* $\mathrm{II}^{2,0}$ is a holomorphic quadratic differential. Thus *constant mean curvature surfaces in* 3*-dimensional space forms are globally isothermic with* $q = \mathrm{II}^{2,0}$.

*Remark.* For $M = S^n$, there are many isothermic surfaces. Indeed, in this case, they comprise an integrable system [12]. However, we must confess that we know of no examples away from this conformally flat setting. It would be interesting to know if any CMC surfaces in 3-manifolds with 4-dimensional isometry group are (globally) isothermic (the survey of Fernández–Mira [17] is a good starting place).

**Definition** (Retraction form)**.** Let $x\colon \Sigma \to M$ be an immersion, $q \in H^0(K^2)$ and $Y$ a conformal vector field on $M$. Define $\eta_Y \in \Omega^1_\Sigma$ by

$$\eta_Y(U) = 2\operatorname{Re} q(Y^\top, U)$$

and so the *retraction form* $\eta^q_x \in \Omega^1_\Sigma \otimes \mathfrak{c}^*_M$, for $\mathfrak{c}_M$ the Lie algebra of conformal vector fields on $M$, by

$$\eta^q_x(Y) = \eta_Y.$$

*Remarks.*

1. *$\eta^q$ is equivariant for the co-adjoint action of the conformal diffeomorphism group of $M$*: if $g\colon M \to M$ is a conformal diffeomorphism of $M$ then
$$\eta^q_{gx} = \operatorname{Ad}^*(g)\eta^q_x, \tag{3.2}$$
thanks to (2.7) (which holds for any Lie group acting on $M$).
2. For $p \in M$, set $\operatorname{stab}_{\mathfrak{c}_M}(p) = \{Y \in \mathfrak{c}_M \mid Y(p) = 0\}$, the infinitesimal stabiliser of $p$. Thus, if $s \in \Sigma$ and $Y \in \operatorname{stab}_{\mathfrak{c}_M}(x(s))$ is a conformal vector field vanishing at $x(s)$, $\eta^q_x(Y) = 2\operatorname{Re} q(Y^\top, -)$ also vanishes at $s$ and we conclude that *$\eta^q_x$ takes values in the bundle $\operatorname{stab}_{\mathfrak{c}_M}(x)^\circ$ of annihilators of infinitesimal stabilisers along $x$.*

Here is the point of this construction:

**Proposition 3.1.** *If $(x, q)\colon \Sigma \to M$ is globally isothermic then $\mathrm{d}\eta^q_x = 0$.*

*Proof.* We must show that $\mathrm{d}\eta_Y = 0$, for $Y \in \mathfrak{c}_M$. For this, write $Q = 2\operatorname{Re} q$ and view $Q$ as a $T^*\Sigma$-valued 1-form. Then
$$\eta_Y = \langle Q, Y^\top\rangle$$
so that
$$\mathrm{d}\eta_Y = \langle \mathrm{d}^{\nabla^\Sigma} Q, Y^\top\rangle - \langle Q \wedge \nabla^\Sigma Y^\top\rangle,$$
where $\langle\,,\,\rangle$ is the evaluation pairing on $T^*\Sigma \times T\Sigma$ which we use to multiply coefficients in the wedge product. Using (2.2), this reads
$$\mathrm{d}\eta_Y = \langle \mathrm{d}^{\nabla^\Sigma} Q, Y^\top\rangle - \langle Q \wedge (\nabla^M Y)^\top\rangle - \langle Q \wedge A^{Y^\perp}\rangle.$$
Since $Y$ is conformal, $(\nabla^M Y)^\top$ commutes with the complex structure $J_x$ of $\Sigma$ while $J^*_x Q = -Q$ so that $\langle Q \wedge (\nabla^M Y)^\top\rangle = 0$. Thus, for any immersion $x$,
$$\mathrm{d}\eta_Y = \langle \mathrm{d}^{\nabla^\Sigma} Q, Y^\top\rangle - \langle Q \wedge A^{Y^\perp}\rangle. \tag{3.3}$$
However, the holomorphicity of $q$ is equivalent to $\mathrm{d}^{\nabla^\Sigma} Q = 0$ while the second summand vanishes by the isothermic condition (3.1). □

*Remark.* When the conformal group acts transitively so that evaluation $\mathfrak{c}_M \to T_pM$ surjects for all $p \in M$, we see from (3.3) that the converse also holds: if $\mathrm{d}\eta^q_x = 0$ then $(x, q)$ is globally isothermic.

This theory comes alive when we take $M = S^n$. Now $\mathfrak{c}_{S^n} \cong \mathfrak{so}(n+1, 1)$ is a simple Lie algebra and we may use the Killing form to identify $\mathfrak{c}^*_{S^n}$ with $\mathfrak{c}_{S^n}$. We therefore view $\eta^q_x$ as a $\mathfrak{c}_{S^n}$-valued 1-form which, in view of a remark above, takes values in $\operatorname{stab}_{\mathfrak{c}_{S^n}}(x)^\perp \leq \mathfrak{c}_{S^n}$. Moreover, for any $p \in S^n$, $\operatorname{stab}_{\mathfrak{c}_{S^n}}(p)^\perp$ is abelian (it comprises the infinitesimal translations of $\mathbb{R}^n = S^n \setminus \{p\}$) so that $\eta^q_x$ takes values in a bundle of abelian subalgebras. From this, we learn two things:

1. *$(x, q)\colon \Sigma \to S^n$ is globally isothermic if and only if $\mathrm{d} + t\eta^q_x$ is a flat connection for each $t \in \mathbb{R}$*:
$$R^{\mathrm{d}+t\eta^q_x} = t\,\mathrm{d}\eta^q_x + \tfrac{1}{2}t^2[\eta^q_x \wedge \eta^q_x] = t\,\mathrm{d}\eta^q_x.$$
This gives an entry point to the well-developed integrable systems theory of isothermic surfaces, see [6, 7, 9, 11, 12, 27], among many others, building on the classical results [1, 2, 5, 11, 13–15], for more on this.

2. The key structure of $S^n$ that was needed for the final part of this analysis amounts to the requirement of a transitive action of a semisimple Lie group with parabolic infinitesimal stabilisers which have abelian nilradicals. Manifolds of this kind are the symmetric $R$-spaces and a surprising amount of the theory of isothermic surfaces goes through in this setting [7].

## 4. Conservation laws again

Let $x\colon \Sigma \to M$ be a CMC surface in a 3-dimensional space form. Then $M$ can be conformally embedded as an open subset of $S^3$ and, by Liouville's theorem, $\mathfrak{c}_M = \mathfrak{c}_{S^3}$ so that, in particular, $\mathfrak{g}_M \leq \mathfrak{c}_{S^3}$. We want to compare the moment class (2.6) $\mu_x \in H^1(\Sigma, \mathbb{R}) \otimes \mathfrak{g}_M^*$ with the cohomology class $[\eta_x]$ of $\eta_x := \eta_x^{\mathrm{II}^{2,0}}$ which lies in $H^1(\Sigma, \mathbb{R}) \otimes \mathfrak{c}_{S^3}$.

Our main result is the following sequence of observations:

1. There is a $G_M$-invariant symmetric bilinear form on $\mathfrak{g}_M$ of signature $(3,3)$ and so a $G_M$-equivariant isomorphism $S\colon \mathfrak{g}_M \to \mathfrak{g}_M^*$.
2. For CMC $x\colon \Sigma \to M$, we have $[\eta_x] \in H^1(\Sigma, \mathbb{R}) \otimes \mathfrak{g}_M$.
3. $\mu_x = S[\eta_x]$.

In this way, we arrive at an explicit representative of $\mu_x$.

For this, we start by taking a unified approach to space forms by viewing them as conic sections.

4.1. **Conic sections in general.** Contemplate $\mathbb{R}^{n+1,1}$, an $(n+2)$-dimensional real vector space with a symmetric bilinear form $(\ ,\ )$ of signature $(n+1,1)$. Denote by $\mathcal{L}$ the lightcone of $\mathbb{R}^{n+1,1}$.

Let $\mathfrak{g} = \mathfrak{so}(n+1,1)$ which we shall frequently (and silently) identify with $\wedge^2\mathbb{R}^{n+1,1}$ via:

$$(a \wedge b)c = (a,c)b - (b,c)a,$$

for $a, b, c \in \mathbb{R}^{n+1,1}$.

Let $\mathfrak{q} \in \mathbb{R}^{n+1,1}$ be non-zero, define the affine hyperplane

$$E_{\mathfrak{q}} := \{v \in \mathbb{R}^{n+1,1} \mid (v, \mathfrak{q}) = -1\}$$

and then the conic section $M_{\mathfrak{q}} := \mathcal{L} \cap E_{\mathfrak{q}}$. Equip $M_{\mathfrak{q}}$ with the (positive definite) metric induced from $\mathbb{R}^{n+1,1}$. Then $M_{\mathfrak{q}}$ is an $n$-dimensional space form[2] of curvature $-(\mathfrak{q}, \mathfrak{q})$ (see, for example, [18, §1.1.4]).

For $p \in M_{\mathfrak{q}}$, the normal bundle to $M_{\mathfrak{q}}$ in $\mathbb{R}^{n+1,1}$ is spanned by $p, \mathfrak{q}$ so that $T_pM_{\mathfrak{q}} = \operatorname{span}\{p, \mathfrak{q}\}^{\perp} \leq \mathbb{R}^{n+1,1}$. We realise $\mathfrak{g}$ as the conformal vector fields $\mathfrak{c}_{M_{\mathfrak{q}}}$ of $M_{\mathfrak{q}}$ by

$$Y(p) = -Yp - (Yp, \mathfrak{q})p$$

where the minus sign ensures that $Y \mapsto (p \mapsto Y(p))\colon \mathfrak{g} \cong \mathfrak{c}_{M_{\mathfrak{q}}}$ is an isomorphism of Lie algebras. Note that this isomorphism identifies $\mathfrak{g}_{M_{\mathfrak{q}}}$ with $\operatorname{stab}(\mathfrak{q}) = \wedge^2\mathfrak{q}^{\perp} \leq \mathfrak{g}$.

We equip $\mathfrak{g}$ with the invariant symmetric bilinear form[3] $(\ ,\ )_{\mathfrak{g}}$ given by

$$(Y_1, Y_2)_{\mathfrak{g}} = \tfrac{1}{2} \operatorname{trace} Y_1 Y_2$$

[2] Strictly, when $(\mathfrak{q}, \mathfrak{q}) > 0$, $M_q$ has two components (it is a hyperboloid of two sheets) each of which is a space form.

[3] This is a non-negative multiple of the Killing form of $\mathfrak{g}$.

and remark that a short calculation gives

$$(Y(p), v) = (Y, p \wedge v)_{\mathfrak{g}}, \tag{4.1}$$

for all $v \in T_p M_{\mathfrak{q}}$.

With this in hand, let $(x, q)\colon \Sigma \to M_{\mathfrak{q}}$ be an isothermic surface and use I to define $Q^{\#} \in \Gamma \operatorname{Sym}_0 T\Sigma$ by

$$Q(U, V) = I(Q^{\#}U, V),$$

where, as usual, $Q = 2\operatorname{Re} q$. Then, for $Y \in \mathfrak{c}_{M_{\mathfrak{q}}}$,

$$\eta_Y(U) = Q(Y^{\top}, U) = \mathrm{I}(Y^{\top}, Q^{\#}U) = (Y \circ x, \mathrm{d}x \circ Q^{\#}U).$$

From (4.1), this gives:

$$\eta^q_x(Y) = (Y, x \wedge \mathrm{d}x \circ Q^{\#})_{\mathfrak{g}}$$

so that, identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ via $(\,,\,)_{\mathfrak{g}}$, we conclude:

$$\eta^q_x = x \wedge \mathrm{d}x \circ Q^{\#} \in \Omega^1_{\Sigma} \otimes \mathfrak{g}, \tag{4.2}$$

(c.f. [10, Equation (3)]).

Finally, for future use, we define a family of $\mathfrak{g}_{M_{\mathfrak{q}}}$-valued 1-forms on $M_{\mathfrak{q}}$ as follows: fix $\mathfrak{o} \in E_{\mathfrak{q}}$, let $i\colon M_{\mathfrak{q}} \to \mathbb{R}^{n+1,1}$ be the inclusion and set $\alpha^{\mathfrak{o}} = (i - \mathfrak{o}) \wedge \mathrm{d}i$. Note that $i - \mathfrak{o}$ and so $\mathrm{d}i$ take values in $\mathfrak{q}^{\perp}$ so that $\alpha^{\mathfrak{o}}$ takes values in $\wedge^2\mathfrak{q}^{\perp} = \mathfrak{g}_{M_{\mathfrak{q}}}$ as required. Explicitly,

$$\alpha^{\mathfrak{o}}_p(v) := (p - \mathfrak{o}) \wedge v, \tag{4.3}$$

for $v \in T_p M_{\mathfrak{q}}$. We have

**Lemma 4.1.** *With $\alpha^{\mathfrak{o}}$ defined as above:*

1. $\mathrm{d}\alpha^{\mathfrak{o}}(V, W) = 2V \wedge W$*, for* $V, W \in \Gamma T\Sigma$.
2. *Modulo exact forms, $\alpha^{\mathfrak{o}}$ is independent of $\mathfrak{o} \in E_{\mathfrak{q}}$.*

*Proof.* By the Leibniz rule,

$$\mathrm{d}\alpha^{\mathfrak{o}} = \mathrm{d}i \curlywedge \mathrm{d}i$$

where $\curlywedge$ is wedge product of $\mathbb{R}^{n+1,1}$-valued 1-forms using the exterior product of $\mathbb{R}^{n+1,1}$ to multiply coefficients to yield a $\wedge^2\mathbb{R}^{n+1,1}$-valued 2-form. Since we are already identifying $TM_{\mathfrak{q}}$ with $\mathrm{d}i(TM_{\mathfrak{q}})$, this reads

$$\mathrm{d}\alpha^{\mathfrak{o}}(V, W) = \mathrm{d}i(V) \wedge \mathrm{d}i(W) - \mathrm{d}i(W) \wedge \mathrm{d}i(V) = 2V \wedge W,$$

which settles Item (1).

For Item (2), note that for $v \in \mathfrak{q}^{\perp}$,

$$\alpha^{\mathfrak{o}+v} = \alpha^{\mathfrak{o}} + \mathrm{d}((i - \mathfrak{o}) \wedge v).$$

□

4.2. **3-dimensional conic sections.** Let us focus our attention on the case $n = 3$ of principal interest to us. Here there is a new ingredient: $\dim \mathfrak{q}^{\perp} = 4$ so that wedge product $\wedge^2\mathfrak{q}^{\perp} \times \wedge^2\mathfrak{q}^{\perp} \to \wedge^4\mathfrak{q}^{\perp} \cong \mathbb{R}$ is a symmetric bilinear form of signature $(3, 3)$ and so induces an isomorphism $S\colon \mathfrak{g}_{M_{\mathfrak{q}}} \to \mathfrak{g}^*_{M_{\mathfrak{q}}}$.

In more detail, fix a volume form $\det \in (\wedge^5\mathbb{R}^{4,1})^*$ and make the following:

**Definition.** Let $S\colon \wedge^2\mathfrak{q}^{\perp} \to (\wedge^2\mathfrak{q}^{\perp})^*$ be given by

$$S(Y_1)(Y_2) = \det(\mathfrak{o} \wedge Y_1 \wedge Y_2), \tag{4.4}$$

for $Y_1, Y_2 \in \wedge^2\mathfrak{q}^{\perp}$. Here $\mathfrak{o}$ is any element of $E_{\mathfrak{q}}$ and $S$ is independent of that choice.

*Remarks.*

1. Note that $S$ is equivariant for the action of $\mathrm{SL}(\mathfrak{q}^\perp)$ and so, in particular, viewed as an isomorphism $\mathfrak{g}_{M_\mathfrak{q}} \cong \mathfrak{g}^*_{M_\mathfrak{q}}$, *$S$ intertwines the adjoint and co-adjoint actions of $G_{M_\mathfrak{q}}$.*
2. When $K = -(\mathfrak{q}, \mathfrak{q}) \neq 0$, $\mathfrak{q}^\perp$ is isomorphic to $\mathbb{R}^4$ or $\mathbb{R}^{3,1}$ according to the sign of $K$. Now $\mathfrak{g}_{M_\mathfrak{q}}$ is semisimple and we can equivariantly identify $\mathfrak{g}_{M_\mathfrak{q}}$ with its dual via (minus) the Killing form. In this setting, we see from (4.4) that $S\colon \mathfrak{g}_{M_\mathfrak{q}} \to \mathfrak{g}_{M_\mathfrak{q}}$ is essentially the Hodge star operator.
3. When $K = 0$, $\mathfrak{g}_{M_\mathfrak{q}}$ is the Euclidean Lie algebra and far from semisimple: it is the semi-direct product $\mathfrak{e}(3) = \mathfrak{so}(3) \oplus \mathbb{R}^3$. In general, the adjoint and co-adjoint representations of $\mathfrak{e}(n)$ are very different and the existence of $S$ when $n = 3$ came as a surprise to us.

For $p \in M_\mathfrak{q}$, orient $N_p M_\mathfrak{q}$ so that $p \wedge \mathfrak{q} > 0$ whence $\mathrm{vol}_{M_\mathfrak{q}|p} = i_{p\wedge\mathfrak{q}} \det$. We now have:

**Lemma 4.2.** *Let $v, w \in T_p M_\mathfrak{q}$ and $Y \in \mathfrak{g}_{M_\mathfrak{q}}$. Then*

$$S(v \wedge w)(Y) = i_{Y(p)} \mathrm{vol}_{M_\mathfrak{q}}(v, w). \tag{4.5}$$

*Proof.* Unravelling the definitions, we see that (4.5) amounts to

$$\mathfrak{o} \wedge v \wedge w \wedge Y = -p \wedge \mathfrak{q} \wedge Yp \wedge v \wedge w. \tag{4.6}$$

It suffices to prove this for decomposable $Y \in \bigwedge^2 \mathfrak{q}^\perp$ so let $Y = a \wedge b$, for $a, b \in \mathfrak{q}^\perp$. Then

$$a = a^\top - (a, p)(\mathfrak{q} + (\mathfrak{q}, \mathfrak{q})p),$$

with $a^\top \in T_p M_\mathfrak{q}$, and similarly for $b$ so that

$$\begin{aligned}
\mathfrak{o} \wedge v \wedge w \wedge Y = p \wedge v \wedge w \wedge Y &= p \wedge v \wedge w \wedge a \wedge b \\
&= -p \wedge v \wedge w \wedge a \wedge (b, p)\mathfrak{q} - p \wedge v \wedge w \wedge (a, p)\mathfrak{q} \wedge b \\
&= -p \wedge \mathfrak{q} \wedge v \wedge w \wedge \big((a, p)b - (b, p)a\big) \\
&= -p \wedge \mathfrak{q} \wedge Yp \wedge v \wedge w,
\end{aligned}$$

as required. Here we used $\bigwedge^5 \mathfrak{q}^\perp = 0$ in the first line to replace $\mathfrak{o}$ with $p$ and $\bigwedge^4 T_p M_\mathfrak{q} = 0$ in the second line to replace $a$ or $b$ with its component along $\mathfrak{q}$. □

As a corollary, we find an explicit representative for $\alpha_Y$:

**Corollary 4.3.** *For $Y \in \mathfrak{g}_{M_\mathfrak{q}}$, we have*

$$S(\alpha^\mathfrak{o})(Y) \equiv 2\alpha_Y \mod \mathrm{d}\Omega^0_{M_\mathfrak{q}}. \tag{4.7}$$

*Proof.* We must show that

$$\mathrm{d}\big(S(\alpha^\mathfrak{o})(Y)\big) = 2 i_Y \mathrm{vol}_{M_\mathfrak{q}}.$$

However, for $V, W \in \Gamma T M_\mathfrak{q}$,

$$\mathrm{d}\big(S(\alpha^\mathfrak{o})(Y)\big)(V, W) = S(\mathrm{d}\alpha^\mathfrak{o}(V, W))(Y) = 2S(V \wedge W)(Y) = 2 i_Y \mathrm{vol}_{M_\mathfrak{q}}(V, W),$$

where we have used Lemma 4.1 for the second equality and Lemma 4.2 for the last. □

4.3. **CMC surfaces in space forms.** Let $x\colon \Sigma \to M_{\mathfrak{q}}$ be a CMC surface in a 3-dimensional space form. We have seen that $x$ is isothermic with $q = \mathrm{II}^{2,0}$ so that, from (4.2),

$$\eta^q_x = x \wedge \mathrm{d}x \circ A^N_0.$$

Moreover, $\mathrm{d}x \circ A^N_0 = -\,\mathrm{d}(N + Hx) \in \Omega^1_\Sigma \otimes \mathfrak{q}^\perp$ from which we learn two things. First,

$$\eta^q_x \mathfrak{q} = d(N + Hx)$$

is exact so that $[\eta^q_x] \in H^1(\Sigma, \mathbb{R}) \otimes \mathfrak{g}_{M_\mathfrak{q}}$. Second, we have

$$\eta^q_x \equiv \mathrm{d}x \wedge (N + Hx) \mod \mathrm{d}\Omega^0_\Sigma \otimes \mathfrak{g}_{M_\mathfrak{q}}.$$

We are now in a position to prove our main result: $S[\eta^q_x] = \mu_x$. First note that, by Lemma 4.2, for $Y \in \mathfrak{g}_{M_\mathfrak{q}}$,

$$S(\mathrm{d}x \wedge N)(Y) = \mathrm{vol}_{M_\mathfrak{q}}(Y \circ x, \mathrm{d}x, N) = \mathrm{vol}_{M_\mathfrak{q}}(Y^\top, \mathrm{d}x, N) = i_{Y^\top}\mathrm{vol}_x.$$

On the other hand, notice that $x^*\alpha^{\mathfrak{o}} = (x - \mathfrak{o}) \wedge \mathrm{d}x$ so that, working modulo exact forms,

$$S(\mathrm{d}x \wedge x)(Y) \equiv S(\mathrm{d}x \wedge (x - \mathfrak{o}))(Y) = -S(x^*\alpha^{\mathfrak{o}})(Y) \equiv -2x^*\alpha_Y,$$

by Corollary 4.3. Putting this all together, we get

$$S[\eta^q_x](Y) = [i_{Y^\top}\mathrm{vol}_x - 2Hx^*\alpha_Y] = \mu_x(Y).$$

To summarise:

**Theorem 4.4.** *Let $x\colon \Sigma \to M_\mathfrak{q}$ be a CMC surface in a 3-dimensional space form with retraction form $\eta^q_x$. Then*

1. $[\eta^q_x] \in H^1(\Sigma, \mathbb{R}) \otimes \mathfrak{g}_{M_\mathfrak{q}}$.
2. $S[\eta^q_x] = \mu_x$.

4.4. **Application: parallel CMC surfaces.** Let $x\colon \Sigma \to M$ have constant mean curvature $H$ in a 3-dimensional space form $M$ of sectional curvature $K$. Then, when $H^2 + K > 0$, and away from umbilics of $x$, one can find a parallel CMC surface $\check{x}\colon \Sigma \to M$ for which the mean curvature, induced conformal structure and Hopf differential coincide with those of $x$. When $M = \mathbb{R}^3$, this is a classical result of Bonnet (c.f [4, p. 266]) and $\check{x} = x + N/H$. For $K \neq 0$, the parallel surfaces come in congruent pairs and are described in Palais–Terng [26] or [18, §2.7; 19, Part II, §5].

We will give a self-contained account of this result and use it to reveal a symmetry of the moment class: $\mu_x = \mu_{\check{x}}$.

So fix $\mathfrak{q}$ with $(\mathfrak{q}, \mathfrak{q}) = -K$ and let $x\colon \Sigma \to M_\mathfrak{q}$ be an immersion with constant mean curvature $H$. Let $m \in \mathbb{R}$ solve

$$m^2 - 2Hm - K = 0, \text{ equivalently, } (m - H)^2 = H^2 + K. \tag{4.8}$$

That $m$ be real is equivalent to $H^2 + K \geq 0$ and we assume henceforth that $m \neq 0, H$, so that, in particular, $H^2 + K > 0$.

We now define $\check{x}\colon \Sigma \to \mathbb{R}^{4,1}$ by

$$\check{x} = \tfrac{1}{m-H}(N + Hx + \mathfrak{q}/m) \tag{4.9}$$

and note that, as a consequence of (4.8), we have

$$(\check{x}, \check{x}) = 0, \qquad (\check{x}, \mathfrak{q}) = -1$$

so that $\check{x}\colon \Sigma \to M_\mathfrak{q}$. We have:

**Proposition 4.5.** *$\check{x}\colon \Sigma \to M_{\mathfrak{q}}$ is parallel to $x$ and immerses away from the umbilics of $x$. Where $\check{x}$ immerses:*

1. *the metric $\check{\mathrm{I}}$ induced by $\check{x}$ is conformal to $\mathrm{I}$;*
2. *$\check{x}$ has (oriented) normal $\check{N}$ given by*
$$\check{N} = \tfrac{1}{m-H}(Kx - HN - \mathfrak{q}); \tag{4.10}$$
3. *$\check{x}$ has constant mean curvature $H$;*
4. *the Hopf differentials of $x$ and $\check{x}$ coincide: $q = \check{q}$.*

Before proving this, we pause to extract an interesting result. For any CMC surface $x$, we have already seen that
$$\eta^q_x = \mathrm{d}(N + Hx) \wedge x.$$
In the case at hand, (4.9) gives
$$\eta^q_x = (m - H)\,\mathrm{d}\check{x} \wedge x.$$
On the other hand, (4.9) and (4.10), along with (4.8), yield
$$\check{N} + H\check{x} = \tfrac{1}{m-H}((H^2 + K)x + (1/m - 1)\mathfrak{q}) = (m - H)x + \tfrac{1/m-1}{m-H}\mathfrak{q}.$$
Thus matters are completely symmetric in $x$ and $\check{x}$:
$$\eta^{\check{q}}_{\check{x}} = (m - H)\,\mathrm{d}x \wedge \check{x}$$
so that we see
$$\eta^q_x = \eta^{\check{q}}_{\check{x}} + (m - H)\,\mathrm{d}(\check{x} \wedge x).$$
The two retraction forms are therefore cohomologous and so, in view of Theorem 4.4, we conclude:

**Theorem 4.6.** *The moment classes of a CMC surface $x$ and its parallel surface $\check{x}$ coincide: $\mu_x = \mu_{\check{x}}$.*

*Remarks* (for the experts).

1. CMC surfaces in space forms and their parallel surfaces are a particular case of a more general theory of *special isothermic surfaces* developed by Bianchi [1] and Darboux [14]. In this setting, the parallel CMC surfaces arise as *complementary isothermic surfaces*. See [10] for a modern perspective on this.
2. In view of the above remark, that the two retraction forms are cohomologous can be also understood from a more general perspective: in fact, a complementary surface is a special kind of Darboux transform and one can show [8] that *any* Darboux pair $x, \check{x}\colon \Sigma \to S^n$ have cohomologous retraction forms.

Now let us take care of unfinished business:

*Proof of Proposition 4.5.* That $\check{x}$ is a parallel surface to $x$ in $M_{\mathfrak{q}}$ follows from the fact that it is a constant coefficient linear combination of $x$, $N$ and $\mathfrak{q}$.

Next, differentiate (4.9) to see that
$$\mathrm{d}\check{x} = -\tfrac{1}{m-H}\,\mathrm{d}x \circ A^N_0 \tag{4.11}$$
and recall that $A^N_0$ is conformal ($(A^N_0)^2 = -\det A^N_0$) and so bijects except at its zeros which are the umbilic points of $x$. Thus $\check{x}$ immerses off the umbilic locus of $x$ and, moreover, $\check{\mathrm{I}} = -\det A^N_0 \mathrm{I}$ there, settling Item 1. For the rest of the proof, we restrict attention to the complement of the umbilic locus of $x$ so that $\check{x}$ immerses.

For Item 2 , one readily checks, using (4.8) that $\check{N}$ defined by (4.10) has unit length and is orthogonal to $\check{x}, \mathrm{d}\check{x}(T\Sigma) = \mathrm{d}x(T\Sigma), \mathfrak{q}$. Moreover, (4.8) and (4.11) yield

$$\check{x}^*(i_{\check{N}}\mathrm{vol}_{M\mathfrak{q}}) = -\tfrac{\det A_0^N}{(m-H)^2}\mathrm{vol}_x$$

which is a positive multiple of $\mathrm{vol}_x$ so that $\check{N}$ is the correctly oriented unit normal to $\check{x}$.

For the rest, we compute the shape operator $\check{A}^{\check{N}}$ of $\check{x}$:

$$\begin{aligned}\mathrm{d}\check{N} &= \tfrac{1}{m-H}\,\mathrm{d}(Kx - HN) = \tfrac{1}{m-H}\,\mathrm{d}x(K + HA^N)\\ &= \tfrac{1}{m-H}\,\mathrm{d}x(H^2 + K + HA_0^N)\\ &= -\,\mathrm{d}\check{x}\big((A_0^N)^{-1}(H^2 + K + HA_0^N)\big) = -\,\mathrm{d}\check{x}\big(H + (H^2+K)(A_0^N)^{-1}\big),\end{aligned}$$

where we used (4.11) to reach the last line. Since $(A_0^N)^{-1}$ is trace-free, we immediately conclude

$$\check{H} = H, \qquad \check{A}_0^{\check{N}} = (H^2+K)(A_0^N)^{-1},$$

settling Item 3. Further, from (4.8) and (4.11), we now get

$$\check{\mathrm{II}}_0 = \check{\mathrm{I}}(\check{A}_0^{\check{N}}, -) = \tfrac{H^2+K}{(m-H)^2}\mathrm{I}(-, A_0^N) = \mathrm{II}_0$$

and so we are done. □

*Remark.* When $K \neq 0$, there are two non-zero solutions $m_\pm$ to (4.8) with $m_+ - H = H - m_-$. It is an easy exercise to see that the corresponding parallel surfaces $\check{x}_\pm$ are related by minus the reflection in the hyperplane orthogonal to $\mathfrak{q}$. Geometrically, when $K > 0$, this means that $\check{x}_\pm$ are antipodal while, when $K < 0$, $\check{x}_\pm$ lie in different sheets of the double-sheeted hyperboloid $M_\mathfrak{q}$. In the latter case, choose $m$ with $|m|$ maximal to get $\check{x}$ in the same sheet as $x$.

4.5. **Flux, torque and generalisations.** We conclude this section by using our results to write down explicit representatives of the moment class in classical terms. When $K = 0$, this will recover the flux form of Meeks et al. [25] and provide an analogous formula for the torque form. Moreover, we offer, apparently novel, generalisations of these to the case $K \neq 0$.

First we take $K = 0$ and take $\mathfrak{o} \in M_\mathfrak{q}$. We identify $x_0 \colon \Sigma \to \mathbb{R}^3 = \mathrm{span}\{\mathfrak{o}, \mathfrak{q}\}^\perp$ with $x \colon \Sigma \to M_\mathfrak{q}$ via

$$x = \mathfrak{o} + x_0 + \tfrac{1}{2}(x_0, x_0)\mathfrak{q}$$

and unit normals to these by

$$N = N_0 + (x_0, N_0)\mathfrak{q}.$$

When $x$ has constant mean curvature $H$, we have

$$\eta_x^q \equiv \mathrm{d}x_0 \wedge (N_0 + Hx_0) + \big((N_0 \wedge \mathrm{d}x_0)x_0 + H(x_0, x_0)\,\mathrm{d}x_0\big) \wedge \mathfrak{q}, \tag{4.12}$$

modulo exact terms. Now, for $v \in \mathbb{R}^3$, let $Y \in \mathfrak{e}(3)$ be given by $Y = \mathfrak{q} \wedge v$ so that $Y$ is the constant vector field $Y(p) = v$ on $\mathbb{R}^3$. Then (4.12) gives

$$S(\eta_x^q)(Y) \equiv \det(\mathfrak{o} \wedge \mathfrak{q} \wedge \eta_x^q \wedge v) = \mathrm{vol}_{\mathbb{R}^3}(\mathrm{d}x_0, N_0 + Hx_0, v) = (\mathrm{d}x_0 \times (N_0 + Hx_0), v).$$

Thus, identifying the constant vector fields in $\mathfrak{e}(3)$ with $\mathbb{R}^3$ via evaluation and using the metric to identify $\mathbb{R}^3$ with its dual, we conclude

**Proposition 4.7.** *Let $x_0 \colon \Sigma \to \mathbb{R}^3$ have constant mean curvature $H$. The restriction of the moment class of $x_0$ to the constant vector fields is represented by the* flux form $\mathrm{d}x_0 \times (N_0 + Hx_0) \in \Omega^1_\Sigma \otimes \mathbb{R}^3$*:*

$$\mu_{x_0|\mathbb{R}^3} = [\mathrm{d}x_0 \times (N_0 + Hx_0)].$$

Similarly, the infinitesimal rotations around zero are $\mathfrak{so}(3) = \wedge^2\mathbb{R}^3 \leq \wedge^2\mathfrak{q}^\perp = \mathfrak{e}(3)$. For $Y = v \wedge w \in \mathfrak{so}(3)$, we have

$$\begin{aligned} S(\eta^q_x)(Y) &\equiv \det(\mathfrak{o} \wedge \eta^q_x \wedge v \wedge w) = -\mathrm{vol}_{\mathbb{R}^3}\big(\big((N_0 \wedge \mathrm{d}x_0)x_0 + H(x_0, x_0)\,\mathrm{d}x_0\big), v, w\big) \\ &= -\big(\big((N_0 \wedge \mathrm{d}x_0)x_0 + H(x_0, x_0)\,\mathrm{d}x_0\big), v \times w\big). \end{aligned}$$

Thus, identifying $\mathfrak{so}(3)^* \cong \mathbb{R}^3$ via $\langle u, v \wedge w\rangle = (u, v \times w)$, we have

**Proposition 4.8.** *Let $x_0 \colon \Sigma \to \mathbb{R}^3$ have constant mean curvature $H$. The restriction of the moment class to $\mathfrak{so}(3)$ is represented by the* torque form $-(N_0 \wedge \mathrm{d}x_0)x_0 - H(x_0, x_0)\,\mathrm{d}x_0 \in \Omega^1_\Sigma \otimes \mathbb{R}^3$*:*

$$\mu_{x_0\,|\,\mathfrak{so}(3)} = [-(N_0 \wedge \mathrm{d}x_0)x_0 - H(x_0, x_0)\,\mathrm{d}x_0].$$

When $K \neq 0$, we get more uniform statements. Firstly, $\mathfrak{g}_{M_\mathfrak{q}} = \wedge^2\mathfrak{q}^\perp$ is identified with its dual using the metric[4] on $\mathfrak{q}^\perp$ induced from that of $\mathfrak{q}^\perp$. Secondly, take $\mathfrak{o} = \mathfrak{q}/K$ and, for any $x \colon \Sigma \to M_\mathfrak{q}$, write

$$x = x_0 + \mathfrak{o}$$

to get $x_0 \colon \Sigma \to \mathfrak{q}^\perp$ taking values in a (pseudo-)sphere: $(x_0, x_0) = 1/K$ with normal $N_0 = N$ (since $N$ is already orthogonal to $\mathfrak{o}$ in this case). Then, when $x_0$ has constant mean curvature $H$, starting from

$$\eta^q_x = (m - H)\,\mathrm{d}x_0 \wedge \check{x}_0,$$

we conclude after a computation:

**Proposition 4.9.** *Let $x_0$ have constant mean curvature $H$ in a (pseudo-)sphere of curvature $K$ in $\mathbb{R}^{p,q}$, where $(p, q) = (4, 0)$ or $(3, 1)$ according to the sign of $K$. Then the moment class of $x_0$ is represented by the* moment form $(Kx_0 - HN) \wedge *\mathrm{d}x_0 \in \Omega^1_\Sigma \otimes \wedge^2\mathbb{R}^{p,q}$*:*

$$\mu_{x_0} = [(Kx_0 - HN) \wedge *\mathrm{d}x_0],$$

*where $*$ is the Hodge star operator of $\Sigma$.*

## 5. Noether's theorem

This section attempts to partially resolve the somewhat mysterious relationship between the moment class

$$\mu_x \in H^{k-1}(\Sigma, \mathbb{R}) \otimes \mathfrak{g}^*_M$$

for a constant mean curvature hypersurface $x \colon \Sigma^k \to M^{k+1}$ in a Riemannian manifold in the case $k = 2$ and the retraction class

$$[\eta^q_x] \in H^1(\Sigma, \mathbb{R}) \otimes \mathfrak{c}^*_M$$

for a globally isothermic surface $(x, q) \colon \Sigma^2 \to M^n$ into a conformal manifold. We shall show that both classes have a variational origin as Noether currents.

We begin with a brief description of Noether analysis suitable for our discussion based on [16]. Let $\Sigma^k$, $M^n$ be manifolds and $\mathcal{F} \subset C^\infty(\Sigma, M)$ an open subset which, in our examples, will be the set of immersions $\Sigma \to M$.

**Definition.** A *Lagrangian* is a smooth map

$$\mathcal{L} \colon \mathcal{F} \to W,$$

where $W$ is a manifold.

[4] This is the negative of $(\,,\,)_\mathfrak{g}$ discussed above.

In this context one is interested in the *critical locus* of $\mathcal{L}$, that is, maps $x \in \mathcal{F}$ for which $\mathrm{d}_x^{\mathcal{F}}\mathcal{L}$ is not surjective.

In many but not all cases, Lagrangians $\mathcal{L}\colon \mathcal{F} \to \mathbb{R}$ factorise

$$\mathcal{L}(x) = \int_\Sigma \ell_x$$

via a *Lagrangian density* $\ell\colon \mathcal{F} \to \Omega^k_\Sigma$. In this setting, $x \in \mathcal{F}$ is critical if and only if

$$\mathrm{d}_x^{\mathcal{F}}\ell(\dot{x}) = 0,$$

for all tangent vectors (variations) $\dot{x} \in T_x\mathcal{F} = \Gamma\, x^*TM$. Thus, the critical points of $\mathcal{L}$ are determined by the $\Omega^k_\Sigma$-valued 1-form

$$\mathrm{d}^{\mathcal{F}}\ell \in \Omega^1_{\mathcal{F}}(\Omega^k_\Sigma)$$

which motivates

**Definition.** A *variational operator* is an $\Omega^k_\Sigma$-valued 1-form on $\mathcal{F}$, that is,

$$\mathrm{D} \in \Omega^1_{\mathcal{F}}(\Omega^k_\Sigma).$$

Thus, for each $x \in \mathcal{F}$ we have a linear map

$$\mathrm{D}_x\colon \Gamma\, x^*TM \to \Omega^k_\Sigma$$

which we assume to be a differential operator. We call $x \in \mathcal{F}$ a *critical point* of D if

$$\mathrm{D}_x(V) = 0$$

for all $V \in \Gamma\, x^*TM$.

We note from the discussion above that, for factorisable Lagrangians $\mathcal{L}$, the variational operator D is the derivative $\mathrm{d}^{\mathcal{F}}\ell$ of the Lagrangian density $\ell$.

**Lemma 5.1** (c.f. [16, §2.4])**.** *Let* $\mathrm{D} \in \Omega^1_{\mathcal{F}}(\Omega^k_\Sigma)$ *be a variational operator for which* $\mathrm{D}_x$ *is first order, for every* $x \in \mathcal{F}$*. Then there exist unique* 1*-forms*

$$\beta \in \Omega^1_{\mathcal{F}}(\Omega^{k-1}_\Sigma) \quad \text{and} \quad E \in \Omega^1_{\mathcal{F}}(\Omega^k_\Sigma)$$

*such that for* $x \in \mathcal{F}$ *we have*

- $\beta_x\colon \Gamma\, x^*TM \to \Omega^{k-1}_\Sigma$ *and* $E_x\colon \Gamma\, x^*TM \to \Omega^k_\Sigma$ *are* $C^\infty(\Sigma,\mathbb{R})$*-linear and so tensorial.*
- *For all* $V \in \Gamma\, x^*TM$,

$$\mathrm{D}_x(V) = \mathrm{d}(\beta_x(V)) + E_x(V). \tag{5.1}$$

*The* 1*-form* $E \in \Omega^1(\mathcal{F}, \Omega^k_\Sigma)$ *is called the* Euler-Lagrange operator *and the* 1*-form* $\beta \in \Omega^1_{\mathcal{F}}(\Omega^{k-1}_\Sigma)$ *the* variational 1-form*.*

*Proof.* Since $\mathrm{D}_x$ is a first order linear differential operator, we have

$$\mathrm{D}_x(fV) = f\mathrm{D}_xV + S_{\mathrm{d}f}V$$

where the bundle homomorphism $S\colon T^*\Sigma \to \mathrm{Hom}(x^*TM, \wedge^k T^*\Sigma))$ is the symbol of $\mathrm{D}_x$. Contraction then gives a bundle homomorphism $\beta_x\colon x^*TM \to \wedge^{k-1}T^*\Sigma$. For $V \in \Gamma\, x^*TM$ one then easily checks that

$$E_x(V) := \mathrm{D}_xV - \mathrm{d}(\beta_x(V))$$

is $C^\infty(\Sigma,\mathbb{R})$-linear as required. Uniqueness follows since any other decomposition of D by $\tilde\beta$, $\tilde E$ gives $\mathrm{d}f \wedge (\beta_x(V) - \tilde\beta_x(V)) = 0$, for all $f \in C^\infty(\Sigma,\mathbb{R})$ and $V \in \Gamma\, x^*TM$. $\square$

We thus get a version of the Euler-Lagrange condition for critical points in our setup:

**Corollary 5.2.** *$x \in \mathcal{F}$ is a critical point for the variational operator* $\mathrm{D}$ *if and only if*

$$E_x = 0.$$

*Proof.* Let $V \in \Gamma\, x^*TM$ be compactly supported. Then (5.1) and Stokes' theorem give:

$$\int_\Sigma \mathrm{D}_x(V) = \int_\Sigma \mathrm{d}(\beta_x(V)) + \int_\Sigma E_x(V) = \int_\Sigma E_x(V).$$

□

Consider a Lagrangian $\mathcal{L}$ which factorises via a Lagrangian density $\ell$ and assume we have a Lie group $G$ acting on $\mathcal{F}$ leaving $\ell$ (and thus also $\mathcal{L}$) invariant, that is, $\ell_{gx} = \ell_x$ for all $g \in G$. Differentiating with respect to $g$ we obtain $\mathrm{d}^{\mathcal{F}}_x \ell(\mathcal{V}) = 0$ for the fundamental vector fields $\mathcal{V} \in \Gamma\, T\mathcal{F}$ of the group action.

**Definition.** Let $\mathrm{D} \in \Omega^1_{\mathcal{F}}(\Omega^k_\Sigma)$ be a variational operator. A vector field $\mathcal{V} \in \Gamma\, T\mathcal{F}$ is a *symmetry* of $\mathrm{D}$ if

$$\mathrm{D}(\mathcal{V}) = 0.$$

**Theorem 5.3** (Noether)**.** *If $\mathcal{V} \in \Gamma\, T\mathcal{F}$ is a symmetry of the variational operator* $\mathrm{D}$ *and $x \in \mathcal{F}$ a critical point of* $\mathrm{D}$*, then $\beta_x(\mathcal{V}_x) \in \Omega^{k-1}_\Sigma$ is closed.*

*We call $\beta_x(\mathcal{V}_x)$ the* Noether current *of* $\mathrm{D}$ *for the symmetry $\mathcal{V}$.*

*Proof.* Since $x$ is critical and $\mathcal{V}$ is a symmetry, both $E_x(\mathcal{V}_x)$ and $\mathrm{D}_x(\mathcal{V}_x)$ vanish and so the result is immediate from (5.1). □

In the remainder of this section we apply our Noether analysis to CMC hypersurfaces and globally isothermic surfaces to show that the moment and retraction classes arise as Noether currents.

5.1. **Constant mean curvature hypersurfaces.** Let $\mathcal{F}$ be the space of immersions $x\colon \Sigma^k \to M^n$ into a Riemannian manifold. We additionally assume

- $\Sigma = \partial\hat{\Sigma}$ is the boundary of a manifold $\hat{\Sigma}^{k+1}$;
- $x$ extends to an immersion $\hat{x}\colon \hat{\Sigma} \to M$ so that $\hat{x}_{|\Sigma} = x$.

Consider the Lagrangian

$$\mathcal{L}(x) = \int_\Sigma \mathrm{vol}_x + \lambda \int_{\hat{\Sigma}} \mathrm{vol}_{\hat{x}}$$

with a Lagrange multiplier $\lambda \in \mathbb{R}$. Critical points of $\mathcal{L}$ extremise the induced volume on $\Sigma$ constrained by the induced enclosed volume on $\hat{\Sigma}$. For a variation $V = \mathrm{d}x(V^\top) + V^\perp \in \Gamma\, x^*TM$, we obtain

$$\mathrm{d}^{\mathcal{F}}_x \mathrm{vol}(V) = \mathrm{d}(i_{V^\top} \mathrm{vol}_x) - k(\mathbf{H}, V^\perp)\mathrm{vol}_x$$

with $\mathbf{H} \in \Gamma\, N\Sigma$ the mean curvature vector field.

Now suppose that $n = k+1$, that is, $x$ is a hypersurface immersion, and that $\hat{V} \in \Gamma\, \hat{x}^*TM$ extends $V$. Then

$$\mathrm{d}^{\mathcal{F}}_{\hat{x}} \mathrm{vol}(\hat{V}) = \mathrm{d}(i_{\hat{V}} \mathrm{vol}_{\hat{x}})$$

since there is no normal component. We therefore arrive at

$$\mathrm{d}^{\mathcal{F}}_x \mathcal{L}(V) = \int_\Sigma \mathrm{d}(i_{V^\top}\mathrm{vol}_x) - k(\mathbf{H}, V^\perp)\mathrm{vol}_x + \lambda(V^\perp, N)\mathrm{vol}_x$$

where we used $i_{\hat{V}}\mathrm{vol}_{\hat{x}|\Sigma} = (V^\perp, N)\mathrm{vol}_x$ for the unit normal field $N \in \Gamma N\Sigma$ of $x\colon \Sigma \to M$.

We can now read off the ingredients of Lemma 5.1: the variational operator $\mathrm{D} \in \Omega^1_{\mathcal{F}}(\Omega^k_\Sigma)$ for $\mathcal{L}$ reads

$$\mathrm{D}_x(V) = \mathrm{d}(i_{V^\top}\mathrm{vol}_x) - k(\mathbf{H}, V^\perp)\mathrm{vol}_x + \lambda(V^\perp, N)\mathrm{vol}_x, \tag{5.2}$$

for $V \in T\mathcal{F}_x = \Gamma\, x^*TM$, while

$$\beta_x(V) = i_{V^\top}\mathrm{vol}_x$$
$$E_x(V) = (-k(\mathbf{H}, V^\perp) + \lambda(V^\perp, N))\mathrm{vol}_x.$$

In particular, $x \in \mathcal{F}$ is a critical point of $\mathcal{L}$ if and only if

$$k\mathbf{H} = \lambda N \quad \text{equivalently} \quad \lambda = kH,$$

that is, $x$ has constant mean curvature $H$.

Now let $V = Y \circ x$ for a Killing field $Y$ on $M$. Then $V$ is not, in general, a symmetry of $\mathcal{L}$ since we have formulated the volume as a boundary integral. However, $V$ *is* a symmetry for the area functional so that, for any immersion $x$ we have the identity (2.1)

$$\mathrm{d}(i_{Y^\top}\mathrm{vol}_x) = k(Y \circ x, \mathbf{H})\mathrm{vol}_x$$

of Lemma 2.1, which, as we have seen, yields the moment class $\mu_x$ under appropriate conditions on the cohomology of $M$.

5.2. **Isothermic surfaces.** Let $\Sigma^2$ be an oriented compact surface without boundary. We collect some basic facts about Teichmüller space following Bohle–Peters–Pinkall [3]. Consider

$$\mathcal{T} = \{J \in \Gamma\ \mathrm{End}(T\Sigma) \mid J^2 = -1\}/\,\mathrm{Diff}_0(\Sigma),$$

the space of equivalence classes of complex structures $J$ on $\Sigma$ modulo diffeomorphisms isotopic to the identity. Then:

- The tangent space at $[J]$ to $\mathcal{T}$ is
$$T_{[J]}\mathcal{T} = \{T \in \Gamma\ \mathrm{End}(T\Sigma) \mid TJ = -JT\}/\{L_X J \mid X \in \Gamma\, T\Sigma\}.$$
- The cotangent space at $[J]$ to $\mathcal{T}$ is
$$T^*_{[J]}\mathcal{T} = \{\sigma \in \Gamma\ \mathrm{Hom}(T\Sigma, T^*\Sigma) \mid J^t\sigma = \sigma J,\ \mathrm{d}^\nabla\sigma = 0\}$$
where $\nabla$ is any torsion-free connection with $\nabla J = 0$. As we remarked above, the map
$$H^0(K^2_\Sigma) \to T^*_{[J]}\mathcal{T}\colon q \mapsto \sigma = 2\,\mathrm{Re}\, q$$
is a linear isomorphism once we interpret $\sigma \in T^*_{[J]}\mathcal{T}$ as a trace-free symmetric bilinear form $\sigma \in \Gamma\, S^2_0 T^*\Sigma$.
- The $L^2$-pairing
$$T^*_{[J]}\mathcal{T} \times T_{[J]}\mathcal{T} \to \mathbb{R}\colon (\sigma, [T]) \mapsto \int_\Sigma \langle \sigma \wedge T\rangle$$
is well-defined and non-degenerate. We view $\sigma$ and $T$ as $T^*\Sigma$ respectively $T\Sigma$ valued 1-forms and the wedge product is over the evaluation pairing $\langle\ ,\ \rangle\colon T^*\Sigma \times T\Sigma \to \mathbb{R}$.

Now let $\mathcal{F}$ be the space of immersions $x\colon \Sigma^2 \to M^n$ of a compact oriented surface into a *conformal* manifold. The conformal structure on $\Sigma$ induced by $x$ is equivalent to a complex structure $J_x$ given by anticlockwise rotation by $\pi/2$. Consider the Lagrangian

$$\mathcal{L}\colon \mathcal{F} \to \mathcal{T}, \quad \mathcal{L}(x) = [J_x].$$

It is shown in [3] that the critical points of $\mathcal{L}$, that is, the points $x$ where $\mathrm{d}^{\mathcal{F}}_x \mathcal{L}$ fails to surject, are precisely the globally isothermic immersion $(x, q)\colon \Sigma \to M$. In more detail, from [3], for $V \in \Gamma\, x^*TM$,

$$\mathrm{d}^{\mathcal{F}}_x \mathcal{L}(V) = [L_{V^\top} J_x + 2A_0^{V^\perp} J_x].$$

Thus, $x \in \mathcal{F}$ is a critical point of $\mathcal{L}$ if and only if there exists a non-zero $\sigma \in T_{[J_x]}\mathcal{T}^*$ with

$$\int_\Sigma \langle \sigma \wedge L_{V^\top} J_x + 2A_0^{V^\perp} J_x \rangle = 0$$

for all $V \in \Gamma\, x^*TN$. Hence the critical points of our Lagrangian $\mathcal{L}$ are determined via the variational operator $\mathrm{D}_x\colon \Gamma\, x^*TM \to \Omega^2_\Sigma$ given by

$$\mathrm{D}_x(V) = \langle \sigma \wedge L_{V^\top} J_x + 2A_0^{V^\perp} J_x \rangle.$$

Now $\langle \sigma \wedge L_{V^\top} J_x \rangle = 2\langle \sigma \wedge \nabla(J_x V^\top) \rangle$ so the Leibniz rule and $\mathrm{d}^\nabla \sigma = 0$ give

$$\mathrm{D}_x(V) = -2\, \mathrm{d}\langle \sigma, J_x V^\top \rangle + 2\langle \sigma \wedge A_0^{V^\perp} J_x \rangle.$$

Let us rewrite this in more familiar terms. Set $Q = J_x^t \sigma$ and recall that $A_0$ anticommutes with $J_x$ to conclude that

$$\mathrm{D}_x(V) = -2\, \mathrm{d}\langle Q, V^\top \rangle - 2\langle Q \wedge A_0^{V^\top} \rangle.$$

From this we read off the variational 1-form $\beta_x(V) = -2\langle Q, V^\top \rangle$ and the Euler–Lagrange operator $E_x(V) = -2\langle Q \wedge A_0^{V^\perp} \rangle$. Thus we learn:

1. $x \in \mathcal{F}$ is a critical point of $\mathcal{L}$ if and only if $\langle Q \wedge A_0^{V^\perp} \rangle = 0$ for all $V \in \Gamma\, x^*TM$. With $q = Q^{2,0}$, this is precisely the condition (3.1) that $(x, q)$ be globally isothermic.
2. If $x$ is a critical point and $V = Y \circ x$ for a conformal vector field $Y \in \mathfrak{c}_M$ (which is clearly a symmetry for $\mathrm{D}_x$) then $\beta_x(Y \circ x) = -2\langle Q, Y^\top \rangle = -2\eta_Y$ is essentially the retraction form and closed:
$$\mathrm{d}\eta^q_x = 0.$$

This gives another proof of Proposition 3.1 and shows that, up to a factor of $-\frac{1}{2}$, $\eta^q_x$ is the Noether current of our Lagrangian.

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