Contest with a Jury

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Contests with a Jury*

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Abstract

We study contests with a jury, i.e. a set of agents each of whom selects a player and the winner of the contest is the player selected by most jurors. We find that the presence of a jury creates a novel pivotality effect: marginal effort is only positive in scenarios where the jury is split among candidates, i.e. there is a pivotal juror, and since increasing jury size decreases the probability that there is a pivotal juror, increasing jury size decreases effort. However, there is also an uncertainty effect: increasing jury size increases the probability that the player who exerted the most effort wins, which leads to players exerting more effort. We explore optimal jury size and how a jury affects contests. We find, among others, that increasing the jury size may reduce discrimination in contests and that the presence of a jury fosters crowding out of players.

JEL Classification: C72, D72, D74, D82.

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1 Introduction

In competitive environments, whether in political contests, sports events, funding competitions, or procurement contracts, the structure of the process plays a crucial role in determining the effort exerted by participants. Central to this structure is the process by which a winner is chosen, typically through a panel of jurors such as voters, academic referees, or government officials. Previous research on contests has primarily focused on single-juror decision making through a contest success function (see the seminal work of Tullock (1980) and more recently

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Corchón and Serena (2018) and references therein). This paper aims to address this gap by examining how the presence of a jury affects participants' behaviour in terms of effort provision and the overall characteristics of contest outcomes.

A jury is a set of agents that, based on players' effort levels, each select a player. The winner of the contest is then the player that was selected by most jurors. There are several features that appear when the winner of a contest is decided by jury instead of a single juror or contest success function. Most notably is the fact that a jury gives rise to a novel **pivotality** effect that cannot be replicated by just a contest success function. To understand this effect, consider a contest between two players. Competition happens via effort choice and once both players have chosen their effort levels, a jury of three jurors chooses the winner, each juror represented by a contest success function, by simple majority. When choosing optimal effort levels, each player considers what is the marginal gain of increasing (or decreasing) effort. Consider the four possible states of the world depending on the various vote splits that may happen: (v_1, v_2) with $v_1 + v_2 = 3$ and $v_1, v_2 \geq 0$ where v_i represents the votes received by player $i \in \{1, 2\}$.

Conditional on a state such that $v_1 \geq 2$, player 1 does not increase its profit by increasing its effort marginally. If $v_1 = 0$, then marginal increases in effort have negligible effects in the probability of winning as the marginal increase in effort has to "convince" two jurors to change their vote. It is thus only in the state such that $v_1 = 1$ where a marginal increase in effort is useful for player 1. The same logic applies from player's 2 point of view. The result of this is that marginal effort is only useful in situations where one juror is voting for player 1, one juror is voting for player 2, and the third juror "'swings", i.e. there is a pivotal juror.

In the traditional contest with only one juror there is no pivotality effect as the juror is always pivotal. Moreover, as the number of jurors increases, the probability that there is a pivotal juror decreases, and thus the incentives for players to exert effort changes. That is, jury size and composition, have a crucial effect on equilibrium efforts.

The presence of a jury induces another effect, which we call the **uncertainty** effect. Increasing the jury size has the result that, cateris paribus, it is more likely that the player who exerted the most effort wins. This is the Condorcet Jury Theory at play. According to this theorem, given a pool of voters choosing between two options, one of which gives a better payoff but there is uncertainty about which of the two options is the better one, where each voter is more likely to vote to the correct than to the other option, the higher the number of voters the more likely it is that simple majority selects the right alternative. Increasing the jury size thus reduces uncertainty about the contest, which under most cases increases returns to effort.

In our results, we find that there is always a finite effort maximizing jury size. The effort maximizing jury size balances out the pivotality effect with the uncertainty effect. We find that in most cases the relation between jury size and the sum of efforts of players is concave, meaning that for small juries, increasing jury size increases the sum of efforts while for high jury sizes, increases in the jury size decrease the sum of efforts.

Furthermore, we explore how the presence of a jury affects discrimination. Consider a situation where all jurors discriminate against a player who happens to be better, in that it has a lower cost of effort, than the other player. We find that increasing the jury size, even if the new jurors also discriminate in the same way as existing jurors, can dampen, and even eliminate, discrimination. The mechanism is the following. Increasing the jury size can lead to an increase in equilibrium efforts., and since the player being discriminated against has a lower cost of effort, this player increases its effort more than the other player. The overall effect is then ambiguous, and there are situations where the extra increase in effort not only offsets the fact that there are now more jurors discriminating against this player, it turns around the contest in favour of the player being discriminated against.

On top of this, we also find that the presence of a jury may crowd out players from the contest, even the players that a lower cost of effort. Consider a situation with three players, one of which has a lower cost function than the other two. In this case, there can be equilibria where the two higher cost players exert positive effort and the low cost player does not exert any effort. Because of the pivotality effect, marginal increases in effort starting from zero effort provide no benefit because the chances that the marginal effort leads enough juries to change their vote is negligible. This does not happen in traditional contents. Moreover, even when considering non negligible increases in effort, the cost in effort needed to have a significant probability of winning given the effort level of the two other contestants may be too high, even for the low cost player. Thus, this player finds it optimal to not participate in the contest, i.e. to exert zero effort.

As already mentioned, the presence of a jury creates two effects that go in opposite directions, and while the uncertainty effect can be replicated by a single contest success function, the pivotality effect cannot. The pivotality effect and how it affects contests is our main contribution. Moreover, contests with a jury are not comparable to contests with multiple battlefields (discussed in depth later one), as in those the strategic problem is one of resource allocation: how to distribute effort across the different battlefields.

The rest of the paper is organized as follows. In the remaining of this section we present a literature review. In section 2 we present a simple example that captures the basis of the pivotality effect. In section 3 we present the main model, our results and alternative assumptions. Finally, section 4 concludes.

1.1 Literature

Contest theory can be traced back to the work of Tullock (1980), which introduced the concept of rent-seeking contests and the idea that individuals expend resources to secure a valuable prize. Another early work is by Hirshleifer (1989), who distinguished between different types of success functions and explored their implications for players' effort levels. Following on these papers, Skaperdas (1996) introduced the concept of endogenous prizes and their impact on contestant behaviour. Konrad (2009) explored dynamic contests and the role of time and cumulative efforts in determining outcomes. Other research also examined the impact of asymmetric information, such as Moldovanu and Sela (2001). Clark and Riis (1998) deal with all-pay auctions where there is more than one price to understand how different payment structures affect effort provision.

Simultaneous multibattle contests, i.e. contest that are won if a majority of battlefields (i.e. single contests) is won, were initially studied by Szentes and Rosenthal's (2003a, 2003b). They study situations where players aim to win a majority of objects in simultaneous all-pay auctions. However, their papers consider a limit case of our model, where the probability of winning an auction is deterministic. This equates to a situation where jurors behaviour is deterministic. We instead assume that jurors behaviour is smooth on player's effort levels. Klumpp and Polborn (2006) compare simultaneous multicontest battles to US primaries, but focus on symmetric candidates and symmetric equilibria, whereas we allow players to be asymmetric, which naturally gives rise to asymmetric equilibria and allow us to study issues such as discrimination and crowding out of more efficient players.

Apart from the fact that we allow for smooth functions and non-symmetric players and equilibria, previous work on contests on battlefields considers the number of battlefields fixed, whereas we are concerned about how jury size affects equilibrium variables and we study what is the jury size that maximizes the combined effort of players. Moreover, note that in these previous papers the pivotatility effect is not present, in Szentes and Rosenthal's (2003a, 2003b) jurors' behaviour is deterministic and thus there is no probability of pivotality effect as contest outcomes are deterministic. In Klumpp and Polborn (2006) players are symmetric and jurors treat all players symmetrically which, as we shall see in Section 2, is the only cutting edge case where the pivotality effect is not present.

In terms of sequential multi-battle contests, where participants compete in a sequential series of interrelated battles or rounds, Klumpp and Polborn (2006) paper shows the impact of early successes on subsequent effort levels. Alcalde and Dahm (2007) propose a serial contest that integrates relative efforts akin to Tullock's contest and absolute effort differences from difference-form contests. Altman et al (2012) investigates the effects of carry over benefits, where performance in one battle affects the probability of success in subsequent battles. Hortala-Vallve and Llorente-Saguer (2012) examine a Colonel Blotto game where opposing parties have varying relative intensities, and characterizes the conditions under which pure strategy equilibria exist. Fu et al (2015) examine multi-stage R&D contests where there contest entry is an endogenous choice. Morgan et al (2018) explore self-selection dynamics where entry into larger or more meritocratic contests varies non-monotonically with ability. Other relevant papers are those of Harris and Vickers (1987), Snyder (1989), Malueg and Yates (2010), Kovenock and Roberson (2010) and Fu et al (2015).

In terms of experimental work, Knight and Schiff's (2010) applied paper on dynamic electoral competition highlights the role of cumulative advantages and strategic momentum in political campaigns. Mago and Sheremeta (2017) experimentally study subjects in simultaneous and sequential multi-battle contests and find incomplete bidding in simultaneous contests and overbidding in sequential contests. Chowdhury et al (2021) investigates the impact of battlefield salience in Colonel Blotto games and finds that subjects deviate from Nash equilibrium predictions when battlefield have salient targets but they do not otherwise. Stephenson (2024) examine contests with complementary prizes where agents allocate fixed budgets across multiple battlefields and shows existence of a unique pure strategy Nash equilibrium under highly sensitive battlefield success functions. Hortala-Vallve and Llorente-Saguer (2015) experientially test a Colonel Blotto game with heterogeneous and asymmetric battlefield valuations and that learning improves aggregate welfare despite initial deviations from theoretical predictions. Deck et al (2017) shows experimentally that subjects prioritize specific winning combinations rather than all battles.

2 An Example

This section considers a simple example that shows the pivotality effect in its simplest form. Assume two players compete to win a price worth 1 unit. To compete for the price, each player $i \in \{1, 2\}$ simultaneously chooses how much effort to exert $e_i \geq 0$. Effort has a cost given by e_i with $\alpha_i > 0$ for $i \in \{1, 2\}$. Once all players have chosen their effort levels, a jury decides who is the winner of the price.

There are $N \geq 1$ members of the jury (jurors henceforth), where N is an odd number. Each juror simultaneously chooses a player. Given effort levels, jurors choice of players are independent an identically distributed according to the Tullock contests success function (CSF), whereby the probability that a juror chooses player i is given by p_i with

$$p_i = \begin{cases} \frac{e_i^m}{e_i^m + e_j^m} & \text{if } e_i + e_j > 0, \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Note that we do not include efforts as an argument in p_i simply for analytical convenience.

The winner of the price is the player that is chosen by more than half the jury (i.e. at least $\frac{N+1}{2}$ jurors). Given this, the payoff of each player i as a function of effort levels is given by π_i where

$$\pi_i(e_i, e_j) \quad = \quad \sum_{k=\frac{N+1}{2}}^{N} \binom{N}{k} p_i^k (1-p_i)^{N-k} - \alpha_i e_i.$$

Note that if there is only one juror then the setting is the standard Tullock Contest. A result in statistics that relates the binomial cumulative density function with the incomplete Beta function is (see Wadsworth (1960) or Abramowitz and Stegun (1965), for completeness the proof is shown in the appendix):

$$\sum_{k=\frac{N+1}{2}}^{N} \binom{N}{k} p_i^k (1-p_i)^{N-k} \quad = \quad \frac{N}{\frac{N-1}{2}!2} \int_0^{p_i} x^{\frac{N-1}{2}} (1-x)^{\frac{N-1}{2}} \mathrm{d}x.$$

Therefore, the critical points of π_i, which are found by setting $\frac{\partial \pi_i}{\partial e_i} = 0$, are given by

$$\frac{N!}{\frac{N-1}{2}!2} p_i^{\frac{N-1}{2}} (1-p_i)^{\frac{N-1}{2}} \frac{\partial p_i}{\partial e_i} \quad = \quad \alpha_i. \tag{1}$$

In the standard Tullock setting with only one juror we have that $\frac{\partial p_i}{\partial e_i} = c_i$. Thus, the addition of a jury of more than one juror adds the term $\frac{N!}{\frac{N-1}{2}!2} p_i^{\frac{N-1}{2}} (1-p_i)^{\frac{N-1}{2}}$ which is equal to the probability that the jury is exactly one vote away from awarding the price to either player. That is, $\frac{N-1}{2}$ jurors are voting for player i and $\frac{N-1}{2}$ jurors are voting for player j, while the left over jury member is still undecided. Using the terminology from the voting literature, $\frac{N!}{\frac{N-1}{2}!2} p_i^{\frac{N-1}{2}} (1-p_i)^{\frac{N-1}{2}}$ is the probability that there is one pivotal juror.

Second order conditions are shown to hold in the general setting in the next section. Given equation (1), we have that in any equilibrium their optimal effort levels are related via the equation

$$\frac{\alpha_1}{\alpha_2} \quad = \quad \frac{\frac{\partial p_1}{\partial e_1}}{\frac{\partial p_2}{\partial e_2}}.$$

Given that $p_i = \frac{e_i^m}{e_i^m + e_j^m}$ and $p_1 = 1 - p_2$, we obtain

$$\frac{\alpha_1}{\alpha_2} \quad = \quad \frac{e_2}{e_1}. \tag{2}$$

Define $k = \frac{\alpha_1}{\alpha_2}$ so that in equilibrium $e_2 = ke_1$. Equations (1) and (2) lead to

$$\begin{aligned}
\alpha_1 &= \frac{N!}{\frac{N-1}{2}!2}\left(\frac{e_1^m}{e_1^m + k^m e_1^m}\right)^{\frac{N-1}{2}}\left(\frac{k^m e_1^m}{e_1^m + ke_1^m}\right)^{\frac{N-1}{2}}\frac{me_1^m k^m e_1^m}{(e_1^m + ke_1^m)^2}\frac{1}{e_1}, \\
\alpha_1 e_1 &= \frac{N!}{\frac{N-1}{2}!2}\left(\frac{k^m}{(1+k^m)^2}\right)^{\frac{N-1}{2}}\frac{mk^m}{(1+k^m)^2}, \\
e_1 &= \frac{N!}{\frac{N-1}{2}!2}\left(\frac{k^m}{(1+k^m)^2}\right)^{\frac{N+1}{2}}\frac{m}{\alpha_i},
\end{aligned}$$

and $e_2 = ke_1$.

From (2) we have that in equilibrium

$$p_1 = \frac{c_2^m}{c_1^m + c_2^m},$$

and $p_2 = 1 - p_1$.

Next we turn out attention to exploring how jury size affects the efforts of both players in equilibrium. To this end, write $e(N)$ as the sum of equilibrium effort levels of both players when there are N jurors. Define $D(N) = e(N+2) - e(N)$. We have that

$$D(N) = \left(\frac{N+2}{N+1}\frac{4k^m}{(1+k^m)^2} - 1\right)e(N).$$

Notice that $\frac{4k^m}{(1+k^m)^2} \leq 1$ with equality only when $k = 1$. Therefore, if $c_1 \neq c_2$ then there exists a finite N that maximizes equilibrium effort. Since $D(N)$ is monotone decreasing in N, the effort maximizing jury size is either the odd floor or the odd ceiling of the N^* that solves $D(N^*) = 0$.

If $\alpha_1 = \alpha_2$ the contests is similar to the one considered in Klumpp and Polborn (2006), and as it can be noted the pivotality effect is not present: $\frac{4k^m}{(1+k^m)^2} = 1$ for all $m > 0$, and thus $D(N)$ is always positive for all N. This is a cutting edge that ceases to hold as soon as either players are not symmetric, jurors do not treat players symmetrically, or non-symmetric equilibria are considered. We drop all these symmetry assumptions which are present in Klumpp and Polborn (2006) (note that they do consider asymmetries, but only for the limit case of $N \to \infty$).

In terms of optimal jury size, if $\alpha_1 = \alpha_2$ then effort maximizing jury size is the one that makes players' profits equal to zero in equilibrium (as the sum of efforts cannot be greater than the amount that leads to zero players' profits).

Figure 1 shows how $e(N)$, i.e. the sum of equilibrium efforts for a given N, changes as the jury size changes. This figure also shows the function $D(N)$. In the example from this

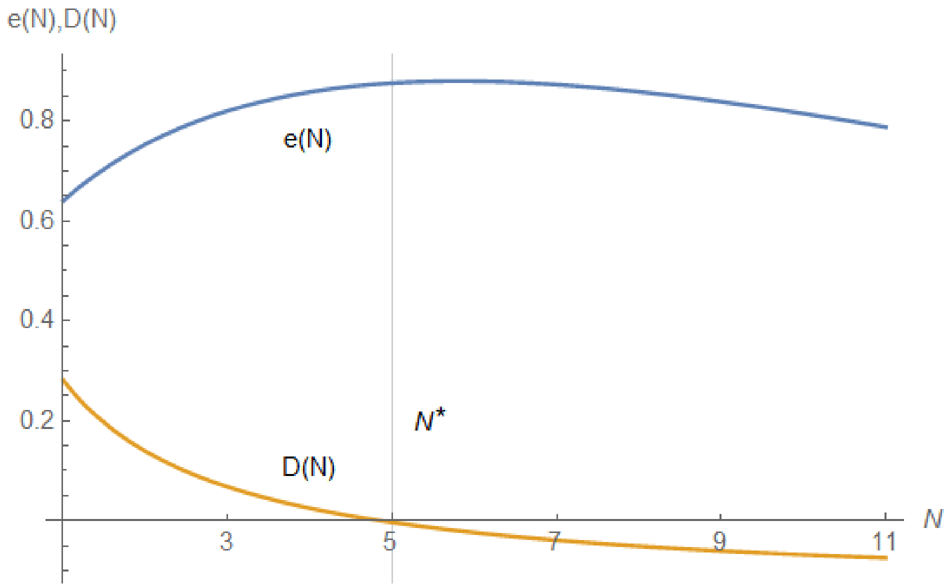

Figure 1: Example with $\alpha_1 = 1$, $\alpha_2 = 0.2$, and $m = 0.9$. Effort maximizing jury size is $N = 5$.

figure, the jury maximizing size is 5. Note that $e(N)$ is maximized at some N between 5 and 7, but we are only allowing N to take on odd values in this section.

Figure 2 shows how player's profits change as jury size changes. As it may be hinted at in the figure, the effort maximizing jury size coincides with the profit minimizing jury size. This is explored in the general model in the next section.

The reason why there can be an optimal interior effort maximizing jury size that is that there are two incentives at play. On the one hand, increasing the jury size reduces **uncertainty**, i.e. the player with a higher effort is more likely to win, which leads to an increase in effort. However, increasing the jury size also leads to a reduction of the probability that a juror is **pivotal** and, thus, a reduction in effort. With low jury size, the uncertainty effect dominates and increasing the number of jurors leads to more effort. However, as jury size is increased it comes a point where the pivotal effect starts to dominate and increasing the number of jurors leads to a decrease in equilibrium effort.

3 The Model

Assume 2 players compete to win a price worth $v > 0$. Section 3.8 explores the case where there are more than two contestants and show that our qualitative results remain unchanged when only two contestants are considered. To compete for the price, each player i simultaneously chooses how much effort to exert $e_i \geq 0$. Effort has a cost given by the function $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ where the argument is the effort of player i and with $c_i' > 0$ and $c_i'' \leq 0$ for

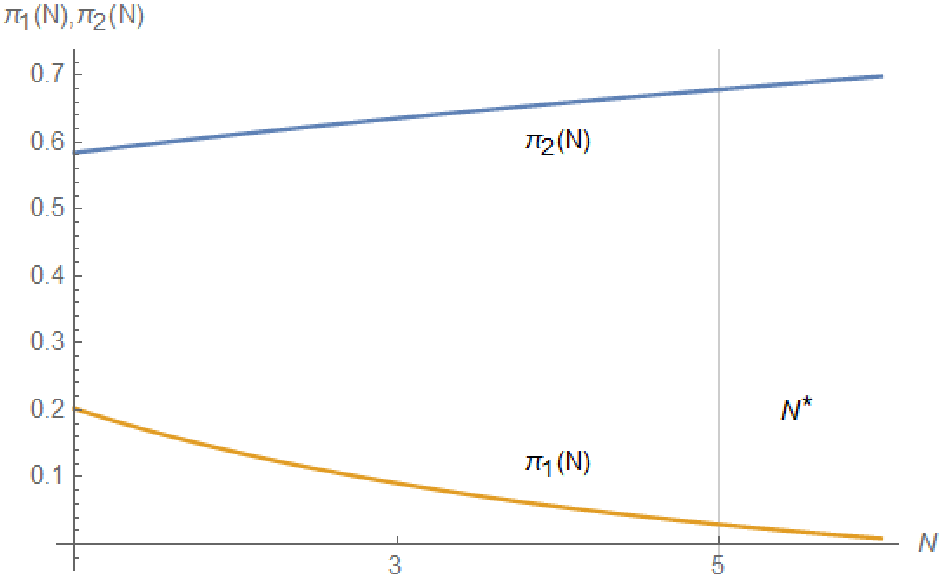

Figure 2: Example with $\alpha_1 = 1$, $\alpha_2 = 0.2$, and $m = 0.9$. Effort maximizing jury size is $N = 5$.

$i \in \{1, 2\}$. Once all players have chosen their effort levels, a jury decides who is the winner of the price.

There are $N \geq 1$ members of the jury (jurors henceforth), where N is an odd number. The case where jury size can take on even vales is considered in Section 3.7 where it is shown that restricting N to be odd does not affect the results but greatly simplifies the analysis. Each juror simultaneously chooses a player. Given effort levels, jurors choice of players are independent an identically distributed where the probability that a juror chooses player i is given by the differentiable function $p_i : \mathbb{R}^2_+ \rightarrow [0, 1]$ with $p_1 + p_2 = 1$. The arguments in p_i are the efforts of players i and j and $p_i(0, 0) = \frac{1}{2}$, $\frac{\partial p_i}{\partial e_i} > 0$, $\frac{\partial p_i}{\partial e_j} < 0$, $\frac{\partial^2 p_i}{\partial^2 e_i} < 0$, $\frac{\partial^2 p_i}{\partial^2 e_j} > 0$ for all $i \in \{1, 2\}$. Whenever it is convenient we do not include efforts as an argument in p_i. In Section 3.7 we allow jurors to be strategic and show that out main results do not change.

The winner of the price is the player who was chosen by most jurors, i.e. a simple majority voting rule. In section 3.5 we consider super majority voting rules and show that our qualitative results do not depend on the particular voting rule.

3.1 Equilibrium

Throughout this paper we focus on pure strategy equilibria. The complexity of mixed strategy equilibria in contest is well known (see for instance Ewerhart (2015) an references therein) and not the purpose of this paper.

The payoff of player i as a function of effort levels is given by π_i where

$$\pi_i = v \sum_{k=\frac{N+1}{2}}^{N} \binom{N}{k} p_i^k (1-p_i)^{N-k} - c_i(e_i).$$

Using the result proven in the appendix, we have

$$\sum_{k=\frac{N+1}{2}}^{N} \binom{N}{k} p_i^k (1-p_i)^{N-k} = \frac{N!}{\frac{N-1}{2}!2} \int_0^{p_i} x^{\frac{N-1}{2}} (1-x)^{\frac{N-1}{2}} \mathrm{d}x.$$

Therefore, the critical points of π_i are given by

$$v \frac{N!}{\frac{N-1}{2}!2} p_i^{\frac{N-1}{2}} (1-p_i)^{\frac{N-1}{2}} \frac{\partial p_i}{\partial e_i} = c_i'(e_i). \tag{3}$$

The second order condition for $N = 1$ is $v\frac{\partial^2 p_i}{\partial^2 e_i} - c_i''(e_i)$ and, thus, π_i is concave as desired. For $N \geq 3$ the second order condition is given by

$$\frac{\partial^2 \pi_i}{\partial^2 e_i} = v \frac{N!}{\frac{N-1}{2}!2} p_i^{\frac{N-3}{2}} (1-p_i)^{\frac{N-3}{2}} \left(\frac{N-1}{2}(1-2p_i)\frac{\partial p_i}{\partial e_i} + p_i(1-p_i)\frac{\partial^2 p_i}{\partial^2 e_i} \right) - c_i''(e_i).$$

Given e_j, note that at any effort level $e_i > e_i^*$ where e_i^* is such that $p(e_i^*, e_j) = \frac{1}{2}$, the equation above is negative and, therefore, concave. Moreover, at effort level $e_i = 0$ the equation above is also concave. Therefore, if a critical point found via equation 3 is such that $p_i \geq \frac{1}{2}$, then that critical point is a maximum. On top of that, among the critical points found via 3 and such that $p_i < \frac{1}{2}$, there is at least a maximum.

Given equation (3), we have that in any equilibrium optimal effort levels are related via the equation

$$\frac{c_1'(e_1)}{c_2'(e_2)} = \frac{\frac{\partial p_1}{\partial e_1}}{\frac{\partial p_2}{\partial e_2}}. \tag{4}$$

The equilibrium efforts of the contest is given implicitly by equations (3) and (4).

Note that in equation (4) the jury size N is not present. While N affects the level of effort exerted by both players, it does not affect how these effort levels are related in equilibrium. Therefore, for all jury size N equation 4 defines implicitly a mapping $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that in equilibrium $e_2 = f(e_1)$. We have just proven our first result:

Proposition 1. *There exists a mapping $f : \mathbb{R}_+ \to \mathbb{R}_+$ that relates both player's equilibrium efforts, and this mapping is independent of the jury size N.*

Using implicit differentiation in equation 4 we have that

$$f'(e_i) = \frac{-c_j'(e_j)\frac{\partial^2 p_i}{\partial^2 e_i} - c_i''(e_i)\frac{\partial p_i}{\partial e_j} - c_i'(e_i)\frac{\partial^2 p_i}{\partial e_i \partial e_j}}{c_i'(e_i)\frac{\partial^2 p_i}{\partial^2 e_j} + c_j''(e_j)\frac{\partial p_i}{\partial e_i} + c_j'(e_j)\frac{\partial^2 p_i}{\partial e_i \partial e_j}}.$$

We assume that the mapping from Proposition 1 is strictly increasing. This means that in equilibrium if a change of parameters leads to the effort of one player to increase, it also increases the effort of the other player.

Assumption 1. *Functions p_i and c_i for all $i \in \{1,2\}$ and all $e_i \in \{1,2\}$ are such that*

$$\frac{-c_j'(e_j)\frac{\partial^2 p_i}{\partial^2 e_i} - c_i''(e_i)\frac{\partial p_i}{\partial e_j} - c_i'(e_i)\frac{\partial^2 p_i}{\partial e_i \partial e_j}}{c_i'(e_i)\frac{\partial^2 p_i}{\partial^2 e_j} + c_j''(e_j)\frac{\partial p_i}{\partial e_i} + c_j'(e_j)\frac{\partial^2 p_i}{\partial e_i \partial e_j}} > 0.$$

A sufficient condition for assumption 1 is that the cross-derivative $\frac{\partial^2 p_i}{\partial e_i \partial e_j}$ is positive or, alternative, that its value is not too high. As we shall show later on, this assumption is satisfied for all parametrizations of the Tullcok contests success function.

3.2 Effort Maximizing Jury Size

Let $e_i(N)$ be the equilibrium effort of player i when there are N jurors and π_i^N the profit of player i when there are N jurors. The problem of choosing the jury size to maximize the sum of equilibrium efforts is given by

$$\begin{aligned}
\max_N e_1(N) + e_2(N) &= \max_N e_1(N) + f(e_1(N)) \\
&= \max_N c_1(e_1(N)) + c_2(f(e_1(N))) \\
&= \min_N v - c_1(e_1(N)) - c_2(f(e_1(N))) \\
&= \min_N \pi_1^N(e_1(N), f(e_1(N)) + \pi_2^N(e_1(N), f(e_1(N)),
\end{aligned}$$

where above we have used the fact that in equilibrium $e_2 = f(e_1)$, where f is a strictly increasing function, and that the cost functions c_i for $i \in \{1,2\}$ are also strictly increasing.

Therefore, the problem of choosing the jury size to maximize equilibrium effort is equivalent to choosing the jury size to minimize players' profits. Since equilibrium efforts are related via the mapping from proposition 1, and that mapping does not depend on the size of the jury, the optimal jury size is either an interior solution to the problem of minimizing player's profits, or a corner solution such that at least one player has zero profit. This constitutes the proof of our next result.

Proposition 2. *The jury size that maximizes the sum of equilibrium effort levels is either the solution to the problem of minimizing the players' profits, or the jury size such that one player has zero profit.*

Proposition 2 has clear implications in terms of rent dissipation (measured as the ratio of the sum of players' costs to the value of the price). Since

$$\frac{c_1(e_1(N))+c_2(e_2(N))}{v}=\frac{v-\pi_1^N(e_1(N),e_2(N))-\pi_2^N(e_1(N),e_2(N))}{v},$$

Proposition 2 implies that the effort maximizing jury size is the one that maximizes rent dissipation.

Corollary 1. *The effort maximizing jury size is the one that maximizes rent dissipation.*

To explore the optimal jury size, let $e(N)=e_1(N)+e_2(N)$. Furthermore, define

$$D_i(N) \quad = \quad \pi_i^{N+2}(e(N))-\pi_i^N(e(N)).$$

If the sign of $D_i(N)$ is positive then at jury size $N+2$ and effort levels $e(N)$ player i can increase its profit by exerting more effort: $e_i(N+2)>e_i(N)$. That is, $D_i(N)>0$ means that at jury size N an increase in the jury size of 2 increases the equilibrium effort of player i.

In an abuse of notation write p_i and $\frac{\partial p_i}{\partial e_i}$ as the equilibrium value of p_i and $\frac{\partial p_i}{\partial e_i}$ when there are N jurors. We have

$$\begin{aligned}
D_i(N) &= \pi_i^{N+2}(e(N))-\pi_i^N(e(N))\\
&= v\frac{(N+2)!}{\frac{N+1}{2}!2}p_i^{\frac{N+1}{2}}(1-p_i)^{\frac{N+1}{2}}\frac{\partial p_i}{\partial e_i}-v\frac{N!}{\frac{N-1}{2}!2}p_i^{\frac{N-1}{2}}(1-p_i)^{\frac{N-1}{2}}\frac{\partial p_i}{\partial e_i}\\
&= \left(\frac{N+2}{N+1}4p_i(1-p_i)-1\right)v\frac{N!}{\frac{N-1}{2}!2}p_i^{\frac{N-1}{2}}(1-p_i)^{\frac{N-1}{2}}\frac{\partial p_i}{\partial e_i}.
\end{aligned}$$

That is,

$$D_i(N) \quad \propto \quad \frac{N+2}{N+1}4p_i(1-p_i)-1. \tag{5}$$

Note that the sign of $D_i(N)$ does not depend on the identity of juror $i\in\{1,2\}$. Moreover, note that $\frac{N+2}{N+1}>1$ and the expression in equation 5 is strictly decreasing in N with limit 1 as N increases to infinity. On top of that, $4p_i(1-p_i)\leq 1$ with equality only when $p_i=\frac{1}{2}$. Thus if $D_i(N)>0$ then both players' equilibrium efforts increase in jury size goes from N to $N+2$. On top of that, as the jury size grows, the sign of $D_i(N)$ becomes negative unless $p_i(1-p_i)$ converges to $\frac{1}{4}$, i.e. p_i converges to $\frac{1}{2}$. Therefore, if $\lim_{N\to\infty}p_i\neq\frac{1}{2}$ then there exists a finite optimal jury size.

On top of that, if $\lim_{N\to\infty}p_i=\frac{1}{2}$ then $\lim_{N\to\infty}v\frac{N!}{\frac{N-1}{2}!2}p_i^{\frac{N-1}{2}}(1-p_i)^{\frac{N-1}{2}}=\infty$. Consequently, from the first order condition in equation 3, we have that equilibrium efforts must tend to infinity, which leads to negative equilibrium profits, a contradiction. Therefore, when $\lim_{N\to\infty} p_i = \frac{1}{2}$ the optimal jury size is a corner solution such that at least one players has profit zero, which can be achieved by a finite jury size.

We collect the facts proven above in the next proposition:

Proposition 3. *There exists a finite optimal jury size. Moreover, changing the jury size either increases both equilibrium effort levels or decreases both equilibrium effort levels.*

The intuition why the optimal jury size would, if we disregard negative payoff considerations, tend to infinity when $\lim_{N\to\infty} p_i = \frac{1}{2}$, i.e. when the contest is most contested, is that the probability that juror is pivotal vanishes at speeds orders of magnitude slower compared to the uncertainty effect. In particular, in the limit the pivotality effect never dominates the uncertainty effect and thus the optimal jury size, as long as player's profits are non-negative, tends to infinity.

3.3 Tullock Contests

In this section we introduce a parametric specification for the cost functions and the jurors behaviour. Whenever we refer to a Tullock contest, we refer to a contests where the cost function and jurors' behaviour follows the parametrization below.

In a Tullcok contest the cost of player i is given by $c_i(e_i) = \gamma_i + \alpha_i c_i^{\beta_i}$ with $\gamma_i \geq 0$, $\alpha_i > 0$ and $\beta_i \geq 1$ for all $i \in \{1, 2\}$.

Moreover, in a Tullcok contest the probability that a juror chooses player i is given by the Tullock contest success function (TCSF):

$$p_i(e_i, e_j) = \begin{cases} \frac{n_i e_i^{m_i}}{n_i e_i^{m_i} + n_j e_j^{m_j}} & \text{if } n_i e_i^{m_i} + n_j e_j^{m_j} \neq 0, \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

with $n_i > 0$ and $m_i > 0$ for all $i \in \{1, 2\}$.

In this case, the mapping f from Proposition 1 is given by

$$\begin{aligned} e_2 &= \left(\frac{m_2\alpha_1\beta_1}{m_1\alpha_2\beta_2}\right)^{\frac{1}{\beta_2}} e_1^{\frac{\beta_1}{\beta_2}} \\ &= K e_1^L, \end{aligned}$$

with $K = \left(\frac{m_2\alpha_1\beta_1}{m_1\alpha_2\beta_2}\right)^{\frac{1}{\beta_2}} > 0$ and $L = \frac{\beta_1}{\beta_2} > 0$. Thus, f is strictly increasing as required by assumption 1.

3.4 Contests with Discrimination

In this subsection we study a contest where there is discrimination against a contestant and show how increasing the jury size, even when new jurors also show discrimination in the same way as existing jurors, can help mitigate the effects of discrimination if the player that is being discriminated against is better (in the sense that it has a lower cost function). The key mechanism is that increasing the jury size can lead to both payers exerting more effort, and since the cost of effort is lower for the player that is being discriminated against, this player increases her effort more so than the other player. This extra increase in effort may not only offset the fact that there are now more jurors who discriminate against the player, but it can lead to a net increase in the probability that the player being discriminated against wins.

To be precise we use the following two definitions.

Definition 1. *We say that there is* ***severe discrimination*** *against player i if for all $a > 0$ whenever $e_1 = e_2 = a$ then $p_i < \frac{1}{2}$. We say there is* ***discrimination*** *against player i there exists some $a > 0$ such that if $e_1 = e_2 = a$ then $p_i < \frac{1}{2}$.*

That is, with severe discrimination against i then for all effort levels, when both players exert the same effort, player i has a lower probability of winning the player j. Given our assumptions on p_i this means that $p_i \geq \frac{1}{2}$ implies $e_i > e_j$. Discrimination, not necessarily severe, occurs if for some effort levels, but not necessarily all effort levels, when both players exert the same effort player i has a lower probability of winning the player j. Note that it is possible to have discrimination against i and discrimination against j at the same time (but necessarily at different effort levels), but it is not possible to have strict discrimination against i and against j at the same time.

The interpretation of severe discrimination and discrimination is that, for example, for some professions there could be age discrimination against young people at senior levels but not at junior levels (discrimination against young), whereas in other professions there could be age discrimination against young people across all seniority levels (severe discrimination against young). In other professions, there could be discrimination against young people for senior positions and discrimination against older people for junior positions (discrimination against young and against older people).

Definition 2. *We say that player i is* ***better*** *than player j if for all $a > 0$ we have $c_i(a) < c_j(a)$.*

That is, given any effort level player i's cost is lower than that of player j.

If i is severely discriminated against and i is better or no player is better than the other player, then severe discrimination either helps the better player win or does not influence whether the better player wins. For that reason, we focus on the case where if a player is severely discriminated against, this player is also better than the other player.

In the Tullock contest, a way to represent this situation is by assuming $\gamma_1 = \gamma_2 = 0$, $\alpha_1 = \beta_1 = n_1 = n_1 = 1$, $\gamma_2 = \alpha_2 = m_2 = 1$, $\beta_2 > 1$ and $n_2 > 1$. Thus, $p_1 = \frac{e_1}{e_1+n_2e_2}$ with $n_2 > 1$. That is, player 1 is severely discriminated against. Moreover, the cost function of player 1 is $c_1(e_1) = e_1$ whereas that of player 2 is $c_2(e_2) = 1 + e_2^{\beta_2}$ with $\beta_2 > 1$. That is, player 1 is better than 2.

Figure 3 shows and example where the probability with which player 1 wins goes from less than 50% to more than 50% when the jury size its increased from 1 to its effort maximizing value.

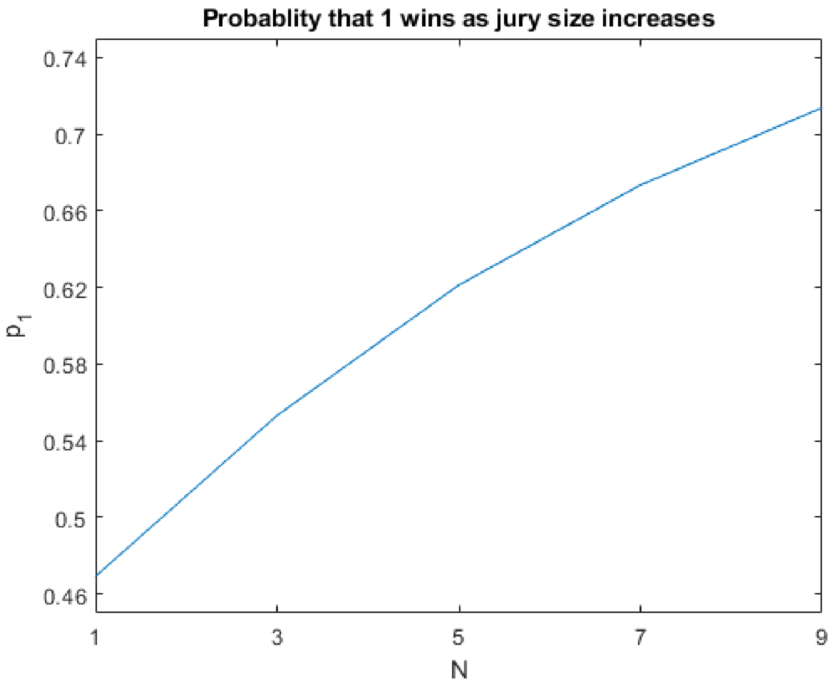

Figure 3: Example with $v = 10$, $\gamma_1 = 0$, $\alpha_1 = \beta_1 = n_1 = n_1 = 1$, $\gamma_2 = \alpha_2 = m_2 = 1$, $\beta_2 = 3$ and $n_2 = 3$. Effort maximizing jury size is $N = 9$.

As can be seen in Figure 3, with one juror (i.e. the standard Tullock contest), the player that is severely discriminated against has a lower than 50% probability of winning. As the jury size increases, this probability increases until it goes past 50% for 9 jurors. In particular, with 9 jurors player's 1 probability of winning is 0.714, which is similar to 0.702 (the probability with which this player wins if there is only one juror but no discrimination of any kind: $n_2 = 1$). That is, the size of the jury can help mitigate or even eliminate the effects of severe discrimination. This is our next result which we have just proven via example:

Proposition 4. *Consider a scenario where there is severe discrimination against a player. Increasing the jury size, even when additional jurors also exhibit discrimination against that player, can increase the probability that the player being discriminated against wins.*

An observation is that if jury size eliminates the effects of severe discrimination it can naturally eliminate the effects of discrimination.

Note that by proposition 2 maximizing the sum of efforts is equivalent to minimizing the sum of the profits of the players. Therefore, even though the effects of discrimination against player 1 are reduced when the jury size is increased, this player (and in fact both players) enjoy a lower expected profit than when there is only one juror.

3.5 Other Voting Rules and Even Number of Jurors

Suppose that N can take even values and the winner of the contest is player 1 if it was chosen by at least a fraction $q \in (0, 1]$ of the jurors, otherwise player 2 wins. We refer to q as the voting rule. For example, if $q = \frac{1}{2}$ then the voting rule is simple majority (this is the case we have looked at so far) where ties go to player 1, whereas if $q = 1$ then the voting rule is unanimity for 1 with player 2 as the status quo.

The payoff of player 1 as a function of effort levels is given by

$$\pi_1 = v \sum_{k=\lceil Nq \rceil}^{N} \binom{N}{k} p_1^k (1 - p_1)^{N-k} - c_1(e_1),$$

and similarly for player 2.

Using the equivalence shown in the appendix, the critical points of π_i for $i \in \{1, 2\}$ are given by

$$v \frac{N!}{(\lceil Nq \rceil - 1)!(N - \lceil Nq \rceil)!} p_i^{\lceil Nq \rceil - 1} (1 - p_i)^{N - \lceil Nq \rceil} \frac{\partial p_i}{\partial e_i} = c_i'(e_i).$$

Equilibrium effort levels are then related via the equation

$$\frac{c_1'(e_1)}{c_2'(e_2)} = \frac{\frac{\partial p_1}{\partial e_1}}{\frac{\partial p_2}{\partial e_2}}.$$

Note that this is the same equation as with with simple majority. Furthermore, note that the voting rule q is not present. While N and q affect the equilibrium levels of effort, they do not affect how these effort levels are related. Just as with simple majority, the mapping f relates equilibrium efforts $e_2 = f(e_1)$ for all N and q.

Increasing the jury size by 2 jurors (we choose 2 extra jurors instead of 1 to keep this section in line with the main model), increases the votes that i needs to win the contest by $a = \lceil q(N+2) \rceil - \lceil qN \rceil \in \{0,1,2\}$. Proceeding as in the case with $q = \frac{1}{2}$, we have

$$\begin{aligned} D_i(N) &= v \frac{(N+2)!}{(\lceil Nq \rceil - 1 + a)!(N - \lceil Nq \rceil + (2-a))!} \\ &\quad \times p_i^{\lceil Nq \rceil - 1 + a}(1-p_i)^{N - \lceil Nq \rceil + (2-a)} \frac{\partial p_i}{\partial e_i} \\ &\quad - v \frac{N!}{(\lceil Nq \rceil - 1)!(N - \lceil Nq \rceil)!} p_i^{\lceil Nq \rceil - 1}(1-p_i)^{N - \lceil Nq \rceil} \frac{\partial p_i}{\partial e_i} \\ &= \left(\frac{(N+2)(N+1)}{G(N,q)} 4 p_i^a (1-p_i)^{2-a} - 1 \right) \\ &\quad \times v \frac{N!}{(\lceil Nq \rceil - 1)!(N - \lceil Nq \rceil)!} p_i^{\lceil Nq \rceil - 1}(1-p_i)^{N - \lceil Nq \rceil} \frac{\partial p_i}{\partial e_i}, \end{aligned}$$

where

$$G(N,q) = \begin{cases} 4(N - \lceil Nq \rceil + 2)(N - \lceil Nq \rceil + 1) & \text{if } a = 0, \\ 4\lceil Nq \rceil (N - \lceil Nq \rceil + 1) & \text{if } a = 1, \\ 4(\lceil Nq \rceil + 1)\lceil Nq \rceil & \text{if } a = 2, \end{cases}$$

with $a = \lceil q(N+2) \rceil - \lceil qN \rceil$.

That is,

$$D_i(N) \propto \frac{(N+2)(N+1)}{G(N,q)} 4 p_i^a (1-p_i)^{2-a} - 1.$$

Note that the equation above is qualitatively the same as equation 5. Thus, the incentives and results around effort maximizing jury size are the same as those already presented and thus the voting rule or the restriction to only odd jury sizes does not affect the results. We have proven our next result.

Proposition 5. *Both propositions 1 and 3 hold for general voting rule q.*

3.6 Effort Maximizing Voting Rule

In this subsection we ask the following question, what is the voting rule that maximizes the sum of efforts? Given the complexities of dealing with a general voting rule whose outcome depends on the rounding of real numbers, we explore this issue numerically.

Consider a Tullock contest with $v = 1$, $\gamma_1 = \gamma_2 = 0$, $\alpha_1 = \beta_1 = n_1 = m_1 = 1$, $\beta_2 = n_2 = m_2 = 1$, and $\alpha_2 = \alpha > 0$. With these parameter values we have that $p_i = \frac{e_i}{e_1 + e_2}$, $c_1(e_1) = e_1$ and $c_2(e_2) = \alpha e_2$. Assume that there are $N = 3$ jurors.

Figure 4 shows how the sum of efforts changes for different voting rule q for two different values of the cost function of player 2. When both players have the same cost, $\alpha = 1$, simple majority maximizes the sum of efforts whereas when player 2 has a higher cost function, $\alpha = 5$, then the voting rule where there is unanimity required for 1 to win outperforms any other voting rule. That is, when the contest is unbalanced in that one player is better than the other one, the effort maximizing voting rule is one where the better player needs more votes to win. This observation is consistent for different parameter values and, since there is no formal proof for it, it is stated as a remark.

Remark 1. *Assume there is no discrimination. In a contest where both players have the same cost function, the effort maxmimizing voting rule is simple majority. If there is one player that is better than the other one, the effort maximizing voting rule is one where the better player needs more votes to win than the other player.*

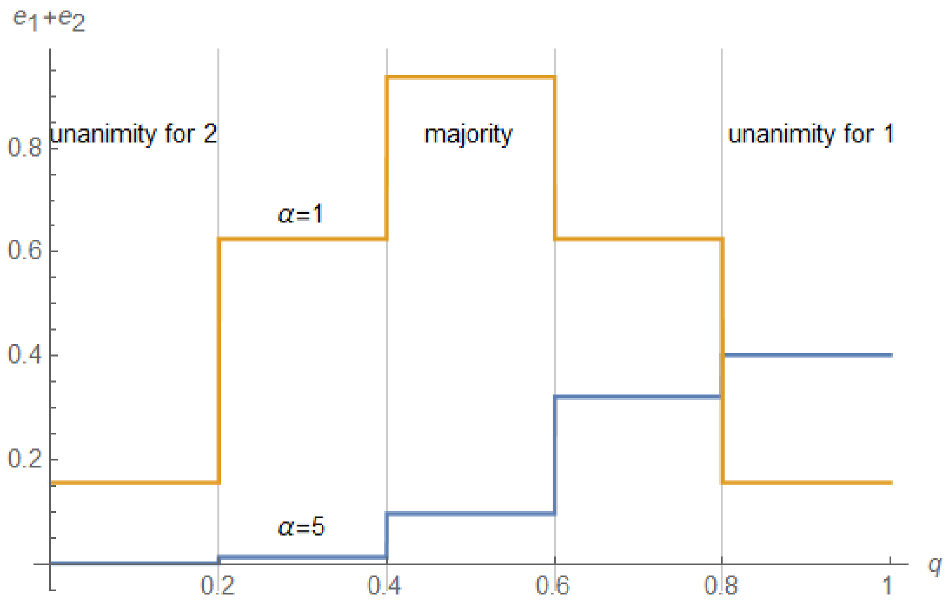

Figure 4: Example with $N = 5$, $v = 1$, $\gamma_1 = \gamma_2 = 0$, $\alpha_1 = \beta_1 = n_1 = n_1 = 1$, $\gamma_2 = m_2 = n_2 = 1$ and $\alpha_2 = \alpha$.

The intuition behind the remark is that in contests where competition is uneven aggregate effort is lower than in contest where there is an even competition between players. The reason for this is that, first, the player with the higher costs has a lower return to effort given the high effort exerted by the other players and, second, the player with the lower costs does not need to exert a high level of effort to win the contest given the costs of the other player. Hence, a voting rule that balances the contest is the one that maximizes aggregate effort. The way the balancing is done is via requiring the player with the lower cost function more votes to win than the other player.

3.7 Strategic Jurors

We now explore in more detail jurors' behaviour by considering the case where jurors can be strategic. A way to motivate jurors' strategic behaviour is by considering a situation where jurors want to choose the player that exerted more effort, i.e. they want player i to win the contest if $e_i > e_j$ for $i \in \{1, 2\}$, and are indifferent about who wins if $e_1 = e_2$.

Since effort may not be directly observable, as effort and output are usually imperfectly correlated, consider that each juror $k \in \{1, \ldots, N\}$ does not know effort choices. Instead, they have a symmetric prior (i.e. their prior distribution about e_1 is the same as that of e_2), and, on top of that, they receive an independent and identically distributed signal $\sigma_k \in \{1, 2\}$ such that

$$P\left(e_i > e_j | \sigma_k = i\right) \quad > \quad \frac{1}{2}. \tag{6}$$

A way to generate such signal is to have

$$P\left(\sigma_k = i | e_1, e_2\right) \quad = \quad p_i.$$

Since $\frac{\partial p_i}{\partial e_i} > 0$ and $\frac{\partial p_i}{\partial e_j} < 0$, we have that signal σ_k contains information about the effort of both players. Moreover, since jurors' prior is symmetric, then if $\sigma_k = i$ the probability that player i exerted more effort than j is higher than $\frac{1}{2}$, whereas if $\sigma_k = j$ this probability is less than $\frac{1}{2}$.

Consider thus a situation where jurors want to choose the player that exerted the most effort and use their signal σ_k from equation 6 to do so. In the main text it was assumed that jurors' vote follows p_i, which with strategic jurors it is equivalent to stating that they vote according to their signal. The question is then whether a situation where each juror chooses a candidate according to its signal is an equilibrium. This is referred to in the voting literature as an informative equilibrium (see for instance Austen-Smith and Banks (1996)). The answer is positive.

Proposition 6. *Assume jurors are strategic, have a symmetric common prior, and receive an informative iid signal about effort choices, then there is an informative equilibrium where every juror votes following its signal.*

The proof follows from Austen-Smith and Banks (1996) and is thus omitted. The intuition is that a juror must behave as if pivotal (i.e. the vote of the other jurors is split 50/50) as otherwise its vote is irrelevant. If a juror is pivotal and if jurors vote according to their signal then there are as many signal for candidate 1 as there are for candidate 2 when not counting a player's own signal. Since the prior is symmetric and signals are iid then the posterior when considering being pivotal and after observing a player's own signal is simply the player's own signal. Given this posterior, it is a best response to vote following the signal as the signal has a more than 50% chance of being correct.

Note that, just as in Austen-Smith and Banks (1996), if we used a non-majority voting rule then there are equilibria where some jurors are not sincere to offset whichever bias the voting rule might introduce. For example, if there are 7 jurors and player 1 wins if and only if this player is chosen by 6 jurors, there is no equilibrium where all jurors are sincere but there is an equilibrium where 4 jurors vote for player 1 and the other 3 vote sincerely. This contest is then equivalent to one where the voting rule is simple majority and there are only 3 jurors.

Thus, for all voting rules there is an equilibrium where some jurors are sincere, and the rest vote to offset the bias introduced by the voting rule. Since sincere voting is essentially what we assumed on the offset, the assumption is withoutloss of generality if the appropriate equilibrium is selected.

3.8 More than Two Contestants

Assume for this section that there are $C \in \{3, \dots\}$ players and the winner of the contest is the player that is chosen by most jurors, where for simplicity in case of a tie a winner is chosen at random uniformly among tied players. Given effort levels, define a state of the world as a vector $v \in \{1, \dots, N\}^C$ such that $\sum_i v_i = N$. That is, v_i are the votes received by player i.

If there are more than two contestants, deriving analytical results is no longer possible. The reason is that with more than two contestant the number of votes needed to win the contest depends on the whole distribution of votes all players receive. As an example, if there are 3 players and 7 jurors then player 1 wins in any of the following states of the world: $(3, 2, 2)$, $(4, 2, 1)$, $(4, 1, 2)$, $(5, 2, 0)$, $(5, 0, 2)$, $(5, 1, 1)$, $(6, 1, 0)$, $(6, 0, 1)$, $(7, 0, 0)$, and $(3, 3, 1)$ and $(3, 1, 3)$ with probability $\frac{1}{2}$. As the number of players and/or the number of jurors increases the number of states increases exponentially and, more importantly, there is no known analytically tractable way to enumerate them for general N and C.

Intuitively, however, our results carry through to the case with more than two contestants, as both the uncertainty effect and pivotality effect are still present and, thus, increasing the jury size involves a trade off of these two effects. Nevertheless, with three or more players there is equilibria multiplicity, and in particular, there can be equilibria where there is crowding out. Crowding out equilibria are those where a contestant, not necessarily the one with lower costs, exerts zero effort in equilibrium.

To illustrate these observations, we solve numerically a Tullock contest with 3 players and jury sizes $N = 1$ and $N = 3$. Assume for simplicity that $v = 1$, $\gamma_1 = \gamma_2 = 0$, $\alpha_1 = \beta_1 = n_1 = m_1 = 1$, $\alpha_2 = \beta_2 = n_2 = m_2 = 1$, and $\alpha_3 = \alpha > 0$ and $\beta_3 = n_3 = m_3 = 1$. With these parameter values we have that $p_i = \frac{e_i}{\sum_j e_j}$ and $c_i(e_i) = e_i$ for all $i \in \{1, 2\}$ and $c_3(e_3) = \alpha e_3$.

Let v_i the votes received by player i. The payoff of each player is given by

$$\begin{aligned}\pi_1 &= P(v_1 > v_2, v_1 > v_3) + \frac{1}{3}P(v_1 = v_2 = v_3) - e_1,\\ \pi_2 &= P(v_2 > v_1, v_2 > v_3) + \frac{1}{3}P(v_1 = v_2 = v_3) - e_2,\\ \pi_3 &= P(v_3 > v_1, v_3 > v_2) + \frac{1}{3}P(v_1 = v_2 = v_3) - \alpha e_3,\end{aligned}$$

note that with jury sizes 1 and 3 there can never be a 2-way tie.

With a one person jury ($N = 1$) it can be easily shown that there is a unique pure strategy equilibrium given by $e_1 = e_2 = \frac{2\alpha}{(2+\alpha)^2}$ and $e_3 = \frac{2(2-\alpha)}{(2+\alpha)^2}$. Notice that if $\alpha < 1$, i.e. player 3 has a lower cost function than players 1 and 2, then this player exerts more effort, whereas if $\alpha > 1$ then the opposite happens. This is a standard result the traditional Tullock contests without a jury.

However, if the jury size is instead $N = 3$, we have that there is equilibria multiplicity. To being with, there is an interior equilibrium where all three players exert positive effort. The equilibrium efforts are shown in Figure 5 for different values of α.

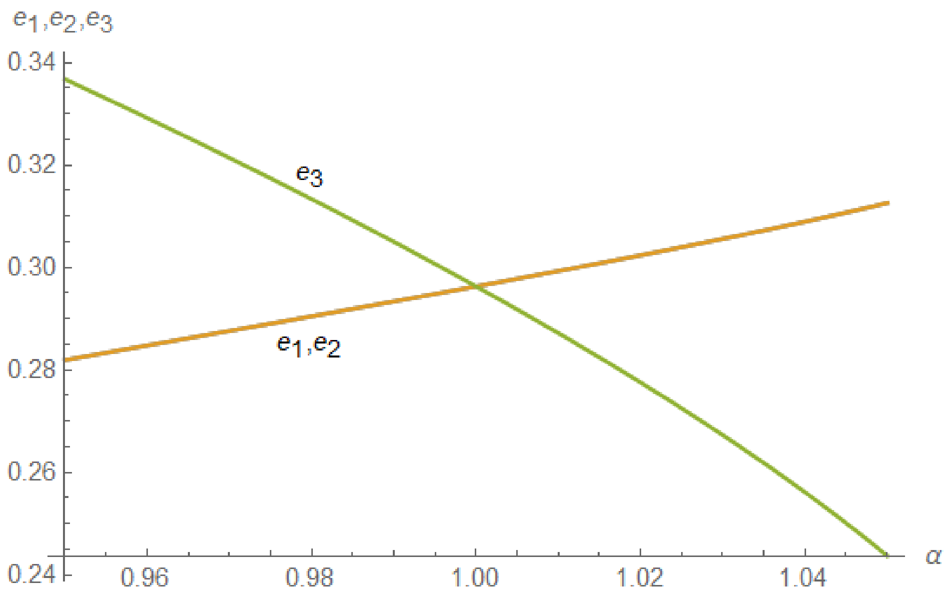

Figure 5: Example with three players and $N = 3$, $v = 1$, $\gamma_1 = \gamma_2 = \gamma_3 = 0$, $\alpha_1 = \beta_1 = n_1 = m_1 = 1$, $\alpha_2 = \beta_2 = n_2 = m_2 = 1$, and $\alpha_3 = \alpha$ and $\beta_3 = n_3 = m_3 = 1$.

On top of that, with $N = 3$ there are also equilibria with crowding out where one of the players exerts zero effort in equilibrium. In terms of crowding out players 1 or 2, if $0.646 < \alpha <$

1.194 there are two equilibria where $e_3 = \frac{6\alpha}{(1+\alpha)^4}$ and either $(e_1, e_2) = \left(\frac{6\alpha^2}{(1+\alpha)^4}, 0\right)$ or $(e_1, e_2) = \left(0, \frac{6\alpha^2}{(1+\alpha)^4}\right)$. A lower bound on α is needed as otherwise player 3's cost function is significantly lower than that of player 1 and player 1 would obtain negative profits in equilibrium. An upper bound on α is needed as otherwise player 3's cost function is significantly higher than that of players 1 and 2, and player 2 finds it profitable to enter the competition and exert a positive effort level.

In terms of crowding out player 3, for $\alpha > 0.906$ there is an equilibrium where players 1 and 2 choose $e_1 = e_2 = \frac{3}{8}$ and player 3 chooses $e_3 = 0$. Notice that there is crowding out of player 3 even in cases where this player has a lower cost function than that of players 1 and 2.

The reason why there are equilibria with crowding out, even when the player crowded out is the one with the lowest cost function is due to the pivotality effect. To see this consider a situation where player 3 is not exerting any effort and players 1 and 2 are exerting positive effort equal to the two player equilibrium, i.e. the equilibrium effort levels in a situation where there is no player 3. Player's 3 marginal profit at zero effort level is negative, the reason is that increasing effort marginally only improves payoff in cases there is a pivotal juror for player 3 (the pivotality effect), but since player 3 is currently exerting zero effort and the other two players are exerting positive effort, the probability of being in a pivotal state for player 3 is zero. Thus, a marginal increase in effort for player 3 will lead to negative profits.

With respect to a non-marginal increase in effort, since players 1 and 2 are exerting the two-player equilibrium effort levels, player 3 exerting a positive effort level will lead to it having negative profits since the cost of effort of having a significant probability of winning the contest is too high given the potential gains. Figure 6 shows the profits of player 3 as a function of its own effort level when $\alpha = 0.95$ and players 1 and 2 are exerting the equilibrium effort levels when there is crowding out of player 3. As it can be noted in the graph, player 3's profits are negative for all effort levels and, thus, its optimal choice is to stay out of the contest even though it has a lower cost function than that of players 1 and 2.

Increasing the player count past 3 or using different parameter values leads to the same observations just made for three players. We collect these in the following proposition, whose proof is via the example just shown.

Proposition 7. *With three or more players there can be equilibria multiplicity. Moreover, there can be crowding out equilibria where only two players, not necessarily those with a lower cost function, exert positive effort.*

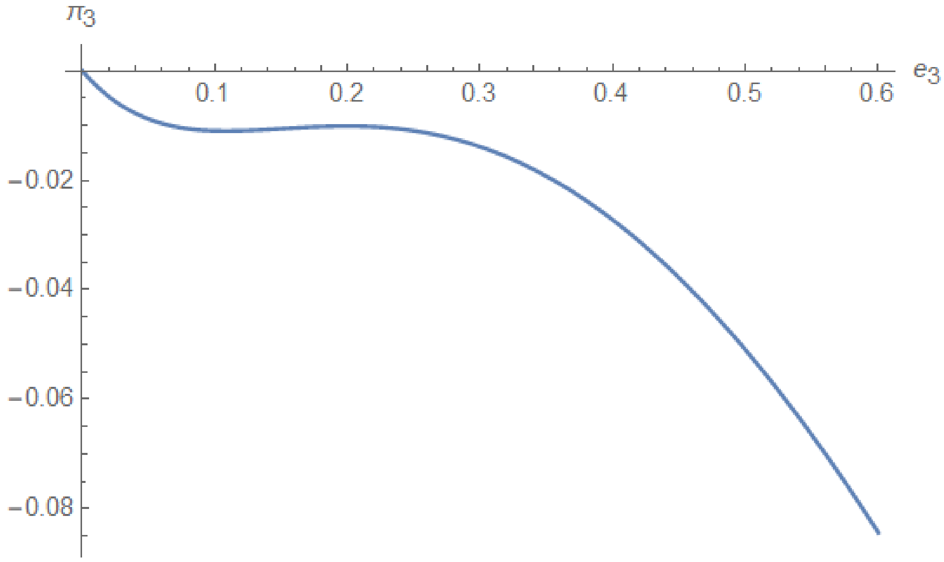

Figure 6: Profits of player 3 as a function of its own effort when $\alpha = 0.95$, $e_1 = e_2 = \frac{3}{8}$ and $N = 3$, $v = 1$, $\gamma_1 = \gamma_2 = \gamma_3 = 0$, $\alpha_1 = \beta_1 = n_1 = m_1 = 1$, $\alpha_2 = \beta_2 = n_2 = m_2 = 1$, and $\alpha_3 = \alpha$ and $\beta_3 = n_3 = m_3 = 1$.

3.9 Sequential Effort Choice

In this section we explore the case where players choose effort levels sequentially. Player 1 chooses its effort e_1 first and then, after observing e_1, player 2 chooses effort level e_2. To simplify, assume for this section a Tullock contest with $v = 1$, $\gamma_1 = \gamma_2 = 0$, $\alpha_1 = \beta_1 = n_1 = m_1 = 1$, $\beta_2 = n_2 = m_2 = 1$, and $\alpha_2 = \alpha > 0$. With these parameter values we have that $p_i = \frac{e_i}{\sum_j e_j}$, $c_1(e_1) = e_1$ and $c_2(e_2) = \alpha e_2$.

By backwards induction, the best response of player 2 given the effort level chosen by player 1 is equivalent to the simultaneous move case:

$$e_2 \quad = \quad \frac{N!}{\frac{N-1}{2}!2} p_1^{\frac{N+1}{2}} (1 - p_1)^{\frac{N+1}{2}} \frac{1}{\alpha}. \tag{7}$$

Since in equilibrium e_2 depends on the chosen level of effort of player 1, we can use implicit differentiation above to obtain the following after some algebra:

$$\frac{\mathrm{d}e_2}{\mathrm{d}e_1} \quad = \quad \frac{\frac{N+1}{2}(1 - 2p_1)\frac{1-p_1}{p_1}}{1 + \frac{N+1}{2}(1 - 2p_1)}. \tag{8}$$

The first order condition of player 1 is

$$\frac{N!}{\frac{N-1}{2}!2} p_1^{\frac{N-1}{2}} (1 - p_1)^{\frac{N-1}{2}} \left[\frac{\partial p_1}{\partial e_1} + \frac{\partial p_1}{\partial e_2} \frac{\mathrm{d}e_2}{\mathrm{d}e_1} \right] \quad = \quad 1,$$

where $\frac{de_2}{de_1}$ is given in equation (8). Substituting we obtain

$$e_1 = \frac{N!}{\frac{N-1}{2}!2} p_1^{\frac{N+1}{2}} (1-p_1)^{\frac{N+1}{2}} \left[1 - \frac{\frac{N+1}{2}(1-2p_1)}{1+\frac{N+1}{2}(1-2p_1)} \right]$$

Combining with the best response of player 2 in equation (7) we have that in equilibrium

$$\frac{1}{\alpha}\frac{p_1}{1-p_1} = \frac{1}{1+\frac{N+1}{2}(2p_1-1)}.$$

Solving the equation above gives the equilibrium value for p_1:

$$p_1 \quad = \quad \frac{1+\alpha+\frac{N+1}{2} - \sqrt{(1+\alpha+\frac{N+1}{2})^2 - 8\alpha\frac{N+1}{2}}}{4\frac{N+1}{2}},$$

the positive root leads to a minimum in the player's optimization problem and is thus ignored.

In the standard Tullock contest with only one juror, $N = 1$, we have that $p_1 = \frac{\alpha}{2}$ (for $\alpha > 2$ there is no pure strategy equilibrium). This means that the player with the lower cost function exerts more effort and has a higher probability of winning the contest, irrespective of which player goes first. That is, there is no first mover advantage. This is a known fact in the contests literature, see for example Linster (1993) or Morgan (2003).

When $N > 1$, we find that the equilibrium value of p_1 is real if and only if $\alpha \leq 1$. This means that there is no equilibrium in pure strategies for some N whenever $\alpha > 1$. The reason is that the presence of a jury when player 1 acts first and has a lower cost function leads to this player choosing a effort level such that player 2 is better off not competing against, thus best responding with $e_2 = 0$. Therefore, the only possibility for an equilibrium is mixed strategies.

If $\alpha \leq 1$ then our findings still hold. Figure 7 shows the sum of equilibrium efforts as a function of the jury size when $\alpha = 0.95$. As with simultaneous competition, for low jury sizes increasing the jury size increases the sum efforts because the uncertainty effect dominates the pivotality effect. For high jury sizes the pivotality effect dominates and equilibrium efforts decrease.

In terms of the differences between sequential competition and simultaneous competition, it is known that the comparison of the two types of competition is a complex issuer. Which type of contest leads to more aggregate effort depends non-trivially on the characteristics of the contests such as players and contest success function. For more on this see for instance Morgan (2003), Serena (2017) and Hinnosaar (2024). Figure 8 shows an example where the simultaneous contest leads to higher aggregate effort, and thus a higher rent dissipation.

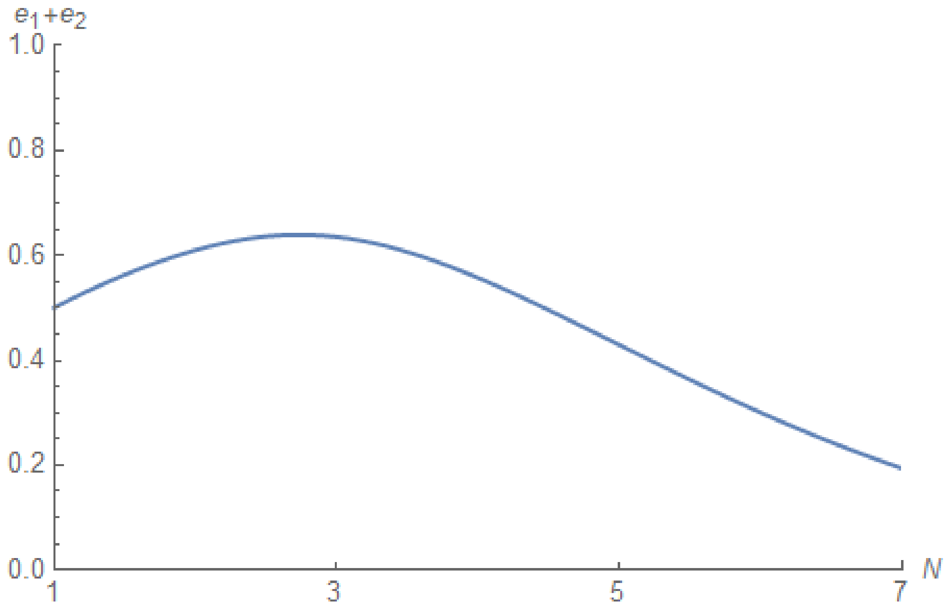

Figure 7: Sum of efforts with sequential competition, player 1 chooses first, with $\gamma_1 = \gamma_2 = 0$, $\alpha_1 = \beta_1 = n_1 = m_1 = 1$, $\beta_2 = n_2 = m_2 = 1$, and $\alpha_2 = \alpha = 0.95$. Effort maximizing jury size is $N = 3$.

4 Conclusions

We consider contests where there is a jury. A jury introduces two distinct effects on a contest: an uncertainty effect and a novel pivotality effect. The pivotality effect arises because marginal increases in effort increase marginal profit only when there is a pivotal juror. As jury size increases, the likelihood of a pivotal juror decreases, thereby decreasing the returns of effort. The uncertainty effect works in the opposite way, increasing jury size reduces uncertainty about the winner of the contest given effort levels, thus increasing returns to effort.

Among others, we have shown that there exists a finite effort maximizing jury sizes and that maximizing equilibrium effort equates to maximizing rent dissipation. We also shown that the presence of a jury can mitigate discrimination against contestants. Furthermore, we have shown that in contest where one of the candidates is better than the other, it may be beneficial to use a voting rule where the better players needs more votes than the other player to win.

Our results have implications in terms of contest design, rent dissipation and discrimination among others. Future research could explore experimentally and/or empirically the pivotality effect and, more in general, how the presence of a jury affects contest along its various dimensions.

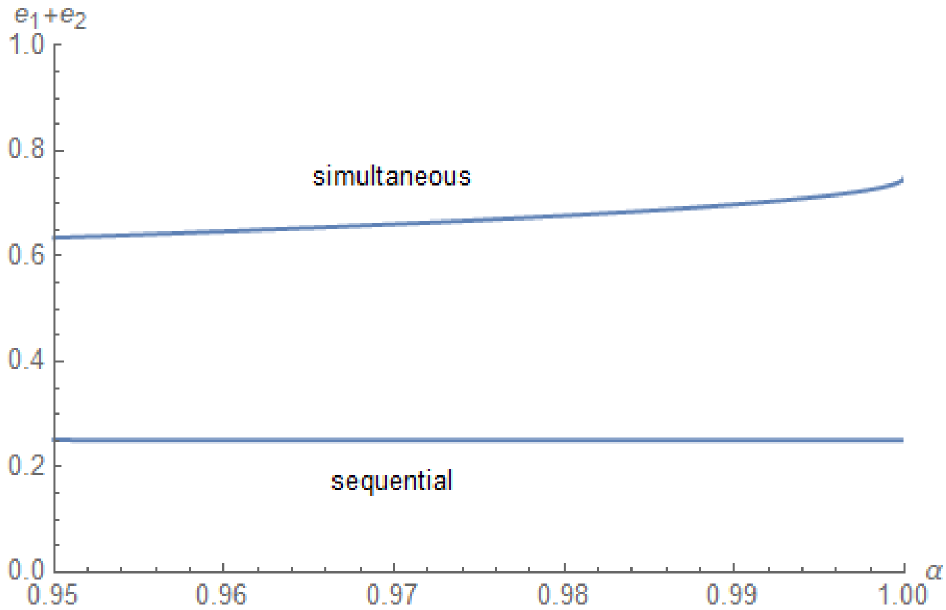

Figure 8: Sum of efforts with simultaneous and sequential competition, player 1 chooses first, with $N = 3$, $\gamma_1 = \gamma_2 = 0$, $\alpha_1 = \beta_1 = n_1 = m_1 = 1$, $\beta_2 = n_2 = m_2 = 1$, and $\alpha_2 = \alpha = 0.95$.

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Appendix

We want to show the following equality for all $N \in \{1, 2, \ldots\}$, $p \in (0, 1)$ and $q \in [0, 1]$:

$$\sum_{k=\lceil Nq\rceil}^{N} \binom{N}{k} p^k (1-p)^{N-k} \quad = \quad \frac{N!}{(\lceil Nq\rceil - 1)!(N - \lceil Nq\rceil)!} \int_0^p x^{\lceil Nq\rceil - 1}(1-x)^{N-\lceil Nq\rceil}\mathrm{d}x.$$

Define $S = \sum_{k=\lceil Nq\rceil}^{N} \binom{N}{k} p^k (1-p)^{N-k}$. On the right hand side in the equation above, set $u = (1-x)^{N-\lceil Nq\rceil}$ and $\mathrm{d}v = x^{\lceil Nq\rceil - 1}\mathrm{d}x$ and integrate by parts to obtain

$$\begin{aligned}\frac{N!}{(\lceil Nq\rceil - 1)!(N - \lceil Nq\rceil)!} \quad & \int_0^p x^{\lceil Nq\rceil - 1}(1-x)^{N-\lceil Nq\rceil}\mathrm{d}x = \\ & \frac{N!}{\lceil Nq\rceil!(N - \lceil Nq\rceil)!} p^{\lceil Nq\rceil}(1-p)^{N-\lceil Nq\rceil} + \\ & \frac{N!}{\lceil Nq\rceil!(N - \lceil Nq\rceil - 1)!} \int_0^p x^{\lceil Nq\rceil}(1-x)^{N-\lceil Nq\rceil - 1}\mathrm{d}x.\end{aligned}$$

Notice how the first term on the right hand size equals the first term in the series S. On the integral on the right hand side, integrate by parts again setting $u = (1-x)^{N-\lceil Nq\rceil - 1}$ and $\mathrm{d}v = x^{\lceil Nq\rceil}\mathrm{d}x$ to obtain

$$\begin{aligned}\frac{N!}{\lceil Nq\rceil!(N - \lceil Nq\rceil - 1)!} \quad & \int_0^p x^{\lceil Nq\rceil}(1-x)^{N-\lceil Nq\rceil - 1}\mathrm{d}x = \\ & \frac{N!}{(\lceil Nq\rceil + 1)!(N - \lceil Nq\rceil - 1)!} p^{\lceil Nq\rceil + 1}(1-p)^{N-\lceil Nq\rceil - 1} + \\ & \frac{N!}{(\lceil Nq\rceil + 1)!(N - \lceil Nq\rceil - 2)!} \int_0^p x^{\lceil Nq\rceil + 1}(1-x)^{N-\lceil Nq\rceil - 2}\mathrm{d}x.\end{aligned}$$

The first term on the right hand size equals the second term in the series S. Continue in this fashion until, after the s-th integration by parts with $0 < s < N - \lceil Nq \rceil$, we arrive at the integral

$$\frac{N!}{(\lceil Nq \rceil + s - 1)!(N - \lceil Nq \rceil - s)!} \int_0^p x^{\lceil Nq \rceil + s - 1}(1-x)^{N - \lceil Nq \rceil - s} \mathrm{d}x.$$

Integrate by parts setting $u = (1-x)^{N-\lceil Nq \rceil - s}$ and $\mathrm{d}v = x^{\lceil Nq \rceil + s - 1}\mathrm{d}x$ to obtain

$$\begin{aligned} \frac{N!}{(\lceil Nq \rceil + s - 1)!(N - \lceil Nq \rceil - s)!} \quad & \int_0^p x^{\lceil Nq \rceil + s - 1}(1-x)^{N - \lceil Nq \rceil - s} \mathrm{d}x = \\ & \frac{N!}{(\lceil Nq \rceil + s)!(N - \lceil Nq \rceil - s)!} \\ & \times p^{\lceil Nq \rceil + k}(1-p)^{N - \lceil Nq \rceil - s} \\ & + \frac{N!}{(\lceil Nq \rceil + s)!(N - \lceil Nq \rceil - s - 1)!} \\ & \times \int_0^p x^{\lceil Nq \rceil + s}(1-x)^{N - \lceil Nq \rceil - s - 1} \mathrm{d}x. \end{aligned}$$

Notice how the first term on the right hand size equals the $(s+1)$-th term in the series S. Continue in this fashion until, after the $(N - \lceil Nq \rceil)$-th integration by parts, we are left with the integral $\frac{N!}{(N-1)!} \int_0^p x^{N-1} \mathrm{d}x$, which is simply p^N, i.e. the last term in the series S.