Lawvere Theories Enriched over a General Base

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Abstract

We generalise the correspondence between Lawvere theories and finitary monads on **Set** in two ways. First, we allow our theories to be enriched in a category V that is locally finitely presentable as a symmetric monoidal closed category: symmetry is convenient but not necessary. And second, we allow the arities of our theories to be finitely presentable objects of a locally finitely presentable V-category A. We call the resulting notion that of a Lawvere A-theory. We extend the correspondence for ordinary Lawvere theories to one between Lawvere A-theories and finitary V-monads on A. We illustrate this with examples leading up to that of the Lawvere **Cat**-theory for cartesian closed categories, i.e., the **Set**-enriched theory on the category **Cat** for which the models are all small cartesian closed categories. We also briefly investigate change of base.

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1 Introduction

The relationship between Lawvere theories, equationally defined algebraic theories and finitary monads on **Set** is one of the deepest relationships in category theory. The notions of Lawvere theory and finitary monad on **Set** are equivalent; every equationally defined algebraic theory gives rise to a Lawvere theory given by the clone of the equational theory, and every Lawvere theory arises from an equationally defined algebraic theory [1]. In mathematics, the relationship yields companion approaches to universal algebra [1, 2] with its usual list of examples. And in computer science, if one makes a routine generalisation from finite sets to countable sets, almost all the monads on **Set** introduced by Moggi in [15, 16] to model computational effects arise as the Lawvere theories generated by computationally natural equationally defined algebraic theories, which is how the computational effects appear in practice [6]. The recognition of the various monads as natural Lawvere theories has led and is leading to a deeper analysis of the semantics of such effects [17, 20, 18, 19].

Lawvere theories were introduced in the early 1960's precisely because of their relationship with equationally defined algebraic theories [13]. Soon afterwards, the relationship between Lawvere theories and monads on **Set** was established [14]. The notion of monad generalises trivially to base categories other than **Set**, whereas the notions of Lawvere theory and equationally defined algebraic theory do not immediately generalise. There were ideas in the air for generalisations of the latter notions [14], and it was recognised that monads on categories other than **Set** arise from a generalised notion of algebraic structure, but a generalised formal correspondence does not seem to have been published at the time.

A generation later, after the underlying results of enriched category theory had been developed [8, 7], a precise formulation of the notion of V-enriched algebraic structure was given and a correspondence with finitary V-monads on a locally finitely presentable V-category A was proved [9, 22]. The notion was soon used in computer science, for instance in [5, 11, 12]. But algebraic structure is not an invariant of a finitary V-monad, i.e., an arbitrary finitary V-monad arises from many different algebraic presentations, and the various presentations are often equally natural: that is the case even when V is **Set** as any group-theorist could assert. And, because of a delicate inductive step, the details of examples of algebraic structure were often remarkably complicated to give in practice, much more so than for usual presentations of equationally defined algebraic structure relative to **Set**. So the lack of a generalised notion of Lawvere theory still affected researchers trying to calculate details.

Eventually, in [21], the correspondence between Lawvere theories and finitary monads was generalised to one between V-enriched Lawvere theories and finitary V-monads on the base category V subject to cocompleteness, size and coherence conditions. Taking V to be **Cat**, that allowed the study of **Cat**-enriched algebraic structure on categories in terms of Lawvere **Cat**-theories. Again making the routine generalisation from finitariness to countability, taking V to be ω**Cpo** in analysing computational effects, the notion of Lawvere

V-theory proved to be fundamental, allowing a study of recursion [6] and in particular allowing for the incorporation of partiality into the study of the various other effects [6]. Implicit in the definition of Lawvere V-theory was a simplified formulation of the notion of V-enriched algebraic structure.

In this paper, we generalise the correspondence between Lawvere theories and finitary monads a step further. We first choose a category V in which to enrich, and then we choose a base V-category A. We then define a notion that we call a Lawvere A-theory and we extend the above correspondence to one between Lawvere A-theories and finitary V-enriched monads on the V-category A. For instance, taking V to be **Set** and A to be the **Set**-enriched category, i.e., the ordinary category, **Cat**, we can consider structure on **Cat** as an ordinary category. This allows us to capture structures that we could not capture when A was identified with V as in the past. For instance, we can consider cartesian closed structure in this setting, which was impossible before because of the contravariance involved with closedness [3]. The techniques we develop here may also help with sophisticated computational effects such as probabilistic nondeterminism, in which one considers the category of dcpo's as an ω**Cpo**-enriched category [6], but that requires further investigation of size. Our definition of Lawvere A-theory implies a simplified formulation of V-enriched algebraic structure on A: we do not give a precise formulation here, but we illustrate by example that our definition allows less complicated formulation of the equations.

In regard to technique, the constructions of this paper, yielding the correspondence between Lawvere A-theories and V-monads on A, are essentially the same as in the past [21]. What is not obvious is how to define the notion of Lawvere A-theory. An ordinary Lawvere theory was defined to be a small category L with finite products and an identity-on-objects strict finite product preserving functor $J : \mathbf{Nat}^{op} \longrightarrow L$, where **Nat** is the category of natural numbers and all functions between them. Here, our definition is quite different: taking both V and A to be Set, our definition of Lawvere A-theory consists of a small category L together with an identity-on-objects functor $J : \mathbf{Nat}^{op} \longrightarrow L$ that strictly preserves finite limits. So we do not assert that L has finite products, but we do assert that J preserves all finite limits of $\mathbf{Nat}^{op}$ rather than just its finite products. It is routine to verify that one of our Lawvere theories is one of Lawvere's ones, as one can readily deduce that L inherits finite products from $\mathbf{Nat}^{op}$. But the converse requires more thought: we prove it in Section 2.

We use cartesian closed structure on categories as a leading example to illustrate the ideas of the paper. The category **Cat** is a convenient base category to illustrate the ideas as it is not itself monadic over **Set**, and thus algebraic structure over **Cat** is quite different to algebraic structure over **Set**. And cartesian closed structure is convenient because it is a familiar notion and because the category of small cartesian closed categories is monadic over **Cat** while not being 2-monadic: so V and A are different, with V being **Set** and A being **Cat**. It is merely meant to act as illustration.

We could, in principle, attempt to take the correspondence we develop here even further: one can speak of a monad *in* any 2-category, and some of the constructions of this paper extend to that level of generality [23]. But the issue of size becomes particularly awkward there, and, even not accounting for size, an appropriately generalised notion of Lawvere theory does not appear in [23] and the details of the axiomatics would make it awkward.

In Section 2, we introduce our definition of Lawvere A-theory and the V-category of models of a theory. In Section 3, we analyse the example of cartesian closed structure, where V is **Set** and A is **Cat**. In Section 4, we show, in general, how to recover a Lawvere A-theory from its V-category of models: this gives a construction of a finitary V-monad on A from a Lawvere A-theory and shows that the definition of Lawvere A-theory is invariant with respect to its V-category of models. In Section 5, we start with a finitary V-monad on A, construct a Lawvere A-theory from it, and show how to recover the V-monad. Combining this with the work of Section 4 yields the correspondence we seek between Lawvere A-theories and finitary V-monads on A. Finally, in Section 6, we consider change of base, i.e., given a monoidal functor $\Phi\colon V \to V'$, we consider the relationship between the models of a Lawvere A-theory and those of the induced Lawvere Φ-$\mathbf{Cat}(A)$-theory: this is important for examples such as those where V is **Cat** but some structure, such as finite product structure, is **Cat**-enriched, while other structure, such as closed structure, is not.

2 Lawvere A-theories and their models

In this section, we introduce the notions of Lawvere A-theory and V-category of models of a theory, and we show that ordinary Lawvere theories, more generally the enriched Lawvere theories of [21], are special cases, with the respective definitions of model agreeing. To give the definitions necessarily involves sophisticated enriched category theory: we shall do our best to keep it comprehensible, but we recommend the less category-theoretic reader focus on the examples of **Set** and **Cat**.

We assume that V is locally finitely presentable as a symmetric monoidal closed category and that A is a locally finitely presentable V-category: symmetry of V is not necessary here, but it is convenient for exposition, includes all our leading examples, and corresponds to most of the relevant literature. The precise definitions of these notions can be found in [8, 7], but in order to understand the point of this paper, one only needs to know examples that appear in the computer science literature [2, 10, 22]. **Set** and **Cat** are locally finitely presentable as symmetric monoidal closed categories. Locally finitely presentable **Set**-categories are exactly ordinary locally finitely presentable categories, such as **Set**, $\mathbf{Set}^k$, **Poset** and $\mathbf{Cat}_o$. **Cat** is a locally finitely presentable **Cat**-category, and that statement extends to V given axiomatically as above.

We write A_f for a skeleton of the full sub-V-category of A given by the finitely presentable objects of A, and we let $\iota\colon A_f \to A$ denote the inclusion V-functor. Following the canonical reference for enriched categories [8], we denote the composite V-functor

$$A \xrightarrow{Y} [A^{op}, V] \xrightarrow{[\iota^{op}, V]} [A_f^{op}, V]$$

by $\tilde{\iota}$, where Y is an enriched version of the Yoneda embedding. For example, up to coherent isomorphism, the category $\mathbf{Set}_f$ is the category $\mathbf{Nat}$, whose objects are natural numbers and whose arrows are all functions between them. The functor $\tilde{\iota}$ sends a set X to the functor $\mathbf{Set}(\iota-, X)$. For a more complex example, $\mathbf{Cat}_f$ is the category of finitely presentable categories, i.e., those categories that are freely generated on a finite graph or are given by coequalising a pair of functors between such freely generated categories.

We next need the idea of a finite cotensor. This generalises the notion of a finite power. A V-category A has *finite cotensors* if for every finitely presentable X in V and every Z in A, there exists an object Z^X of A together with a natural isomorphism

$$[X, A(-, Z)] \cong A(-, Z^X)$$

For example, in the case $V = \mathbf{Set}$, a finite cotensor means that X is finite and Z^X is a product of X copies of Z. In the case $A = V$, the cotensor Z^X is given by the exponential $[X, Z]$. We write $\mathbf{FC}(A, V)$ for the full sub-V-category of $[A, V]$ determined by those V-functors that preserve finite cotensors.

Finally, we need the notion of a *finite enriched limit.* The formal definition is complicated, so we shall not give it directly but rather use a characterisation theorem that makes the notion much easier to grasp [8]: a V-category admits all finite V-limits if and only if it admits all finite conical limits and all finite cotensors Z^X. Here, the notion of conical limit is exactly as one would expect, bearing in mind that enrichment means one wants an isomorphism in V between the object of cones over a digram and the homobject of comparison maps, rather than a mere bijection of sets [8]. We write $\mathbf{FL}(A, V)$ for the full sub-V-category of $[A, V]$ determined by those V-functors that preserve finite V-limits. The V-functor ι preserves all finite V-colimits, and representable V-functors preserve V-limits, so $\tilde{\iota}$ factors through $\mathbf{FL}(A_f^{op}, V)$. So we sometimes consider $\tilde{\iota}$ as a V-functor from A to $\mathbf{FL}(A_f^{op}, V)$. The central result of Gabriel-Ulmer duality, generalised to enriched categories, asserts that $\tilde{\iota}$ induces an equivalence $A \simeq \mathbf{FL}(A_f^{op}, V)$ of V-categories [7]. Since $\mathbf{FL}(A_f^{op}, V)$ is a full sub-V-category of $\mathbf{FC}(A_f^{op}, V)$, we also sometimes consider $\tilde{\iota}$ as a V-functor from A to $\mathbf{FC}(A_f^{op}, V)$.

Finally, we can write the central definition of the paper. We assume V and A satisfy the axiomatic structure described above, i.e., A is a locally finitely presentable V-category for appropriate V.

Definition 2.1 *A* Lawvere A-theory *is a small V-category L together with an identity-on-objects strict finite V-limit preserving V-functor $J\colon A_f^{op} \to L$.*

So the objects of L are *exactly* the objects of A_f^{op}. One understands them in this setting to be generalised *arities*, and one understands the arrows of L to be operations. This should become clearer when we study examples. But to see the distinction between preservation of limits and preservation of cotensors in our definition, consider the example of $V = Set$ and $A = Cat$, and note that the triangle category is a pushout in Cat_f constructed from two copies of the arrow category together with the unit category 1.

A map of Lawvere A-theories from L to L' is an identity-on-objects V-functor from L to L' that commutes with the V-functors from A_f^{op}. Together with the usual composition of V-functors, Lawvere A-theories and their maps yield an ordinary category we denote by $\mathbf{Law}_A$.

Definition 2.2 *Given a Lawvere A-theory L with $J\colon A_f^{op} \to L$, define its V-category of models by the following pullback in the category V-$\mathbf{Cat}$ of locally small V-categories.*

$$\begin{tikzcd} \mathbf{Mod}(L) \arrow[r, "P_L"] \arrow[d, "U_L"'] \arrow[dr, phantom, "\lrcorner", very near start] & {[L,V]} \arrow[d, "{[J,V]}"] \\ A \arrow[r, "\tilde{\iota}"'] & {[A_f^{op},V]} \end{tikzcd}$$

We call objects of $\mathbf{Mod}(L)$ models *of L.*

So a model consists of an object X of A together with a functor $M : L \longrightarrow V$ whose behaviour when restricted to A_f^{op} is completely determined by A. Thus a model is determined by X together with data and axioms arising from those maps in L that are not already in A_f^{op}.

There is a subtle 2-categorical point here that is particularly convenient for us. The pullback defining $\mathbf{Mod}(L)$ is unusual in that it is also a *bipullback* [3], meaning that if one systematically replaces equality of diagrams in V-$\mathbf{Cat}$ by coherent isomorphism, this pullback still satisfies the systematically weakened version of the universal property. That can readily be checked directly, but axiomatically, it holds because the V-functor $[J, V]$ satisfies an isomorphism lifting property. We shall henceforth largely gloss over this point for the sake of exposition.

It will be easier to explain examples and to characterise the definition in special cases if we first give an alternative definition of the V-category of models as provided by the following proposition.

Proposition 2.3 *For any Lawvere A-theory L with $J\colon A_f^{op} \to L$, the following diagram forms a pullback in V-$\mathbf{Cat}$.*

$$\begin{tikzcd} \mathbf{Mod}(L) \arrow[r] \arrow[d, "U_L"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{FC}(L,V) \arrow[d, "{\mathbf{FC}(J,V)}"] \\ A \arrow[r, "\tilde{\iota}"'] & \mathbf{FC}(A_f^{op},V) \end{tikzcd}$$

Proof First observe that L has finite cotensors: J is the identity on objects, so every object of L lies uniquely in the image of J; moreover, J strictly preserves finite cotensors, hence the result. Now note that the square

$$
\begin{tikzcd}
\mathbf{FC}(L,V) \arrow[r, "\text{inclusion}"] \arrow[d, "\mathbf{FC}(J,V)"'] \arrow[dr, phantom, "\lrcorner", near start] & {[L,V]} \arrow[d, "{[J,V]}"] \\
\mathbf{FC}(A_f^{op},V) \arrow[r, "\text{inclusion}"'] & {[A_f^{op},V]}
\end{tikzcd}
$$

is a pullback: if M is a V-functor from L to V such that $M \circ J$ preserves finite V-cotensors, it follows from the above construction of pullbacks in L that M preserves them. The lemma now follows from the definition of $\mathbf{Mod}(L)$ and generalities about pullbacks.

We now compare our definitions with those already in the literature. An ordinary Lawvere theory [1] is usually defined to be a small category L with finite products together with an identity-on-objects strict finite product preserving functor from $\mathbf{Nat}^{op}$ to L. A model in $\mathbf{Set}$ is defined to be a finite product preserving functor from L to $\mathbf{Set}$. Note that one assumes that L has finite products and that the functor from $\mathbf{Nat}^{op}$ to L strictly preserves finite products, whereas in our general definition, we asked for strict preservation of finite limits but made no further assumption of existence of any kind of limits in L.

Theorem 2.4 *An ordinary Lawvere theory is a Lawvere* $\mathbf{Set}$*-theory and conversely. Moreover, the two definitions of the category of models agree.*

Proof Let L be any ordinary Lawvere theory. It corresponds to a finitary monad T. Moreover, L is isomorphic to the restriction of $\mathbf{Kl}(T)^{op}$ to the natural numbers, and the functor $J\colon \mathbf{Nat}^{op} \to L$ is given by the restriction of the canonical functor $\mathbf{Set} \longrightarrow \mathbf{Kl}(T)$. So $J\colon \mathbf{Nat}^{op} \to L$ strictly preserves all finite limits of $\mathbf{Nat}$, as the corresponding finite colimits are strictly preserved both by the inclusion into $\mathbf{Set}$ and by the canonical functor into $\mathbf{Kl}(T)$. So every ordinary Lawvere theory is a Lawvere $\mathbf{Set}$-theory in the above sense. The converse is trivially true. For the statement about models, first observe that $\mathbf{Set}_f^{op}$ is the free $\mathbf{Set}$-category with finite cotensors, i.e., finite powers, on 1. So $\tilde{\iota}$ yields a canonical equivalence $\mathbf{Set} \simeq \mathbf{FC}(\mathbf{Set}_f^{op}, \mathbf{Set})$. So Proposition 2.3 implies $\mathbf{Mod}(L) \simeq \mathbf{FC}(L, \mathbf{Set})$. But all finite products on $\mathbf{Set}^{op}$, hence also on L, are given by finite powers of copies of 1, i.e., by finite cotensors, and so preservation of finite powers is equivalent, in this setting, to preservation of finite products, hence the result.

Enriching this result, in [21], given V satisfying the axioms we have here, a Lawvere V-theory was defined to be a small V-category L with finite V-cotensors together with an identity-on-objects strict finite V-cotensor preserving V-functor $J\colon V_f^{op} \to L$. The V-category of models of such a Lawvere V-theory was defined to be $\mathbf{FC}(L,V)$.

Theorem 2.5 *If A is V, Lawvere A-theories are precisely Lawvere V-theories defined as above. Moreover, the two definitions of the V-category of models agree.*

Proof The proof of the correspondence is given by a simple enrichment of the proof of Theorem 2.4. Similarly for the statement about models.

3 Examples

In this section, we give three examples of Lawvere A-theories, developing our leading example of the Lawvere **Cat**-theory for cartesian closed categories. Our first two examples, those of categories with a terminal object and categories with binary products, may be seen as examples of the enriched Lawvere theories of [21] as both V and A are **Cat**. But our final example, that of cartesian closed structure for categories, is genuinely new in that, although A is **Cat**, this example is not **Cat**-enriched but is only **Set**-enriched owing to the contravariance inherent in the notion of closedness [3].

3.1 Categories with a terminal object

Let 0 denote the empty category. Let 1 denote the trivial one object category. Let **2** denote the category $\{d \to c\}$. And let Δ denote the diagonal functor.

Put $A = V = \mathbf{Cat}$, and let L be the **Cat**-category with finite **Cat**-cotensors freely generated by adding arrows $\tau\colon 0 \to 1$ and $\sigma\colon 1 \to \mathbf{2}$ to $\mathbf{Cat}_f^{op}$ subject to commutativity of the following diagrams:

$$\begin{tikzcd} 1 \arrow[r, "\sigma"] \arrow[dr, "id"'] & \mathbf{2} \arrow[d, "d^{op}"] \\ & 1 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "\sigma"] \arrow[d, "!_0^{op}"'] & \mathbf{2} \arrow[d, "c^{op}"] \\ 0 \arrow[r, "\tau"'] & 1 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r, "\tau"] \arrow[d, "\tau"'] & 1 \arrow[d, "\sigma"] \\ 1 \arrow[r, "!_{\mathbf{2}}^{op}"'] & \mathbf{2} \end{tikzcd}$$

This is the Lawvere **Cat**-theory for a category with an assigned terminal object, i.e., the category of models of this Lawvere **Cat**-theory is equivalent to the 2-category of small categories with an assigned terminal object.

By Theorem 2.5, to give a model M of L is equivalent to giving a finite **Cat**-cotensor preserving **Cat**-functor from L to **Cat**. So, for any model M, the following diagrams must commute in **Cat**:

$$\begin{tikzcd} M1 \arrow[r, "M\sigma"] \arrow[dr, "id"'] & (M1)^{\mathbf{2}} \arrow[d, "\mathrm{dom}"] \\ & M1 \end{tikzcd} \qquad \begin{tikzcd} M1 \arrow[r, "M\sigma"] \arrow[d, "!_{M1}"'] & (M1)^{\mathbf{2}} \arrow[d, "\mathrm{cod}"] \\ 1 \arrow[r, "M\tau"'] & M1 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "M\tau"] \arrow[d, "M\tau"'] & M1 \arrow[d, "M\sigma"] \\ M1 \arrow[r, "\Delta"'] & (M1)^{\mathbf{2}} \end{tikzcd}$$

So the category $M1$ has an object t determined by $M\tau$. The first two diagrams assert that $M\sigma$ sends an object x of $M1$ to an arrow from x to t. The third diagram asserts that $M\sigma$ sends the object t to the identity map on t. From this together with functoriality of $M\sigma$ and *cod*, one can deduce uniqueness of the map from an arbitrary object x into t [21].

For the converse construction, given a category C with a terminal object t, let M from L to **Cat** send 1 to C and, more generally, send 1^X to C^X; let $M\tau$ choose t, and for any object x of C, let $M\sigma$ send x to the unique arrow from x to t. These constructions make the diagrams commute and respect finite cotensors, so, by construction of L and by definition of a model, determine a model. It is routine to verify that the two constructions are mutually inverse.

3.2 Categories with binary products

Let 2 denote the discrete category on two objects a and b. Let **Cone** denote the category given by 2 together with a cone π over it. Let $a \times b$ denote the vertex. Let **DoubleCone** denote the category given by **Cone** together with a cone μ over it. Let m denote the vertex. Mildly overloading notation, let μ: **Cone** $\to$ **DoubleCone** send π_a and π_b to μ_a and μ_b, respectively. Similarly, for any arrow $f\colon x \to y$ in C, let $f\colon \mathbf{2} \to C$ send d, c and the arrow $d \to c$ to x, y and f respectively. For example, $\mu_{a\times b}\colon \mathbf{2} \to$ **DoubleCone** sends d, c and the arrow $d \to c$ to m, $a \times b$ and $\mu_{a\times b}$ respectively. Let *inc* denote the inclusion of 2 into **Cone** and of **Cone** into **DoubleCone**.

Put $A = V =$ **Cat**, and let L be freely generated by adding arrows β: 2 $\to$ **Cone** and α: **Cone** $\to$ **DoubleCone** to $\mathbf{Cat}_f^{op}$ and by insisting that the following diagrams commute:

$$\begin{tikzcd} 2 \arrow[r, "\beta"] \arrow[dr, "id"'] & \mathbf{Cone} \arrow[d, "inc^{op}"] \\ & 2 \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Cone} \arrow[r, "\alpha"] \arrow[dr, "id"'] & \mathbf{DoubleCone} \arrow[d, "\mu^{op}"] \\ & \mathbf{Cone} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Cone} \arrow[r, "\alpha"] \arrow[d, "inc^{op}"'] & \mathbf{DoubleCone} \arrow[d, "inc^{op}"] \\ 2 \arrow[r, "\beta"'] & \mathbf{Cone} \end{tikzcd}$$

$$\begin{tikzcd} 2 \arrow[r, "\beta"] & \mathbf{Cone} \arrow[r, "\alpha"] & \mathbf{DoubleCone} \arrow[r, shift left, "\mu_{a\times b}^{op}"] \arrow[r, shift right, "\Delta(a\times b)^{op}"'] & \mathbf{2} \end{tikzcd}$$

By essentially the same argument as in Subsection 3.1, this is the Lawvere **Cat**-theory for a category with binary products.

3.3 Cartesian closed categories

For our final example of a Lawvere A-theory, consider cartesian closed categories. The category of small cartesian closed categories is not given by the case of $A = V = \mathbf{Cat}$, which is covered in [21], owing to the contravariance necessarily involved with closedness [3]. But it is still an example for us, taking V to be **Set** and A to be **Cat**. In principle, one way to see that is by applying Corollary 5.2 to the example of cartesian closed structure in [3]. But the spirit of this paper is to see such structure directly as a model of a generalised Lawvere theory. So we shall outline what is required here, leaving most of the syntactic detail to the enthusiastic reader.

By Subsections 3.1 and 3.2 and using the work on change-of-base in Section 6, one obtains the Lawvere $\mathbf{Cat}_o$-theory for a category with finite products. We now seek to add closedness to that. It is not obvious that one can do that. For each pair of objects (x, y) of a category with finite products C, we need an object $[x, y]$ and a unit map $\eta : y \longrightarrow [x, y \times x]$. That is no problem, similar to the data in Subsections 3.1 and 3.2. But then one needs to assert that for each arrow of the form $x \times y \longrightarrow z$, one obtains a Currying. But that is a problem: the structure of a Lawvere $\mathbf{Cat}_0$-theory only allows us to start with an *arbitrary* arrow, not one with a domain of a particular form.

The way to resolve that, cf [9], is by describing closed structure less directly: given an object x, one asks for an endofunctor $[x, -]$ on C, then one asks for a unit and a counit that makes $[x, -]$ a right adjoint to $- \times x$, then one imposes naturality axioms and the triangle equations. To describe all that in detail is lengthy albeit routine, with each piece of data requiring analysis similar to that in Subsection 3.2, with the added complexity here of needing to assert functoriality explicitly.

For instance, we introduce an arrow $[-, -]_{ob}$: $1 + 1 \rightarrow 1$ and an arrow $[-, -]_{ar}$: $1 + \mathbf{2} \rightarrow \mathbf{2}$ to represent the object and arrow parts respectively of the functor $[x, -]$ for each object x. The following diagrams represent the condition that the domain object and the codomain object determined by $[x, -]_{ar}$ are as expected:

$$\begin{tikzcd}
1+\mathbf{2} \arrow[r, "{[-,-]_{ar}}"] \arrow[d, "(id+d)^{op}"'] & \mathbf{2} \arrow[d, "d^{op}"] & & 1+\mathbf{2} \arrow[r, "{[-,-]_{ar}}"] \arrow[d, "(id+c)^{op}"'] & \mathbf{2} \arrow[d, "c^{op}"] \\
1+1 \arrow[r, "{[-,-]_{ob}}"'] & 1 & & 1+1 \arrow[r, "{[-,-]_{ob}}"'] & 1
\end{tikzcd}$$

It follows from Definition 2.2 that for any model M there exists a $C \in \mathbf{Cat}_o$ such that $M \circ J = \mathbf{Cat}_o(\iota -, C)$. So the first diagram yields the following diagram in **Set**:

$$\begin{tikzcd}
\mathrm{ob}(C) \times \mathbf{Cat}_o(\mathbf{2}, C) \arrow[r, "{M[-,-]_{ar}}"] \arrow[d, "id \times \mathrm{dom}"] & \mathbf{Cat}_o(\mathbf{2}, C) \arrow[d, "\mathrm{dom}"] \\
\mathrm{ob}(C) \times \mathrm{ob}(C) \arrow[r, "{M[-,-]_{ob}}"'] & \mathrm{ob}(C)
\end{tikzcd}$$

The second diagram is dual.

We relegate the rest of the operations and diagrams to the enthusiastic reader: in principle, they are not difficult, following the above explanation; but the details are lengthy and require concentration. The cognoscenti may observe that the details are simpler than those generated by the algebraic structure of [9] as we do not require the delicate and complicated induction needed there.

4 Invariance of Lawvere A-theories

In this section, given any Lawvere A-theory, we prove that the forgetful V-functor $U_L : \mathbf{Mod}(L) \longrightarrow A$ is finitarily V-monadic, yielding a finitary V-monad T_L on A. We further show how one can reconstruct L from T_L.

First observe that for any Lawvere A-theory L, since A is locally finitely presentable, so equivalent to $\mathbf{FL}(A_f^{op}, V)$, and since representables preserve finite limits as does J, there is a canonical V-functor J' such that the following square commutes up to isomorphism:

$$\begin{tikzcd}
L^{op} \arrow[rr, "Y"] \arrow[d, dotted, "J'"'] & & {[L, V]} \arrow[d, "{[J, V]}"] \\
A \arrow[r, "\simeq"'] & \mathbf{FL}(A_f^{op}, V) \arrow[r, "\text{inclusion}"'] & {[A_f^{op}, V]}
\end{tikzcd}$$

One can make a slightly stronger statement: if one is willing to replace Y by a V-functor that is isomorphic to it, one can force the diagram actually to commute; although a minor point, that is convenient for us.

Applying the universal property determines a V-functor J'' as follows:

$$\begin{tikzcd}
L^{op} \arrow[dr, dotted, "J''"] \arrow[drr, bend left, "Y"] \arrow[ddr, "J'"'] & & \\
& \mathbf{Mod}(L) \arrow[r, "P_L"] \arrow[d, "U_L"] \arrow[dr, phantom, "\lrcorner", very near start] & {[L, V]} \arrow[d, "{[J, V]}"] \\
& A \arrow[r, "\tilde{\iota}"'] & {[A_f^{op}, V]}
\end{tikzcd}$$

Since $\tilde{\iota}$ is fully faithful, so is P_L, and, since Y is also fully faithful, so is J''.

Proposition 4.1 *For any Lawvere A-theory L and for any objects X of A_f and M of $\mathbf{Mod}(L)$, $\mathbf{Mod}(L)(J''J^{op}X, M)$ and $A(\iota X, U_L M)$ are canonically V-naturally isomorphic in V.*

Proof By fully faithfulness of P_L, and by the enriched Yoneda lemma, with I the unit of V and since $L(JX, -) = P_L J'' J^{op} X$, and finally as $(P_L M)JX = ([J,V]P_L M)X = (\tilde{\iota} U_L M)X = A(\iota X, U_L M)$, we have the following string of V-natural correspondences:

$$\begin{array}{c}
J''J^{op}X \longrightarrow M \\
\hline
P_L J''J^{op}X \longrightarrow P_L M \\
\hline
I \longrightarrow (P_L M)JX \\
\hline
\iota X \longrightarrow U_L M
\end{array}$$

Recall that U_L is defined by a pullback in V-$\mathbf{Cat}$. So its defining diagram commutes exactly rather than just up to coherent isomorphism. That strictness is convenient, but we need care in order to maintain it. So, in the following, when we speak of a left Kan extension along a fully faithful inclusion V-functor, we shall assume that it is chosen to make the induced triangle commute exactly: a Kan extension along a fully faithful V-functor always makes the triangle commute up to coherent isomorphism [8], and when that V-functor is an inclusion, we can choose the Kan extension to make the triangle commute exactly.

Theorem 4.2 *U_L has a left V-adjoint given by the left Kan extension of $J'' \circ J^{op}$ along ι.*

Proof Let F_L be the left Kan extension of $J'' \circ J^{op}$ along ι. It has a right adjoint H that sends a model M to $\mathbf{Mod}(L)(J''J^{op}-, M)$. By Proposition 4.1, $HM \cong \mathbf{Mod}(L)(J''J^{op}-, M) \cong A(\iota -, U_L M) \cong U_L M$

Theorem 4.3 *U_L is finitary V-monadic.*

Proof By Theorem 4.2, U_L has a left V-adjoint. Let f, g be a U_L-split coequaliser pair in $\mathbf{Mod}(L)$. Since $[L, V]$ is cocomplete, $P_L f$ and $P_L g$ have a coequaliser, and the coequaliser can be chosen so that it is strictly preserved by $[J, V]$. Since a split coequaliser of $U_L f$ and $U_L g$ is also preserved by $\tilde{\iota}$, f and g have a coequaliser in $\mathbf{Mod}(L)$ and U_L strictly preserves it. So by Beck's monadicity theorem [1] and by remarks on enrichment of monadicity in [9], U_L is V-monadic. Finitariness of U_L follows from that of $[J, V]$ and $\tilde{\iota}$.

We define T_L to be the finitary V-monad induced by a Lawvere A-theory L by Theorem 4.3. By the next corollary, we can reconstruct L from the monadic V-functor U_L.

Corollary 4.4 *One rediscovers (L^{op}, J^{op}, J'') as the (identity-on-objects,fully faithful) factorisation of $F_L \circ \iota$.*

$$\begin{tikzcd} L^{op} \arrow[r, "J''"] & \mathbf{Mod}(L) \\ A_f \arrow[u, "J^{op}"] \arrow[r, "\iota"'] & A \arrow[u, "F_L"'] \end{tikzcd}$$

Proof The diagram commutes by the construction of F_L in Theorem 4.2. Moreover, J^{op} is identity-on-objects and J'' is fully faithful.

5 Lawvere A-theories and finitary V-monads

In this section, we give an equivalence between the category of Lawvere A-theories and that of finitary V-monads on A. We first construct a Lawvere A-theory L_T from an arbitrary finitary V-monad T on A. We then show that the construction of Section 4 allows us to reconstruct T from L_T. Finally, we observe that the two constructions extend to an equivalence between the category of Lawvere A-theories and that of finitary V-monads on A.

For a finitary V-monad T on A, define (K_T, J_T, ι_T) by taking the (identity-on-objects,fully faithful) factorisation of $F^T \circ \iota$:

$$\begin{tikzcd} K_T \arrow[r, "\iota_T"] & T\text{-}\mathbf{Alg} \\ A_f \arrow[u, "J_T"] \arrow[r, "\iota"'] & A \arrow[u, "F^T"'] \end{tikzcd}$$

Since ι and F^T preserve finite V-colimits and ι_T reflects finite V-colimits, J_T is an identity-on-objects strict finite V-colimit preserving V-functor. So we define L_T to be K_T^{op}.

Note the similarity between this definition and Corollary 4.4. Also observe that, letting F_T be the canonical left V-adjoint from A to the Kleisli V-category $\mathbf{Kl}(T)$, we could equally have defined (K_T, J_T, ι_T) by taking the (identity-on-objects,fully faithful) factorisation of $F_T \circ \iota$:

$$\begin{tikzcd} K_T \arrow[r, "\iota_T"] & \mathbf{Kl}(T) \\ A_f \arrow[u, "J_T"] \arrow[r, "\iota"'] & A \arrow[u, "F_T"'] \end{tikzcd}$$

This formulation agrees more closely with Theorem 2.4 but would make for slightly greater complication in our ongoing exposition.

Theorem 5.1 *For a finitary V-monad T on A, let $F^T \dashv G^T$ be the canonical V-adjunction between the Eilenberg-Moore V-category T-**Alg** and A, and let Q^T send a T-algebra α to T-$\mathbf{Alg}(\iota_T -, \alpha)$. Then, if we allow Q^T to be replaced by a canonically isomorphic functor, the following square yields a pullback:*

$$\begin{tikzcd}
T\text{-}\mathbf{Alg} \arrow[r, "Q^T"] \arrow[d, "G^T"'] \arrow[dr, phantom, "\lrcorner"] & {[L_T, V]} \arrow[d, "{[J_T^{op}, V]}"] \\
A \arrow[r, "\tilde{\iota}"'] & {[A_f^{op}, V]}
\end{tikzcd}$$

Proof Since $\iota_T \circ J_T = F^T \circ \iota$ and we have a V-adjunction T-$\mathbf{Alg}(F^T\iota-, -) \cong A(\iota-, G^T-)$, the square commutes up to isomorphism. As the V-functor $[J_T^{op}, V]$ satisfies the isomorphism lifting property, Q^T is isomorphic to a V-functor that makes the diagram commute exactly.

Now let $a \in A$ and $M\colon L_T \to V$ satisfy $A(\iota-, a) \cong MJ_T^{op}$. Using the isomorphism, the funtoriality data of M yields maps

$$A(\iota m, T\iota n) \longrightarrow [A(\iota n, a), A(\iota m, a)]$$

V-natural in m and n. By V-naturality in m and by density of A_f in A, these are equivalent to maps

$$A(\iota n, a) \longrightarrow A(T\iota n, a)$$

V-natural in n, which in turn correspond to the components of a map of the form

$$\int^{n \in A_f} A(\iota n, a) \otimes T\iota n \longrightarrow a$$

with the V-naturality corresponding to the property of being a cocone. So, as $Ta = \int^{n\in A_f} A(\iota n, a) \otimes T\iota n$, the functoriality data of M yields a map $\alpha\colon Ta \to a$, cf [8, 21]. It is a T-algebra and satisfies $G^T\alpha = a$. It is routine to verify that α is the unique T-algebra such that $Q^T\alpha$ is canonically isomorphic to T-$\mathbf{Alg}(\iota_T -, \alpha)$. Conjugating with respect to isomorphisms of V-functors, one can obtain strict commutavity. Functoriality is routine.

We remark that this theorem yields an alternative proof of the fact that the V-category of algebras for a finitary V-monad on a locally finitely presentable V-category is itself locally finitely presentable. For the fully faithfulness of Q^T shows that K_T is dense in T-**Alg**, with the objects of K_T all finitely presentable in A and hence in T-**Alg**. As T-**Alg** is also V-cocomplete, it is locally finitely presentable.

Corollary 5.2 *The construction of T_L from an arbitrary Lawvere V-theory L and that of L from an arbitrary finitary V-monad T on A extend canonically to an equivalence of categories $\mathbf{Law}_A \simeq \mathbf{Mnd}_f(A)$. Moreover, the V-categories $\mathbf{Mod}(L)$ and T_L-**Alg** are canonically isomorphic.*

Proof By Theorem 5.1, $T \cong T_{L_T}$ for an arbitrary finitary V-monad T on A. Conversely, given an arbitrary Lawvere A-theory L, the Lawvere A-theory L_{T_L} is defined to be the (identity-on-objects,fully faithful) factorisation of $F^{T_L} \circ \iota\colon A_f \to T_L$-**Alg**. By Corollary 4.4 and since $\mathbf{Mod}(L) \cong T_L$-**Alg**, this factorisation agrees with L, and so L_{T_L} is isomorphic to L. The two constructions routinely extend to an equivalence of categories.

The final line of Corollary 5.2 is delicate. Although there exists a canonical isomorphism as stated, it is not true that, taking V and A to be Set, one has an isomorphism between the usual category of models of a Lawvere theory and the category of algebras for the corresponding monad. That lack of an isomorphism is consistent with our result because our category of models is only equivalent, rather than ismorphic, to Lawvere's category.

6 Change-of-base

In this section, we briefly discuss change of base category V in which to enrich. Recall that that is central to analysis of our leading example, that of cartesian closed structure, in Section 3. Changing V affects V-categories A, Lawvere A-theories, and models. We first show that applying the forgetful **Set**-functor $V(I, -)\colon V \to \mathbf{Set}$ respects the definitions of Lawvere A-theory L and $\mathbf{Mod}(L)$. We then extend the analysis to any finitary symmetric monoidal closed adjunction.

Theorem 6.1 *For any Lawvere A-theory L with $J\colon A_f^{op} \to L$, the data $J_o\colon (A_f^{op})_o \to L_o$ forms a Lawvere A_o-theory, for which there is a canonical isomorphism* $\mathbf{Mod}(L)_o \cong \mathbf{Mod}(L_o)$.

Proof For any finitary V-monad T on A, the underlying ordinary category T-$\mathbf{Alg}_o$ of the V-category T-**Alg** is isomorphic to the ordinary category T_o-**Alg** determined by the ordinary monad T_o on A_o [9]. It follows from the definition that T is finitary if and only if T_o is. So, by Corollary 5.2, we have $\mathbf{Mod}(L_T)_o \cong T$-$\mathbf{Alg}_o \cong T_o$-$\mathbf{Alg} \cong \mathbf{Mod}(L_{T_o})$.

Since A is locally finitely presentable, $(A_f)_o \cong (A_o)_f$ [7]. Moreover, the underlying ordinary functor of a fully faithful V-functor is necessarily fully faithful. So the following diagram agrees with the (identity-on-objects,fully faithful) factorisation that defines $(L_{T_0})^{op}$:

$$\begin{tikzcd}
(L_T^{op})_o \arrow[r, "(\iota_T)_o"] & T\text{-}\mathbf{Alg}_o \\
(A_f)_o \arrow[u, "(J_T)_o"] \arrow[r, "\iota_o"'] & A_o \arrow[u, "(F^T)_o"']
\end{tikzcd}$$

So $(L_T)_o \cong L_{T_o}$.

Now let $V = (V_o, \otimes, I, \alpha, \lambda, \rho, \gamma)$ and $V' = (V'_o, \otimes', I', \alpha', \lambda', \rho', \gamma')$ be locally finitary presentable as symmetric monoidal closed categories and assume that $\Psi \dashv \Phi \colon V \to V'$ is a finitary symmetric monoidal closed adjunction [4, 7].

Example 6.2 *The forgetful* $\mathbf{Set}$*-functor* $V(I, -)\colon V \to \mathbf{Set}$ *generates a finitary symmetric monoidal closed adjunction.*

We may define a 2-functor $\mathbf{\Phi\text{-}Cat}\colon V\text{-}\mathbf{Cat} \to V'\text{-}\mathbf{Cat}$ as follows. Let L be a V-category whose composition and identities are given by $c_L(x,y,z)\colon L(y,z) \otimes L(x,y) \to L(x,z)$ and $i_L(x)\colon I \to L(x,x)$ for each $x,y,z \in \operatorname{ob} L$. Then, $\mathbf{\Phi\text{-}Cat}(L)$ is the V'-category whose objects, hom, composition and identities are given by $\operatorname{ob} L$, $\Phi(L(x,y))$, $\Phi c_L(x,y,z) \circ \phi_2(x,y,z)$ and $\Phi i_L(x) \circ \phi_0$ where $\phi_2(x,y,z)\colon \Phi(L(y,z)) \otimes' \Phi(L(x,y)) \to \Phi(L(y,z) \otimes L(x,y))$ and $\phi_0\colon I' \to \Phi I$ are given canonically by the monoidal functor Φ [8]. Our final result is essentially equivalent to Theorem 6.1.

Corollary 6.3 *For any Lawvere A-theory L with $J\colon A_f^{op} \to L$, the data* Φ-$\mathbf{Cat}(L)$ *and* Φ-$\mathbf{Cat}(J)$ *form a Lawvere* Φ-$\mathbf{Cat}(A)$*-theory for which there is a canonical isomorphism* $\Phi\text{-}\mathbf{Cat}(\mathbf{Mod}(L)) \cong \mathbf{Mod}(\Phi\text{-}\mathbf{Cat}(L))$.

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