Gradient Flows in Asymmetric Metric Spaces

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Abstract

This article is concerned with gradient flows in asymmetric metric spaces, that is, spaces with a topology induced by an asymmetric metric. Such asymmetry appears naturally in many applications, e.g., in mathematical models for materials with hysteresis. A framework of asymmetric gradient flows is established under the assumption that the metric is weakly lower semicontinuous in the second argument (and not necessarily on the first), and an existence theorem for gradient flows defined on an asymmetric metric space is given.

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1 Introduction

The traditional definition of gradient flows in a Hilbert space has been recently extended to metric spaces [1]. However, for some applications, the symmetry of a metric is too restrictive an assumption. For example, there are various

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problems in continuum mechanics where asymmetric metrics occur quite naturally. For this reason, we develop in this article a framework for gradient flows in asymmetric metric spaces.

Recently, Rossi, Mielke and Savaré [2] gave a general existence theorem for a class of doubly nonlinear evolution equations, where the metric can be asymmetric. In particular, gradient flows are covered. The result we present here is much more special in the sense that it is restricted to gradient flows. However, our focus is on weakening the assumptions on the metric. Specifically, it is assumed in [2] that the metric is weakly lower-semicontinuous in both arguments, and various examples are given where this assumption is appropriate. However, if one thinks of a time-discretisation of a gradient flow, then it seems natural to require lower semicontinuity in the second argument of the metric, but not necessarily in the first one. This is the situation we study in this article. An example of such a metric, in a setting inspired by asymmetric gradient flows, can be found in [3]. A further potential class of applications are models with time independent energies that can capture hysteretic effects. One example is due to Abeyaratne, Chu and James [4]. They consider the kinetics of transitions between two martensitic variants in a material where the evolution of the volume fraction of one of the variants is governed by a gradient flow; the energy in this model is time-independent, but has many small-scale wiggles, which lead to hysteresis. If one considers the full multivariant system including the austenitic phase, then it may be important to consider an asymmetric metric, as pointed out by [5]. This is the situation we consider here.

We show the existence of asymmetric gradient flows if the metric is weakly lower semicontinuous in the second argument and an additional asymmetric topological condition is satisfied (Theorem 4.21). The key ingredient of our proof is a generalised version of Helly's Theorem (Theorem 4.20), which may be of independent interest. This asymmetric Helly-type theorem we give here is a natural extension of an asymmetric Arzelà-Ascoli Theorem [6].

We restrict the analysis to quadratic dissipation; in particular, we do not study rate independent problems, which are characterised by 1-homogeneous dissipation in terms of the asymmetric distance. This restriction is motivated by the fact that for rate-independent models in asymmetric situations, a number of existence results are available (e.g., for the evolution of shape memory alloys [5]). Mainik and Mielke [7] discuss asymmetric rate-independent models for phase transformations in shape memory alloys, brittle fracture and delamination and develop a framework for rate-dependent models.

Since the main emphasis in this paper is on weakening symmetry assumptions, the difference between different asymmetric topologies becomes more pronounced than in other papers, which is why we include a discussion of the topological background in Section 2. Section 3 describes some further potential pitfalls where symmetric arguments break down.

Once the asymmetric framework is set up properly and the asymmetric Helly-type Theorem 4.20 is established, many ideas from the symmetric case carry over. We take inspiration from [7] as well as from the work of Ambrosio, Gigli and Savaré [1]; ideas from both approaches will be combined to study rate-dependent processes described by gradient flows.

For the reader's convenience, we quickly recall the fundamental ideas leading to a definition of a gradient flow in a metric space; details can be found in [1]. For a curve ν and a functional ϕ on a Hilbert space,

$$\nu' = -D\phi \circ \nu \tag{1}$$

describes a gradient flow, wherever the gradient $D\phi$ of the functional and the derivative ν' of the curve exist. If ν is a solution to the gradient flow (1), then ν' and $D\phi \circ \nu$ are antiparallel and $|\nu'| = |D\phi \circ \nu|$. Precisely under these conditions, one obtains

$$(\phi \circ \nu)' = \langle D\phi \circ \nu, \nu' \rangle = -\left|D\phi \circ \nu\right| \left|\nu'\right| = -\frac{1}{2}\left|\nu'\right|^2 - \frac{1}{2}\left|D\phi \circ \nu\right|^2. \tag{2}$$

Equation (2) remains valid if the last two equalities are replaced by estimates from below. Thus, the reverse inequality characterises gradient flows: (1) is equivalent to

$$(\phi \circ \nu)' \leqslant -\frac{1}{2}\left|\nu'\right|^2 - \frac{1}{2}\left|D\phi \circ \nu\right|^2, \tag{3}$$

which can be interpreted in purely metric terms; Definition 3.10 spells this out for the asymmetric case.

This article is organised as follows. Asymmetric metric spaces are defined in Section 2. In Section 3, gradient flows in these spaces are introduced. Section 4 contains the main result of this article, namely an existence proof for gradient flows for functionals on asymmetric metric spaces (Theorem 4.21).

Notation: we write $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{R}^+ := (0, \infty)$, $\mathbb{R}_0^+ := [0, \infty)$, $\overline{\mathbb{R}}_0^+ := \mathbb{R}_0^+ \cup \{\infty\}$ and $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$. We denote topological closures by the symbol cl. Almost everywhere, abbreviated a.e., is to be understood w.r.t. the Lebesgue measure on $\mathbb{R}$.

2 Asymmetric metric spaces

Definition 2.1 (Asymmetric metric spaces) *Let S be a set. A function $d\colon S \times S \to \mathbb{R}_0^+$ is an* asymmetric metric *and (S, d) an* asymmetric metric space *if the following hold:*

(i) For $x,\, y \in S$*, one has* $d\,(x, y) = 0$ *if, and only if,* $x = y$ *(definiteness),*
(ii) for $x, y, z \in S$*, one has* $d\,(x, z) \leqslant d\,(x, y) + d\,(y, z)$ *(triangle inequality).*

Obviously, an asymmetric metric lacks the symmetry condition of a metric. The study of asymmetric metrics, often called quasi-metrics, has a long history, going back at least to [8,9]. Not only applications in science and engineering suggest that the symmetry requirement of a metric is often too restrictive; Gromov points out the limiting effects of this assumption [10, Introduction].

We present one simple example of an asymmetric metric space, which serves as a prototype of admissible metrics. We refer to [6] for further examples.

Example 1 (Sorgenfrey asymmetric metric) *The* Sorgenfrey asymmetric metric *is the function* $d_s \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}_0^+$ *given by*

$$d_s\,(x, y) := \begin{cases} y - x & \text{\textit{if} } y \geqslant x \\ 1 & \text{\textit{otherwise.}} \end{cases}$$

For the reader's convenience, we recall the basic topological framework [6]. Henceforth, (S, d) will denote an asymmetric metric space.

Definition 2.2 (Forward and backward topologies) *The* forward topology *induced by* d*, or the* $d^{>}$-topology*, is generated by the* forward open balls

$$B^{>}\,(x, \epsilon) := \left\{ y \in S \;\middle|\; d\,(x, y) < \epsilon \right\},\ \text{\textit{where} } x \in S \text{ \textit{and} } \epsilon > 0.$$

Likewise, the backward topology *induced by* d*, or the* $d^{<}$-topology*, is generated by the* backward open balls

$$B^{<}\,(x, \epsilon) := \left\{ y \in S \;\middle|\; d\,(y, x) < \epsilon \right\},\ \text{\textit{where} } x \in S \text{ \textit{and} } \epsilon > 0.$$

From now on, the prefixes "$d^{>}$" and "$d^{<}$" distinguish topological notions with respect to the $d^{>}$- and $d^{<}$-topologies. We focus on forward notions and refrain from formulating the equivalent backward notions unless confusion could arise.

One can see that these topologies are T_1, that is, finite point sets are closed [11, Section 1]. However, as illustrated by the following example, they need not be Hausdorff.

Example 2 (Sorgenfrey-like asymmetric metric) *The function* $d \colon \mathbb{R}^n \times$

$\mathbb{R}^n \to \mathbb{R}_0^+$, *with* $n \geqslant 2$, *defined for* $x := (x^1, \ldots, x^n)$ *and* $y := (y^1, \ldots, y^n)$ *by*

$$d(x,y) := \begin{cases} 0 & \text{if } x = y \\ y^1 - x^1 & \text{if } x^1 < y^1 \\ 1 & \text{otherwise} \end{cases}$$

is an asymmetric metric. The forward topology induced by this asymmetric metric is not Hausdorff since whenever $x, y \in \mathbb{R}^n$ *with* $x \neq y$ *and* $x^1 = y^1$, *one has*

$$B^{>}(x,\epsilon) \cap B^{>}(y,\epsilon) = B^{>}(x,\epsilon) \setminus \{x\} = B^{>}(y,\epsilon) \setminus \{y\} \neq \emptyset.$$

Moreover, *backward Hausdorffness* (i.e., Hausdorffness of the backward topology) does not imply *forward Hausdorffness*, and vice versa. For example, let $S := ([0,1] \times \{0\}) \cup (\{0\} \times (0,1))$ and let d be the asymmetric metric of Example 2.

Henceforth we assume that (S,d) is forward Hausdorff; the Sorgenfrey asymmetric metric (Example 1) provides a prototype.

Definition 2.3 (Boundedness) *Let* $Y \subset S$. *If there exists* $x \in S$ *and* $r > 0$ *such that* $Y \subset B^{>}(x,r)$, *then* Y *is* forward bounded *or* $d^{>}$-bounded.

Forward-boundedness does not imply backward-boundedness, and vice versa. For example, let d_s be the Sorgenfrey asymmetric metric (Example 1): it is easy to verify that $\mathbb{N}$ is $d_s^{<}$-bounded but not $d_s^{>}$-bounded.

Definition 2.4 (Cauchy Sequence) *A sequence* $(x_k)_{k\in\mathbb{N}}$ *is* forward Cauchy *if for every* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that* $m \geqslant n \geqslant N$ *implies* $d(x_n, x_m) < \epsilon$.

Definition 2.5 (Convergence) *A sequence* $(x_k)_{k\in\mathbb{N}}$ forward converges *to* $x \in X$ *if* $\lim_{k\to\infty} d(x, x_k) = 0$. *Notation:* $x = \mathrm{d}^{>}\text{-}\lim_{k\to\infty} x_k$.

Definition 2.6 (Continuity and Lipschitz continuity) *Let* (S_1, d_1) *and* (S_2, d_2) *be asymmetric metric spaces. A function* $f \colon S_1 \to S_2$ *is* forward-continuous *at* $x \in S_1$, *if, for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $y \in B^{>}(x,\delta)$ *implies* $f(y) \in B^{>}(f(x), \varepsilon)$.

The function f *is* forward-Lipschitz *if there exists an* $L \geqslant 0$ *such that for* $x, y \in S_1$, *it holds that* $d_2(f(x), f(y)) \leqslant L d_1(x,y)$. *(On the other hand if* $d_2(f(x), f(y)) \leqslant L d_1(y,x)$ *then* f *is* backward-Lipschitz.*)*

There are four natural notions of continuity for a function $f \colon S_1 \to S_2$ at $x \in S_1$. The justification for the restriction to two notions is that the other two notions agree for continuity on the entire domain. We remark that the composition of two forward-continuous functions is forward-continuous; the composition of two backward-continuous functions is also forward-continuous.

Since, as is obvious, asymmetric metric spaces are first countable, as a consequence sequential continuity implies continuity. Likewise the sequential closure of a set is the closure of the set.

We mention a useful class of functions:

Definition 2.7 (Distance functions) *For fixed $x \in S$, the* forward distance function *induced by d is defined by $d_x^{>} : S \longrightarrow \mathbb{R}_0^+, y \mapsto d(x, y)$. If d is symmetric, then we drop the superscript and write $d_x : S \longrightarrow \mathbb{R}_0^+$.*

Remark 1 *In contrast to the symmetric situation, forward distance functions need be neither forward- nor backward-Lipschitz, as Example 1 shows: for any $L \geqslant 0$,*

$$\begin{aligned} L+1 &= |d_s(0, L+1) - d_s(0,0)| &\nleq L d_s(L+1, 0) &= L, \\ L+1 &= |d_s(0,0) - d_s(-L-1, 0)| &\nleq L d_s(0, -L-1) &= L. \end{aligned}$$

On the other hand with $(\mathbb{R}, d_s)$ as domain and target space, the forward distance function $d_x^{>}$ is backward Lipschitz on the interval $(-\infty, x)$ and on the interval (x, ∞).

Remark 2 (Semi-continuity of the distance functions) *A simple argument shows that forward distance functions are forward upper semicontinuous (u.s.c.) and backward lower semicontinuous (l.s.c.). However they need be neither forward l.s.c. nor backward u.s.c.; in the setting of Example 2, let $x_k := \left(\frac{1}{k}, 0, \ldots, 0\right)$. Note that $d(x, x_k) \to 0$ for every x with $x^1 = 0$. Thus, for every such $x \neq 0$, one has*

$$1 = d_0^{>}(x) > \liminf_{k \to \infty} d_0^{>}(x_k) = 0.$$

The other counterexample is similar.

3 Gradient flows in asymmetric metric spaces

Recall that (S, d) is a forward-Hausdorff asymmetric metric space. In the following let $(a, b) \subset \mathbb{R}$.

We begin by establishing suitable notions of continuity for curves $\nu\colon (a,b) \to (S,d)$, by equipping (a,b) with a suitable topology. We remark that equipping (a,b) with any (symmetric) metrisable topology would destroy asymmetric properties of d, since forward- and backward quantities are then necessarily comparable. To avoid this we equip (a,b) with an asymmetric metric whose forward or backward topology is not (symmetrically) metrisable. We choose the Sorgenfrey asymmetric metric d_s (Example 1). The Sorgenfrey asymmetric metric is particularly appealing since it is one-sided Euclidean and the forward topology it generates is the lower limit topology [12, Counterexample 51]. Other choices are possible, and of course simpler cases such as the Euclidean metric are included.

Definition 3.1 (Continuous curves) *The notion of forward continuity from Definition 2.6 is adapted for curves $\nu\colon ((a,b), d_s) \to (S,d)$ as follows. The curve ν is* forward continuous, *$\nu \in C^{\rhd}((a,b),(S,d))$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $s \in (t-\delta, t)$, one has $d(\nu(s), \nu(t)) < \epsilon$ and for every $s \in (t, t+\delta)$, one has $d(\nu(t), \nu(s)) < \epsilon$. Note that forward continuity is equivalent to the requirement that*

$$\lim_{s \nearrow t} d(\nu(s), \nu(t)) = 0 \text{ and } \lim_{s \searrow t} d(\nu(t), \nu(s)) = 0. \tag{4}$$

We remark that this definition combines the forward-continuity at two points $t-\delta$ and t. Uniform continuity is not required in the definition of forward-continuous curves. The following definition extends the definition of absolutely continuous curves in metric spaces [1, Definition 1.1.1]. Again, this definition differs from the symmetric case by the introduction of asymmetry via the requirement $s \leqslant t$.

Definition 3.2 (Absolute continuity) *A curve $\nu\colon (a,b) \to (S,d)$ is* forward absolutely continuous, *$\nu \in AC^{\rhd}((a,b),(S,d))$, if there exists $m \in L^1\left((a,b), \mathbb{R}_0^+\right)$ such that for $s,t \in (a,b)$ with $s \leqslant t$,*

$$d(\nu(s), \nu(t)) \leqslant \int_s^t m(u)\, \mathrm{d}u. \tag{5}$$

It is immediate that forward absolute continuity implies forward continuity. We state one auxiliary statement whose proof is also straightforward.

Lemma 3.3 (Composition of a.c. curves and Lipschitz functions) *Let $\nu \in AC^{\rhd}((a,b),(S,d))$, and suppose $f\colon (S,d) \to (S',d')$ is forward-Lipschitz. Then $f \circ \nu \in AC^{\rhd}((a,b),(S',d'))$.*

In a Hilbert space, there is a natural notion of a gradient. An extension of this notion to metric spaces is the notion of metric derivative, analysed by Ambrosio et al. [1, Section 1.1]. We extend this approach to asymmetric metric spaces.

Definition 3.4 (Metric derivative) *Let* $\nu\colon (a,b)\to S$ *be a curve. The* forward metric derivative *of* ν *at* $t\in(a,b)$, $|\nu^{\triangleright}|(t)$, *is defined whenever the following limits exist and agree; in this case,*

$$|\nu^{\triangleright}|(t) := \lim_{s\nearrow t}\frac{d(\nu(s),\nu(t))}{t-s} = \lim_{s\searrow t}\frac{d(\nu(t),\nu(s))}{s-t}.$$

Theorem 3.5 (Existence of metric derivatives) *Let* $\nu\in AC^{\triangleright}((a,b),(S,d))$. *Then* $|\nu^{\triangleright}|$ *exists a.e. in* (a,b).

Proof For $a<r<s<t<b$,

$$\left(d^{>}_{\nu(r)}\circ\nu\right)(t)-\left(d^{>}_{\nu(r)}\circ\nu\right)(s) = d(\nu(r),\nu(t))-d(\nu(r),\nu(s)) \leqslant d(\nu(s),\nu(t)).$$

Further, by (4), for $\epsilon>0$ there exists $r'<s$ such that

$$d(\nu(r),\nu(s))<\epsilon \text{ for every } r'<r<s. \tag{6}$$

Since this is true for arbitrary s and ϵ, in the limit $s\to t$, we obtain

$$0\leqslant \sup_{r<t}\lim_{s\nearrow t}\frac{\left(d^{>}_{\nu(r)}\circ\nu\right)(t)-\left(d^{>}_{\nu(r)}\circ\nu\right)(s)}{t-s}\leqslant\frac{d(\nu(s),\nu(t))}{t-s}\leqslant\frac{\int_s^t m(u)\,\mathrm{d}u}{t-s}$$

(we remark that in the symmetric case, the limit is non-negative, while here only the supremum of the limit has to be non-negative. This explains why the argument deviates slightly from the symmetric one.) A very similar argument shows that for $a<r<t<s<b$,

$$0\leqslant \sup_{r<t}\lim_{s\searrow t}\frac{\left(d^{>}_{\nu(r)}\circ\nu\right)(s)-\left(d^{>}_{\nu(r)}\circ\nu\right)(t)}{s-t}\leqslant\frac{d(\nu(t),\nu(s))}{s-t}\leqslant\frac{\int_t^s m(u)\,\mathrm{d}u}{s-t}.$$

It follows that the difference quotients are bounded uniformly in s, and thus the limit exists for a.e. $t\in(a,b)$. We write

$$m^{\triangleright}_{-}(t) := \sup_{r\in(a,t)}\left(d^{>}_{\nu(r)}\circ\nu\right)'(t),$$

which is finite a.e. by the previous consideration. We claim that $m_-^{\triangleright} = |\nu^{\triangleright}|$ a.e.. To prove this it suffices to show that for a.e. $t \in (a,b)$,

$$\limsup_{s \nearrow t} \frac{d\left(\nu\left(s\right), \nu\left(t\right)\right)}{t-s} \leqslant m_-^{\triangleright}\left(t\right) \leqslant \liminf_{s \nearrow t} \frac{d\left(\nu\left(s\right), \nu\left(t\right)\right)}{t-s}, \tag{7}$$

$$\limsup_{s \searrow t} \frac{d\left(\nu\left(t\right), \nu\left(s\right)\right)}{s-t} \leqslant m_-^{\triangleright}\left(t\right) \leqslant \liminf_{s \searrow t} \frac{d\left(\nu\left(t\right), \nu\left(s\right)\right)}{s-t}. \tag{8}$$

We prove the first pair of inequalities; the proof of the other pair is similar.

For $a < r < s < t < b$,

$$d\left(\nu\left(s\right), \nu\left(t\right)\right) \geqslant d\left(\nu\left(r\right), \nu\left(t\right)\right) - d\left(\nu\left(r\right), \nu\left(s\right)\right), \tag{9}$$

with equality in particular for $r = s$. Since $\nu \in C^{\triangleright}\left((a,b), (S,d)\right)$, by (4),

$$\begin{aligned} d\left(\nu\left(s\right), \nu\left(t\right)\right) &= \sup_{r \in (a,s)} \left(d\left(\nu\left(r\right), \nu\left(t\right)\right) - d\left(\nu\left(r\right), \nu\left(s\right)\right)\right) \\ &= \sup_{r \in (a,s)} \left(\left(d_{\nu(r)}^{>} \circ \nu\right)\left(t\right) - \left(d_{\nu(r)}^{>} \circ \nu\right)\left(s\right)\right). \end{aligned}$$

Since the composition in the equation above is absolutely continuous in the interval under consideration,

$$\begin{aligned} d\left(\nu\left(s\right), \nu\left(t\right)\right) &= \sup_{r \in (a,s)} \int_s^t \left(d_{\nu(r)}^{>} \circ \nu\right)'\left(u\right) \, \mathrm{d}u \\ &\leqslant \int_s^t \sup_{r \in (a,u)} \left(d_{\nu(r)}^{>} \circ \nu\right)'\left(u\right) \, \mathrm{d}u = \int_s^t m_-^{\triangleright}\left(u\right) \, \mathrm{d}u. \end{aligned}$$

Thus, if t is a Lebesgue point of $m_-^{\triangleright}$,

$$\limsup_{s \nearrow t} \frac{d\left(\nu\left(s\right), \nu\left(t\right)\right)}{t-s} \leqslant \limsup_{s \nearrow t} \frac{\int_s^t m_-^{\triangleright}\left(u\right) \, \mathrm{d}u}{t-s} = m_-^{\triangleright}\left(t\right).$$

To show the reverse inequality, let t be a Lebesgue point of $m_-^{\triangleright}$. It follows from (9) that for $a < r < s < t$,

$$\liminf_{s \nearrow t} \frac{d\left(\nu\left(s\right), \nu\left(t\right)\right)}{t-s} \geqslant \liminf_{s \nearrow t} \frac{\left(d_{\nu(r)}^{>} \circ \nu\right)\left(t\right) - \left(d_{\nu(r)}^{>} \circ \nu\right)\left(s\right)}{t-s} = \left(d_{\nu(r)}^{>} \circ \nu\right)'\left(t\right).$$

We take the supremum with respect to $r \in (a,t)$ on both sides and obtain

$$\liminf_{s \nearrow t} \frac{d\left(\nu\left(s\right), \nu\left(t\right)\right)}{t-s} \geqslant \sup_{r \in (a,t)} \left(d_{\nu(r)}^{>} \circ \nu\right)'\left(t\right) = m_-^{\triangleright}\left(t\right).$$

□

Remark 3 *For* $\nu \in AC^{\triangleright}\left((a,b), (S,d)\right)$, *it follows from* (7) *that for every* $m \in L^1\left((a,b), \mathbb{R}_0^+\right)$ *satisfying* (5), $|\nu^{\triangleright}| \leqslant m$ *a.e..*

For a function $\phi\colon S \to \overline{\mathbb{R}}$, we write $\phi^+(x) := \max(\phi(x), 0)$ and denote its effective domain by

$$D(\phi) := \{x \in S \mid \phi(x) < \infty\}. \tag{10}$$

We extend the notion of norms of gradients on Hilbert spaces through upper gradients and (the weaker notion of) local slopes (see [1, Section 1.2].)

Definition 3.6 (Upper gradient) *A function $g\colon S \to \overline{\mathbb{R}}_0^+$ is a* forward upper gradient *for $\phi\colon S \to \mathbb{R}$ if for every $\nu \in AC^{\rhd}((a,b), s)$, $g \circ \nu$ is Borel, and*

$$|\phi \circ \nu(t) - \phi \circ \nu(s)| \leqslant \int_s^t (g \circ \nu)(r) \, |\nu^{\rhd}|(r) \, \mathrm{d}r. \tag{11}$$

Definition 3.7 (Local slope) *The* forward local slope *of ϕ, $|\partial^{>}\phi| : D(\phi) \to \overline{\mathbb{R}}_0^+$, is defined by*

$$|\partial^{>}\phi|(x) := \begin{cases} 0 & \text{if } x \text{ is an isolated point of } S, \\ \mathrm{d}^{>}\text{- }\limsup_{y \to x} \frac{(\phi(x) - \phi(y))^+}{d(x,y)} & \text{otherwise.} \end{cases}$$

The next theorem shows that local slopes behave like norms of the gradient on non-increasing curves [1, Definition 1.2.4].

Theorem 3.8 (Chain rule) *Let $\nu \in AC^{\rhd}((a,b), (S,d))$ and $\phi \circ \nu$ be a.e. non-increasing. Then a.e.,*

$$\left|(\phi \circ \nu)'\right| \leqslant (|\partial^{>}\phi| \circ \nu) \, |\nu^{\rhd}|. \tag{12}$$

The proof is similar to the one of [1, Theorem 1.2.5] and thus omitted.

Definition 3.9 (Relaxed local slope) *The $d^{>}$-relaxed forward local slope of ϕ, $|\partial^{\rhd}\phi| : d^{>}\text{-cl}(D(\phi)) \to \overline{\mathbb{R}}_0^+$, is defined by*

$$|\partial^{\rhd}\phi|(u) := \inf_{\substack{(u_n)_{n\in\mathbb{N}} \\ d^{>}\text{- }\lim_{n\to\infty} u_n = u \\ \sup_{n\in\mathbb{N}} \phi(u_n) < \infty}} \liminf_{n \to \infty} |\partial^{>}\phi|(u_n).$$

3.4 Gradient flows

Recall that gradient flows in Hilbert spaces are characterised by (3),

$$(\phi \circ \nu)' \leqslant -\frac{1}{2}|\nu'|^2 - \frac{1}{2}|D\phi \circ \nu|^2 ,$$

which involves only notions that we have extended to asymmetric metric spaces: metric derivatives (Definition 3.4) extend the notion of norm of the derivative of a curve; upper gradients (Definition 3.6) and local slopes (Definition 3.7) extend the notion of norm of the gradient of a function. This motivates the following definitions [1, Sections 1.2 and 2.2].

Definition 3.10 (Gradient flows I) *A curve $\nu \in AC^{\rhd}((a,b),(S,d))$ is a* forward gradient flow *on ϕ if $\phi \circ \nu$ is non-increasing and a.e.,*

$$(\phi \circ \nu)' \leqslant -\frac{1}{2}|\nu^{\rhd}|^2 - \frac{1}{2}\left(|\partial^{>}\phi| \circ \nu\right)^2 . \tag{13}$$

Definition 3.11 (Gradient flows II) *A curve $\nu \in AC^{\rhd}((a,b),(S,d))$ is a* forward gradient flow *on ϕ* w.r.t. its upper gradient *g if $\phi \circ \nu$ is non-increasing and a.e.,*

$$(\phi \circ \nu)' \leqslant -\frac{1}{2}|\nu^{\rhd}|^2 - \frac{1}{2}(g \circ \nu)^2 . \tag{14}$$

4 Variational approximation of gradient flows

In this section, we introduce the forward Moreau-Yosida approximation.

Definition 4.1 (Forward Moreau-Yosida approximation) *For $\phi\colon S \to \overline{\mathbb{R}}$ and $\tau \in \mathbb{R}^+$, we define $\Phi_\tau\colon S \times S \to \overline{\mathbb{R}}$ by*

$$\Phi_\tau(u,v) := \frac{1}{2\tau}d^2(u,v) + \phi(v) .$$

Then the forward Moreau-Yosida approximation *of ϕ is defined by*

$$\phi_\tau(u) := \inf_{v \in S} \Phi_\tau(u,v) . \tag{15}$$

4.1 Assumptions

We make the following assumptions about the forward-Hausdorff asymmetric metric space (S,d) and the functional $\phi\colon S \to \overline{\mathbb{R}}$ whose effective domain is non-empty. Assumptions 4.2–4.4 are invoked in the Helly Theorem of Subsection 4.4; Assumptions 4.5–4.7 are required in Subsection 4.2 while Assumptions 4.4, 4.7 and 4.8 appear in Subsection 4.5.

Assumption 4.2 (Completeness and Hausdorffness) *The asymmetric metric space (S, d) is forward complete, that is, every forward Cauchy sequence is forward convergent. In addition, we assume that (S, d) is a Hausdorff space in the forward topology.*

We remark that this definition of completeness is the correct one in the sense that one recovers the expected statement: if every forward Cauchy sequence has a forward convergent subsequence, then the space (S, d) is forward complete [6, Lemma 4.3].

Assumption 4.3 (Backward convergence implies forward convergence) *We assume that backward convergence implies forward convergence. That is, if a sequence $(u_n)_{n\in\mathbb{N}}$ backward converges, $d^{<}\text{-}\lim u_n = u$, then it also forward converges.*

We remark that this asymmetric condition also appears in an asymmetric version of an Arzelà-Ascoli theorem [6, Theorem 5.12]. It is not hard to see that if a sequence backward converges and forward converges, then the two limits have to agree [6, Lemma 3.1]. Also, Assumption 4.3 implies that the backward limit is unique ([6, Corollary 3.2]), so the space (S, d) is automatically backward Hausdorff, which is why we only assume forward Hausdorffness in Assumption 4.2.

Assumption 4.4 (Lower semi-continuity of the asymmetric metric) *The asymmetric metric d is l.s.c. in the second argument, i.e., $d(\cdot, u) \leqslant \liminf d(\cdot, u_n)$ for every $(u_n)_{n\in\mathbb{N}}$ with $d^{>}\text{-}\lim u_n = u$.*

We introduce the notation

$$\tau_\star := \sup\left\{\tau > 0 \;\middle|\; \text{there exists } u \in S \text{ such that } \phi_\tau(u) > -\infty\right\}, \tag{16}$$

Assumption 4.5 (Coercivity) *The functional ϕ is such that $\tau_\star > 0$ and there exists $u_\star \in S$ such that $\phi_{\tau_\star}(u_\star) > -\infty$.*

Assumption 4.6 (Compactness) *If $(u_n)_{n\in\mathbb{N}} \subset S$ with $\sup_{n\in\mathbb{N}} \phi(u_n) < \infty$, and $\sup_{n,m\in\mathbb{N}} d(u_n, u_m) < \infty$, then $(u_n)_{n\in\mathbb{N}}$ admits a $d^{>}$-convergent subsequence.*

Assumption 4.7 (Lower semi-continuity) *The functional ϕ is $d^{>}$-l.s.c.: $\liminf_{n\to\infty} \phi(u_n) \geqslant \phi(u)$ whenever $(u_n)_{n\in\mathbb{N}} \subset S$ with $d^{>}\text{-}\lim_{n\to\infty} u_n = u$.*

Assumption 4.8 (Relaxed forward slope) *The $d^{>}$-relaxed forward local slope of ϕ is a forward upper gradient for ϕ.*

Next, we study Moreau-Yosida approximations of functionals (Definition 4.1) in greater detail. It is convenient to introduce

$$
\begin{aligned}
&J_\tau(u) := \arg\min \Phi_\tau(u,\cdot) := \left\{ v \in S \;\middle|\; \Phi_\tau(u,v) \leqslant \Phi_\tau(u,w) \text{ for all } w \in S \right\}, \\
&J_0(u) := \{u\}.
\end{aligned}
$$

In this subsection, the asymmetric assumptions are not significant (unlike in the subsequent subsections). Thus, many arguments are the same as in the symmetric case; one only has to check that symmetry is not involved in the proof. We nevertheless include the proofs so that the reader can readily verify the claims.

We show, for $\tau \in (0, \tau_\star)$, that $J_\tau(u)$ is non-empty (Proposition 4.11), and estimate the forward local slope of ϕ on $J_\tau(u)$ (Proposition 4.12).

Lemma 4.9 *Let $u_1, u_2 \in S$. Then $0 < \tau_2 < \tau_1$ implies*

$$
\phi_{\tau_1}(u_1) - \phi_{\tau_2}(u_2) \leqslant \frac{1}{\tau_1 - \tau_2} d^2(u_1, u_2).
$$

Proof From the triangle and Young's inequalities, for $u_1, u_2, v \in S$ and $\epsilon > 0$,

$$
d^2(u_1, v) \leqslant \left(1 + \frac{1}{\epsilon}\right) d^2(u_1, u_2) + (1+\epsilon)\, d^2(u_2, v).
$$

With the choice $\epsilon = \frac{\tau_1 - \tau_2}{\tau_1 + \tau_2}$, one obtains

$$
\frac{1}{2\tau_1} d^2(u_1, v) \leqslant \frac{1}{\tau_1 - \tau_2} d^2(u_1, u_2) + \frac{1}{\tau_1 + \tau_2} d^2(u_2, v). \tag{17}
$$

Thus, we can estimate $\Phi_{\tau_2}(u_2, v)$ as follows: using (17) in the first inequality and $\tau_2 < \tau_1$ in the last two inequalities:

$$\begin{aligned}
\Phi_{\tau_2}(u_2, v) &= \frac{1}{2\tau_2} d^2(u_2, v) + \phi(v) \\
&= \frac{\tau_1 - \tau_2}{2\tau_2(\tau_1 + \tau_2)} d^2(u_2, v) + \frac{1}{\tau_1 + \tau_2} d^2(u_2, v) + \phi(v) \\
&\geqslant \frac{\tau_1 - \tau_2}{2\tau_2(\tau_1 + \tau_2)} d^2(u_2, v) - \frac{1}{\tau_1 - \tau_2} d^2(u_1, u_2) \\
&\qquad + \left(\frac{1}{2\tau_1} d^2(u_1, v) + \phi(v)\right) \\
&\geqslant \frac{\tau_1 - \tau_2}{4\tau_1\tau_2} d^2(u_2, v) - \frac{1}{\tau_1 - \tau_2} d^2(u_1, u_2) + \phi_{\tau_1}(u_1) \qquad (18) \\
&\geqslant \phi_{\tau_1}(u_1) - \frac{1}{\tau_1 - \tau_2} d^2(u_1, u_2).
\end{aligned}$$

Taking the infimum with respect to $v \in S$, one obtains

$$\phi_{\tau_2}(u_2) - \phi_{\tau_1}(u_1) \geqslant -\frac{1}{\tau_1 - \tau_2} d^2(u_1, u_2).$$

□

Corollary 4.10 *Let $0 < \tau_2 < \tau_1 \leqslant \tau_\star$, where τ_1 is such that there is $u_1 \in S$ with $\phi_{\tau_1}(u_1) > -\infty$. Then, the following estimate holds for every $u_2, u_3 \in S$:*

$$d^2(u_2, u_3) \leqslant \frac{4\tau_1\tau_2}{\tau_1 - \tau_2}\left(\Phi_{\tau_2}(u_2, u_3) - \phi_{\tau_1}(u_1) + \frac{1}{\tau_1 - \tau_2} d^2(u_1, u_2)\right). \qquad (19)$$

This shows immediately that the sublevels of $\Phi_{\tau_2}(u_2, \cdot)$ are bounded.

Proof Estimate (19) is (18) for $v = u_3$. We remark that by definition of $\tau_\star$ (Assumption 4.5), a time τ_1 and $u_1 \in S$ with $\phi_{\tau_1}(u_1) > -\infty$ exist. □

Proposition 4.11 *For $u \in S$ and $\tau \in (0, \tau_\star)$, $\Phi_\tau(u, \cdot)$ admits a minimiser and thus $J_\tau(u)$ is non-empty.*

Proof This follows by the direct method from the calculus of variations; given $\tau_2 := \tau$, there exists by definition of $\tau_\star$ a $\tau_1 > \tau_2$ such that there is a $u_1 \in S$ with $\phi_{\tau_1}(u_1) > -\infty$. By (19) with $u_2 = u$ and $u_3 = v$, one sees that $d(u, v)$ and thus $\phi(v) = \inf_{v \in S} \Phi(u, v)$ are bounded from above on sublevels of $\Phi_\tau(u, v)$; the latter are $d^>$-sequentially compact by the assumptions on lower semicontinuity (Assumption 4.7) and compactness (Assumption 4.6). □

Proposition 4.12 (Slope estimate) *For $u \in S$ and $\tau \in (0, \tau_\star)$, the forward local slope of ϕ evaluated at $u_\tau \in J_\tau(u)$ is bounded from above:*

$$|\partial^> \phi|(u_\tau) \leqslant \frac{d(u, u_\tau)}{\tau}. \qquad (20)$$

In particular $u_\tau \in D\left(\left|\partial^{>}\phi\right|\right)$.

Proof For $u_\tau \in J_\tau(u)$ and $v \in S$, one has

$$\frac{1}{2\tau} d^2(u, u_\tau) + \phi(u_\tau) \leqslant \frac{1}{2\tau} d^2(u, v) + \phi(v).$$

Two applications of the triangle inequality yield

$$\begin{aligned}
\phi(u_\tau) - \phi(v) &\leqslant \frac{1}{2\tau}\left(d^2(u, v) - d^2(u, u_\tau)\right) \\
&= \frac{1}{2\tau}\left(d(u, v) - d(u, u_\tau)\right)\left(d(u, v) + d(u, u_\tau)\right) \\
&\leqslant \frac{1}{2\tau} d(u_\tau, v)\left(2d(u, u_\tau) + d(u_\tau, v)\right).
\end{aligned}$$

Thus, by Definition 3.7 of local slopes and the forward u.s.c. of the forward distance function (Remark 2),

$$\begin{aligned}
\left|\partial^{>}\phi\right|(u_\tau) &= \mathrm{d}^{>}\text{-}\limsup_{v \to u_\tau} \frac{\left(\phi(u_\tau) - \phi(v)\right)^+}{d(u_\tau, v)} \\
&\leqslant \frac{1}{\tau} d(u, u_\tau) + \frac{1}{2\tau}\, \mathrm{d}^{>}\text{-}\limsup_{v \to u_\tau} d(u_\tau, v) = \frac{d(u, u_\tau)}{\tau}.
\end{aligned}$$

□

Lemma 4.13 *Let* $u \in S$ *and* $\tau > 0$. *If* $J_\tau(u)$ *is non-empty, let* $u_\tau \in J_\tau(u)$. *Then:*

(i) $\phi(u) \geqslant \phi_\tau(u)$ *and* $\tau \mapsto \phi_\tau(u)$ *is non-increasing.*
(ii) Wherever defined, $\tau \mapsto d(u, u_\tau)$ *is non-decreasing for any choice of* $u_\tau \in J_\tau(u)$.
(iii) $\phi(u) \geqslant \phi(u_\tau)$ *and, wherever defined,* $\tau \mapsto \phi(u_\tau)$ *is non-increasing.*
(iv) If $u \in d^{>}\text{-cl}\left(D(\phi)\right)$ *then* $\tau \mapsto d(u, u_\tau)$ *is continuous at* 0 *for any choice of* $u_\tau \in J_\tau(u)$.

Proof (i). For $\tau > 0$, by the definition (15) of the Moreau-Yosida approximation,

$$\phi(u) = \Phi_\tau(u, u) \geqslant \inf_{v \in S} \Phi_\tau(u, v) = \phi_\tau(u).$$

The second part follows from Lemma 4.9 by setting $u_1 = u_2$.

(ii). Let $0 < \tau_1 < \tau_2$. Then, for $u_{\tau_1} \in J_{\tau_1}(u)$ and $u_{\tau_2} \in J_{\tau_2}(u)$,

$$\begin{aligned}
\frac{1}{2\tau_1} d^2(u, u_{\tau_1}) + \phi(u_{\tau_1}) &= \Phi_{\tau_1}(u, u_{\tau_1}) = \inf_{v \in S} \Phi_{\tau_1}(u, v) \\
&\leqslant \Phi_{\tau_1}(u, u_{\tau_2}) = \frac{1}{2\tau_1} d^2(u, u_{\tau_2}) + \phi(u_{\tau_2}).
\end{aligned} \tag{21}$$

By switching the indices, we derive in the same way

$$\frac{1}{2\tau_2} d^2(u, u_{\tau_2}) + \phi(u_{\tau_2}) \leqslant \frac{1}{2\tau_2} d^2(u, u_{\tau_1}) + \phi(u_{\tau_1}). \tag{22}$$

Estimates (21) and (22) together yield

$$\frac{1}{2\tau_1}\left(d^2(u, u_{\tau_1}) - d^2(u, u_{\tau_2})\right) \leqslant \frac{1}{2\tau_2}\left(d^2(u, u_{\tau_1}) - d^2(u, u_{\tau_2})\right).$$

Since $0 < \tau_1 < \tau_2$, it follows that $d(u, u_{\tau_1}) \leqslant d(u, u_{\tau_2})$.

(iii). From (22) and $u_{\tau_j}^{>} \in J_{\tau_j}^{>}(u)$ for $j = 1, 2$,

$$0 \leqslant \frac{1}{2\tau_2}\left(d^2(u, u_{\tau_2}) - d^2(u, u_{\tau_1})\right) \leqslant \phi(u_{\tau_1}) - \phi(u_{\tau_2}).$$

Thus $\phi(u_{\tau_1}) \geqslant \phi(u_{\tau_2})$. Furthermore, since $u_{\tau_1} \in J_{\tau_1}(u)$,

$$\phi(u) = \Phi_{\tau_1}(u, u) \geqslant \Phi_{\tau_1}(u, u_{\tau_1}) = \frac{1}{2\tau_1} d^2(u, u_{\tau_1}) + \phi(u_{\tau_1}) \geqslant \phi(u_{\tau_1}).$$

Thus $\phi(u) \geqslant \phi(u_{\tau_1}) \geqslant \phi(u_{\tau_2})$.

(iv). We use again that $\phi_\tau(u) \leqslant \Phi_\tau(u, v)$ for $v \in S$. Hence,

$$\frac{1}{2\tau} d^2(u, u_\tau) + \phi(u_\tau) \leqslant \frac{1}{2\tau} d^2(u, v) + \phi(v).$$

In particular, for fixed $v \in D(\phi)$ as defined in (10), in the limit $\tau \searrow 0$, this becomes

$$\limsup_{\tau \searrow 0} d^2(u, u_\tau) \leqslant -2 \liminf_{\tau \searrow 0} \tau \phi(u_\tau) + d^2(u, v).$$

From (iii), $\phi(u_\tau)$ is bounded from below as $\tau \searrow 0$ since we assume that this expression is well-defined on $(0, \tau_\star)$; we obtain

$$\limsup_{\tau \searrow 0} d^2(u, u_\tau) \leqslant d^2(u, v). \tag{23}$$

Choosing a sequence $(v_n)_{n \in \mathbb{N}} \subset D(\phi)$ such that $\mathrm{d}^{>}\text{-}\lim_{n\to\infty} v_n = u$, we find from (23) that $\lim_{\tau \searrow 0} d(u, u_\tau) = 0$. Since $d(u, u_\tau)|_{\tau=0} = 0$, this implies continuity for $\tau = 0$. $\square$

Lemma 4.14 *For $u \in S$, $\tau \in (0, \tau_\star)$ and any choice of $u_\tau \in J_\tau(u)$,*

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \phi_\tau(u) = -\frac{d^2(u, u_\tau)}{2\tau^2} \quad a.e. \tag{24}$$

and

$$\phi(u) - \phi(u_\tau) \geqslant \frac{1}{2\tau} d^2(u, u_\tau) + \int_0^\tau \frac{1}{2r^2} d^2(u, u_r)\, \mathrm{d}r. \tag{25}$$

Proof From Proposition 4.11, $J_\tau(u)$ is non-empty for any $u \in S$ and $\tau \in (0, \tau_\star)$. For $u_\tau \in J_\tau(u)$, one finds

$$\begin{aligned}\phi_{\tau'}(u) - \phi_\tau(u) &\leqslant \Phi_{\tau'}(u, u_\tau) - \Phi_\tau(u, u_\tau) \\ &= \frac{1}{2}\left(\frac{1}{\tau'} - \frac{1}{\tau}\right) d^2(u, u_\tau) = \frac{\tau - \tau'}{2\tau\tau'} d^2(u, u_\tau).\end{aligned} \tag{26}$$

Thus,

$$\begin{aligned}\lim_{\tau' \nearrow \tau} \frac{\phi_{\tau'}(u) - \phi_\tau(u)}{\tau' - \tau} &\geqslant - \lim_{\tau' \nearrow \tau} \frac{1}{2\tau\tau'} d^2(u, u_\tau) = -\frac{1}{2\tau^2} d^2(u, u_\tau), \\ \lim_{\tau' \searrow \tau} \frac{\phi_{\tau'}(u) - \phi_\tau(u)}{\tau' - \tau} &\leqslant - \lim_{\tau' \searrow \tau} \frac{1}{2\tau\tau'} d^2(u, u_\tau) = -\frac{1}{2\tau^2} d^2(u, u_\tau).\end{aligned}$$

These limits exist for every $t \in (a, b)$ and agree for Lebesgue points; this proves (24). From Lemma 4.13 (i), $\tau \mapsto \phi_\tau(u)$ is non-increasing. Thus $\tau' \mapsto \frac{\mathrm{d}}{\mathrm{d}\tau}\phi_\tau(u)\Big|_{\tau=\tau'}$ exists a.e., is in $L^1(\mathbb{R}, \mathbb{R})$ and for $0 < \tau' \leqslant \tau$,

$$\phi_{\tau'}(u) - \phi_\tau(u) \geqslant - \int_{\tau'}^{\tau} \frac{\mathrm{d}}{\mathrm{d}r} \phi_r^{>}(u)\, \mathrm{d}r.$$

Also from Lemma 4.13(i), $\phi(u) \geqslant \phi_{\tau'}(u)$. Thus, in the limit $\tau' \searrow 0$,

$$\phi(u) - \phi_\tau(u) \geqslant - \int_0^{\tau} \frac{\mathrm{d}}{\mathrm{d}r} \phi_r^{>}(u)\, \mathrm{d}r.$$

We use the definition of $\phi(u_\tau)$, insert (24) and obtain

$$\phi(u) - \phi(u_\tau) \geqslant \frac{1}{2\tau} d^2(u, u_\tau) + \int_0^{\tau} \frac{1}{2r^2} d^2(u, u_r)\, \mathrm{d}r.$$

□

4.3 Interpolations

Let $T > 0$ be arbitrary, but fixed. For $N \in \mathbb{N}$, we consider a set of *(time) increments* $\tau(N) := \left\{\tau_n \;\middle|\; \tau_n > 0, n = 1, \dots, N\right\}$ with $\sum_{n=1}^{N} \tau_n = T$. Then,

$$\mathcal{P}_{\tau(N)} := \left\{t_{\tau,n} \;\middle|\; t_{\tau,0} := 0; t_{\tau,n} := t_{\tau,n-1} + \tau_n, n = 1, \dots, N\right\}$$

is a partition of $[0, T]$. We also define $|\tau| := \sup_{n=1,\dots,N} \tau_n$.

Proposition 4.11 establishes the existence of solutions to the forward Moreau-Yosida approximation (15) at discrete time steps when $|\tau| < \tau_\star$. This shows the existence of functions $U_\tau \colon \mathcal{P}_{\tau(N)} \to S$, $t_{\tau,n} \mapsto U_{\tau,n}$ which satisfy

$$U_{\tau,n} \in J_{\tau_n}(U_{\tau,n-1}) \text{ for } n = 1, \dots, N. \tag{27}$$

We introduce interpolations of such functions.

Definition 4.15 (Piecewise-constant interpolation) *Given* $U_\tau \colon \mathcal{P}_{\tau(N)} \to S$ *satisfying* (27), *its* piecewise-constant interpolation $\overline{U}_\tau \colon [0,T] \to S$ *is defined by*

$$\begin{aligned} \overline{U}_\tau\left(t_{\tau,n}\right) &:= U_{\tau,n}, \qquad n = 0,\ldots,N, \\ \overline{U}_\tau\left(t\right) &:= U_{\tau,n-1}, \quad t \in \left(t_{\tau,n-1}, t_{\tau,n}\right), \ n = 1,\ldots,N; \end{aligned}$$

and the piecewise-constant function $\left|d\overline{U}_\tau\right| : (0,T] \to \mathbb{R}_0^+$ *by*

$$\left|d\overline{U}_\tau\right|(t) := \frac{d\left(U_{\tau,n-1}, U_{\tau,n}\right)}{t_{\tau,n} - t_{\tau,n-1}} \qquad \text{for } t \in [t_{\tau,n-1}, t_{\tau,n}), \ n = 1,\ldots,N. \tag{28}$$

Definition 4.16 (Forward De Giorgi interpolation) *Let* $U_\tau \colon \mathcal{P}_{\tau(N)} \to S$ *that satisfies* (27) *be given. A* forward De Giorgi interpolation *of* U_τ *is an interpolation* $\widetilde{U}_\tau \colon [0,T] \to S$ *satisfying*

$$\begin{aligned} \widetilde{U}_\tau\left(t_{\tau,n}\right) &:= U_{\tau,n}, \qquad\qquad\qquad n = 0,\ldots,N, \\ \widetilde{U}_\tau\left(t\right) &\in J_{t-t_{\tau,n-1}}\left(U_{\tau,n-1}\right), \quad t \in \left(t_{\tau,n-1}, t_{\tau,n}\right), \ n = 1,\ldots,N. \end{aligned}$$

Given such $\widetilde{U}_\tau$, *we define* $\left|d\widetilde{U}_\tau\right| : (0,T] \to \mathbb{R}_0^+$ *by*

$$\left|d\widetilde{U}_\tau\right|(t) := \frac{d\left(U_{\tau,n-1}, \widetilde{U}_\tau\left(t\right)\right)}{t - t_{\tau,n-1}} \qquad \text{for } t \in (t_{\tau,n-1}, t_{\tau,n}], \ n = 1,\ldots,N. \tag{29}$$

We immediately obtain the following Corollary to Proposition 4.12:

Corollary 4.17 *For a forward De Giorgi interpolation,*

$$|\partial^{>}\phi|\left(\widetilde{U}_\tau\right) \leqslant \left|d\widetilde{U}_\tau\right|.$$

Next we prove an a priori energy estimate.

Lemma 4.18 (A priori energy estimate) *Let* $|\tau| \in (0, \tau_\star)$. *Then, for* $i,j = 0,\ldots,N$ *with* $i \leqslant j$ *we have*

$$\phi\left(U_{\tau,i}\right) - \phi\left(U_{\tau,j}\right) \geqslant \frac{1}{2}\int_{t_{\tau,i}}^{t_{\tau,j}} \left|d\overline{U}_\tau\right|^2(t)\ \mathrm{d}t + \frac{1}{2}\int_{t_{\tau,i}}^{t_{\tau,j}} \left|d\widetilde{U}_\tau\right|^2(t)\ \mathrm{d}t. \tag{30}$$

Moreover, for any $u_\star \in S$ *and constants* $K, L > 0$, *there exists a constant* C, *depending only on* $u_\star$, $\tau_\star$, K, L *and* T, *such that if*

$$\phi\left(U_{\tau,0}\right) \leqslant K, \qquad d^2\left(u_\star, U_{\tau,0}\right) \leqslant L, \qquad |\tau| < \frac{\tau_\star}{4},$$

then for $n = 1, \dots, N$ and $t \in [0, T]$,

$$\begin{aligned} d^2\left(u_\star, U_{\tau,n}\right) &\leqslant C, &(31)\\ \frac{1}{2}\sum_{j=1}^{n}\frac{1}{\tau_j}d^2\left(U_{\tau,j-1}, U_{\tau,j}\right) &\leqslant \phi\left(U_{\tau,0}\right) - \phi\left(U_{\tau,n}\right) \leqslant C, &(32)\\ d^2\left(\overline{U}_\tau\left(t\right), \widetilde{U}_\tau\left(t\right)\right) &\leqslant 2\left|\tau\right| C. &(33)\end{aligned}$$

Proof Substituting $U_{\tau,n-1}$ for u and $U_{\tau,n}$ for u_τ in (25) and using (28) and (29), we obtain

$$\begin{aligned} &\phi\left(U_{\tau,n-1}\right) - \phi\left(U_{\tau,n}\right)\\ &\quad\geqslant \frac{1}{2\tau_n}d^2\left(U_{\tau,n-1}, U_{\tau,n}\right) + \frac{1}{2}\int_{t_{\tau,n-1}}^{t_{\tau,n}}\frac{d^2\left(U_{\tau,n-1}, \widetilde{U}_\tau\left(t\right)\right)}{\left(t - t_{\tau,n-1}\right)^2}\,\mathrm{d}t &(34)\\ &\quad= \frac{1}{2}\int_{t_{\tau,n-1}}^{t_{\tau,n}}\left|d\overline{U}_\tau\right|^2(t)\,\mathrm{d}t + \frac{1}{2}\int_{t_{\tau,n-1}}^{t_{\tau,n}}\left|d\widetilde{U}_\tau\right|^2(t)\,\mathrm{d}t.\end{aligned}$$

Summing from $n = i + 1$ to $n = j$ proves (30). On the other hand, neglecting the (non-negative) integral in (34) and summing from $n = 1$ to $n = N$ gives

$$\phi\left(U_{\tau,0}\right) - \phi\left(U_{\tau,N}\right) \geqslant \frac{1}{2}\sum_{j=1}^{N}\frac{d^2\left(U_{\tau,j-1}, U_{\tau,j}\right)}{\tau_j}, \tag{35}$$

which is the first inequality in (32). For $\tau' \in (0, \tau_\star)$ we have the trivial bound

$$-\infty < \phi_{\tau'}\left(u_\star\right) \leqslant \frac{1}{2\tau'}d^2\left(u_\star, U_{\tau,n}\right) + \phi\left(U_{\tau,n}\right). \tag{36}$$

The combination of (35) and (36) for $n = N$ yields (32).

Let $\tau' \in (0, \tau_\star)$. Using (i) a telescoping series, (ii) the inequality $d^2(a, b) - d^2(a, c) \leqslant 2d(a, b)\,d(c, b)$ (which is trivial if $d(a, b) < d(a, c)$ and follows from the binomial identity and the triangle inequality otherwise), (iii) Young's inequality, (iv) estimate (35) with $N = n$ and (v) estimate (36), we derive,

$$\begin{aligned}
&d^2(u_\star, U_{\tau,n}) - d^2(u_\star, U_{\tau,0}) \\
&\quad = \sum_{j=1}^{n} \left(d^2(u_\star, U_{\tau,j}) - d^2(u_\star, U_{\tau,j-1}) \right) \\
&\quad \leqslant 2 \sum_{j=1}^{n} d(u_\star, U_{\tau,j}) \cdot d(U_{\tau,j-1}, U_{\tau,j}) \\
&\quad \leqslant \frac{\tau'}{2} \sum_{j=1}^{n} \frac{d^2(U_{\tau,j-1}, U_{\tau,j})}{\tau_j} + \frac{2}{\tau'} \sum_{j=1}^{n} \tau_j d^2(u_\star, U_{\tau,j}) \\
&\quad \leqslant \tau' \left(\phi(U_{\tau,0}) - \phi(U_{\tau,n}) \right) + \frac{2}{\tau'} \sum_{j=1}^{n} \tau_j d^2(u_\star, U_{\tau,j}) \\
&\quad \leqslant \tau' \left(\phi(U_{\tau,0}) - \phi_{\tau'}(u_\star) + \frac{1}{2\tau'} d^2(u_\star, U_{\tau,n}) \right) + \frac{2}{\tau'} \sum_{j=1}^{n} \tau_j d^2(u_\star, U_{\tau,j}) .
\end{aligned}$$

Thus,

$$\begin{aligned}
d^2(u_\star, U_{\tau,n}) &\leqslant 2 \left(d^2(u_\star, U_{\tau,0}) + \tau' \phi(U_{\tau,0}) - \tau' \phi_{\tau'}(u_\star) \right) \\
&\qquad + \frac{4}{\tau'} \sum_{j=1}^{n} \tau_j d^2(u_\star, U_{\tau,j}) \\
&\leqslant 2 \left(L + \tau' K - \tau' \phi_{\tau'}(u_\star) \right) + \frac{4}{\tau'} \sum_{j=1}^{n} \tau_j d^2(u_\star, U_{\tau,j}) .
\end{aligned}$$

We obtain (31) by applying a discrete Gronwall inequality [1, Lemma 3.2.4]. This requires that τ' be picked to be in $(4|\tau|, \tau_\star)$ which is possible since $|\tau| < \frac{\tau_\star}{4}$.

Finally, (33) is an immediate consequence of Lemma 4.13 (ii) and (32): For $t \in [t_{\tau,j-1}, t_{\tau,j})$, $j = 1, \ldots, N$,

$$\begin{aligned}
d^2\left(\overline{U}_\tau(t), \widetilde{U}_\tau(t)\right) = d^2\left(U_{\tau,j-1}, \widetilde{U}_\tau(t)\right) &\leqslant d^2(U_{\tau,j-1}, U_{\tau,j}) \\
&\leqslant |\tau| \sum_{j=1}^{n} \frac{d^2(U_{\tau,j-1}, U_{\tau,j})}{\tau_j} \leqslant 2|\tau| C.
\end{aligned}$$

□

4.4 A Helly-type theorem

We now come to the extension of the Helly-type theorem mentioned in the introduction (see [7, Theorem 3.2] for a related result with different assumptions). The result presented here is a natural extension of the asymmetric Arzelà-Ascoli theorem of Collins and Zimmer [6, Theorem 5.12]; both require an assumption about backward convergence implying forward convergence (or vice versa).

Definition 4.19 (Variation and bounded variation) *For* $U\colon [0,T] \to (S,d)$ *the* forward variation *of* U *is*

$$\mathrm{Var}^{\rhd}(U,[0,T]) := \sup_N \sup_{\tau(N)} \sum_{\substack{i=0\\ t_{\tau,i}\in\mathcal{P}_{\tau(N)}}}^{N-1} d\left(U(t_{\tau,i}), U(t_{\tau,i+1})\right).$$

U *is of* forward bounded variation*,* $U \in BV^{\rhd}\left([0,T],(S,d)\right)$*, if* $\mathrm{Var}^{\rhd}(U,[0,T])$ *is finite.*

Theorem 4.20 *Let (S,d) be an asymmetric metric space satisfying Assumption 4.2, 4.3 and 4.4. Let the forward variations of* $(U_n\colon [0,T] \to (S,d))_{n\in\mathbb{N}}$ *be uniformly bounded in* n*. Then there exists* $U \in BV^{\rhd}\left([0,T],(S,d)\right)$ *and a subsequence (not relabeled) such that* $d^{>}\text{-}\lim U_n = U$ *pointwise.*

Proof For $t \in [0,T]$, let $\varphi_n(t) := \mathrm{Var}^{\rhd}(U_n,[0,t])$. Since, by Lemma 4.18 (see (32)), φ is bounded independently of n, by the classical Helly theorem there exists $\varphi\colon [0,T] \to \mathbb{R}_0^+$ such that up to a subsequence (not relabeled) φ_n converges pointwise to φ.

Since φ is monotone and bounded, it has at most countably many discontinuities. Choose a dense subset $Q \subset [0,T]$ such that the discontinuities are in Q. Fix $t \in Q$. Since (S,d) is forward-complete and $U_n(t)$ is forward bounded, there exists $U(t)$ and a subsequence (not relabelled) such that

$$d\left(U(t),U_n(t)\right) \to 0 \qquad t \in Q. \tag{37}$$

By a Cantor diagonal argument there exists a joint subsequence for every $t \in Q$.

Now fix $t \in [0,T] \setminus Q$. Consider the subsequence constructed so far, which we also denote by U_n. Since $U_n(t)$ is forward bounded there exists a forward accumulation point $U(t)$, i.e., $d\left(U(t),U_n(t)\right) \to 0$. Next we show that this accumulation point is unique:

Consider a sequence $(t_j)_{j\in\mathbb{N}} \subset Q$ such that $t_j \nearrow t$. Then, since $\mathrm{Var}^{\rhd}(U_n,[t_j,t]) \to 0$ uniformly in n, $d\left(U_n(t_j),U_n(t)\right) \to 0$ uniformly in n. Using this fact, Assumption 4.4, the triangle inequality and (37) we obtain

$$\begin{aligned} d\left(U(t_j),U(t)\right) &\leqslant d\left(U(t_j),U_n(t)\right) \\ &\leqslant d\left(U(t_j),U_n(t_j)\right) + d\left(U_n(t_j),U_n(t)\right) \to 0. \end{aligned}$$

Thus $d^{<}\text{-}\lim U(t_j) = U(t)$. By Assumption 4.3 the result follows. □

We now formulate the main result which shows that the interpolations defined in Section 4.3 converge to a forward gradient flow. (Compare [1, Theorem 2.3.3] for gradient flows in metric spaces and [7, Theorem 3.2] for rate-independent evolutions.)

Theorem 4.21 *Let (S, d) be an asymmetric metric space satisfying Assumptions 4.2, 4.3 and 4.4; let $\phi\colon (S,d) \to \overline{\mathbb{R}}$ satisfy Assumptions 4.6, 4.7, 4.5 and 4.8. Let $T > 0$ and $\left(\mathcal{P}_{\tau(N)}\right)_{N\in\mathbb{N}}$ be a sequence of partitions of $[0,T]$ such that $\lim_{N\to\infty} |\tau(N)| = 0$. Let $u_0 \in S$ and $\left(U_{\tau(N)}\colon \mathcal{P}_{\tau(N)} \to S\right)_{N\in\mathbb{N}}$ satisfy $U_{\tau(N)}(0) = u_0$ and (27); let $\left(\overline{U}_{\tau(N)}\right)_{N\in\mathbb{N}}$ and $\left(\widetilde{U}_{\tau(N)}\right)_{N\in\mathbb{N}}$ be their piecewise-constant and forward De Giorgi interpolations, respectively. Then*

(i) There exist a limit function $U \in AC^{\rhd}([0,T],(S,d))$ and subsequences (not relabelled) such that $d^{>}$-$\lim_{N\to\infty} \overline{U}_{\tau(N)} = d^{>}$-$\lim_{N\to\infty} \widetilde{U}_{\tau(N)} = U$ pointwise.

(ii) The limit function U is a forward gradient flow for ϕ w.r.t. $|\partial^{>}_{d^{>}}\phi|$ and satisfies

$$\phi(u_0) - \phi(U(t)) = \frac{1}{2}\int_0^t |U^{\rhd}|^2(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t |\partial^{>}_{d^{>}}\phi|^2(U(s))\,\mathrm{d}s. \tag{38}$$

(iii) The following hold:

$$\lim_{N\to\infty} \phi\left(\overline{U}_{\tau(N)}(t)\right) = \phi(U(t)) \quad \textit{for } t \in [0,T], \tag{39}$$

$$\lim_{N\to\infty} |\partial^{>}_{d^{>}}\phi|\left(\overline{U}_{\tau(N)}\right) = |\partial^{>}_{d^{>}}\phi|(U) \;\; \textit{in } L^2([0,T]), \tag{40}$$

$$\lim_{N\to\infty} \left|d\overline{U}_{\tau(N)}\right| = |U^{\rhd}| \qquad \textit{in } L^2([0,T]). \tag{41}$$

Proof (i). The existence of limit functions $\overline{U}, \widetilde{U}\colon [0,T] \to S$ such that, up to a subsequence,

$$d^{>}\text{-}\lim_{N\to\infty} \overline{U}_{\tau(N)} = \overline{U} \text{ and } d^{>}\text{-}\lim_{N\to\infty} \widetilde{U}_{\tau(N)} = \widetilde{U}$$

follows from the extension of Helly's theorem proved earlier (Theorem 4.20). Since we do not require lower semicontinuity of d in both arguments, we deviate slightly from the symmetric argument and infer that for $t \in [0,T]$, from Assumption 4.4 and (33),

$$\begin{aligned} d\left(\overline{U}(t), \widetilde{U}(t)\right) &\leqslant d\left(\overline{U}(t), \overline{U}_{\tau(N)}(t)\right) + d\left(\overline{U}_{\tau(N)}(t), \widetilde{U}(t)\right) \\ &\leqslant d\left(\overline{U}(t), \overline{U}_{\tau(N)}(t)\right) + d\left(\overline{U}_{\tau(N)}(t), \widetilde{U}_{\tau(N)}(t)\right) \to 0 \end{aligned}$$

as $N \to \infty$. Thus $\overline{U} = \widetilde{U} =: U$.

Next we establish the regularity of U, following the arguments of [1, Corollary 3.3.4]: From (32) $\left(\left|d\overline{U}_{\tau(N)}\right|\right)_{N\in\mathbb{N}}$ is bounded in $L^2([0,T])$, and thus has a weak limit, say, A. For each set of time increments τ and for fixed $0 \leqslant s < t \leqslant T$ we define $s_\tau := \max\left\{r \in \mathcal{P}_\tau \,\middle|\, r \leqslant s\right\}$ and $t_\tau := \min\left\{r \in \mathcal{P}_\tau \,\middle|\, t \leqslant r\right\}$. Then

$$d\left(\overline{U}_{\tau(N)}(s), \overline{U}_{\tau(N)}(t)\right) \leqslant \int_{s_{\tau(N)}}^{t_{\tau(N)}} \left|d\overline{U}_{\tau(N)}\right|(r)\ \mathrm{d}r.$$

Since A is the weak limit of (a subsequence of) $\left(\left|d\overline{U}_{\tau(N)}\right|\right)_{N\in\mathbb{N}}$,

$$\liminf_{N\to\infty} d\left(\overline{U}_{\tau(N)}(s), \overline{U}_{\tau(N)}(t)\right) \leqslant \int_s^t A(r)\ \mathrm{d}r.$$

By Assumption 4.4 this yields

$$d(U(s), U(t)) \leqslant \int_s^t A(r)\ \mathrm{d}r,$$

implying, $U \in AC^{\triangleright}([0,T],(S,d))$.

(ii). Observe that since $U \in AC^{\triangleright}([0,T],(S,d))$, from Theorem 3.5, $|U^{\triangleright}|$ exists a.e.. From Remark 3 it follows that a.e.,

$$|U^{\triangleright}| \leqslant A. \tag{42}$$

Since $\mathrm{d}^{>}\text{-}\lim \overline{U}_{\tau(N)} = U$ pointwise up to a subsequence (not relabelled) and ϕ is $d^{>}$-l.s.c. (Assumption 4.7),

$$\phi(U(t)) \leqslant \liminf_{N\to\infty} \phi\left(\overline{U}_{\tau(N)}(t)\right).$$

After extraction of a further subsequence (again, not relabelled), we obtain

$$\phi(U(t)) \leqslant \lim_{N\to\infty} \phi\left(\overline{U}_{\tau(N)}(t)\right). \tag{43}$$

Corollary 4.17 yields

$$|\partial^{\triangleright}\phi|(U(t)) \leqslant \liminf_{N\to\infty} |\partial^{>}\phi|\left(\widetilde{U}_{\tau(N)}(t)\right) \leqslant \liminf_{N\to\infty} \left|d\widetilde{U}_{\tau(N)}\right|(t). \tag{44}$$

We now proceed as in [1, Section 3.4]; using (42), (43), (44) and Fatou's Lemma,

$$\begin{aligned}
&\frac{1}{2}\int_0^t |U^{\rhd}|^2(s)\ \mathrm{d}s + \frac{1}{2}\int_0^t |\partial_{d^>}^{>}\phi|^2(U(s))\ \mathrm{d}s + \phi(U(t)) \\
&\leqslant \frac{1}{2}\int_0^t A^2(s)\ \mathrm{d}s + \frac{1}{2}\int_0^t \liminf_{N\to\infty}\left|d\widetilde{U}_{\tau(N)}\right|^2 \mathrm{d}s + \lim_{N\to\infty}\phi\left(\overline{U}_{\tau(N)}(t)\right) \\
&\leqslant \liminf_{N\to\infty}\left(\frac{1}{2}\int_0^t \left|d\overline{U}_{\tau(N)}\right|^2(s)\ \mathrm{d}s + \frac{1}{2}\int_0^t \left|d\widetilde{U}_{\tau(N)}\right|^2 \mathrm{d}s\right) \\
&\quad + \lim_{N\to\infty}\phi\left(\overline{U}_{\tau(N)}(t)\right) \\
&\leqslant \phi(u_0)\,.
\end{aligned} \tag{45}$$

On the other hand, since, by Assumption 4.8, $|\partial_{d^>}^{>}\phi|$ is a forward upper gradient for ϕ, we obtain the reverse inequality,

$$\phi(u_0) \leqslant \phi(U(t)) + \int_0^t |\partial^{\rhd}\phi|(U(s))\cdot|U^{\rhd}|(s)\ \mathrm{d}s. \tag{46}$$

Estimates (45) and (46) imply (38) and that for a.e. $t\in[0,T]$,

$$\begin{aligned}
|U^{\rhd}|(t) &= |\partial^{\rhd}\phi|(U(t))\,,\\
\phi(u_0) - \phi(U(t)) &= \int_0^t |\partial^{\rhd}\phi|(U(s))\cdot|U^{\rhd}|(s)\ \mathrm{d}s,
\end{aligned}$$

which shows that $t\mapsto\phi(U(t))$ is locally absolutely continuous and a.e.,

$$(\phi\circ U)' = -\,|\partial^{\rhd}\phi|\circ U\ \cdot\ |U^{\rhd}| = -\frac{1}{2}|U^{\rhd}|^2 - \frac{1}{2}\left(|\partial^{\rhd}\phi|\circ U\right)^2.$$

(iii). Again estimates (45) and (46) imply (39). Along with the definition of A they imply (40), and along with (44) imply (41). □

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