Stability Properties and Nonlinear Mappings of Two and Three-Layer Stratified Flows

By L. Chumakova, F. E. Menzaque, P. A. Milewski, R. R. Rosales, E. G. Tabak, and C. V. Turner

Two and three-layer models of stratified flows in hydrostatic balance are studied. For the former, nonlinear transformations are found that map [*baroclinic*] two-layer flows with either rigid top and bottom lids or vertical periodicity, into [*barotropic*] single-layer, shallow water free-surface flows. We have previously shown that two-layer flows with Richardson number greater than one are nonlinearly stable, in the following sense: when the system is well-posed at a given time, it remains well-posed through the nonlinear evolution. Here, we give a general necessary condition for the nonlinear stability of systems of mixed type. For three-layer flows with vertical periodicity, the domains of local stability are determined and the system is shown not to satisfy the necessary condition for nonlinear stability. This means that there are wave-motions that evolve into shear unstable flows.

1. Introduction

The relevance of multilayer models for stratified flows arises from two main sources: On the one hand, some natural flows are very well approximated by a set of layers. Examples include the layer of fresh water overlying salty

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waters when sea-ice melts or near a river outflow into the sea, the oceanic and atmospheric mixed layers, and even the full troposphere, thought of as a relatively uniform layer (in terms of potential temperature) underlying the stratosphere. On the other hand, multilayer settings can work as conceptual models for continuously stratified flows. Examples ubiquitous in the literature include the modeling of the first baroclinic mode of both atmosphere and ocean as a two-layer flow. Flows with more layers are used in isopycnal general circulation models as a natural discretization of continuously stratified flows [2].

From a more theoretical perspective, the consideration of a continuosly stratified profile as a limiting case of a multilayer one allows one to extend to continuosly stratified flows some powerful tools arising in discrete systems of conservation laws. This has been pursued in [3] to study fully nonlinear, breaking simple waves, and to establish a criterion for local stability (more precisely, well-posedness) based of the system's type: hyperbolic when stable, elliptic otherwise.

Stratified flows are succeptible to shear-instabilities, leading to local mixing and homogenization. Classical results for the instability to shear of continuously stratified flows can be found in [11, 4]. The extension of these results to characterize the well-posedness of unsteady, nonplanar flows, has been studied in [3]; the possibility of nonlinear instability of unsteady flows has been shown in [9, 10].

The characterization of stabilty in terms of the system's type can be used to inquire on the *nonlinear* stabilty of a flow. We use the words "nonlinear stability" to mean that a flow whose dynamics is initially well posed remains so throughout its smooth evolution (generically, waves will eventually break and further evolution requires a closure). It was shown in [12] that two-layer flows with Richardson number bigger than one are nonlinearly stable (the proof presented in the present article is an alternative one, based on the mapping between two-layers and standard free-surface shallow waters). We establish a more general necessary condition for nonlinear stabilty of systems of mixed type. Two-layer flows satisfy this condition, but three-layer flows do not.

Section 2 concentrates on two-layer flows, discussing their nonlinear stability, and showing the surprizing map between two-layer flows with either rigid top and bottom lids or two-layer vertical periodicity, and single-layer, free-surface shallow-water flows. This map is a fully nonlinear extension of the well-known similarity at the linear level between baroclinic and barotropic modes. Two-layer flows with either rigid lids or vertical periodicity are the simplest (and most commonly used) instances of baroclinic modes: when one-layer expands, the other shrinks, and the corresponding fluid velocities point in opposite directions. Single-layer, free surface flows, on the other hand, are the prototype representatives of barotropic modes, with a depth-independent velocity field.

Section 3 proves a general result on nonlinear stability of systems of mixed type. Our result relates nonlinear stability to the invariance of the tangent plane to the sonic surface under the action of the matrix specifying the system's dynamics.

Two-layer flows are special in more than one way. In particular, they give rise to systems of only two conservation laws, and so have Riemann invariants that make them essentially integrable up to wave breaking. Section 4 studies three-layer flows, the simplest among the multilayered flows of a more "general" structure. We characterize their stable domain numerically, and propose some analytical formulas that seem to describe this domain well. We also show that a strong theorem on nonlinear stability similar to that for two-layer flows does not apply to the three-layer case.

2. Two-layer shallow water flows

2.1. Flows bounded by rigid lids

The simplest scenario for internal waves in a stratified flow has two-layers of incompressible fluid with slightly different densities, between two horizontal rigid lids. This is the case studied in [12, 8, 1, 13]. The nondimensionalized equations describing the flow are

$$h_t + uh_x + hu_x = 0 \tag{1}$$

$$u_t + \frac{1-3h}{1-h}uu_x + \left((1-h) - \frac{1}{(1-h)^2}u^2\right)h_x = 0. \tag{2}$$

Here the velocity of the lower layer is given by $u_1 = \sqrt{g'H}u$, where $g' = g\frac{\Delta\rho}{\rho_1}$ is the reduced gravity constant, and H is the distance between the two rigid lids. The height of the lower layer is $h_1 = Hh$. The variables u_2 and h_2 for the upper layer follow from the constancy of the total height $H = h_1 + h_2$ and the volume flow $Q = h_1u_1 + h_2u_2$, which is set to be zero by the choice of an appropriate frame of reference. The equations (1, 2) form a system of mixed type, with characteristics

$$\frac{dx}{dt} = \frac{1-2h}{1-h}u \pm \sqrt{h\frac{(1-h)^2 - u^2}{1-h}},$$

that are real when

$$\frac{(1-h)^2}{u^2} > 1 \tag{3}$$

and complex otherwise. Our characterization of stability in [3] identifies real eigenvalues (i.e., the system's hyperbolicity) with local stability. In the adopted frame where $Q = 0$, the quantity

$$Ri = \frac{\frac{g}{\rho_1}\left(\frac{\rho_1-\rho_2}{H}\right)}{\left(\frac{u_1-u_2}{H}\right)^2} = \frac{(1-h)^2}{u^2} \tag{4}$$

is the Richardson number for two-layer flows. The system is hyperbolic when $Ri > 1$. It was proved in [12] that the elliptic domain $Ri < 1$, unstable to shear, is unreachable from hyperbolic initial data. Here we show the same result from a different perspective, building a map between two-layer flows and their single-layer counterpart, for which nonlinear stability is well-known.

We reformulate the problem in terms of the Riemann invariants. First, in terms of the variables

$$v = 1-2h, \quad r = \frac{1}{\sqrt{Ri}} = \frac{u}{1-h},$$

the equations adopt the symmetric form

$$v_t + \left(\frac{1}{2}r(v^2-1)\right)_x = 0 \tag{5}$$

$$r_t + \left(\frac{1}{2}v(r^2-1)\right)_x = 0, \tag{6}$$

with characteristics

$$\lambda^{\pm} = vr \pm \frac{1}{2}\sqrt{(1-v^2)(1-r^2)}. \tag{7}$$

To compute the Riemann invariants, one multiplies the system on the left by the corresponding left eigenvectors

$$\left(\frac{1}{\sqrt{1-v^2}}, \mp\frac{1}{\sqrt{1-r^2}}\right),$$

yielding

$$R_t^{\pm} + \lambda^{\pm}R_x^{\pm} = 0,$$

where $dR^{\pm} = (\frac{dv}{\sqrt{1-v^2}}, \mp\frac{dr}{\sqrt{1-r^2}})$, so

$$R^{\pm} = \arcsin(v) \mp \arcsin(r). \tag{8}$$

The characteristic speeds can be written in terms of the Riemann invariants:

$$\lambda^{+} = \frac{3}{4}\cos(R^{+}) - \frac{1}{4}\cos(R^{-}) \tag{9}$$

$$\lambda^{-} = \frac{1}{4}\cos(R^{+}) - \frac{3}{4}\cos(R^{-}). \tag{10}$$

This suggest replacing the Riemann invariants by their cosines,

$$R = \cos(R^{+}) \quad \text{and} \quad L = \cos(R^{-}),$$

in terms of which the system can be written in the simple form

$$R_t + \left(\frac{3}{4}R - \frac{1}{4}L\right) R_x = 0 \tag{11}$$

$$L_t - \left(\frac{3}{4}L - \frac{1}{4}R\right) L_x = 0. \tag{12}$$

Here we can apply the theorem proved in [12], valid for general systems of two conservation laws, that hyperbolic initial data will remain hyperbolic for all times (up to breaking), provided that the characteristic speeds are smooth functions of the Riemann invariants. Yet the following observation provides an alternative proof.

2.2. Transformation to one-layer flows

Surprizingly, the characteristic form (11, 12) is precisely the same one as for the single-layer shallow water equations

$$h_t + (hu)_x = 0 \tag{13}$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}h^2\right)_x = 0. \tag{14}$$

For these, the Riemann invariants R and L are given by

$$R = 2\sqrt{h} + u, \quad L = 2\sqrt{h} - u,$$

with inverse

$$h = \left(\frac{R+L}{4}\right)^2, \quad u = \frac{R-L}{2}.$$

Substituting these into the characteristic speeds

$$\lambda^{\pm} = u \pm \sqrt{h},$$

yields precisely the same form as in the previous case (11, 12).

This coincidence supplies an explicit one-to-one correspondence between smooth solutions to the single-layer shallow water equations and two-layer flows: a solution in either of the two settings can be written in terms of the Riemann invariants, and then re-interpreted in the other setting, by writing the Riemann invariants in terms of the corresponding set of physical variables.

Instability of the shallow-water equations corresponds to the height h becoming negative. Because it is well-known that this cannot happen from smooth data with positive h, the nonlinear-stability of two-layer flows is established. Notice, however, that the map is between smooth solutions. Therefore, stability can only be established up to the time of wave breaking.

The implication of this map between single and two-layer flows goes beyond the proof of stability of the latter. Two-layer flows are often used as surrogates for the first baroclinic mode of both ocean and atmosphere. It is well-known that, at the linear level, the barotropic and first baroclinic modes (and, in fact, all others as well) behave in exactly the same way, once their times are scaled appropriately. Here we show that this analogy extends to fully nonlinear solutions, up to their breaking time.

Moreover, as we show below, multilayer flows with two-layer periodicity are also nonlinearly equivalent to shallow waters.

2.3. Vertically periodic layers

The equations describing N-layer Boussinesq flows can be written in the form [3]

$$\begin{aligned} S_t^j - ((1-S^j)u^j)_x &= 0, \\ u_t^j + u^j u_x^j + M_x^j &= 0, \\ \Delta_2 M^j &= S^j, \end{aligned} \tag{15}$$

where $j = 1, \dots N$, with constraints

$$\sum_{j=1}^{N} S^j = 0 \tag{16}$$

$$\sum_{j=1}^{N} u^j(-1+S^j) = 0. \tag{17}$$

The nondimensionalized thickness of each layer is $h^j = 1 - S^j, u^j$ is the corresponding mean velocity, the density differences between layers have been normalized to 1, and M^j is the discrete Montgomery potential, given by

$$M^j = \frac{1}{2}\left(p^{j+\frac{1}{2}} + p^{j-\frac{1}{2}}\right) + g\rho^j \frac{1}{2}\left(z^{j+\frac{1}{2}} + z^{j-\frac{1}{2}}\right). \tag{18}$$

The variable $p^{j+\frac{1}{2}}$ represents the pressure at the interface between layers, $z^{j+\frac{1}{2}}$ its height, and $\Delta_2 M^j$ stands for the discrete second difference $M^{j+1} - 2M^j + M^{j-1}$. The mean layer thickness (one in this nondimensionalization) and its effect on the Montgomery potential have been removed from the dynamical variables S and M. Hence, one may consider flows that are a vertically periodic perturbation of a background stratification; such that

$$S^{j+N} = S^j \quad u^{j+N} = u^j \quad M^{j+N} = M^j. \tag{19}$$

In physical terms this correponds to infinitely many homogeneous layers where the density jump amongst layers has been adopted constant for simplicity [3], and where the thickness has period N in the vertical. This is a appropriate description of all baroclinic modes.

In particular, for flows with two-layer periodicity ($N = 2$), we have $S^1 = -S^2 \equiv S$,

$$S = 2(M^2 - M^1),$$

and

$$S(u^2 - u^1) = -(u^1 + u^2).$$

Introducing $w = u^1 - u^2$, the strength of the vortex sheet at the interface between layers,

$$u^1 = \frac{S+1}{2} w, \quad u^2 = \frac{S-1}{2} w,$$

the system of equations reduces to

$$S_t + \left(\frac{1}{2}(S^2 - 1)w\right)_x = 0, \tag{20}$$

$$w_t + \left(\frac{1}{2}(w^2 - 1)S\right)_x = 0. \tag{21}$$

This system is the same as the rigid lid system (5, 6) with the equivalence $S \leftrightarrow v$ and $w \leftrightarrow r$, hence providing a map between solutions of the two systems.

2.4. A numerical example

To illustrate the explicit nonlinear mapping among solutions of the three systems (two-layer flows with rigid lids, multilayer flows with two-layer periodicity and single-layer shallow water) we consider a simple wave with $L = 0$, and R satisfying the Hopf equation

$$R_t + \frac{3}{4} R R_x = 0.$$

A solution for $R(x, t)$, with initial data $R(x, 0) = \frac{2}{3}(\frac{3}{4} + \frac{1}{2}\sin(x))$ is provided in Figure 1, together with its translation in Figure 2 into the three physical settings.

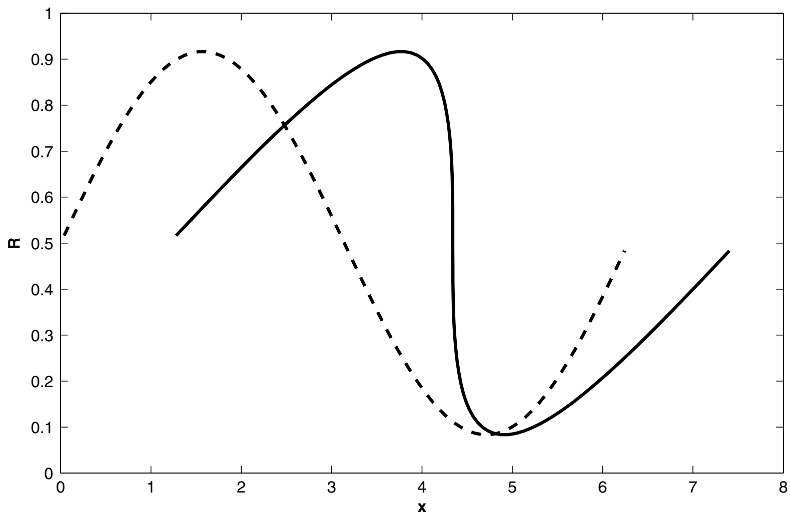

Figure 1. Simple wave with $L = 0$ and R initially sinusoidal, up to the breaking time.

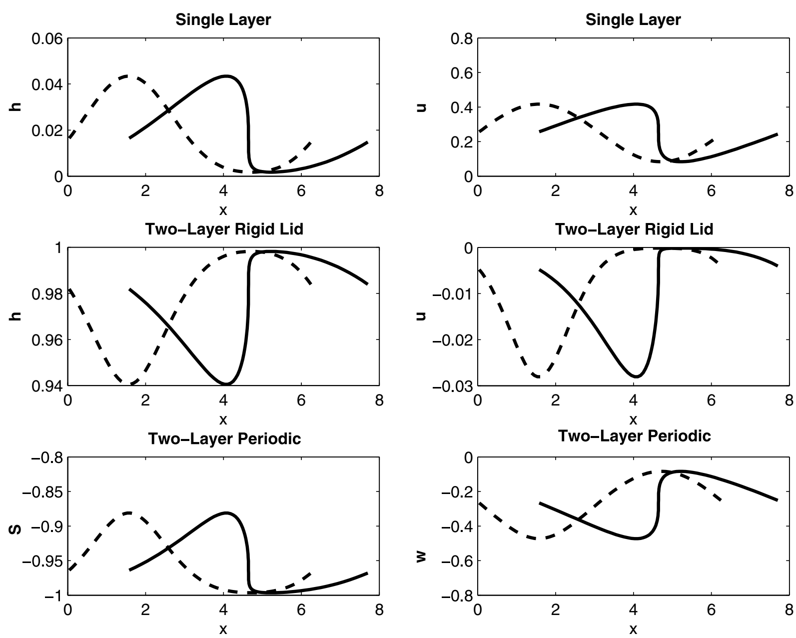

Figure 2. Simple wave with $L = 0$ and R initially sinusoidal, up to the breaking time, mapped into three different flows: a single shallow water layer, two-layers with top and bottom rigid lids, and two vertically periodic layers.

This solution breaks at time $t = 4$, a generic behavior for shallow water waves. After breaking, the Riemann invariant formulation is no longer valid and the three physical scenarios need not have an explicit correspondence: physically relevant jump conditions would differ in the three cases [5]. In the Appendix we provide a method of identifying all possible conserved quantities of the systems, from which one could choose physically relevant shock conditions.

3. A criterion for nonlinear stability

The criterion for stabilty of a system of conservation laws based on its type is local in time: if, at time t, the system is hyperbolic at every point x, the evolution is locally well posed, and we denote it stable. A global in time criterion, if available, would state under which conditions a system that is everywhere hyperbolic at a given time will remain hyperbolic at least for a finite time interval. We would then qualify the system as nonlinearly stable, because the nonlinear evolution of the system does not bring about instabilities. This issue may be further complicated by the possible breaking of waves, which is an "overturning instability" fundamentally different from the one due to shear. A breaking wave remains hyperbolic even past the breaking point, while shear instability arises as the system turns elliptic.

In [12], we proved that two-layer systems are nonlinearly stable up to breaking, and showed a sufficient criterion for nonlinear stability of general 2×2 autonomous systems of conservation laws. Here we derive a necessary criterion for systems of conservation laws of any size to be nonlinearly stable.

A necessary condition for a system of mixed type

$$u_t + A(u)u_x = 0,$$

to be nonlinearly stable is that at every point u on the sonic surface $S(u) = 0$ (the surface in phase space where the system changes type) with a degenerate eigenvalue, the plane T tangent to the sonic surface must include the eigenvector of A corresponding to that eigenvalue.

On the sonic surface, at least one eigenvalue of A will be repeated (have algebraic multiplicity greater than one). A *degenerate eigenvalue* λ has algebraic multiplicity 2 and geometric multiplicity 1 (only one eigenvector). This is the generic case along the sonic surface.

The proof of this criterion involves two steps. First, we show that for the system to be nonlineary stable, T needs to be invariant under the action of A. This follows from considering the solution in phase-space at the time when it touches the sonic surface. At this time, generically, the solution is sonic at one point. In phase space, it is represented by a curve $u(x)$ tangent to the sonic surface at that point, where x plays the role of a parameter along the curve. At the point of intersection, $u_x \in T$. If T were not invariant under A, the curve $u(x)$ could be chosen so that its tangent at the contact point is transformed by A into a vector not in T. Then, u_t will be transversal to $S(u) = 0$, leading u into the elliptic domain.

In the second step, we show that for an hyperplane T of codimension 1 to be invariant under the action of a matrix A with a degenerate eigenvalue λ, it needs to include the eigenvector r corresponding to λ. This follows from considering the plane R spanned by r and the generalized eigenvector s defined by

$$(A - \lambda I)s = r, \quad Ar = \lambda r. \tag{22}$$

By assumption $AT \subset T$, and, clearly $AR \subset R$. Hence

$$A(T \cap R) \subset (T \cap R). \tag{23}$$

Because T has codimension 1 and R is two-dimensional, $T \cap R$ is either one- or two-dimensional. If it is two-dimensional then $T \cap R$ is R itself, whereas if it is one-dimensional, then (23) implies that it is spanned by the eigenvector of A, which is r. In either case, r is included in T, which concludes the proof.

To illustrate this criterion we apply it to the two-layer case which we know to be nonlinearly stable. Consider (1, 2) on the sonic curve $u = 1 - h$. There, the degenerate eigenvalue λ is $1 - 2h$ and the right eigenvector $r = (1, -1)^T$ is tangent to the sonic curve, as required.

4. Three-layer flows

The case with two-layers is somewhat special: it leads to a system of two equations in two unknowns, which has therefore Riemann invariants. Moreover, the discrete Laplacian Δ_2 behaves like a first-order difference for two-layers, which makes the equations structurally different from those with more layers. It appears natural, therefore, to study next a three-layer system, which is the simplest among the "general" multilayer flows. Here Riemann invariants are not to be expected, and hence it is not clear a priori whether a nonlinear stability result will hold.

4.1. Formulation

To reduce the system (15) with $N = 3$ of six equations to a 4×4 system using the restrictions (16, 17), we define the new variables

$$w^{12} = u^2 - u^1, \quad w^{23} = u^3 - u^2, \quad w^{31} = u^1 - u^3, \tag{24}$$

which can be inverted using (17), to yield

$$
\begin{aligned}
u^1 &= \frac{1}{3}[(S^2-1)w^{12} - (S^3-1)w^{31}],\\
u^2 &= \frac{1}{3}[(S^3-1)w^{23} - (S^1-1)w^{12}],\\
u^3 &= \frac{1}{3}[(S^1-1)w^{31} - (S^2-1)w^{23}].
\end{aligned}
\tag{25}
$$

The differences of the M's can also be easily inverted:

$$
\begin{aligned}
M^2 - M^1 &= -\frac{1}{3}(S^2 - S^1),\\
M^3 - M^2 &= -\frac{1}{3}(S^3 - S^2),\\
M^1 - M^3 &= -\frac{1}{3}(S^1 - S^3).
\end{aligned}
\tag{26}
$$

Clearly only two of the w's and two of the S's are independent, because $w^{12} + w^{23} + w^{31} = 0$ and $S^1 + S^2 + S^3 = 0$; hence the reduction from 6 to 4 equations

$$
\begin{aligned}
S^1_t - ((1-S^1)u^1)_x &= 0,\\
S^2_t - ((1-S^2)u^2)_x &= 0,\\
w^{12}_t + u^2u^2_x - u^1u^1_x - \frac{1}{3}(S^2 - S^1)_x &= 0,\\
w^{23}_t + u^3u^3_x - u^2u^2_x - \frac{1}{3}(S^3 - S^2)_x &= 0.
\end{aligned}
$$

with $u's$ given above. The system has the form $V_t + AV_x = 0$, where $V = (S^1\, S^2\, w^{12}\, w^{23})^T$.

4.2. Stability criteria

Because we have an analytic expression for the matrix A, we can find the location of the three-dimensional sonic surface. In practice this is difficult to do analytically except in special cases. An example where the surface can be described analytically is the section with $S_1 = S_2 = 0$, corresponding to a point (x, t) where the three-layers have the same width. This stability region is the hexagon displayed on the left in Figure 3, given by the inequalities

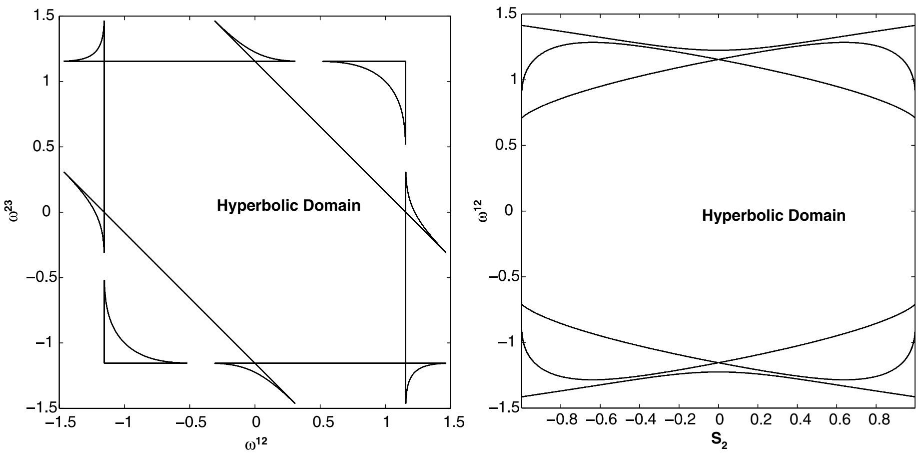

Figure 3. Two sections of the three-dimensional boundary of the stable domain for the three-layer system. On the left we show the section $S_1 = S_2 = 0$. On the right, the section is $S_1 = \omega^{23} = 0$. (Outside the central hyperbolic regions, there are other areas where the system is hyperbolic. These stable domains, however, are not physically meaningful: they correspond to two interfaces becoming unstable which, in a layered formulation, results in the cancellation of the two instabilities.

$$\begin{aligned}\frac{w^{12^2}}{(1-S^1)+(1-S^2)} &= \frac{1}{Ri_{12}} \le \frac{2}{3}\\ \frac{w^{23^2}}{(1-S^2)+(1-S^3)} &= \frac{1}{Ri_{23}} \le \frac{2}{3}\\ \frac{w^{31^2}}{(1-S^3)+(1-S^1)} &= \frac{1}{Ri_{31}} \le \frac{2}{3}.\end{aligned} \tag{27}$$

The Richardson number for each layer pair, given be the expression

$$Ri_{12} = \frac{g'(h_1+h_2)}{(u_1-u_2)^2}, \tag{28}$$

in terms of the dimensional variables, can be shown to equal the quotient E_p/E_k of potential energy barrier to mixing 2 consecutive layers divided by the kinetic energy available for mixing. Recall that, in the two-layer case, stability is equivalent to $Ri_{12} \ge 1$, while in the continuous case, a sufficient condition for linear stability in terms of energy quotients is $Ri \ge \frac{1}{2}$, corresponding to the classical $1/4$ criterion in terms of the conventional definition of the Richardson number [4, 3]. On the right in Figure 3 is another section of the sonic surface for $S_1 = \omega^{23} = 0$.

Two-layer flows are nonlinearly stable: when the initial data are in the hyperbolic domain, they remain there for all times. This result was linked to the fact that the system describing two-layer flows has Riemann invariants.

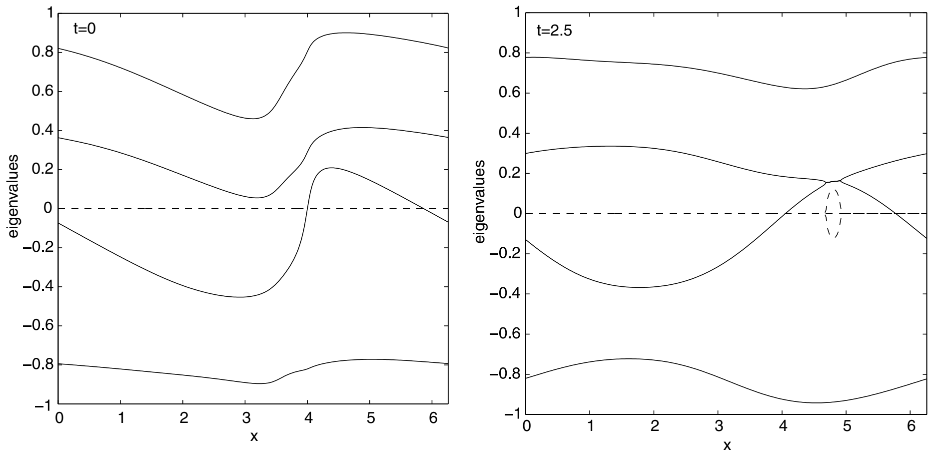

Figure 4. Numerical simulation of a three-layer flow showing that initially hyperbolic data can evolve into the elliptic region. We show the four eigenvalues of the system as a function of position. The dashed lines correspond to the imaginary part of the eigenvalues. Left: Eigenvalues for the initial data. Right: Eigenvalues at $t = 2.5$.

This is not the case for three-layer flows, for which, in fact, such a strong nonlinear stability result *does not* hold. This can be seen by providing a counterexample to the necessary condition for nonlinear stability of Section 3. We have verified numerically that typical points on the sonic boundary on the right panel of Figure 3 do not satisfy the necessary condition, providing the required counterexample.

To illustrate the fact that initially hyperbolic data (that is, with shears below the threshold for local instability) can lead to solutions which cross the sonic surface at a later time, we show in Figure 4 a numerical example where two or the four eigenvalues collide and become complex, making the problem elliptic. This is an example of a shear instability arising from the nonlinear wave motion.

5. Conclusions

This article considers two-layer flows, as well as multilayered flows which are vertically periodic, with two and three-layer periodicity.

We show that two-layer flows can be mapped into single-layer, free-surface flows. This extends the known linear analogy between barotropic and first-baroclinic waves into the nonlinear realm. Moreover, another map shows that multilayer flows with two-layer periodicity are also equivalent to one-layer shallow water.

Multilayered models are systems of conservation laws, for which the natural characterization of local stability is the system's type: a hyperbolic system is stable, whereas an elliptic one, ill-posed in time [6], is unstable. In this framework, we have established a new nonlinear stability criterion for general systems of conservation laws, a necessary condition for nonlinear stability based on the local geometry of the sonic surface: in phase space, its tangent plane needs to contain the eigenvector corresponding to the degenerate eigenvalue. Using this condition, it is shown that contrary to two-layer flows, three-layer flows are not nonlinearly stable.

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Appendix: Conserved Quantities

A single layer shallow water flow must conserve mass and momentum even when shocks form, so the conservation form in (13, 14), which preserves the integrals of h and hu, is the correct one. For two-layer flows, the situation is far more complex. If the two fluids are miscible, then the mass of the individual layers needs not be conserved; and their densities will change due to entrainment at shocks. As for the momentum of the individual layers, it is not clear a priori even for immiscible fluids how they will evolve at shocks, because the momentum exchange between layers at the location of the jump cannot be calculated without further hypothesis [7].

To explore possible closures at shocks for our two-layer system, we may start by asking which candidate conserved quantities are consistent with the equations (11, 12) in smooth parts of the flow. In other words, we want to write the system in the conservation form $F_t + G_x = 0$, where $F(R, L)$ is a conserved quantity, and $G(R, L)$ its associated flux. Expanding $F_t = F_R R_t + F_L L_t$ and $G_x = G_R R_x + G_L L_x$, we find that

$$F_R(3R - L) = 4G_R, \tag{A.1}$$

$$F_L(3L - R) = -4G_L. \tag{A.2}$$

so all conserved quantities F must satisfy the PDE

$$2(R + L)F_{RL} = F_R + F_L, \tag{A.3}$$

and reciprocally all solutions to (A.3) represent conserved quantities consistent with the flow evolution in smooth parts.

For shallow water, which has the same Riemann invariant form, $h = \frac{1}{4}(R + L)^2$ and $u = \frac{1}{2}(R - L)$. Therefore, possible conserved quantities include the height h, velocity u momentum hu, and energy $h^2 + hu^2$, which, when translated through the map to two-layer and multilayer with two-layer periodicity flows gives more complicated expressions.

Carrying out a similar procedure for F in terms of h and u for two-layer flows with rigid lid gives the following equation for the candidate conserved quantity $F(h, u)$:

$$hF_{hh} - \frac{2hu}{1-h}F_{uh} + \left(\frac{u^2}{(1-h)^2} + h - 1\right)F_{uu} = 0. \tag{A.4}$$

Solutions here include h, u, the energy $h^2 + hu^2/(1-h)$, and $r = u/(1-h)$. The momentum hu of the lower layer, on the other hand, is not conserved even by the smooth evolution.

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