Integral points on surfaces

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Abstract

This thesis studies integer solutions to polynomial equations, using modern algebraic geometry. Our focus lies on understanding the arithmetic properties of geometric objects defined by these equations, which in turn sheds light on the solutions to the original polynomial equations.

To begin with we study two different families of cubic polynomials, which can be seen as generalisations of the sum of three cubes conjecture. In particular, we consider whether the existence of local solutions is not sufficient for the existence of global solutions, i.e. do any of these equations fail the integral Hasse principle? To do so we use the integral Brauer–Manin obstruction. In the first case we show that the integral Brauer–Manin obstruction does not yield any failure of the integral Hasse principle for nearly all these equations.

In studying the second family of cubic equations (which is joint work with Julian Lyczak and Vladimir Mitankin) we show that the integral Brauer–Manin obstruction can give failures of the integral Hasse principle, providing the first examples of such equations failing the integral Hasse principle. Furthermore, we also quantify the frequency of these failures.

We proceed by studying a family of polynomial equations which admit a special geometric structure, namely a fibration of torsors under norm 1 tori. Here we give sufficient conditions for the existence of integer solutions using the descent-fibration method.

Finally, we move on to studying singular del Pezzo surfaces over finite fields in an attempt to provide an analogue of a conjecture of Lang. Explicitly, we want to know if there is a smooth point on these surfaces. We show that away from characteristic 2, this is always true, however there are counterexamples in the case of characteristic 2.

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Contents

Notation

- K will be reserved for a number field.
- Ω_K is the set of places of K.
- Ω_K^∞ is the set of infinite places of K.
- We denote by K_v the completion of K at $v \in \Omega_K$.
- $\mathcal{O}_K$ denotes the ring over integers of K.
- We denote by $\mathcal{O}_{K_v}$ the completion of $\mathcal{O}_K$ at the place v.
- Let G be a group and M a (left) G-module. We denote by M^G the submodule of M whose elements are invariant under the action of G i.e.

$$M^G = \{m \in M : g\dot{m} = m \text{ for all } g \in G\}.$$

- We impose that a scheme is always noetherian.
- Given a scheme S over a field F we denote $F[S]$ the global sections of S i.e. $F[S] := \mathcal{O}_S(S)$. Moreover, if S is integral we define $F(S)$ to be the local ring of the generic point of S.
- A variety over a field F is a separated scheme of finite type over $\operatorname{Spec} F$.
- If F is a field we denote by F^{sep} a separable closure of F. If F is perfect we denote by $\bar{F}$ an algebraic closure of F
- Let S be a scheme over a field F. We denote by $\bar{S}$ the scheme

$$\bar{S} := \begin{cases} S \times_{\operatorname{Spec} F} \operatorname{Spec} \bar{F} & \text{if } F \text{ is perfect,} \\ S \times_{\operatorname{Spec} F} \operatorname{Spec} F^{\text{sep}} & \text{otherwise.} \end{cases}$$

Chapter 1

Introduction

This thesis studies integer solutions to diophantine equations. The study of such equations dates back to Diophantus in 200 AD and now forms the basis of many fundamental questions within mathematics. For example, in 1900, Hilbert posed 23 of the most significant problems in mathematics, with the 10th problem asking if it is possible to provide a general algorithm that, for any given diophantine equation can decide whether the equation has a solution with integer values. However, this was subsequently shown to be false by the Davis–Putnam–Robinson–Matiyasevich theorem [Mat70].

This leads to the question: Can one find such an algorithm for subfamilies of diophantine equations? One such way is using local solutions as there is a finite time algorithm for checking if a diophantine equation has local solutions, i.e. solutions in the p-adic integers $\mathbb{Z}_p$ for all primes p and $\mathbb{R}$. Clearly, the existence of local solutions is a necessary condition for the existence of integer solutions, but when is it sufficient? We say a diophantine equation satisfies the *integral Hasse principle* if the existence of local solutions implies the existence of a solution in the integers. As expected there are counterexamples to the integral Hasse principle; e.g. a famous example of Selmer [Sel51].

In 1970, Manin [Man71] introduced the *Brauer–Manin obstruction* which attempts to detect if the integral Hasse principle fails for homogeneous diophantine equations. Manin's construction was later adapted by Colliot-Thélène and Xu to include inhomogeneous equations and is formally called the *integral Brauer–Manin obstruction*. The idea behind both of these constructions is given a diophantine equation denoted by $\mathcal{U}$, we make an intermediary set $\mathcal{U}(\mathbb{A}_\mathbb{Z})^{\mathrm{Br}}$ between the set of local solutions $\mathcal{U}(\mathbb{A}_\mathbb{Z})$ and the set of integer solutions $\mathcal{U}(\mathbb{Z})$, i.e. we have the following inclusions

$$\mathcal{U}(\mathbb{Z}) \subseteq \mathcal{U}(\mathbb{A}_\mathbb{Z})^{\mathrm{Br}} \subseteq \mathcal{U}(\mathbb{A}_\mathbb{Z}).$$

We then have a failure of the integral Hasse principle, or rather an *integral Brauer–Manin obstruction to the integral Hasse principle*, if $\mathcal{U}(\mathbb{A}_\mathbb{Z}) \neq \emptyset$ but $\mathcal{U}(\mathbb{A}_\mathbb{Z})^{\mathrm{Br}} = \emptyset$.

These constructions are more easily explained through Grothendieck's language of algebraic geometry, where solutions to polynomial equations correspond to points on geometric objects called schemes. This is the language we will use throughout the rest of this thesis.

Overview of Chapters 3 and 4

Chapters 3 and 4 are inspired by a conjecture of Heath-Brown, which was initially a question of Mordell [Mor53] and is commonly referred to as the "sum of three cubes conjecture".

This conjecture predicts that any integer not congruent to 4 or 5 modulo 9 can be expressed as the sum of three integer cubes. Geometrically, this family of equations defines a family of affine cubic surfaces and the sum of three cubes conjecture predicts that all these affine cubic surfaces have integral points.

In Chapter 3, we use the integral Brauer–Manin obstruction to study the affine cubic surfaces

$$f(u_1) + f(u_2) + f(u_3) = n$$

where $f \in \mathbb{Z}[x]$ is a fixed monic cubic polynomial and $n \in \mathbb{Z}$. Note that when $f = x^3$, we recover the case of the sum of three cubes conjecture and this particular case was studied by Colliot-Thélène and Wittenberg [CTW12]. We show that for fixed f there is no integral Brauer–Manin obstruction for all but finitely many $n \in \mathbb{Z}$, by taking advantage of the geometry of these surfaces. Explicitly, we prove this result by showing that the Brauer group is trivial for all but finitely many $n \in \mathbb{Z}$. Following this we study in depth the conjecture of whether the sum of three tetrahedral numbers can always represent an integer; specifically, a tetrahedral number is of the form $u(u+1)(u+2)/6$. We want to determine for which integers n the equation

$$u_1(u_1+1)(u_1+2) + u_2(u_2+1)(u_2+2) + u_3(u_3+1)(u_3+2) = 6n \tag{1.0.1}$$

has a solution $(u_1, u_2, u_3) \in \mathbb{Z}^3$. Here we show that the equations (1.0.1) always have local solutions but have no integral Brauer–Manin obstruction for any choice of $n \in \mathbb{Z}$.

In Chapter 4 we examine diagonal cubic surfaces in joint work with Vladimir Mitankin and Julian Lyczak. Explicitly, we study the equations

$$\mathcal{U} : a_1u_1^3 + a_2u_2^3 + a_3u_3^3 = a_0$$

where $(a_0, a_1, a_2, a_3) \in \mathbb{Z}^4$. These surfaces were first studied by Colliot-Thélène and Wittenberg for the cases $a_1 = a_2 = 1$ and $a_3 = 1$ or 2. They were unable to find any examples failing the integral Hasse principle, however we are able show the existence of certain diagonal affine cubic surfaces failing the integral Hasse principle but not failing the Hasse principle for rational points. From there we study the frequency of these failures. We begin with the analogue of the sum of three cubes conjecture i.e. we fix a_1, a_2, a_3 and we vary a_0. Let $\mathbb{Z}^n_{\text{prim}}$ be the set of n-tuples in $\mathbb{Z}^n$ with non-zero coprime coordinates. Let $\mathbb{Z}_{\neq 0}$ denote the non-zero integers, then for any real $B \geq 1$ and $(a_1, a_2, a_3) \in \mathbb{Z}^3_{\text{prim}}$ define

$$N_{a_1,a_2,a_3}(B) = \#\big\{a_0 \in [-B, B] \cap \mathbb{Z}_{\neq 0} \ : \ \mathcal{U}(\mathbb{A}_\mathbb{Z}) \neq \emptyset \text{ but } \mathcal{U}(\mathbb{A}_\mathbb{Z})^{\text{Br}} = \emptyset\big\}.$$

We show that if we assume $a_1a_2a_3 \not\equiv 2 \bmod \mathbb{Q}^{*3}$, we have

$$N_{a_1,a_2,a_3}(B) \ll_{a_1a_2a_3} B^{1/3},$$

as $B \to \infty$. We then continue by varying the cubic form while $a_0 \neq 0$ stays fixed. Let

$$N_{a_0}(B) = \#\big\{(a_1, a_2, a_3) \in [-B, B]^3 \cap \mathbb{Z}^3_{\text{prim}} \ : \ \mathcal{U}(\mathbb{A}_\mathbb{Z}) \neq \emptyset \text{ but } \mathcal{U}(\mathbb{A}_\mathbb{Z})^{\text{Br}} = \emptyset\big\}.$$

We establish the following upper bound

$$N_{a_0}(B) \ll B^{3/2},$$

as $B \to \infty$. Lastly, we vary all four coefficients of $\mathcal{U}$. For this purpose let

$$N(B) = \#\{(a_0, a_1, a_2, a_3) \in [-B, B]^4 \cap \mathbb{Z}^4_{\text{prim}} \ : \ \mathcal{U}(\mathbb{A}_\mathbb{Z}) \neq \emptyset \text{ but } \mathcal{U}(\mathbb{A}_\mathbb{Z})^{\text{Br}} = \emptyset\}.$$

Our final result gives

$$\frac{B^2}{\log B} \ll N(B) \ll B^2(\log B)^6$$

as $B \to \infty$.

Overview of Chapter 5

We say that the integral Brauer–Manin obstruction completely controls the integral Hasse principle if the existence of local solutions and no integral Brauer–Manin obstruction implies the existence of an integer solution. It is useful to understand when the integral Brauer–Manin obstruction completely controls the integral Hasse principle, however there are not many tools which can determine such results. In [SD95] Swinnerton-Dyer developed the descent-fibration method, which consists of combining the fibration method with descent on genus 1 curves and attempts to establish such results. In Chapter 5, we study the surfaces

$$ap_A(t)x^2 + bp_B(t)y^2 = 1$$

where p_A and p_B are products of linear polynomials and a, b are integers. Here we have a fibration of torsors under norm 1 tori and we use an adaptation of Harpaz [Har17] of the descent-fibration method to determine sufficient conditions under the assumption of Schinzel's hypothesis for the integral Brauer–Manin obstruction to completely control the integral Hasse principle.

Overview of Chapter 6

A beautiful family of varieties are Fano varieties. Such varieties give the first non-trivial examples of many of the research topics within geometry and number theory. A conjecture of Lang predicts that over certain fields called C_1-fields (an example of such a field is a finite field) these varieties always have a rational point. However, what about if we consider mildly singular Fano varieties over C_1-fields, where the natural question would be to ask for the existence of smooth points? In Chapter 6, we study this question for singular Fano varieties of dimension 2, which are called singular del Pezzo surfaces. These surfaces can be broken down into subfamilies depending on the degree of the surface and we show away from degree 2 such a statement is true, however we do have counterexamples in the case of degree 2 in characteristic 2.

Chapter 2

Background

2.1 Blow-ups

In the study of birational geometry of surfaces, blow-ups are fundamental because any birational morphism between smooth surfaces can be factored into a sequence blow-ups along closed points. Moreover, any singularity on a surface can be resolved by blowing up in arbitrary characteristic. The characteristic 0 case was done by Zariski in [Zar39] and characteristic p by Abhyankar in [Abh56].

Definition 2.1.1. Let S be a scheme, and $Z \subset S$ a closed subscheme. Then the *blow-up* of S along Z is a scheme S' together with a morphism $\pi : S' \to S$ such that

1. the inverse image $E := \pi^{-1}(Z)$ is an effective Cartier divisor called the *exceptional divisor* of the blow-up,

2. π is the universal such object namely if $\psi : Y \to S$ is any morphism such that the inverse image of Z is an effective Cartier divisor then ψ factors uniquely through π.

Remark 2.1.2. Let $\pi : S' \to S$ be the blow-up of S along a closed subscheme Z and $E := \pi^{-1}(Z)$ then $\pi : S' \backslash E \to S \backslash Z$ is an isomorphism [Har04, §II, Prop. 7.13].

2.1.1 Galois action on blow-ups

For a scheme S over a field F we have an action of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ on $\bar{S}$. Explicitly, any $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/F)$ acts on $\bar{S}$ via

$$\mathrm{id}_S \times \sigma^* : S \times_{\operatorname{Spec} F} \operatorname{Spec} F^{\mathrm{sep}} \to S \times_{\operatorname{Spec} F} \operatorname{Spec} F^{\mathrm{sep}}, \quad (s, x) \mapsto (s, \sigma^* x),$$

where σ^* is the induced morphism from $\sigma : F^{\mathrm{sep}} \to F^{\mathrm{sep}}$. The aim of the following proposition is to understand how this action changes once we blow-up S. This will allow us to understand arithmetic information about S from its blow-up.

Proposition 2.1.3. *Let S be a scheme over a field F and $Z \subset S$ a closed subscheme. Consider the blow-up $\pi : S' \to S$ of S along Z. Then the induced morphism $\bar{S}' \to \bar{S}$ is $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-equivariant.*

Proof. Given $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/F)$ we have a commutative diagram

$$\begin{array}{ccc} S' \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, F^{\mathrm{sep}} & \xrightarrow{\mathrm{id}_{S'} \times \sigma^*} & S' \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, F^{\mathrm{sep}} \\ \downarrow{\scriptstyle \pi \times \mathrm{id}_{F^{\mathrm{sep}}}} & & \downarrow{\scriptstyle \pi \times \mathrm{id}_{F^{\mathrm{sep}}}} \\ S \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, F^{\mathrm{sep}} & \xrightarrow[\mathrm{id}_S \times \sigma^*]{} & S \times_{\mathrm{Spec}\, F} \mathrm{Spec}\, F^{\mathrm{sep}} \end{array}$$

showing the map $\bar{S}' \to \bar{S}$ is $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-equivariant. ☐

2.1.2 Resolution of singularities and rational double points

We now discuss resolution of singularities on surfaces with a focus on rational double point singularities.

Definition 2.1.4. Let S be a surface over a field F. We say that $\pi : S' \to S$ is a *resolution of singularities* if S' is a smooth surface and π is a proper birational morphism. We say that the resolution is *minimal* if every other resolution factors uniquely through it, hence it is unique up to isomorphism. We call the minimal resolution, the *minimal desingularisation* of S.

Remark 2.1.5. Note any singular point on S can be resolved by blowing up [Zar39] (when the characteristic F is 0) and [Abh56] (when the characteristic F is positive).

Definition 2.1.6. Let S be a surface over a field F. We say that $s \in S$ is a *rational singularity* if there exists a resolution of singularities $\pi : S' \to S$ such that the stalk $(R^i\pi_*\mathcal{O}_{S'})_s = 0$ for $i > 0$. We call π the *resolution of* s. Moreover, we say S has *rational singularities* if there exists a resolution of singularities $\pi : S' \to S$ such that $(R^i\pi_*\mathcal{O}_{S'})_s = 0$ for all $s \in S$ and $i > 0$ (equivalently, $R^i\pi_*\mathcal{O}_{S'} = 0$ for $i > 0$).

Definition 2.1.7. Let S be a surface over a field F. We say that $s \in S$ is a *rational double point singularity* if s is a rational singularity and the multiplicity of the maximal ideal $\mathfrak{m}_s$ in the stalk $\mathcal{O}_{S,s}$ is 2 (see [ZS60, Ch. VII, §10, pg.294] for definition of multiplicity of ideals). Moreover, we say S has *only rational double point singularities* if S has rational singularities and for each $s \in S$ the multiplicity of $\mathfrak{m}_s$ in the stalk $\mathcal{O}_{S,s}$ is 1 or 2.

Proposition 2.1.8 ([DPT80, §IV Thm. 1, §V Prop. 1,§V Thms. 1,2])**.** *Let S be a normal surface over a field F. Then $s \in S$ is a rational double point singularity if and only if the dual graph of the resolution of the point s on S is a Coxeter–Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8 (see Table 2.2.1). Moreover, all geometric irreducible components of the exceptional locus are isomorphic to $\mathbb{P}^1$ and have self-intersection -2.*

Definition 2.1.9. Let S be a surface over a field F with only rational double point singularities. For a rational double point singularity $s \in S$ we define the *type* of s to be the Coxeter-Dynkin diagram type of the dual graph of the resolution of s.

Definition 2.1.10. Let S be a surface over a field F with only rational double point singularities. The *resolution graph* of $\bar{S}$ is the union of all dual graphs of resolutions of rational double point singularities on $\bar{S}$.

Definition 2.1.11. Let S be a surface over a field F with only rational double point singularities. Denote by $\pi : S' \to S$ the minimal desingularisation of S. A *line* on S is the image of a (-1)-curve under the morphism π.

We now deduce some facts about surfaces with only rational double point singularities.

Lemma 2.1.12. *Let S be a singular surface with only rational double point singularities over a field F and $S' \to S$ its minimal desingularisation. Then the resolution graph Γ on $\bar{S}'$ is a disjoint union of connected graphs Γ_i, where each Γ_i is of one of types stated in Proposition 2.1.8.*

Proof. We can assume that F is algebraically closed. Suppose $x, y \in S$ are distinct singular points and Γ_x, Γ_y are the resolution graphs of x and y respectively. Let $\pi_x : S_1 \to S$ (resp. $\pi_y : S_2 \to S_1$) be the resolution of the singularity x (resp. y) and $\pi := \pi_y \circ \pi_x$. It is now sufficient to show that Γ_x and Γ_y do not intersect on S_2. As $\pi_x : S_1 \backslash \Gamma_x \to S \backslash \{x\}$, is an isomorphism we have that $y \notin \Gamma_x$ and $\pi_y^{-1}(\Gamma_x) \cap \pi_y^{-1}(\{y\}) = \emptyset$. As $\pi_y : S_2 \backslash \Gamma_y \to S_1 \backslash \{y\}$ is an isomorphism we have that $\pi_y^{-1}(\Gamma_x) = \Gamma_x$ and $\pi_y^{-1}(\{y\}) = \Gamma_y$, hence each distinct singular point gives rise to a connected component of the resolution graph of S. □

Proposition 2.1.13. *Let S be a singular surface with only rational double point singularities over a field F. Then the action of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ on $\bar{S}$ preserves the singularity type of points, i.e. if two singular points $x, y \in \bar{S}$ have different singularity types, then there does not exist $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/F)$ such that $\sigma(x) = y$.*

Proof. Suppose there exists singular points $x, y \in \bar{S}$ of different singularity types such that there exists $\sigma \in \mathrm{Gal}(F^{\mathrm{sep}}/k)$ where $\sigma(x) = y$. Consider the desingularisation of these points $\pi : \bar{S}' \to \bar{S}$ where the resolution graph of x and y is Γ_x and Γ_y respectively. By Lemma 2.1.12 we have that $\Gamma_x \cap \Gamma_y = \emptyset$. By Proposition 2.1.3 the morphism π is $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-equivariant, hence $\sigma(\Gamma_x) \neq \Gamma_x$. As $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ sends (-2)-curves to (-2)-curves and defines a graph automorphism of $\Gamma_x \sqcup \Gamma_y$ the only possibility is $\sigma(\Gamma_x) = \Gamma_y$. However, the intersection matrix of the curves in Γ_x is not equal to the intersection matrix of the curves in Γ_y (up to a change of basis). This provides a contradiction as $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ preserves intersection pairings. □

2.1.3 Picard group of blow-ups

We now turn to understanding how the Picard group and its intersection pairing change for a smooth surface under blowing up a point.

Definition 2.1.14. Let S be a smooth surface and $\pi : S' \to S$, the blow-up of S at a closed point $x \in S$. If D is an effective divisor on S, we define the *strict transform* D^{strict} of D to be the closure of $\pi^{-1}(D \backslash \{x\})$ in S'.

Definition 2.1.15. Let S be a smooth projective surface and $n \in \mathbb{Z}$. A $(-n)$*-curve* on S is a smooth geometrically integral curve $E \subset S$ of genus 0 such that $E^2 = -n$.

Proposition 2.1.16 ([Liu02, Prop. 2.18, Prop. 2.23, Prop. 2.24])**.** *Let S be a smooth surface over a field F and let $\pi : S' \to S$ be the blow-up of S along $x \in S(F)$. Let $E := \pi^{-1}(x)$ be the exceptional divisor of π. Then the natural morphism*

$$\operatorname{Pic} S \oplus \mathbb{Z} \to \operatorname{Pic} S', (D, E) \mapsto D + E$$

is an isomorphism. Moreover,

1. *the canonical divisor $K_{S'}$ of S' is $K_{S'} = \pi^* K_S + E$,*

2. E is a (-1)-curve,

3. for any two divisors D_1, D_2 on S, we have $\pi^*(D_1) \cdot \pi^*(D_2) = D_1 \cdot D_2$,

4. for any effective divisor D on S which contains x with multiplicity r, we have $\pi^*(D) = D^{\text{strict}} + rE$.

2.2 Del Pezzo and weak del Pezzo surfaces

Del Pezzo surfaces have been of great interest to geometers and number theorists for many years. This is due to the fact that their configuration of lines allows one to extract geometric and number theoretic information. Moreover, over an algebraically closed field these surfaces are birational to the projective plane, meaning over such a field they are easy to understand. However, once one is over a non-algebraically closed field these surfaces give examples of non-trivial phenomenon.

Definition 2.2.1. Let X be a smooth projective geometrically integral variety over a field F. We say that X is Fano if its anticanonical divisor $-K_X$ is ample. If $\dim X = 2$ we call X a *del Pezzo surface* and we define the degree of X to be K_X^2.

Definition 2.2.2. Let F be an algebraically closed field. We say a collection of distinct points $\{P_1, \dots, P_r\}$ of $\mathbb{P}^2_F$ where $1 \leq r \leq 8$ are in *general position* if none of the following hold

1. three of the P_i lie on a line,
2. six of the P_i lie on a conic,
3. eight of the P_i lie on a singular cubic, with one of the points lying at the singular point.

Theorem 2.2.3 ([Man86, Thm. 24.4], Classification of del Pezzo surfaces)**.** *Let X be a del Pezzo surface of degree d over an algebraically closed field. Then $1 \leq d \leq 9$ and one of the follow holds*

1. $X \cong \mathbb{P}^2$ (degree 9),

2. $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ (degree 8),

3. X is the blow-up of $9 - d$ points on $\mathbb{P}^2$ in general position (degree $9 - d$).

Moreover, every del Pezzo surface arises in this way.

If we do not blow-up points in general position, but rather in "almost general position" the resulting surface will be a weak del Pezzo surfaces.

Definition 2.2.4. A divisor D on a projective surface X, is called *nef* if for any irreducible curve $C \subseteq X$, the intersection number $C \cdot D \geq 0$. Furthermore, D is called *big* if $D^2 > 0$.

Definition 2.2.5. A *weak del Pezzo surface* is a smooth projective geometrically integral surface X with $-K_X$ nef and big. Its degree is defined to be K_X^2.

Definition 2.2.6. A *blow down structure* on a surface X is a sequence of blow-ups

$$\pi_X : X := X_N \xrightarrow{\pi_N} X_{N-1} \xrightarrow{\pi_{N-1}} \dots X_1 \xrightarrow{\pi_1} X_0$$

where X_0 contains no (-1)-curves and each $\pi_i : X_i \to X_{i-1}$ is the blow-up along a point P_i. We say P_i is *infinitely near* to P_{i-1} if $\pi_i(P_i) = P_{i-1}$.

Remark 2.2.7. For a blow down structure $\pi_X : X \to \mathbb{P}^2$ we can call P_i a point in $\mathbb{P}^2$. This can be done by identifying these points with points in the bubble space of $\mathbb{P}^2$, [Dol12, Def. 7.3.3]. We refer the reader to [Dol12, § 7.3.2] for background on bubble spaces and allow for this abuse of notation by calling P_i a point of $\mathbb{P}^2$.

Definition 2.2.8. Let F be an algebraically closed field. We say a collection of points $\{P_1, \dots, P_r\}$ of $\mathbb{P}^2_F$ (possibly infinitely near) where $1 \leq r \leq 8$ are in *almost general position* if none of the following happen:

1. four of the P_i lie on a line,
2. seven of the P_i lie on a conic,
3. the corresponding blow down structure to $\{P_1, \dots, P_r\}$ has P_i lying on a (-2)-curve of X_{i-1}.

Theorem 2.2.9 ([CT88, Prop. 0.4], Classification of weak del Pezzo surfaces)**.** *Let X be a weak del Pezzo surface of degree d over an algebraically closed field F. Then $1 \leq d \leq 9$ and either*

1. $X \cong \mathbb{P}^2$ *(degree 9),*
2. $X \cong \mathbb{P}^1_k \times \mathbb{P}^1_k$ *(degree 8),*
3. $X \cong \mathbb{F}_2$ *"a Hirzebruch surface" (degree 8),*
4. X *is the blow-up of* $\mathbb{P}^2_k$ *in* $9-d$ *points in almost general position (degree* $9-d$*).*

Conversely, every weak del Pezzo surface arises this way.

Remark 2.2.10. Given a (weak) del Pezzo surface X over a field F, such that $X \times \bar{F}$, where $\bar{F}$ is an algebraic closure of F, is realised as a sequence of blow-ups of $\mathbb{P}^2$, a result of Coombes [Coo88, Thm. 1] shows that $X \times F^{\text{sep}}$ is also realised as a sequence of blow-ups of $\mathbb{P}^2$ in F^{sep}-rational points.

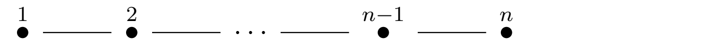
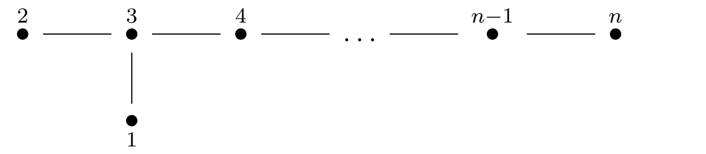
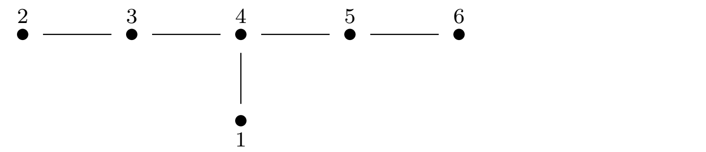
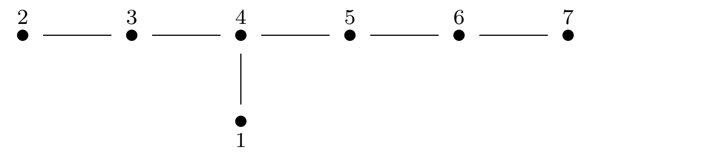
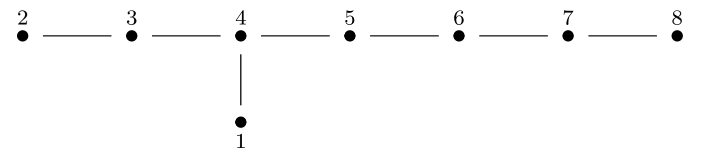
Picard group of (weak) del Pezzo surfaces

A useful way to study the Picard group of (weak) del Pezzo surfaces is via root systems. We now explain this relation.

Definition 2.2.11. A *Cartan matrix* is a symmetric matrix $C := (c_{ij})$ such that $c_{ii} = -2$ and $c_{ij} \geq 0$ for $i \neq j$.

For a Cartan matrix C we can write $C = A - 2I_n$ where I_n is the identity matrix. Then A is the incidence matrix for a union of Coxeter-Dynkin diagrams which are shown in Table 2.2.1.

Table 2.2.1: *Coxeter–Dynkin Diagrams*

Type	Graph
A_n	
D_n	
E_6	
E_7	
E_8	

Definition 2.2.12. A lattice with a quadratic form defined by a Cartan matrix is called a *root lattice.*

Let $I^{1,N} = \mathbb{Z}^{N+1}$ equipped with the symmetric bilinear form defined by the diagonal matrix $\operatorname{diag}(1, -1, ..., -1)$ with respect to the standard basis

$$e_0 = (1, 0, ..., 0),\ e_1 = (0, 1, 0, ..., 0), \ldots, e_N = (0, ..., 0, 1)$$

of $\mathbb{Z}^{N+1}$. Denote by k_N the vector

$$k_N = -3e_0 + \sum_{i=1}^{N} e_i \in I^{1,N}.$$

We define the E_N*-lattice* as the sublattice of $I^{1,N}$, defined by

$$E_N := (\mathbb{Z}k_N)^{\perp}.$$

Note that E_N is a root lattice for $3 \leq N \leq 8$ [Dol12, Lemma 8.2.6].

Definition 2.2.13. A vector $\alpha \in E_N$ is called a *root* if $\alpha^2 = -2$.

Definition 2.2.14. A vector $v \in I^{1,N}$ is called *exceptional* if $k_N \cdot v = -1$ and $v^2 = -1$.

Suppose X is a (weak) del Pezzo surface of degree $d \leq 7$ over a field F, i.e. $\bar{X}$ is the blow-up of $9 - d$ points $\{P_1, \ldots, P_{9-d}\}$ in (almost) general position in $\mathbb{P}^2_{\bar{F}}$. It then follows that $\operatorname{Pic} \bar{X} \cong \mathbb{Z}^{10-d}$, with generators $l_0, l_1, \ldots, l_{9-d}$ where

1. l_0 is the pullback of a line in $\mathbb{P}^2_{\bar{F}}$,
2. l_i is the exceptional curve of the blow-up corresponding to the point P_i.

We now see the relation to root lattices.

Proposition 2.2.15 ([CT88, Prop. 0.4], [Man86, Ch. IV, Prop. 25.1])**.** *Let X be a (weak) del Pezzo of degree $d \leq 6$, over a field F with intersection pairing*

$$\langle -, - \rangle : \operatorname{Pic} \bar{X} \times \operatorname{Pic} \bar{X} \to \mathbb{Z}.$$

Then

1. *there is an isomorphism $\pi : \operatorname{Pic} \bar{X} \to I^{1,9-d}$ such that $\operatorname{Pic} \bar{X}$ has a basis $l_0, \ldots, l_{9-d}$, satisfying*

$$l_0^2 = 1,\ l_i^2 = -1 \text{ for } i \in \{1, \ldots, 9-d\},\ \langle l_i, l_j \rangle = 0 \text{ for all } i \neq j,$$

2. *the anticanonical divisor $-K_X$ of X is given by*

$$-K_X = 3l_0 - \sum_{i=1}^{d} l_i,$$

3. *the isomorphism π induces an isomorphism of root lattices $(\mathbb{Z}K_X)^\perp \cong E_N$ where E_N is the root system*

$$E_N = \begin{cases} A_1 \times A_2 & \text{if } d = 6, \\ A_4 & \text{if } d = 5, \\ D_5 & \text{if } d = 4, \\ E_6 & \text{if } d = 3, \\ E_7 & \text{if } d = 2, \\ E_8 & \text{if } d = 1. \end{cases}$$

Remark 2.2.16. Under the isomorphism $\pi : \operatorname{Pic} \bar{X} \to I^{1,N}$ from Proposition 2.2.15 the image of a (-1)-curve will be an exceptional vector and the image of a (-2)-curve will be a root.

Graph of negative curves

For any del Pezzo surface of degree d over an algebraically closed field the graph of negative curves is independent of the surface. However, for weak del Pezzo surfaces the story is very different because there are such surfaces of the same degree with a different number or configuration of negative curves. In this section we discuss how one can determine the graph of negative curves on a weak del Pezzo surfaces of degree $d \leq 6$.

Proposition 2.2.17 ([DPT80, Thm. III.2 and Corollary])**.** *Let X a weak del Pezzo of degree $d \leq 6$ over an algebraically closed field. Denote by $R \subseteq \operatorname{Pic} X$ the subset of (-2)-curves of X. A class $\lambda \in \operatorname{Pic} X$ satisfying*

1. *$\lambda^2 = -1$, and,*

2. $\lambda \cdot K_X \leq 0$,

is an irreducible effective divisor if and only if $\langle \lambda, \alpha \rangle \geq 0$ *for all* $\alpha \in R$.

Proposition 2.2.18 ([Dol12, Prop. 8.2.7])**.** *Let* $3 \leq N \leq 8$ *and*

$$\alpha_{ij} = l_i - l_j, \ 1 \leq i < j \leq N,$$
$$\alpha_{ijk} = l_0 - l_i - l_j - l_k, \ 1 \leq i < j < k \leq N.$$

Then any root in E_N *is equal to* $\pm\alpha$ *where* α *is one of the vectors*

1. $N = 3 : \alpha_{ij}, \alpha_{123}$.
2. $N = 4 : \alpha_{ij}, \alpha_{ijk}$.
3. $N = 5 : \alpha_{ij}, \alpha_{ijk}$.
4. $N = 6 : \alpha_{ij}, \alpha_{ijk}, 2l_0 - l_1 - \ldots - l_6$.
5. $N = 7 : \alpha_{ij}, \alpha_{ijk}, 2l_0 - (l_1 + \ldots + l_7) + l_i,$ *for* $1 \leq i \leq 7$.
6. $N = 8 : \alpha_{ij}, \alpha_{ijk}$,

$$\begin{aligned} &2l_0 - (l_1 + \ldots + l_8) + l_i + l_j && \text{for } 1 \leq i < j \leq 8, \\ &3l_0 - (l_1 + \ldots + l_8) - l_i && \text{for } 1 \leq i \leq 8. \end{aligned}$$

Moreover, the number of roots for each N *is as follows*

N	3	4	5	6	7	8
#	8	20	40	72	126	240

Proposition 2.2.19 ([Dol12, Prop. 8.2.19])**.** *Let* $3 \leq N \leq 8$ *and* $i \in \{1, \ldots, N\}, 1 \leq j < k \leq N$. *The set of exceptional vectors in* $I^{1,N}$ *is as follows:*

1. $N = 3 : l_i, l_0 - l_j - l_k$.
2. $N = 4 : l_i, l_0 - l_j - l_k$.
3. $N = 5 : l_i, l_0 - l_j - l_k, 2l_0 - l_1 - \ldots - l_5$.
4. $N = 6 : l_i, l_0 - l_j - l_k, 2l_0 - l_1 - \ldots - l_6 + l_i$.
5. $N = 7 : l_i, l_0 - l_j - l_k, 2l_0 - l_1 - \ldots - l_7 + l_j + l_k, -k_7 - l_i$.
6. $N = 8 : l_i, l_0 - l_j - l_k$,

$$\begin{aligned} &2l_0 - (l_1 + \ldots + l_8) + l_s + l_j + l_k && \text{for } 1 \leq s < j < k \leq N, \\ &-k_8 + l_j - l_k, && \\ &-k_8 + l_0 - l_s - l_j - l_k && \text{for } 1 \leq s < j < k \leq N, \\ &-k_8 + 2l_0 - l_{i_1} - \ldots - l_{i_6} && \text{for } i_j \in \{1, \ldots, 8\}, \\ &-2k_8 - l_i. && \end{aligned}$$

Moreover, the number of exceptional vectors for each N is as follows:

N	3	4	5	6	7	8
#	6	10	16	27	56	240

For the rest of Subsection 2.2 we assume $3 \leq N \leq 8$.

Definition 2.2.20. For any root $\alpha \in I^{1,N}$ we define r_α to be the map

$$I_{1,N} \to I^{1,N}, v \mapsto v + (v, \alpha)\alpha$$

where $\langle -, - \rangle$ is the intersection pairing on $I^{1,N}$. We call r_α a *reflection*.

Definition 2.2.21. A basis $\boldsymbol{\alpha}$ for $E_N \subseteq I^{1,N}$ is called a *canonical root basis* if the symmetric pairing restricted to E_N defines a Cartan Matrix.

Definition 2.2.22. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$ be a canonical root basis. We define the *Weyl group* of E_N which is denoted by $W(E_N)$, to be the subgroup of $\mathrm{Aut}(E_N)$ generated by reflections r_{α_i} for $i \in \{1, \ldots, N\}$.

Determining configuration of (-1)-curves

We now provide an algorithm (Algorithm 2.2.23) to determine the number and configuration of (-1)-curves on a weak del Pezzo surface X of degree $d \leq 6$ over an algebraically closed field when given the configuration of (-2)-curves on X.

Algorithm 2.2.23. We first choose a basis for $\mathrm{Pic}\, X$ as in Proposition 2.2.15 and fix an isomorphism $\pi : \mathrm{Pic}\, X \to I^{1,9-d}$. The image of the sublattice $\mathcal{R} \subseteq \mathrm{Pic}\, X$ generated by (-2)-curves will be a subroot lattice $\mathfrak{R}$ of E_{9-d}. Note there is a choice of isomorphism $\mathrm{Pic}\, X \to I^{1,9-d}$, however the choice only differs by an element of $W(E_{9-d})$ [Dol12, Prop. 8.2.34]. Hence, we consider $\mathfrak{R}$ up to the action of $W(E_{9-d})$. See [Osh07, §10] for a classification of subroot systems of E_{9-d} up to an action of $W(E_{9-d})$.

Input: Root lattice $\mathfrak{R} \subseteq I^{1,9-d}$.

1. **Step 1: Choice of subroot lattice**
   - Choose representatives $\mathcal{A} := \{\mathfrak{R}_1, \ldots, \mathfrak{R}_l\}$ for each class of subroot lattices of the same type as $\mathfrak{R}$, (i.e. $\mathfrak{R}$ and $\mathfrak{R}_i$ have the same Cartan matrix with respects to the quadratic form $\langle -, - \rangle$ induced on $I^{1,9-d}$ by the isomorphism π for $i \in \{1, \ldots, l\}$) up to $W(E_{9-d})$-equivalence.
2. **Step 2: Find exceptional curves corresponding to (-1)-curves**
   - For each $\mathfrak{R}' \in \mathcal{A}$ enumerate through all exceptional vectors $e \in I^{1,9-d}$ and compute the pairing $\langle e, r \rangle$ for each $r \in \mathfrak{R}'$. By Proposition 2.2.17 if
   $$\langle e, r \rangle \geq 0 \text{ for all } r \in \mathfrak{R}' \tag{2.2.1}$$
   and $\mathfrak{R}'$ is $W(E_{9-d})$-equivalent to $\mathfrak{R}$ then e is the image of a (-1)-curve under π. Create a list $\mathrm{Exc}(\mathcal{R}_i)$ of all exceptional vectors in $I^{1,9-d}$ satisfying (2.2.1) where $\mathfrak{R}' = \mathfrak{R}_i$ and $i \in \{1, \ldots, l\}$.
3. **Step 3: Intersection matrices**
   - For $i \in \{1, \ldots, l\}$ compute the intersection matrix M_i for the vectors in $\mathrm{Exc}(\mathcal{R}_i)$.

   Output: The set $\mathrm{Exc}(\mathcal{R}_i)$ and matrix M_i for $i \in \{1, \ldots, l\}$.

Galois action on the Picard group

It will be become useful to study the action of $\Gamma := \mathrm{Gal}(F^{\mathrm{sep}}/F)$ on $\mathrm{Pic}\,\bar{X}$ where X is a (weak) del Pezzo surface over a field F. This is because this action will allow us to extract arithmetic information about X. The action of Γ on $\mathrm{Pic}\,\bar{X}$ is induced by the action of Γ on $\bar{X}$. Moreover, we have that the action of Γ on $\mathrm{Pic}\,\bar{X}$ fixes the canonical divisor, preserves the intersection pairing and sends (-1) (resp. (-2))-curves to (-1) (resp. (-2))-curves [Man86, Thm. 23.8].

As $\mathrm{Pic}\,\bar{X}$ is a reduced, torsion free and free of finite rank, there exists an extension M/F, such that $\mathrm{Gal}(F^{\mathrm{sep}}/M)$ acts trivially on $\mathrm{Pic}\,\bar{X}$. The existence of such an extension can be found by picking generators for $\mathrm{Pic}\,\bar{X}$ and taking the intersection of each stabilizer subgroup of the generators.

Definition 2.2.24. Let S be a smooth projective surface over a field F with $\mathrm{Pic}\,\bar{S}$ reduced, torsion free and free of finite rank. Let M be the minimal separable and normal extension of F such that

$$\mathrm{Pic}\,S_M \to \mathrm{Pic}\,\bar{S}$$

is an isomorphism. We call M the *splitting field of* S.

Remark 2.2.25. Suppose the degree of X is $d \leq 6$. If M is the splitting field of X, we have that the action of Γ on $\mathrm{Pic}\,\bar{X}$ factors through the action of $\mathrm{Gal}(M/F)$ on $\mathrm{Pic}\,X_M$, i.e. we get a Galois representation (well defined up to conjugation)

$$\mathrm{Gal}(M/F) \to \mathrm{Aut}(\mathrm{Pic}\,\bar{X}).$$

Moreover, any automorphism of $\bar{X}$ induces an automorphism of $\mathrm{Pic}\,\bar{X}$ which leaves the canonical divisor invariant. Hence, we have an action on $\mathbb{Z}K_X^{\perp}$ which induces the following

$$\mathrm{Gal}(M/F) \to \mathrm{Aut}(\mathrm{Pic}\,\bar{X}) \to \mathrm{Aut}(\mathbb{Z}K_X^{\perp}) \cong W(E_N).$$

Let C be the subgroup of $W(E_N)$ which sends K_X to K_X, (-1)-curves to (-1)-curves and (-2)-curves to (-2)-curves (see [Dol12, Prop. 8.2.32] for a description of C). Hence, the Galois action on $\bar{X}$ has to factor through this subgroup of the Weyl group. When X is a smooth del Pezzo surface we have that $C = W(E_N)$ [Dol83, §4, Thm. 1].

It will be useful to understand which negative curves are defined over the ground field. The following well known statement is taken from Daniel Loughran's PhD thesis [Lou11]. However, as this thesis is not readily available online we recite the statement and proof here

Lemma 2.2.26. *Let X be a smooth projective surface over a field F. Then any negative curve on X is the unique effective curve in its divisor class. In particular, if E is a negative curve on $\bar{X}$ whose divisor class is invariant under the action of* $\mathrm{Gal}(F^{\mathrm{sep}}/F)$*, then E is in fact defined over F.*

Proof. Suppose E is a negative curve on X and E' is an effective divisor in $\mathrm{Pic}\,X$ linearly equivalent, but not equal to E. This implies $E \not\subset E'$. Hence, $E' \cdot E \geq 0$. However, $E \cdot E' = E^2 < 0$ as E is a negative curve, giving a contradiction. To prove the second part of the lemma, it suffices to note that the action of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ sends effective divisors to effective divisors. □

2.3 Singular del Pezzo surfaces

In Chapter 6 we will be interested in studying singular del Pezzo surfaces. In this section we formally define such surfaces and describe their relation to weak del Pezzo surfaces.

Definition 2.3.1. A *singular del Pezzo surface* is a normal projective surface X with only rational double point singularities, whose anticanonical divisor $-K_X$ is ample. Its degree is defined to be K_X^2.

Definition 2.3.2 ([Dol12, pg. 247])**.** Let S be a smooth projective variety over a field F and let D be a Cartier divisor on S. We define the *graded algebra associated to* D as

$$R(S,D) := \bigoplus_{r=0}^{\infty} \mathrm{H}^0(S, \mathcal{O}_S(rD)).$$

Moreover, if the algebra $R(S,D)$ is finitely generated, we define the variety S_D to be $S_D := \operatorname{Proj} R(S,D)$ and there is a rational map $\phi_D : S \dashrightarrow S_D$.

Remark 2.3.3 ([Dol12, pg. 247])**.** For any $r > 0$ the rational map ϕ_r given by the complete linear system $|rD|$ factors as follows

$$\phi_{D,r} : S \overset{\phi_D}{\dashrightarrow} S_D \overset{\tau_{D,r}}{\dashrightarrow} \mathbb{P}\big(\mathrm{H}^0(S, \mathcal{O}_S(rD))^\vee\big).$$

Consider the case where S is a weak del Pezzo surface and $D = -K_S$. Here S_{-K_S} is a surface with an ample anticanonical divisor which is called the *anticanonical model of* S and ϕ_{-K_S} is a morphism which contracts the (-2)-curves to rational double point singularities and is an isomorphism outside of the (-2)-curves, ϕ_{-K_S} is called the *anticanonical morphism* [DPT80, §V, Thms. 1, 2 and Cor. 3]. Hence, S_{-K_S} is a singular del Pezzo surface. Conversely, the minimal desingularisation of any singular del Pezzo surface is a weak del Pezzo surface [Dol12, Thm. 8.1.15].

We now discuss possibilities for $\tau_{-K_S,r}$ for S which are geometrically a blow-up of the projective plane and minimal r such that $|-rK_S|$ is defines a morphism.

Theorem 2.3.4 ([Dol12, Thm. 3.8.2])**.** *Let X be a weak del Pezzo surface of degree d over a field F which is geometrically a blow-up of $\mathbb{P}^2$ in $9-d$ points. If*

1. *$d \geq 3$ then $|-K_X|$ defines a morphism $X \to \mathbb{P}^d$ and $\tau_{-K_X,1}$ defines a closed embedding of $X_{-K_X} \hookrightarrow \mathbb{P}^d$ which is an isomorphism onto a degree d surface,*
2. *$d = 2$ then $|-K_X|$ defines a morphism $X \to \mathbb{P}^2$ where $\tau_{-K_X,1}$ is a double cover of X_{-K_X} over $\mathbb{P}^2$.*
3. *$d = 1$ then $|-2K_X|$ defines a morphism $X \to \mathbb{P}^3$ where $\tau_{-K_X,2} : X_{-K_X} \to Q \subset \mathbb{P}^3$ is a double cover over a quadric cone Q.*

Using the relationship between weak del Pezzo surfaces and singular del Pezzo surfaces we now derive a useful fact about rational points on singular del Pezzo surfaces.

Proposition 2.3.5. *Let X be a singular del Pezzo surface over a field F and $\pi : X' \to X$ the minimal desingularisation of X. If there exists $p \in X'(F)$ not lying on a (-2)-curve, then X has a smooth rational point.*

Proof. Denote by $\text{Sing}(X)$ the set of singular points on X. Then $\pi^{-1}(\text{Sing}(X))$ is the set of (-2)-curves on X'. As $\pi : X' \backslash \pi^{-1}(\text{Sing}(X)) \to X \backslash \text{Sing}(X)$ is an isomorphism and p lies away from $\pi^{-1}(\text{Sing}(X))$, we conclude that X has a smooth rational point. □

Remark 2.3.6. A particular case of Proposition 2.3.5, is when F is perfect and we have a (-1)-curve on $\bar{X}'$, which is fixed by the action of $\text{Gal}(\bar{F}/F)$ with a rational point lying away from any (-2)-curve. Using Lemma 2.2.26, we can deduce there is a rational point on X' lying away from any (-2)-curve.

2.4 Rational points

Algebraic geometry was born to study polynomial equations. After Grothendieck revolutionised the area with the introduction of schemes it may feel that the origins of algebraic geometry are lost. However, if one studies schemes using the functor of points it again puts to the forefront the study of solutions to polynomial equations.

Definition 2.4.1 (Functor of points)**.** Let S be a scheme over a field F, then a F-point of S is a morphism $\text{Spec}\, F \to S$ which is a section of the structure morphism $X \to \text{Spec}\, F$. More generally, if S is a scheme then for a scheme T, the T-valued points of S are $S(T) := \text{Mor}(T, S)$.

We can view $\text{Mor}(-, S)$ as a functor

$$\text{Mor}(-, S) : \text{Sch}^{\text{op}} \to \textbf{Sets}, \quad T \mapsto \text{Mor}(T, S).$$

Yoneda's Lemma tells us that this functor completely determines the scheme S.

Example 2.4.2. Let F be a field and $S := \text{Spec}(F[x_1, \ldots, x_n]/(f_1, \ldots, f_r))$. Intuitively, the F-rational points on S are $\mathbf{x} \in F^n$ such that

$$f_1(\mathbf{x}) = \ldots = f_r(\mathbf{x}) = 0.$$

Taking the functor of points point of view, F-points on S are morphisms $\text{Spec}\, F \to S$. However, these morphisms just correspond to ring homomorphisms

$$F[x_1, \ldots, x_n]/(f_1, \ldots, f_r) \to F.$$

Clearly, any such ring homomorphism corresponds to $\mathbf{x} \in F^n$ mapping $g \in F[x_1, \ldots, x_n]$ to $g(\mathbf{x})$. For such a map to be well defined we require $\mathbf{x}$ to be in the zero locus of the f_i for $i \in \{1, \ldots, r\}$, hence we return back to our intuitive definition of a rational point on S.

To study integral points on schemes, there is a choice of model involved. We first define what we mean by a model then give an example of where the choice of model gives different results regarding integral points.

Definition 2.4.3. Let R be an integral domain with field of fractions F. Let S be a finite type scheme over F, an *integral model* of S over R is a pair $(\mathcal{S}, f)$ where $\mathcal{S}$ is finite type scheme over R and f is an isomorphism between S and the generic fibre of $\mathcal{S}$.

Example 2.4.4. Consider the scheme

$$U : x^2 + 3y^2 + 5z^3 = 1 \subseteq \mathbb{A}^3_{\mathbb{Q}}.$$

Then two choices of integral models of U over $\mathbb{Z}$ are

$$\mathcal{U} : x^2 + 3y^2 + 5z^2 = 1 \subseteq \mathbb{A}^3_{\mathbb{Z}}, \quad \mathcal{U}' : 4x^2 + 3y^2 + 5z^2 = 1 \subseteq \mathbb{A}^3_{\mathbb{Z}}.$$

It is clear that $\mathcal{U}(\mathbb{Z}) \neq \emptyset$, however $\mathcal{U}'(\mathbb{Z}) = \emptyset$.

We will predominantly want to study schemes $\mathcal{S}$ over the ring of integers $\mathcal{O}_K$ of a number field K, and we will be interested in studying the set $\mathcal{S}(\mathcal{O}_K)$. There is no finite time algorithm to determine this set in general, so one instead tries to determine this global information using local information. Formally, for each place $v \in \Omega_K$ we ask if $S(\mathcal{O}_{K_v}) \neq \emptyset$? This is due to the fact that there is an inclusion $S(\mathcal{O}_K) \hookrightarrow S(\mathcal{O}_{K_v})$. We now present Hensel's Lemma which shows that checking if $S(\mathcal{O}_{K_v}) \neq \emptyset$ (for finite primes v) can be done in a finite amount of time.

Theorem 2.4.5 ([Poo17, Thm. 3.5.63], Hensel's Lemma)**.** *Let R be a complete notherian local ring with maximal ideal $\mathfrak{m}$ and $\mathcal{S}$ a scheme over R. If the special fibre of $\mathcal{S}$ has a smooth point, then $\mathcal{S}(R) \neq \emptyset$.*

The next natural question to ask is, if $S(\mathcal{O}_{K_v}) \neq \emptyset$ for all $v \in \Omega_K$ does this imply $S(\mathcal{O}_K) \neq \emptyset$?

Definition 2.4.6 (Integral adeles)**.** Let U be a scheme over K and $\mathcal{U}$ an integral model of U over $\mathcal{O}_K$. We define the *integral adelic points on* $\mathcal{U}$ as

$$\mathcal{U}(\mathbb{A}_{\mathcal{O}_K}) := \prod_{v \in \Omega_K^\infty} U(K_v) \times \prod_{v \in \Omega_K \setminus \Omega_K^\infty} \mathcal{U}(\mathcal{O}_{K_v}).$$

Definition 2.4.7. Let $\mathcal{U}$ be a scheme over $\mathcal{O}_K$. We say $\mathcal{U}$ satisfies the *integral Hasse principle* if $\mathcal{U}(\mathbb{A}_{\mathcal{O}_K}) \neq \emptyset$ implies $\mathcal{U}(\mathcal{O}_K) \neq \emptyset$.

Remark 2.4.8. Recall the definition of the Hasse principle for rational points. Let U be a scheme over K and denote by

$$U(\mathbb{A}_K) := \prod_{v \in \Omega_K}{}' U(K_v)$$

where $\prod'$ denotes the restricted product. The elements of $U(\mathbb{A}_K)$ are *adelic points* on U. We say that U satisfies the *Hasse principle* if $U(\mathbb{A}_K) \neq \emptyset$ implies $U(K) \neq \emptyset$. By the valuative criterion for properness [Har04, Ch. 2, Thm. 4.7] if U is proper then the integral Hasse principle is equivalent to the Hasse principle for rational points.

Example 2.4.9. Consider the curve

$$C : 3x^3 + 4y^3 + 5z^3 = 0 \subseteq \mathbb{P}^2_{\mathbb{Q}}$$

Selmer shows in [Sel51] that $C(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$, however $C(\mathbb{Q}) = \emptyset$. Hence, this is an example of the Hasse principle failing.

2.5 Brauer–Manin obstruction

In 1970, Manin introduced the Brauer–Manin obstruction [Man71] which tries to explain why the Hasse principle for rational points fails. However, we will focus on a variant of this construction by Colliot-Thélène and Xu [CTX09], which attempts to explain when the integral Hasse principle fails. First we will develop some background on Brauer groups.

2.5.1 Brauer group of a field

Definition 2.5.1. A *central simple algebra* over a field F is a finite dimensional F-algebra A which is simple (i.e. the two sided ideals of A are 0 and A) and central as a F-algebra (i.e. the centre of A is the image of $F \hookrightarrow A$).

Notation 2.5.2. We denote by $\mathrm{M}_n(F)$ the F-algebra of n by n matrices. Moreover, for a F-algebra A we denote by $\mathrm{M}_n(A) := \mathrm{M}_n(F) \otimes_F A$.

Definition 2.5.3. Let F be a field and A, B two central simple algebras over F. We say that A and B are are *equivalent* $(A \sim B)$ if there exists $m, n \in \mathbb{Z}_{>0}$ such that

$$\mathrm{M}_m(A) \cong \mathrm{M}_n(B).$$

Definition 2.5.4. Let A be a central simple algebra over a field F. We define the *opposite algebra to A* as the algebra A^{opp} where the underlying set and addition of A^{opp} is the same as in A, however multiplication is reversed. Explicitly, denote by $+_A, \cdot_A, +_{A^{\mathrm{opp}}}, \cdot_{A^{\mathrm{opp}}}$ the addition and multiplication laws in A, A^{opp} respectively. Then given $\alpha, \beta \in A, A^{\mathrm{opp}}$ we have

$$\alpha +_{A^{\mathrm{opp}}} \beta = \alpha +_A \beta \quad \alpha \cdot_{A^{\mathrm{opp}}} \beta = \beta \cdot_A \alpha.$$

Remark 2.5.5.

1. For $m, n \in \mathbb{Z}_{>0}$ we have $\mathrm{M}_n(F) \otimes_F \mathrm{M}_m(F) \cong \mathrm{M}_{mn}(F)$.

2. For a central simple algebra A, we have $A \otimes A^{\mathrm{opp}} \cong \mathrm{M}_n(F)$ for some $n \in \mathbb{Z}_{>0}$.

Proposition 2.5.6 ([Mil20, Ch. IV, pg. 130])**.** *Let F be a field. We define the* Brauer group of F *to be*

$$\mathrm{Br}\, F := \{\textit{Central simple algebras over } F\}/\sim .$$

Then $\mathrm{Br}\, F$ is an abelian group with the following operations

1. *Multiplication:* $[A] \cdot [B] = [A \otimes B]$,

2. *Inverse:* $[A]^{-1} = [A^{opp}]$,

3. *Identity:* $[\mathrm{M}_n(F)]$ *for every* $n \in \mathbb{Z}_{>0}$.

We can give a reinterpretation of the Brauer group in terms of Galois cohomology. This reinterpretation will be useful when we consider a more general definition for schemes and allows us to easily compute the Brauer group of special types of fields.

Theorem 2.5.7 ([Mil20, Ch. IV, Cor. 3.16])**.** *Let F be a field then*

$$\mathrm{Br}\, F \cong \mathrm{H}^2(F, (F^{\mathrm{sep}})^*).$$

Definition 2.5.8 ([GS06, Def. 6.2.1])**.** A field F is said to satisfy condition C_1 if every homogeneous polynomial $f \in F[x_1, \ldots, x_n]$ of degree $d < n$ has a non-trivial zero in F^n. We call such fields C_1*-fields*.

Proposition 2.5.9 ([GS06, Prop. 6.2.3])**.** *If F is a C_1-field then for any finite extension M/F we have* $\operatorname{Br} M = 0$.

Example 2.5.10.

1. If F is a separably closed field then $\operatorname{Br} F = 0$ by Proposition 2.5.9.

2. Let F be the function field of an algebraic curve over an algebraically closed field. Then $\operatorname{Br} F = 0$ by Tsen's theorem [GS06, Thm. 6.2.8] and Proposition 2.5.9.

3. Let K be a number field and $v \in \Omega_{K_v}$, then either

$$\operatorname{Br} K_v = \begin{cases} \mathbb{Q}/\mathbb{Z} & \text{if } v \text{ is finite,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } K_v = \mathbb{R}, \\ 0 & \text{if } K_v = \mathbb{C}. \end{cases}$$

   This case follows from the fact that for all places $v \in \Omega_K$ there exists an injective homomorphism of groups $\operatorname{inv}_v : \operatorname{Br} K_v \to \mathbb{Q}/\mathbb{Z}$ called the *invariant map* [Har20, Thm. 8.9], which has

$$\operatorname{im}(\operatorname{inv}_v) = \begin{cases} \mathbb{Q}/\mathbb{Z} & \text{if } v \text{ is finite,} \\ \{0, 1/2\} & \text{if } K_v = \mathbb{R}, \\ 0 & \text{if } K_v = \mathbb{C}. \end{cases}$$

For this thesis we will focus on the Brauer group of a field F where F is a global or local field. In this case the Brauer group is generated by cyclic algebras [Poo17, Thm. 1.5.34 (iii), Thm. 1.5.36 (iii)] which are defined in general as follows.

Definition 2.5.11. Let L/F be a cyclic field extension and $\sigma \in \operatorname{Gal}(L/F)$. For $a \in K^*$, define the K-vector space

$$(a, L/F, \sigma) := \bigoplus_{i=0}^{n-1} K x^i$$

endowed with a multiplication operation defined by the following properties

$$x^n = a, \quad cx^i \cdot dx^j = c\sigma^i(d)x^{i+j} \text{ for } c, d \in L.$$

We call $(a, L/F, \sigma)$ a *cyclic algebra*.

Remark 2.5.12. Suppose $[L : F] = n$ and F contains a primitive nth root of unity ζ_n. By Kummer theory [Neu13, Ch. IV, §3, Prop. 3.2] we can write L as $L = F(\sqrt[n]{b})$ for some $b \in F$ where $\sigma(\sqrt[n]{b}) = \zeta_n \sqrt[n]{b}$. In this case we will reduce our notation to $(a, b)_{\zeta_n}$.

2.5.2 Brauer groups of schemes

We now want to generalise the definition of the Brauer group of field to general schemes from the view point of Theorem 2.5.7.

Definition 2.5.13. Let S be a scheme. The *Brauer–Grothendieck group* of S is

$$\operatorname{Br} S := \mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_m).$$

We will refer to $\operatorname{Br} S$ as the *Brauer group* of a scheme S.

Remark 2.5.14 ([Mil16, §III, Ex. 1.2])**.** When S is the spectrum of a field F we have an isomorphism

$$\mathrm{H}^2_{\text{ét}}(\operatorname{Spec} F, \mathbb{G}_m) \cong \mathrm{H}^2(F, (F^{\text{sep}})^*)$$

Remark 2.5.15. There is an alternative view point one can take for the Brauer group of schemes. Namely, the Brauer–Azumaya group $\operatorname{Br}_{\text{Az}}$ (see [CTS21, §3.1]). Thanks to the following theorem, this is the same as working with the Brauer–Grothendieck group for all cases we care about throughout this thesis. In particular, we will determine the Brauer group of certain smooth schemes and in this case $\operatorname{Br} S$ is torsion [CTS21, Thm. 3.5.5].

Theorem 2.5.16 ([CTS21, Thm. 4.2.1, Gabber])**.** *Let S be a quasi-projective scheme over an affine scheme, then there is an isomorphism*

$$\operatorname{Br}_{\text{Az}} S \to (\operatorname{Br} S)_{tors}.$$

For a scheme S over a field F, it is useful to break the Brauer group into pieces. More formally, we define a filtration on $\operatorname{Br} S$, separating elements which become trivial after a finite field extension (algebraic elements) and ones that do not (transcendental elements).

Definition 2.5.17. Let S be a scheme over a field F. We define the filtration

$$0 \subseteq \operatorname{Br}_0 S \subseteq \operatorname{Br}_1 S \subseteq \operatorname{Br} S$$

where $\operatorname{Br}_0 := \operatorname{im}(\operatorname{Br} F \to \operatorname{Br} S)$ and $\operatorname{Br}_1 S := \ker(\operatorname{Br} S \to \operatorname{Br} \bar{S})$. We call $\operatorname{Br}_1 S$ the *algebraic Brauer group* and $\operatorname{Br} S / \operatorname{Br}_1 S$ the *transcendental Brauer group.*

If S is a geometrically locally factorial and integral scheme over a field F we have an injection $\operatorname{Br} S \to \operatorname{Br} F(S)$. However, we would like to determine which elements of $\operatorname{Br} F(S)$ pullback to $\operatorname{Br} S$. This can be done for certain types of schemes using the following proposition.

Proposition 2.5.18 ([CTS21, Thm. 3.7.3])**.** *Let S be an excellent, regular and integral scheme over a field F of characteristic 0. Then there exists an exact sequence*

$$0 \to \operatorname{Br} S \to \operatorname{Br} F(S) \xrightarrow{\partial_D} \bigoplus_D \mathrm{H}^1_{\text{ét}}(\kappa(D), \mathbb{Q}/\mathbb{Z})$$

where D ranges over irreducible divisors of S and $\kappa(D)$ denotes the residue field at the generic point of D. We call the maps ∂_D residue maps.

Definition 2.5.19. Let S be as in Proposition 2.5.18. Let D be an irreducible divisor of S with associated residue map ∂_D. Given $A \in \operatorname{Br} F(S)$ we say that A is *unramified at D* if $\partial_D(A) = 0$; otherwise we say that A is *ramified at D.* Moreover, if A is unramified at all irreducible divisors of S we say that A is *unramified*; otherwise we say that A is *ramified.*

When considering cyclic algebras of order $n \in \mathbb{Z}_{\geq 2}$, finding the image of the residue maps can be easier. This is explained in the following lemma when $F(S)$ contains a primitive nth root of unity. In this case for any irreducible divisor D on S we have an isomorphism

$$\mathrm{H}^1_{\text{ét}}(\kappa(D), \mathbb{Z}/n\mathbb{Z}) \cong \kappa(D)^*/(\kappa(D)^*)^n.$$

by Kummer theory.

Lemma 2.5.20. *Let S be as in Proposition 2.5.18. Fix an integer $n \in \mathbb{Z}_{\geq 2}$ and assume $F(S)$ contains a primitive nth root of unity ζ_n. Then for any irreducible divisor D on S the image of the cyclic algebra $(f,g)_\zeta$ under the residue map ∂_D is given by*

$$\partial_D(f,g)_{\zeta_n} = (-1)^{\mathrm{val}_{\eta_D}(f)\,\mathrm{val}_{\eta_D}(g)} \overline{\left(\frac{f^{\mathrm{val}_{\eta_D}(g)}}{g^{\mathrm{val}_{\eta_D}(f)}}\right)} \in \kappa(D)^*/(\kappa(D)^*)^n$$

where η_D is the generic point of D, val_{η_D} denotes the associated valuation of η_D on the stalk $\mathcal{O}_{D,\eta_D}$ and bar denotes the image of the map $\mathcal{O}_{S,\eta_D} \to \kappa(D)$.

Proof. There is a *tame symbol* [GS06, Ex. 7.1.5, Prop. 7.5.1]

$$\begin{aligned} \delta_D : F(S)^* \times F(S)^* &\to \kappa(D)^*, \\ (f,g) &\mapsto (-1)^{\mathrm{val}_{\eta_D}(f)\,\mathrm{val}_{\eta_D}(g)} \overline{\left(\frac{f^{\mathrm{val}_{\eta_D}(g)}}{g^{\mathrm{val}_{\eta_D}(f)}}\right)}. \end{aligned}$$

The tame symbol gives rise to the following commutative diagram [JS17, Prop. 5.1]

$$\begin{array}{ccc} F(S)^* \times F(S)^* & \xrightarrow{\quad \delta_D \quad} & \kappa(D) \\ \downarrow R & & \downarrow \pi \\ \mathrm{Br}\, F(S)[n] & \xrightarrow{\quad \partial_D \quad} & \mathrm{H}^1_{\text{ét}}(\kappa(D), \mathbb{Z}/n\mathbb{Z}) \end{array}$$

where $R : F(S)^* \times F(S)^* \to \mathrm{Br}\, F(S)[n], (f,g) \mapsto (f,g)_{\zeta_n}$ and π is just the projection map $\pi : \kappa(D)^* \to \kappa(D)^*/(\kappa(D)^*)^n$. The result now follows. ☐

2.5.3 Algebraic Brauer group of affine log K3 surfaces

In Chapters 3 and 4 we will study certain affine log $K3$ surfaces. We first recall the definition of a log $K3$ surface.

Definition 2.5.21 ([Har17, Def. 2.4])**.** Let U be a smooth geometrically integral surface over a field F and let $i : U \hookrightarrow X$ be a smooth compactification defined over F, such that $D := X \backslash U$ is a strict normal crossing divisor. A *log K3 structure* on U is a triple (U, D, i) such that

$$-K_X = D.$$

A *log K3 surface* is a smooth geometrically integral, simply connected surface U equipped with a log K3 structure (U, D, i).

We will be interested in the cases where the log $K3$ surface is the complement of a smooth irreducible anticanonical divisor D on a smooth del Pezzo surface X of degree $d \leq 7$ over a number field K. In particular, we want to compute the Brauer group of these surfaces. The first step is to compute the algebraic Brauer and in general, one can compute $\operatorname{Br}_1 U$ modulo constants following [BL19, Prop. 2.1]. Explicitly, using the facts $\mathrm{H}^0_{\text{ét}}(\bar{X}, \mathbb{G}_m) = \mathrm{H}^0_{\text{ét}}(\bar{U}, \mathbb{G}_m) = \bar{K}^*$ and $\mathrm{H}^3(K, K^*) = 0$, the Hochschild–Serre spectral sequence gives us natural group isomorphisms

$$\operatorname{Br} X / \operatorname{Br}_0 X \cong \mathrm{H}^1(K, \operatorname{Pic} \bar{X}) \quad \text{and} \quad \operatorname{Br}_1 U / \operatorname{Br}_0 U \cong \mathrm{H}^1(K, \operatorname{Pic} \bar{U}).$$

As D is irreducible we have the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{i} \operatorname{Pic} \bar{X} \xrightarrow{\pi} \operatorname{Pic} \bar{U} \to 0,$$

where i is defined by $1 \mapsto [D] = -K_X$. The canonical class K_X is primitive in $\operatorname{Pic} \bar{X} \cong \mathbb{Z}^{10-d}$, so $\operatorname{Pic} \bar{U}$ and $\operatorname{Pic} \bar{X}$ are both torsion-free and of finite rank. Hence, there exists an extension L/K such that $\operatorname{Gal}(\bar{K}/L)$ acts trivially on $\operatorname{Pic} \bar{U}$, i.e. this is the splitting field of X. As $\operatorname{Gal}(L/K)$ is torsion the inflation-restriction sequence gives us $\operatorname{Br}_1 U / \operatorname{Br}_0 U \cong \mathrm{H}^1(\operatorname{Gal}(L/K), \operatorname{Pic} U_L)$. In general, $\operatorname{Gal}(L/K)$ can only act in finitely many different ways on $\operatorname{Pic} X_L$ as the action factors through the Weyl group by Remark 2.2.25. We then enumerate through all possibilities for the Galois action on $\operatorname{Pic} U_L$ and compute the Galois cohomology group $\mathrm{H}^1(\operatorname{Gal}(L/K), \operatorname{Pic} U_L)$.

2.5.4 Integral Brauer–Manin obstruction

In this section we recall the integral Brauer–Manin set up, as introduced by Colliot-Thélène and Xu [CTX09]. Throughout we denote by K a number field and $\mathcal{O}_K$ the ring of integers of K.

Let $\mathcal{U}$ be a scheme over $\mathcal{O}_K$ and let U denote the base change of $\mathcal{U}$ to K. We have the following commutative diagram for each $\alpha \in \operatorname{Br} U$, where the bottom row is exact by the Brauer–Hasse–Noether theorem [Har20, Thm. 14.11]:

$$\begin{array}{ccccccccc} & & \mathcal{U}(\mathcal{O}_K) & \longrightarrow & \mathcal{U}(\mathbb{A}_{\mathcal{O}_K}) & & & & \\ & & \downarrow \alpha & & \downarrow \alpha & & & & \\ 0 & \longrightarrow & \operatorname{Br} K & \longrightarrow & \bigoplus_{v \in \Omega_K} \operatorname{Br} K_v & \xrightarrow{\sum_{v \in \Omega_K} \operatorname{inv}_v} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

By [CTS21, Prop. 13.3.1] for every $\alpha \in \operatorname{Br} U$ there exists a finite set of places S_α including all archimedean ones, such that the invariant map

$$\operatorname{inv}_v \alpha : \mathcal{U}(\mathcal{O}_{K_v}) \to \mathbb{Q}/\mathbb{Z}$$

is zero for all $v \notin S_\alpha$. Hence, there exists a well defined pairing

$$\mathcal{U}(\mathbb{A}_{\mathcal{O}_K}) \times \operatorname{Br} U \to \mathbb{Q}/\mathbb{Z}, \ ((x_v)_v, \alpha) \mapsto \sum_{v \in \Omega} \operatorname{inv}_v(\alpha(x_v)).$$

The following set

$$\mathcal{U}(\mathbb{A}_{\mathcal{O}_K})^{\operatorname{Br}} := \left\{ (x_v)_v \in \mathcal{U}(\mathbb{A}_{\mathcal{O}_K}) : \sum_{v \in \Omega} \operatorname{inv}_v(\alpha(x_v)) = 0 \ \forall \alpha \in \operatorname{Br} U \right\}$$

is the *integral Brauer–Manin set* and there are inclusions

$$\mathcal{U}(\mathcal{O}_K) \subseteq \mathcal{U}(\mathbb{A}_{\mathcal{O}_K})^{\mathrm{Br}} \subseteq \mathcal{U}(\mathbb{A}_{\mathcal{O}_K}).$$

We say there is an *integral Brauer–Manin obstruction to the integral Hasse principle* when

$$\mathcal{U}(\mathbb{A}_{\mathcal{O}_K}) \neq \emptyset \quad \text{but} \quad \mathcal{U}(\mathbb{A}_{\mathcal{O}_K})^{\mathrm{Br}} = \emptyset.$$

Remark 2.5.22. If $\mathcal{U}$ were proper over $\operatorname{Spec} \mathcal{O}_K$ then we recover the construction of the Brauer–Manin obstruction originally given by Manin in [Man71].

The Brauer–Manin obstruction allows one to study density questions about the set of rational points.

Definition 2.5.23. Let U be a scheme over a number field K. We say U satisfies *weak approximation* if the diagonal image

$$U(k) \hookrightarrow \prod_{v \in \Omega_K} U(K_v)$$

is dense. We say U satisfies *strong approximation* if $U(K) \hookrightarrow U(\mathbb{A}_K)$ is dense.

The definition of strong approximation is too strong to hold for large classes of varieties, for example if U is affine then $U(K)$ is discrete in $U(\mathbb{A}_K)$ [Poo17, pg. 53], because of this we weaken our definition of strong approximation to exclude some finite set of primes.

Definition 2.5.24. Let S be finite set of places of K. Then for a scheme U over K the S-adeles of U are

$$U(\mathbb{A}_K^S) := \{(x_v)_v \in \prod_{v \notin S} U(K_v) : v(x_v) \geq 0 \text{ for all but finitely many } v\}.$$

where $U(\mathbb{A}_K^S)$ has a natural restricted product topology. We say that U satisfies *strong approximation away from* S if $U(K) \hookrightarrow U(\mathbb{A}_K^S)$ is dense.

Remark 2.5.25. The closure of $U(k)$ in $U(\mathbb{A}_k)$ lies in $U(\mathbb{A}_k)^{\mathrm{Br}}$ [CTS21, Thm. 13.3.2]. This leads to the following definition.

Definition 2.5.26. We say that there is a *Brauer–Manin obstruction to strong approximation* if $U(\mathbb{A}_K)^{\mathrm{Br}} \neq U(\mathbb{A}_K)$.

We now introduce analogues of strong approximation and Brauer–Manin obstruction to strong approximation for integral points. Here we denote by $\mathcal{U}$ a scheme over $\mathcal{O}_K$.

Definition 2.5.27. We say that $\mathcal{U}$ satisfies *integral strong approximation off* ∞ if the diagonal image $\mathcal{U}(\mathcal{O}_K) \hookrightarrow \mathrm{pr}_\infty(\mathcal{U}(\mathbb{A}_{\mathcal{O}_K}))$ is dense, where $\mathrm{pr}_\infty : \mathcal{U}(\mathbb{A}_{\mathcal{O}_K}) \to \prod_{v \nmid \infty} \mathcal{U}(\mathcal{O}_v)$ means the projection to the finite adeles $\prod_{v \nmid \infty} \mathcal{U}(\mathcal{O}_v)$.

Definition 2.5.28. There is a *Brauer–Manin obstruction integral strong approximation off* ∞ if $\mathrm{pr}_\infty\big(\mathcal{U}(\mathbb{A}_{\mathcal{O}_K})^{\mathrm{Br}}\big)$ is a strict subset of the finite integral adeles $\prod_{v \nmid \infty} \mathcal{U}(\mathcal{O}_v)$.

2.6 Thin sets and Hilbert's irreducibility Theorem

In this section we introduce Hilbert's irreducibility theorem in modern terminology, using thin sets; a notion introduced by Serre [Ser92, Ch. 3 §1]. Throughout Section 2.6, F will be a field of characteristic 0 and an *algebraic variety* over F will be an integral and quasi-projective variety. Denote by V an algebraic variety over F.

2.6.1 Thin sets

Definition 2.6.1. A subset $A \subset V(F)$ is said to be of

1. *type I* if there is a proper closed subscheme $W \subset V$, with $A \subset W(F)$.
2. *type II* if there is an algebraic variety V' with $\dim V = \dim V'$, and a generically surjective morphism $\pi : V' \to V$ of degree greater than or equal to 2, with $A \subset \pi(V'(F))$.

Definition 2.6.2. A subset $A \subset V(F)$ is called *thin* if it is contained in a finite union of sets of type I or II.

Let V be as above, $F(V)$ be its function field and

$$f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in k(V)[x]$$

be an irreducible polynomial over $F(V)$. Let $G \subset S_n$ be the Galois group of f, viewed as a group of permutations on the roots of f. If $t \in V(F)$ and t is not a pole of any of the a_i then $a_i(t) \in F$ is well defined. We define the specialization of f at t as

$$f_t(x) = x^n + a_{n-1}(t)x^{n-1} + \ldots + a_0(t) \in k[x].$$

Theorem 2.6.3 (Hilbert's irreducibility theorem, [Ser92, Prop. 3.3.5])**.** *There exists a thin set $A \subset V(k)$ such that, if $t \notin A$, then*

1. *t is not a pole of any of the a_i,*
2. *$f_t(x)$ is irreducible over k,*
3. *the Galois group of f_t is G.*

Note that there is a subset $B \subseteq A$ such that for $t \notin B$ the Galois group of f_t embeds into the Galois group of f. The elements of B are exactly those which f_t is separable. We show this in the case $V = \mathbb{A}^1_{\mathbb{Q}}$.

Proposition 2.6.4 ([Con])**.** *Let $f(x,t) \in \mathbb{Q}(t)[t]$ be a monic a polynomial such that $F := \mathbb{Q}(t)[x]/(f)$ is a Galois extension of $\mathbb{Q}(t)$. Pick a specialisation $t_0 \in \mathbb{Q}$ of t such that $f(x,t_0)$ is separable and denote by K the splitting field of $f(x,t_0)$. Then* $\mathrm{Gal}(K/\mathbb{Q}) \hookrightarrow \mathrm{Gal}(F/\mathbb{Q}(t))$.

Proof. Let $R = \mathbb{Q}[t]$, $F = \mathrm{Frac}(R)$ and $f(t,x) \in \mathbb{Q}[t,x]$ be irreducible. Furthermore, let F' be F adjoining a x-root of $f(t,x)$ and R' the integral closure of R in F'. If a specialisation $t = t_0 \in \mathbb{Q}$ is such that $f(t_0,x)$ is separable in $\mathbb{Q}[t]$ then the prime ideal $(t-t_0) \subset R$ is unramified in R'. Let E be the Galois closure of F'/F and I the integral closure of R in E. Then as $(t-t_0)$ is unramified in R' it is unramified in I. Pick a prime $\mathfrak{I} \subset I$ above

$(t - t_0)$. Note that $(I/\mathfrak{J})/(R/(t - t_0))$ is a finite extension. It is clear that $R/(t - t_0) = \mathbb{Q}$. Denote by K the field $K := I/\mathfrak{J}$. As $K/\mathbb{Q}$ is an extension of fields of characteristic 0 it is separable and as E/F is Galois $K/\mathbb{Q}$ is normal, hence $K/\mathbb{Q}$ is also Galois. Furthermore, K is the Galois closure of $f(t_0, x)$ over $\mathbb{Q}$. As $\mathrm{Gal}(K/\mathbb{Q})$ is isomorphic to the decomposition group $D_{\mathfrak{J}}$ there is an embedding $\mathrm{Gal}(K/\mathbb{Q}) \hookrightarrow \mathrm{Gal}(E/F)$. □

2.7 Torsors

A useful way to study integral and rational points on varieties is using torsors. They naturally arise when considering certain varieties, for example elliptic curves. Moreover, they give us concrete geometric interpretation for certain cohomology groups/sets. Conversely, these cohomology groups/sets can give us information about torsors.

2.7.1 Torsors as sets

We first describe the basic case of torsors which gives some intuition about how one should think about these objects. This subsection is taken from [Poo17, §5.12].

Definition 2.7.1. Let G be a group. A (right) *G-torsor* is a right G-set Y where G acts freely and transitively, i.e. Y is isomorphic to G as G-sets where G acts on itself by right translation.

Remark 2.7.2. The idea is a G-torsor is a group which has forgotten its identity. One can see this analogy with a vector space and affine space, i.e. $\mathbb{A}^n_{\mathbb{C}}$ is the $\mathbb{C}$-vector space $\mathbb{C}^n$ but we don't distinguish the identity element.

Example 2.7.3. Let V be a vector space and $G \subseteq V$ a vector subspace. Let $Y := x + G$ where $x \in V$. Then Y is non-canonically isomorphic to G, since for any $y \in Y$ the map from G to Y defined by translating G by y is an isomorphism of G-sets.

2.7.2 Torsors over a variety

Definition 2.7.4. Let X be a scheme and G a smooth algebraic group over X. Then a *G-torsor* over X is a faithfully flat scheme Y over X, with a (right) action of G

$$\alpha : Y \times G \to Y, \quad (y, g) \mapsto yg$$

such that the map

$$(\pi, \alpha) : Y \times_X G \to Y \times_X Y, \quad (y, g) \mapsto (y, yg),$$

(where $\pi : Y \times_X G \to Y$ is the projection map) is an isomorphism.

Remark 2.7.5. Alternatively, if we restrict to the case where X is a variety over a field F and either G is finite or G and Y are affine, then the last condition can be rephrased as:

1. For all $(y, g) \in Y \times G$ we have $f(yg) = f(y)$ i.e. G preserves the fibres of f,
2. For all geometric fibres, $G(\bar{F})$ acts freely and transitively, i.e. let $t \in X(\bar{F})$ then $G(\bar{F})$ acts freely and transitively on $Y_t(\bar{F})$ (this is the set theoretic definition we saw).

Throughout the rest of Section 2.7 X will be a variety over a field F and G a smooth algebraic group over F.

Definition 2.7.6. Let $Y \to X$ and $Y' \to X$ be G-torsors over X. Then a *morphism of G-torsors* is a morphism of F-varieties

$$\phi : Y \to Y'$$

such that $\phi(yg) = \phi(y)g$ for all $y \in Y(\bar{k})$ and $g \in G(\bar{k})$. An *isomorphism of G-torsors* is a morphism of G-torsors which is also an isomorphism of varieties.

Definition 2.7.7. The *trivial torsor* is $Y := X \times_F G \to X$ with a right action by translation of G on the second component. Moreover, we say a G-torsor $Y \to X$ is *trivial* if $Y \to X$ is isomorphic to the trivial torsor.

2.7.3 Torsors over a field

In this subsection we will give examples of G-torsors $Y \to \operatorname{Spec} F$ where F is a perfect field. Note that the condition of the action $Y \times G \to Y$ being defined over F means that this action is compatible with the left action of Galois, i.e. for all $\gamma \in \operatorname{Gal}(\bar{F}/F)$ we have

$$\gamma(yg) = \gamma(y)\gamma(g)$$

for all $y \in Y(\bar{F}), g \in G(\bar{F})$.

Remark 2.7.8. The trivial torsor in this case is just $Y = G$ with a right action by translation. Moreover, a G-torsor $Y \to \operatorname{Spec} F$ is the trivial torsor if and only if $Y(F) \neq \emptyset$. This is because, if $Y \to \operatorname{Spec} F$ is the trivial G-torsor, as $G(F) \neq \emptyset$ we have $Y(F) \neq \emptyset$. For the reverse direction, as $Y(F) \neq \emptyset$, pick $y_0 \in Y(F)$ then we have a F-isomorphism

$$G \to Y, \quad g \mapsto y_0 g.$$

Example 2.7.9 ([Sko01, pg. 55])**.** Let C/F be a genus 1 curve with Jacobian J/F. Pick $c \in C(\bar{F})$. Then the $\bar{F}$-morphism

$$\bar{J} \to \bar{C}, \quad y \mapsto y + c$$

descends to a F-morphism $D \to C$ where $D \to C$ is a J-torsor.

Example 2.7.10. Consider the variety $G : V(x^2 + y^2 - 1) \subseteq \mathbb{A}^2_{\mathbb{Q}}$. This is an algebraic group over $\mathbb{Q}$ via the action

$$G \times G \to G, \quad ((x, y), (w, t)) \mapsto (xw - 2ty, tx + yw).$$

Hence, this is the trivial G-torsor over $\mathbb{Q}$.

Example 2.7.11. Let $X = \operatorname{Spec} \mathbb{Q}, Y = V(x^2 + y^2 - r^2)$ and $G = V(w^2 + t^2 - 1)$ where $r \in \mathbb{Z}$. Then Y is a G-torsor over $\mathbb{Q}$ via the action

$$Y \times G \to Y, \quad ((x, y), (w, t)) \mapsto (xw - 2ty, tx + yw).$$

Note that as $Y(\mathbb{Q}) \neq \emptyset$, we have Y is isomorphic to the trivial torsor.

Example 2.7.12. Let $X = \operatorname{Spec}\mathbb{Q}$ and $Y = V(x^2 + 2y^2 + 3)$. Let $G = V(x^2 + 2y^3 - 1)$, then Y is a G-tosor over X via

$$Y \times G \to Y, \quad ((x, y), (w, t)) \mapsto (xw - 2ty, tx + yw).$$

Note that Y is not a trivial G-torsor. This is because $Y(\mathbb{Q}) = \emptyset$.

Example 2.7.13. Let L/F be a finite extension. Fix a basis $w_1, \ldots, w_n$ of L/F. Then we have a F-group

$$G : \mathrm{Norm}_{L/F}(x_1w_1 + \ldots + x_nw_n) = 1.$$

For $a \in F^*$ we have a F-variety defined by the affine equation

$$Y : \mathrm{Norm}_{L/F}(x_1w_1 + \ldots + x_nw_n) = a.$$

Then Y is a G-torsor over $\operatorname{Spec} F$ via the following action. Let $y = (y_1, \ldots, y_n) \in Y(\bar{F}), g = (g_1, \ldots, g_n) \in G(\bar{F})$. As $w_1, \ldots, w_n$ is a basis of $L \otimes \bar{F}$

$$(y_1, \ldots, y_n)(g_1, \ldots, g_n) = h_1w_1 + \ldots + h_nw_n.$$

We set $yg = (h_1, \ldots, h_n)$. This is a G-torsor because of the multiplicative properties of the norm map. Moreover, Y is trivial if and only if a is a norm for the extension L/F.

If we have a finite group (not necessarily constant) G and $Y \to \operatorname{Spec} F$ is a G-torsor, then Y has to be zero dimensional. This is because G acts freely and transitively on the (geometric) fibres. Hence, Y is the spectrum of a F-algebra, call it A. Then $\bar{F}$-points on Y are $\mathrm{Mor}_F(\operatorname{Spec}\bar{F}, Y)$ which correspond $\mathrm{Hom}_F(A, \bar{F})$. As Y is a G-torsor we require $\mathrm{Hom}_F(A, \bar{F})$ to have a free and transitive action of $G(\bar{F})$.

Example 2.7.14. Assume $\mathrm{char}(F) \neq 2$. Let $L := F[x]/(x^2 - a) = F(\sqrt{a})$ where $a \in F^* \backslash F^{*,2}$. We have $\mathrm{Hom}_F(L, \bar{F}) = \{\sigma_0, \sigma_1\}$ where $\sigma_i(\sqrt{a}) = (-1)^i\sqrt{a}$. Let G be the group $\mathbb{Z}/2\mathbb{Z}$ which acts on $Y = \operatorname{Spec} L$ via $\sqrt{a} \mapsto -\sqrt{a}$. Then $Y \to X$ is a G-torsor. The trivial G-torsor is $\operatorname{Spec} F \sqcup \operatorname{Spec} F$.

Example 2.7.15. Let $Y = \operatorname{Spec}\mathbb{Q}(\sqrt[3]{2}) \to \operatorname{Spec}\mathbb{Q}$. Then $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}(\sqrt[3]{2}), \bar{\mathbb{Q}}) = \{\sigma_0, \sigma_1, \sigma_2\}$ where $\sigma_i(\sqrt[3]{2}) = \sqrt[3]{2}\zeta^i$ and $\zeta \in \bar{\mathbb{Q}}$ satisfies $\zeta^2 + \zeta + 1$. We have a free and transitive action on $\mathrm{Hom}_{\mathbb{Q}}(\sqrt[3]{2}), \bar{\mathbb{Q}})$ by

$$\sigma_0 \mapsto \sigma_1, \sigma_1 \mapsto \sigma_2, \sigma_2 \mapsto \sigma_0.$$

However, this is not a $\mathbb{Z}/3\mathbb{Z}$-torsor. Suppose Y is $\mathbb{Z}/3\mathbb{Z}$-torsor, then there exists $g \in \mathbb{Z}/3\mathbb{Z}$ such that $\sigma_0 g = \sigma_1$. Let $\sigma \in \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acting via complex conjugation. As Y is a G-torsor over $\operatorname{Spec}\mathbb{Q}$ we require that the action of G is compatible with the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. In particular, $\sigma(\sigma_0 g) = \sigma(\sigma_0)\sigma(g)$ and as $\sigma(\sigma_0 g) = \sigma(\sigma_1)$ it follows that $\sigma(\sigma_1) = \sigma(\sigma_0)\sigma(g) = \sigma_0 g = \sigma_1$. Using the fact that $\sigma(\sigma_1) = \sigma_2$ we deduce that $\sigma_2 = \sigma_1$ which is a contradiction. Hence, Y is not a $\mathbb{Z}/3\mathbb{Z}$-torsor. In fact, it is a μ_3 torsor (as we require a non-trivial action of Galois on the group). The trivial μ_3 torsor is $\operatorname{Spec}\mathbb{Q} \sqcup \operatorname{Spec}\mathbb{Q}[x]/(x^2 + x + 1)$.

Example 2.7.16. Let L/F be a Galois extension of fields and G the constant group associated to $\mathrm{Gal}(L/F)$. Then the left action of $\mathrm{Gal}(L/F)$ on L induces a right action of G on $\operatorname{Spec} L$. Then $\operatorname{Spec} L$ is a connected G-torsor over $\operatorname{Spec} F$. One can show the converse is also true, i.e.

$$\{\text{Connected } G\text{-torsors over } \operatorname{Spec} F\} \longleftrightarrow \{G\text{-extensions of } F\}.$$

2.7.4 Torsors over higher dimensional bases

We now consider torsors over higher dimensional bases. Note that a G-torsor $Y \to X$ is trivial if and only if there is a section to the morphism $Y \to X$.

Example 2.7.17. Consider the case $X = \mathbb{A}^1_{\mathbb{C}} \backslash \{0\}$ and $Y = V(y^2 - x) \backslash \{(0,0)\} \subseteq \mathbb{A}^2_{\mathbb{C}}$. Let $G = \mathbb{Z}/2\mathbb{Z}$ and the morphism $f : Y \to X$ defined by $(x,y) \mapsto x$. Let pt $\in X$ be a closed point i.e. pt : $x = \alpha$ where $\alpha \in \mathbb{C}^*$. Then the fibre looks like $\{(\alpha, \sqrt{\alpha}), (\alpha, -\sqrt{\alpha})\}$. Now this is a G-torsor via the action of $(x,y) \mapsto (x,-y)$ on Y. If we added back the origin, this is no longer a G-torsor as G does not act freely on the fibre above the origin.

Example 2.7.18. Let $X : \mathbb{V}(f(x_0, \ldots, x_n)) \subset \mathbb{P}^n_{\mathbb{C}}$ and Y be the associated affine cone to X in $\mathbb{A}^{n+1} \backslash \{0\}$. Then we have a morphism $f : Y \to X$ and the fibres above a point $[x_0 : \ldots : x_n]$ are $\lambda(x_0, \ldots, x_n)$ where $\lambda \in \mathbb{C}^*$, i.e. this is a $\mathbb{G}_m$-torsor. It is clear that this torsor is trivial.

Example 2.7.19. Let G be smooth algebraic group over F and H a smooth closed subgroup of G over F. Consider the projection map $\phi : G \to C := G/H$. Note that C is not necessarily an algebraic group as H might not be a normal subgroup. Then for every $c \in C(F)$ the fibre is a H-torsor over Spec F. A special case of this can be seen for isogenies of elliptic curves. For example, let E be an elliptic curve with $C := E(F)[2] \cong \mathbb{Z}/2\mathbb{Z}$ then for the 2-isogeny

$$\psi : E \to D := E/C$$

the fibres above geometric points are always a $\mathbb{Z}/2\mathbb{Z}$-torsor.

2.7.5 Classifying torsors

We have seen that geometrically a G-torsor be viewed as a "family" of copies of G's where we are forgetting the identity element and this is close to the notion of a G-bundle. With G-bundles you have trivialising covers, and if you restricted to the Zariski topology for G-torsors then we wouldn't get this. However, if we use a more refined topology, namely the fppf site (fidèlement plat de présentation finie, i.e. flat and locally finite presentation) we get the required result.

Theorem 2.7.20 ([Mil16, Prop. 4.1])**.** *Let X be a variety over a field F and Y a G-torsor over X where G is a smooth algebraic group. Then there exists a family of fppf morphisms $f_i : U_i \to X$ with $\bigcup_i f_i(U_i) = X$ (fppf cover for X) such that for any i, the G-torsor $Y \times_X U_i$ over U_i is trivial.*

Now using fppf cohomology we can also classify G-torsors up to isomorphism.

Theorem 2.7.21 ([Mil16, Cor. 4.7])**.** *Let G be an affine commutative group scheme over X. Then there is a one-to-one correspondence*

$$\mathrm{H}^1_{fppf}(X,G) \leftrightarrow \{G\text{-torsors over } X\}/\cong$$

Remark 2.7.22. Given a point $P \in X(F)$ and a G-torsor $Y \to X$ we have that the fibre Y_P is a G-torsor. Hence, we have a map of cohomology groups

$$\mathrm{H}^1_{\mathrm{fppf}}(X,G) \to \mathrm{H}^1_{\mathrm{fppf}}(P,G), [Y \to X] \mapsto [Y_P \to \mathrm{Spec}\, F].$$

Remark 2.7.23. When G is smooth and affine over X we can restrict to the étale topology [Sta24, Tag 0497], this will be the case throughout the rest of the thesis.

Chapter 3

Integral points on symmetric affine cubic surfaces

3.1 Introduction

Many interesting Diophantine problems involve finding solutions to inhomogeneous equations, for example a conjecture by Heath-Brown [HB92] states that if n is an integer that is not congruent to 4 or 5 modulo 9, then there exists infinitely many integral solutions to the equation

$$u_1^3 + u_2^3 + u_3^3 = n. \tag{3.1.1}$$

The existence of integral solutions to (3.1.1) was first studied by Mordell in [Mor53]. Later Miller and Woollett [MW55] showed the existence of integral solutions for $0 \leq n \leq 100$, with n not congruent to 4 or 5 mod 9. Colliot-Thélène and Wittenberg [CTW12, Thm. 4.1a] showed that (3.1.1) has no integral Brauer-Manin obstruction for any $n \in \mathbb{Z}$. We ask if a more general statement holds, specifically: given a monic cubic polynomial $f(u) \in \mathbb{Z}[u]$ and an integer n, can the affine surface

$$\mathcal{U}_n : f(u_1) + f(u_2) + f(u_3) = n$$

have an integral Brauer-Manin obstruction?

Results

We prove the following result, which extends our understanding of such questions.

Theorem 3.1.1. *Let $f(u) \in \mathbb{Z}[u]$ be a monic cubic polynomial. Then for all but finitely many $n \in \mathbb{Z}$, the affine surface*

$$\mathcal{U}_n : f(u_1) + f(u_2) + f(u_3) = n \subset \mathbb{A}^3_{\mathbb{Z}} \tag{3.1.2}$$

has no integral Brauer-Manin obstruction.

3.2 Conic bundles on cubic surfaces

In this section we explain some basic properties about conic bundles and give their relation to smooth cubic surfaces. For the rest of Section 3.2 we assume k is a field of characteristic not equal to 2 or 3.

Definition 3.2.1. A *conic bundle* over k is a smooth projective surface X over k together with a dominant morphism $\pi : X \to \mathbb{P}^1_k$, all of whose fibres are isomorphic to either a smooth conic or a union of two lines.

Remark 3.2.2. The Picard group of a conic bundle X is free and finitely generated. Indeed, as $\operatorname{Pic} X$ is a subgroup of $\operatorname{Pic} \bar{X}$ it is sufficient to show that $\operatorname{Pic} \bar{X}$ is free and finitely generated. Any conic bundle over an algebraically closed field is rational (over $\mathbb{P}^1$) [Isk79, §2, Prop. 1b], using [Liu02, §9, Thm. 2.2] and [Har77, Ex. 8.5a] it follows that $\operatorname{Pic} \bar{X}$ is free and finitely generated.

Definition 3.2.3. A conic C over k is called *split* if it is either smooth or isomorphic to a union of two rational lines over k.

Lemma 3.2.4. *Let G be a finite group acting on a finite dimensional vector space V over $\mathbb{R}$. Denote by V^G the vector subspace of V spanned by vectors which are invariant under the action of G. If the action of G is non-trivial then $\dim_{\mathbb{R}} V^G < \dim_{\mathbb{R}} V$.*

Proof. The action of G on V defines a representation $\rho : G \to \mathrm{GL}(V)$, and as V is a finite dimensional vector space. We have that V^G is a vector subspace of V, hence if $\dim V^G = \dim V$, this implies that ρ is the trivial representation, i.e. G acts trivially on V. □

Lemma 3.2.5. *Let $\pi : X \to \mathbb{P}^1_k$ be a conic bundle over k, such that $X(k) \neq \emptyset$. If every singular fibre of π over $\overline{k}$ is defined over k and split, then X is split over k.*

Proof. Let $L \supseteq k$ be a field extension. Using [FLS18, Lemma 2.1] we have that for $X_L := X \times_k L$

$$\operatorname{rank} \operatorname{Pic}(X_L) = 2 + \#\{\text{closed point } p \in \mathbb{P}^1_L \ : \pi^{-1}(p) \text{ singular and split over } \kappa(p)\}$$

where $\kappa(p)$ is the residue field of the point p. Under the assumption that each singular fibre is defined and split over k, $\operatorname{rank} \operatorname{Pic} X = \operatorname{rank} \operatorname{Pic} \bar{X}$. Moreover, as X is a projective scheme over a field k with $X(k) \neq \emptyset$, we can use [CTS21, Rem. 5.4.3] to show

$$\operatorname{Pic} X = \left(\operatorname{Pic} \bar{X}\right)^{\mathrm{Gal}(\bar{k}/k)}.$$

By definition $\dim_{\mathbb{R}}(\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}) = \operatorname{rank} \operatorname{Pic} X$. The absolute Galois group acts on $\operatorname{Pic} \bar{X}$ via a finite subgroup $G \subset \mathrm{Gal}(\bar{k}/k)$. Using the fact that $\dim_{\mathbb{R}}(\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}) = \dim_{\mathbb{R}}\big(\operatorname{Pic}(\bar{X}) \otimes_{\mathbb{Z}} \mathbb{R}\big)$, Lemma 3.2.4 shows that the action of $\mathrm{Gal}(\bar{k}/k)$ on $\operatorname{Pic}(\bar{X}) \otimes_{\mathbb{Z}} \mathbb{R}$ is trivial, hence $\operatorname{Pic} X = \operatorname{Pic} \bar{X}$. □

Lemma 3.2.6. *Let $\pi : X \to \mathbb{P}^1_k$ be a conic bundle over k. If π admits a section over k, then every singular fibre of π is split over k.*

Proof. Let s be a section of π. As s is a well-defined map it will meet each fibre exactly once. Moreover, as X is smooth, any section will intersect each fibre at a smooth point [Liu02, §9 Cor. 1.32]. As the section s can only meet singular fibres at smooth points, i.e. it intersects only one of the lines in the singular fibre, this line is Galois-invariant and the singular fibre is split over k. □

Remark 3.2.7. If X is a smooth cubic surface defined over k with a line $L \subset X$, then X has a conic bundle structure. Up to a change of variables we can assume $L := \{x_1 = 0, x_3 = 0\}$. As X is a smooth cubic surface we can write the defining equation for X as $x_1C_1 + x_3C_2$ where each C_i is a form of degree 2 in variables x_0, x_1, x_2, x_3. Consider the pencil of planes which contain L

$$P_{s,t} : sx_1 - tx_3 = 0.$$

Then the residual intersection of $P_{s,t}$ with X is a conic which we denote by $C_{s,t}$. As X is smooth C_1 and C_2 have no common intersection on L, hence at a point $Q := [x_0 : x_1 : x_2 : x_3]$ if $(x_1, x_3) = (0, 0)$ then $(C_1, C_2) \neq (0, 0)$. We can then define a morphism $\pi : X \to \mathbb{P}^1_k$ by

$$Q := [x_0, x_1, x_2, x_3] \mapsto \begin{cases} [x_1 : x_3] & \text{if } (x_1, x_3) \neq (0, 0), \\ [-C_2(Q) : C_1(Q)] & \text{if } (C_1(Q), C_2(Q)) \neq (0, 0), \end{cases}$$

i.e. π is the projection away from the line L. Moreover, this is a dominant morphism and the fibre above each closed point of $\mathbb{P}^1_k$ is a conic, hence we have a conic bundle structure. If we consider the geometric point $[t : s] \in \mathbb{P}^1_{\bar{k}}$, the fibre above this point is the conic $C_{s,t}$. Consider the representative matrix of the generic fibre of π, which we denote by T. Denote by $\Delta(s, t)$ the determinant of T; this is a function in s and t. Then the geometric points $[\beta : \alpha]$ of $\mathbb{P}^1_{\bar{k}}$ which have a singular fibre are exactly those that satisfy $\Delta(\alpha, \beta) = 0$. We call π the *conic bundle map of X associated to L*. Alternatively, we can view π as the morphism given from the linear system $|-K_X - L|$.

3.3 Geometry of the compactification

When calculating the integral Brauer-Manin obstruction for the affine surfaces (3.1.2) we need to calculate the Brauer group of $U_n := \mathcal{U}_n \times_{\mathbb{Z}} \mathbb{Q}$. If the compactification X_n of U_n in $\mathbb{P}^3$ is smooth, by Grothedieck's purity theorem [CTS21, Thm. 3.7.1] there is an inclusion $\operatorname{Br} X_n \hookrightarrow \operatorname{Br} U_n$. Therefore, the first step in calculating $\operatorname{Br} U_n$ is to calculate $\operatorname{Br} X_n$. Throughout Section 3.3 fix a field k of characteristic not equal to 2 or 3 and a polynomial $f(u) = u^3 + a_2u^2 + a_1u^2 + a_0 \in k[u]$. For $n \in k$ denote by U_n the affine cubic surface

$$U_n : f(u_1) + f(u_2) + f(u_3) = n \subset \mathbb{A}^3_k.$$

By completing the cube and making a change of variables we can assume that

$$U_n : F(1, u_1) + F(1, u_2) + F(1, u_3) = n \subset \mathbb{A}^3_k$$

where $F(x_0, x_i) = x_i^3 + ax_ix_0^2 + bx_0^3 \in k[x_0, x_i]$. The compactification of U_n is

$$X_n : F(x_0, x_1) + F(x_0, x_2) + F(x_0, x_3) = nx_0^3 \subset \mathbb{P}^3_k$$

where $u_i = x_i/x_0$ for $i = 1, 2, 3$. We require X_n to be smooth to apply Grothedieck's purity theorem; in Section 3.3.1 we show that outside of a finite set of $n \in k$, the cubic surface X_n is always smooth. Since Theorem 3.7.4 asserts outside a finite set of $n \in \mathbb{Z}$, the affine cubic surface (3.1.2) has no integral Brauer-Manin obstruction, hence we may assume X_n is smooth.

3.3.1 Smooth compactifications

Lemma 3.3.1. *Define*

$$X : F(x_0, x_1) + F(x_0, x_2) + F(x_0, x_3) - nx_0^3 \subset \mathbb{P}_k^3 \times \mathbb{A}_k^1$$

where $\mathbb{A}_k^1$ *has the coordinate* n. *Then the projection map* $\pi : X \to \mathbb{A}_k^1$ *has a smooth generic fibre.*

Proof. The generic fibre is $X_\eta := X_n \times_k k(n) \subset \mathbb{P}^3_{k(n)}$. Let

$$G(x_0, x_1, x_2, x_3) := x_1^3 + x_2^3 + x_3^3 + a(x_1 + x_2 + x_3)x_0^2 + (3b - n)x_0^3.$$

If there exists a singular point $p = [x_0 : x_1 : x_2 : x_3]$ lying on X_η, then by definition of smoothness we see that the partial derivatives of G will vanish when evaluated at p. Explicitly,

$$\frac{\partial G}{\partial x_i}(p) = 3x_i^2 + ax_0^2 = 0 \text{ for } i = 1, 2, 3$$
$$\frac{\partial G}{\partial x_0}(p) = 2a(x_1 + x_2 + x_3)x_0 + 3(3b - n)x_0^2 = 0.$$

As $\frac{\partial G}{\partial x_0}(p) = 0$ this implies that $(2a(x_1 + x_2 + x_3) + 3(3b - n)x_0)x_0 = 0$, hence either

$$x_0 = 0 \text{ or } 2a(x_1 + x_2 + x_3) + 3(3b - n)x_0 = 0.$$

If $x_0 = 0$, this implies $x_1 = x_2 = x_3 = 0$, hence such a singular point cannot exist. We can now assume that $x_0 = 1$, using the partial derivative $\frac{\partial G}{\partial x_i}$ we see $x_i = \pm\sqrt{-a/3}$ for $i = 1, 2, 3$. If $x_1 = x_2 = x_3 = \sqrt{-a/3}$, then

$$G(1, \sqrt{-a/3}, \sqrt{-a/3}, \sqrt{-a/3}) = 3(\sqrt{-a/3})^3 + 3a\sqrt{-a/3} + (3b - n) = 0,$$

hence $n = 3b - \frac{2}{3}\sqrt{3}(-a)^{3/2}$. As $3b - \frac{2}{3}\sqrt{3}(-a)^{3/2} \in \bar{k}$ but $n \notin \bar{k}$, this is a contradiction. The other possibilities for x_1, x_2, x_3 lead to the same contradiction, hence we see that the generic fibre X_η is smooth over $k(n)$. □

Lemma 3.3.2. *If the cubic polynomial* $f_1(x) = x^3 + ax + 3b - n \in k[x]$ *is not separable then the surface* X_n *is singular.*

Proof. f_1 is not separable if and only if the discriminant of f_1 is zero i.e. $4a^3 + 27(3b - n)^2 = 0$. Then the surface X_n has a singular point $[x_0 : x_1 : x_2 : x_3] = [1 : \sqrt{-a/3} : \sqrt{-a/3} : -\sqrt{-a/3}]$. □

Proposition 3.3.3. *There are only finitely many choices of* $n \in k$ *such that* X_n *is singular.*

Proof. We saw in Lemma 3.3.1 that the generic fibre of the projection map $\pi : X \to \mathbb{A}_k^1$ is smooth. This means that the smooth locus of the morphism π is an open subset of $\mathbb{A}_k^1$ which contains the generic point, thus there can only be finitely many singular fibres. □

For the rest of Section 3.3, fix a choice of $n \in k$ is such that X_n is smooth.

3.3.2 Conic bundle structure on X_n

We can now describe explicitly the conic bundle structure on X_n over a possible finite extension l of k. Let $f_1(x) = x^3 + ax + 3b - n$. If $f_1(x)$ is irreducible, let $l := k[x]/(f_1(x))$ and let r be the image of x in l. Otherwise, let $l := k$ and let r be a root of $f_1(x)$ in l. Then $X_n \times_k l$ contains a line L, where

$$L : x_2 + x_3 = 0,\ x_1 - rx_0 = 0.$$

We then run the construction given in Remark 3.2.7 and this defines the conic bundle map π of X_n associated to the line L. The points of $\mathbb{P}^1_l$ over which π has singular fibres are

$$p = \begin{cases} (t), \\ (s+t), \\ \big(at^3 + 3r^2st^2 - 3r^2s^2t + (4a+3r^2)s^3\big). \end{cases}$$

The singular fibres of π over $\bar{k}$ are defined over the splitting field of

$$ax^3 + 3r^2x^2 - 3r^2x + (4a + 3r^2)$$

over l.

Remark 3.3.4. To fix some notation for the rest of the section, we label nine lines on $\bar{X}_n$. Let r_1, r_2, r_3 be roots of $f_1(x)$ in $\bar{k}$. The surface $\bar{X}_n$ contains the lines

$$\begin{aligned}
L_{1,1} &: x_2 + x_3 = x_1 - r_1x_0 = 0,\\
L_{2,1} &: x_1 + x_3 = x_2 - r_1x_0 = 0,\\
L_{3,1} &: x_1 + x_2 = x_2 - r_1x_0 = 0,\\
L_{1,2} &: x_2 + x_3 = x_1 - r_2x_0 = 0,\\
L_{2,2} &: x_1 + x_3 = x_2 - r_2x_0 = 0,\\
L_{3,2} &: x_1 + x_2 = x_3 - r_2x_0 = 0,\\
L_{1,3} &: x_2 + x_3 = x_1 - r_3x_0 = 0,\\
L_{2,3} &: x_1 + x_3 = x_2 - r_3x_0 = 0,\\
L_{3,3} &: x_1 + x_2 = x_3 - r_3x_0 = 0.
\end{aligned}$$

Note that each set of lines $[L_{i,1}, L_{i,2}, L_{i,3}]$ for $i = 1, 2, 3$, define three coplanar lines, meeting at an Eckardt point.

3.3.3 Splitting field

Lemma 3.3.5. *Denote by L the splitting field of f_1 over k. The conic bundle map associated to $L_{1,1}$ on X_n admits a section over L, namely $L_{2,2}$.*

Proof. It is sufficient to show that $L_{2,2}$ does not intersect $L_{1,1}$. Let P be the plane $x_2 + x_3 = 0$, this plane contains $L_{1,j}$ for $j \in \{1, 2, 3\}$. The line $L_{2,2}$ intersects P at the point $p = [1 : r_2 : r_2 : -r_2]$. Moreover, if $p \in L_{1,1}$ this would imply that $r_1 = r_2$. This is only possible if f_1 is not separable, by Lemma 3.3.2 this implies X_n is not smooth contradicting our assumption on X_n. □

Proposition 3.3.6. *X_n is split over the splitting field of the two polynomials*

$$f_1(x) = x^3 + ax + 3b - n \in k[x],\ g_1(x) = ax^3 + 3r^2(x^2 - x + 1) + 4a \in k(r)[x]$$

where if f_1 is reducible over k, r is a root of f_1; if f_1 is irreducible over k, r is the image x in the field extension $k[x]/(f_1(x))$.

Proof. Let F be the splitting field of f_1 and g_1 over k and $\pi : X_n \to \mathbb{P}^1_K$ be the conic bundle map associated to $L_{1,1}$ on X_n. All singular fibres over $\bar{k}$ are defined over F and by Lemma 3.3.5 there exists a section of π, hence by Lemma 3.2.6 every fibre of π is split. By Lemma 3.2.5 X_n is split over L. □

Even though we have polynomials that define the splitting field for X_n, one of the polynomials is defined over k and the other over a possible finite extension of k. To understand the Galois action on the lines of X_n it will be easier if both polynomials are always defined over k; the rest of the results in Section 3.3.3 are devoted to obtaining two such polynomials over k for all but finitely many choices of $n \in \mathbb{Z}$. In the case $a = 0$ we see that $f_1(x) = x^3 + 3b - n$ and $g_1(x) = 3r^2(x^2 - x + 1)$ and as the splitting field of f_1 will contain a third root of unity ζ_3 we have that $x^2 - x + 1$ will factor as $(x + \zeta_3)(x + \zeta_3^2)$, hence the splitting field of f_1 and g_1 is the same as the splitting field of f_1. For the rest of Section 3.3.3 we assume a and $4a^3 + 27(3b - n)^2$ are non-zero. Let $l := k(r) = k[x]/(f_1(x))$ if f_1 is irreducible or $l := k$ where r is a root of f_1 if f_1 is reducible.

Lemma 3.3.7. *Denote by ξ_1 the l-algebra homomorphism*

$$\xi_1 : l[x] \to l[y],\ x \mapsto -ay^2 - \left(3\,r^2 - a\right)y + (4a + 3r^2).$$

The image of the ideal generated by $f_2(x) = x^3 - 12ax^2 + 36a^2x + 27(3b - n)^2 + 4a^3$ is contained in the ideal generated by $g_1(y)$.

Proof. The image of $f_2(x)$ under ξ_1 is $f_2(\xi_1(x))$. By applying Euclidean division to $f_2(\xi_1(x))$ and $g_1(y)$ we see that $f_2(\xi_1(x)) = p(y)g_1(y) + q(y)$ where

$$\begin{aligned} p(y) &= -a^3y^3 + 18\,ar^4 - 6\,a^2r^2 + 5\,a^3 - 3\left(2\,a^2r^2 - a^3\right)y^2 - 3\left(3\,ar^4 - 5\,a^2r^2 + a^3\right)y, \\ q(y) &= -27\,r^6 - 54\,ar^4 - 27\,a^2r^2 + 243\,b^2 - 162\,bn + 27\,n^2. \end{aligned}$$

Factoring $q(y)$ we see that $q(y) = (-27\,r^3 - 27\,ar + 81\,b - 27\,n)(r^3 + ar + 3\,b - n)$, hence $q(y) = 0$ in $l[y]$ as r is root of f_1. Thus $\xi_1((f_2)) \subseteq (g_1)$. □

Lemma 3.3.8. *Denote by ξ_2 the l-algebra homomorphism $\xi_2 : l[y] \to l[x]$ where y is mapped to*

$$\frac{\left(6\,ar^2 + 4\,a^2 - 9cr\right)x^2 + \left(36\,a^2r^2 + 28\,a^3 - 54\,acr + 27\,c^2\right)x}{3\,(4\,a^3 + 27c^2)a} + \frac{r^2}{a} + \frac{4}{3}$$

where $c := 3b - n$. Then the image of the ideal generated by $g_1(y)$ is contained in the ideal generated by $f_2(x)$. In particular, The l-algebra morphisms

$$\xi_1 : l[x]/(f_2(x)) \to l[y]/(g_1(y)) \text{ and } \xi_2 : l[y]/(g_1(y)) \to l[x]/(f_2(x))$$

are well defined and inverses of each other.

Proof. The first part of the statement was done by a SageMath computation and shows ξ_2 is well defined. The fact that ξ_1 is well defined follows Lemma 3.3.7. It is then easy to check they are inverses of each other. □

Proposition 3.3.9. *Consider the cubic polynomials*

$$f_2(x) = x^3 - 12ax^2 + 36a^2x + 27(3b-n)^2 + 4a^3 \text{ and } g_1(x) = ax^3 + 3r^2(x^2 - x + 1) + 4a.$$

defined over l. If a and $4a^3 + 27(3b-n)^2$ are non-zero, then there exists a l-algebra isomorphism

$$l[x]/(f_2(x)) \xrightarrow{\sim} l[y]/(g_1(y)).$$

and the compositum of the splitting fields of f_1 and f_2 is isomorphic to the compositum of the splitting fields of f_1 and g_1.

Proof. The statement follows from Lemma 3.3.8. □

3.3.4 Brauer group of compactification

We have shown that the splitting field K of X_n is the compositum of the splitting fields of f_1 and g_1. We are now in a position to determine the Brauer group of X_n. In the case $a = 0$ this is completely determined by Colliot-Thélène and Wittenberg [CTW12, Prop. 2.1]. We determine the Brauer group when $a \neq 0$.

Proposition 3.3.10. *Let $F(x_0, x_i) = x_i^3 + ax_ix_0^2 + bx_0^3 \in \mathbb{Q}(a,b,n)[x_0, x_1]$, and denote by $X_{a,b,n}$ the cubic surface*

$$X_{a,b,n} : F(x_0, x_1) + F(x_0, x_2) + F(x_0, x_2) = nx_0^3$$

over $\mathbb{Q}(a,b,n)$. Denote by L the splitting field of $X_{a,b,n}$. Then $\mathrm{Gal}(L/\mathbb{Q}(a,b,n)) \cong S_3 \times S_3$.

Proof. Note that if we specialise a, b and n to values in $\mathbb{Q}$ such that X_n is smooth with splitting field K, then as abstract groups $\mathrm{Gal}(K/\mathbb{Q})$ is a subgroup of $\mathrm{Gal}(L/\mathbb{Q}(a,b,n))$, by Proposition 2.6.4. Choose $a = 19, b = 8$ and $n = 5$. Using Magma, we see that X_n is smooth and has splitting field K such that $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3 \times S_3$, hence $[L : \mathbb{Q}(a,b,n)] \geq 36$. Furthermore, by Proposition 3.3.6, $[L : \mathbb{Q}(a,b,n)] \leq 36$, hence $[L : \mathbb{Q}(a,b,n)] = 36$. As $S_3 \times S_3 \leq \mathrm{Gal}(L/\mathbb{Q}(a,b,c,n))$ and the degree of the extension of L is 36 we can deduce that $\mathrm{Gal}(L/\mathbb{Q}(a,b,c,n)) \cong S_3 \times S_3$. □

Remark 3.3.11 (Specialisations)**.** There is an isomorphism $\mathrm{Gal}(L/\mathbb{Q}(a,b,n)) \to N \subset W(E_6)$ where N is a subgroup of $W(E_6)$. Moreover, N has an action on the 27 exceptional vectors in $E_6 \otimes \mathbb{Q}$ and this is the action of $\mathrm{Gal}(L/\mathbb{Q}(a,b,n))$ on the 27 lines of $X_{a,b,n}$. When we specialise to some values in a, b, n such that f_1 and g_1 are separable and X_n is a smooth cubic surface we have a homomorphism $\mathrm{Gal}(K/\mathbb{Q}) \hookrightarrow \mathrm{Gal}(L/\mathbb{Q}(a,b,n)) \to N$ where the image of $\mathrm{Gal}(K/\mathbb{Q})$ inside N is some subgroup G of N and the Galois action on the 27 lines of X_n is the action of G on the 27 exceptional vectors of $E_6 \otimes \mathbb{Q}$. Fix a choice of $a, b, n \in \mathbb{Q}$ such that X_n is smooth and the compositum of the splitting fields of f_1 and g_1 generate a $S_3 \times S_3$ extension of $\mathbb{Q}$. Then the sets $\{L_{1,i} : i = 1,2,3\}$,$\{L_{2,i} : i = 1,2,3\}$ and $\{L_{3,i} : i = 1,2,3\}$ form Galois orbits of size three. Enumerating the subgroups of $W(E_6)$ which have three orbits of size three and are isomorphic to $S_3 \times S_3$ leads to two possibilities for the orbit types, namely $[3,3,3,18]$ and $[3,3,3,3,3,3,9]$. Moreover, using

Magma we can show that for the case $a = 19, b = 8$ and $n = 5$ the Galois orbit type is $[3,3,3,18]$ hence, the Galois orbit type of $X_{a,b,n}$ is $[3,3,3,18]$. Now we know every possible Galois action on any smooth specialisation of $X_{a,b,n}$ such that f_1 and g_1 are separable. This is described fully in Table 3.3.1. We also use Magma to compute $\mathrm{H}^1(\mathbb{Q}, \operatorname{Pic} \bar{X})$ for any smooth cubic surface X over $\mathbb{Q}$ for the Galois actions listed in Table 3.3.1.

Table 3.3.1 describes the possible Galois actions on the surface $\bar{X}_n$ with splitting field K over $\mathbb{Q}$. The first column describes $\mathrm{Gal}(K/\mathbb{Q})$, the second column describes the possible Galois action on the lines i.e. $[1^3, 2^{12}]$ means there are 3 orbits of size 1 and 12 orbits of size 2. Then the final column describes the first Galois cohomology of $\operatorname{Pic} \bar{X}_n$.

Table 3.3.1: *Possible Galois Types*

Galois Group	Orbit Type	$\mathrm{H}^1(\mathbb{Q}, \operatorname{Pic} \bar{X}_n)$
C_1, C_2	$[1^{27}], [1^{15}, 2^6]$	0
C_2	$[1^3, 2^{12}]$	$(\mathbb{Z}/2\mathbb{Z})^2$
C_2, C_3, C_3	$[1^3, 2^{12}], [1^9, 3^6], [3^9]$	0
C_3	$[3^9]$	$(\mathbb{Z}/3\mathbb{Z})^2$
C_2^2	$[1^3, 2^6, 4^3]$	$\mathbb{Z}/2\mathbb{Z}$
S_3	$[3^3, 6^3]$	$(\mathbb{Z}/2\mathbb{Z})^2$
S_3, C_6, C_6, S_3, S_3	$[1^9, 3^6], [1^3, 2^3, 6^3], [3^5, 6^2], [1^3, 2^3, 6^3], [3^3, 6^3]$	0
S_3	$[3^3, 6^3]$	$\mathbb{Z}/3\mathbb{Z}$
C_3^2	$[3^3, 9^2]$	$\mathbb{Z}/3\mathbb{Z}$
$C_2 \times S_3$	$[3^3, 6, 12]$	$\mathbb{Z}/2\mathbb{Z}$
$C_2 \times S_3, C_3 \rtimes S_3$	$[1^3, 2^3, 6^3], [3^3, 18]$	0
$C_3 \times S_3$	$[3^3, 9^2]$	$\mathbb{Z}/3\mathbb{Z}$
$C_3 \times S_3, S_3^2$	$[3^3, 18], [3^3, 18]$	0

Proposition 3.3.12. *Choose $n \in \mathbb{Q}$ such that X_n is smooth over $\mathbb{Q}$. Denote by K the splitting field of X_n. If* $\mathrm{Gal}(K/\mathbb{Q}) \cong C_2 \times S_3$ *and f_1 is reducible or* $\mathrm{Gal}(K/\mathbb{Q}) \cong S_3 \times S_3$ *then* $\operatorname{Br} X_n = \operatorname{Br} \mathbb{Q}$.

Proof. The cubic surface X_n always has a rational point, namely $[x_0 : x_1 : x_2 : x_3] = [0 : 1 : -1 : 0]$, hence it is everywhere locally soluble. By the Hochschild-Serre spectral sequence we can conclude that $\operatorname{Br} X / \operatorname{Br} \mathbb{Q} \cong \mathrm{H}^1(\mathbb{Q}, \operatorname{Pic} \bar{X}_n)$, and then using the results from Table 3.3.1 and the fact the Galois orbit type of lines on $X_{a,b,n}$ is $[3,3,3,18]$ the proposition follows. □

Notation 3.3.13. Denote by $\Delta_i(n)$ the discriminant of f_i, for a fixed choice of $n \in \mathbb{Z}$. As $a, b, n \in \mathbb{Z}$, this implies that $\Delta_i(n) \in \mathbb{Z}$ for $i = 1, 2$. Furthermore, denote by $\Delta_3(n)$ the quotient $\Delta_2(n)/\Delta_1(n)$ if $4a^3 + 27(3b - n)^2 \neq 0$ or $\Delta_1(n)/\Delta_2(n) = 0$ otherwise.

Lemma 3.3.14. *Let*

$$A_1 := \{n \in \mathbb{Z} : \Delta_1(n), \Delta_2(n) \text{ or } \Delta_3(n) \text{ is a square or either of } f_1 \text{ or } f_2 \text{ is reducible}\}$$

then for all $n \notin A_1$. Then the compositum of the splitting field of f_1 and f_2 is a $S_3 \times S_3$-extension of $\mathbb{Q}$.

Proof. By Propositions 3.3.10 and 2.6.4 the Galois group of the splitting field K of f_1 and f_2 over $\mathbb{Q}$ is a subgroup of $S_3 \times S_3$, so it is sufficient to show that K has degree 36 for $n \notin A_1$. Choose $n \in \mathbb{Z}$ such that $n \notin A_1$. Then $\mathbb{Q}(r_i) = \mathbb{Q}[x]/(f_i)$ is a degree 3 extension of $\mathbb{Q}$. Moreover, as $\Delta_3(n)$ is not a square $[\mathbb{Q}(r_1, r_2) : \mathbb{Q}] = 9$. As $\Delta_i(n)$ is not a square

$$\left[\mathbb{Q}\left(\sqrt{\Delta_i(n)}\right) : \mathbb{Q}\right] = 2 \text{ for } i \in \{1, 2\}.$$

If $\sqrt{\Delta_i(n)} \in \mathbb{Q}(\sqrt{\Delta_j(n)})$ for $i, j \in \{1, 2\}$ and $i \neq j$ then

$$x_n + y_n\sqrt{\Delta_j(n)} = \sqrt{\Delta_i(n)} \text{ for } x_n, y_n \in \mathbb{Q},$$
$$x_n^2 + y_n^2\Delta_j(n) + 2x_n y_n\sqrt{\Delta_j(n)} = \Delta_i(n).$$

By comparing coefficients $2x_n y_n = 0$, and either $x_n = 0$ or $y_n = 0$. If $x_n = 0$, $\Delta_3(n)$ is a square in $\mathbb{Q}$ and if $y_n = 0$ then, $x_n^2 = \Delta_i$ both of which are contradictions to the fact $n \notin A_1$. Hence,

$$\left[\mathbb{Q}\left(\sqrt{\Delta_1(n)}, \sqrt{\Delta_2}(n)\right) : \mathbb{Q}\right] = 4.$$

Then by applying the tower law we can conclude that

$$\left[\mathbb{Q}\left(r_1, r_2, \sqrt{\Delta_1(n)}, \sqrt{\Delta_2(n)}\right) : \mathbb{Q}\right] = 36.$$ □

Lemma 3.3.15. *Let*

$$A_2 := \{n \in \mathbb{Z} : \Delta_1(n), \Delta_2(n) \textit{ or } \Delta_3(n) \textit{ is a square or } f_2 \textit{ is reducible}\}$$

then for all $n \notin A_2$ the compositum of the splitting field of f_1 and f_2, denoted by K, is a $S_3 \times S_3$ or $C_2 \times S_3$ extension of $\mathbb{Q}$.

Proof. Fix $n \notin A_2$. If f_1 is irreducible then by Lemma 3.3.14 K is a $S_3 \times S_3$ extension of $\mathbb{Q}$. If f_1 is reducible, by a similar argument to Lemma 3.3.14 K is a $C_2 \times S_3$ extension of $\mathbb{Q}$. □

Proposition 3.3.16. *Fix $n \in \mathbb{Z}$ such that X_n is smooth and $n \notin A_2$, then* $\operatorname{Br} X_n = \operatorname{Br} \mathbb{Q}$.

Proof. Note that if $n \notin A_2$ then $4a^3 + 27(3b - n)^2 \neq 0$, as otherwise f_2 would be reducible. Then by Proposition 3.3.9 the compositum of the splitting fields of f_1 and g_1 is isomorphic to the splitting field of f_1 and f_2. Then the statement follows from Proposition 3.3.12. □

For the rest of the section denote by $A_3 := A_2 \cup \{n \in \mathbb{Z} : X_n \text{ is not smooth}\}$.

3.3.5 Finiteness results for thin sets

Section 3.3.5 is dedicated to showing that if $a, b \in \mathbb{Z}$ are fixed and a is non-zero, then for all but finitely many choices of $n \in \mathbb{Z}$, $\operatorname{Br} X_n = \operatorname{Br} \mathbb{Q}$. By Proposition 3.3.16 it will be sufficient to show that for all but finitely many $n \in \mathbb{Z}$ the polynomials f_1 and f_2 generate either a $S_3 \times S_3$-extension or a $C_2 \times S_3$-extension such that f_1 is reducible.

Proposition 3.3.17. *Fix $a, b \in \mathbb{Z}$ with a non-zero. Then there exists a thin set $A \subset \mathbb{A}^1_{\mathbb{Q}}(\mathbb{Q})$, such that for $n \notin A$, the cubic surface X_n over $\mathbb{Q}$ is smooth and has the property* $\operatorname{Br} X_n = \operatorname{Br} \mathbb{Q}$.

Proof. By Proposition 3.3.3 the set $B := \{n \in \mathbb{Q} : X_n \text{ is not smooth}\}$ is finite. Using Theorem 2.6.3 there is a thin set C of $n \in \mathbb{Q}$ such that the compositum of the splitting field of f_1 and f_2 generate an $S_3 \times S_3$ extension of $\mathbb{Q}$. Then the proposition follows from using Proposition 3.3.16 and letting $A := (B \cup C) \cap \{n : n \in \mathbb{Z}\}$. □

Remark 3.3.18. The assumption that a is non-zero in Proposition 3.3.17 is a necessary condition as if $a = 0$ then by [CTW12, Prop. 2.1] $\operatorname{Br} X_n / \operatorname{Br} \mathbb{Q} \cong \mathbb{Z}/3\mathbb{Z}$ for all $n \in \mathbb{Q}$ such that f_1 is irreducible.

Lemma 3.3.19. *$\Delta_1(n)$ is a square only for finitely many $n \in \mathbb{Z}$.*

Proof. Suppose $\Delta_1(n)$ is a square for infinitely many choices of $n \in \mathbb{Z}$, i.e. there exists an infinite family $(n, z_n) \in \mathbb{Z}^2$ such that

$$\Delta_1(n) = -\left(4\,a^3 + 27\,(3\,b - n)^2\right) = z_n^2. \tag{3.3.1}$$

It is clear there are only finitely many pairs $(n, 0)$ in such a family (n, z_n), hence we restrict our considerations to n such that $z_n \in \mathbb{Z}\backslash\{0\}$. If $a > 0$ such a z_n cannot exist, if $a < 0$ let $a = -a'$ and we can rewrite (3.3.1) as $4(a')^3 - 27(3b - n)^2 = z_n^2$. Let

$$N : \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right] \to \mathbb{Z}$$

be the norm map associated to the field extension $\mathbb{Q}(\sqrt{-3})$. Then the existence of a family (n, z_n) implies the existence of infinitely many $(x_n, y_n) \in \mathbb{Z}^2$ such that

$$N(x_n + \alpha y_n) = 4(a')^3 \tag{3.3.2}$$

where $\alpha = \frac{1+\sqrt{-3}}{2}$. As $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ is a unique factorization domain with only finitely many units, there are only finitely many $x_n, y_n \in \mathbb{Z}$ that satisfy (3.3.2), a contradiction. □

Lemma 3.3.20. *$\Delta_2(n)$ is a square only for finitely many $n \in \mathbb{Z}$.*

Proof. Suppose $\Delta_2(n)$ is a square for infinitely many choices of $n \in \mathbb{Z}$, i.e. there exists an infinite family $(n, z_n) \in \mathbb{Z}^2$ such that

$$\Delta_2(n) = -243\left(4\,a^3 + 27\,(3\,b - n)^2\right)\left(4\,a^3 + 3\,(3\,b - n)^2\right) = z_n^2. \tag{3.3.3}$$

As in Lemma 3.3.19 we can assume $z_n \in \mathbb{Z}\backslash\{0\}$. The statement (3.3.3) is equivalent to

$$-(4a^3 + 27(3b - n)^2)(4a^3 + 3(3b - n)^2) = 3t_n^2 \tag{3.3.4}$$

for $t_n \in \mathbb{Z}\backslash\{0\}$. By expanding brackets, we know that $-(16\,a^6 + 120\,a^3(3b - n)^2 + 81\,(3b - n)^4) = 3t_n^2$. Hence, 3 divides a and clearly, we require $a < 0$ as $t_n^2 > 0$. Let $-a = 3a'$ then by (3.3.4) and dividing through by 3, we obtain $(36(a')^3 - 9(3b-n)^2)((3b-n)^2 - 36(a')^3) = t_n^2$. Let $x = (36(a')^3 - 9(3b - n)^2)$ and $y = ((3b - n)^2 - 36(a')^3)$, then for large enough n we have that $y > 0$ and $x < 0$, hence such an infinite family $(n, z_n) \in \mathbb{Z}^2$ cannot exist. □

Lemma 3.3.21. *$\Delta_3(n)$ is a square for finitely many $n \in \mathbb{Z}$.*

Proof. Suppose $\Delta_3(n)$ is a square in $\mathbb{Q}$ for infinitely many $n \in \mathbb{Z}$, i.e. there exists an infinite family $(n, z_n) \in \mathbb{Z} \times \mathbb{Q}$ such that

$$\Delta_3(n) = \begin{cases} \frac{-243\left(4\,a^3+27\,(3\,b-n)^2\right)\left(4\,a^3+3\,(3\,b-n)^2\right)}{-\left(4\,a^3+27\,(3\,b-n)^2\right)} = z_n^2 & \text{if } 4\,a^3 + 27\,(3\,b-n)^2 \neq 0, \\ 0 & \text{if } 4\,a^3 + 27\,(3\,b-n)^2 = 0. \end{cases} \tag{3.3.5}$$

Clearly, $\Delta_3(n) = 0$ for only finitely many choices of n, so we can focus on the cases where $z_n \in \mathbb{Q}^\times$. Then (3.3.5) is equivalent to $\Delta_3(n) = 4\,a^3 + 3\,(3\,b-n)^2 = 3t_n^2$ where $t_n \in \mathbb{Z}\backslash\{0\}$. Clearly 3 divides a, hence $36\,(a')^3 + (3\,b-n)^2 = t_n^2$ where $a' := a/3$. Let $N := 36\,(a')^3$ and $x_n := 3\,b-n$ then $t_n^2 - x_n^2 = N$ and as $N \in \mathbb{Z}$ it has only finitely many factors, so there are only finitely many choices for x_n and t_n. ☐

Lemma 3.3.22. *Fix $a, b \in \mathbb{Z}$ with a non-zero. Denote by C the affine curve*

$$C : x^3 - 12ax^2 + 36a^2x + 27(3b-t)^2 + 4a^3 = 0 \subset \mathbb{A}^2_{\mathbb{Q}}$$

in variables t and x. Then the compactification $\tilde{C}$ in $\mathbb{P}^2$ of C is smooth.

Proof. Let $F(t, x, y) = x^3 - 12ax^2y + 36a^2xy^2 + 27y(3by-t)^2 + 4a^3y^3$, then the partial derivatives of F are

$$\begin{aligned} \frac{\partial F}{\partial t} &= -54\,(3\,by-t)y, \\ \frac{\partial F}{\partial x} &= 36\,a^2y^2 - 24\,axy + 3\,x^2, \\ \frac{\partial F}{\partial y} &= 12\,a^3y^2 + 72\,a^2xy - 12\,ax^2 + 162\,(3\,by-t)by + 27\,(3\,by-t)^2. \end{aligned}$$

Assume that $\tilde{C}$ has a non-smooth point $p = [t, x, y]$. Then $F(p) = 0$ and all the partial derivatives of F will vanish at p. As $\frac{\partial F}{\partial t}(p) = 0$, then either $y = 0$ or $t = 3by$. If $y = 0$ clearly $x = 0$, as $\frac{\partial F}{\partial y}(p) = x = y = 0$ this implies $t = 0$ which is a contradiction. Suppose $t = 3by$, as $\frac{\partial F}{\partial x}(p) = 0$ this implies $x = 6ay$ or $x = 2ay$. However, as we require $F(p) = 0$, if $t = 3by$ and $x = 6ay$ or $x = 2ay$ this implies $a = 0$ which is a contradiction. Hence, $\tilde{C}$ is smooth. ☐

Proposition 3.3.23. *Let $f(t, x) = x^3 - 12ax^2 + 36a^2x + 27(3b-t)^2 + 4a^3$ such that $a, b \in \mathbb{Z}$ and a is non-zero. Then for all but finitely many $n \in \mathbb{Z}$, the polynomial $f(n, x) \in \mathbb{Z}[x]$ is irreducible.*

Proof. By Lemma 3.3.22 and the genus-degree formula, the curve $f(t, x) \subset \mathbb{A}^2_{\mathbb{Q}}$ has genus 1. Every specialisation $t = n \in \mathbb{Z}$ gives rise to a monic cubic polynomial $f(n, x) \in \mathbb{Q}[x]$. By Gauss's Lemma [Lan02, Thm. 2.3], we see that $f(n, x)$ is irreducible over $\mathbb{Q}[x]$ if and only if $f(n, x)$ is irreducible over $\mathbb{Z}[x]$. Hence, we can apply Siegel's Theorem on curves [Sie14, Thm 3.2] to show there are only finitely many specialisations $t = n \in \mathbb{Z}$ such that $f(n, x)$ is reducible. ☐

Corollary 3.3.24. *Fix $a, b \in \mathbb{Z}$ such that a is non-zero. Then for all but finitely many choices of $n \in \mathbb{Z}$, $\operatorname{Br} X_n = \operatorname{Br} \mathbb{Q}$.*

Proof. By Proposition 3.3.16 it is sufficient to show the set A_3 is finite, this follows from Lemmas 3.3.19, 3.3.20, 3.3.21 and Propositions 3.3.3, 3.3.23. ☐

3.4 Geometry of the affine surfaces

When studying the geometry of the affine cubic surfaces $\mathcal{U}_n$, one notices that the geometry is similar in certain aspects to that of the sum of three cubes. An example of this is the presence of transcendental Brauer group elements in certain cases. We adapt the argument given by Wittenberg and Colliot-Thélène's in [CTW12, Prop. 3.1] to prove that for certain cases there are no transcendental elements in $\operatorname{Br} U_n$. Furthermore, the Brauer group of the compactification of U_n coincides with $\operatorname{Br} U_n$.

3.4.1 Brauer group of the affine surfaces

Lemma 3.4.1 ([CTW12, Lemma 3.2]). *Let $D \subset \mathbb{P}^2_{\mathbb{Q}}$ be a plane curve with equation $x_1^3 + x_2^3 + x_3^3 = 0$ and denote by $P_1, P_2 \in D(\mathbb{Q})$ the points with coordinates $P_1 = [1 : -1 : 0]$, $P_2 = [-1 : 0 : 1]$. Then*

1. *The image of $\mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1_{\text{ét}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ evaluated at the point P_1 is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.*
2. *An element of $\mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z})$ which is zero at P_1 and P_2 is zero.*

Proposition 3.4.2. *Fix choice of $n \in \mathbb{Z}$ such that f_1 has a non-square discriminant and the cubic surface X_n is smooth. Then the natural map $\operatorname{Br} X_n \to \operatorname{Br} U_n$ is an isomorphism.*

Proof. Denote by $D \subset X$ the hyperplane section defined by $x_0 = 0$. We can now apply Grothedieck's purity theorem [CTS21, Thm. 3.7.1], which gives an exact sequence

$$0 \to \operatorname{Br} X_n \to \operatorname{Br} U_n \xrightarrow{\partial_D} \mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z}).$$

We want to show that the image of the residue map ∂_D is 0. Let $A \in \operatorname{Br} U_n$ but $A \notin \operatorname{Br} X_n$ and let $m = \partial_D(A) \in \mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z})$. Denote by P_1, P_2, P_3 the points in $X_n(\mathbb{Q})$ with coordinates in $[x_0 : x_1 : x_2 : x_3]$ given by $P_1 = [0 : 1 - 1 : 0], P_2 = [0 : -1 : 0 : 1]$. Let $k = \mathbb{Q}[x]/(f_1(x))$ where r is the image of x and $L_i := L_{i,1} \subset X_n \times k$ for $i = 1, 2$. Each line L_i on $X_n \times k$ intersects the divisor $D \times k$ at a unique point $P_i' := P_i \times k$. This gives the following commutative diagram for each i

$$\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{Br} X_n & \longrightarrow & \operatorname{Br} U_n & \xrightarrow{\partial_D} & \mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z}) \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha_i} \\
0 & \longrightarrow & \operatorname{Br} L_i & \longrightarrow & \operatorname{Br}(L_i \backslash P_i') & \longrightarrow & \mathrm{H}^1_{\text{ét}}(P_i', \mathbb{Q}/\mathbb{Z})
\end{array}$$

Let $A_i \in \operatorname{Br}(L_i \backslash P_i')$ be the restriction of A to $L_i \backslash P_i'$. As $L_i \backslash P_i' \cong \mathbb{A}^1_k$ we have that $\operatorname{Br}(L_i \backslash P_i') = \operatorname{Br}(k)$ hence the class of A_i is constant, and it's residue at the point P_i' is zero. One can now deduce that $\operatorname{im}(\partial_D) \subset \ker(\alpha_i)$ for all i. As $P_i \in X_n(\mathbb{Q})$ we have that $m(P_i) \in \mathrm{H}^1_{\text{ét}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ then the image of $m(P_i)$ under the restriction map

$$\operatorname{Res}_{k/\mathbb{Q}} : \mathrm{H}^1_{\text{ét}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1_{\text{ét}}(k, \mathbb{Q}/\mathbb{Z})$$

is $m(P_i')$. The map $\operatorname{Res}_{k/\mathbb{Q}}$ is injective as k is either trivial or a non-Galois cubic extension of $\mathbb{Q}$, as we require $m(P_i') = 0$ for $i = 1, 2$ the injectivity of the restriction map implies $m(P_i) = 0$. Using Lemma 3.4.1 we can deduce that $m = 0$. $\square$

3.5 Main theorems

Theorem 3.5.1. *Let $f(x) = u^3 + au + b \in \mathbb{Z}[u]$ where a is non-zero and let $n \in \mathbb{Z}$ such that $n \notin A_3$. Consider the affine surface $\mathcal{U}_n$ over $\mathbb{Z}$ defined by*

$$\mathcal{U}_n : f(u_1) + f(u_2) + f(u_3) = n.$$

Set $U_n := \mathcal{U}_n \times \mathbb{Q}$, then $\operatorname{Br} U_n = \operatorname{Br} \mathbb{Q}$ and there is no-integral Brauer-Manin obstruction to the Hasse principle on $\mathcal{U}_n$.

Proof. In Proposition 3.3.24 we showed that if $n \notin A_3$ then $\operatorname{Br} X_n = \operatorname{Br} \mathbb{Q}$. Furthermore, it follows from Proposition 3.4.2 for such choices of n we have $\operatorname{Br} U_n = \operatorname{Br} X_n = \operatorname{Br} \mathbb{Q}$. If $\mathcal{U}_n$ is not everywhere locally soluble for some n we automatically have

$$\mathcal{U}_n(\mathbb{Z}) = \emptyset \text{ and } \mathcal{U}_n(\mathbb{A}_{\mathbb{Z}}) = \emptyset.$$

In this case we say there is no integral Brauer-Manin obstruction to the Hasse principle. We now assume $\mathcal{U}_n$ is everywhere locally soluble. As $\operatorname{Br} U_n = \operatorname{Br} \mathbb{Q}$ the statement follows. □

Remark 3.5.2. If $n \in A_3 \backslash A_1$ then f_1 is reducible. This implies $\mathcal{U}_n(\mathbb{Z}) \neq 0$, hence in this case the integral Hasse principle holds.

Theorem 3.5.3. *Let $f(u) = u^3 + a_2u^2 + a_1u + a_0 \in \mathbb{Z}[u]$ and consider the affine cubic surface over $\mathbb{Z}$*

$$\mathcal{U}_n : f(u_1) + f(u_2) + f(u_3) = n$$

for some $n \in \mathbb{Z}$. If $3a_1 - a_2^2 = 0$ then there is no integral Brauer-Manin obstruction to the Hasse principle.

Proof. Suppose $a_2 = 0$, as $3a_1 - a_2^2 = 0$ then $a_1 = 0$, hence we can write $\mathcal{U}_n$ as

$$\mathcal{U}_n : u_1^3 + u_2^3 + u_3^3 = n - 3a_0. \tag{3.5.1}$$

If $a_2 \neq 0$, 3 divides a_2 and the change of variables $u_i \to u_i - \frac{a_2}{3}$ for $i = 1, 2, 3$ defines an isomorphism $\mathcal{U}_n \to \mathcal{U}'_n$ over $\mathbb{Z}$, where

$$\mathcal{U}'_n : u_1^3 + u_2^3 + u_3^3 = n - 2a_2 - a_2a_1 + 3a_0. \tag{3.5.2}$$

Then by [CTW12, Thm. 4.1a] (3.5.1) and (3.5.2) never have an integral Brauer-Manin obstruction for any choice of $n \in \mathbb{Z}$. □

Proof of Theorem 3.1.1. Let $f(u) = u^3 + a_2u^2 + a_1u + a_0 \in \mathbb{Z}[u]$ and consider the affine cubic surface over $\mathbb{Z}$

$$U_n : f(u_1) + f(u_2) + f(u_3) = n$$

for some $n \in \mathbb{Z}$. By Theorem 3.5.3 we may assume $3a_1 - a_2^2 \neq 0$. Denote $U_n := \mathcal{U}_n \times_{\mathbb{Z}} \mathbb{Q}$, apply the change of variables $u_i \to \frac{u_i - a_2}{3}$ and clearing denominators, we can write U_n as

$$U_n : u_1^3 + u_2^3 + u_3^3 + a(u_1 + u_2 + u_3) = 27n - 3(2a_2^3 - 9a_1a_2 + 27a_0) \tag{3.5.3}$$

where $a := 9a_1a_3 - 3a_2^2 \in \mathbb{Z}\backslash\{0\}$. Then the statement follows from Theorem 3.5.1 and Corollary 3.3.24. □

3.6 Sum of three tetrahedral numbers

Zhi-Wei Sun asks in [Sun] if any integer n can be represented as the sum of three tetrahedral numbers. We show that for any choice of n, such an equation is everywhere locally soluble and there is no integral Brauer-Manin obstruction. Throughout Section 3.6 k is a field of characteristic not equal to 2 or 3. Let $F(x_i, x_0) = x_i(x_i + x_0)(x_i + 2x_0) \in k[x_i, x_0]$ and p_0, p_1, p_2, p_3 be the geometric points on $\mathbb{P}^3_k$

$$\begin{aligned}
p_0 &: x_1 = x_2 = x_3 = \left(-1 + \frac{\sqrt{3}}{3}\right)x_0, \\
p_2 &: x_1 = x_2 = \left(-1 + \frac{\sqrt{3}}{3}\right)x_0, x_3 = \left(-1 - \frac{\sqrt{3}}{3}\right)x_0, \\
p_1 &: x_1 = x_2 = x_3 = \left(-1 - \frac{\sqrt{3}}{3}\right)x_0, \\
p_3 &: x_1 = x_2 = \left(-1 - \frac{\sqrt{3}}{3}\right)x_0, x_3 = \left(-1 + \frac{\sqrt{3}}{3}\right)x_0.
\end{aligned}$$

To begin with, we consider the family of cubic surfaces

$$X_n : F(x_1, x_0) + F(x_2, x_0) + F(x_3, x_0) - 6nx_0^3 \subset \mathbb{P}^3_{\mathbb{Q}}$$

where $n \in \mathbb{Q}$.

Lemma 3.6.1. *Denote by X the surface*

$$X : F(x_0, x_1) + F(x_0, x_2) + F(x_0, x_3) = 6nx_0^3$$

defined over k. If X has a singular point then either $n = \frac{\sqrt{3}}{9}, \frac{-\sqrt{3}}{9}, \frac{\sqrt{3}}{27}$ or $\frac{-\sqrt{3}}{27}$.

Proof. Let $G(x_0, x_1, x_2, x_3) := F(x_0, x_1) + F(x_0, x_2) + F(x_0, x_3) - 6nx_0^3$. Assume there is a non-smooth point $p = [x_0 : x_1 : x_2 : x_3]$ on X_n, then the partial derivatives of G will vanish at p i.e.

$$\begin{aligned}
&\frac{\partial G}{\partial x_i}(p) = 2x_0^2 + 6x_0x_i + 3x_i^2 = 0, \\
&\frac{\partial G}{\partial x_0}(p) = -18nx_0^2 + 4x_0x_1 + 4x_0x_2 + 4x_0x_3 + 3x_1^2 + 3x_2^2 + 3x_3^2 = 0.
\end{aligned}$$

for $i = 1, 2, 3$. We can see if $x_0 = 0$ for such a point p this would imply $x_i = 0$ for $i = 1, 2, 3$ which cannot happen, hence we can assume $x_0 \neq 0$. Using the symmetry of the defining equation for X and the fact $\frac{\partial G}{\partial x_i}(p) = 0$ we can assume that p is either p_0, p_1, p_2 or p_3. As $\frac{\partial G}{\partial x_0}(p) = 0$ this implies $n = \frac{\sqrt{3}}{9}, \frac{-\sqrt{3}}{9}, \frac{\sqrt{3}}{27}$ or $\frac{-\sqrt{3}}{27}$. □

Proposition 3.6.2. *X_n is smooth for all $n \in \mathbb{Q}$.*

Proof. This follows from Lemma 3.6.1. □

3.6.1 Finding $\mathbb{Z}_p$ points on $\mathcal{U}_n$

Throughout Section 3.6.1, p will be a prime and $\mathbb{F}_q$ a finite field of size $q = p^k$ for some integer k. Denote by $\mathcal{X}_n$ the integral model of X_n defined by the same equation over $\mathbb{Z}_p$ and denote by $\mathcal{X}_{n,p}$ the special fibre of $\mathcal{X}_n$. If the special fibre is smooth we say $\mathcal{X}_n$ has *good reduction* at p, otherwise we say $\mathcal{X}_n$ has *bad reduction* at p.

Theorem 3.6.3 (Weil, [Man86, Thm. 27.1]). *Let X be a smooth surface over $\mathbb{F}_q$, if $\bar{X}$ is rational, then*

$$\#X(\mathbb{F}_q) = q^2 + \mathrm{Tr}(\phi^*)q + 1$$

where $\mathrm{Tr}\,\phi^*$ denotes the trace of the Frobenius automorphism ϕ in the representation of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ on $\mathrm{Pic}(\bar{V})$.

Remark 3.6.4. For a smooth cubic surface X over a field k we have that $\mathrm{Pic}\,\bar{X} \cong \mathbb{Z}^7$. If $k = \mathbb{F}_q$ it is then clear that $|\,\mathrm{Tr}(\phi^*)| \leq 7$. Thus, for a smooth cubic surface X over $\mathbb{F}_q$ we have a lower bound $\#X(\mathbb{F}_q) \geq 1 - 7q + q^2$.

Theorem 3.6.5 (Hasse, [Sil09, Thm. V.1.1]). *Let E be an elliptic curve over $\mathbb{F}_q$ then*

$$\#E(\mathbb{F}_q) \leq 2\sqrt{q} + q + 1.$$

Lemma 3.6.6. *For all primes p which divide $6n$, $\mathcal{X}_{n,p}$ has a smooth $\mathbb{F}_p$-point.*

Proof. Consider the defining equation for $\mathcal{X}_{n,p}$

$$G(x_0, x_1, x_2, x_3) = F(x_0, x_1) + F(x_0, x_2) + F(x_0, x_3) \equiv 0 \pmod{p}.$$

The point $[x_0 : x_1 : x_2 : x_3] = [1 : -1 : 0 : 0]$ is a smooth $\mathbb{F}_p$-point. □

Lemma 3.6.7. *Let $p > 3$ and $p \equiv 2 \bmod 3$ then if*

1. $n \equiv \frac{-\sqrt{3}}{9}$ *mod p the point* $[1 : -1 + \frac{\sqrt{3}}{3} : -1 + \frac{\sqrt{3}}{3} : -1 - \frac{2\sqrt{3}}{3}] \in \mathcal{X}_{n,p}(\mathbb{F}_p)$ *is smooth,*
2. $n \equiv \frac{\sqrt{3}}{9}$ *mod p the point* $[1 : -1 - \frac{\sqrt{3}}{3} : -1 - \frac{\sqrt{3}}{3} : -1 + \frac{2\sqrt{3}}{3}] \in \mathcal{X}_{n,p}(\mathbb{F}_p)$ *is smooth,*
3. $n \equiv \frac{\sqrt{3}}{27}$ *mod p the point* $[1 : -1 - \frac{\sqrt{3}}{3} : -1 + \frac{\sqrt{3}}{3} : -1 + \frac{2\sqrt{3}}{3}] \in \mathcal{X}_{n,p}(\mathbb{F}_p)$ *is smooth,*
4. $n \equiv \frac{-\sqrt{3}}{27}$ *mod p the point* $[1 : -1 - \frac{\sqrt{3}}{3} : -1 + \frac{\sqrt{3}}{3} : -1 - \frac{2\sqrt{3}}{3}] \in \mathcal{X}_{n,p}(\mathbb{F}_p)$ *is smooth.*

Proof. Denote by F the defining equation for $\mathcal{X}_{n,p}$, then $\frac{\partial F}{\partial x_1} \equiv 3 \bmod p$ for each point. □

Lemma 3.6.8. *Let $p > 3$ and p be a prime of bad reduction. Then there exists a smooth $\mathbb{F}_p$-point on $\mathcal{X}_{n,p}$.*

Proof. Using Lemma 3.6.1 we see that for p to be a bad prime 3 needs to be square mod p this is equivalent to $p \equiv 2 \bmod 3$ and $n \bmod p$ has four possible forms which are dependent on $\sqrt{3} \bmod p$, namely $n \equiv \frac{-\sqrt{3}}{9}, \frac{\sqrt{3}}{9}, \frac{\sqrt{3}}{27}$ or $\frac{-\sqrt{3}}{27} \bmod p$. By Lemma 3.6.7 in each case there exists a smooth $\mathbb{F}_p$ point on $\mathcal{X}_{n,p}$ by Lemma 3.6.7. □

Lemma 3.6.9. *Let $p \geq 11$ be a prime of good reduction. Then $\mathcal{X}_{n,p}$ has a smooth $\mathbb{F}_p$-point which does not lie on the prime divisor $x_0 = 0$.*

Proof. As p is prime of good reduction we can apply Theorem 3.6.3

$$\#\mathcal{X}_{n,p}(\mathbb{F}_p) = 1 + \mathrm{Tr}\,\phi^* p + p^2 \geq 1 - 7p + p^2.$$

Denote by $\mathcal{X}'_{n,p}(\mathbb{F}_p)$ the points away from the hypersection defined by $x_0 = 0$, using Theorem 3.6.5, along the divisor $x_0 = 0$ to get we get that

$$\#\mathcal{X}'_{n,p}(\mathbb{F}_p) \geq 1 - 7p + p^2 - (1 + p + 2\sqrt{p}).$$

It is easy to check that for $p \geq 11$, $\mathcal{X}'_{n,p}(\mathbb{F}_p) \geq 1$. □

Proposition 3.6.10. *There exists a smooth $\mathbb{F}_p$-point on the special fibre of $\mathcal{X} \to \mathrm{Spec}\,\mathbb{Z}_p$ which lies away from the divisor $x_0 = 0$.*

Proof. For primes $p \geq 11$, $p = 2$ or 3 this follows from Lemmas 3.6.6, 3.6.8 and 3.6.9. The only cases left to consider are $p = 5$ or 7. Let $p = 5$. If $n = 1, 4 \bmod 5$ then there exists a smooth point $[1 : x : -x : 0]$ where $x^2 = n \bmod 5$. If $n = 2 \bmod 5$ then $[1 : 1 : 1 : 0]$ is a smooth point and if $n = 3 \bmod 5$ then $[1 : 2 : 2 : 0]$ is a smooth point. Let $p = 7$. If $n = 1, 2, 4 \bmod 7$ then there exists a smooth point $[1 : x : -x : 0]$ where $x^2 = n \bmod 7$. If $n = 3 \bmod 7$ then $[1 : 3 : 0 : 0]$ is a smooth point, if $n = 5 \bmod 7$ then $[1 : 4 : 4 : 0]$ is a smooth point and if $n = 6 \bmod 7$ then $[1 : 3 : 3 : 0]$ is a smooth point. □

Theorem 3.6.11. *Let v be a place of $\mathbb{Q}$. Then $\mathcal{U}_n(\mathbb{Z}_v) \neq \emptyset$.*

Proof. For the real place v, clearly $\mathcal{U}_n(\mathbb{Z}_v) \neq 0$. Let p be a prime. Then by Proposition 3.6.10 we see that $\mathcal{X}_{n,p}$ has a smooth $\mathbb{F}_p$-point away from the divisor $x_0 = 0$. Using Hensel's lemma we can lift this point to a $\mathbb{Z}_p$-point $\mathcal{X}_n$ which lies in the image of $\mathcal{U}_n(\mathbb{Z}_p) \hookrightarrow \mathcal{X}_n(\mathbb{Z}_p)$. □

3.6.2 Brauer group of U_n

Using Proposition 3.3.6 we see that the splitting field K of X_n for any n is defined by the cubics $f_1(x) = x^3 - x - 6n$ and $f_2(x) = x^3 + 12x^2 - 36x + 972n^2 - 4$. We can now determine the Brauer group of $\mathcal{U}_n \times_{\mathbb{Z}} \mathbb{Q}$.

Lemma 3.6.12. *Fix $n \in \mathbb{Z}$, denote by K the compositum of the splitting fields of f_1 and f_2. If f_1 is irreducible then $[K : \mathbb{Q}] = 36$.*

Proof. As f_2 has no solutions modulo 81 and is a degree 3 polynomial we have that, f_2 is irreducible for any choice of $n \in \mathbb{Z}$. In addition to this, f_2 has a discriminant equal to $-2^4 3^6(2187n^4 + 1206n^2 - 37)$, which is never a square for any $n \in \mathbb{Z}$. Moreover, $f_1(x)$ has discriminant $4(1 - 243n^2)$, which is also never a square for any choice of $n \in \mathbb{Z}$. The quotient of the discriminants of f_1 and f_2 is never a square, as it is not a square in $\mathbb{Q}_3$, for any $n \in \mathbb{Z}$. Hence, $[K : \mathbb{Q}] = 36$. □

Proposition 3.6.13. *Fix $n \in \mathbb{Z}$ such that f_1 is irreducible then* $\mathrm{Br}\,U_n = \mathrm{Br}\,X_n = \mathrm{Br}\,\mathbb{Q}$.

Proof. Using Lemma 3.6.12 and Proposition 3.3.16 then $\mathrm{Br}\,X_n = \mathrm{Br}\,\mathbb{Q}$. As the discriminant of f_1 is never a square for any $n \in \mathbb{Z}$ we can apply Proposition 3.4.2 to deduce that $\mathrm{Br}\,X_n = \mathrm{Br}\,U_n$. □

Theorem 3.6.14. *There is no integral Brauer-Manin obstruction to the Hasse principle on the affine surface*

$$\mathcal{U}_n : u_1(u_1+1)(u_1+2) + u_2(u_2+1)(u_2+2) + u_3(u_3+1)(u_3+2) = 6n$$

for all $n \in \mathbb{Z}$.

Proof. If $f_1(u) = u^3 - u - 6n$ is reducible by Gauss's Lemma [Lan02, Thm. 2.3] there exists $\alpha \in \mathbb{Z}$ such that $f_1(\alpha) = 0$. Hence, $(u_1, u_2, u_3) = (\alpha+1, 0, 0) \in \mathcal{U}_n(\mathbb{Z})$ i.e. there is no integral Brauer-Manin obstruction in this case. Suppose f_1 is irreducible, by Proposition 3.6.13 $\operatorname{Br} U_n = \operatorname{Br} \mathbb{Q}$, and by Theorem 3.6.11 we have $\mathcal{U}_n(\mathbb{A}_\mathbb{Z}) \neq \emptyset$, hence $\mathcal{U}_n(\mathbb{A}_\mathbb{Z})^{\operatorname{Br}} = \mathcal{U}_n(\mathbb{A}_\mathbb{Z}) \neq \emptyset$. □

Remark 3.6.15. Even in the case where f_1 is reducible we still have $\operatorname{Br} X_n = \operatorname{Br} U_n = \operatorname{Br} \mathbb{Q}$ by Propositions 3.3.16, 3.4.2. In regards to the integral Brauer-Manin this fact is useless as you have an obvious integral point, however if one was studying weak approximation on the surfaces U_n this does become useful, see Remark 3.7.3.

3.6.3 Rationality of the sum of three tetrahedral numbers

For a more complete understanding of the equation defined by the sum of three tetrahedral numbers we show that these surfaces are non-rational when there is not an obvious integral solution.

Definition 3.6.16. We say that a variety V over a field F is F-rational if there exists a birational map $\mathbb{P}^m \dashrightarrow V$ defined over F, for some positive integer m.

Definition 3.6.17. A smooth surface over a perfect field k is said to be k-minimal if there is no nonempty $\operatorname{Gal}(\bar{k}/k)$-stable set of pairwise non-intersecting exceptional curves.

Theorem 3.6.18 ([Man86, Thm. 37.1])**.** *Every minimal smooth cubic surface is non-rational.*

Lemma 3.6.19. *If* n *is an integer such that* $f_1 = x^3 - x - 6n$ *is irreducible then,* X_n *is a non-rational surface over* $\mathbb{Q}$.

Proof. By Theorem 3.6.18 it is sufficient to prove for such a choice of n the cubic surface X_n is minimal. X_n has orbit type $[3, 3, 3, 18]$ where the Galois orbits of size 3 meet at an Eckardt point (see Remark 3.3.4), i.e. they are not pairwise skew. The Galois orbit of size 18 cannot be pairwise skew as any set containing more than 6 lines of a smooth cubic surface contains intersecting lines. □

Theorem 3.6.20. *If* n *is an integer such that* $f_1 = x^3 - x - 6n$ *is irreducible then* U_n *is non-rational over* $\mathbb{Q}$.

Proof. U_n is birational to X_n as it is a Zariski open subset of X_n. Using Lemma 3.6.19 we can deduce U_n is non-rational. □

3.7 Counter-example to weak and strong approximation

Even though in Theorem 3.1.1 we showed the surfaces $\mathcal{U}_n$ almost never have an integral Brauer-Manin obstruction the surface $U_n := \mathcal{U}_n \times_{\mathbb{Z}} \mathbb{Q}$ may still fail strong approximation. We show an example where U_n fails weak approximation, hence fails strong approximation.

Lemma 3.7.1 ([CTS21, Prop. 13.2.3])**.** *If U and U' are birational smooth varieties over a number field K such that U and U' are everywhere locally soluble then U satisfies weak approximation if and only if U' satisfies weak approximation.*

Remark 3.7.2. It should be noted that strong approximation is not a birational invariant. For example if one takes $U = \mathbb{A}^1_{\mathbb{Q}}, U' := \mathbb{G}_{m,\mathbb{Q}}$ and $S \neq \emptyset$ then U satisfies strong approximation off S whereas U' does not. The failure of strong approximation on U' can be explained by a result of Minčhev [Min89] and the fact that $\pi^{\text{ét}}(\bar{U}') = \hat{\mathbb{Z}}$.

Remark 3.7.3. Let $f(u) = u^3 + a_2u^2 + a_1u + a_0$ such that $3a_1 - a_2^2 \neq 0$ and denote by U_n be the surface

$$U_n : f(u_1) + f(u_2) + f(u_n) = n \subset \mathbb{A}^3_{\mathbb{Q}}.$$

The proof of Theorem 3.1.1 shows that for all but finitely $n \in \mathbb{Z}$ we have $\operatorname{Br} U_n = \operatorname{Br} X_n = \operatorname{Br} \mathbb{Q}$, where X_n is the compactification of U_n One should then expect the surfaces U_n to rarely fail weak approximation. Consider the case where the compactificaiton X_n is smooth and $\operatorname{Br} X_n = \operatorname{Br} \mathbb{Q}$. Note that $X_n(\mathbb{Q}) \neq \emptyset$ then as X_n is a smooth cubic surface the set $X(\mathbb{Q})$ will be Zariski dense in X_n, this implies $U_n(\mathbb{Q}) \neq \emptyset$. Under a conjecture of Colliot-Thélène [CTS21, Conj. 14.1.2], $X(\mathbb{Q})$ is dense in $X_n(\mathbb{A}_{\mathbb{Q}})^{\text{Br}} = X_n(\mathbb{A}_{\mathbb{Q}})$, i.e. if this conjecture holds X_n satisfies weak approximation. As X_n and U_n are smooth birational varieties which are everywhere locally soluble, under conjecture [CTS21, Conj. 14.1.2], U_n would also satisfy weak approximation. In particular, the example of sum of three tetrahedral numbers given in Section 3.6 will satisfy weak approximation if conjecture [CTS21, Conj. 14.1.2] is true.

3.7.1 Example

Theorem 3.7.4 ([CTS21, Cor. 11.3.5])**.** *Let $X \to \mathbb{P}^1_K$ be a conic bundle. Fix $m \in \mathbb{P}^1_K$ a closed point such that $\pi^{-1}(m)$ is smooth. Let $S \subset \mathbb{P}^1_K$ be the finite set of closed points with singular fibres. We define $\mathbb{P}^1 \supset \mathbb{A}^1_K = \mathbb{P}^1_K \backslash \{m\}$ with coordinate t. Then*

$$\operatorname{Br} X / \operatorname{Br} K = \pi^*(B)$$

where $B \subset \operatorname{Br}(K(\mathbb{P}^1)) = \operatorname{Br}(K(t))$ is a finite subgroup. Moreover, we have an explicit description of the elements in B. A closed point $p \in S$ is the zero set of some monic irreducible polynomial $P(t) \in K[t]$. Let $K(p) = K[t]/(P(t))$ be the residue field of p and denote τ_p to be the image of t. Then there exists a quadratic extension $F_p = K(p)(\sqrt{a_p})$ such that the fibre above p splits over F_p into two transversal lines, now we can define the set $W \subset \mathbb{F}_2^{|S|}$ of vectors $\varepsilon = (\varepsilon_p)_{p \in S}$ such that

$$\prod_{p \in S} \left(\mathrm{N}_{K(p)/K}(a_p)\right)^{\varepsilon_p} = 1 \in K^*/(K^*)^2.$$

Then there is an injective map $W \to \operatorname{Br} K(t)$ which sends ε to

$$A_\varepsilon = \sum_{p \in S} \varepsilon_p \operatorname{Cores}_{k(p)/k}(t - \tau_p, a_p)$$

where $(t-\tau_p, a_p)$ *represents a quaternion algebra in* $\mathrm{Br}(K(p)(t))$.

Example 3.7.5. We now present an example which fails both strong and weak approximation, namely the affine suface

$$U_{50} : u_1^3 + u_2^3 + u_3^3 - 15(u_1 + u_2 + u_3) + 50 = 0.$$

Consider the cubic surface X_{50} which is the compactification of U_{50} in $\mathbb{P}^3$. Then X_{50} contains a line, namely $L = \{x_1 + x_2 = 0, x_3 + 5x_0 = 0\}$, hence X_{50} admits a conic bundle structure. We apply a change of variables $x_1 + x_2 \mapsto x_1, x_3 + 5x_0 \mapsto x_3$ so we get a smooth cubic surface $X_{50}^{'}$ which is isomorphic to X_{50} over $\mathbb{Q}$ and where $L' := \{x_1 = x_3 = 0\}$ is the image of L under this isomorphism. Using the method described in Remark 3.2.7 we can construct the conic bundle map associated to the line L', $\pi : X'_{50} \to \mathbb{P}^1_{\mathbb{Q}}$. The closed points of $\mathbb{P}^1_{\mathbb{Q}}$ which correspond to singular fibres of π are $\{(t), (s-t), (s+t), (s^2 - 4st + t^2)\}$. We now use Theorem 3.7.4 to construct elements of $\mathrm{Br}\, X'_{50}/\,\mathrm{Br}\,\mathbb{Q}$.

1. The residue field of the point $[0 : 1]$ is $\mathbb{Q}$ and the fibre above this point is $60x_0^2 - 15x_0x_1 + x_1^2 = 0$ which splits over $\mathbb{Q}(\sqrt{-15})$,
2. The residue field of the point $[1 : -1]$ is $\mathbb{Q}$ and the fibre above this point is split over $\mathbb{Q}$,
3. The residue field of the point $[1 : 1]$ is $\mathbb{Q}$ and the fibre above this point is $45x_0^2 - 15x_0x_1 + 2x_1^2 - 3x_1x_2 + 3x_2^2 = 0$ which splits over $\mathbb{Q}(\sqrt{-15})$,
4. The residue field of the point $p = (s^2 - 4st + t^2)$ is $\mathbb{Q}(\sqrt{12})$. The fibres above the two $\mathbb{Q}(\sqrt{12})$-points are split over $\mathbb{Q}(\sqrt{-15})$.

We now can apply Theorem 3.7.4 and can see a non-trivial choice for ε is:

$$\varepsilon_p = \begin{cases} 1 \text{ for } p = [0:1], \\ 0 \ \text{ for } p = [1:-1], \\ 1 \text{ for } p = [1:1], \\ 0 \text{ for } p = (s^2 - 4st + t^2). \end{cases}$$

Then

$$\prod_{p \in S} \big(\mathrm{N}_{k(p)/k}(a_p)\big)^{\varepsilon_p} = (-15)^2 = 1 \ \big(\mathrm{mod}\ (\mathbb{Q}^\times)^2\big).$$

Using Magma we can check that $\mathrm{Br}\, X_{50}^{'}/\,\mathrm{Br}\,\mathbb{Q} \cong \mathbb{Z}/2\mathbb{Z}$, hence we can define the generator for $\mathrm{Br}\, X'_{50}/\,\mathrm{Br}\,\mathbb{Q}$ specifically the pullback of $A_\varepsilon = (x(x-1), -15)$ where $x = x_1/x_3$, this substitution acts as the pullback map, i.e.

$$\mathrm{Br}\, X_{50}^{'}/\,\mathrm{Br}\,\mathbb{Q} \text{ is generated by } \alpha' = \left(\left(\frac{x_1}{x_3} - 1\right)\frac{x_1}{x_3}, -15\right).$$

We can then deduce

$$\mathrm{Br}\, X_{50}/\,\mathrm{Br}\,\mathbb{Q} \text{ is generated by } \alpha = \left(\left(\frac{x_1 + x_2}{x_3 + 5x_0} - 1\right)\frac{x_1 + x_2}{x_3 + 5x_0}, -15\right).$$

Let $p = \infty$ and $\boldsymbol{x}_1 = [-1/10 : 1/2 : -1 : 1]$ and $\boldsymbol{x}_2 = [1/2 : 7/6 : 11/6 : 1]$ on $X_{50}(\mathbb{R})$, and we shall denote $\alpha(\boldsymbol{x}_i)$ for the image of $\boldsymbol{x}_i$ under the evaluation map $\mathrm{ev}_{\alpha,\infty}$. Then we have

$$\mathrm{inv}_\infty(\alpha(\boldsymbol{x}_i)) = \begin{cases} 1 & \text{if } \alpha(\boldsymbol{x}_i) \text{ is split over } \mathbb{R} \\ -1 & \text{if } \alpha(\boldsymbol{x}_i) \text{ is not split over } \mathbb{R}. \end{cases}$$

We see that $\alpha(\boldsymbol{x}_1) = (2, -15)$ which is split over $\mathbb{R}$ and $\alpha(\boldsymbol{x}_2) = (-6/49, -15)$ which is not split over $\mathbb{R}$, hence inv_∞ takes both values. As X_{50} is a smooth projective variety such that $X_{50}(\mathbb{Q}) \neq \emptyset$, [CTS21, Prop. 13.3.12] tells us that X_{50} fails weak approximation. It is clear that $U_{50}(\mathbb{Q}) \neq \emptyset$ and U_{50} is birational to X_{50}, hence by Lemma 3.7.1 the failure of weak approximation on X_{50} implies that U_{50} fails weak approximation. Finally, as strong approximation implies weak approximation U_{50} must also fail strong approximation.

Chapter 4

Integral points on diagonal cubic surfaces

4.1 Introduction

This chapter is devoted to the study of integers represented by diagonal ternary cubic forms and was joint work with V. Mitankin and J. Lyczak. A special case of this corresponds to the conjecture stated in Chapter 3 regarding integral solutions to the equation (3.1.1). Our main results give examples of diagonal ternary cubic forms representing an integer over $\mathbb{Q}$ but for which the integral Hasse principle fails. In fact, we give two infinite families of such examples in §4.5 and §4.7.

To set up the framework for our investigation, let a_0, a_1, a_2, a_3 be non-zero integers such that $\gcd(a_1, a_2, a_3) = 1$ and denote by U the smooth affine surface over $\mathbb{Q}$ given by

$$U: \quad a_1u_1^3 + a_2u_2^3 + a_3u_3^3 = a_0. \tag{4.1.1}$$

We fix an integral model $\mathcal{U}$ of U over $\mathbb{Z}$, defined by the same equation.

Results

To study the Brauer–Manin obstruction on U, as given as in (4.1.1), we first need to calculate $\operatorname{Br} U$.

Theorem 4.1.1. *Assume that $a_0, a_1, a_2, a_3 \in \mathbb{Q}^*$ are cube-free. Then the algebraic part of $\operatorname{Br} U$ is isomorphic to $\operatorname{Br} X$. Moreover,*

$$\operatorname{Br} U = \begin{cases} \operatorname{Br} X \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } a_1a_2a_3 \equiv 2 \bmod \mathbb{Q}^{*3}, \\ \operatorname{Br} X & \text{otherwise.} \end{cases}$$

The condition $a_1a_2a_3 \equiv 2 \bmod \mathbb{Q}^{*3}$ appears from the fact that the Jacobian of the curve $D := X \backslash U$ has non-trivial rational 2-torsion points. The correspondence between these points and elements of order 2 in $\operatorname{Br} U$ is seen in Lemma 4.2.13 and in end of the proof of Theorem 4.1.1.

To explore the frequency of integral Hasse failures and to measure how often integral strong approximation off ∞ holds in the family (4.1.1), we vary a_0, the coefficients of the cubic form a_1, a_2, a_3, and all a_0, a_1, a_2, a_3 in a equal-sided box. In view of Remark 4.1.5, the number of everywhere locally soluble $\mathcal{U}$ up to height B in the three counting problems is of magnitude B, B^3, B^4, respectively. We give in Propositions 4.3.2 and 4.3.3 a sufficient criterion for the lack of a Brauer–Manin obstruction to the integral Hasse principle and for the presence of a Brauer–Manin obstruction to integral strong approximation off ∞. Counting surfaces failing this criterion allows us to obtain upper bounds for the amount of Hasse failures and simultaneously to estimate the number of surfaces satisfying strong approximation off ∞ in all three natural counting problems.

We begin with the analogue of the sum of three cubes conjecture, i.e. we fix a_1, a_2, a_3 and we vary a_0. Set $\mathbb{Z}^n_{\text{prim}}$ for the set of n-tuples in $\mathbb{Z}^n$ with non-zero coprime coordinates. If $\mathbb{Z}_{\neq 0}$ stands for the non-zero integers, for any real $B \geq 1$ and $(a_1, a_2, a_3) \in \mathbb{Z}^3_{\text{prim}}$ define

$$N_{a_1,a_2,a_3}(B) = \#\big\{a_0 \in [-B, B] \cap \mathbb{Z}_{\neq 0} \ : \ \mathcal{U}(\mathbb{A}_\mathbb{Z}) \neq \emptyset \text{ but } \mathcal{U}(\mathbb{A}_\mathbb{Z})^{\text{Br}} = \emptyset\big\},$$
$$N^{'}_{a_1,a_2,a_3}(B) = \#\{a_0 \in [-B, B] \cap \mathbb{Z}_{\neq 0} \ : \ \mathcal{U}(\mathbb{A}_\mathbb{Z}) \neq \emptyset \text{ and } \mathcal{U} \text{ satisfies ISA off } \infty\}.$$

We exclude the case a_0 because in this case the surface is singular, however including this case would not effecting the counting results. Our next result provides upper bounds for these quantities.

Theorem 4.1.2. *Assume that* $a_1a_2a_3 \not\equiv 2 \bmod \mathbb{Q}^{*3}$. *We then have*

$$N_{a_1,a_2,a_3}(B),\ N^{'}_{a_1,a_2,a_3}(B) \ll_{a_1a_2a_3} B^{1/3},$$

as B goes to infinity.

Theorem 4.1.2 shows how rare Hasse failures are if the cubic form is fixed, thus revealing the difficulty of finding explicit examples of them. In fact, $N_{a_1,a_2,a_3}(B)$ is zero for the two specific families $(a_1, a_2, a_3) = (1, 1, 1)$ and $(1, 1, 2)$ studied by Colliot-Thélène and Wittenberg [CTW12]. Hence, finding the magnitude of $N_{a_1,a_2,a_3}(B)$, or even a lower bound for it, amounts to selecting specific choices of (a_1, a_2, a_3) for which Hasse failures exist. Such choices are extremely rare, as we shall see in the next results.

We continue by varying the cubic form while $a_0 \neq 0$ stays fixed. Let

$$N_{a_0}(B) = \#\big\{(a_1, a_2, a_3) \in [-B, B]^3 \cap \mathbb{Z}^3_{\text{prim}} \ : \ \mathcal{U}(\mathbb{A}_\mathbb{Z}) \neq \emptyset \text{ but } \mathcal{U}(\mathbb{A}_\mathbb{Z})^{\text{Br}} = \emptyset\big\},$$
$$N^{'}_{a_0}(B) = \#\big\{(a_1, a_2, a_3) \in [-B, B]^3 \cap \mathbb{Z}^3_{\text{prim}} \ : \ \mathcal{U}(\mathbb{A}_\mathbb{Z}) \neq \emptyset \text{ and } \mathcal{U} \text{ satisfies ISA off } \infty\big\}.$$

Our methods establish upper bounds for these quantities, given in the next theorem.

Theorem 4.1.3. *We have*

$$N_{a_0}(B),\ N^{'}_{a_0}(B) \ll B^{3/2},$$

as B goes to infinity.

Lastly, we vary all four coefficients of $\mathcal{U}$. For this purpose let

$$N(B) = \#\big\{(a_0, a_1, a_2, a_3) \in [-B, B]^4 \cap \mathbb{Z}^4_{\text{prim}} \ : \ \mathcal{U}(\mathbb{A}_\mathbb{Z}) \neq \emptyset \text{ but } \mathcal{U}(\mathbb{A}_\mathbb{Z})^{\text{Br}} = \emptyset\big\},$$
$$N^{'}(B) = \#\big\{(a_0, a_1, a_2, a_3) \in [-B, B]^4 \cap \mathbb{Z}^4_{\text{prim}} \ : \ \mathcal{U}(\mathbb{A}_\mathbb{Z}) \neq \emptyset \text{ and } \mathcal{U} \text{ satisfies ISA off } \infty\big\}.$$

Our final result delivers upper and lower bounds for $N(B)$, of the same magnitude modulo a small power of $\log B$, and an upper bound for $N^{'}(B)$.

Theorem 4.1.4. *We have*

$$\frac{B^2}{\log B} \ll N(B) \ll B^2(\log B)^6 \quad \textit{and} \quad N^{'}(B) \ll B^2(\log B)^2,$$

as B goes to infinity.

We construct in §4.5 and in §4.7 the first examples of diagonal affine cubic surfaces with a Brauer–Manin obstruction to the integral Hasse principle. Moreover, all of the surfaces in §4.5 and §4.7 have a non-empty set of rational points (see Remarks 4.5.1 and 4.7.4) and thus our examples of integral Hasse failures do not follow trivially from Hasse failures for rational points. Colliot-Thélène, Kanevsky and Sansuc constructed an infinite family of cubic surfaces, failing the Hasse principle for rational points in [CTKS87, §7, Prop. 5]. Each of their surfaces produces integral Hasse failures by taking away exactly one of the hyperplanes corresponding to the zero locus of a coordinate. It is relatively easy to see that the number of failures coming from Colliot-Thélène, Kanevsky and Sansuc's family counted by $N(B)$ is at most $B/(\log B)^2$. Hence, our lower bound does not follow from the results in [CTKS87].

The lower bound of Theorem 4.1.4 is obtained by counting surfaces of the family featured in §4.5. At the same time, the counter-examples to the integral Hasse principle appearing in §4.7 are interesting on their own right. Our approach in §4.7 builds on the work of Colliot-Thélène, Kanevsky and Sansuc [CTKS87] and does not require any knowledge of explicit representatives of Brauer elements, unlike in §4.5. This is particularly handy, as explicit representatives of Brauer elements are genuinely hard to get hold of.

Finally, we note that the number of surfaces considered in Theorem 4.1.3, whose transcendental Brauer group is non-trivial, is negligible compared to the upper bounds established there. This is shown in Proposition 4.6.1. At the same time, all possible failures of the integral Hasse principle and integral strong approximation off ∞ counted in $N_{a_1,a_2,a_3}(B)$, $N^{'}_{a_1,a_2,a_3}(B)$ and in $N(B)$, $N^{'}(B)$ may come from transcendental Brauer elements.

Questions about local-global principles are non-trivial only for varieties with a non-empty adelic space. We discuss the amount of everywhere locally soluble surfaces in the family (4.1.1) in the next remark. For this purpose, with notation as in the introduction, for any real $B \geq 1$ let

$$\begin{aligned} N^{\mathrm{ELS}}_{a_1,a_2,a_3}(B) &= \#\{a_0 \in [-B,B] \cap \mathbb{Z}_{\neq 0} \ : \ \mathcal{U}(\mathbb{A}_\mathbb{Z}) \neq \emptyset\} \text{ for } (a_1,a_2,a_3) \in \mathbb{Z}^3_{\text{prim}}, \\ N^{\mathrm{ELS}}_{a_0}(B) &= \#\big\{(a_1,a_2,a_3) \in [-B,B]^3 \cap \mathbb{Z}^3_{\text{prim}} \ : \ \mathcal{U}(\mathbb{A}_\mathbb{Z}) \neq \emptyset\big\} \text{ for } a_0 \in \mathbb{Z}_{\neq 0}, \\ N^{\mathrm{ELS}}(B) &= \#\big\{(a_0,a_1,a_2,a_3) \in [-B,B]^4 \cap \mathbb{Z}^4_{\text{prim}} \ : \ \mathcal{U}(\mathbb{A}_\mathbb{Z}) \neq \emptyset\big\}. \end{aligned}$$

Remark 4.1.5. The limits

$$\lim_{B\to\infty} \frac{N^{\mathrm{ELS}}_{a_1,a_2,a_3}(B)}{B}, \quad \lim_{B\to\infty} \frac{N^{\mathrm{ELS}}_{a_0}(B)}{B^3}, \quad \lim_{B\to\infty} \frac{N^{\mathrm{ELS}}(B)}{B^4}$$

exist and each of them equals a positive constant. This may be verified, for example, with the help of [BBL16, Lem. 3.1] by checking that all three assumptions of its statement are satisfied. Its second assumption is easily seen along the lines of the proof of [BBL16, Thm. 2.2]. The local conditions from the proof of [BBL16, Thm. 2.2] modified for $\mathbb{Z}_p$-points on $\mathcal{U}$ also imply the first assumption of [BBL16, Lem. 3.1] in view of the properties of the Haar measure on $\mathbb{Z}_p^n$. The last assumption of [BBL16, Lem. 3.1] follows from a completely analogous analysis to the one featured in the proof of [Mit17, Thm. 1.1] as a prime has to divide at least two of the a_i for the lack of $\mathbb{Z}_p$-points.

Notation 4.1.6. We fix a choice of *(smooth) compactification* $X \subset \mathbb{P}^3$ *of* U, given in (4.1.1), which is

$$X\colon \quad a_1x_1^3 + a_2x_2^3 + a_3x_3^3 = a_0x_0^3 \subseteq \mathbb{P}^3.$$

Furthermore, we fix D for the divisor on X defined by $D := \{x_0 = 0\}$, unless otherwise stated, and hence $U = X \setminus D$. Integral models of U, X and D will be denoted by $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{D}$, respectively. They are assumed to be given by the defining equation of U, X and D, respectively, unless otherwise stated.

4.2 Brauer groups of diagonal affine cubic surfaces

This section is dedicated to finding the Brauer group of the diagonal affine cubic surfaces U over $\mathbb{Q}$ given in (4.1.1). Recall that X is its compactification and D is the boundary divisor of X such that $U = X \setminus D$.

4.2.1 Algebraic Brauer group

In this subsection we compute the algebraic Brauer group of U over a number field K that does not contain a primitive third root of unity.

Proposition 4.2.1. *Let $U \to X$ over K be a smooth compactification of the affine diagonal cubic given by* (4.1.1).

(1) *The natural map* $\operatorname{Br} X \to \operatorname{Br}_1 U$ *is an isomorphism,*

(2) $\operatorname{Br}_1 U/\operatorname{Br}_0 U = \begin{cases} 0 & \text{if a cross ratio } a_i a_j/a_h a_\ell \in K^{*3}, \\ \mathbb{Z}/3\mathbb{Z} & \text{otherwise.} \end{cases}$

(3) $\operatorname{Br}_1 U/\operatorname{Br}_0 U$ *is generated by*

$$\mathcal{A}' := \operatorname{Cores}_{K(\omega)/K} \mathcal{A}$$

where $\mathcal{A}$ is an element of $\operatorname{Br}_1 U_{K(\omega)}/\operatorname{Br}_0 U_{K(\omega)}$.

Proof. Let $X_{\alpha_1,\alpha_2,\alpha_3}$ be the surface

$$X_{\alpha_1,\alpha_2,\alpha_3} : x_0^3 + \alpha_1 x_1^3 + \alpha_2 x_2^3 + \alpha_3 x_3^3 = 0 \subset \mathbb{P}^3$$

over the field $E := K(\alpha_1, \alpha_2, \alpha_3)$ where α_i are purely transcendental elements, the splitting field of this surface is given by the extension $E' := E(\omega, \sqrt[3]{\alpha_1}, \sqrt[3]{\alpha_2}, \sqrt[3]{\alpha_3})$ where $\omega \in \bar{K}$ such that $\omega^2 + \omega + 1 = 0$. There exists $\beta \in E'$ such that $E' = E(\beta)$, with minimal polynomial $f \in E[x]$. For any L/K obtained by specialising α_i in K, where f specialises to a separable polynomial, gives an embedding $\operatorname{Gal}(L/K) \hookrightarrow \operatorname{Gal}(E'/E)$ by Proposition 2.6.4. This shows that the splitting field for X is given by $L = K\left(\omega, \sqrt[3]{a_1/a_0}, \sqrt[3]{a_2/a_0}, \sqrt[3]{a_3/a_0}\right)$ and $\operatorname{Gal}(L/K) \hookrightarrow \operatorname{Gal}(E(\omega, \sqrt[3]{\alpha_1}, \sqrt[3]{\alpha_2}, \sqrt[3]{\alpha_3})/E)$. We get a chain of inclusions

$$\operatorname{Gal}(K(\omega)/K) \subseteq \operatorname{Gal}(L/K) \hookrightarrow \operatorname{Gal}(E'/E) \cong C_3^3 \rtimes C_2$$

which act compatibly on $\operatorname{Pic} \bar{X} \xrightarrow{\sim} \operatorname{Pic} \bar{X}_{\alpha_1,\alpha_2,\alpha_3}$. From here we can determine the image of the representation $\operatorname{Gal}(\bar{K}/K) \to W(E_6)$ induced by the action of $\operatorname{Gal}(\bar{K}/K)$ on $\operatorname{Pic} \bar{X}$.

We enumerate all subgroups N of $W(E_6)$ isomorphic to $C_3^3 \rtimes C_2$, up to conjugacy, and compute $\mathrm{H}^1(N', (\operatorname{Pic} \bar{X})^{N'})$ for all subgroups $N' \subseteq N$. There are 8 even order subgroups and 4 odd order subgroups where $\mathrm{H}^1(N', (\operatorname{Pic} \bar{X})^{N'}) = \mathrm{H}^1(N', (\operatorname{Pic} \bar{U})^{N'})$ and 4 odd order subgroups where $\mathrm{H}^1(N', (\operatorname{Pic} \bar{X})^{N'}) \neq \mathrm{H}^1(N', (\operatorname{Pic} \bar{U})^{N'})$.

As K does not contain a primitive third root of unity $2 \mid \#N'$, hence

$$\operatorname{Br} X/\operatorname{Br}_0 X \cong \mathrm{H}^1(N', (\operatorname{Pic}\bar{X})^{N'}) \xrightarrow{\sim} \mathrm{H}^1(N', (\operatorname{Pic}\bar{U})^{N'}) \cong \operatorname{Br}_1 U/\operatorname{Br}_0 U.$$

As $\operatorname{Br}_0 X \xrightarrow{\sim} \operatorname{Br}_0 U$ we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Br}_0 X & \longrightarrow & \operatorname{Br} X & \longrightarrow & \operatorname{Br} X/\operatorname{Br}_0 X & \longrightarrow & 0 \\ & & \downarrow{\cong} & & \downarrow & & \downarrow{\cong} & & \\ 0 & \longrightarrow & \operatorname{Br}_0 U & \longrightarrow & \operatorname{Br}_1 U & \longrightarrow & \operatorname{Br}_1 U/\operatorname{Br}_0 U & \longrightarrow & 0 \end{array}$$

Then by the snake lemma we can deduce that $\operatorname{Br} X \xrightarrow{\sim} \operatorname{Br}_1 U$.

For the last statement we use that $\operatorname{Br} X_{K(\omega)}/\operatorname{Br}_0 X_{K(\omega)}$ is either 0, $\mathbb{Z}/3\mathbb{Z}$ or $(\mathbb{Z}/3\mathbb{Z})^2$ [CTKS87, §1, Prop. 1]. Since both $\operatorname{Br} X/\operatorname{Br}_0 X$ and $\operatorname{Br} X_{K(\omega)}/\operatorname{Br}_0 X_{K(\omega)}$ are 3-torsion, and $[K(\omega)\colon K] = 2$ we apply [CTS21, Thm. 3.8.5], which gives that $\operatorname{Res}_{K(\omega)/K}$ is an isomorphism

$$\operatorname{Res}_{K(\omega)/K} : \operatorname{Br} X/\operatorname{Br}_0 X \to \left(\operatorname{Br} X_{K(\omega)}/\operatorname{Br}_0 X_{K(\omega)}\right)^{\operatorname{Gal}(K(\omega)/K)},$$

whose inverse is given by $-\operatorname{Cores}_{K(\omega)/K}$. Hence a generator $\mathcal{A}' \in \operatorname{Br} X/\operatorname{Br}_0 X$ corresponds to a unique Galois invariant element $\mathcal{A} \in \operatorname{Br} X_{K(\omega)}/\operatorname{Br}_0 X_{K(\omega)}$. □

Remark 4.2.2. The condition K does not contain a primitive third root of unity is necessary in Proposition 4.2.1. Consider the cubic surface

$$X'\colon \quad x_0^3 + x_1^3 + x_2^3 + ax_3^3 = 0 \subset \mathbb{P}^3_{\mathbb{Q}(\omega)}.$$

where a is cube-free. In this case $\operatorname{Br} X'/\operatorname{Br}\mathbb{Q}(\omega) \cong (\mathbb{Z}/3\mathbb{Z})^2$ [CTKS87, §1, Prop. 1]. By [BL19, Lem. 2.2] for any smooth and irreducible anticanonical curve D' the surface $U' := X' \backslash D'$ has $\operatorname{Br}_1 U'/\operatorname{Br}\mathbb{Q}(\omega) \cong (\mathbb{Z}/3\mathbb{Z})^3$.

4.2.2 Transcendental Brauer group

In Proposition 4.2.1 we showed that $\operatorname{Br}_1 U = \operatorname{Br} X$. To establish Theorem 4.1.1, namely

$$\operatorname{Br} U = \begin{cases} \operatorname{Br} X \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } a_1a_2a_3 \equiv 2 \bmod \mathbb{Q}^{*3}, \\ \operatorname{Br} X & \text{otherwise,} \end{cases}$$

we need to determine the transcendental Brauer group of U. For this we adapt some results by Jahnel and Schindler [JS17, Thm. 4.9, Cor 4.10, Rem 4.11] on degree four del Pezzo surfaces to geometrically rational surfaces. Their proofs can be used almost verbatim in this situation. From there we use a method of Colliot-Thélène and Wittenberg [CTW12, §3,5].

Lemma 4.2.3. *Let F be a field of characteristic 0 and S a smooth geometrically rational projective surface over F. Denote by H a F-rational hyperplane and V the affine surface $V := S\backslash H$. Suppose $C := H \cap S$ is smooth then the natural morphism*

$$\operatorname{Br}(\bar{V}) \to \mathrm{H}^1_{\text{ét}}(\bar{C}, \mathbb{Q}/\mathbb{Z}).$$

is an isomorphism.

Proof. As $\bar{S}$ is rational $\mathrm{H}^2_{\text{ét}}(\bar{S}, \mathbb{G}_m) = \mathrm{H}^3_{\text{ét}}(\bar{S}, \mathbb{G}_m) = 0$. Then the statement follows from Grothendieck's purity theorem [CTS21, Thm. 3.7.1]. □

Proposition 4.2.4. *Let V and C be as in Lemma 4.2.3. Then there is an injection*

$$(\operatorname{Br} V / \operatorname{Br}_1 V)[n] \hookrightarrow \operatorname{Hom}(J(C)(\bar{F})[n], \mathbb{Q}/\mathbb{Z})^{\operatorname{Gal}(\bar{F}/F)}.$$

In particular, if C is a genus one curve then a class of order n in $\operatorname{Br} V/\operatorname{Br}_1 V$ induces a F-rational n-isogeny

$$J(C) \twoheadrightarrow C'$$

to an elliptic curve C' with a F-point of order n.

Proof. From the Kummer sequence of étale sheaves we can deduce $\mathrm{H}^1_{\text{ét}}(\bar{C}, \mu_n) \cong \operatorname{Pic}(\bar{C})[n]$ and $\mathrm{H}^2_{\text{ét}}(\bar{C}, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$. As C is a smooth curve

$$\operatorname{Pic}(\bar{C})[n] \cong \operatorname{Pic}^0(\bar{C})[n] \cong J(C)(\bar{F})[n].$$

Applying Poincaré duality [Mil16, §VI Cor. 11.2], which is a perfect pairing and Galois invariant

$$\mathrm{H}^1_{\text{ét}}(\bar{C}, \mu_n) \times \mathrm{H}^1_{\text{ét}}(\bar{C}, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{H}^2_{\text{ét}}(\bar{C}, \mu_n) \cong \mathbb{Z}/n\mathbb{Z} \cong \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}$$

induces the following isomorphism

$$\mathrm{H}^1_{\text{ét}}(\bar{C}, \mathbb{Z}/n\mathbb{Z}) \cong \operatorname{Hom}(\mathrm{H}^1_{\text{ét}}(\bar{C}, \mu_n), \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}) \cong \operatorname{Hom}(J(C)(\bar{F})[n], \mathbb{Q}/\mathbb{Z}).$$

By definition there is a canonical injection $\operatorname{Br} V/\operatorname{Br}_1 V \hookrightarrow \operatorname{Br}(\bar{V})^{\operatorname{Gal}(\bar{F}/F)}$, hence by Lemma 4.2.3 we have an inclusion

$$(\operatorname{Br} V / \operatorname{Br}_1 V)[n] \hookrightarrow \operatorname{Hom}(J(C)(\bar{F})[n], \mathbb{Q}/\mathbb{Z})^{\operatorname{Gal}(\bar{F}/F)}.$$ □

We also need the following result on étale cohomology.

Lemma 4.2.5. *Let F be a field that does not contain a primitive third root of unity and $L := F(\sqrt[3]{a}) = F[x]/(x^3 - a)$ for some $a \in F$. Then the restriction map*

$$\mathrm{H}^1_{\text{ét}}(F, \mathbb{Z}/3\mathbb{Z}) \to \mathrm{H}^1_{\text{ét}}(L, \mathbb{Z}/3\mathbb{Z})$$

is injective.

Proof. Elements of $\mathrm{H}^1(F, \mathbb{Z}/n\mathbb{Z})$ can be interpreted as isomorphism classes (E, s) where E is a cyclic field extension of degree 3 of F and s generates $\operatorname{Gal}(E/F)$. Then (E, s) is in the kernel of the restriction map $\mathrm{H}^1_{\text{ét}}(F, \mathbb{Z}/3\mathbb{Z}) \to \mathrm{H}^1_{\text{ét}}(L, \mathbb{Z}/3\mathbb{Z})$ if and only if E/F can be embedded inside L/F. As L/F is either trivial or a degree 3 non-Galois extension, it does not contain any non-trivial cyclic extensions of degree 3, hence $E = F$. □

4.2.3 Jacobian of diagonal cubic curves

As we saw in Proposition 4.2.4, it is crucial to study the Jacobian of D. This section is dedicated to finding which cyclic p-isogenies $J(D)$ can have. Note that if F is a field of characteristic not equal to 2 or 3, then the Jacobian of D/F is the curve

$$J(D): \quad x_1^3 + x_2^3 + a_1 a_2 a_3 x_3^3 = 0.$$

By fixing the point $\mathcal{O} := [1 : -1 : 0]$ on $J(D)$, the 3-torsion $J(D)(\bar{F})[3]$ is defined by $x_1 x_2 x_3 = 0$ and we can rewrite the defining equation for $J(D)$ in Weierstrass form

$$J(D) : y^2 z = x^3 - 27(4a_1a_2a_3)^2 z^3.$$

Lemma 4.2.6 ([CLR21, Table 5]). *Let D be the plane curve*

$$D : x_1^3 + x_2^3 + a_3x_3^3 = 0 \subset \mathbb{P}^2_{\mathbb{Q}}$$

where a_3 is a cube-free integer. Then the $\mathbb{Q}$-isogeny classes of D are shown in Table 4.2.1.

Table 4.2.1: *$\mathbb{Q}$-isogeny classes of curves isogenus to D*

Isogeny Classes			
	Isogenous curves	Degree of isogeny	Torsion subgroup
$a_3 = 1$	D	1	$\mathbb{Z}/3\mathbb{Z}$
	$y^2 + y = x^3 - 270x - 1708$	3	Trivial
	$y^2 + y = x^3$	3	$\mathbb{Z}/3\mathbb{Z}$
	$y^2 + y = x^3 - 30x + 63$	9	$\mathbb{Z}/3\mathbb{Z}$
$a_3 = 2$	D	1	$\mathbb{Z}/2\mathbb{Z}$
	$y^2 = x^3 - 135x - 594$	2	$\mathbb{Z}/2\mathbb{Z}$
	$y^2 = x^3 + 1$	3	$\mathbb{Z}/6\mathbb{Z}$
	$y^2 = x^3 - 15x + 22$	6	$\mathbb{Z}/6\mathbb{Z}$
$a_3 \neq 1, 2$	D	1	Trivial
	$y^2 = x^3 + (4a_3)^2$	3	$\mathbb{Z}/3\mathbb{Z}$

Lemma 4.2.7. *Let K be a number field not containing a primitive third root of unity and D the elliptic curve over K defined by*

$$D\colon \quad x_1^3 + x_2^3 + a_3x_3^3 = 0 \subseteq \mathbb{P}^2_K.$$

(1) *If $a_3 \in K^{*3}$ then D has two cyclic 3-isogenies over K,*

(2) *Otherwise, D has one cyclic 3-isogeny over K.*

Proof. Writing D in Weierstrass form $D : y^2 = x^3 - 27(4a_3)^2$ we can consider the action of $\mathrm{Gal}(\bar{K}/K)$ on $D[3]$. Denote by x_i the roots of $x^3 - 27(4^3a_3^2)$ for $i = 1, 2, 3$. The subgroups of $D[3]$ written in (x, y) coordinates are

$$S_1 := \{\mathcal{O}, (0, 12a_3\sqrt{-3}), (0, -12a_3\sqrt{-3})\}, \qquad S_2 := \{\mathcal{O}, (x_1, 36a_3), (x_1, -36a_3)\},$$
$$S_3 := \{\mathcal{O}, (x_2, 36a_3), (x_2, -36a_3)\}, \qquad S_4 := \{\mathcal{O}, (x_3, 36a_3), (x_3, -36a_3)\}.$$

The subgroup S_1 is Galois invariant so D has at least one cyclic 3-isogeny over K. If $a_3 \notin K^{*3}$ then S_2, S_3 and S_4 are permuted by $\mathrm{Gal}(\bar{K}/K)$, hence $D \to D/S_1$ is the unique (up to isomorphism) cyclic 3-isogeny of D. If $a_3 \in K^{*3}$ then there exists $i = 2, 3$ or 4 such that S_i is Galois invariant. Without loss of generality we can assume $i = 1$, then S_2 and S_3 will be permuted by $\mathrm{Gal}(\bar{K}/K)$ as K does not contain a primitive third root of unity, hence the statement. □

4.2.4 Proof of Theorem 4.1.1

Proposition 4.2.8. *The following hold.*

(1) *If $a_1a_2a_3 \equiv 2$ mod $\mathbb{Q}^{*3}$ then $(\mathrm{Br}\, U/\mathrm{Br}_1\, U)[p] = 0$ for any prime p where $p \neq 2, 3$.*

(2) *If $a_1a_2a_3 \not\equiv 2$ mod $\mathbb{Q}^{*3}$ then $(\mathrm{Br}\, U/\mathrm{Br}_1\, U)[p] = 0$ for any prime p where $p \neq 3$.*

Proof. This follows automatically from Proposition 4.2.4 and Lemma 4.2.6. □

Propositions 4.2.10 and 4.2.12 deal with the 3-torsion in $\operatorname{Br} U/\operatorname{Br}_1 U$. We will show that in all cases $(\operatorname{Br} U/\operatorname{Br}_1 U)[3] = 0$. Without loss of generality we can assume $a_1 = 1$, and we first consider the case where $a_2 = 1$.

Lemma 4.2.9. *Let $D \subset \mathbb{P}^2_{\mathbb{Q}}$ be the smooth plane cubic with the equation $x_1^3 + x_2^3 + a_3 x_3^3 = 0$ where $a_3 \in \mathbb{Z}_{>0}$ is cube-free and not 1 or 2. Define $k_2 := \mathbb{Q}(\sqrt[3]{a_3}) = \mathbb{Q}[x]/(x^3 - a_3)$ and $P_1 := [1 : -1 : 0] \in D(\mathbb{Q}), P_2 := \left[1 : 0 : -1/\sqrt[3]{a_3}\right] \in D(k_2)$.*

(1) *The kernel of the map $\mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1_{\text{ét}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ evaluating at the point P_1 is isomorphic to $\mathbb{Z}/3\mathbb{Z}$.*

(2) *The kernel of the map $\mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1_{\text{ét}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \times \mathrm{H}^1_{\text{ét}}(k_2, \mathbb{Q}/\mathbb{Z})$ evaluating at P_1 and P_2 respectively is trivial.*

Proof. The only other curve in the $\mathbb{Q}$-isogeny class of D is a 3-isogenous curve by Lemma 4.2.6. Writing D in Weierstrass normal form $D : y^2 = x^3 + d$ where $d := -27(4a_3)^2$ we can explicitly write down the 3-isogeny $\psi : D \to E$

$$\psi : D \to E, \ (x, y) \mapsto \left(\frac{y^2 + 3d}{x^2}, \frac{y(x^3 - 8d)}{x^3} \right).$$

The kernel of ψ is $\{\mathcal{O}, [0 : \sqrt{d} : 1], [0 : -\sqrt{d} : 1]\}$ i.e. the kernel is isomorphic to μ_3 as a Galois module over $\mathbb{Q}$. The image of ψ in E is $\mathbb{Z}/3\mathbb{Z}$, this gives rise to the following exact sequence,

$$0 \to \mu_3 \to D[3] \to \mathbb{Z}/3\mathbb{Z} \to 0.$$

Applying the $\operatorname{Hom}(\mu_3, -)$ functor to this exact sequence results in a long exact sequence

$$0 \to \operatorname{Hom}(\mu_3, \mu_3) \to \operatorname{Hom}(\mu_3, D[3]) \to \operatorname{Hom}(\mu_3, \mathbb{Z}/3\mathbb{Z}) \to \operatorname{Ext}(\mu_3, \mu_3) \to \dots$$

As $\operatorname{Hom}(\mu_3, \mathbb{Z}/3\mathbb{Z}) = 0$ this implies $\operatorname{Hom}(\mu_3, \mu_3) = \operatorname{Hom}(\mu_3, D[3])$. As $\operatorname{Hom}(\mu_3, \mu_3) = \mathbb{Z}/3\mathbb{Z}$, it follows $\operatorname{Hom}(\mu_3, D[3]) = \mathbb{Z}/3\mathbb{Z}$. The kernel of the map

$$\mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1_{\text{ét}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}),$$

which evaluates a class at the point P_1, corresponds to $\operatorname{Hom}(\mu_3, D)$ where this is as homomorphisms of algebraic groups over $\mathbb{Q}$. We previously saw $\operatorname{Hom}(\mu_3, D) = \mathbb{Z}/3\mathbb{Z}$, so the evaluation at a point P_1 induces a decomposition

$$\mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z}) = \mathrm{H}^1_{\text{ét}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \oplus \mathbb{Z}/3\mathbb{Z},$$

by [CTW12, Rmk. 3.3]. This completes the proof of (1). Pick $m \in \mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z})$ which generates the factor of $\mathbb{Z}/3\mathbb{Z}$, with corresponding exact sequence

$$0 \to \mathbb{Z}/3\mathbb{Z} \to D/\mu_3 \xrightarrow{\hat{\psi}} D \to 0$$

induced by the dual isogeny map $\hat{\psi}$ to ψ. We now show that $m(P_2) \neq 0$ to establish (2). Suppose $m(P_2) = 0$ this implies that for the $\mathbb{Z}/3\mathbb{Z}$-torsor $\hat{\psi} : D/\mu_3 \to D$ the fibre above P_2 is a trivial torsor i.e. $\hat{\psi}^{-1}(P_2)$ consists of 3 distinct k_2-points. However, computing the 3-torsion for the elliptic curve $D/\mu_3 : y^3 = x^3 + (4c)^2$ we see $\#D/\mu_3(k_2)[3] = 3$, hence the fibre above P_2 is irreducible as $P_2 \in D(k_2)[3]$ and the three k_2-points on D/μ_3 correspond to the fibre of $\hat{\psi}$ above P_1. □

Proposition 4.2.10. *Assume that $a_1 = a_2 = 1$ and a_3 is a cube-free integer. If $a_3 \neq 2$, then the natural morphism $\operatorname{Br} X \to \operatorname{Br} U$ is an isomorphism.*

Proof. The case where $a_3 = 1$ was dealt with in [CTW12, Prop. 3.1], from now on we can assume $a_3 \neq 1$. By applying Grothedieck's purity theorem [CTS21, Thm. 3.7.1], we get an exact sequence

$$0 \to \operatorname{Br} X \to \operatorname{Br} U \xrightarrow{\partial_D} \mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z}).$$

Let $A \in \operatorname{Br}(U)$ such that $A \notin \operatorname{Im}(\operatorname{Br} X \to \operatorname{Br} U)$ and $m = \partial_D(A) \in \mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z})$. Define $k_1 = \mathbb{Q}(\sqrt[3]{a_3/a_0}), k_2 = \mathbb{Q}(\sqrt[3]{a_3})$, $k_3 = k_2(\sqrt[3]{a_0})$ and $P_1, P_2 \in D$ as in Lemma 4.2.9. Then X_{k_1} contains the line

$$L_1 : x_1 + x_2 = x_3 - \sqrt[3]{\frac{a_3}{a_0}} x_0 = 0,$$

which intersects the curve D_{k_1} at the point $P_1 \otimes_{\mathbb{Q}} k_1 = [1 : -1 : 0]$. This gives the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Br} X_{k_1} & \longrightarrow & \operatorname{Br} U_{k_1} & \xrightarrow{\partial_D} & \mathrm{H}^1_{\text{ét}}(D_{k_1}, \mathbb{Q}/\mathbb{Z}) \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \alpha_1} \\ 0 & \longrightarrow & \operatorname{Br} L_1 & \longrightarrow & \operatorname{Br}(L_1 \backslash P_1) & \longrightarrow & \mathrm{H}^1_{\text{ét}}(P_1 \otimes k_1, \mathbb{Q}/\mathbb{Z}) \end{array}$$

where the map α_1 evaluates classes of $\mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z})$ at the point P_1. Let $A' \in \operatorname{Br}(L_1 \backslash P_1)$ be the restriction of A to $L_1 \backslash P_1$. As $L \backslash P_1 \cong \mathbb{A}^1_{k_1}$ we have $\operatorname{Br}(L_1 \backslash P_1) = \operatorname{Br}(k_1)$, hence the class of A' is constant and its residue at the point $P_1 \otimes_{\mathbb{Q}} k_1$ is 0. One can now deduce that $\operatorname{Im}(\partial_D) \subset \ker(\alpha_1)$ i.e. $m(P_1 \otimes k_1) = 0$. Similarly, X_{k_3} contains a line

$$L_2 : x_1 + \sqrt[3]{a_3} x_3 = x_2 - \sqrt[3]{a_0} x_0 = 0.$$

We see that D_{k_3} intersects L_2 at the point $P_2 \otimes_k k_3 = [1 : 0 : -1/\sqrt[3]{a_3}]$. Applying a similar procedure as above gives the condition $m(P_2 \otimes k_3) = 0$. As $P_1 \in D(\mathbb{Q})$ (resp. $P_2 \in D(k_2)$) we have $m(P_1) \in \mathrm{H}^1_{\text{ét}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ (resp. $m(P_2) \in \mathrm{H}^1_{\text{ét}}(k_2, \mathbb{Q}/\mathbb{Z})$). Moreover, there are injective restriction maps

$$\phi_1 : \mathrm{H}^1_{\text{ét}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1_{\text{ét}}(k_1, \mathbb{Q}/\mathbb{Z}), \quad \phi_2 : \mathrm{H}^1_{\text{ét}}(k_2, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1_{\text{ét}}(k_3, \mathbb{Q}/\mathbb{Z})$$

by Lemma 4.2.5. Under the condition that $m(P_1 \otimes k_1) = m(P_2 \otimes k_3) = 0$ and using the fact that ϕ_1 and ϕ_2 are injective implies that m is in the kernel of the map

$$\mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1_{\text{ét}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \times \mathrm{H}^1_{\text{ét}}(k_2, \mathbb{Q}/\mathbb{Z})$$

evaluating an element of $\mathrm{H}^1_{\text{ét}}(D, \mathbb{Q}/\mathbb{Z})$ at the points P_1 and P_2 respectively onto each factor. Using Lemma 4.2.9 we can deduce $m = 0$. □

Next we now deal with $(\operatorname{Br} U / \operatorname{Br}_1 U)[3]$ where $a_1 = 1$ and $a_2, a_3 \in \mathbb{Q}^* \setminus \mathbb{Q}^{*3}$. For what follows $K := \mathbb{Q}(\sqrt[3]{a_2})$ and note that a_3 could lie in K^{*3}.

Lemma 4.2.11. *Let D be the curve $x_1^3 + x_3^3 + a_3 x_3^3 = 0 \subset \mathbb{P}^2_K$. Then*

(1) *If $a_3 \in K^{*3}$ i.e. there exists $\alpha \in K$ such that $\alpha^3 = a_3$ then $P_1 := [1 : -1 : 0], P_2 := [1 : 0 : -1/\alpha] \in D(K)$, and the kernel of*

$$\mathrm{H}^1_{\text{ét}}(D, \mathbb{Z}/3\mathbb{Z}) \to \mathrm{H}^1_{\text{ét}}(K, \mathbb{Z}/3\mathbb{Z}) \times \mathrm{H}^1_{\text{ét}}(K, \mathbb{Z}/3\mathbb{Z}), m \mapsto (m(P_1), m(P_2))$$

is trivial.

(2) *If* $a_3 \notin K^{*3}$, *let* $L = K(\alpha) = K[x]/(x^3 - a_3)$, $P_1 := [1 : -1 : 0] \in D(K)$ *and* $P_2 := [1 : 0 : -1/\alpha] \in D(L)$, *then the kernel of the map*

$$\mathrm{H}^1_{\text{ét}}(D, \mathbb{Z}/3\mathbb{Z}) \to \mathrm{H}^1_{\text{ét}}(K, \mathbb{Z}/3\mathbb{Z}) \times \mathrm{H}^1_{\text{ét}}(L, \mathbb{Z}/3\mathbb{Z}), m \mapsto (m(P_1), m(P_2)).$$

is trivial.

Proof. For case (1) (resp. (2)) arguing as in [CTW12, Lem. 3.2] (resp. Lemma 4.2.9) and using Lemma 4.2.11 we can deduce that the kernel of the evaluation at P_1 map

$$\mathrm{H}^1_{\text{ét}}(D, \mathbb{Z}/3\mathbb{Z}) \to \mathrm{H}^1_{\text{ét}}(K, \mathbb{Z}/3\mathbb{Z})$$

is isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Hence, we have a decomposition

$$\mathrm{H}^1_{\text{ét}}(D, \mathbb{Z}/3\mathbb{Z}) = \mathrm{H}^1_{\text{ét}}(K, \mathbb{Z}/3\mathbb{Z}) \oplus \mathbb{Z}/3\mathbb{Z}.$$

Pick the generator of the $\mathbb{Z}/3\mathbb{Z}$ factor which corresponds to a short exact sequence

$$0 \to \mathbb{Z}/3\mathbb{Z} \to D/\mu_3 \to \mu_3 \to 0.$$

To show that the kernel of map in (1) and (2) is trivial it is sufficient to note that $\#D_K/\mu_3(K)[3] = 3$ in case (1) and $\#D/\mu_3(L)[3] = 3$ in case (2) i.e. the fibre above P_2 of $D/\mu_3 \to D$ is irreducible for both cases. □

Proposition 4.2.12. *Assume that* $a_1 = 1$ *and* a_2, a_3 *are cube-free. Then*

$$(\operatorname{Br} U / \operatorname{Br}_1 U)[3] = 0.$$

Proof. We can assume $a_2, a_3 \neq 1$ as this is dealt with in [CTW12, Prop. 3.1, 3.4] and Proposition 4.2.10. Let $K := \mathbb{Q}[x]/(x^3 - a_2)$ then U_K is isomorphic to the surface

$$W\colon \quad u_1^3 + u_2^3 + a_3 u_3^3 = a_0$$

over K. It is then sufficient to show $(\operatorname{Br} W / \operatorname{Br}_1 W)[3] = 0$. Denote by Y the compactification of W in $\mathbb{P}^3_K$ and $D := Y \setminus W$. By Grothendieck's Purity theorem [CTS21, Thm. 3.7.1]

$$0 \to \operatorname{Br} Y[3] \to \operatorname{Br} W[3] \xrightarrow{\partial_D} \mathrm{H}^1_{\text{ét}}(D, \mathbb{Z}/3\mathbb{Z}).$$

Let $A \in \operatorname{Br} W[3]$ such that $A \notin \operatorname{Im}(\operatorname{Br} Y \to \operatorname{Br} W)$ and $m := \partial_D(A)$. Let $k_1 := K(\sqrt[3]{a_0/a_3}), k_2 := K(\sqrt[3]{a_3})$. Applying the argument from Proposition 4.2.10 it follows that evaluating m at the points $P_1 := [1 : -1 : 0] \in D(K)$ and $P_2 := [1 : 0 : -1/\sqrt[3]{a_3}] \in D(k_2)$ leads to the condition $m(P_1) = m(P_2) = 0$. By Lemma 4.2.11 this implies $m = 0$. □

Having dealt with the 3-torsion in the transcendental Brauer group we now deal with the 2-torsion. The following Lemma was given to the author by Olivier Wittenberg.

Lemma 4.2.13. *Let* $D \subseteq \mathbb{P}^2$ *be a smooth genus 1 curve over a field* F *of characteristic* 0. *Then there is a decomposition*

$$\mathrm{H}^1_{\text{ét}}(D, \mathbb{Z}/2\mathbb{Z}) = \mathrm{H}^1_{\text{ét}}(F, \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{H}^1_{\text{ét}}(\bar{D}, \mathbb{Z}/2\mathbb{Z})^{\operatorname{Gal}(\bar{F}/F)}.$$

Proof. Consider the 5-term exact sequence coming from the Hochschild-Serre spectral sequence

$$0 \longrightarrow \mathrm{H}^1_{\text{ét}}(F,\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\xi_1} \mathrm{H}^1_{\text{ét}}(D,\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathrm{H}^1_{\text{ét}}(\bar{D},\mathbb{Z}/2\mathbb{Z})^{\mathrm{Gal}(\bar{F}/F)} \longrightarrow \mathrm{H}^2_{\text{ét}}(F,\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\xi_2} \mathrm{H}^2_{\text{ét}}(D,\mathbb{Z}/2\mathbb{Z}).$$

Let $L \subset \mathbb{P}^2_F$ be a line not tangent to D, which can be chosen by Bertini's Theorem [Jou83, Thm. 6.3(4)]. By Bezout's Theorem D and L intersect at 3 distinct points geometrically. Consider the map

$$\sigma_i : \mathrm{H}^i_{\text{ét}}(D,\mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^i_{\text{ét}}(F,\mathbb{Z}/\mathbb{Z}), m \mapsto \sum_{p\in L\cap D} \mathrm{Cores}_{F(p)/F}\, m(p) \text{ for } i = 1,2.$$

Composing with the map ξ_i

$$\mathrm{H}^i_{\text{ét}}(F,\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\xi_i} \mathrm{H}^i_{\text{ét}}(D,\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sigma_i} \mathrm{H}^i_{\text{ét}}(F,\mathbb{Z}/\mathbb{Z})$$

gives multiplication by 3, hence an isomorphism.

This shows that the map $\mathrm{H}^1_{\text{ét}}(\bar{D},\mathbb{Z}/2\mathbb{Z})^{\mathrm{Gal}(\bar{F}/F)} \to \mathrm{H}^2_{\text{ét}}(F,\mathbb{Z}/2\mathbb{Z})$ is surjective. Which results in the following exact sequence with a left splitting

$$0 \longrightarrow \mathrm{H}^1_{\text{ét}}(F,\mathbb{Z}/2\mathbb{Z}) \underset{\curvearrowleft\,\sigma_1/3}{\xrightarrow{\xi_1}} \mathrm{H}^1_{\text{ét}}(D,\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathrm{H}^1_{\text{ét}}(\bar{D},\mathbb{Z}/2\mathbb{Z})^{\mathrm{Gal}(\bar{F}/F)} \longrightarrow 0$$

and leads to the sought after decomposition

$$\mathrm{H}^1_{\text{ét}}(D,\mathbb{Z}/2\mathbb{Z}) = \mathrm{H}^1_{\text{ét}}(F,\mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{H}^1_{\text{ét}}(\bar{D},\mathbb{Z}/2\mathbb{Z})^{\mathrm{Gal}(\bar{F}/F)}. \qquad \square$$

Proof of Theorem 4.1.1. By Proposition 4.2.1 $\mathrm{Br}\,X \xrightarrow{\sim} \mathrm{Br}_1\,U$. If $a_1a_2a_3 \not\equiv 2 \bmod \mathbb{Q}^{*3}$ then by Propositions 4.2.8, 4.2.10 and 4.2.12 we have $\mathrm{Br}_1\,U = \mathrm{Br}\,U$, hence $\mathrm{Br}\,X \xrightarrow{\sim} \mathrm{Br}\,U$. Suppose $a_1a_2a_3 \equiv 2 \bmod \mathbb{Q}^{*3}$ then it is sufficient to show that $\mathrm{Br}\,X$ and $\mathrm{Br}\,U$ differ by an order 2 element by Propositions 4.2.8 and 4.2.12. Consider the plane curve

$$D\colon \quad a_1x_1^3 + a_2x_2^3 + a_3x_3^3 = 0 \subseteq \mathbb{P}^2_{\mathbb{Q}}.$$

The Jacobian of D has a rational 2-torsion point if and only if $a_1a_2a_3 \equiv 2 \bmod \mathbb{Q}^{*3}$. Using [CTW12, Lem. 5.4] and Lemma 4.2.13 we can compute $(\mathrm{Br}\,U/\,\mathrm{Br}\,X)[2]$ as

$$\ker\left(\mathrm{H}^1_{\text{ét}}(D,\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sigma_1/3} \mathrm{H}^1_{\text{ét}}(\mathbb{Q},\mathbb{Z}/\mathbb{Z})\right) = \mathrm{H}^1_{\text{ét}}(\bar{D},\mathbb{Z}/2\mathbb{Z})^{\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})} = J(D)(\mathbb{Q})[2].$$

Hence, we can conclude that $(\mathrm{Br}\,U/\,\mathrm{Br}_1\,U)[2] \cong \mathbb{Z}/2\mathbb{Z}$. $\square$

4.2.5 Explicit generators

For future computations it will be useful to write down explicit elements for the Brauer group of diagonal affine cubic surfaces. As we have seen in the generic case this Brauer group is isomorphic to the Brauer group of its compactification which was studied by Colliot-Thélène, Sansuc and Kanevsky [CTKS87]. Unfortunately, there is no uniform generator for the family of all diagonal cubic surfaces [Uem14, Thm. 1.2]. However, in specific subfamilies one is able to write down generators. In particular, this is possible if $a_1 = a_2 = 1$, which we will assume throughout this subsection. Colliot-Thélène and Wittenberg have made an extensive study in [CTW12] of the case where $a_3 = 1$ or $a_3 = 2$. Note that the compactification X of U always has a non-empty set of rational points, namely $[0 : 1 : -1 : 0] \in X(\mathbb{Q})$.

Lemma 4.2.14 ([CTKS87, §1, Prop. 1]). *If $a_1 = a_2 = 1$ and a_0, a_3 are cube-free, then*

$$\operatorname{Br} X / \operatorname{Br} \mathbb{Q} = \begin{cases} 0 & \text{if } a_0 a_3 \text{ or } a_0/a_3 \text{ are in } \mathbb{Q}^{*3}, \\ \mathbb{Z}/3\mathbb{Z} & \text{otherwise.} \end{cases}$$

In particular, if $\operatorname{Br} X / \operatorname{Br} \mathbb{Q} \cong \mathbb{Z}/3\mathbb{Z}$, then $\operatorname{Br} X / \operatorname{Br} \mathbb{Q}$ is generated by the cyclic algebra

$$\mathcal{A}' = \operatorname{Cores}_{\mathbb{Q}(\omega)/\mathbb{Q}} \mathcal{A} \in \operatorname{Br} \mathbb{Q}(X), \text{ where } \mathcal{A}' = \left(\frac{a_0}{a_3}, \frac{x_1 + \omega x_2}{x_1 + x_2} \right)_\omega .$$

Lemma 4.2.15. *Assume that $a_1 = a_2 = 1$ and a_0, a_3 are cube-free. Then*

$$\operatorname{Br} U = \begin{cases} \operatorname{Br} X \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } a_3 \equiv 2 \bmod \mathbb{Q}^{*3}, \\ \operatorname{Br} X & \text{otherwise.} \end{cases}$$

In particular, if $a_3 \equiv 2 \bmod \mathbb{Q}^{*3}$, then the $\mathbb{Z}/2\mathbb{Z}$ factor is generated by the transcendental element

$$(a_0(x_1 + x_2 + 2x_3), -3(x_1 + x_2 + 2x_3)(x_1 + x_2)) \in \operatorname{Br} \mathbb{Q}(U).$$

Proof. The main statement is proved in Theorem 4.1.1 and the explicit generator for the $\mathbb{Z}/2\mathbb{Z}$ factor was determined in [CTW12, Prop. 3.4]. ☐

Remark 4.2.16. Consider the case $a_1 = 1$ and $a_2 a_3 \equiv 2 \bmod \mathbb{Q}^{*3}$. Over the field extension $K := \mathbb{Q}[x]/(x^3 - a_2)$ the surface U becomes isomorphic to the surface

$$U' : \quad u_1^3 + u_2^3 + 2u_3^3 = a_0.$$

By Lemma 4.2.13 and [CTW12, Lem. 5.4] we can deduce $(\operatorname{Br} U' / \operatorname{Br}_1 U') \cong \mathbb{Z}/2\mathbb{Z}$. Hence, the two torsion element in $\operatorname{Br} U$ can be written as the corestriction from K to $\mathbb{Q}$ of the two torsion element in Lemma 4.2.15.

4.3 Hilbert symbols

This section describes how to compute the local invariant map. We begin by defining Hilbert symbols and describe their relation to the invariant map. From there we give an algorithm from [CTKS87] which will enable us to compute the Brauer–Manin set for generic diagonal affine cubic surfaces. Here U is as defined in (4.1.1) and we keep the notational convention set up earlier.

4.3.1 Construction of the Hilbert symbol

Let K be a number field and v a place of K. Assume K_v contains a primitive nth root of unity ω_n. There exists a pairing [Ser79, Chap XIV, §2]

$$(\cdot,\cdot)_{\omega_n} : K_v^*/K_v^{*n} \times K_v^*/K_v^{*n} \to \operatorname{Br} K_v,\ (a,b) \mapsto (a,b)_{\omega_n}.$$

In $\operatorname{Br} K_v$ we have the relations

$$(aa',b)_{\omega_n} = (a,b)_{\omega_n} + (a',b)_{\omega_n}, \quad (a,b)_{\omega_n} = -(b,a)_{\omega_n}, \quad (a^n,b)_{\omega_n} = 0. \tag{4.3.1}$$

We define the *Hilbert symbol*

$$(a,b)_{\omega_n,v} = \operatorname{inv}_v(a,b)_{\omega_n} \in \mathbb{Q}/\mathbb{Z}.$$

For any non-zero prime ideal $\mathfrak{p}$ of K, which does not lie over any of the prime divisors of n, with associated place v

$$(u,u')_{\omega_n,v} = 0 \quad \text{and} \quad (u,\pi)_{\omega_n,v} = 0 \Leftrightarrow u \in K_v^n, \tag{4.3.2}$$

for $u,u' \in O_{K_v}^*$ and $\pi \in O_{K_v}$ a uniformiser. For $n = 3$ we recall some formulae from [CTKS87, §4] who identify $\frac{1}{3}\mathbb{Z}/\mathbb{Z}$ with $\mathbb{Z}/3\mathbb{Z}$, which agrees with [Ser79, Chap XIV, §2]. Throughout the rest of the paper we make the same identification. Let ω be a fixed root of x^2+x+1 and we fix an isomorphism $k := \mathbb{Q}(\omega) \cong \mathbb{Q}[x]/(x^2+x+1)$ by sending x to ω.

1. Let p be a prime where $p \equiv 1 \bmod 3$. As ideals in $\mathcal{O}_k$ we have $(p) = \mathfrak{p}_1\mathfrak{p}_2$. Denote by v a place corresponding to the prime ideal $\mathfrak{p}_1$ or $\mathfrak{p}_2$ extending from the p-adic valuation on $\mathbb{Q}$. If u is a unit in the ring of integers of $\mathbb{Q}_p \cong k_v$, then we have the formula
$$(u,p)_{\omega,v} = -i \bmod 3 \tag{4.3.3}$$
where $u^{\frac{p-1}{3}} \equiv \omega^i \bmod p$. In particular $(\omega,p)_v \equiv -\frac{p-1}{3} \bmod 3$.

2. Let p be a prime where $p \equiv 2 \bmod 3$. As ideals in $\mathcal{O}_k$ we have $(p) = \mathfrak{p}$. Denote by v a place corresponding to the prime ideal $\mathfrak{p}$ extending from the p-adic valuation on $\mathbb{Q}$. If u is a unit in the ring of integers of $\mathbb{Q}_p(\omega) \cong k_v$ we have the formula
$$(u,p)_{\omega,v} = -i \bmod 3 \tag{4.3.4}$$
where $u^{\frac{p^2-1}{3}} \equiv \omega \bmod p$. In particular we have $(\omega,p)_{\omega,v} \equiv -\frac{p^2-1}{3} \bmod 3$.

3. Let $p = 3$, then as ideals in $\mathcal{O}_k$ we have $(3) = \mathfrak{p}^2$. Note $\mathcal{O}_k = \mathbb{Z}[\omega]$ and the prime ideal $\mathfrak{p}$ is generated by $\lambda' = 2\omega + 1$ which satisfies $\lambda'^2 = -3$. We choose the uniformizer $\lambda = \lambda'^2\omega + \lambda' = -3\omega + (2\omega+1) = 1-\omega$, with minimal polynomial $\lambda^2 - 3\lambda + 3 = 0$ and the following relations
$$\begin{aligned} \omega = 1-\lambda, &\quad 3 \equiv -\lambda^2 - \lambda \bmod \lambda^4, \\ 3 = -\omega^2\lambda^2, &\quad 2 \equiv -1 - \lambda^2 - \lambda^3 \bmod \lambda^4. \end{aligned} \tag{4.3.5}$$
The relations for $(-,-)_{\mathfrak{p}}$ are given in [CTKS87, p. 34], however we give a summary of the most important facts. Any element in $\mathbb{Z}_3[\theta]$ can be written as $\pm\lambda^e u$ where u

is a 1-unit. We will write a 1-unit u_b as $1+b_1\lambda+b_2\lambda^2+\dots$ with $b_i\in\mathbb{Z}$. To compute the symbol at $\mathfrak{p}$ with associated place v, we will only need the following information

$$(u_b,u_c)_{\omega,v} = b_1c_1(b_1-c_1)-b_1c_2+b_2c_1, \tag{4.3.6}$$

$$(\lambda,u_b)_{\omega,v} = \frac{b_1-b_1^2}{3}+b_1b_2-b_3. \tag{4.3.7}$$

Remark 4.3.1. Note the relation (4.3.6) only depends on b_i, c_i modulo 3 and not on any b_i, c_i with $i\geq 3$. For the relation (4.3.7) we need to know b_1 modulo 9, b_2, b_3 modulo 3 and none of the b_i with $i\geq 4$.

4.3.2 Conditions for a Brauer–Manin obstruction

We are now in position to establish sufficient conditions for the lack of a Brauer–Manin obstruction to the integral Hasse principle on $\mathcal{U}$. This is done in Propositions 4.3.2 and 4.3.3. We write $\mathcal{B}$ for both a generator of $\operatorname{Br}X/\operatorname{Br}_0 X$ and its restriction to U.

Proposition 4.3.2. *Assume that there exists a prime $p\neq 3$ such that $p\mid a_0$ but $p^3\nmid a_0$ and $p\nmid a_1a_2a_3$. If $\mathcal{U}(\mathbb{Z}_p)\neq\emptyset$, then $\operatorname{inv}_p\mathcal{B}:\mathcal{U}(\mathbb{Z}_p)\to\frac{1}{3}\mathbb{Z}/\mathbb{Z}$ is surjective.*

Proof. Under the conditions of the proposition we have $\operatorname{Br}X/\operatorname{Br}_0 X\cong\mathbb{Z}/3\mathbb{Z}$. Consider the composition

$$\mathcal{U}(\mathbb{Z}_p)\to\mathcal{X}(\mathbb{Z}_p)\xrightarrow{\text{red}}E(\mathbb{F}_p)$$

where E is the elliptic curve

$$a_1x_1^3+a_2x_2^3+a_3x_3^3=0$$

Clearly, this composition is surjective. The invariant map $\operatorname{inv}_p\mathcal{B}$ factors as the surjective map red and a surjective homomorphism $E(\mathbb{F}_p)\to\frac{1}{3}\mathbb{Z}/\mathbb{Z}$, [Jah14, Chap 4, Thm 6.4 c)i)]. By functoriality we can deduce that $\operatorname{inv}_p\mathcal{B}:\mathcal{U}(\mathbb{Z}_p)\to\frac{1}{3}\mathbb{Z}/\mathbb{Z}$ is surjective. □

Proposition 4.3.3. *Assume that there exists a prime $p\geq 17$ such that $p\mid a_i$ for some $i\in\{1,2,3\}$ but $p^3\nmid a_i$ and $p\nmid a_j$ for $i\neq j$ where $j\in[0,3]$. Then $\operatorname{inv}_p\mathcal{B}:\mathcal{U}(\mathbb{Z}_p)\to\frac{1}{3}\mathbb{Z}/\mathbb{Z}$ is surjective.*

Proof. The proof is very similar to the one of Proposition 4.3.2 with the only difference that the local invariant map now factors through the composition

$$\mathcal{U}(\mathbb{Z}_p)\to\mathcal{X}(\mathbb{Z}_p)\xrightarrow{\text{red}}(E\setminus D)(\mathbb{F}_p),$$

where if $i=1$ the elliptic curve is $E: a_2x_2^3+a_3x_3^3-a_0x_0^3=0$ and D is the divisor on E given by the vanishing locus of x_0. As before this composition is surjective and hence the proof boils down to establishing that the homomorphism $(E\setminus D)(\mathbb{F}_p)\to\frac{1}{3}\mathbb{Z}/\mathbb{Z}$ is surjective. This follows from the Hasse–Weil bound and [Jah14, Chap 4, Thm 6.4 c)i)] as there are at least $(p+1-2\sqrt{p})/3-3$ points with a given value of the local invariant map. This number is clearly positive provided that $p\geq 17$, which confirms our claim. □

4.3.3 Computing invariant maps for generic families

We give an overview of the work in [CTKS87] which describes how to compute the Brauer–Manin set for diagonal cubic surfaces. Assume that $a_i \in \mathbb{Z}_{\neq 0}$ are cube-free. Dividing the defining equation of X by a_0 gives

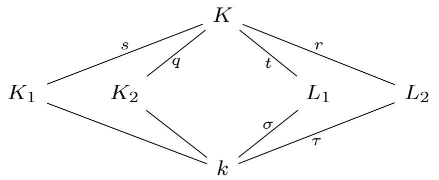
$$X\colon \quad x_0^3 + \lambda x_1^3 + \mu x_2^3 + \lambda\mu\nu x_3^3 = 0,$$

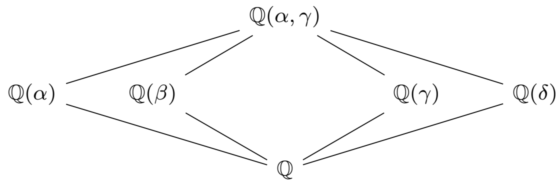
where $\lambda, \mu, \nu \in \mathbb{Q}^*$ are $\lambda = a_1/a_0$, $\mu = a_2/a_0$ and $\nu = -a_3 a_0/(a_1 a_2)$. Pick $\alpha, \gamma, \omega \in \bar{\mathbb{Q}}$, such that $\alpha^3 = \lambda$, $\gamma^3 = \nu$, $\omega^2 + \omega + 1 = 0$. Define $K := \mathbb{Q}(\omega, \alpha, \gamma)$ which is Galois over $k := \mathbb{Q}(\omega)$, with Galois group G which is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^2$ and generated by s, t defined by

$${}^s\alpha = \alpha, \quad {}^t\alpha = \omega\alpha, \quad {}^s\gamma = \omega\gamma, \quad {}^t\gamma = \gamma.$$

Let $\beta := \alpha\gamma$ and $\delta := \alpha/\gamma$. This defines the following Galois extensions

$$\begin{aligned}
&\alpha = \sqrt[3]{\lambda} && \langle s\rangle = \mathrm{Gal}(K/K_1)\\
&\beta = \sqrt[3]{\lambda\nu} = \alpha\gamma && \langle q\rangle = \mathrm{Gal}(K/K_2), q = st^2 \text{ so } {}^q\beta = \beta \text{ and } {}^q\delta = \omega^2\delta\\
&\gamma = \sqrt[3]{\nu} && \langle t\rangle = \mathrm{Gal}(K/L_1)\\
&\delta = \sqrt[3]{\lambda/\nu} = \alpha/\gamma && \langle r\rangle = \mathrm{Gal}(K/L_2), r = st \text{ so } {}^r\beta = \omega^2\beta \text{ and } {}^r\delta = \delta.
\end{aligned}$$





Remark 4.3.4. Let S be a smooth, projective variety over a number field K and v a place of K. If S is K_v-rational we have $\mathrm{Br}\, S_{K_v} = \mathrm{Br}\, K_v$ i.e. the invariant map at v is constant for any element in $\mathrm{Br}\, S$. In the case of diagonal cubic surfaces X/K, Colliot-Thélène, Kanevsky and Sansuc give a necessary and sufficient condition for X to be K_v-rational [CTKS87, §5, Lem. 8]. Namely, let F be a field of characteristic not equal to 3, then X/F is F-rational if and only if $X(F) \neq \emptyset$ and $a_0 a_1/a_2 a_3$ is a cube in F^*.

In the case $\mathrm{Br}\, X/\mathrm{Br}_0\, X \cong \mathbb{Z}/3\mathbb{Z}$, as in [CTKS87, §3] we choose a generator $\mathcal{A} \in \mathrm{Br}\, X_k$ such that $\mathcal{A}' := \mathrm{Cores}_{k/\mathbb{Q}}\, \mathcal{A}$ generates $\mathrm{Br}\, X/\mathrm{Br}_0\, X$. Then for a place v

$$\mathrm{inv}_v\, \mathcal{A}' = \sum_{w|v} \mathrm{inv}_w\, \mathcal{A}.$$

Let v be a finite place of $\mathbb{Q}$ and w a place of k above v. Moreover, let w' be a place of K lying above w. Table 4.3.1 describes how to compute $\mathrm{inv}_w\, \mathcal{A}(P_v)$ at a point $P_v \in X(\mathbb{Q}_v) \subseteq X(k_w)$ which is dependent on the decomposition group $G^v = \mathrm{Gal}(K_{w'}/k_w)$.

Here

$$f = \frac{x_0 + \alpha\omega x_1}{x_0 + \alpha\omega^2 x_1} \text{ and } h = \frac{x_2 + \beta\omega x_3}{x_2 + \beta x_3}.$$

$\varepsilon \in K^*, \eta \in K_1^*$ satisfy the following equations

$$\mathrm{N}_{K/L_1}(\varepsilon) = -\mu \text{ and } \eta/{}^r\eta = -\mu/\mathrm{N}_{K/K_1}(\varepsilon). \tag{4.3.8}$$

Furthermore, $\xi(P_v) \in K_{w'}^*$ [CTKS87, p. 39] satisfies

$$(1-t)(\xi(P_v)) = g(P_v)/\varepsilon.$$

Table 4.3.1: *Computing invariant maps of X*

Condition on a_0, a_1, a_2, a_3	Condition on λ, ν	G^v	$\mathrm{inv}_w \mathcal{A}(P_v)$
$a_0/a_1, a_2/a_3 \in \mathbb{Q}_v^{*3}$, or $a_0a_3/a_1a_2, a_0a_2/a_1a_3 \in \mathbb{Q}_v^{*3}$	$\lambda, \nu \in \mathbb{Q}_v^{*3}$	$\langle e \rangle$	$= 0$
$a_0a_3/a_1a_2 \in \mathbb{Q}_v^{*3}$	$\nu \in \mathbb{Q}_v^{*3}$	$\langle t \rangle$	$= 0$
$a_0a_2/a_1a_3 \in \mathbb{Q}_v^{*3}$	$\lambda/\nu \in \mathbb{Q}_v^{*3}$	$\langle r \rangle$	$({}^q\varepsilon/\varepsilon \cdot \eta, \nu)_{\omega,w}$
$a_1/a_0 \in \mathbb{Q}_v^{*3}$	$\lambda \in \mathbb{Q}_v^{*3}$	$\langle s \rangle$	$(f(P_v)/\eta, \nu)_{\omega,w}$
$a_3/a_2 \in \mathbb{Q}_v^{*3}$	$\lambda\nu \in \mathbb{Q}_v^{*3}$	$\langle q \rangle$	$({}^r\varepsilon/\varepsilon \cdot 1/h(P_v) \cdot {}^t\eta, \lambda)_{\omega,w}$
Otherwise	Otherwise	G	$(N_s(\xi(P_v))f(P_v)/\eta, \nu)_{\omega,w}$

Remark 4.3.5. Suppose we choose ε such that $\varepsilon = \varepsilon_\beta \varepsilon_\delta$ where $\varepsilon_\beta \in K_2^*$ and $\varepsilon_\delta \in L_2^*$ then we can choose $\eta = 1$ [CTKS87, p. 30]. If $\varepsilon = \varepsilon'$ or $\varepsilon = 1/\varepsilon''$ where ε' and ε'' are products of integral elements of $\mathbb{Q}(\beta)$ and $\mathbb{Q}(\delta)$ then $\mathrm{inv}_w \mathcal{A} = 0$ for all places w of good reduction on X_k [CTKS87, p. 31].

The last situation is clearly satisfied if $\mu \in \mathrm{N}_{\mathbb{Q}(\beta)/\mathbb{Q}} \mathbb{Q}(\beta)^*$. Equivalently one can show that $\mu \in \mathrm{N}_{k(\beta)_{w'}/k_w} \mathbb{Q}(\beta)^*$ for all $w \in \Omega_k$ and any place w' of $k(\beta)$ lying above w. This local condition is satisfied in the following cases

(i) w is a place of good reduction for X_k;

(ii) ν is a cube in k_w^*;

(iii) λ/ν is a cube in k_w^*, or

(iv) μ/ν is a cube in k_w^*, but $w \neq w_3$ where w_3 is the unique place of k dividing 3.

All these statements can be found in Proposition 4 in [CTKS87].

4.4 A three coefficient family

We shall focus in this section on the affine diagonal cubic surfaces given in (4.1.1) with $a_1 = a_2$. It is convenient for the remainder of the section to set $K = \mathbb{Q}(\omega)$. The data collected here will be used to construct an explicit family of Hasse failures in §4.5, allowing us to prove the lower bound in Theorem 4.1.4. As U is isomorphic over $\mathbb{Q}$ to the surface

$$U: \quad u_1^3 + u_2^3 + (a_3/a_1)u_3^3 = a_0/a_1,$$

we can use the results from §4.2.5 for the Brauer group of U. Since we are no longer concerned with $\mathrm{Br}\, X$, we will abuse notation by using $\mathcal{A}$ and $\mathcal{A}'$ for their images under the natural map $\mathrm{Br}\, X \to \mathrm{Br}\, U$. This should cause no confusion in computing local invariant maps by functoriality. We will be primarily interested in the algebraic Brauer element from Lemma 4.2.14, that is

$$\mathcal{A}' = \mathrm{Cores}_{K/\mathbb{Q}} \mathcal{A} \in \mathrm{Br}\, \mathbb{Q}(U), \text{ where } \quad \mathcal{A} = \left(\frac{a_0}{a_3}, \frac{x_1 + \omega x_2}{x_1 + x_2} \right)_\omega \in \mathrm{Br}\, U_K,$$

which generates $\mathrm{Br}\, U/\mathrm{Br}\, \mathbb{Q}$ by Lemmas 4.2.14 and 4.2.15 unless $a_3/a_1 \equiv 2 \bmod \mathbb{Q}^{*3}$. We need the following lemma in order to evaluate the local invariant map of $\mathcal{A}$.

Lemma 4.4.1. *Let $p \neq 3$ be a finite prime and fix $\mathfrak{p} \mid p$. If σ generates $\mathrm{Gal}(K/\mathbb{Q})$, then*

$$\mathrm{inv}_p \mathcal{A}' = \begin{cases} \mathrm{inv}_{\mathfrak{p}}(1+\sigma)\mathcal{A}, & \text{if } p \equiv 1 \bmod 3, \\ \mathrm{inv}_{\mathfrak{p}} \mathcal{A} & \text{if } p \equiv 2 \bmod 3. \end{cases}$$

Moreover, $(1+\sigma)\mathcal{A}$ can be explicitly expressed as

$$(1+\sigma)\mathcal{A} = \left(\frac{a_0}{a_3}, \frac{u_1+\omega u_2}{u_1+\omega^2 u_2}\right)_\omega .$$

Proof. If $p \equiv 2 \bmod 3$ there is a single prime $\mathfrak{p}$ above p in K and thus the claim follows from [BSD04, Lem. 5.i]. If $p \equiv 1 \bmod 3$, then $(p) = \mathfrak{p}\sigma(\mathfrak{p})$ is split in O_K. It follows from [BSD04, Lem. 5.i] that $\mathrm{inv}_p \mathcal{A}' = \mathrm{inv}_{\mathfrak{p}} \mathcal{A} + \mathrm{inv}_{\sigma(\mathfrak{p})} \mathcal{A}$. It thus suffices to show that $\mathrm{inv}_{\sigma(\mathfrak{p})} \mathcal{A} = \mathrm{inv}_{\mathfrak{p}} \sigma(\mathcal{A})$ since the local invariant map is a homomorphism. To see this, we apply a similar analysis to the one appearing in [GLN22, Lem. 4.2.]. Consider the diagram

$$\begin{array}{ccccc} \mathrm{Br}\, U_K & & \xrightarrow{\quad\sigma\quad} & & \mathrm{Br}\, U_K \\ \downarrow{\scriptstyle x_p} & & & & \downarrow{\scriptstyle x_p} \\ \mathrm{Br}\, K_{\mathfrak{p}} & \xrightarrow[\simeq]{} & \mathrm{Br}\, \mathbb{Q}_p & \xleftarrow[\simeq]{} & \mathrm{Br}\, K_{\sigma(\mathfrak{p})} \\ \downarrow{\scriptstyle \mathrm{inv}_{\mathfrak{p}}} & & \downarrow{\scriptstyle \mathrm{inv}_p} & & \downarrow{\scriptstyle \mathrm{inv}_{\sigma(\mathfrak{p})}} \\ \mathbb{Q}/\mathbb{Z} & \xrightarrow[\mathrm{id}]{} & \mathbb{Q}/\mathbb{Z} & \xleftarrow[\mathrm{id}]{} & \mathbb{Q}/\mathbb{Z} \end{array}$$

As explained in the proof of [GLN22, Lem. 4.2.] it commutes. The commutativity of the top square follows from the definition of the embeddings $K \to K_{\mathfrak{p}}$ and $K \to K_{\sigma(\mathfrak{p})}$, and that of the bottom squares by [Neu13, Prop. II.1.4]. As a conclusion we see that $\mathrm{inv}_{\mathfrak{p}} \mathcal{A} = \mathrm{inv}_{\sigma(\mathfrak{p})} \sigma(\mathcal{A})$ is now implied by chasing the above diagram.

We have so far shown that $\mathrm{inv}_p \mathcal{A}' = \mathrm{inv}_{\sigma(\mathfrak{p})}(1+\sigma)\mathcal{A}$. Since σ generates $\mathrm{Gal}(K/\mathbb{Q})$, we have $\sigma^2 = 1$ and hence $(1+\sigma)\mathcal{A}$ is Galois invariant. Thus $\mathrm{inv}_{\sigma(\mathfrak{p})}(1+\sigma)(\mathcal{A}) = \mathrm{inv}_{\sigma^2(\mathfrak{p})} \sigma(1+\sigma)(\mathcal{A}) = \mathrm{inv}_{\mathfrak{p}}(1+\sigma)(\mathcal{A})$ by the above commutative diagram, which confirms the first part of the statement. Finally, observe that $(1+\sigma)\mathcal{A}$ is given by

$$\begin{aligned} (1+\sigma)(\mathcal{A}) &= \left(\frac{a_0}{a_3}, \frac{u_1+\omega u_2}{u_1+u_2}\right)_\omega + \left(\frac{a_0}{a_3}, \frac{u_1+\omega^2 u_2}{u_1+u_2}\right)_{\omega^2} \\ &= \left(\frac{a_0}{a_3}, \frac{u_1+\omega u_2}{u_1+u_2}\right)_\omega + \left(\frac{a_0}{a_3}, \frac{u_1+u_2}{u_1+\omega^2 u_2}\right)_\omega = \left(\frac{a_0}{a_3}, \frac{u_1+\omega u_2}{u_1+\omega^2 u_2}\right)_\omega . \end{aligned}$$

This completes the proof of Lemma 4.4.1. □

Assume now that $\mathcal{U}(\mathbb{Z}_p) \neq \emptyset$. We proceed by computing the local invariant map of $\mathcal{A}'$ in various cases depending on p and the coefficients of $\mathcal{U}$.

Lemma 4.4.2. *We have* $\mathrm{inv}_\infty \mathcal{A}' = 0$.

Proof. The claim follows from Remark 4.3.4 and the fact that $a_0/a_3 \in \mathbb{R}^{*3}$. □

Lemma 4.4.3. *If $p \neq 3$ and $a_0/a_3 \in \mathbb{Q}_p^{*3}$, then* $\mathrm{inv}_p \mathcal{A}' = 0$.

Proof. This is a consequence of (4.3.1) and Lemma 4.4.1. □

Lemma 4.4.4. *Assume that $p \neq 3$ and $p \nmid a_0 a_1 a_3$. Then* $\mathrm{inv}_p \mathcal{A}' = 0$.

Proof. The condition on a_0, a_1, a_3 implies that a_0/a_3 is a a unit of $\mathbb{Z}_p$. If $a_0/a_3 \in \mathbb{Z}_p^{*3}$, then Lemma 4.4.3 implies that the local invariant map vanishes. Note that this assumption holds to any $p \equiv 2 \bmod 3$. Assume now that $p \equiv 1 \bmod 3$ and $a_0/a_3 \in \mathbb{Z}_p^* \setminus \mathbb{Z}_p^{*3}$. Since $p \nmid a_0a_3$ we have $a_0 - a_3u^3 \neq 0 \bmod p$ and hence none of $u_1 + \omega u_2$ and $u_1 + \omega^2 u_2$ vanishes mod $\mathfrak{p}$ because $\mathcal{U}$ extends to the following two integral models over $K_\mathfrak{p}$:

$$a_1(u_1 + \omega u_2)\big(u_1^2 - \omega u_1 u_2 + \omega^2 u_2^2\big) = a_0 - a_3 u_3^3,$$
$$a_1\big(u_1 + \omega^2 u_2\big)\big(u_1^2 - \omega^2 u_1 u_2 + \omega u_2^2\big) = a_0 - a_3 u_3^3.$$

This shows that both entries of $(1+\sigma)(\mathcal{A})$ are units of $O_\mathfrak{p}$ and proves our claim in view of Lemma 4.4.1 and (4.3.2). □

Lemma 4.4.5. *If* $p \equiv 2 \bmod 3$ *and* $a_0/a_3 \in \mathbb{Z}_p^*$, *then* $\operatorname{inv}_p \mathcal{A}' = 0$.

Proof. The proof follows from the fact that a_0/a_3 is a unit of $\mathbb{Z}_p$ while for $p \equiv 2 \bmod 3$ any unit of $\mathbb{Z}_p$ is a cube. The claim then follows from Lemma 4.4.3. □

Lemma 4.4.6. *Assume that* $(a_0, a_1, a_3) \equiv (2, 8, 5) \bmod 9$. *Then* $\operatorname{inv}_3 \mathcal{A}' = 2/3$.

Proof. Let $\mathfrak{p}$ be the unique prime ideal above 3 in the ring of integers of $\mathbb{Q}_3(\omega)$, it is generated by $\lambda = 1 - \omega$. Our proof rests upon (4.3.6), which confirms the claim of Lemma 4.4.6 provided that

$$\frac{a_0}{a_3} \bmod \mathbb{Z}_3^{*3} \text{ expands as } 1 + b_2\lambda^2 + \dots,$$
$$\frac{u_1 + \omega u_2}{u_1 + u_2} \bmod \mathbb{Z}_3^{*3} \text{ expands as } 1 + c_1\lambda + \dots$$

and $b_2 c_1 \equiv 2 \bmod 3$.

Our assumptions imply that $a_0/a_3 \equiv 4 \bmod 9$ and thus it has expansion $4 = 1 + 3 = 1 - \lambda^2 - \lambda^3 + \dots$ according to (4.3.5). Hence $b_2 = -1$.

The conditions on a_0, a_1, a_3 force any point in $\mathcal{U}(\mathbb{Z}_3)$ to obey either $u_1^3 \equiv u_2^3 \equiv 1 \bmod 9$ or $u_1^3 \equiv u_2^3 \equiv -1 \bmod 9$. If $u_1^3 \equiv u_2^3 \equiv 1 \bmod 9$ or equivalently $u_1 \equiv u_2 \equiv 1 \bmod 3$, we have $u_1 + \omega u_2 = 1 + \omega + 3(k + m\omega)$ for some $k, m \in \mathbb{Z}$. Multiplying both $u_1 + \omega u_2$ and $u_1 + u_2$ by -1, which is a cube of $\mathbb{Z}_3^{*3}$, shows that $-u_1 - \omega u_2 \equiv 1 - 3 + \lambda \equiv 1 + \lambda \bmod \lambda^2$ and $-u_1 - u_2 \equiv 1 \bmod \lambda^2$ by (4.3.5). Finally, multiplying $(u_1 + \omega u_2)/(u_1 + u_2)$ by $(u_1 + u_2)^3$ now confirms that for any choice of $\mathbb{Z}_3$-point with $u_1 \equiv u_2 \equiv 1 \bmod 3$ the coefficients b_2, c_1 above satisfy $b_2 = -1$ and $c_1 = 1$ whose product is congruent to 2 mod 3. The same analysis without the need of multiplying $u_1 + \omega u_2$ and $u_1 + u_2$ by -1 yields an identical conclusion if $u_1^3 \equiv u_2^3 \equiv -1 \bmod 9$ and hence the claim. □

Lemma 4.4.7. *Assume that* $(a_0, a_1, a_3) = (b, a, -2b)$ *and* $2 \nmid ab$. *Then* $\operatorname{inv}_2 \mathcal{A}' \in \{0, 2/3\}$.

Proof. Since $2 \nmid ab$ the reduction of $\mathcal{U}$ mod 2 is given by

$$u_1^3 + u_2^3 \equiv 1 \bmod 2.$$

It is clear that any $\mathbb{Z}_2$-point must obey $u_1 \equiv 0 \bmod 2$ or $u_2 \equiv 0 \bmod 2$ but u_1, u_2 do not vanish mod 2 simultaneously. If $u_2 \equiv 0 \bmod 2$ then $\operatorname{inv}_2 \mathcal{A}' = 0$ as the second entry of $\mathcal{A}'$ becomes 1 mod 2 and thus it is a cube of $\mathbb{Z}_2$. On the other hand, if $u_1 \equiv 0 \bmod 2$, then the reduction of u_2 in $\mathbb{F}_4$ is 1 as the only elements of $\mathbb{F}_4$ that come from reduction of $\mathbb{Z}_2$ elements are 0 and 1. Thus the second entry of $\mathcal{A}'$ is ω and hence $\operatorname{inv}_p \mathcal{A}' = 2/3$ by (4.3.4) confirming the claim in the statement. □

4.5 Lower bound

Keep notation as in Section 4.4. To prove the lower bound of Theorem 4.1.4 we restrict to the subfamily given by

$$U\colon \quad au_1^3 + au_2^3 - 2bu_3^3 = b, \tag{4.5.1}$$

with positive coprime a, b such that $a \equiv 17 \bmod 18$, $b \equiv 11 \bmod 18$ and if $p \mid ab$, then $p \equiv 2 \bmod 3$.

Remark 4.5.1. The compactification of (4.5.1) has the obvious rational point $(0 : 1 : -1 : 0)$. Projective cubic surfaces with a rational point are unirational [Kol02, Thm. 1], thus $U(\mathbb{Q}) \neq \emptyset$.

Local solubility

We claim that $\mathcal{U}(\mathbb{Z}_p) \neq \emptyset$ for all p. It is clear that $\mathcal{U}(\mathbb{R}) \neq \emptyset$. For $p \nmid ab$ such that $p \neq 3, 7$ this follows by setting $u_3 = 0$. Then the reduction of $\mathcal{U}$ mod p is an elliptic curve minus the divisor at ∞ which has a smooth $\mathbb{F}_p$-point by the Hasse-Weil bound and hence it has a $\mathbb{Z}_p$-point by Hensel. If $p \nmid ab$ and $p = 7$ either a/b is a cube mod 7, in which case one may set $u_2 = u_3 = 0$, or $a/b \equiv \pm 2, \pm 3 \bmod 7$, where local solubility once more is easily verified by setting $u_3 = 1$ and looking at

$$\frac{a}{b}\left(u_1^3 + u_2^3\right) \equiv 3 \bmod 7.$$

Finally, local solubility at $p = 3$ is implied by the congruence conditions on a, b as setting $u_2 = -1$ and $u_3 = 0$ reduces the defining equation of $\mathcal{U}$ mod 9 to $u_1^3 \equiv -1 \bmod 9$. A unit of $\mathbb{Z}_3$ is a cube if and only if it is congruent to ± 1 mod 9 and hence the claim of local solubility at 3. If $p \mid ab$, then $p \equiv 2 \bmod 3$ and thus any unit mod p is a cube. As $(a, b) = 1$ and $2 \nmid ab$ this is sufficient to deduce local solubility at such primes.

Values of the local invariant map

We will now show that each $\mathcal{U}$ has a Brauer–Manin obstruction to the integral Hasse principle as the sum of local invariant maps of $\mathcal{A}'$ is never 0. If $p = \infty$ the local invariant map of $\mathcal{A}'$ vanishes by Lemma 4.4.2. The same holds for $p \nmid 2ab$ provided that $p \neq 3$ by Lemma 4.4.4. It also holds to any $p \mid ab$. Indeed, $2 \nmid ab$ and thus -2^{-1}, which is the first entry of $\mathcal{A}'$, is a unit modulo any prime $p \mid ab$. By assumption any such prime satisfies $p \equiv 2 \bmod 3$ and thus Lemma 4.4.5 is applicable. The local invariant map equals $2/3$ at $p = 3$ by Lemma 4.4.6. It remains to show that it does not equal $1/3$ at $p = 2$ which follows from Lemma 4.4.7. This confirms that each $\mathcal{U}$ has a Brauer–Manin obstruction to the integral Hasse principle as the sum of local invariant maps of each adelic point is either $1/3$ or $2/3$.

4.5.1 Lower bound

Let $S(B)$ denote the number of $a, b \leq B$ as in (4.5.1), that is

$$S(B) = \sum_{\substack{a \leq B,\ a \equiv 17 \bmod 18 \\ p \mid a \implies p \equiv 2 \bmod 3}} \sum_{\substack{b \leq B,\ b \equiv 11 \bmod 18 \\ p \mid b \implies p \equiv 2 \bmod 3 \\ (b,a)=1}} 1.$$

We begin by encoding the coprimality condition $(a,b)=1$ using its indicator function $\sum_{d|(a,b)}\mu(d)$. We then use the orthogonality of Dirichlet's characters to see that

$$S(B)=\sum_{\chi,\psi \bmod 18}\frac{\chi(17^{-1})\psi(11^{-1})}{36}\sum_{\substack{d\le B\\ p|d\implies p\equiv 2 \bmod 3}}\mu(d)\chi(d)\psi(d)\sum_{\substack{a,b\le B/d\\ p|ab\implies p\equiv 2 \bmod 3}}\chi(a)\psi(b).$$

Here $11^{-1},17^{-1}$ are the inverses of $11,17 \bmod 18$, respectively.

We split the sum over d in two sums, corresponding to the ranges $1\le d\le B^{1/2}$ and $B^{1/2}<d\le B$. If $B^{1/2}<d\le B$, we may apply the trivial bound $B/d+O(1)$ to the sums over a and over b. Bounding trivially the remaining sums over d then gives

$$S(B)=\sum_{\chi,\psi \bmod 18}\frac{\bar\chi(17)\bar\psi(11)}{36}\sum_{\substack{d\le B^{1/2}\\ p|d\implies p\equiv 2 \bmod 3}}\mu(d)\chi(d)\psi(d)\sum_{\substack{a,b\le B/d\\ p|ab\implies p\equiv 2 \bmod 3}}\chi(a)\psi(b)$$
$$+O\Big(B^{3/2}\Big). \tag{4.5.2}$$

It is clear that the two inside sums over a and over b are identical. For a real $x\ge 1$, let

$$T_\chi(x)=\sum_{\substack{a\le x\\ p|a\implies p\equiv 2 \bmod 3}}\chi(a).$$

Consider the Dirichlet's series $F(s,\chi)$ of this sum. We employ the standard notation $s=\sigma+it$. If $\sigma>1$, the function $F(s,\chi)$ can be written as an Euler product

$$F(s,\chi)=\sum_{p|a\implies p\equiv 2 \bmod 3}\frac{\chi(a)}{a^s}=\prod_{p\equiv 2 \bmod 3}\left(1-\frac{\chi(p)}{p^s}\right)^{-1}.$$

The quadratic character mod 3 may be used to detect congruences of that modulus. Let $\chi_1(\cdot)=\chi(\cdot)\big(\frac{\cdot}{3}\big)$. As $\chi(3)=\chi_1(3)=0$, the binomial series expansion shows that

$$\begin{aligned}F(s,\chi)&=\prod_p\left(1-\frac{1}{2}\Big(1-\Big(\frac{p}{3}\Big)\Big)\frac{\chi(p)}{p^s}\right)^{-1}=\prod_p\left(1-\frac{\chi(p)}{2p^s}+\frac{\chi_1(p)}{2p^s}\right)^{-1}\\&=\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1/2}\left(1-\frac{\chi_1(p)}{p^s}\right)^{1/2}E_p(s)=\frac{L(s,\chi)^{1/2}}{L(s,\chi_1)^{1/2}}E(s).\end{aligned} \tag{4.5.3}$$

Here $E_p(s)=1+O(1/p^{2s})$ and $E(s)=\prod_p E_p(s)$. We conclude that $F(s,\chi)$ admits an analytic continuation to $\sigma>1/2$. Moreover, since we are working with characters of fixed modulus, if χ is non-trivial and if $L(s,\chi_1)$ has a Siegel zero, then for T sufficiently big this zero must lie to the left of $1-1/\log T$. Thus, if χ is a non-trivial character mod 18, the function $F(s,\chi)$ is holomorphic in the region $\sigma\ge 1-1/\log T$ for big T.

Select $T=(\log(B/d))^2$, $\eta_1=1/T$ and $\eta_2=1/(\log T)^2$. Perron's formula [Tit86, p.70, Lem 3.19] now yields

$$T_\chi(B/d)=\frac{1}{2\pi i}\int_{1+\eta_1-iT}^{1+\eta_1+iT}F(s,\chi)\frac{(B/d)^s}{s}ds+O\left(\frac{B}{d\log(B/d)}\right).$$

We move the integration to the contour from $1+\eta_1+iT$ to $1-\eta_2+iT$ on the left, followed by $1-\eta_2+iT$ downwards to $1-\eta_2-iT$ and finally from $1-\eta_2-iT$ to $1+\eta_1-iT$ on the right. If χ is non-trivial, we can apply Cauchy's theorem to see that

$$T_\chi(B/d) = -\frac{1}{2\pi i}\left(\int_{1+\eta_1+iT}^{1-\eta_2+iT} + \int_{1-\eta_2+iT}^{1-\eta_2-iT} + \int_{1-\eta_2-iT}^{1+\eta_1-iT}\right) + O\left(\frac{B}{d\log(B/d)}\right)$$

with $F(s,\chi)\frac{(B/d)^s}{s}$ the underlying function in the three integrals above.

It follows from the triangle inequality that $|F(s,\chi)| \leq |\zeta(s)|$. On the other hand, there exists an absolute positive constant c such that $|\zeta(s)| \ll (\log T)^{c_1}$ for some $c_1 > 0$ in the region $\sigma \geq 1 - c/2\log T$ [MV07, Thm. 6.7, p.174], in which we are provided that T is sufficiently big. Bounding the two integrals over the real segments trivially now gives

$$\int_{1+\eta_1+iT}^{1-\eta_2+iT} + \int_{1-\eta_2-iT}^{1+\eta_1-iT} \ll \frac{(B/d)^{1+\eta_1}(\log T)^{c_1}}{\log(B/d)T} \ll \frac{B}{d\log(B/d)}.$$

Finally, bounding the integral over the imaginary line trivially shows that

$$\int_{1-\eta_2+iT}^{1-\eta_2-iT} \ll \frac{B^{1-\eta_2}(\log T)^{c_1+1}}{d^{1-\eta_2}} \ll \frac{B}{d\log(B/d)}.$$

If χ is non-trivial, we have then verified that

$$T_\chi(B/d) \ll \frac{B}{d\log(B/d)}.$$

On the other hand, if $\chi = \chi_0$ is the trivial character mod 18, we can use a Tauberian theorem [Ten15, Thm. II.7.28] to see that

$$T_{\chi_0}(B/d) = \sum_{\substack{a \leq B/d \\ p|a \implies p\equiv 5 \bmod 6}} 1 \asymp \frac{B}{d(\log B/d)^{1/2}}.$$

Substituting the above estimates for $T_\chi(B/d)$ and $T_\psi(B/d)$ in (4.5.2) shows that

$$S(B) \asymp B^2 \sum_{\substack{d \leq B^{1/2} \\ p|d \implies p\equiv 5 \bmod 6}} \frac{\mu(d)}{d^2}\frac{1}{\log(B/d)}.$$

It remains to observe that if $1 \leq d \leq B^{1/2}$, we have $1/\log(B/d) \gg 1/\log B$ and therefore

$$S(B) \gg \frac{B^2}{\log B} \sum_{\substack{d \leq B^{1/2} \\ p|d \implies p\equiv 5 \bmod 6}} \frac{\mu(d)}{d^2}.$$

The remaining sum over d is clearly convergent. Completing that sum gives $S(B) \gg B^2/\log B$, which verifies the claimed lower bound in Theorem 4.1.4. $\square$

4.6 Upper bounds

We begin with an estimate of the number of surfaces U as defined in (4.1.1) with a non-trivial transcendental Brauer group. Let

$$M_{a_0}^{\mathrm{Tr}}(B) = \#\big\{(a_1,a_2,a_3) \in [-B,B]^3 \cap \mathbb{Z}^3_{\mathrm{prim}} \ : \ \operatorname{Br} U/\operatorname{Br}_1 U \text{ non-trivial}\big\},$$
$$M^{\mathrm{Tr}}(B) = \#\left\{(a_0,a_1,a_2,a_3) \in [-B,B]^4 \cap \mathbb{Z}^4_{\mathrm{prim}} \ : \ \begin{array}{c}(a_1,a_2,a_3)=1,\\ \operatorname{Br} U/\operatorname{Br}_1 U \text{ non-trivial}\end{array}\right\}.$$

Proposition 4.6.1. *We have*

$$M_{a_0}^{\mathrm{Tr}}(B) \asymp B(\log B)^6 \quad \textit{and} \quad M^{\mathrm{Tr}}(B) \asymp B^2(\log B)^6.$$

Proof. It follows from Theorem 4.1.1 that U has a non-trivial transcendental Brauer group if and only if $a_1a_2a_3 \equiv 2 \bmod \mathbb{Q}^{*3}$. We will prove the result in the statement for $M_{a_0}^{\mathrm{Tr}}(B)$, as the one for $M^{\mathrm{Tr}}(B)$ immediately follows from it. Indeed,

$$M^{\mathrm{Tr}}(B) = M_{a_0}^{\mathrm{Tr}}(B) \sum_{a_0 \le B} 1 = BM_{a_0}^{\mathrm{Tr}}(B) + O\big(M_{a_0}^{\mathrm{Tr}}(B)\big).$$

Note that the number of $a_1,a_2,a_3 \le B$ whose product $a_1a_2a_3 \in 2\mathbb{Q}^{*3}$ is of the same magnitude as the number of those $a_1,a_2,a_3 \le B$ whose product is a cube. This follows, for example, from the fact that in the former problem at least one of a_1,a_2,a_3 has to be even. Thus any solution to $a_1a_2a_3 = 2n^3$ must come from a solution of $a_1a_2a_3 = n^3$ by multiplying one of the latter a_i by 2. But each solution of of $a_1a_2a_3 = n^3$ produces only finitely many solutions to $a_1a_2a_3 = 2n^3$ via doubling a coordinate and a permutation.

The signs of the a_i are immaterial for the order of magnitude of $M_{a_0}^{\mathrm{Tr}}(B)$, as they only change the leading constant in the asymptotic. It thus suffices to count $0 < a_1,a_2,a_3 \le B$ with $\gcd(a_1,a_2,a_3) = 1$ and $a_1a_2a_3 \in \mathbb{Q}^{*3}$. We thus have

$$M_{a_0}^{\mathrm{Tr}}(B) \asymp \sum_{\substack{a_1,a_2,a_3 \le B \\ (a_1,a_2,a_3)=1,\ a_1a_2a_3 \in \mathbb{Q}^{*3}}} 1 = \sum_{\substack{0<a_1,a_2,a_3 \le B \\ (a_1,a_2,a_3)=1,\ a_1a_2a_3 = n^3}} 1.$$

As explained in [HBM99], a condition $n \le B$ is redundant in the above problem. Then

$$M_{a_0}^{\mathrm{Tr}}(B) \asymp \sum_{\substack{0<a_1,a_2,a_3,n \le B \\ (a_1,a_2,a_3)=1,\ a_1a_2a_3 = n^3}} 1.$$

The last sum has been investigated in the main result of [HBM99], and it is asymptotically a constant times $B(\log B)^6$. This proves the claim in the statement of the proposition. □

We will also need the following two simple lemmas that will be applied at several instances in the study of the upper bounds considered here.

Lemma 4.6.2. *Let $X, Y, Z \ge 1$ be real numbers. Then*

$$\sum_{k\ell \le X,\ km \le Y,\ \ell m \le Z} 1 \ll (XYZ)^{1/2}.$$

Proof. Without loss of generality we may assume that $X \geq Y \geq Z$. Let S denote the triple sum in the statement. Treating the sum over m first shows that

$$S = \sum_{k\ell \leq X,\ k \leq Y,\ \ell \leq Z} \left(\min\left\{ \frac{Y}{k}, \frac{Z}{\ell} \right\} + O(1) \right). \tag{4.6.1}$$

Let R be the sum corresponding to the error term above. Summing over one of k, ℓ first and then over the other gives

$$R = \begin{cases} X \log(YZ/X) + O(X) & \text{if } X < YZ, \\ YZ + O(Y) & \text{if } X \geq YZ. \end{cases}$$

As $\log(x) \leq x^{1/2}$ whenever $x > 0$, we see that in both cases $R \ll (XYZ)^{1/2}$.

We then consider separately the two contributions coming from the minimum, depending on if $Y/k < Z/\ell$ or vice versa. By doing so we get two identical sums S_1 and S_2 with arguments $1/k$ and $1/\ell$, respectively. We analyse S_1, the argument for S_2 being the same.

$$\begin{aligned} S_1 &= Y \sum_{Y/Z < k \leq Y} \frac{1}{k} \sum_{\ell \leq \min\{kZ/Y, X/k\}} 1 = Y \sum_{Y/Z < k \leq Y} \frac{1}{k} \left(\min\left\{ \frac{kZ}{Y}, \frac{X}{k} \right\} + O(1) \right) \\ &= Z \sum_{Y/Z < k \leq \sqrt{XY/Z}} 1 + XY \sum_{\sqrt{XY/Z} < k \leq Y} \frac{1}{k^2} + O(Y \log Z), \end{aligned}$$

as $\sum_{k \leq x} 1/k = \log x + O(1)$. The asymptotic formula for the first sum on the last line is $(XYZ)^{1/2} + O(Y)$, while the second sum is non-empty only when $X < YZ$ and it is convergent in that case. By completing it we produce an error term of size $O(X)$, which in view of $X < YZ$ is $O((XYZ)^{1/2})$. At the same time the completed sum equals $(XYZ)^{1/2} + O(Z)$. This altogether gives

$$S_1 \ll (XYZ)^{1/2} + Y \log Z.$$

All error terms so far are $O((XYZ)^{1/2})$ as $\log Z \ll Z^{1/2} \ll (XZ/Y)^{1/2}$. The same argument with Z and Y swapped applies to S_2, yielding

$$S \ll (XYZ)^{1/2},$$

since $S = S_1 + S_2 + O(R)$ by (4.6.1). This completes the proof. □

A careful reading of the proof by chasing the error terms shows that, in fact, the upper bound presented here is attained by S. This is particularly visible if $X = YZ$, where the sum obtained from the summands with $m = 1$ is asymptotically $YZ = (XYZ)^{1/2}$.

Lemma 4.6.3. *Let $X, Y, Z, W \geq 1$ be real numbers. The following holds.*

$$\sum_{\substack{k\ell m \leq X,\ kns \leq Y \\ \ell nt \leq Z,\ mst \leq W}} 1 \ll (XYZW)^{1/2} (\log(XYZW))^2.$$

Proof. Write the sextuple sum in the statement as

$$\sum_{k\ell m \leq X} \sum_{ns \leq Y/k,\ nt \leq Z/\ell,\ st \leq W/m} 1.$$

Lemma 4.6.3 is now a straightforward application of Lemma 4.6.2 and $\sum_{r \leq X} \tau_3(r)/r^{1/2} \ll X^{1/2}(\log X)^2$, which follows from example from [SMC95, II.12] combined with partial summation, where $\tau_3(n)$ denotes the 3-divisor function. □

4.6.1 Upper bounds

We are now in position to prove the upper bounds in this article. Propositions 4.3.2 and 4.3.3 show that, under the arithmetic conditions given there, the Brauer–Manin set obtained from algebraic Brauer elements is non-empty but at the same time it is strictly smaller that the integral adelic set $\mathcal{U}(\mathbb{A}_\mathbb{Z})$. In view of Remark 4.3.4, the local invariant map at ∞ of any Brauer element is constant, as every real number is a cube. Thus the projection to the finite adeles $\prod_{p\neq\infty}\mathcal{U}(\mathbb{Z}_p)$ preserves strict inclusions. Any surface satisfying the arithmetic conditions of Propositions 4.3.2 and 4.3.3 then fails integral strong approximation off ∞ and at the same time has no algebraic Brauer–Manin obstruction to the integral Hasse principle.

Therefore, in view of Proposition 4.6.1 it suffices to count surfaces failing the arithmetic conditions on a_0, a_1, a_2, a_3 given in Propositions 4.3.2 and 4.3.3. This will give upper bounds for the number of surfaces in the family which have a Brauer–Manin obstruction to the integral Hasse principle or which satisfy strong approximation off ∞. We proceed with establishing the upper bounds in Theorems 4.1.4, 4.1.3 and 4.1.2 in this order.

Proof of Theorem 4.1.4. Propositions 4.3.2 and 4.3.3 imply that if the local invariant map does not surject at a given prime, then the coefficients of $\mathcal{U}$ must factorise as

$$\begin{aligned} a_0 &= \pm r_0 b_0^3 u_{01}u_{02}u_{03}v_{01}v_{10}^2v_{02}v_{20}^2v_{03}v_{30}^2,\\ a_1 &= \pm r_1 b_1^3 u_{01}u_{12}u_{13}v_{10}v_{01}^2v_{12}v_{21}^2v_{13}v_{31}^2,\\ a_2 &= \pm r_2 b_2^3 u_{02}u_{12}u_{23}v_{20}v_{02}^2v_{21}v_{12}^2v_{23}v_{32}^2,\\ a_3 &= \pm r_3 b_3^3 u_{03}u_{13}u_{23}v_{30}v_{03}^2v_{31}v_{13}^2v_{32}v_{23}^2. \end{aligned} \tag{4.6.2}$$

Here all variables on the right hand side are positive integers such that if $p \mid r_i$, then $p = 3$ for $i = 0$ and $p < 17$ for $i = 1, 2, 3$ while if $p \mid a_i/r_i$ then $p \neq 3$ for $i = 0$ and $p \geq 17$ for $i = 1, 2, 3$. Moreover, $a_i/(r_i b_i^3)$ are cube-free and u_{ij}, v_{ij}, v_{ji} are squarefree for any $i, j \in \{0, 1, 2, 3\}$.

The definition of r_i implies that for any $\alpha > 0$, we have

$$\sum_{r_i\leq X}\frac{1}{r_i^\alpha} \ll 1, \quad i = 0, 1, 2, 3. \tag{4.6.3}$$

Indeed, let $\varepsilon(n)$ be the indicator function for the set $\{n \in \mathbb{Z}_{\geq 0} \,:\, p \mid n \implies p < 17\}$. This function is non-negative and multiplicative and hence

$$\sum_{n\leq X}\frac{\varepsilon(n)}{n^\alpha} \ll \sum_{n=1}^{\infty}\frac{\varepsilon(n)}{n^\alpha} = \prod_{p<17}\left(1-\frac{1}{p^\alpha}\right)^{-1} \ll 1.$$

Since the signs of a_i are immaterial to the count, assuming that all of them are positive only changes $N(B)$ or $N'(B)$ by a constant. Forgetting $(a_1, a_2, a_3) = 1$ now gives

$$N(B) \ll M^{\mathrm{Tr}}(B) + \sum_{a_0,a_1,a_2,a_3\leq B}^{\star} 1 \quad\text{and}\quad N'(B) \ll \sum_{a_0,a_1,a_2,a_3\leq B}^{\star} 1,$$

where the $*$ in the sum above means that the a_i counted satisfy (4.6.2). We separate the sum over u_{ij}, which allows us to apply Lemma 4.6.3 in the summation range

$$u_{01}u_{02}u_{03} \leq \frac{B}{r_0 b_0^3 v_{01}v_{10}^2v_{02}v_{20}^2v_{03}v_{30}^2}, \quad u_{01}u_{12}u_{13} \leq \frac{B}{r_1 b_1^3 v_{10}v_{01}^2v_{12}v_{21}^2v_{13}v_{31}^2},$$
$$u_{02}u_{12}u_{23} \leq \frac{B}{r_2 b_2^3 v_{20}v_{02}^2v_{21}v_{12}^2v_{23}v_{32}^2}, \quad u_{03}u_{13}u_{23} \leq \frac{B}{r_3 b_3^3 v_{30}v_{03}^2v_{31}v_{13}^2v_{32}v_{23}^2}.$$

By doing so and by bounding all logarithms trivially, we get

$$\sum_{a_0,a_1,a_2,a_3\le B}^{\star} 1 \ll B^2(\log B)^2 \sum \frac{1}{\prod_{i=0}^3 r_i^{1/2} b_i^{3/2} \prod_{j\neq k} v_{jk}^{3/2}}.$$

The sums over b_i and v_{jk} are convergent and we may ignore them. We then deduce from (4.6.3) that the above sum is $O(1)$ and by Proposition 4.6.1

$$N(B) \ll B^2(\log B)^6 \quad \text{and} \quad N^{'}(B) \ll B^2(\log B)^2.$$

This completes the proof of Theorem 4.1.4. □

Proof of Theorem 4.1.3. The proof is very similar to the one of the upper bound in Theorem 4.1.4. If $\mathcal{U}$ is counted in $N_{a_0}(B)$ or in $N^{'}_{a_0}(B)$, then Propositions 4.3.2 and 4.3.3 imply that

$$\begin{aligned} a_1 &= \pm r_1 b_1^3 u_{12} u_{13} v_{12} v_{21}^2 v_{13} v_{31}^2, \\ a_2 &= \pm r_2 b_2^3 u_{12} u_{23} v_{21} v_{12}^2 v_{23} v_{32}^2, \\ a_3 &= \pm r_3 b_3^3 u_{13} u_{23} v_{31} v_{13}^2 v_{32} v_{23}^2, \end{aligned}$$

where $r_i, b_i > 0$ and if $p \mid r_i$, then either $p \mid a_0$ or $p < 17$, the u and v variables are all positive, squarefree and $(a_1, a_2, a_3) = 1$. Recalling Proposition 4.6.1, ignoring signs and the coprimality condition and then summing over u_{12}, u_{13}, u_{23} in view of Lemma 4.6.2 gives

$$N_{a_0}(B),\ N^{'}_{a_0}(B) \ll B(\log B)^6 + B^{3/2} \sum_{r_i, b_i, v_{jk}} \frac{1}{(r_1r_2r_3)^{1/2}(b_1b_2b_3)^{3/2} \prod_{j,k} v_{j,k}^{3/2}}.$$

Once more, we may ignore the sums over b_i and v_{jk} as they are convergent. We them apply (4.6.3) to see that

$$N_{a_0}(B),\ N^{'}_{a_0}(B) \ll B^{3/2}.$$

This verifies the claimed upper bounds in Theorem 4.1.3. □

Proof of Theorem 4.1.2. As $a_1a_2a_3 \not\equiv 2 \bmod \mathbb{Q}^{*3}$, the transcendental part of $\operatorname{Br} U$ is trivial by Theorem 4.1.1. Similarly to the other upper bounds, $N_{a_1,a_2,a_3}(B)$ and $N_{a_1,a_2,a_3}(B)$ then only count those $\mathcal{U}$ for which

$$a_0 = \pm u b^3,$$

where if $p \mid u$, then $p \mid 3a_1a_2a_3$. Summing over b first and then over u, while taking into account the definition of u and the analogue of (4.6.3) for it, gives

$$N_{a_1,a_2,a_3}(B),\ N^{'}_{a_1,a_2,a_3}(B) \ll B^{1/3} \sum_{u\le B} \frac{\mathbf{1}_{p\mid u \implies p\mid 3a_1a_2a_3}}{u^{1/3}} \ll_{a_1a_2a_3} B^{1/3}.$$

This completes the proof of Theorem 4.1.2. □

4.7 Examples of Brauer–Manin obstructions

The family of all affine diagonal cubic surfaces does not have a uniform generator of the Brauer group [Uem14, Thm. 2]. Getting a sharp lower bound on how many surfaces in this family have a Brauer–Manin obstruction thus requires a different approach than the one in Section 4.3. To get around the issue of lacking a uniform generator, we showcase the results from Section 4.3 by giving instances of Brauer–Manin obstructions to the integral Hasse principle and to strong approximation without the need of having explicit representatives of Brauer elements. However, applying the method outlined here to the counting results is challenging as it requires controlling uniformly the parameters ε and η, introduced in 4.3.

4.7.1 An interesting family

Definition 4.7.1. We define $\mathcal{X}_{\ell,p,q} \subseteq \mathbb{P}^3_{\mathbb{Z}}$ by

$$x_0^3 + \ell x_1^3 + pqx_2^3 + q\ell^2 x_3^3 = 0,$$

where

- ℓ, p and q are distinct primes,
- $q \equiv 8 \mod 9$,
- $\{p + 9\mathbb{Z}, \ell + 9\mathbb{Z}\} = \{2 + 9\mathbb{Z}, 5 + 9\mathbb{Z}\}$.

We define $\mathcal{U}_{\ell,p,q} \subseteq \mathbb{P}^3_{\mathbb{Z}}$ as the complement of the curve $x_1 = 0$. We will write $S = \{3, \ell, p, q\}$ for the primes of bad reduction on $\mathcal{X}_{\ell,p,q}$.

In accordance with the notation of §4.3.3, we will consistently write $\lambda = \ell$, $\mu = pq$ and $\nu = \ell/p$. Furthermore, recall that $\beta^3 = \lambda\nu = \ell^2/p$.

Note that $\mathcal{X}_{\ell,p,q}$ is k_v-rational for $v \in \{3, \ell, p\}$, or equivalently, there is a cross ratio of the four coefficients which is a cube in k_v. Indeed, $\ell \cdot q\ell^2/pq \in \mathbb{Z}_\ell$ and $\ell \cdot pq/q\ell^2 \in \mathbb{Z}_q$ are cubes, since their valuations are multiples of 3 and $\ell \equiv q \equiv 2 \mod 3$. Rationality at 3 follows from $(q\ell^2 \cdot pq)/\ell \equiv p\ell \equiv 1 \mod 9$.

The following two results show that the surfaces $\mathcal{U}_{\ell,p,q}$ are everywhere locally soluble and always have an obstruction to strong/weak approximation. Moreover, $\mathcal{U}_{\ell,p,q}$ should have a rational point (see Remark 4.7.4), but there may be an integral Brauer–Manin obstruction to the integral Hasse principle on a different choice of integral model.

Lemma 4.7.2. *The affine cubic $\mathcal{U}_{\ell,p,q}$ is everywhere integrally locally soluble.*

Proof. Let $\mathcal{X} := \mathcal{X}_{\ell,p,q}$ and $\mathcal{U} := \mathcal{X} \setminus \mathcal{C}$ where $\mathcal{C}$ is the curve given by $x_1 = 0$ on $\mathcal{X}$. Local solubility at the archimedean place is immediate. For $v \notin \{3, \ell, p, q\}$ $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{C}$ have good reduction. The Hasse–Weil bound for genus 1 curves and Weil's Theorem [Man86, Thm. 27.1], for smooth cubic surfaces gives

$$\#\mathcal{X}(\mathbb{F}_v) \geq q_v^2 - 2q_v + 1 \quad \text{and} \quad \#\mathcal{C}(\mathbb{F}_v) \leq q_v + 1 + 2\sqrt{q_v}.$$

We find $\mathcal{U}(\mathbb{F}_v) > 0$ for $q \geq 5$ and by Hensel's lemma we can lift such a $\mathbb{F}_v$-point to a $\mathbb{Z}_v$-point. If $q_v \equiv 2 \mod 3$ we have that any unit is a cube. This implies for $v \neq \ell$ we have $(-\sqrt[3]{\ell}, 1, 0, 0) \in \mathcal{U}(\mathbb{Z}_v)$. For $v = \ell$ it shows we can lift $(\sqrt[3]{pq}, -1, 0) \in \mathcal{U}(\mathbb{F}_\ell)$ to a $\mathbb{Z}_\ell$-point.

We are left with $v = 3$. We have either $p \equiv 2 \mod 9$ and $\ell \equiv 5 \mod 9$, or $p \equiv 5 \mod 9$ and $\ell \equiv 2 \mod 9$, in both cases we can lift the point $(1, -1, 1) \in \mathcal{U}(\mathbb{Z}/9\mathbb{Z})$ to a $\mathbb{Z}_3$-point. ☐

Proposition 4.7.3. *We have* $\operatorname{Br} U_{\ell,p,q}/\operatorname{Br}\mathbb{Q} \cong \mathbb{Z}/3\mathbb{Z}$. *Moreover, for a generator* $\mathcal{A}$ *the following holds*

$$\operatorname{inv}_v \mathcal{A}' \colon \begin{cases} U_{\ell,p,q}(\mathbb{Q}_v) \to \frac{1}{3}\mathbb{Z}/\mathbb{Z} \text{ is constant} & \text{if } v \neq p, \\ \mathcal{U}_{\ell,p,q}(\mathbb{Z}_v) \to \frac{1}{3}\mathbb{Z}/\mathbb{Z} \text{ is surjective} & \text{if } v = p. \end{cases}$$

Proof. The first statement follows from Proposition 4.2.1(2).

Using Proposition 4.3.2 we see that

$$\operatorname{inv}_v \mathcal{A}' \colon \mathcal{U}_{\ell,p,q}(\mathbb{Z}_p) \to \frac{1}{3}\mathbb{Z}/\mathbb{Z}$$

is surjective. This proves the claim for $v = p$.

For the remaining primes v we have that $v \notin S$ is a prime of good reduction or $\mathcal{X}_{\ell,p,q}$ is k_v-rational. In those cases the invariant map is constant, see for example [CTKS87, Lem. 5]. □

Remark 4.7.4. For $X_{\ell,p,q} := \mathcal{X}_{\ell,p,q} \times \mathbb{Q}$ we have $X_{\ell,p,q}(\mathbb{A}_\mathbb{Q})^{\mathrm{Br}} \neq \emptyset$ [CTKS87, §5, Prop. 2] and under a conjecture by Colliot-Thélène [CTS21, Conj. 14.1.12] we have $X_{\ell,p,q}(\mathbb{Q}) \neq \emptyset$. As $X_{\ell,p,q}$ is a smooth cubic surface $X_{\ell,p,q}(\mathbb{Q})$ is dense in $X_{\ell,p,q}$, hence the affine surface $U_{\ell,p,q} := \mathcal{U}_{\ell,p,q} \times \mathbb{Q}$ always has $U_{\ell,p,q}(\mathbb{Q}) \neq \emptyset$. For example, specialising $(\ell, p, q) = (2, 5, 17)$ we can find the rational point $(-1/2, 1/2, -1/2) \in U_{2,5,17}(\mathbb{Q})$. Moreover, we see in Proposition 4.7.12 that a different choice of integral model for $U_{2,5,17}$ has an integral Brauer–Manin obstruction. Note from Proposition 4.7.3, we see that the invariant map is surjective at $v = p$, hence there is a Brauer–Manin obstruction to strong/weak approximation for all $U_{\ell,p,q}$.

The exact value of $\operatorname{inv}_v \mathcal{A}'$ for $v \neq p$ depends on the choice of normalisation of $\mathcal{A}'$. To simplify the computation of the Brauer–Manin obstruction we make the following convenient choice by taking an $\varepsilon \in \mathbb{Q}(\beta)$ satisfying (4.3.8) where ε a priori only lies in $\mathbb{Q}(\omega, \alpha, \gamma)$.

Lemma 4.7.5. *There exists an* $\varepsilon \in \mathbb{Q}(\beta)$ *such that* $\mathrm{N}_{\mathbb{Q}(\beta)/\mathbb{Q}}\, \varepsilon = \mu$.

Proof. By Remark 4.3.5 we need to show that $\mu \in k$ is a local norm at all places $w \in \Omega_k$. Let w_3, w_ℓ, w_p, w_q be the unique places of k above the indicated rational primes. Case (i) of Remark 4.3.5 deals with the places $w \notin \{w_3, w_\ell, w_p, w_q\}$. For the places $w \in \{w_\ell, w_q\}$ we use Case (ii) of the same remark. For the place $w = w_p$ we note that $\ell, q \in k_w$ are cubes, and $\ell^2/p = \beta^3$ is a norm. Hence pq is also a norm. By the reciprocity law it follows that μ is also a local norm at $w = w_3$. □

To obtain a counterexample to the Hasse principle, we would want $\operatorname{inv}_p \colon \mathcal{U}_{\ell,p,q}(\mathbb{Z}_p) \to \frac{1}{3}\mathbb{Z}/\mathbb{Z}$ to no longer be surjective. We will show that it at least assumes a single value on a large subset of these points.

Proposition 4.7.6. *For a point* $P_p = [x_0 : x_1 : x_2 : x_3] \in \mathcal{X}_{\ell,p,q}(\mathbb{Z}_p)$ *such that* $x_0 = 0 \mod p$ *and* $x_1 \not\equiv 0 \mod p$, *we have*

$$\operatorname{inv}_p \mathcal{A}'(P_p) = \begin{cases} 2/3 & \text{if } p \equiv 2 \mod 9; \\ 1/3 & \text{if } p \equiv 5 \mod 9. \end{cases}$$

Proof. Let w_p be the unique prime of k above p. As $\nu = \ell/p$ and $\lambda/\nu = p$ are not cubes in $\mathbb{Q}_p$ but $\lambda = \ell \in \mathbb{Q}^{*3}$, we fall into the case $G^{\langle p \rangle} = \langle s \rangle$ from Table 4.3.1. Hence, we can compute $\operatorname{inv}_p \mathcal{A}'(P_p) = \operatorname{inv}_{w_p} \mathcal{A}(P_p)$ as

$$\left(\frac{x_0 + \alpha\omega x_1}{x_0 + \alpha\omega^2 x_1}, \ell/p\right)_{\omega, w_p} = \left(\frac{\alpha\omega}{\alpha\omega^2}, 1/p\right)_{\omega, w_p} = (\omega, p)_{\omega, w_p} = -\frac{p^2-1}{9} \in \frac{1}{3}\mathbb{Z}/\mathbb{Z}. \qquad \square$$

This shows that $\sum_v \operatorname{inv}_v \mathcal{A}'$ is constant on $\mathcal{U}'(\mathbb{A}_\mathbb{Z})$ where $\mathcal{U}'$ is given by

$$\mathcal{U}'\colon \quad p^3u_1^3 + pqu_2^3 + q\ell^2u_3^3 = \ell.$$

To determine whether there is a Brauer–Manin obstruction, that is $\sum_v \operatorname{inv}_v \mathcal{A}' \neq 0$, we will need to determine the constant value of $\operatorname{inv}_v \mathcal{A}'$ for all v in the set of bad primes for X and $\mathcal{A}'$,

$$S_{\mathcal{A}'} := \{v\colon v \mid w \text{ and } w(\varepsilon) \neq 0\} \cup S.$$

By Remark 4.3.5 we have $\operatorname{inv}_v \mathcal{A}'$ is identically 0 for all other primes.

In the next section we will construct a family where we have $S_{\mathcal{A}'} = S$, and hence $\operatorname{inv}_v \mathcal{A}'$ is identically zero for $v \notin \{3, \ell, p, q\}$.

4.7.2 A family of counterexamples to the Hasse principle

The places where ε is not integral might still show up in the Brauer–Manin obstruction. However, in some cases we can ensure that there are no such primes.

Lemma 4.7.7. *Restrict to $\ell = 2$ and $p = 5$ and consider the surface $\mathcal{U}_{2,5,q}$. There exists an $\varepsilon \in \mathcal{O}_{\mathbb{Q}(\beta)}$ for which $\mathrm{N}_{\mathbb{Q}(\beta)/\mathbb{Q}}(\varepsilon) = -\mu$. In particular, such an ε is a unit away from 5 and q.*

Proof. Let us write $M_2 = \mathbb{Q}(\beta)$ where $\beta^3 = 4/5$. In $\mathcal{O}_{M_2}$ we can see that both (5) and (q) are divisible by prime ideals $\mathfrak{p}_5$ and $\mathfrak{p}_q$ of inertia degree 1. Since $\mathrm{Cl}(M_2) = \{0\}$ we see that the ideal $\mathfrak{p}_5\mathfrak{p}_q$ is generated by an element $\varepsilon' \in \mathcal{O}_{M_2}$. By definition of $\mathfrak{p}_5$ and $\mathfrak{p}_q$ we see that $\mathrm{Norm}_{M_2/\mathbb{Q}}((\varepsilon')) = (5q)$. Hence $\mathrm{Norm}(\varepsilon') = 5qu$ for some unit $u \in \mathbb{Z}^*$. So hence either ε' or $-\varepsilon'$ is the ε we are looking for. $\square$

Remark 4.7.8. Note that β itself is not integral; a basis for the free $\mathbb{Z}$-module $\mathcal{O}_{M_2}$ is for example given by $1, 5\beta, \frac{5\beta^2+10\beta+2}{6}$. So we can always write $\varepsilon = \frac{a_2\beta^2+a_1\beta+a_0}{6}$ with $a_i \in \mathbb{Z}$.

For this reason we restrict to $\ell = 2$ and $p = 5$, but many other families could be considered. Hence, we now consider the cubic surface

$$\mathcal{X} = \mathcal{X}_{2,5,q}\colon x_0^3 + 2x_1^3 + 5qx_2^3 + 4qx_3^3 = 0,$$

where q is a prime that is $q \equiv 8 \mod 9$. Using the notation from Subsection 4.3.3 we have $\lambda = 2, \mu = 5q, \nu = 2/5$ and $\beta^3 = 4/5$. It follows from Lemma 4.7.2 that $\mathcal{X}(\mathbb{A}_\mathbb{Z}) \neq \emptyset$.

Proposition 4.7.9. *We have that the invariant maps for $\mathcal{U}_{2,5,q}$ at the places $v \neq p$ are constant and satisfy*

$$\operatorname{inv}_v \mathcal{A}' = \begin{cases} 0 & \text{if } v \neq 3, p; \\ 1/3 & \text{if } v = 3. \end{cases}$$

Proof. We already saw in Proposition 4.7.3 that all invariant maps apart from $v = p$ are constant.

For the primes of good reduction $v \notin S$ we refer to [CTKS87, p. 31]. For the three remaining primes we compute the invariant map at a single point using Table 4.3.1.

<u>$v = 3$</u>

Note that $\lambda, \nu, \lambda/\nu$ are not cubes in $\mathbb{Q}_3$ but $\lambda\nu = 4/5$ is a cube in $\mathbb{Q}_3$, thus we fall into the case $G^{\langle 3 \rangle} = \langle q \rangle$. Since $\varepsilon \in \mathbb{Q}(\beta)$ is a 2-unit we have for all $P_3 \in X(\mathbb{Q}_3)$

$$\operatorname{inv}_3 \mathcal{A}'(P_3) = \operatorname{inv}_{w_3} \mathcal{A}(P_3) = (r(\varepsilon)/\varepsilon, 2)_{\omega,w_3} - (h(P_3), 2)_{\omega,w_3} = -(h(P_3), 2)_{\omega,w_3},$$

where w_3 is the unique place of k dividing 3. As inv_3 is constant on $X(\mathbb{Q}_3)$, it is sufficient to compute the invariant map at one point $P_3 := [-\sqrt[3]{q} : 0 : 1 : -1] \in X(\mathbb{Q}_3)$. This gives

$$(h(P_3), 2)_{\omega,w_3} = \left(\frac{1-\omega\beta}{1-\beta}, 2\right)_{\omega,w_3} = 2/3.$$

<u>$v = \ell = 2$</u>

Let w_2 be the one place of k above 2. Note that λ and ν are not cubes in $\mathbb{Q}_2$ but $\lambda/\nu = 5$ is a cube in $\mathbb{Q}_2$, thus we fall into the case $G^{\langle 2 \rangle} = \langle r \rangle$. By Lemma 4.7.7, $\varepsilon \in \mathbb{Q}(\beta)$, hence $q(\varepsilon) = \varepsilon$ and for all $P_2 \in X(\mathbb{Q}_2)$

$$\operatorname{inv}_2 \mathcal{A}'(P_2) = (q(\varepsilon)/\varepsilon, 2/5)_{\omega,w_2} = (1, 2/5)_{\omega,w_2} = 0.$$

<u>$v = q$</u>

As ν is a cube in $\mathbb{Q}_q$ we fall into the case $G^{\langle q \rangle} = \langle t \rangle$, hence $\operatorname{inv}_q = 0$. □

Proposition 4.7.10. *Let $P_5 = [x_0 : x_1 : x_2 : x_3] \in \mathcal{X}(\mathbb{Z}_5)$ such that $x_0 = 0 \mod 5$ and $x_1 \not\equiv 0 \mod 5$, then*

$$\operatorname{inv}_5 \mathcal{A}'(P_5) = 1/3.$$

Proof. This is precisely Proposition 4.7.6 for the case $\ell = 2$ and $p = 5$. □

Corollary 4.7.11. *Let $(P_v)_{v\in\Omega_\mathbb{Q}} \in \mathcal{X}(\mathbb{A}_\mathbb{Z})$ such that $P_5 = [x_0 : x_1 : x_2 : x_3]$ where $x_0 \equiv 0$ mod 5 and $x_1 \not\equiv 0$ mod 5. Then*

$$\sum_{v\in\Omega_\mathbb{Q}} \operatorname{inv}_v \mathcal{A}'(P_v) = 2/3.$$

Proof. As $\mathcal{A}'$ is of order 3 in $\operatorname{Br} X$ we have $\operatorname{inv}_\infty \mathcal{A}'$ is constant as $\operatorname{Br} \mathbb{R} \cong \mathbb{Z}/2\mathbb{Z}$. By our choice of normalisation we have $\operatorname{inv}_\infty \mathcal{A}'(P_\infty) = 0$ for all $P_\infty \in X(\mathbb{R})$. Then by Propositions 4.7.9 and 4.7.10 we have

$$\sum_{v\in\Omega_\mathbb{Q}} \operatorname{inv}_v \mathcal{A}'(P_v) = 2/3.$$ □

The above result shows that strong approximation on $\mathcal{U}_{2,5,q}$ fails. We can use this result to produce affine diagonal cubics for which the Brauer–Manin obstruction obstructs the integral Hasse principle.

Proposition 4.7.12. *Let $q \equiv 8 \mod 9$ be a prime number. Consider the surface $\mathcal{U}'$ given by*

$$5^3u_1^3 + 5qu_2^3 + 4qu_3^3 = 2 \subset \mathbb{A}^3_\mathbb{Z}.$$

We have $\mathcal{U}'(\mathbb{A}_\mathbb{Z}) \neq \emptyset$, but $\mathcal{U}'(\mathbb{A}_\mathbb{Z})^{Br} = \emptyset$. In particular, $\mathcal{U}'(\mathbb{Z}) = \emptyset$.

Proof. Let
$$i : \mathcal{U}' \to \mathcal{X}, (u_1, u_2, u_3) \mapsto [5u_1 : u_2 : u_3 : 1]$$
and $U' := \mathcal{U}' \times_{\mathbb{Z}} \mathbb{Q}$. Since $\mathcal{U}_{2,5,q}$ and $\mathcal{U}'$ are isomorphic over $\mathbb{Z}[1/5]$ the local solubility is immediate away from 5. At the place 5 we can take the point $(0, 0, 1/\sqrt[3]{2q}) \in \mathcal{U}'(\mathbb{Z}_5)$.

Since the surfaces are isomorphic over $\mathbb{Q}$ we have that $i^*\mathcal{A}'$ generates $\operatorname{Br} U'/\operatorname{Br}_0 U'$. By functoriality and Corollary 4.7.11 we see that for all $(P_v)_{v\in\Omega_{\mathbb{Q}}} \in \mathcal{U}'(\mathbb{A}_{\mathbb{Z}})$
$$\sum_{v\in\Omega_{\mathbb{Q}}} \operatorname{inv}_v i^*\mathcal{A}'(P_v) = 2/3.$$
Hence, $\mathcal{U}'(\mathbb{A}_{\mathbb{Z}})^{\mathrm{Br}} = \emptyset$. □

Chapter 5

Integral points on affine surfaces fibered over the affine line

5.1 Introduction

In [SD95] Swinnerton-Dyer pioneered the descent-fibration method which consists of combining the fibration method with descent on genus 1 curves. Later Colliot-Thélène, Skorobogatov and Swinnerton-Dyer [CTSSD98], Skorobogtov and Swinnerton-Dyer [SSD05], Wittenberg [Wit07], Harpaz and Skorobogatov [HS16] adapted the method further to a wider range of settings. The idea behind the method is: given a genus 1 fibration $f : X \to Y$, find a rational point on the base where the fibre above this point is everywhere locally soluble and such that a suitable Selmer group associated to this fibre is small enough forcing this fibre to have a rational point. The adaption we are interested in for this chapter is one of Harpaz [Har19], where he uses Swinnerton-Dyer's method on a fibration of torsors under norm 1 tori over the projective line to study integral points on conic log $K3$ surfaces. In this chapter, similarly to Harpaz we will study a fibration of torsors under norm 1 tori; however the contrast between our adaptations are twofold

1. In this chapter we have better control of certain "Selmer" and "dual Selmer" groups, compared to that of [Har19].
2. In [Har19], Harpaz is able to gain unconditional results. However, in this chapter our results are conditional on the Schinzel Hypothesis (H) (see Section 5.5).

5.2 Results

Theorem 5.2.1. *Let S_0 be a finite set of places of $\mathbb{Q}$ containing ∞. Let J be a non empty set and for each $i \in J$, let $c_i, d_i \in \mathbb{Z}_{S_0}$ be coprime elements such that $\Delta_{i,j} := c_i\frac{d_j}{c_j} - d_i \neq 0$ for $i \neq j$. Denote by $p_i(t) = c_it + d_i$ and $p_{J'} = \prod_{i \in J'} p_i$ for $J' \subseteq J$. For $a, b \in \mathbb{Z}_{S_0} \backslash \{0\}$ and a partition $J = A \cup B$, consider the scheme*

$$\mathcal{U} : ap_A(t)x^2 + bp_B(t)y^2 = 1 \subseteq \mathbb{A}^3_{\mathbb{Z}_{S_0}}$$

over $\mathbb{Z}_{S_0}$. Let $\pi : \mathcal{U} \to \mathbb{A}^1, (x, y, t) \mapsto t$. Under the assumption of Schinzel's Hypothesis (H), if there exists a S_0-integral adelic point $(P_v) = (x_v, y_v, t_v) \in \mathcal{U}(\mathbb{A}_{S_0})$ such that

1. *Condition (D) (See Section 5.4) holds*

2. $\mathrm{val}_v(dp_J(t_v)) \leq 1$ *for all* $v \notin S_0$ *and* $\mathrm{val}_2(dp_J(t_v)) = 1$ *if* $2 \notin S_0$

3. *There exists a place* $v \in S_0$ *for which* $-dp_J(t_v)$ *is a non-zero square at* $\mathbb{Q}_v$

4. $P_v \in \mathcal{U}(\mathbb{A}_{\mathbb{Z}_{S_0}})^{\mathrm{Br}^{vert}(X,\pi)}$

then $\mathcal{U}(\mathbb{Z}_{S_0}) \neq \emptyset$.

Remark 5.2.2. By a variety V over a field k we mean a scheme of finite type over k which is smooth, separated and geometrically integral. Given an integral scheme S over a field F we denote by $F(S)$ for the function field of S.

5.3 Norm one tori

We give a brief recap of norm 1 tori, if the reader wants a more detailed exposition we refer them to [Har19, §2.2]. For the rest of Section 5.3 fix a number field K, a finite set of places S_0 containing all archimedian places and let $d \in \mathcal{O}_{S_0}$ be a non-zero S_0-integer. Denote by $\mathcal{T}_d$ the $\mathcal{O}_{S_0}$-scheme

$$\mathcal{T}_d : x_0^2 - dx_1^2 = 1.$$

For every $a \mid d$ we denote by $\mathcal{U}_{a,b}$ the $\mathcal{O}_{S_0}$-scheme

$$\mathcal{U}_{a,b} : ax_0^2 + bx_1^2 = 1$$

where $b = -\frac{d}{a}$ and $(a, b) = 1$ in $\mathcal{O}_{S_0}$. The action of $\mathcal{T}_d$ on $\mathcal{U}_{a,b}$ exhibits $\mathcal{U}_{a,b}$ as a torsor under $\mathcal{T}_d$. Let $L := k(\sqrt{d})$ and $S := S_0 \cup \{\text{all places of } K \text{ which ramify in } L\}$. Further, let $T \subseteq \Omega_L$ be the set of places of L lying above S.

5.3.1 Selmer and Tate Shafarevich groups

Throughout Section 5.3, let $\mathcal{T} := \mathcal{T}_d \times_{\mathrm{Spec}\,\mathcal{O}_{S_0}} \mathrm{Spec}\,\mathcal{O}_S$ and $\hat{\mathcal{T}} := \mathrm{Hom}_{\mathrm{Sh}_{\text{ét}}}(\mathcal{T}, \mathbb{G}_m)$ where $\mathrm{Sh}_{\text{ét}}$ denotes the category of étale sheaves.

Definition 5.3.1. Denote by $Ш^1(\mathcal{T}, S)$ the kernel

$$Ш^1(\mathcal{T}, S) := \ker\left(\mathrm{H}^1_{\text{ét}}(\mathcal{O}_S, \mathcal{T}) \to \prod_{v \in S} \mathrm{H}^1_{\text{ét}}(k_v, \mathcal{T} \otimes_{\mathcal{O}_S} K_v)\right).$$

Remark 5.3.2. Given a class $[T] \in \mathrm{H}^1_{\text{ét}}(\mathcal{O}_S, \mathcal{T})$ we always have $T(\mathcal{O}_v) \neq \emptyset$ for $v \notin S$ i.e. the image of the map

$$\mathrm{H}^1_{\text{ét}}(\mathcal{O}_S, \mathcal{T}) \to \mathrm{H}^1_{\text{ét}}(K_v, \mathcal{T} \otimes_{\mathcal{O}_S} K_v)$$

is zero. This is because the map $\mathrm{H}^1_{\text{ét}}(\mathcal{O}_v, \mathcal{T}) \to \mathrm{H}^1_{\text{ét}}(K_v, \mathcal{T} \otimes_{\mathcal{O}_S} K_v)$ factors through $f : \mathrm{H}^1_{\text{ét}}(\mathcal{O}_v, \mathcal{T}) \to \mathrm{H}^1(\mathbb{F}_v, \mathcal{T}')$ where $\mathcal{T}'$ is the special fibre of $\mathcal{T} \to \mathrm{Spec}\,\mathcal{O}_v$. The map f is an isomorphism and by Lang's Theorem [Spr98, Thm. 4.4.17] $\mathrm{H}^1(\mathbb{F}_v, \mathcal{T}') = 0$.

Definition 5.3.3. Denote by $Ш^2(\hat{\mathcal{T}}, S)$ the kernel

$$Ш^2(\hat{\mathcal{T}}, S) := \ker\left(\mathrm{H}^2_{\text{ét}}(\mathcal{O}_S, \hat{\mathcal{T}}) \to \prod_{v \in S} \mathrm{H}^2_{\text{ét}}(K_v, \hat{\mathcal{T}} \otimes_{\mathcal{O}_S} K_v)\right).$$

Remark 5.3.4. There exists:

1. A perfect pairing $Ш^1(\mathcal{T}, S) \times Ш^2(\hat{\mathcal{T}}, S) \to \mathbb{Q}/\mathbb{Z}$ [Har17, pg. 9].
2. The short exact sequence of étale sheaves
$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathcal{T} \xrightarrow{\times 2} \mathcal{T} \to 0,$$
and the corresponding sequence in étale cohomology give rise to a map
$$\phi_1 : \mathrm{H}^1_{\text{ét}}(\mathcal{O}_S, \mathcal{T}) \to \mathrm{H}^1_{\text{ét}}(\mathcal{O}_S, \mathcal{T}).$$
3. The short exact sequence of étale sheaves
$$0 \to \hat{\mathcal{T}} \xrightarrow{2} \hat{\mathcal{T}} \to \mathbb{Z}/2\mathbb{Z} \to 0,$$
and the corresponding sequence in étale cohomology give rise to a map
$$\phi_2 : \mathrm{H}^1_{\text{ét}}(\mathcal{O}_S, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^2_{\text{ét}}(\mathcal{O}_S, \hat{\mathcal{T}}).$$

Definition 5.3.5. We define the *Selmer group* of $\mathcal{T}$ over $\mathcal{O}_S$ to be
$$\mathrm{Sel}(\mathcal{T}, S) := \{x \in \mathrm{H}^1(\mathcal{O}_S, \mathcal{T}) : \phi_1(x) \in Ш^1(\mathcal{T}, S)\},$$
and the *dual Selmer group* of $\mathcal{T}$ over $\mathcal{O}_S$ to be
$$\mathrm{Sel}(\hat{\mathcal{T}}, S) := \{x \in \mathrm{H}^1(\mathcal{O}_S, \mathbb{Z}/2\mathbb{Z}) : \phi_2(x) \in Ш^2(\hat{\mathcal{T}}, S)\}.$$

5.3.2 Duality and its consequences

Throughout Section 5.3.2, let S' be any finite set of places containing S such that $\mathrm{Pic}\, \mathcal{O}_{S'} = 0$.

Definition 5.3.6. For a $v \in S'$ denote by V^v,V_v
$$V^v, V_v := \mathrm{H}^1_{\text{ét}}(K_v, \mathbb{Z}/2\mathbb{Z}) \cong K_v^*/(K_v^*)^2.$$

Definition 5.3.7. We define $I^{S'}$ and $I_{S'}$ to be
$$I^{S'}, I_{S'} := \mathcal{O}_{S'}^*/(O_{S'}^*)^2,$$

Definition 5.3.8. We define a non-degenerate pairing
$$\langle -, - \rangle_v : V_v \times V^v \to \mathbb{Z}/2\mathbb{Z}, \quad (a, b) \mapsto \langle a, b \rangle_v.$$
where $\langle -, - \rangle_v$ is the Hilbert symbol at the place v. Moreover, we can extend this pairing to a non-degenerate pairing
$$\langle -, - \rangle_{S'} : V_{S'} \times V^{S'} \to \mathbb{Z}/2\mathbb{Z}, \quad \Big((v_1, \dots, v_{|S'|}), (v'_1, \dots, v'_{|S'|})\Big) \mapsto \sum_{v \in S'} \langle v_i, v'_i \rangle_v.$$

Remark 5.3.9. As S' contains all the real places, all places above 2 and $\operatorname{Pic}\mathcal{O}_{S'} = 0$, we have that $I_{S'}, I^{S'} \cong \mathrm{H}^1_{\text{ét}}(\mathcal{O}_{S'}, \mathbb{Z}/2\mathbb{Z})$ and localisation maps

$$I_{S'} := \mathrm{H}^1_{et}(\mathcal{O}_{S'}, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow \mathrm{H}^1_{\text{ét}}(K_v, \mathbb{Z}/2\mathbb{Z}) =: V_{S'}$$
$$I^{S'} := \mathrm{H}^1_{et}(\mathcal{O}_{S'}, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow \mathrm{H}^1_{\text{ét}}(k_v, \mathbb{Z}/2\mathbb{Z}) =: V^{S'}$$

Moreover, $I^{S'}$ is the orthogonal complement of $I_{S'}$ via $\langle -, - \rangle_{S'}$ and vice versa.

Definition 5.3.10. For each $v \in S'$ we define the subspace $W^v \subseteq V^v$ to be

$$W^v := \begin{cases} [d] & \text{if } v \in S, \\ \operatorname{im}\big(\mathcal{O}_v^*/(\mathcal{O}_v^*)^2 \to K_v^*/(K_v^*)^2\big) & \text{if } v \in S' \setminus S. \end{cases}$$

We then define $W_v \subseteq V_v$ to be the orthogonal complement of W^v with respect to $\langle -, - \rangle_v$. Let

$$W_{S'} := \bigoplus_{v \in S'} W_v \qquad W^{S'} := \bigoplus_{v \in S'} W^v.$$

Remark 5.3.11. By construction we see that $W_{S'}$ is the orthogonal complement of $W^{S'}$ with respect to $\langle -, - \rangle_{S'}$ and vice versa. Furthermore, we have induced pairings

$$\langle -, - \rangle^1_{S'} : I_{S'} \times W^{S'} \to \mathbb{Z}/2\mathbb{Z},\ \langle -, - \rangle^2_{S'} : W_{S'} \times I^{S'} \to \mathbb{Z}/2\mathbb{Z}.$$

Proposition 5.3.12 ([Har19, Prop. 2.2.4])**.** *The Selmer group* $\operatorname{Sel}(\mathcal{T}, S)$ *can be identified with*

1. $I_{S'} \cap W_{S'}$,
2. *the left kernel of* $\langle -, - \rangle^1_{S'}$,
3. *the left kernel of* $\langle -, - \rangle^2_{S'}$.

The dual Selmer group $\operatorname{Sel}(\hat{\mathcal{T}}, S)$ *can be identified with*

1. $I^{S'} \cap W^{S'}$,
2. *the right kernel of* $\langle -, - \rangle^1_{S'}$,
3. *the right kernel of* $\langle -, - \rangle^2_{S'}$.

The following is a key Proposition we will use to establish Theorem 5.2.1.

Proposition 5.3.13 ([Har19, Prop. 2.3.1, Corollary 2.3.2])**.** *If for all* $v \notin S_0$ *we have* $\operatorname{val}_v(d) \leq 1$ *and* $\operatorname{val}_v(d) = 1$ *if* v *lies above* 2, *then for all* $a \mid d$ *the* $\mathcal{O}_{S_0}$*-scheme* $\mathcal{U}_{a,b}$ *has a* $\mathcal{O}_{S_0}$*-point if and only if the* $\mathcal{O}_S$ *scheme* $\mathcal{U}_{a,b} \times_{\operatorname{Spec}\mathcal{O}_{S_0}} \operatorname{Spec}\mathcal{O}_S$ *has an* $\mathcal{O}_S$*-point. Moreover, if* $\operatorname{Sel}(\hat{\mathcal{T}}, S)$ *is generated by* $[d]$ *then for every* $a \mid d$ *the* $\mathcal{O}_{S_0}$ *scheme* $\mathcal{U}_{a,b}$ *satisfies the* S_0*-integral Hasse principle.*

5.4 Condition (D)

We formulate Condition (D), this property is crucial as it allows one to be able to kill certain elements in the Selmer and dual Selmer group. Through Section 5.4 a, b, d_i, c_i and p_i will be as in Theorem 5.2.1.

Definition 5.4.1. Denote by G the abelian group

$$G := \mathbb{Q}^*/(\mathbb{Q}^*)^2 \bigoplus \mathrm{Span}_{\mathbb{F}_2}\{[p_i] : i \in J\}$$

where $[p_i]$ is a formal symbol. We will write elements of G as $[c][p_{J'}]$ where and $J' \subseteq J$ and

$$[c][p_{J'}] := c \prod_{i \in J'} p_i.$$

Notation 5.4.2. Let $J' \subseteq J$ then we denote by $(J')^c$ the complement of J' in J i.e. $(J')^c := J \backslash J'$. Moreover, let $d = ab$ and $J' \subseteq J$, then for $i \in J$ denote by

$$D_i^{J'} := \begin{cases} p_{J'}\left(-\frac{d_i}{c_i}\right) & \text{if } i \notin J', \\ dp_{(J')^c}\left(-\frac{d_i}{c_i}\right) & \text{if } i \in J'. \end{cases} \quad \hat{D}_i^{J'} := \begin{cases} p_{J'}\left(-\frac{d_i}{c_i}\right) & \text{if } i \notin J', \\ -dp_{(J')^c}\left(-\frac{d_i}{c_i}\right) & \text{if } i \in J'. \end{cases}$$

Definition 5.4.3. Let $i \in J$. We denote by G_i is the subgroup of G containing the elements $[c][p_{J'}]$ such that

$$[cD_i^{J'}] \in \langle [aD_i^A] \rangle$$

Definition 5.4.4. Let $i \in J$. We denote by G^i is the subgroup of G containing the elements $[c][p_{J'}]$ such that

$$[c\hat{D}_i^{J'}] \in \langle [aD_i^A] \rangle.$$

Definition 5.4.5. Denote by G_D the intersection $G_D := \bigcap_{i \in J} G_i$. Similarly, denote by G^D the intersection $G^D := \bigcap_{i \in J} G^i$.

Condition 5.4.6 (Condition (D))**.** The group G_D is generated by $[a][p_A]$ and $[d][p_J]$ and G^D is generated by $[-d][p_J]$.

5.5 Schinzel Hypothesis

Conjecture 5.5.1 (Schinzel's Hypothesis (H))**.** *For every finite collection $\{f_1, \ldots, f_k\}$ of non-constant irreducible polynomials over the integers with positive leading coefficients, one of the following conditions holds:*

1. *There is an integer m which is called a* fixed divisor*, that always divides the product $f_1(n) \ldots f_k(n)$ for $n \in \mathbb{Z}$. Or, equivalently: There exists a prime p such that for every n there exists an i such that $f_i(n) \equiv 0 \pmod p$.*

2. *There are infinitely many positive integers n such that $f_1(n), \ldots, f_k(n)$ are simultaneously prime numbers.*

Remark 5.5.2. Consider a finite collection of linear polynomials $\{f_1(t), \ldots, f_k(t)\}$ where $f_i(t) = c_i t + d_i$ and

$$\Delta_{j,k} := c_j d_k - c_k d_j \neq 0$$

for $j \neq k$. Then under Hypothesis (H) there exists infinitely many integers n such that $f_1(n), \ldots, f_k(n)$ are distinct primes.

We will require a slightly different version of (H) due to Serre.

Conjecture 5.5.3 (Serre (H_1))**.** *Let K be a number field, $f_i(t)$ irreducible polynomials over K. Let S be a finite set of places of K, containing all*

1. *archimedean places,*
2. *finite places v where one of f_i does not have v-integral coefficients or has all its coefficients divisible by v,*
3. *all finite places above a prime p less than or equal to the degree of the polynomial $N_{K/\mathbb{Q}}(f_i(t))$.*

Given elements $\lambda_v \in K_v$ at finite places $v \in S$ one may find $\lambda \in K$, integral away from S, arbitrarily close to each λ_v for the v-adic topology for $v \in S$ finite, arbitrarily big in the archimedean completions K_v, and such that for each i, $f_i(\lambda) \in K$ is a unit in K_w for all place $w \notin S$ except perhaps one place w_i, where it is a uniformizer.

Remark 5.5.4. Conjecture 5.5.1 implies Conjecture 5.5.3, [CTSD94, Lemma 4.1]. Because of this we will denote by (H) the conjectures (H) and (H_1) as they are equivalent.

5.6 Vertical Brauer group

In this section we will determine the vertical Brauer group of the scheme

$$U : ap_A(t)x^2 + bp_B(t)y^2 = 1 \subseteq \mathbb{A}^3_{\mathbb{Q}}$$

with respect to the morphism $\pi : U \to \mathbb{A}^1, (x, y, t) \mapsto t$ and keeping notation as in Theorem 5.2.1.

Definition 5.6.1. Let $f : X \to Y$ be a morphism of varieties over a field F. The *vertical Brauer group* $\mathrm{Br}^{\mathrm{vert}}(X, f)$ is defined to be

$$\mathrm{Br}^{\mathrm{vert}}(X, f) := \mathrm{Br}(X) \cap f^* \mathrm{Br}(F(Y)) \subseteq \mathrm{Br}(F(X)).$$

Definition 5.6.2. A scheme S of finite type over a perfect field k is called *split* if S contains an irreducible component of multiplicity 1 which is geometrically irreducible.

We now determine the vertical Brauer group of U over k a field of characteristic 0.

Definition 5.6.3. For each $i \in J$, let $\mathcal{A}_i \in \mathrm{Br}\, k(U)$ be

$$\mathcal{A}_i := \begin{cases} (ap_A(-\frac{d_i}{c_i}), p_i(t)) & \text{if } i \notin A, \\ (bp_B(-\frac{d_i}{c_i}), p_i(t)) & \text{if } i \in A. \end{cases}$$

Lemma 5.6.4. *The class $\mathcal{A}_i$ lies in* $\mathrm{Br}^{Vert}(U, \pi)$.

Proof. As the left entry of $\mathcal{A}_i$ is constant, for any prime divisor $D \subseteq U$ with generic point η_D, we have

$$\mathrm{val}_{\eta_D}\left(ap_A\left(-\frac{d_i}{c_i}\right)\right) = \mathrm{val}_{\eta_D}\left(bp_B\left(-\frac{d_i}{c_i}\right)\right) = 0. \qquad (5.6.1)$$

Let $Z_i \subseteq U$ be the vanishing locus of p_i, by (5.6.1) $\mathcal{A}_i$ is unramified outside of Z_i. Suppose $i \notin A$, for $D \subseteq Z_i$ we have

$$\partial(\mathcal{A}_i)_{\eta_D} = \frac{1}{ap_A(\frac{-d_i}{c_i})^{\mathrm{val}_{\eta_D}(p_i(t))}}.$$

As $bp_B(t) \mid_{Z_i}= 0$ then $ap_A(t)|_{Z_i}$ is a square in $k[Z_i]$. Moreover, as $ap_A(t)|_{Z_i} = ap_A(-\frac{d_i}{c_i})$, we have $\mathcal{A}_i$ is unramified along Z_i. The case for $i \in A$ is similar. It is then clear that $\mathcal{A}_i \in \mathrm{Br}^{\mathrm{Vert}}(U, \pi)$. □

Remark 5.6.5. Let $\mathcal{M} := \{t \in \mathbb{A}^1_k(k) : \text{there exists an } i \text{ where } p_i(t) = 0\}$. Pick $t \in \mathcal{M}$ where $p_i(t) = 0$ for $i \notin A$, then the fibre of π above t is

$$ap_A(t)x^2 = 1.$$

Suppose $ap_A(t)$ is not a square over k, in this case the fibre U_t is split after a quadratic extension. One can define a morphism

$$\mathbb{A}^1_L \to U_t,\ y \mapsto (1/\alpha, y)$$

where $L := k(\alpha) = k[w]/(w^2 - ap_A(t))$. If $ap_A(t)$ is a square one can take $k(\alpha) = k$ and we have two copies of $\mathbb{A}^1$, namely $(1/\alpha, y)$ and $(-1/\alpha, y)$, each are geometrically irreducible, i.e. the fibre in this case is split.

For the rest of Section 5.6, let $m \in \mathbb{A}^1$ be a closed point, U_m the fibre of $\pi : U \to \mathbb{A}^1$ above m and $\mathcal{A} \in \mathrm{Br}\, k(t)$ such that $\pi^*\mathcal{A} \in \mathrm{Br}^{\mathrm{vert}}(U, \pi)$.

Lemma 5.6.6. *Suppose the fibre U_m is smooth, then $\partial_m(\mathcal{A}) = 0$.*

Proof. As the fibre is smooth, hence split we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Br}\, k & \longrightarrow & \mathrm{Br}\, k(t) & \xrightarrow{\partial_m} & \mathrm{H}^1_{\text{ét}}(k(m), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\sim} & k(m)^*/(k(m)^*)^2 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Br}\, U & \longrightarrow & \mathrm{Br}\, k(U) & \xrightarrow{\partial_{U_m}} & \mathrm{H}^1_{\text{ét}}(k(U_m), \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{\sim} & k(U_m)^*/(k(U_m)^*)^2. \end{array}$$

Then the image of $\partial_{U_m}(\pi^*\mathcal{A})$ is equal to the image of $\partial_m\mathcal{A}$ under the map

$$k(m)^*/(k(m)^*)^2 \to k(U_m)^*/(k(U_m)^*)^2.$$

By assumption $\partial_{U_m}(\pi^*\mathcal{A}) = 0$, hence $\partial_m\mathcal{A}$ is a square in $k(U_m)^*/(k(U_m)^*)^2$. As U_m is a smooth affine conic it is geometrically integral, so the algebraic closure of $k(m)$ in $k(U_m)$ is $k(m)$, hence $\partial_m\mathcal{A}$ is a square. □

Lemma 5.6.7. *Suppose the fibre U_m is a split singular fibre, then $\partial_m(\mathcal{A}) = 0$.*

Proof. If U_m is split over $k(m)$ then take a component of multiplicity 1, say Y_m. Applying the same argument as Lemma 5.6.6 but with Y_m instead of U_m we get the statement. □

Lemma 5.6.8. *Suppose U_m is a singular fibre that is not split, then $\partial_m(\mathcal{A}) = \partial_m(\mathcal{A}_i)$ for some $i \in J$ or $\partial_m(\mathcal{A}) = 0$.*

Proof. The singular fibres of $\pi : U \to \mathbb{A}^1$ are exactly

$$\mathcal{M} := \{m_i \in \mathbb{A}^1(k) : p_i(m_i) = 0\}.$$

Then for $m_i \in \mathcal{M}$ we have $\partial_{m_i}\mathcal{A}_i = \alpha_i \in k^*/(k^*)^2$, where α_i is such that the fibre U_m splits over $k(\sqrt{\alpha_i})$. Moreover, it is clear that $k(U_m) = k(t)(\sqrt{\alpha_i})$ where t is a purely transcendental element. As $\pi^*\mathcal{A}$ is unramified $\partial_{U_{m_i}}\pi^*\mathcal{A} = 0$, hence either $\partial_{m_i}\mathcal{A} = 0$ or $\partial_{m_i}\mathcal{A} = \alpha_i$ in $k^*/(k^*)^2$ □

Proposition 5.6.9. *We have that* $\operatorname{Br}^{Vert}(U,\pi)/\operatorname{Br} k$ *is generated by the* $\mathcal{A}_i$*, i.e. if* $\mathcal{A} \in \operatorname{Br}^{Vert}(U,\pi)$ *then* $\mathcal{A} = \sum_{i \in J} \varepsilon_i\mathcal{A}_i + \mathcal{B}$ *where* $\varepsilon_i \in \mathbb{Z}/2\mathbb{Z}$ *and* $\mathcal{B} \in \operatorname{Br} k$.

Proof. Let $\mathcal{A} \in \operatorname{Br}^{\mathrm{Vert}}(U,\pi)$ which is the pull back of an element $\mathcal{A}' \in \operatorname{Br} k(t)$. Then $\partial_m(\mathcal{A}') = 0$ for all but finitely many closed points $m \in \mathbb{A}^1$, hence $\mathcal{A}' \in \operatorname{Br}(W)$ where W is a dense open of $\mathbb{A}^1$. It then follows that $\mathcal{A} \in \operatorname{Br}(\pi^{-1}W)$ i.e. non-trivial residues of $\mathcal{A}$ can only appear from vertical divisors $D \subset U$, by definition these correspond to the fibres of $\pi : U \to \mathbb{A}^1$. Let

$$E := \{i : \mathcal{A}' \text{ has non-trivial residue at the closed point } m_i \in \mathbb{A}^1\}.$$

For each $i \in E$ we have that $\mathcal{A}'$ and $\mathcal{A}_i$ have the same residue at m_i, hence $\alpha := \mathcal{A}' - \sum_{i \in E}\mathcal{A}_i \in \operatorname{Br}\mathbb{A}^1 = \operatorname{Br} k$ and $\pi^*(\mathcal{A}' - \sum_{i\in E}\mathcal{A}_i) = \mathcal{A} - \sum_{i \in E}\mathcal{A}_i = 0$ in $\operatorname{Br}^{\mathrm{Vert}} U/\operatorname{Br} k$. □

Notation 5.6.10. Let v be a place of $\mathbb{Q}$. Throughout the rest of Chapter 5, given a quaternion algebra $(a,b) \in \operatorname{Br}\mathbb{Q}_v$ we denote by $\langle a,b\rangle_v$ the image of the invariant map v i.e. $\operatorname{inv}_v(a,b) = \langle a,b\rangle_v$.

5.7 Locally soluble fibres

Throughout the rest of the paper S_0 will be the finite set of places in Theorem 5.2.1.

Definition 5.7.1. Let $S_{\mathrm{bad}} \subseteq \Omega_{\mathbb{Q}}\backslash S_0$ be the set of places outside of S_0 for which

1. $v \mid \Delta_{i,j}$ for some $i \neq j$, or,
2. $v \mid d$, or,
3. $v = 2$, or,
4. v such that $\operatorname{val}_v(p_J(t_v)) > 0$ for every $t_v \in \mathbb{Z}_v$.

Notation 5.7.2. For $a \in \mathbb{Z}_v^*$ for some place v, we shall denote by $[a]_v$ the class of a in $\mathbb{Z}_v^*/(\mathbb{Z}_v^*)^2$.

Lemma 5.7.3. *Let* $v \notin S_0 \cup S_{bad}$ *be a place. Suppose* $t_v \in \mathbb{Z}_v^*$ *and* $i \in J$ *is an element such that* $\operatorname{val}_v(p_i(t_v)) > 0$*. Denote by* $\mathcal{U}_{t_v}$ *of the fibre above* t_v*, then* $\mathcal{U}_{t_v}(\mathbb{Z}_v) \neq \emptyset$ *if and only if*

1. $[ap_A(-\frac{d_i}{c_i})]_v = 0$ *if* $i \notin A$,
2. $[bp_B(-\frac{d_i}{c_i})]_v = 0$ *if* $i \in A$.

Proof. Note as $v \notin S_{\text{bad}}$ we have that $\text{val}_v(p_i(t_v)) > 0$ for a unique $i \in J$. We shall assume that $i \notin A$, then the special fibre of $\mathcal{U}_{t_v} \to \operatorname{Spec} \mathbb{Z}_v$ is defined by the equation

$$\mathcal{U}_{t_v} : ap_A\left(-\frac{d_i}{c_i}\right)y^2 = 1 \pmod{\mathfrak{m}_v}.$$

By Hensel's Lemma $\mathcal{U}_{t_v}(\mathbb{Z}_v) \neq \emptyset$ if and only if $\left[ap_A\left(-\frac{d_i}{c_i}\right)\right]_v = 0$. The case of $i \in A$ is similar. □

Lemma 5.7.4. *Let $v \notin S_0 \cup S_{bad}$ be a place. Then for $i \in J$ we have that*

$$\operatorname{inv}_v \mathcal{A}_i(P_v) = 0$$

for all $P_v = (x_v, y_v, t_v) \in \mathcal{U}(\mathbb{Z}_v)$.

Proof. As $\operatorname{inv}_v \mathcal{A}_i(P_v) = \langle aD_i^A, p_i(t_v)\rangle_v$ and aD_i^A is a v-adic unit, we see $\operatorname{inv}_v \mathcal{A}_i(P_v) \neq 0$ if and only if aD_i^A is a not a square and $\text{val}_v(p_i(t_v)) \equiv 1 \bmod 2$. By Lemma 5.7.3 this is impossible as $\mathcal{U}_{t_v}(\mathbb{Z}_v) \neq \emptyset$. □

5.8 Partial adelic points

Definition 5.8.1. Let $T \subset \Omega_{\mathbb{Q}}$ be a finite set of places containing S_0 then a *partial adelic point over* T is a point $(P_v)_{v\in T} = (x_v, y_v, t_v)_{v\in T}$ in the product

$$(P_v)_{v\in T} \in \prod_{v\in S_0} \mathcal{U}(\mathcal{O}_v) \times \prod_{v\in T\setminus S_0} U(k_v).$$

Moreover, we denote by P_T the partial adelic point $(P_v)_{v\in T}$ over T.

Remark 5.8.2. If $[x][p_{J'}] \in G_i\backslash G_D$, i.e. there exists $i_x \neq i$ such that $cD_{i_x}^{J'} \notin \{1, aD_{i_x}^A\}$, then there exists a place $v_x \notin S_0 \cup S_{\text{bad}}$ such that $cD_{i_x}^{J'}$ is not a square at v_x but $aD_{i_x}^A$ is a square at v_x. For each i and each $x \in G_i\backslash G_D$ choose a prime v_x as above. Then one obtains a finite set of v_x's which we will denote by S_D.

Definition 5.8.3. Let $S := S_0 \cup S_{\text{bad}} \cup S_D$.

Definition 5.8.4. Let T be a finite set of places containing S. A partial adelic point P_T is called *suitable* if the following hold:

1. $dp_J(t_v) \neq 0$ for all $v \in T$,
2. $\text{val}_v(dp_J(t_v)) \leq 1$ for all $v \in T\setminus S_0$,
3. $\text{val}_2(dp_J(t_v)) = 1$ if $2 \in T\setminus S_0$,
4. There is one place $v \in S_0$ such that $-dp_v(t_v) \in \mathbb{Q}_v^*$ is a square,
5. $\sum_v \operatorname{inv}_v \mathcal{A}_i(P_v) = 0$ for every $i \in J$.
6. For every $v_x \in S_D$ the local point $P_v \in \mathcal{U}(\mathbb{Z}_v)$ lies above $t_x \in \mathbb{A}^1(\mathbb{Z}_{v_x})$ as described in Remark 5.8.2.

Proposition 5.8.5. *Under the assumptions of Theorem 5.2.1 there always exists a suitable partial adelic point with respect to T on $\mathcal{U}$.*

Proof. Under the assumptions of Theorem 5.2.1 we have an adelic point satisfying (1)-(4) of Definition 5.8.4. As this adelic point is orthogonal to the vertical Brauer group by Lemma 5.6.4 we have condition (5) is satisfied. For such v_x the invariant map at v_x pairs trivially with $\mathcal{U}(\mathbb{Z}_{v_x})$ by Lemma 5.7.4 we can shift our adelic point without disrupting the properties (1)-(5) and satisfying (6). □

5.9 Admissible fibres

Definition 5.9.1. Given a finite set of places T containing S, and a suitable partial adelic point $P_T \in \mathcal{U}$, we will say that a point $t_0 \in \mathbb{Z}_{S_0}$ is *T-admissible* with respect to P_T if

1. $[p_i(t_0)]_v = [p_i(t_v)]_v$ for all $v \in T$.
2. For each $i \in J$, we have that $p_i(t)$ is a unit outside of T except at one place u_i, where $p_i(t_0)$ is a uniformiser (i.e. $\mathrm{val}_{u_i}(p_i(t_0)) = 1$).
3. The fibre $\mathcal{U}_{t_0}$ has a S_0-integral adelic point i.e. $\mathcal{U}_{t_0}(\mathbb{A}_{S_0}) \neq \emptyset$.

Definition 5.9.2. Let $P_T = (x_v, y_v, t_v)_{v\in T}$ be a suitable partial adelic point then we define $T_0 \subseteq T$ to be the subset of places v which lie in S_0 or $\mathrm{val}_v(dp_J(t_v)) = 1$.

Definition 5.9.3. Let t be a T-admissible point with $\{u_i\}_{i\in J}$ the places appearing in condition (2) of Definition 5.9.1, then we denote by $T(t) := T \cup \{u_i\}_{i\in J}$ and $T_0(t) := T_0 \cup \{u_i\}_{i\in J}$.

Proposition 5.9.4. *Let T be a finite set of places containing S and P_T a suitable partial adelic point with respect to T on $\mathcal{U}$. Then there exists a T-admissible point t on $\mathbb{A}^1_{\mathbb{Z}_{S_0}}$ with respect to P_T.*

Proof. By Conjecture 5.5.3 we can find $t \in \mathbb{Z}_{S_0}$ and places u_i for $i \in J$ such that conditions $(1) - (2)$ of Definition 5.9.1 are satisfied. Now it is sufficient to show that the fibre above t has a S_0-integral adelic point. One applies the implicit function theorem [Ser06, pg. 73] to see that for all $v \in T$ we have $\mathcal{U}_t(\mathbb{Z}_v) \neq \emptyset$. For $v \notin T \cup \{u_i\}_{i\in J}$ the special fibre of $\mathcal{U}_t \to \operatorname{Spec}\mathbb{Z}_v$ is smooth and has a point over $\mathbb{F}_v$ (as it is a smooth affine conic), hence $\mathcal{U}_t(\mathbb{Z}_v) \neq \emptyset$ by Hensel's Lemma. We now focus on $v = u_i$ for some $i \in J$. By quadratic reciprocity

$$\langle aD_i^A, p_i(t)\rangle_{u_i} = \sum_{v\in T} \langle aD_i^A, p_i(t)\rangle_v = \sum_{v\in T} \langle aD_i^A, p_i(t_v)\rangle_v.$$

By property (5) of Definition 5.8.4 we have that this sum is zero. As $p_i(t)$ is a uniformizer at u_i and aD_i^A is a u_i-adic unit, it follows that aD_i^A is a square, hence by Lemma 5.7.3, $\mathcal{U}_t(\mathbb{Z}_{u_i}) \neq \emptyset$. □

5.10 Selmer group of admissible fibres

Given a suitable partial adelic point $(P_v)_{v\in T}$ and a T-admissible point t_0 of P_T we want to be able to control the size of $\mathrm{Sel}(\hat{\mathcal{T}}_{t_0}, T_0(t_0))$. However, the first natural question is:

Question 5.10.1. Given two T-admissible points t_0, t_1 of P_T, how do $\dim_{\mathbb{F}_2} \mathrm{Sel}(\hat{\mathcal{T}}_{t_0}, T_0(t_0))$ and $\dim_{\mathbb{F}_2} \mathrm{Sel}(\hat{\mathcal{T}}_{t_1}, T_0(t_1))$ differ?

In this section we show that the choice of T-admissible point t_0 has no effect on dimension of $\mathrm{Sel}(\hat{\mathcal{T}}_{t_0}, T_0(t_0))$ as a $\mathbb{F}_2$-vector space.

Definition 5.10.2. Let T be a finite set of places of $\mathbb{Q}$ containing S. Let $J_T \subseteq G$ generated by I_T and the symbols $[p_i]$. Similarly, $J^T \subseteq G$ generated by I^T and the symbols $[p_i]$ i.e.

$$J^T, J_T := \mathcal{O}_T^*/(\mathcal{O}_T^*)^2 \bigoplus \mathrm{Span}_{\mathbb{F}_2}\{[p_i] : i \in J\}$$

Definition 5.10.3. For a T-admissible point t_0, we denote by ev_{t_0}, ev^{t_0} the maps

$$ev_{t_0} : J_T \to I_{T(t_0)}, \quad [c][p_{J'}] \mapsto [c][p_{J'}(t_0)]$$

and

$$ev^{t_0} : J^T \to I^{T(t_0)}, \quad [c][p_{J'}] \mapsto [c][p_{J'}(t_0)]$$

Remark 5.10.4. ev_{t_0} and ev^{t_0} are isomorphisms of $\mathbb{F}_2$-vector spaces.

Definition 5.10.5. We will define the *relative Selmer group* and the *relative dual Selmer group* as

$$R_{P_T,t_0} := ev_{t_0}^{-1}(\mathrm{Sel}(\mathcal{T}_{t_0}, T_0(t_0))) \text{ and } \hat{R}_{P_T,t_0} := ev_{t_0}^{-1}\Big(\mathrm{Sel}(\hat{\mathcal{T}}_{t_0}, T_0(t_0))\Big).$$

Remark 5.10.6. Recall Definition 5.3.10. In the rest of the paper we use the following analogous notation: given a T-admissible point t and a place $v \in T(t)$ we denote by $W^v(t) \subseteq V^v$ for the space

$$W^v(t) := \begin{cases} [-dp_J(t)] & \text{if } v \in T_0(t), \\ \mathrm{im}\big(\mathcal{O}_v^*/(\mathcal{O}_v^*)^2 \to K_v^*/(K_v^*)^2\big) & \text{otherwise.} \end{cases}$$

We then define $W_v(t) \subseteq V_v$ to be the orthogonal complement of $W^v(t)$ with respect to $\langle -, - \rangle_v$.

Lemma 5.10.7. *Given a finite set $T \subset \Omega_{\mathbb{Q}}$ containing S and t_0 a T-admissible point with respect to a suitable partial adelic point P_T. Let $i \in J$ and $v \in T_0(t_0)$ where $v \notin S_0 \cup S_{bad}$ and odd, such that $t_0 \equiv -\frac{d_i}{c_i}$ mod $\mathfrak{m}_v$. If $x = [c][p_{J'}] \in J^T$ and $\mathrm{val}_v(c) = 0$ then $ev^{t_0}(x) \in W^v(t_0)$ if and only if*

1. $[ev^{t_0}(x)]_v = 0$ *if* $i \notin J'$,
2. $[ev^{t_0}(x[-d][p_J])]_v = 0$ *if* $i \in J'$.

Furthermore, $ev_{t_0}(x) \in W_v(t_0)$ if and only if

3. $[ev_{t_0}(x)]_v$ *if* $i \notin J'$,
4. $[ev^{t_0}(x[d][p_J])]_v = 0$ *if* $i \in J'$.

Proof. For $i \neq j$

$$p_j(t_0) \equiv c_j t_0 + d_j \equiv -c_j \frac{d_i}{c_i} + d_j \bmod \mathfrak{m}_v.$$

If $p_j(t_0) \equiv 0 \bmod \mathfrak{m}_v$ we have that $\Delta_{i,j} \equiv 0 \bmod \mathfrak{m}_v$ but as $v \notin S_{bad}$, we can deduce $\mathrm{val}_v(p_j(t_0)) = 0$ for $i \neq j$. By assumption on v we have that $p_i(t_0) \equiv 0 \bmod \mathfrak{m}_v$. As P_T is a suitable partial adelic point $\mathrm{val}_v(dp_J(t_v)) \leq 1$ for $v \in T \backslash S_0$ and t_0 approximates t_v for $v \in T$. We can conclude for $v \in T \backslash S_0$ that $\mathrm{val}_v(dp_J(t_0)) \leq 1$. If $v \notin T$ i.e. it is one of the primes u_i^0 (we can assume that each u_i^0 is distinct for each i) using the definition of T-admissible point $\mathrm{val}_{u_i^0}(dp_J) \leq 1$. Moreover, $\mathrm{val}_v(p_i(t_0)) = \mathrm{val}_v(dp_J(t_0)) = 1$ and

$$\mathrm{val}_v(ev^{t_0}(x)) := \begin{cases} 1 & \text{if } i \in J', \\ 0 & \text{if } i \notin J'. \end{cases}$$

Recall $W^v(t_0)$ is the subgroup of V^v generated by $[-dp_J]_v$ and as v is odd $W_v(t_0)$ it is the subgroup of V_v generated by $[dp_J]_v$. Then $ev_{t_0}(x) \in W_v(t_0)$ if and only if pairs (with the pairing from Definition 5.3.8) trivially with $dp_J(t_0)$. Similarly, for $ev^{t_0}(x)$. Assume for now $i \notin J'$. We showed that $\mathrm{val}_v(dp_J(t_0)) = 1$, hence it is a uniformiser of $\mathbb{Z}_v$, so computing the Hilbert symbol

$$\langle ev^{t_0}(x), dp_J(t_0)\rangle_v := \begin{cases} 0 & \text{if } [ev^{t_0}(x)]_v = 0. \\ 1 & \text{otherwise.} \end{cases}$$

This completes (1). If $i \in J'$ then both $ev^{t_0}(x)$ and $dp_J(t_0)$ are uniformisers and by [Ser73, §3, Props. 2i,2iv]

$$\langle cp_{J'}(t_0), dp_J(t_0)\rangle_v = \langle dp_J(t_0), -cdp_{(J')^c}(t_0)\rangle_v = \langle dp_J(t_0), \mathrm{ev}^{t_0}(x[-d][p_J])\rangle_{u_i}$$

Then

$$\langle ev^{t_0}(x), dp_J(t_0)\rangle_v := \begin{cases} 0 & \text{if } [ev^{t_0}(x[-d][p_J])]_v = 0. \\ 1 & \text{otherwise.} \end{cases}$$

This completes (2). The proofs for (3) and (4) are completely analogous to (1) and (2). □

Remark 5.10.8. Taking into account Proposition 5.3.12, we call the conditions from Lemma 5.10.7 Selmer conditions.

Lemma 5.10.9. *Let $x = [c][p_{J'}] \in J^T$. Then $x \in \hat{R}_{P_T,t_0}$ if and only if*

1. *For every $v \in T$: $ev^{t_0}(x) \in W^v(t_0)$.*
2. *For every $i \in J$:*

$$\langle p_i(t_0), ev^{t_0}(x)\rangle_{u_i} = 0 \text{ if } i \notin J',$$
$$\langle p_i(t_0), ev^{t_0}(x[-d][p_J])\rangle_{u_i} = 0 \text{ if } i \in J'.$$

Proof. Let $\alpha := ev^{t_0}(x)$. Recall $\mathrm{val}_{u_i}(p_i(t_0)) = \mathrm{val}_{u_i}(dp_J(t_0)) = 1$, hence

$$dp_J(t_0) = d\left(\prod_{k \neq i} p_k(t_0)\right) p_i(t_0)$$

where $d \prod_{k \neq i} p_k(t_0) \in \mathbb{Z}_{u_i}^*$. Moreover, $\mathrm{val}_{u_i}(\alpha) = 1$ if $i \in J$ and 0 otherwise. If $i \notin J$ then we know α is a unit and by [Ser73, §3 Prop. 2iii,Lemma 1.2]

$$\langle \alpha, dp_J(t_0) \rangle_{u_i} = \langle p_i(t_0), \alpha \rangle_{u_i}.$$

If $i \in J$ then by [Ser73, §3 Prop 2iv]

$$\langle \alpha, dp_J(t_0) \rangle_{u_i} = \langle p_i(t_0), -cdp_{(J')^c}(t_0) \rangle_{u_i} = \langle p_i(t_0), ev^{t_0}(x[-d][p_J]) \rangle_{u_i}.$$

□

Proposition 5.10.10. *Let t_0, t_1 be T-admissible points of the suitable partial adelic point P_T, then $\hat{R}_{P_T,t_0} \cong \hat{R}_{P_T,t_1}$.*

Proof. Suppose $[c][p_{J'}] \in J_T$, let $\alpha_1 := ev^{t_0}(x)$ and $\alpha_2 := ev^{t_1}(x)$. What we want to show is $\alpha_1 \in W^{T(t_0)}$ if and only if $\alpha_2 \in W^{T(t_1)}$. We know if $v \in T$ then $\alpha_1 \in W^v$ if and only if $\alpha_2 \in W^v$, so it is sufficient to show that $\alpha_1 \in W^{u_i^0}$ if and only if $\alpha_2 \in W^{u_i^1}$ for $i \in J$. By Lemma 5.10.9 this is equivalent to showing that t_0 satisfies condition Lemma 5.10.9 (2) if and only t_1 does. Let us first assume $i \notin J'$ then

$$\langle p_i(t_0), ev^{t_0}(x) \rangle_{u_i^0} = \langle p_i(t_0), \prod_{i \in J'} p_j\left(\frac{-d_i}{c_i}\right) \rangle_{u_i^0}.$$

Applying quadratic reciprocity and using the fact that the right entry of the Hilbert symbol in this case is a unit outside of T,

$$\langle p_i(t_0), \prod_{i \in J'} p_j\left(\frac{-d_i}{c_i}\right) \rangle_{u_i^0} = \sum_{v \in T} \langle p_i(t_0), \prod_{i \in J'} p_j\left(\frac{-d_i}{c_i}\right) \rangle_v = \sum_{v \in T} \langle p_i(t_1), \prod_{i \in J'} p_j\left(\frac{-d_i}{c_i}\right) \rangle_v$$

which is equal to $\langle p_i(t_1), ev^{t_1}(x) \rangle_{u_i^1}$. A similar calculation shows if $i \in J'$ then the statement holds using the fact that $-cd$ is a unit outside of T. □

Remark 5.10.11. By Proposition 5.10.10, we see that the choice of T-admissible point does not matter so we can reduce out notation of $\hat{R}_{P_T,t_i}$ to $\hat{R}_{P_T}$.

5.11 Comparing Selmer groups

Let T be a finite set of places and P_T a suitable partial adelic point. Further, let t_0 be a T-admissible point with associated places $\{u_i^0\}_{i \in J}$. The goal of this section is to understand what happens to the size of the Selmer and dual Selmer group when one adds to T an additional place $w \notin S_0$. Let $T_w := T \cup \{w\}$ and let P_{T_w} be an extension of P_T to T_w. We denote by t_1 a T_w-admissible point with respect to P_{T_w} with associated places $\{u_i^1\}_{i \in J}$. We make the following assumption for the rest of Section 5.11. However, in the proof of Theorem 5.2.1 in Section 5.12 this assumption will be satisfied by using (H).

Assumption 5.11.1. There exists $i_w \in J$ such that $p_{i_w}(t_w)$ is a uniformizer.

Lemma 5.11.2. *For every $i \neq j \in J$ and $i \neq i_w$ we have*

$$\langle p_i(t_0), p_j(t_0) \rangle_{u_i^0} + \langle p_i(t_1), p_j(t_1) \rangle_{u_i^1} = 0.$$

Proof. If $i = j \neq i_w$ then the statement is trivial. Now assume that $i \neq i_w$ and $i \neq j$. As $p_i(t_0)$ (respectively $p_i(t_1)$) is a uniformizer at u_i^0 (respectively u_i^1) we have that $c_i t_0 + d_i \equiv 0 \mod u_i^0$ and $c_i t_1 + d_i \equiv 0 \mod u_i^1$ i.e. $t_0 \equiv \frac{-d_i}{c_i} \mod u_i^0$ and $t_1 \equiv \frac{-d_i}{c_i} \mod u_i^1$. Hence,

$$\langle p_i(t_0), p_j(t_0)\rangle_{u_i^0} + \langle p_i(t_1), p_j(t_1)\rangle_{u_i^1} = \langle p_i(t_0), -\frac{d_i}{c_i}c_j + d_j\rangle_{u_i^0} + \langle p_i(t_1), -\frac{d_i}{c_i}c_j + d_j\rangle_{u_i^1}$$

and by quadratic reciprocity

$$\langle p_i(t_0), -\frac{d_i}{c_i}c_j + d_j\rangle_{u_i^0} + \langle p_i(t_1), -\frac{d_i}{c_i}c_j + d_j\rangle_{u_i^1} = 2\sum_{v\in T}\langle p_i(t_0), -\frac{d_i}{c_i}c_j + d_j\rangle_v = 0.$$

□

Lemma 5.11.3. *Let $x = [c][p_{J'}] \in J^T$ and $i \in (J')^c$. If $i = i_w$ assume that $cp_{J'}(t_1)$ is a square at w then*

$$\langle p_i(t_0), p_j(t_0)\rangle_{u_i^0} + \langle p_i(t_1), p_j(t_1)\rangle_{u_i^1} = 0.$$

Proof. Assume $i \neq i_w$ then $p_i(t_1)$ is only a non-unit at u_i^1 outside of T. Consider

$$\langle p_i(t_0), cp_{J'}(t_0)\rangle_{u_i^0} + \langle p_i(t_1), cp_{J'}(t_1)\rangle_{u_i^1}$$

by using the bimultiplicative property of Hilbert symbols we have this sum is equal to

$$\langle p_i(t_0), [c]\rangle_{u_i^0} + \langle p_i(t_1), [c]\rangle_{u_i^1} + \sum_{j\in J'}[\langle p_i(t_0), p_j(t_0)\rangle_{u_i^0} + \langle p_i(t_0), p_j(t_1)\rangle_{u_i^1}].$$

As $i \neq i_w$ and c is a unit outside of T we have that $\langle p_i(t_0), [c]\rangle_{u_i^0} = \langle p_i(t_1), [c]\rangle_{u_i^1}$ by quadratic reciprocity. Then by Lemma 5.11.2 the statement follows for $i \neq i_w$. Now assume $i = i_w$. By approximation conditions for all $v \in T$ we have $\langle p_{i_w}(t_0), cp_{J'}(t_0)\rangle_v = \langle p_{i_w}(t_1), cp_{J'}(t_1)\rangle_v$. By quadratic reciprocity

$$\langle p_{i_w}(t_1), cp_{J'}(t_1)\rangle_{u_{i_w}^1} = \langle p_{i_w}(t_1), cp_{J'}(t_1)\rangle_w + \sum_{v\in T}\langle p_{i_w}(t_1), cp_{J'}(t_1)\rangle_v.$$

As $cp_{J'}(t_1)$ is a square at w we have

$$\langle p_{i_w}(t_1), cp_{J'}(t_1)\rangle_w = 0.$$

Hence,

$$\langle p_{i_w}(t_1), cp_{J'}(t_1)\rangle_{u_{i_w}^1} = \langle p_{i_w}(t_0), cp_{J'}(t_0)\rangle_{u_{i_w}^0}.$$

□

5.11.1 Strictly smaller Selmer groups

We now want to consider the dual Selmer groups of $\mathcal{T}_{t_0}$ and $\mathcal{T}_{t_1}$. Let

$$V_w := \mathrm{H}^1_{\text{ét}}(\mathbb{Q}_w, \mathbb{Z}/2\mathbb{Z}) \quad V^w := \mathrm{H}^1_{\text{ét}}(\mathbb{Q}_w, \mathbb{Z}/2\mathbb{Z}).$$

We have two natural maps

$$\mathrm{loc}_w : J_{T_w} \to V_w \quad \mathrm{loc}^w : J^{T_w} \to V^w.$$

where $\mathrm{loc}_w := \phi_w \circ ev_{t_1}$ where $\phi_w : I_w \hookrightarrow V_w$ and $\mathrm{loc}^w := \phi^w \circ ev^{t_1}$ where $\phi^w : I^w \hookrightarrow V^w$.

Definition 5.11.4. For a finite set of places H containing S and a suitable partial adelic point P_H we define the subgroups $R^0_{P_H} \subseteq R_{P_H}$ and $\hat{R}^0_{P_H} \subseteq \hat{R}_{P_H}$ to be the subgroup consisting of elements $[c][p_{J'}]$ such that $i_w \notin J'$.

Definition 5.11.5. We denote by $P_0 \subseteq V_w$ (resp $P^0 \subseteq V^w$) the image of $R^0_{P_T}$ via loc_w (resp image of $\hat{R}^0_{P_T}$ via loc^w), i.e.

$$P_0 := \mathrm{loc}_w(R^0_{P_T}) \quad P^0 := \mathrm{loc}^w(\hat{R}^0_{P_T}).$$

Definition 5.11.6. We denote by $P_1 \subseteq V_w$ (resp $P^1 \subseteq V^w$) the image of $R^0_{P_{T_w}}$ via loc_w (resp image of $\hat{R}^0_{P_{T_w}}$ via loc^w), i.e.

$$P_1 := \mathrm{loc}_w(R^0_{P_{T_w}}) \quad P^1 := \mathrm{loc}^w(\hat{R}^0_{P_{T_w}}).$$

Lemma 5.11.7. *Let $x = [cp_{J'}] \in J_T$ then*

1. *For $v \in T$ $\mathrm{ev}_{t_0}(x) \in W_v(t_0)$ if and only if $\mathrm{ev}_{t_1}(x) \in W_v(t_1)$,*
2. *For $i \neq i_w \in J$ we have $\mathrm{ev}_{t_0}(x) \in W_{u^i_0}(t_0)$ if and only if $\mathrm{ev}_{t_1}(x) \in W_{u^i_1}(t_1)$.*

Proof. The first condition of the statement follows for approximation conditions in Definition 5.9.1. We saw in Lemma 5.10.7 that $ev_{t_k}(x) \in W_{u^k_i}(t_k)$ for $k = 0, 1$ if and only if $[cD^{J'}_i]_{u^k_i} = 0$. By Lemma 5.11.3

$$\langle p_i(t_0), cD^{J'}_i \rangle_{u^0_i} + \langle p_i(t_1), cD^{J'}_i \rangle_{u^1_i} = 0.$$

Using the fact that $\mathrm{val}_{u^0_i}(p_i(t_0)) = \mathrm{val}_{u^1_i}(p_i(t_1)) = 1$ and $cD^{J'}_i$ is a S-unit, the second condition of the statement follows from Lemma 5.10.7. □

Lemma 5.11.8. *The Hilbert pairing of any element in $P_0 \subseteq V_w$ with any element of $P^1 \subseteq V^w$ is trivial.*

Proof. Let $x_1 = [c_1][p_{J_1}] \in R^0_{P_T}$ and $x_2 = [c_2][p_{J_2}] \in \hat{R}^0_{P_{T_w}}$ then by definition $i_w \notin J_1, J_2$. Then consider

$$\alpha_1 := ev_{t_1}(x_1) \text{ and } \alpha_2 := ev^{t_1}(x_2).$$

What we want to show is $\langle \alpha_1, \alpha_2 \rangle_w = 0$. As $i_w \notin J_1, J_2$ we have both α_1, α_2 are units in $\mathbb{Z}_{u^1_{i_w}}$. Hence, the Hilbert symbol

$$\langle \alpha_1, \alpha_2 \rangle_{u^1_{i_w}} = 0.$$

For $v \in T$ and $v = u^1_i$ for $i \neq i_w$ we have that $\alpha_1 \in W_v(t_1)$ by Lemma 5.11.7 and by definition $\alpha_2 \in W^v(t_1)$. By the orthogonality of $W^v(t_1), W_v(t_1)$ we have that

$$\langle \alpha_1, \alpha_2 \rangle_v = 0.$$

As α_1, α_2 are units outside of $T_w(t_1)$, we have that for $v \notin T_w(t_1)$

$$\langle \alpha_1, \alpha_2 \rangle_v = 0.$$

The only place that is left is w, applying quadratic reciprocity, we find

$$\langle \alpha_1, \alpha_2 \rangle_w = \sum_{v \in T \cup \{u^1_i\}_{i \in J}} \langle \alpha_1, \alpha_2 \rangle_v = 0.$$

□

Lemma 5.11.9. *Let $x = [c][p_{J'}] \in \hat{R}^0_{P_{T_w}}$ be an element such that $loc^w(x) = 0$, then $x \in \hat{R}^0_{P_T}$*

Proof. By definition we have $i_w \notin J'$ as $x \in \hat{R}^0_{P_{T_w}}$. Moreover, as $\text{loc}^w(x) = [cp_{J'}(t_1)]_w = 0$ in $\mathbb{Q}_w^*/(\mathbb{Q}_w^*)^2$, this implies that c is a unit at w, hence $c \in \mathbb{Z}_T^*$. By Selmer conditions we have $ev^{t_1}(x) \in W^v(t_1)$ for all $v \in T$ and $ev^{t_0}(x) \in W^v(t_0)$ for all $v \in T$. Moreover,

$$\begin{cases} \langle p_i(t_1), ev^{t_1}(x)\rangle_{u_i^1} = 0 & \text{if } i \notin J' \\ \langle p_i(t_1), ev^{t_1}(x[-d][p_J])\rangle_{u_i^1} = 0 & \text{if } i \in J \end{cases}$$

What we need to show is that for u_i^0

$$\begin{cases} \langle p_i(t_0), ev^{t_0}(x)\rangle_{u_i^0} = 0 & \text{if } i \notin J' \\ \langle p_i(t_0), ev^{t_0}(x[-d][p_J])\rangle_{u_i^0} = 0 & \text{if } i \in J \end{cases}$$

We first consider the case $i \notin J'$ (this includes $i = i_w$), this follows from Lemma 5.11.3. Now assume $i \in J'$ (which implies $i \neq i_w$), then the statement follows from bimultiplicative property of Hilbert symbols, the fact that $-cd$ are units outside of T and Lemma 5.11.2. □

Lemma 5.11.10 ([Har19, Prop 3.8.11,(2)])**.** *If P_0 and P^0 are non-zero then $P^1 = \{0\}$ and $\hat{R}^0_{P_{T_w}} \subsetneq \hat{R}^0_{P_T}$.*

5.12 Proof of main theorem

In this section we will finish the proof of Theorem 5.2.1. Recall if $S_{\text{split}} \subseteq S_0$ denotes the set of places of S_0 which split in K, then

$$\dim_{\mathbb{F}_2} \text{Sel}(\mathcal{T}, S) - \dim_{\mathbb{F}_2} \text{Sel}(\hat{\mathcal{T}}, S) = |S_{\text{split}}|.$$

Lemma 5.12.1. *Let P_T be a suitable partial adelic point with t an associated admissible point. If $x \in R_{P_T}$ lies in G_i for some i then $x \in G_D$.*

Proof. Suppose there exists $x = [c][p_{J'}] \in G_i \backslash G_D$ i.e. there exists a $i_x \neq i$ with associated place v_x such that $[aD_i^{J'}]_{v_x} \neq 0$. As x satisfies the Selmer conditions at v_x from Lemma 5.10.7 we require $[aD_i^{J'}]_{v_x} = 0$ which is a contradiction. □

Lemma 5.12.2. *Let T be a finite set of places containing S, let P_T be a suitable partial adelic point and t_0 a T-admissible point with respect to P_T with associated places $\{u_i^0\}_{i \in J}$. Let $[x] \in \hat{R}_{P_T}$ be such that $[x] \notin \{0, [-d][p_J]\}$. Then there exists a place w and an extension of P_T to a suitable partial adelic point P_{T_w}, where $T_w := T \cup \{w\}$ such that $\hat{R}_{P_{T_w}} \subsetneq \hat{R}_{P_T}$.*

Proof. Let $x_0 := x = [c_0][p_{J_0}]$. By assumption there exists a place $v \in S_0$ such that $v \in S_{\text{split}}$ we know the difference $\dim_{\mathbb{F}_2} R_{P_T} - \dim_{\mathbb{F}_2} \hat{R}_{P_T} \geq 1$ and by assumption $\dim_{\mathbb{F}_2} \hat{R}_{P_T} \geq 2$. Hence,

$$\dim_{\mathbb{F}_2} R_{P_T} > \dim_{\mathbb{F}_2} \hat{R}_{P_T} \geq 2.$$

and we can deduce there exists $x_1 = [c_1][p_{J'}] \in R_{P_T}$ such that $x_1 \notin \langle [a][p_A], [d][p_J]\rangle \subseteq G$. By Condition (D) there exists $i_x \in J$ such that $c_1 \hat{D}_{i_x}^{J_1} \notin \{1, [aD_{i_x}^A]$. We can assume $i_x \notin J_0 \cup J_1$ as we can replace x_0 with $x_0[-dp_J]$ and x_1 with $x_1[dp_J]$. If $x_1 \in R_{P_T}$ and $x_1 \in G_{i_x}$ by Lemma 5.12.1 then $x_1 \in G_D$ which would contradict Condition (D), hence,

$x_1 \notin G_{i_x}$. Applying Chebotarev's density theorem, there exists a place w such $aD_{i_x}^{A}$ is a square at w but $c_0 D_{i_x}^{J_0}$ and $c_1 \hat{D}_{i_x}^{J_1}$ are non-squares. We can choose $t_w \in \mathbb{Z}_w$ such that $p_i(t_w)$ is a uniformizer at w, using (H). By Lemma 5.7.3 the fibre $\mathcal{U}_{t_w}(\mathbb{Z}_w) \neq \emptyset$ and we may extend our suitable partial adelic point P_T over T to a suitable partial adelic point P_{T_w} over T_w, where P_w is a point above t_w. By Proposition 5.9.4 we can find a T_w-admissible point t_w with respect to P_{T_w}. We denote by $\{u_i^1\}_{i \in J}$ the associated places of t_1 appearing from applying (H). We have that $x_0 \in \hat{R}^0_{P_T}$ and $x_1 \in R^0_{P_T}$ as $i_x \notin J_0 \cup J_1$. By our choice of w we have that $\mathrm{loc}^w(x_0), \mathrm{loc}_w(x_1) \neq 0$. Hence, $P^0, P_0 \neq \{0\}$. By Lemma 5.11.10 we have that $\hat{R}^0_{P_{T_w}} \subsetneq \hat{R}^0_{P_T}$. As

$$\hat{R}_{P_T} = \hat{R}^0_{P_T} \oplus \mathbb{F}_2\langle[-d][p_J]\rangle, \quad \hat{R}_{P_{T_w}} = \hat{R}^0_{P_{T_w}} \oplus \mathbb{F}_2\langle[-d][p_J]\rangle$$

we have $\hat{R}_{P_{T_w}} \subsetneq \hat{R}_{P_T}$. □

Proof of Theorem 5.2.1. We can always find a suitable partial adelic point P_S by Proposition 5.8.5. Invoking Proposition 5.9.4 we can find a S-admissible point t_0 associated to P_S. Then we can apply Lemma 5.12.2 recursively to find a finite set of places $S \subseteq T$ such that we can extend P_S to P_T and for every T-admissible point t_1 associated to P_T has the dual Selmer group $\mathrm{Sel}(\hat{\mathcal{T}}_{t_1}, T_0(t_1))$ is generated by $[-dp_J(t_1)]$, as the choice of T-admissible point does not matter. Applying Proposition 5.3.13 we can deduce $\mathcal{U}_{t_1}(\mathbb{Z}_{S_0}) \neq \emptyset$, hence $\mathcal{U}(\mathbb{Z}_{S_0}) \neq \emptyset$. □

Chapter 6

Smooth points on singular del Pezzo surfaces

6.1 Introduction

In an unpublished paper Lang conjectured the following.

Conjecture 6.1.1. *Every smooth proper separably rationally connected variety over a C_1-field has a rational point.*

The simplest type of C_1-field is a finite field. In [Esn03], Esnault provides a profound result which shows that Lang's conjecture is true over finite fields. In this chapter we consider an analogue of Lang's conjecture for mildly singular Fano varieties, where the most natural question is to ask for the existence of a smooth rational point. One would guess that over an infinite C_1-field, the set of rational points will be dense, hence the interesting case would be finite fields. We specifically consider Fano surfaces over finite fields where the mild singularities are rational double point singularities (Definition 2.1.7); these are singular del Pezzo surfaces. There have already been results in this direction. For example a result of Kollár [Kol02, Thm. 2] shows that every irreducible nonconical cubic hypersurface of dimension n in $\mathbb{P}^{n+1}$ over a finite field has a smooth rational point (see Remark 6.7.3). In an attempt to obtain analogous results for singular del Pezzo surfaces surfaces in this chapter we prove the following.

Theorem 6.1.2. *Let X be a singular del Pezzo surface of degree d over the finite field $\mathbb{F}_q$. Then X has a smooth rational point if*

1. *$d \geq 3$, or $d = 1$, or*
2. *$d = 2$ and X does not have singularity type*
   - (a) *$A_1, A_2, 3A_1, A_4, D_4, A_3 + 2A_1, 2A_2 + A_1$ or $7A_1$ where $q = 2$, or,*
   - (b) *A_2 where $q = 4$.*

Theorem 6.1.2 extends a corollary of Kollár's result to higher degree singular del Pezzo surfaces, degree 1 del Pezzo surfaces and also singular del Pezzo surfaces of degree 2 away from finite fields of size 2 and 4. Moreover, using work of Coray and Tsfasman [CT88] we obtain the following corollary to Theorem 6.1.2 about del Pezzo surfaces of degree $d \geq 4$ which are not an Iskovskikh surface (see [CT88, pg. 74]).

Corollary 6.1.3. *Let X be a singular del Pezzo surface of degree $d \geq 4$ over a finite field k. If X is not an Iskovskikh surface then X is k-rational.*

The following theorem shows that the condition $q \neq 2$ is necessary in Theorem 6.1.2 for singular del Pezzo surfaces of degree 2.

Theorem 6.1.4. *There exists singular del Pezzo surfaces of degree 2 of singularity types $A_1, 3A_1$, D_4 and $7A_1$ over $\mathbb{F}_2$ without a smooth rational point.*

Theorem 6.1.4 shows that this analogue of Lang's conjecture does not hold in general for singular surfaces. However, it would be interesting to classify which varieties over C_1 fields have a smooth rational point. This would be useful in determining (uni)rationality of singular Fano varieties and also determining p-adic solubility of smooth varieties with bad reduction via Hensel's Lemma.

Notation

We denote by $\mathbb{F}_q$ a finite field of cardinality q. Later in Chapter 6 it will be convenient to use notation for a finite field where there is no dependence on q, in this case we will use k.

6.2 Point counts over finite fields

In this section we describe how weak del Pezzo surfaces can be used to study the number of rational points on singular del Pezzo surfaces.

Theorem 6.2.1 (Weil, [Man71, Ch. IV, Thm. 27.1])**.** *Let S be a smooth projective surface over a finite field $\mathbb{F}_q$. If $\bar{S}$ is rational then*

$$\#S(\mathbb{F}_q) = q^2 + \mathrm{Tr}(\phi^*)q + 1$$

where ϕ is the Frobenius endomorphism on $\mathrm{Pic}\,\bar{S}$ and $\mathrm{Tr}(\phi^*)$ is the trace of the representation corresponding to ϕ.

Remark 6.2.2. Smooth del Pezzo surfaces are encapsulated in the definition of a singular del Pezzo surface (Definition 2.3.1). However, by Theorem 6.2.1 they always have a smooth $\mathbb{F}_q$-point, hence we can ignore these surfaces from now on.

Theorem 6.2.3 ([Kap13, Prop. 24])**.** *Let X be a singular del Pezzo of degree $9 - N \leq 6$ with minimal desingularisation $\pi : X' \to X$. Let $\mathcal{R} \subset \mathrm{Pic}\,\bar{X}'$ be the root sublattice generated by (-2)-curves on $\bar{X}'$. Then $\#X(\mathbb{F}_q) = q^2 + q + 1 + qt$ where*

$$t = \mathrm{Tr}(\phi^*|_{E_N}) - \mathrm{Tr}(\phi^*|_{\mathcal{R}})$$

and E_N is as in Proposition 2.2.15.

Remark 6.2.4. If X is a singular del Pezzo with no singular rational points (i.e. all singular points are contained in a Galois orbit of degree greater than 1) then by Theorem 6.2.3 we see that $X(\mathbb{F}_q) \neq \emptyset$, hence X has a smooth rational point.

Corollary 6.2.5. *Let X be a singular del Pezzo over a finite field $\mathbb{F}_q$ with $\delta \geq 0$ singular rational points. If $\delta \not\equiv 1 \bmod q$, then X has a smooth $\mathbb{F}_q$-point. In particular, if $\delta = 0, 2$ then X has a smooth rational point.*

Proof. If $\delta = 0$ the result is trivial. We now assume $\delta > 0$. By Theorem 6.2.3 we have $\#X(\mathbb{F}_q) \equiv 1 \bmod q$. As X has δ singularities defined over $\mathbb{F}_q$, we deduce that $\#X(\mathbb{F}_q) \geq \delta$. As $\delta \not\equiv 1 \bmod q$ the number of rational points on X has a lower bound of $\#X(\mathbb{F}_q) \geq 1+q$. If $q+1 > \delta$ then we clearly have a smooth $\mathbb{F}_q$-point on X. From now on we consider the case $q+1 < \delta$. There exists a natural number $N \geq 1$ such that $1+Nq < \delta < 1+(N+1)q$. Then $\#X(\mathbb{F}_q) \geq \delta > 1+Nq$ hence, $\#X(\mathbb{F}_q) \geq 1+(N+1)q > \delta$ as $\#X(\mathbb{F}_q) \equiv 1 \bmod q$. We now deduce X has a smooth rational point. □

Notation 6.2.6. We define and explain some notation that will be used in Proposition 6.2.7 and Algorithm 6.2.9. Let X be a singular del Pezzo surface over $\mathbb{F}_q$ of degree d, where $d \leq 6$. Denote by X' the minimal desingularisation of X and Γ the graph of negative curves on $\bar{X}'$. Note that the action of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ on $\bar{X}'$ will factor through a finite group $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) = \langle \mathrm{Frob}_q \rangle$ for some $n \in \mathbb{N}$ and defines a graph automorphism of Γ, i.e. the action of Frob_q on $\bar{X}'$ corresponds to an element $g \in \mathrm{Aut}(\Gamma)$. Denote by ϕ_g the homomorphism

$$\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \to \mathrm{Aut}(\Gamma),\ \mathrm{Frob}_q \mapsto g.$$

Let each vertex v_i in Γ correspond to a basis element e_i of the free $\mathbb{Z}$-module $M = \oplus_{i=1}^{n} \mathbb{Z} e_i$ with intersection pairing $\langle -, - \rangle$ defined by:

- $\langle e_i, e_i \rangle = -n$ if v_i corresponds to a $(-n)$-curve,
- $\langle e_i, e_j \rangle =$ number of edges between v_i and v_j if $i \neq j$.

Denote by A the matrix corresponding to the intersection form on M. A has rank $10-d$. One can define an action of $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ on M via

$$\mathrm{Frob}_q \cdot m := \phi_g(\mathrm{Frob}_q) m$$

for $m \in M$. Note that we can write a basis $l_0, \ldots, l_N$, for $\mathrm{Pic}\,\bar{X}'$, such that

$$l_0^2 = 1,\ l_i^2 = -1 \text{ for } i \in [1, N],\ \langle l_i, l_j \rangle = 0 \text{ for all } i \neq j$$

by Proposition 2.2.15. Then by [DPT80, Thm. 3.10] the effective cone of $\mathrm{Pic}\,\bar{X}' \otimes_{\mathbb{Z}} \mathbb{R}$ is generated by negative curves, hence each divisor l_i can be written as a sum $\sum a_j R_j$, where R_j is a (-1) or (-2)-curve. We can define a morphism

$$\psi : \mathrm{Pic}\,\bar{X}' \otimes_{\mathbb{Z}} \mathbb{R} \to M/\ker(A) \otimes_{\mathbb{Z}} \mathbb{R}, \quad l_i \mapsto \sum a_j R_j,$$

where the image of $\sum a_j R_j$ is a choice of representation of l_i as the sum of negative curves.

Proposition 6.2.7. *The map* $\psi : \mathrm{Pic}\,\bar{X}' \otimes_{\mathbb{Z}} \mathbb{R} \to M/\ker(A) \otimes_{\mathbb{Z}} \mathbb{R}$, *is an isomorphism of* $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$*-modules.*

Proof. The map ψ is clearly a $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-module homomorphism and surjective. Now it is sufficient to show that ψ is injective. Suppose $D \in \mathrm{Pic}\,\bar{X}'$ is in the kernel of ψ. Then $\psi(D) \in \ker(A)$, hence $\langle D, D' \rangle = 0$, for all negative curves on $\bar{X}$. As numerical equivalence is the same as linear equivalence in $\mathrm{Pic}\,\bar{X}' \otimes_{\mathbb{Z}} \mathbb{R}$, we have that $D = 0$ in $\mathrm{Pic}\,\bar{X}' \otimes_{\mathbb{Z}} \mathbb{R}$. □

Remark 6.2.8. We will use Proposition 6.2.7 to determine the possible actions of Frobenius on $\mathrm{Pic}\,\bar{X}$. Note that in Proposition 6.2.7 we base change to $\mathbb{R}$ so we are free to change basis. As the choice of basis has no effect on the trace of Frobenius and the Picard group is torsion-free, this has no effect on our calculations in the future.

We now detail an algorithm (Algorithm 6.2.9) to determine the possibilities for $\mathrm{Tr}(\phi^*)$ and $\mathrm{Tr}(\phi^*|_{\mathcal{R}})$ on X'. Moreover, using Theorem 6.2.3 combined with the results from Algorithm 6.2.9 determines the number of rational points on the singular del Pezzo surface X which is the anticanonical model of X'.

Algorithm 6.2.9. The algorithm determines the possibilities for $\mathrm{Tr}(\phi^*)$ and $\mathrm{Tr}(\phi^*|_{\mathcal{R}})$ by determining the possible actions of $\mathrm{Aut}(\Gamma)$ on M/K. This is because all possible actions of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ on $\mathrm{Pic}\,\bar{X}'$ are subsets of the possible actions of $\mathrm{Aut}(\Gamma)$ on M/K. Note that one can find the action of $\mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ on the sublattice $\mathcal{R} \subset \mathrm{Pic}\,\bar{X}'$ as the set of (-2)-curves are linearly independent subset.

Input: Graph of negative curves Γ on $\bar{X}'$.

1. **Step 1: Automorphism group**
    - Determine the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ.
2. **Step 2: Representation of action**
    - For each $\sigma \in \mathrm{Aut}(\Gamma)$ determine the matrix T_σ (resp. R_σ) corresponding to the action of σ on M/K (resp. the image of (-2)-curves under the isomorphism ψ in Proposition 6.2.7).
3. **Step 3: Traces**
    - For each T_σ and R_σ from Step 2, determine $t_\sigma := \mathrm{Tr}(T_\sigma)$ and $r_\sigma := \mathrm{Tr}(R_\sigma)$.

Output: The group $\mathrm{Aut}(\Gamma)$ and $\{\sigma, t_\sigma, r_\sigma\}_{\sigma \in \mathrm{Aut}(\Gamma)}$.

6.3 Toric singular del Pezzo surfaces

In this section we introduce toric varieties and show that any toric singular del Pezzo surface over a finite field has a smooth rational point.

Definition 6.3.1. An *algebraic torus* T over a field F is an algebraic group over F, such that $\bar{T} \cong \mathbb{G}^n_{m,\bar{F}}$ for some $n \in \mathbb{N}$.

Definition 6.3.2. A *toric variety* is a normal variety X over a field F with a faithful action of an algebraic torus T over F which has a dense open orbit.

Proposition 6.3.3. *Let V be a variety over a field F. If $\bar{V}$ is a toric variety over F^{sep}, then V is a toric a variety over F. Moreover, if F is a finite field, then V has a smooth rational point.*

Proof. As $\bar{V}$ is toric, we have a subtorus $T_1 \subseteq \mathrm{Aut}(\bar{X})$. Then T_1 lies in a maximal torus $T' \subseteq \mathrm{Aut}(\bar{V})$. By Grothendieck's theorem [GC66, SGA3, Exp. XIV, Thm 1.1] there exists a maximal torus $T \subseteq \mathrm{Aut}(V)$ such that $\bar{T}$ is maximal in $\mathrm{Aut}(\bar{V})$. As all maximal tori are conjugate and T_1 acts via a dense open orbit, so does T. Hence, V is toric over F. As T acts freely and transitively on U, then U is a T-torsor over $\mathrm{Spec}\,F$. If F is a finite field by Lang's Theorem [Bor91, Thm. 16.3] we have $\mathrm{H}^1_{\text{ét}}(F, T) = 0$, i.e. every T-torsor over $\mathrm{Spec}\,F$ is trivial. Then $U \cong T$ and U is smooth and has a rational point. □

Proposition 6.3.4. *Let X be a singular del Pezzo surface of degree d over a finite field k. Suppose X has singularity type S over $\bar{k}$, where*

1. $d = 7$ *or* 8 *with* $S = A_1$,
2. $d = 6$ *with* $S = A_1$ *and* $\bar{X}$ *has 4 lines,* $S = 2A_1$ *or* $A_2 + A_1$,
3. $d = 5$ *with* $S = 2A_1$ *or* $A_2 + A_1$,
4. $d = 4$ *with* $S = 4A_1, A_2 + 2A_1$ *or* $A_3 + 2A_1$, *or*
5. $d = 3$ *with* $S = 3A_2$.

Then X has a smooth rational point.

Proof. Derenthal gives a description of all possible singular del Pezzo surfaces which are toric [Der06, §1.8]. These are exactly the cases in the proposition. Using Proposition 6.3.3, we deduce that X has a smooth rational point. □

6.3.1 Notation for rest of the chapter

Throughout the rest of the chapter we fix the following notation. In particular, the following describes the notation we use for any graphs.

Notation 6.3.5.

1. When drawing a graph of negative curves of a weak del Pezzo surface the filled in nodes correspond to (-1)-curves and the unfilled nodes correspond to (-2)-curves.
2. We fix k to be a finite field of cardinality q.
3. For a singular del Pezzo surface X, we denote by $\pi : X' \to X$ the minimal desingularisation of X.
4. We denote by $\mathrm{Tr}(\phi^*)$ the trace of Frobenius on $\mathrm{Pic}\,\bar{X}'$ and $\mathrm{Tr}(\phi^* \mid_{\mathcal{R}})$ the trace of Frobenius on the sublattice $\mathcal{R} \subseteq \mathrm{Pic}\,\bar{X}'$ generated by (-2)-curves.

In the preceding statements, we do not explicitly state the singularity type. Rather, we have given detailed descriptions of the singularity type in Table 6.4.1 for degrees 8, 7, 6 and Tables 6.5.1, 6.6.1, 6.8.1 for degrees 5, 4 and 2 respectively. Moreover, we have the following key for our classification tables.

Key. *This is a key for Tables 6.4.1, 6.5.1, 6.6.1 and 6.8.1*

- *The first column of the tables label each type of singular del Pezzo surface. This column is constructed using the classification of subroot systems of E_{9-d} for $d \leq 6$ [Dyn52] and [CT88, Prop. 8.1] for $d = 7$ and 8.*
- *The second column gives the Coxeter–Dynkin diagram type of singular points on $\bar{X}$.*
- *The third column gives the Coxeter–Dynkin diagram type of singular points on $\bar{X}$ which are invariant under the action of $\mathrm{Gal}(\bar{k}/k)$.*

- *The fourth column gives the number of lines on $\bar{X}$. This is done using Algorithm 2.2.23. The significance of this column is due to the fact that two del Pezzo surfaces of degree d can have the same singularity type over $\bar{k}$ but a different number of lines. However, the class of a singular del Pezzo surface over an algebraically closed field is uniquely determined by the Coxeter–Dynkin diagram types for the singular points and the number of lines. To distinguish del Pezzo surfaces of degree d with the same singularity type but with a different number of lines, we use the notation $[-]'$ and $[-]''$.*

- *The fifth column shows for which q the given singularity type for X possesses a smooth rational point. A ✓ is placed in this entry if for any choice of q, this singularity type will have a smooth rational point. If this singularity type does not necessarily possess a smooth rational point we write for which q this can happen. For example, if we put $q \neq 2$ in this column this means this singularity type has a smooth point over $\mathbb{F}_q$ for $q \neq 2$ but not necessarily over $\mathbb{F}_2$. Moreover, we put a N/A in this column if this singularity type does not exist over a perfect field.*

- *The sixth and final column gives the location for the proof of the particular singularity type having a smooth point over the stated fields, or a proof of why such a surface cannot exist.*

6.4 Del Pezzo surfaces of degree 6, 7, and 8

In Table 6.4.1 we give a classification of singular del Pezzo surfaces of degree 6, 7, and 8. Note that all singular Pezzo surfaces of degree 7 and 8 are toric, hence Proposition 6.3.3 shows that these surfaces always have a smooth rational point over a finite field.

Table 6.4.1: *Classification of singular del Pezzo surfaces of degree 6, 7 and 8*

Type	Singular points over algebraic closure	Singular rational points	Lines	Smooth point	Proof
8.1	A_1	A_1	0	✓	Prop. 6.3.4
7.1	A_1	A_1	2	✓	Prop. 6.3.4
6.1	A_1	A_1	4	✓	Lemma 6.4.2
6.2	A_2	A_2	2	✓	Lemma 6.4.3
6.3	$2A_1$	$\emptyset$	2	✓	Remark 6.2.4
6.4	$2A_1$	$2A_1$	2	✓	Corollary 6.2.5
6.5	$A_1 + A_2$	$A_1 + A_2$	1	✓	Corollary 6.2.5

Remark 6.4.1. There are no non-smooth singular del Pezzo surfaces X whose minimal desingularisation $\pi : X' \to X$ is geometrically $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose otherwise, let $x \in X$ be a rational double point singularity on X. Then $\bar{X}'$ would need to contain a collection of (-2)-curves in the fibre of π above x. However, $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ contain no irreducible curves of self intersection -2. This is the reason why these cases are missed in Table 6.4.1. Moreover, the case where the minimal desingularisation of a singular del Pezzo surface is the Hirzebruch surface $\mathbb{F}_2$ is the case of surfaces of Type 8.1.

Lemma 6.4.2. *Let X be a surface of Type* 6.1 *over k, then X has a smooth rational point.*

Proof. Consider the graph of negative curves [CT88, Prop. 8.3, Diagram 2] on $\bar{X}'$. Using Algorithm 6.2.9, we see that $\mathrm{Tr}(\phi^*) \geq 1$. Moreover, as there is only one (-2)-curve on $\bar{X}'$ we have that $\mathrm{Tr}(\phi^* \mid_{\mathcal{R}}) = 1$. Hence, using Theorem 6.2.3 $\#X(k) \geq q^2+1 \geq 2$ as $q \geq 2$. □

Lemma 6.4.3. *Let X be a surface of Type 6.2 over k, then X has a smooth rational point.*

Proof. Using Algorithm 6.2.9, we see that the graph of negative curves Γ [CT88, Prop. 8.3, Diagram 4] on $\bar{X}'$ has an automorphism group $\mathrm{Aut}(\Gamma) \cong C_2$. If the action of Galois on $\mathrm{Pic}\,\bar{X}'$ is trivial then $\mathrm{Tr}(\phi^*) = 4$ and $\mathrm{Tr}(\phi^* \mid_{\mathcal{R}}) = 2$. If Galois acts via the non-trivial automorphism of Γ, then $\mathrm{Tr}(\phi^*) = 2$ and $\mathrm{Tr}(\phi^* \mid_{\mathcal{R}}) = 2$. Hence, using Theorem 6.2.3 $\#X(k) \geq q^2 + 1 > 2$ as $q \geq 2$. □

6.5 Del Pezzo surfaces of degree 5

We now deal with singular del Pezzo surfaces of degree 5.

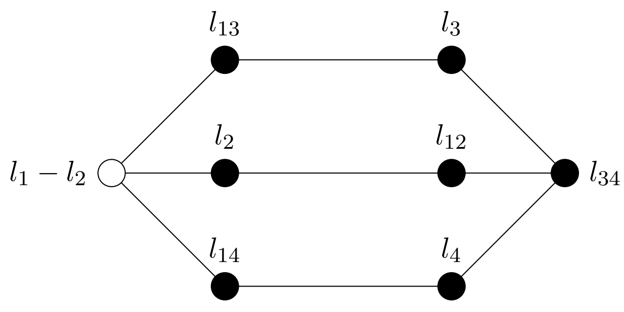
Table 6.5.1: *Classification of singular del Pezzo surfaces of degree 5*

Type	Singular points over algebraic closure	Singular rational points	Lines	Smooth point	Proof
5.1	A_1	A_1	7	✓	Lemma 6.5.2
5.2	$2A_1$	$\emptyset$	5	✓	Remark 6.2.4
5.3	$2A_1$	$2A_1$	5	✓	Prop. 6.3.4
5.4	A_2	A_2	4	✓	Lemma 6.5.3
5.5	$A_1 + A_2$	$A_1 + A_2$	3	✓	Prop. 6.3.4
5.6	A_3	A_3	2	✓	Lemma 6.5.4
5.7	A_4	A_4	1	✓	Lemma 6.5.4

Remark 6.5.1. Note that we can also show that Types 5.3 and 5.5 have a smooth point via Corollary 6.2.5.

Lemma 6.5.2. *Let X be a surface of Type 5.1 over k, then X has a smooth rational point.*

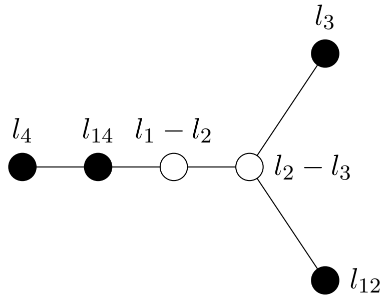
Proof. Consider the graph of negative curves Γ on $\bar{X}'$ as shown in [CT88, Prop. 8.5, Diagram 1]



Any possible automorphism of Γ will fix the node l_{34}, i.e. the action of $\mathrm{Gal}(\bar{k}/k)$ on $\bar{X}'$ will always fix a (-1)-curve. As this curve does not intersect the (-2)-curve $l_1 - l_2$ we can deduce using Proposition 2.3.5 that X has a smooth rational point. □

Lemma 6.5.3. *Let X be a surface of Type 5.4 over k, then X has a smooth rational point.*

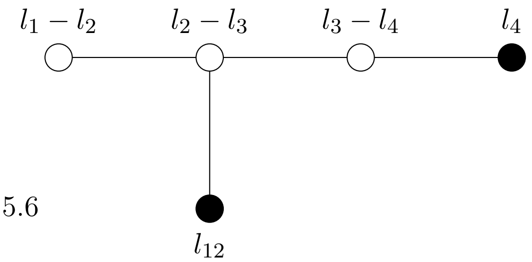
Proof. Consider the graph of negative curves on $\bar{X}'$ as shown in [CT88, Prop. 8.5, Diagram 3]



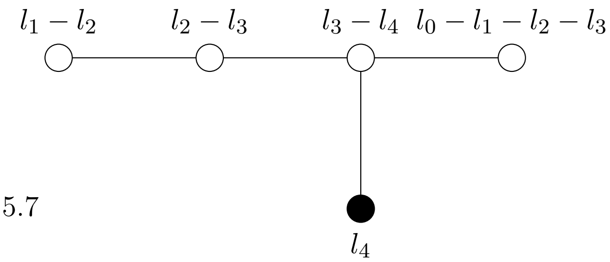
We see that the (-1)-curve l_4 is fixed under the action of $\mathrm{Gal}(\bar{k}/k)$ and doesn't intersect any (-2)-curve, hence we can apply Proposition 2.3.5 to deduce that X has a smooth rational point. ☐

Lemma 6.5.4. *Let X be a surface of Type* 5.6 *or Type* 5.7 *over k, then X has a smooth rational point.*

Proof. Consider the graph of negative curves on $\bar{X}'$ as shown in [CT88, Prop. 8.5, Diagrams 5, 6]Then



Type 5.6



Type 5.7

The (-1)-curve l_4 is fixed under $\mathrm{Gal}(\bar{k}/k)$ in both graphs and only intersects one (-2)-curve. Then we can deduce using Proposition 2.3.5 that X has a smooth rational point. ☐

6.6 Del Pezzo surfaces of degree 4

This section is separated into two parts. The first deals with odd characteristic where we can give geometric reasoning why singular del Pezzo surfaces of degree 4 have a smooth point. The second uses Algorithm 6.2.9, but works in arbitrary characteristic.

Table 6.6.1: *Classification of singular del Pezzo surfaces of degree 4*

Type	Singular points over algebraic closure	Singular rational points	Lines	Smooth point	Proof
4.1	A_1	A_1	12	✓	Prop. 6.6.7
4.2	$[2A_1]'$	$\emptyset$	9	✓	Remark 6.2.4
4.3	$[2A_1]'$	$2A_1$	9	✓	Corollary 6.2.5
4.4	$[2A_1]''$	$\emptyset$	8	✓	Remark 6.2.4
4.5	$[2A_1]''$	$2A_1$	8	✓	Corollary 6.2.5
4.6	A_2	A_2	8	✓	Lemma 6.6.8
4.7	$3A_1$	$\emptyset$	6	N/A	Lemma 6.6.9
4.8	$3A_1$	A_1	6	✓	Lemma 6.6.10
4.9	$3A_1$	$3A_1$	6	✓	Lemma 6.6.11
4.10	$A_1 + A_2$	$A_1 + A_2$	6	✓	Corollary 6.2.5
4.11	$[A_3]'$	A_3	5	✓	Lemma 6.6.11
4.12	$[A_3]''$	A_3	4	✓	Lemma 6.6.12
4.13	$A_1 + A_3$	$A_1 + A_3$	3	✓	Corollary 6.2.5
4.14	$A_2 + 2A_1$	A_2	4	✓	Lemma 6.6.13
4.15	$A_2 + 2A_1$	$A_2 + 2A_1$	4	✓	Lemma 6.6.13
4.16	$4A_1$	$\emptyset$	4	✓	Remark 6.2.4
4.17	$4A_1$	A_1	4	N/A	Lemma 6.6.14
4.18	$4A_1$	$2A_1$	4	✓	Prop. 6.3.4
4.19	$4A_1$	$4A_1$	4	✓	Prop. 6.6.15
4.20	A_4	A_4	3	✓	Lemma 6.6.11
4.21	D_4	D_4	2	✓	Lemma 6.6.16
4.22	$2A_1 + A_3$	A_3	2	✓	Lemma 6.6.17
4.23	$2A_1 + A_3$	$2A_1 + A_3$	2	✓	Lemma 6.6.17
4.24	D_5	D_5	1	✓	Lemma 6.6.11

Remark 6.6.1. Note that we can also show that Type 4.18 has a smooth point via Corollary 6.2.5 and Type 4.19 via Proposition 6.3.4.

6.6.1 Odd characteristic

Throughout Subsection 6.6.1, we shall assume that q is coprime to 2, i.e. k is of odd characteristic.

Lemma 6.6.2 ([Edo08, Table 1])**.** *Let $Q \subset \mathbb{P}^3_k$ a quadric surface. Then the number of rational points on Q is as follows:*

Rank	*Type*	$Q(k)$
1	*Repeated plane*	$q^2 + q + 1$
2	*Pair of distinct planes*	$2q^2 + q + 1$
2	*Geometrically reducible quadric*	$q + 1$
3	*Quadric cone*	$q^2 + q + 1$
4	*Hyperbolic quadric*	$(q + 1)^2$
4	*Elliptic quadric*	$q^2 + 1$

Proposition 6.6.3. *Let X be a singular del Pezzo of degree 4 over k with at least one singular rational point. Then $\#X(\mathbb{F}_q) \geq q^2 - 2q + 1$.*

Proof. By [CTSSD87, Prop. 2.1], we can write X as

$$X = \begin{cases} x_0x_1 - g(x_1, x_2, x_3, x_4) & = 0 \\ f(x_1, x_2, x_3, x_4) & = 0 \end{cases} \subseteq \mathbb{P}^4$$

where f and g are quadratic forms, f is of rank at least 3, and X has a singular point at $[x_0 : x_1 : x_2 : x_3 : x_4] = [1 : 0 : 0 : 0 : 0]$. Considering the affine open $U := X \backslash \mathbb{V}(x_1)$, we can define an isomorphism from U to the affine scheme $V \subset \mathbb{A}^3$, defined by $f(1, x_2, x_3, x_4) = 0$ via

$$\begin{aligned} &U \to V, && (x_0, x_2, x_3, x_4) \mapsto (x_2, x_3, x_4), \\ &V \to U, && (x_2, x_3, x_4) \mapsto (g(1, x_2, x_3, x_4), x_2, x_3, x_4). \end{aligned}$$

Let $\mathbb{V} := \mathbb{V}(f(x_1, x_2, x_3, x_4)) \subset \mathbb{P}^3$ be the compactification of V inside $\mathbb{P}^3$ and let H be the hyperplane $H := \mathbb{V}(x_1)$. The hyperplane section $C := \mathbb{V} \cap H$ is a (possibly singular) conic. As

$$\begin{aligned} \#X(k) &= V(k) + \#(X \cap \mathbb{V}(x_1))(k), \text{ and} \\ \#\mathbb{V}(k) &= \#V(k) + \#C(k) \end{aligned}$$

we deduce that $\#X(k) = \#\mathbb{V}(k) - \#C(k) + \#(X \cap \mathbb{V}(x_1))(k)$. Note that if there is a singular point of X lying away from the hyperplane $\mathbb{V}(x_1)$ the rank of f must be 3. This is because the open affine U would have a singular point, then V also has a singular point, hence so does $\mathbb{V}$. By Lemma 6.6.2 and the fact f has rank at least 3

$$\#X(k) \geq \begin{cases} q^2 - 2q + \delta & \text{if all singular rational points lie on the hyperplane } \mathbb{V}(x_1), \\ q^2 - q + 1 & \text{otherwise.} \end{cases}$$

Hence, the statement. □

Corollary 6.6.4. *Let X be a singular del Pezzo of degree 4 over k with $\delta > 0$ singular rational points. If $\delta \neq 4$ then X has a smooth rational point.*

Proof. Note that by the classification of singular del Pezzo surfaces of degree 4 we have $\delta \leq 4$. By Proposition 6.6.3 and the fact that $q \geq 3$, the number of smooth points on X is at least $q^2 - 2q + 1 - \delta \geq 4 - \delta$. Hence, we have a smooth point under the assumption $\delta \neq 4$. □

Corollary 6.6.5. *Let X be a singular del Pezzo of degree 4 over k and suppose that X has 4 singular rational points. Then X has a smooth rational point.*

Proof. Keeping notation as in Proposition 6.6.3, if all singular points lie on the hyperplane $\mathbb{V}(x_1)$ then $\#X(k) \geq q^2 - 2q + 4$, hence for $q \geq 3$ we always have a smooth point. If there exists a singular point lying away from $\mathbb{V}(x_1)$ then f must have rank 3, hence $\#X(k) \geq q^2 - q + 1$. As $q^2 - q - 3 > 0$ for $q \geq 3$, it follows that X must have a smooth rational point. □

6.6.2 General characteristic

We now move on to k of arbitrary characteristic.

Lemma 6.6.6. *Let X be a surface of Type* 4.1 *over* $\mathbb{F}_2$. *Then* $\mathrm{Tr}(\phi^*) \neq -1$ *or* -2.

Proof. Suppose $\mathrm{Tr}(\phi^*) = -2$. As X has an A_1 singularity, we must have $\mathrm{Tr}(\phi^* |_{\mathcal{R}}) = 1$, hence by Theorem 6.2.3 we have that $\#X(\mathbb{F}_2) = -1$, which is a contradiction. If $\mathrm{Tr}(\phi^*) = -1$, then the order of the action of Frobenius on $\mathrm{Pic}\,\bar{X}'$ must be 6. Considering the graph of negative curves Γ [CT88, Prop. 6.1, Diagram 1] on $\bar{X}'$, we see that all elements of order 6 of $\mathrm{Aut}(\Gamma)$ fix a (-1)-curve on $\bar{X}'$, hence $\#X'(\mathbb{F}_2) \geq 5$. However, if $\mathrm{Tr}(\phi^*) = -1$, then $\#X'(\mathbb{F}_2) = 3$, so we have a contradiction. □

Proposition 6.6.7. *Let X be a surface of Type 4.1 over k. Then X has a smooth rational point.*

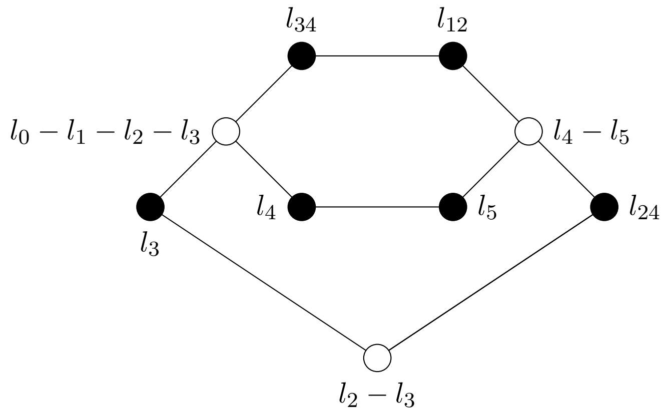
Proof. If we consider all possible traces for the action of Frobenius on $\bar{X}'$ we see that $\mathrm{Tr}(\phi^*) \geq -2$. This is because the action of $\mathrm{Gal}(\bar{k}/k)$ will factor through a conjugacy class of $W(D_4)$. Hence, by simply enumerating all conjugacy classes of $W(D_4)$ we see $\mathrm{Tr}(\phi^*) \geq -2$. If $\mathrm{Tr}(\phi^*) = -2$ and $q \geq 4$ we have that $\#X(k) \geq 5$ by Theorem 6.2.3, hence X has a smooth rational point. The case where $q = 3$ is dealt with in Corollary 6.6.4. Using Lemma 6.6.6 we see that if $q = 2$ then $\mathrm{Tr}(\phi^*) \geq 0$. We conclude by Theorem 6.2.3 that $\#X(k) \geq 3$ and deduce that X has a smooth rational point in this case also. □

Lemma 6.6.8. *Let X be a surface of Type* 4.6 *over k. Then X has a smooth rational point.*

Proof. By considering the graph of negative curves on $\bar{X}'$ and running Algorithm 6.2.9 we see that $\mathrm{Tr}(\phi^*) = 0, 2, 4$ or 6, respectively $\mathrm{Tr}(\phi^* |_{\mathcal{R}}) = 0, 2, 2$ or 2. Using Theorem 6.2.3 $\#X(\mathbb{F}_q) \geq q^2 + 1 \geq 5$ for $q \geq 2$, hence X has a smooth rational point. □

Lemma 6.6.9. *There are no surfaces of Type* 4.7 *over a field F.*

Proof. Suppose X is a singular del Pezzo of Type 4.7. Then the graph of negative curves on $\bar{X}'$ is as follows [CT88, Prop. 6.1, Diagram 6]:



For this singularity type to exist we would need all three (-2)-curves to be permuted by the action of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$. However, it is clear that no such graph automorphism exists. □

Lemma 6.6.10. *Let X be a surface of Type* 4.8 *over k. Then X has a smooth rational point.*

Proof. In this case one can run Algorithm 6.2.9 and see that $\mathrm{Tr}(\phi^*) = 0$ or 2 and $\mathrm{Tr}(\phi^* |_{\mathcal{R}}) = 1$ in both cases. Hence, by Theorem 6.2.3 $\#X(\mathbb{F}_q) \geq q^2 - q + 1 > 1$ and X has a smooth rational point. □

Lemma 6.6.11. *Let X be a surface of Type* 4.9, 4.11, 4.20 *or* 4.24 *over k. Then X has a smooth rational point.*

Proof. Consider the graph of negative curves Γ, on $\bar{X}'$, as shown in [CT88, Prop 6.1, Diagrams 7, 12, 15]. Then one can see that there is always a (-1)-curve fixed by Galois which intersects at most two (-2)-curves. Using Proposition 2.3.5 we conclude that X has a smooth rational point. □

Lemma 6.6.12. *Let X be a surface of Type* 4.12 *over k. Then X has a smooth rational point.*

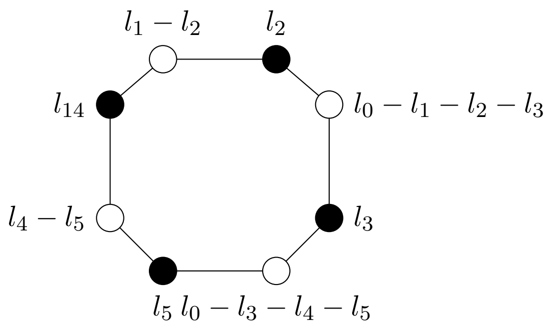
Proof. In this case one can run Algorithm 6.2.9 and see that $\mathrm{Tr}(\phi^*) = 2, 2, 4$ or 6 respectively $\mathrm{Tr}(\phi^*|_{\mathcal{R}}) = 1, 3, 3$ or 3 respectively. Hence, by Theorem 6.2.3 $\#X(k) \geq q^2 - q + 1 > 1$ and X has a smooth rational point. □

Lemma 6.6.13. *Let X be a surface of Type* 4.15 *or* 4.16 *over k. Then X has a smooth rational point.*

Proof. Consider the graph of negative curves Γ on $\bar{X}'$, as shown in [CT88, Prop. 6.1, Diagram 10]. In the case where X is of Type 4.16, it is easy to see that the action of $\mathrm{Gal}(\bar{k}/k)$ on Γ is trivial. Moreover, in this case there exists a (-1)-curve which is defined over k and intersects one (-2)-curve, hence we can apply Proposition 2.3.5 to deduce there is a smooth rational point on X. If we are in the case of Type 4.15 then $\mathrm{Aut}(\Gamma) \cong C_2$, and the Galois action on $\mathrm{Pic}\,\bar{X}'$ must be non-trivial. Then we must have $\mathrm{Tr}(\phi^*) = 0$ and $\mathrm{Tr}(\phi^* |_{\mathcal{R}}) = 0$, thus by Theorem 6.2.3 $\#X(k) = q^2 + 1 > 3$ as $q \geq 2$. □

Lemma 6.6.14. *There are no surfaces of Type 4.17 over a field F.*

Proof. Consider the graph of negative curves [CT88, Prop. 6.1, Diagram 9] on the minimal desingularisation $\bar{X}'$ as shown below.



We see that if the (-2)-curve $l_1 - l_2$ is defined over F, then the other negative curves must also be defined over F, as there are no automorphisms of the above graph that just fix $l_1 - l_2$ and no other (-2)-curve, i.e. the Galois action on $\bar{X}'$ must be trivial. Hence, the Galois action on $\bar{X}$ must also be trivial but this singularity type requires a non trivial Galois action on $\bar{X}$, as three singular points are permuted. □

Proposition 6.6.15. *Let X be a surface of Type* 4.19 *over k. Then X has a smooth rational point.*

Proof. For $q \neq 3$ we use Corollary 6.2.5, whereas for $q = 3$ we use Corollary 6.6.5. □

Lemma 6.6.16. *Let X be a surface of Type* 4.21 *over k. Then X has a smooth rational point.*

Proof. Consider the graph of negative curves Γ on $\bar{X}'$, as shown in [CT88, Prop. 6.1, Diagram 13]. Here $\mathrm{Aut}(\Gamma) \cong C_2$. Denote by $\sigma \in \mathrm{Aut}(\Gamma)$ the non-trivial element. If $\mathrm{Gal}(\bar{k}/k)$ acts trivially on Γ, then by Theorem 6.2.3 $\#X'(k) = q^2 + 6q + 1$, and $\#X(k) \geq q^2 + 2q + 1$, hence X has a smooth point. If the action of $\mathrm{Gal}(\bar{k}/k)$ on Γ factors through the subgroup generated by σ, then $\mathrm{Tr}(\phi^*) = 2$ and $\mathrm{Tr}(\phi^* \mid_{\mathcal{R}}) = 2$. Hence, by Theorem 6.2.3 $\#X(k) = q^2 + 1 > 2$, as $q \geq 2$. □

Lemma 6.6.17. *Let X be a surface of Type* 4.22 *or* 4.23 *over k. Then X has a smooth rational point.*

Proof. Consider the graph Γ of negative curves on the minimal desingularisation X' of X as shown in [CT88, Prop. 6.1, Diagram 14]. If X has Type 4.23, it is easy to see that the action of $\mathrm{Gal}(\bar{k}/k)$ on Γ is trivial. Moreover, in this case there exists a (-1)-curve which is defined over k and intersects one (-2)-curve, hence we can apply Proposition 2.3.5 to deduce there is a smooth rational point on X. If X has type 4.22 then $\mathrm{Aut}(\Gamma) \cong C_2$, and as we are in case 4.23 the Galois action on $\bar{X}'$ is non-trivial. Then we must have $\mathrm{Tr}(\phi^*) = 2$ and $\mathrm{Tr}(\phi^* \mid_{\mathcal{R}}) = 1$, hence by Theorem 6.2.3 $\#X(k) = q^2 + q + 1 > 1$ as $q \geq 2$. □

6.7 Del Pezzo surfaces of degree 3

Theorem 6.7.1 ([Kol02, Thm. 2])**.** *Let F be a field and $X \subset \mathbb{P}^{n+1}$ an irreducible cubic hypersurface of dimension ≥ 2 over F which is not a cone over a $(n-1)$-dimensional cubic. Then X has a F-point if and only if X has a smooth F-point.*

Corollary 6.7.2. *If X is a singular del Pezzo of degree 3 over k, then X has a smooth rational point.*

Proof. By Corollary 6.2.3 there always exists a k-point on X, then by Theorem 6.7.1, X has a smooth rational point. □

Remark 6.7.3. To prove Corollary 6.7.2 where X is a general cubic hypersurface, one can use Chevalley-Warning [War35].

6.8 Del Pezzo surfaces of degree 2

Any singular del Pezzo surface X of degree 2 is of the form

$$X : w^2 + wG_2(x, y, z) + G_4(x, y, z) \subset \mathbb{P}(1, 1, 1, 2)$$

where G_i is a homogeneous polynomial of degree i [CDL+24, Prop. 0.5.4].

Table 6.8.1: *Classification of singular del Pezzo surfaces of degree 2*

Type	Singular points over algebraic closure	Singular rational points	Lines	Smooth point	Proof
2.1	A_1	A_1	44	$q \neq 2$	Corollary 6.8.17
2.2	$2A_1$	$\emptyset$	34	✓	Remark 6.2.4
2.3	$2A_1$	$2A_1$	34	✓	Corollary 6.2.5
2.4	A_2	A_2	32	$q \neq 2, 4$	Prop. 6.8.15
2.5	$[3A_1]'$	$\emptyset$	26	✓	Remark 6.2.4
2.6	$[3A_1]'$	A_1	26	✓	Prop. 6.8.15
2.7	$[3A_1]'$	$3A_1$	26	$q \neq 2$	Prop. 6.8.15
2.8	$[3A_1]''$	$\emptyset$	25	✓	Remark 6.2.4
2.9	$[3A_1]''$	A_1	25	✓	Prop. 6.8.15
2.10	$[3A_1]''$	$3A_1$	25	$q \neq 2$	Prop. 6.8.15
2.11	$A_2 + A_1$	$A_2 + A_1$	24	✓	Corollary 6.2.5
2.12	A_3	A_3	22	✓	Corollary 6.8.19
2.13	$[4A_1]'$	$\emptyset$	20	✓	Remark 6.2.4
2.14	$[4A_1]'$	A_1	20	✓	Prop. 6.8.15
2.15	$[4A_1]'$	$2A_1$	20	✓	Corollary 6.2.5
2.16	$[4A_1]'$	$4A_1$	20	✓	Corollary 6.8.25
2.17	$[4A_1]''$	$\emptyset$	19	✓	Remark 6.2.4
2.18	$[4A_1]''$	A_1	19	✓	Prop. 6.8.15
2.19	$[4A_1]''$	$2A_1$	19	✓	Corollary 6.2.5
2.20	$[4A_1]''$	$4A_1$	19	✓	Corollary 6.8.25
2.21	$A_2 + 2A_1$	A_2	18	✓	Prop. 6.8.15
2.22	$A_2 + 2A_1$	$A_2 + 2A_1$	18	✓	Prop. 6.8.15
2.23	$2A_2$	$\emptyset$	16	✓	Remark 6.2.4
2.24	$2A_2$	$2A_2$	16	✓	Corollary 6.2.5
2.25	$[A_3 + A_1]''$	$A_3 + A_1$	16	✓	Corollary 6.2.5
2.26	$[A_3 + A_1]'$	$A_3 + A_1$	15	✓	Corollary 6.2.5
2.27	A_4	A_4	14	✓	Prop. 6.8.15
2.28	D_4	D_4	14	$q \neq 2$	Prop. 6.8.15
2.29	$[A_3 + 2A_1]'$	A_3	12	✓	Prop. 6.8.15
2.30	$[A_3 + 2A_1]'$	$A_3 + 2A_1$	12	$q \neq 2$	Prop. 6.8.15
2.31	$[A_3 + 2A_1]''$	A_3	11	N/A	Prop. 6.8.14
2.32	$[A_3 + 2A_1]''$	$A_3 + 2A_1$	11	$q \neq 2$	Prop. 6.8.15
2.33	$A_2 + 3A_1$	A_2	13	✓	Prop. 6.8.15
2.34	$A_2 + 3A_1$	$A_2 + A_1$	13	✓	Corollary 6.2.5
2.35	$A_2 + 3A_1$	$A_2 + 3A_1$	13	✓	Prop. 6.8.15
2.36	$5A_1$	$\emptyset$	14	✓	Remark 6.2.4
2.37	$5A_1$	A_1	14	✓	Prop. 6.8.15
2.38	$5A_1$	$2A_1$	14	✓	Corollary 6.2.5
2.39	$5A_1$	$3A_1$	14	$q \neq 2$	Prop. 6.8.15
2.40	$5A_1$	$5A_1$	14	✓	Prop. 6.8.15
2.41	$A_3 + A_2$	$A_3 + A_2$	10	✓	Corollary 6.2.5
2.42	$D_4 + A_1$	$D_4 + A_1$	9	✓	Corollary 6.2.5

2.43	$[A_5]'$	A_5	8	✓	Prop. 6.8.15
2.44	$[A_5]''$	A_5	7	✓	Prop. 6.8.15
2.45	$2A_2 + A_1$	A_1	12	N/A	Prop. 6.8.14
2.46	$2A_2 + A_1$	$2A_2 + A_1$	12	$q \neq 2$	Prop. 6.8.15
2.47	$A_4 + A_1$	$A_4 + A_1$	10	✓	Corollary 6.2.5
2.48	D_5	D_5	8	✓	Prop. 6.8.15
2.49	$6A_1$	$\emptyset$	10	✓	Remark 6.2.4
2.50	$6A_1$	A_1	10	N/A	Prop. 6.8.14
2.51	$6A_1$	$2A_1$	10	✓	Corollary 6.2.5
2.52	$6A_1$	$3A_1$	10	N/A	Prop. 6.8.14
2.53	$6A_1$	$4A_1$	10	N/A	Prop. 6.8.14
2.54	$6A_1$	$6A_1$	10	✓	Prop. 6.8.15
2.55	A_6	A_6	4	✓	Prop. 6.8.15
2.56	$2A_3$	$\emptyset$	6	✓	Remark 6.2.4
2.57	$2A_3$	$2A_3$	6	✓	Corollary 6.2.5
2.58	$3A_2$	$\emptyset$	8	✓	Remark 6.2.4
2.59	$3A_2$	A_2	8	✓	Prop. 6.8.15
2.60	$3A_2$	$3A_2$	8	✓	Prop. 6.8.15
2.61	$A_3 + 3A_1$	A_3	8	N/A	Prop. 6.8.14
2.62	$A_3 + 3A_1$	$A_3 + A_1$	8	✓	Corollary 6.2.5
2.63	$A_3 + 3A_1$	$A_3 + 3A_1$	8	✓	Prop. 6.8.15
2.64	$D_4 + 2A_1$	D_4	6	✓	Prop. 6.8.15
2.65	$D_4 + 2A_1$	$D_4 + 2A_1$	6	✓	Prop. 6.8.15
2.66	D_6	D_6	3	✓	Prop. 6.8.15
2.67	$[A_5 + A_1]'$	$A_5 + A_1$	6	✓	Corollary 6.2.5
2.68	$[A_5 + A_1]''$	$A_5 + A_1$	5	✓	Corollary 6.2.5
2.69	$A_3 + A_2 + A_1$	$A_3 + A_2 + A_1$	7	✓	Prop. 6.8.15
2.70	$A_4 + A_2$	$A_4 + A_2$	6	✓	Corollary 6.2.5
2.71	$D_5 + A_1$	$D_5 + A_1$	5	✓	Corollary 6.2.5
2.72	E_6	E_6	4	✓	Prop. 6.8.15
2.73	$7A_1$	$\emptyset$	7	✓	Remark 6.2.4
2.74	$7A_1$	A_1	7	✓	Prop. 6.8.15
2.75	$7A_1$	$2A_1$	7	✓	Corollary 6.2.5
2.76	$7A_1$	$3A_1$	7	✓	Prop. 6.8.15
2.77	$7A_1$	$4A_1$	7	N/A	Prop. 6.8.14
2.78	$7A_1$	$5A_1$	7	N/A	Prop. 6.8.14
2.79	$7A_1$	$7A_1$	7	$q \neq 2$	Prop. 6.8.15
2.80	$D_4 + 3A_1$	D_4	4	✓	Prop. 6.8.15
2.81	$D_4 + 3A_1$	$D_4 + A_1$	4	✓	Corollary 6.2.5
2.82	$D_4 + 3A_1$	$D_4 + 3A_1$	4	✓	Prop. 6.8.15
2.83	A_7	A_7	2	✓	Prop. 6.8.15
2.84	$2A_3 + A_1$	A_1	4	✓	Prop. 6.8.15
2.85	$2A_3 + A_1$	$2A_3 + A_1$	4	✓	Prop. 6.8.15
2.86	$A_5 + A_2$	$A_5 + A_2$	3	✓	Corollary 6.2.5
2.87	$D_6 + A_1$	$D_6 + A_1$	2	✓	Corollary 6.2.5

2.88	E_7	E_7	1	✓	Prop. 2.3.5

6.8.1 Finite covers

Definition 6.8.1. Let V and W be integral schemes over a field F. We say that a morphism $f : V \to W$ is a *finite cover* if f is finite, flat and surjective. We define the *degree of* f to be the degree of the field extension $F(V)/F(W)$

$$\deg f := [F(V) : F(W)].$$

Definition 6.8.2. Let V, W and f be as in Definition 6.8.1. We say that f is *separable* if the field extension $F(V)/F(W)$ is separable.

Definition 6.8.3. Let $f : T \to S$ be a morphism of schemes which is locally of finite presentation. We say that f is is *unramified* at $s \in S$ if for all $t \in T$ such that $f(t) = s$, we have that

1. the residue field $\kappa(t)$ is a separable algebraic extension of $\kappa(s)$,
2. the induced morphism on stalks $f^{\#} : \mathcal{O}_{s,S} \to \mathcal{O}_{t,T}$ satisfies $f^{\#}(\mathfrak{m}_s)\mathcal{O}_{t,T} = \mathfrak{m}_t$ where $\mathfrak{m}_s$ and $\mathfrak{m}_t$ are the maximal ideals of $\mathcal{O}_{s,S}$ and $\mathcal{O}_{t,T}$ respectively.

Otherwise, we say f is *ramified* at $s \in S$. Moreover, we say f is *unramified* if it is unramified for all $s \in S$, otherwise we call f *ramified.*

Remark 6.8.4. Let $f : V \to W$ be a separable finite cover where V is normal and W is smooth. Then there is a non-empty open subset $U \subseteq W$ where f is unramified [Zar58, Prop. 2]. We call the complement $B := W \backslash U$ the *branch locus* of f. Consider the inverse image $f^{-1}(B)$. We define the *ramification locus of* f to be the reduced subscheme $R := f^{-1}(B)_{\mathrm{Red}}$.

Remark 6.8.5. Let X be a singular del Pezzo of degree 2 over a field F and denote by f anticanonical map associated to X

$$f : X \to \mathbb{P}^2_F, \quad [x : y : z : w] \mapsto [x : y : z].$$

This realises X as a double cover of $\mathbb{P}^2_F$. If $\mathrm{char}(F) \neq 2$ then by completing the square we can assume that X is a of the form $w^2 = G_4(x, y, z)$ where G_4 is a quartic form. In this case the double cover has branch locus $B : G_4(x, y, z) = 0$. If $\mathrm{char}(F) = 2$ the morphism f can be inseparable. In the cases where f is separable, its *branch locus* is given by the plane conic $B : G_2(x, y, z) = 0$ (which can be reducible or non-reduced). The ramification locus R is given by

$$R : \begin{cases} G_4(x, y, z) = 0 & \text{if } \mathrm{char}(F) \neq 2, \\ G_2(x, y, z) = 0, w^2 = G_4(x, y, z) & \text{if } \mathrm{char}(F) = 2. \end{cases}$$

6.8.2 Separable anticanonical morphism

In this section we give cases where the anticanonical morphism associated to a singular del Pezzo surface of degree 2 is separable. The proof is inspired by [DM23, Prop 3.1]. The statement in [DM23] is only for smooth del Pezzo surfaces of degree 2; we show this proof also works for their singular counterparts.

Lemma 6.8.6. *Let F be a field and X, Y integral schemes of finite type over F. Let $f : X \to Y$ be a finite morphism such that $F^{sep}(X)/F^{sep}(Y)$ is separable. Then $F(X)/F(Y)$ is separable.*

Proof. As $F^{\mathrm{sep}}(X)/F^{\mathrm{sep}}(Y)$ and $F^{\mathrm{sep}}(Y)/F(Y)$ are separable $F^{\mathrm{sep}}(X)/F(Y)$ is separable. Then the inclusion of fields $F(Y) \subset F(X) \subset F^{\mathrm{sep}}(X)$ implies via [Lan02, §4, Thm. 4.5] that $F(X)/F(Y)$ is separable. □

Lemma 6.8.7. *Let X be a singular del Pezzo surface of degree d over an algebraically closed field F with minimal desingularisation $\pi : X' \to X$. Let $n := \#\{(-2)\text{-curves on } X'\}$. Then the rank of $\operatorname{Pic} X$ is $10 - d - n$.*

Proof. We have an exact sequence [Bri13, Prop. 1]

$$0 \to \operatorname{Pic} X \xrightarrow{\pi^*} \operatorname{Pic} X' \to \mathcal{R}^\vee \to \operatorname{Br} X \to \operatorname{Br} X'. \tag{6.8.1}$$

As X' is rational and F is algebraically closed $\operatorname{Br} X' = 0$. Moreover, $\operatorname{Br} X$ is torsion [Bri13, Prop. 4], so tensoring (6.8.1) with the flat $\mathbb{Z}$-module $\mathbb{Q}$ results in the exact sequence of $\mathbb{Q}$-vector spaces

$$0 \to \operatorname{Pic} X \otimes \mathbb{Q} \xrightarrow{\pi^*} \operatorname{Pic} X' \otimes \mathbb{Q} \to \mathcal{R}^\vee \otimes \mathbb{Q} \to 0.$$

As the rank of $\operatorname{Pic} X, \operatorname{Pic} X'$ and $\mathcal{R}$ is the same as the dimension of the $\mathbb{Q}$-vector spaces $\operatorname{Pic} X \otimes \mathbb{Q}, \operatorname{Pic} X' \otimes \mathbb{Q}$ and $\mathcal{R} \otimes \mathbb{Q}$ we have that

$$\operatorname{rank}_{\mathbb{Z}} \operatorname{Pic} X + \operatorname{rank}_{\mathbb{Z}} \mathcal{R} = \operatorname{rank}_{\mathbb{Z}} \operatorname{Pic} X'.$$

As X' is the blow-up of $9-d$ points of $\mathbb{P}^2$ the rank of $\operatorname{Pic} X'$ is $10-d$, and as the (-2)-curves in $\operatorname{Pic} X'$ are linearly independent [Dol12, Prop. 8.2.25], we deduce that $\operatorname{rank}_{\mathbb{Z}} \mathcal{R} = n$. The statement now follows. □

Proposition 6.8.8. *Let X be a singular del Pezzo surface of degree 2 over a perfect field F. Let X' be the minimal desingularisation of X. If the number of (-2)-curves on $\bar{X}'$ is less than 7, then the anticanonical map $f : X \to \mathbb{P}^2_F$ is separable.*

Proof. We can assume that F is algebraically closed by Lemma 6.8.6. If the characteristic of F is not 2, then this is clear. Assume now that $\operatorname{char}(F) = 2$ and f is purely inseparable. Pick a prime $\ell \gg 0$ and consider the Kummer sequence of étale sheaves associated to X

$$0 \to \mu_{\ell^n, X} \to \mathbb{G}_{m,X} \to \mathbb{G}_{m,X} \to 0$$

where $n \in \mathbb{N}$. Applying étale cohomology gives an exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mu_{\ell^n}) \longrightarrow \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_m) \xrightarrow{x \mapsto x^{\ell^n}} \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_m) \\ &\longrightarrow \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}) \longrightarrow \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) \xrightarrow{x \mapsto x^{\ell^n}} \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m) \longrightarrow \mathrm{H}^3_{\text{ét}}(X, \mu_{\ell^n}) \longrightarrow \dots \end{aligned}$$

Using $\operatorname{Pic} X = \mathrm{H}^1_{\text{ét}}(X, \mathbb{G}_m)$ and $\operatorname{Br} X = \mathrm{H}^2_{\text{ét}}(X, \mathbb{G}_m)$, the exact sequence reduces to

$$0 \to \operatorname{Pic} X/(\ell^n \operatorname{Pic}(X)) \to \mathrm{H}^2_{\text{ét}}(X, \mu_{\ell^n}) \to \operatorname{Br} X[\ell^n] \to 0.$$

As we chose $\ell \gg 0$ we have that $\operatorname{Br} X[\ell^n] = 0$ for all $n \in \mathbb{N}$ [Bri13, Thm. 6]. Taking inverse limits gives $\operatorname{Pic} X \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong \mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell)$. If f is purely inseparable then f is a homeomorphism in the étale topology. However, ℓ-adic cohomology is invariant for the small étale site [Sta24, Tag 04DY, Prop. 59.45.4], and as the rank of $\mathrm{H}^2_{\text{ét}}(X, \mathbb{Z}_\ell)$ is greater than 1 by Lemma 6.8.7 and the rank of $\mathrm{H}^1_{\text{ét}}(\mathbb{P}^2, \mathbb{Z}_\ell)$ is 1, we obtain a contradiction. □

Remark 6.8.9. The case where X' has seven (-2)-curves the anticanonical morphism can be inseparable. For example, a $7A_1$ singularity type on X which is given by the equation

$$X' : w^2 = G_4(x, y, z) \subset \mathbb{P}(1, 1, 1, 2)$$

where G_4 is homogeneous of degree 4 and X is over a field of characteristic 2, always has an inseparable anticanonical map [CDL+24, pg. 93].

6.8.3 General conic bundles

In this subsection we introduce general conic bundles and their relation to singular del Pezzo surfaces of degree 2. This relation was pointed out to the author by Alexander Kuznetsov on MathOverflow [U].

Definition 6.8.10. A *general conic bundle* is a smooth projective surface S with a dominant morphism $S \to \mathbb{P}^1$ whose geometric generic fibre is a smooth integral curve of genus 0.

Remark 6.8.11. The reason we differentiate between general conic bundles and conic bundles is because the fibres of general conic bundles do not have to be plane conics. This is particularly relevant when working with weak del Pezzo surfaces because a (-2)-curve may be a component of some fibre. The following is taken from [LS22, Lemma 2.8].

Lemma 6.8.12. *Let S be a smooth projective surface over a perfect field F satisfying $\mathrm{H}^1(S, \mathcal{O}_S) = 0$ which contains a curve $C \subset S$ of arithmetic genus 0. Suppose that $-K_S \cdot C = 2$. Then the complete linear system $|C|$ is a pencil. Moreover, if $|C|$ has no fixed component and C is geometrically integral, then $|C|$ induces a general conic bundle structure.*

Proof. It is sufficient to prove the statement over $\bar{F}$. Consider the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0.$$

Applying Zariski cohomology and using $\mathrm{H}^1(S, \mathcal{O}_S) = 0$ we get an exact sequence

$$0 \to \mathrm{H}^0(S, \mathcal{O}_S) \to \mathrm{H}^0(S, \mathcal{O}_S(C)) \to \mathrm{H}^0(S, \mathcal{O}_C(C)) \to 0.$$

As $C^2 = 0$ we can use adjunction [Har77, §V, Prop. 1.5] to show that $\dim \mathrm{H}^0(S, \mathcal{O}_C(C)) = 1$ and as S is projective $\dim \mathrm{H}^0(S, \mathcal{O}_S) = 1$. Applying a dimension argument for vector spaces it is easy to see $\dim \mathrm{H}^0(S, \mathcal{O}_S(C)) = 2$. We can conclude $\dim |C| = 1$, hence a pencil. If $|C|$ has no fixed component, using $C^2 = 0$ and the fact that $|C|$ is a pencil gives that $|C|$ is base point free. As C is integral we can deduce via [Liu02, Prop 4.1] that a generic member is smooth. Hence, the generic fibre of the morphism $S \to \mathbb{P}^1$ is geometrically integral smooth curve of arithmetic genus 0. □

Lemma 6.8.13. *Let X be a singular del Pezzo surface of degree 2 over a field of odd characteristic with one rational double point singularity. Then X' has a general conic bundle structure $\phi : X' \to \mathbb{P}^1$.*

Proof. Denote by x the singular point on X and $E := \pi^{-1}(x)$. Consider the linear system $|-K_{X'} - E|$. It easy to check that

$$(-K_{X'} - E)^2 = 0 \text{ and } -K_{X'} \cdot (-K_{X'} - E) = 2.$$

We can check via adjunction [Har77, §V, Prop. 1.5] that the arithmetic genus of $D \in |-K_{X'} - E|$ is 0. By Lemma 6.8.12, it is now sufficient to prove that the complete linear system $|-K_{X'} - E|$ has no fixed component and a generic element is geometrically integral. All $D \in |-K_{X'} - E|$ correspond to the strict transform of curves on X through the singular point x. Using the anticanonical morphism $f : X \to \mathbb{P}^2$, we see that curves through the singular point on X, get mapped to lines through the singular point on the branch locus. As this family of lines forms a pencil, it is clear there is no fixed component, showing that $|-K_{X'} - E|$, has no fixed component. Moreover, we see that a generic element of $|-K_{X'} - E|$ is geometrically integral because generically each element corresponds to a strict transform of $\mathbb{P}^1$ blown up in a single point. □

Proposition 6.8.14. *There do not exist surfaces X over a perfect field F of the following Types: 2.31, 2.45, 2.50, 2.52, 2.53, 2.61, 2.77, 2.78.*

Proof. All these singularity types require a certain Galois action on the singular points, i.e. the action of $\mathrm{Gal}(\bar{F}/F)$ on $\bar{X}$ permutes the singular points in a particular way. Using Proposition 2.1.3 and the intersection matrices in Appendix A we know such an action cannot not arise. □

Key. *This is a key for Table 6.8.2*

1. *Table 6.8.2 describes the possible traces of Frobenius* $\mathrm{Tr}(\phi^*)$ *on certain weak del Pezzo surfaces X' and also the trace of Frobenius restricted to the sublattice of (-2)-curves, $\mathcal{R} \subset \mathrm{Pic}\,\bar{X}'$, we denote this by* $\mathrm{Tr}(\phi^*|_{\mathcal{R}})$. *We use Algorithm 6.2.9 to determine the possible values for* $\mathrm{Tr}(\phi^*)$ *and* $\mathrm{Tr}(\phi^*|_{\mathcal{R}})$.
2. *The first column describes the Type of the associated singular del Pezzo surface to X', call it X.*
3. *The second column gives the geometric singularity type on X; the third column gives the rational singular points on X.*
4. *Columns 4 and 5 state the possible values of* $\mathrm{Tr}(\phi^*)$ *and* $\mathrm{Tr}(\phi^*|_{\mathcal{R}})$ *respectively.*
5. *Finally, the last column gives the number of smooth rational points on X by combining the results in column's 4 and 5 with Theorem 6.2.3.*

Table 6.8.2: *Trace of Frobenius on weak del Pezzo surfaces of degree 2*

Type	Singular points over algebraic closure	Singular rational points	$\mathrm{Tr}(\phi^*)$	$\mathrm{Tr}(\phi^*\|_{\mathcal{R}})$	$\#X^{\mathrm{sm}}(\mathbb{F}_q)$
2.1	A_1	A_1	8	1	q^2+7q
			6	1	q^2+5q
			5	1	q^2+4q
			4	1	q^2+3q

			3	1	q^2+2q
			2	1	q^2+q
			1	1	q^2
			0	1	q^2-q
			-1	1	q^2-2q
			-2	1	q^2-3q
			-4	1	q^2-5q
2.4	A_2	A_2	8	2	q^2+6q
			6	2	q^2+4q
			5	2	q^2+3q
			4	2	q^2+2q
			3	2	q^2+q
			2	2	q^2
			2	0	q^2+2q
			1	0	q^2+q
			0	0	q^2
			-1	0	q^2-q
			-2	0	q^2-2q
			-4	0	q^2-4q
2.6	$[3A_1]'$	A_1	4	1	q^2+3q
			2	1	q^2+q
			0	1	q^2-q
2.7	$[3A_1]'$	$3A_1$	8	3	q^2+5q-2
			6	3	q^2+3q-2
			4	3	q^2+q-2
			2	3	q^2-q-2
			0	3	q^2-3q-2
2.9	$[3A_1]''$	A_1	4	1	q^2+3q
			3	1	q^2+2q
			2	1	q^2+q
			1	1	q^2
			0	1	q^2-q
2.10	$[3A_1]''$	$3A_1$	8	3	q^2+6q-2
			6	3	q^2+3q-2
			5	3	q^2+2q-2
			4	3	q^2+q-2
			3	3	q^2-2
			2	3	q^2-q-2
			0	3	q^2-3q-2
2.12	A_3	A_3	8	3	q^2+5q
			6	3	q^2+3q
			5	3	q^2+2q
			4	3	q^2+q
			4	1	q^2+3q
			3	3	q^2
			3	1	q^2+2q
			2	3	q^2-q

			2	1	q^2+q
			1	1	q^2
			0	1	q^2-q
			-2	1	q^2-3q
2.14	$[4A_1]'$	A_1	2	1	q^2+q
2.16	$[4A_1]'$	$4A_1$	8	4	q^2+4q-3
			6	4	q^2+2q-3
			4	4	q^2-3
			2	4	q^2-2q-3
2.18	$[4A_1]''$	A_1	2	1	q^2+q
2.20	$[4A_1]''$	$4A_1$	8	4	q^2+4q-4
			6	4	q^2+2q-3
			4	4	q^2-3
			2	4	q^2-2q-3
2.21	A_2+2A_1	A_2	4	2	q^2+2q
			2	2	q^2
			2	0	q^2+2q
			0	0	q^2
2.22	A_2+2A_1	A_2+2A_1	8	4	q^2+4q-2
			6	4	q^2+2q-2
			2	2	q^2-2
			0	2	q^2-2q-2
2.27	A_4	A_4	8	4	q^2+4q
			6	4	q^2+2q
			5	4	q^2+q
			1	0	q^2+q
			0	0	q^2
			-2	0	q^2-2q
2.28	D_4	D_4	8	4	q^2+4q
			6	4	q^2+2q
			4	4	q^2
			4	2	q^2+2q
			2	4	q^2-2q
			2	2	q^2
			2	1	q^2+q
2.29	$[A_3+2A_1]'$	A_3	4	1	q^2+3q
			4	3	q^2+1
			2	3	q^2-q
			2	1	q^2+q
2.30	$[A_3+2A_1]'$	A_3+2A_1	8	5	q^2+3q-2
			6	5	q^2+q-2
			4	3	q^2+q-2
			2	3	q^2-q-2
2.32	$[A_3+2A_1]''$	A_3+2A_1	8	5	q^2+3q-2
			6	5	q^2+q-2
			4	3	q^2+q-2
			2	3	q^2-q-2

2.33	$A_2 + 3A_1$	A_2	2 2	2 0	q^2 $q^2 + 2q$
2.35	$A_2 + 3A_1$	$A_2 + 3A_1$	8 2	5 3	$q^2 + 3q - 3$ $q^2 - q - 3$
2.37	$5A_1$	A_1	4 2 0	1 1 1	$q^2 + 3q$ $q^2 + q$ $q^2 - q$
2.39	$5A_1$	$3A_1$	4	3	$q^2 + q - 2$
2.40	$5A_1$	$5A_1$	8 6 4	5 5 5	$q^2 + 3q - 4$ $q^2 + q - 4$ $q^2 - q - 4$
2.43	$[A_5]'$	A_5	8 6 2 0	5 5 1 1	$q^2 + 3q$ $q^2 + q$ $q^2 + q$ $q^2 - q$
2.44	$[A_5]''$	A_5	8 6 5 3 2 0	5 5 5 1 1 1	$q^2 + 3q$ $q^2 + q$ q^2 $q^2 + 2q$ $q^2 + q$ $q^2 - q$
2.46	$2A_2 + A_1$	$2A_2 + A_1$	8 2 0	5 1 1	$q^2 + 3q - 2$ $q^2 + 2q - 2$ $q^2 - q - 2$
2.48	D_5	D_5	8 6 4 2	5 5 3 3	$q^2 + 3q$ $q^2 + q$ $q^2 + q$ $q^2 - q$
2.54	$6A_1$	$6A_1$	8 6	6 6	$q^2 + 2q - 5$ $q^2 - 5$
2.55	A_6	A_6	8 0	6 0	$q^2 + 2q$ q^2
2.59	$3A_2$	A_2	2 2	2 0	q^2 $q^2 + 2q$
2.60	$3A_2$	$3A_2$	8 0	6 0	$q^2 + 2q - 2$ $q^2 - 2$
2.63	$A_3 + 3A_1$	$A_3 + 3A_1$	8 4	6 4	$q^2 + 2q - 3$ $q^2 - 3$
2.64	$D_4 + 2A_1$	D_4	4 2	2 2	$q^2 + 2q$ q^2
2.65	$D_4 + 2A_1$	$D_4 + 2A_1$	8 6	6 6	$q^2 + 2q - 2$ $q^2 - 2$
2.66	D_6	D_6	8 6	6 6	$q^2 + 2q$ q^2
2.69	$A_3 + A_2 + A_1$	$A_3 + A_2 + A_1$	8 2	6 2	$q^2 + 2q - 2$ $q^2 - 2$

2.72	E_6	E_6	8 2	6 2	q^2+2q q^2
2.74	$7A_1$	A_1	2	1	q^2+q
2.76	$7A_1$	$3A_1$	4	3	q^2+q-2
2.79	$7A_1$	$7A_1$	8	7	q^2+q-6
2.80	D_4+3A_1	D_4	2	1	q^2+q
2.82	D_4+3A_1	D_4+3A_1	8	7	q^2+q-4
2.83	A_7	A_7	8 2	7 1	q^2+q q^2+q
2.84	$2A_3+A_1$	A_1	2	1	q^2+q
2.85	$2A_3+A_1$	$2A_3+A_1$	8 4	7 3	q^2+q-2 q^2+q-2

Proposition 6.8.15. *Let X be a del Pezzo surface of degree 2 not of the following types*

1. *Type 2.1 over $\mathbb{F}_2, \mathbb{F}_3$ and $\mathbb{F}_5$,*
2. *Type 2.4 over $\mathbb{F}_2$ and $\mathbb{F}_4$,*
3. *Types 2.7, 2.10, 2.27, 2.30, 2.32, 2.46, 2.79 over $\mathbb{F}_2$,*
4. *Type 2.12, 2.16, 2.20 over $\mathbb{F}_3$,*

Then X has a smooth rational point.

Proof. If X has no singular points defined over k then X has a smooth rational point via Remark 6.2.4. If X has exactly two singular rational points then X has a smooth point by Corollary 6.2.5. Finally, the remaining cases are dealt with in Table 6.8.2. □

6.8.4 Remaining cases

To establish Theorem 6.1.2, we deal with the cases in odd characteristic.

Proposition 6.8.16. *Let X be a singular del Pezzo of degree 2 over a finite field k where the characteristic of k is odd. If $\bar{X}$ has one singular point which is an A_1-singularity then X has a smooth rational point.*

Proof. Denote by $p \in X$ the singular point and let $\pi : X' \to X$ be the minimal desingularisation of X with E the exceptional divisor lying above p. By Lemma 6.8.12 the linear system $|-K_{X'} - E|$ gives a general conic bundle structure on X', i.e. a morphism $\phi : X' \to \mathbb{P}^1$. As

$$(-K_{X'} - E) \cdot E = 2$$

the curve E meets each fibre twice (counted with multiplicity). If there exists a smooth fibre of ϕ defined over k, then this fibre has a rational point not lying on E, so the image of this rational point under π is smooth. Suppose otherwise, i.e. every fibre above $\mathbb{P}^1(k)$ is singular. The restriction $\phi|_E : E \to \mathbb{P}^1$, is a degree 2 morphism and by Riemann–Hurwitz we have that the degree of the ramification divisor of $\phi|_E$ is 2, i.e. there exists a rational point of E lying on a singular fibre away from this fibre's singular point. Hence, there is a split singular fibre over k. As the fibre has $2q+1$ rational points we have a rational point lying away from E, and the projection of this rational point to X is smooth. □

Corollary 6.8.17. *Let X be a surface with Type 2.1 over k. If $q \neq 2$ then X has a smooth rational point.*

Proof. For $q \neq 2, 3, 5$ the result follows from Proposition 6.8.15. For $q \neq 2$ we can show X has a smooth point by Proposition 6.8.16 □

Proposition 6.8.18. *Let X be a del Pezzo of degree 2 with an A_3-singularity over a finite field k where the characteristic of k is odd. Then X has a smooth point.*

Proof. Let $\pi : X' \to X$ be the minimal desingularisation of X and denote by E_1, E_2, E_3 the geometrically irreducible curves in the exceptional divisor of π. Let $C := -K_{X'} - E_1 - E_2 - E_3$. We have a general conic bundle structure $\phi : X' \to \mathbb{P}^1$ given by the linear system $|C|$ by Lemma 6.8.12. Without loss of generality we can assume that $E_1 \cdot E_2 = E_2 \cdot E_3 = 1$ and $E_1 \cdot E_3 = 0$. Then

$$E_1 \cdot C = 1, \quad E_2 \cdot C = 0, \quad E_3 \cdot C = 1.$$

If there exists a smooth fibre of ϕ defined over k, then it is automatic that X has a smooth point as there will a rational point on X' not lying on E_i for $i = 1, 2, 3$. So suppose that all the fibres of ϕ defined over k are singular; as E_2 is a component of at most one singular fibre, there is at least one singular fibre where E_2 is not a component. We denote this fibre by F. As E_1 and E_2 do not intersect F at its singular point, there is a k-point of X' not lying on any (-2)-curves as F is split. The statement then follows from considering the image of this point under π. □

Corollary 6.8.19. *Let X be a surface of Type 2.12 over a finite field k. Then X has a smooth rational point.*

Proof. For $q \neq 3$ the statement follows from Proposition 6.8.15. For $q = 3$ we use Proposition 6.8.18. □

Lemma 6.8.20 ([Dol12, pg. 452])**.** *Let X be a singular del Pezzo surface of degree 2 over an algebraically closed field F of* $\mathrm{char}(F) \neq 2$. *Suppose that X has 4 singular points of type A_1 defined over F. Then the branch locus of the anticanonical morphism is either a union of*

1. *an irreducible nodal cubic and a line, or*
2. *two smooth conics.*

Remark 6.8.21. We can assume that the two smooth conics in case 2 of Lemma 6.8.20 are distinct. Suppose otherwise, then X has the form

$$X : w^2 = C_1 C_2$$

where C_i is a conic for $i = 1, 2$ and $C_1 = \lambda C_2$ for some $\lambda \in F^*$. It is easy to see in this case X has a singular locus of dimension one, which is a contradiction. Moreover, by Bézout's theorem the 4 points the two conics intersect at in Lemma 6.8.20 are in general position.

Lemma 6.8.22. *Let k be a finite field of odd characteristic and let X be the surface over k defined by*

$$X : w^2 = C_1 C_2 \subset \mathbb{P}(1, 1, 1, 2)$$

where C_1 and C_2 are smooth distinct conics that are Galois conjugates with $\#(C_1 \cap C_2)(k) = 4$. Then X has a smooth rational point.

Proof. The above surface defines a singular del Pezzo surface with a singularity type of $4A_1$. By Corollary 6.2.5 X has a smooth rational point if the cardinality q of k is not 3, hence we can assume $q = 3$. We can compute all possible smooth conics over $k' := k[x]/(x^2 - x - 1)$. The space of plane conics in $\mathbb{P}^2$ is parameterised by $\mathbb{P}^5$. As we specify 4 points in general position in $\mathbb{P}^2$ for which the conics have to pass through, there are $\#\mathbb{P}^1(k') = q^2 + 1$ choices for C_1. By assumption, C_1 is not defined over k, hence we have $q^2 - q$ choices for C_1. Moreover, the $q^2 - q$ come in Galois conjugate pairs, so there are $(q^2 - q)/2$ choices for C_1 and C_2. It is now easy to iterate all possible choices for C_1 and C_2 and see that there always exists $[x : y : z] \in \mathbb{P}^2(k)$ such that $C_2(x, y, z) = C_2(x, y, z) \neq 0$. We can then deduce that X has a smooth rational point as all singular points $[x : y : z : w]$ have the property $C_2(x, y, z) = C_2(x, y, z) = 0$. □

Lemma 6.8.23. *Let k be a finite field of odd characteristic and let X be the surface over k defined by*

$$X : w^2 = C_1C_2 \subset \mathbb{P}(1, 1, 1, 2)$$

where C_1 and C_2 are smooth distinct conics that are defined over k with $\#(C_1 \cap C_2)(k) = 4$. If $q > 3$, then X has a smooth rational point and if $q = 3$ then X does not exist.

Proof. Similarly, to Lemma 6.8.22 cases where $q \neq 3$ are dealt with by Corollary 6.2.5. We can now assume $q = 3$. As the four points in the intersection of C_1 and C_2 are in general position we can assume they are $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$ and $[1 : 1 : 1]$. Hence, C_i for $i = 1, 2$ has the form

$$C_i : a_0^{(i)}xy + a_1^{(i)}xz + a_2^{(i)}yz.$$

As each C_i is smooth we require $a_j^{(i)} \neq 0$ for $j \in \{0, 1, 2\}$. By enumerating through all possibilities for C_i we see that the only choice for $a_j^{(i)}$ is $a_j^{(1)} = 2a_j^{(2)}$ for $j \in \{0, 1, 2\}$. However, this is a contradiction to the existence of X as C_1 and C_2 are distinct. □

Proposition 6.8.24. *Let X be a singular del Pezzo surface of degree 2 over a finite field k of odd characteristic. Suppose X has $4A_1$ singularities defined over k and $\bar{X}$ also has a singularity type of $4A_1$. Then X has a smooth rational point.*

Proof. By Corollary 6.2.5 if $q \neq 3$ then X has a smooth rational point, hence we can assume $q = 3$. Denote by $\pi : X \to \mathbb{P}^2$ the anticanonical morphism associated to X. Note that π gives a one-to-one correspondence between singular points on X and singular points on the branch curve [CDL+24, Prop. 0.4.16], so it is sufficient to show that the branch locus has a smooth rational point. By Lemma 6.8.20 there are two possibilities for the branch locus, we proceed by dealing with each case individually. If the branch locus B of π is geometrically $B = C \cup L$ where C is a cubic and L is a line, by Bezout's Theorem there are at most 3 rational points of B lying on L. However, as $\#L(k) = 4$ we deduce that B has a smooth rational point. We are now left with the situation where the branch locus is geometrically a union of two smooth conics $B = C_1 \cup C_2$. The case where C_1 and C_2 are both defined over k is dealt with in Lemma 6.8.23. The only case left is where C_1 and C_2 are Galois conjugates; however this is dealt with in Lemma 6.8.22. □

Corollary 6.8.25. *Let X be a surface of Type 2.16 or 2.20 over a finite field k. Then X has a smooth rational point.*

Proof. For $q \neq 3$ we use 6.2.5. For $q = 3$ case we use Proposition 6.8.24. □

6.9 Del Pezzo surfaces of degree 1

Let X be a singular del Pezzo surface of degree 1 over a field K. Then X can be written as

$$w^2 = z^3 + G_1(x, y)zw + G_3(x, y)w + G_6(x, y) \subset \mathbb{P}(1, 1, 2, 3)$$

where x, y, z, w have weights 1,1,2,3 respectively, and G_i is homogenous polynomial of degree i [CDL+24, Prop. 0.5.4]. Denote by ϕ the map

$$\phi : X \dashrightarrow \mathbb{P}^1, \quad [x : y : z : w] \mapsto [x : y].$$

The fibre above $[a : b]$ of this morphism is the intersection of X with the hyperplane $ax + by = 0$. The rational map ϕ is not defined where $x = y = 0$. Consider the minimal desingularisation $\pi : X' \to X$ of X and the linear system $| - K_{X'}|$. Then the map given by this linear system is the map $\phi \circ \pi$. This linear system has no fixed component [Dol12, Thm. 8.3.2i] and has a base point above the rational point on X given by $[x : y : z : w] = [0 : 0 : 1 : 1]$.

Lemma 6.9.1. *Let X' be a weak del Pezzo surface of degree 1. Then the base point of the linear system $| - K_{X'}|$ does not lie on any (-2)-curve.*

Proof. Denote by p the base point of the linear system $| - K_{X'}|$ and let E be a (-2)-curve on $\bar{X}'$. As $K_{X'} \cdot E = 0$, either $p \notin E$ or E is a fixed component of the linear system. However, $| - K_{X'}|$ has no fixed component [Dol12, Thm. 8.3.2i], hence $p \notin E$. □

Corollary 6.9.2. *Any singular del Pezzo surface X of degree 1 over a field F has a smooth rational point.*

Proof. This follows from Lemma 6.9.1 and Proposition 2.3.5. □

6.9.1 Proof of Theorem 6.1.2 and Corollary 6.1.3

Proof of Theorem 6.1.2. The result follows from Tables 6.4.1, 6.5.1, 6.6.1, 6.8.1 and Corollary 6.9.2. □

Proof of Corollary 6.1.3. Theorem 6.1.2 shows that a singular del Pezzo surface of degree $d \geq 3$ over a finite field always has a smooth rational point. To prove the corollary, it is sufficient now to apply [CT88, Thm. 9.1] for $d \geq 5$, [CT88, Lemma 7.19a] for $d = 4$. □

6.10 Counterexamples

Before we give some examples of singular del Pezzo surfaces of degree 2 with only singular rational points, we explain how we obtained these examples.

Lemma 6.10.1. *Let X be a singular del Pezzo surface of degree 2 over a finite field k with no smooth rational point. Denote by $\pi : X' \to X$ the minimal desingularisation of X and $f : X \to \mathbb{P}^2$ the anticanonical morphism of X. If f is separable then the number of rational points on the branch locus of f is the same as the number of rational points on X.*

Proof. Denote by R the ramification divisor of f. Then $f\mid_{X\backslash R}: X\backslash R \to \mathbb{P}^2\backslash f(R)$ is étale, hence all singular points on X lie on R. In the case that the singular subscheme of X, denoted by X^{sing}, has $\#X^{\text{sing}}(k) = \#X(k)$, we must have the following two conditions:

$$(X\backslash R)(k) = \emptyset \text{ and } \#X(k) = \#R(k).$$

As a rational point on the ramification curve maps to a rational point on the branch locus of f we have $\#B(k) = \#R(k)$ and deduce that $\#B(k) = \#X(k)$. □

Algorithm 6.10.2. Algorithm to find singular del Pezzo surfaces of degree 2 with n singular rational points over a finite field k of characteristic 2.

Input: An integer n.

1. **Step 1: Find candidates for the branch curve**

   - Find all plane conics over k with exactly n rational points.

2. **Step 2: Find candidate surfaces**

   - For each conic $G_2(x, y, z)$ from Step 1, find all surfaces X of the form

     $$X : w^2 = wG_2(x, y, z) + G_4(x, y, z) \subset \mathbb{P}(1, 1, 1, 2)$$

     which have exactly n rational points. By Lemma 6.10.1 all singular del Pezzo surfaces of degree 2 with n singular rational points will lie in the output from Step 2.

3. **Step 3: Reduce list to singular del Pezzo surfaces**

   - For each X in Step 2, check that X has only rational double point singularities.

Output: Either no such surface exists, in which case the output is empty, or if such a surface does exist, the output will be a singular del Pezzo surface with n singular rational points and no smooth rational point.

Lemma 6.10.3 (A_1-singularity)**.** *Consider the surface*

$$X : w^2 + w(y^2 + yz + z^2) + (x^2yz + xyz^2 + y^4 + y^2z^2 + z^4) \subset \mathbb{P}(1, 1, 1, 2)$$

over $\mathbb{F}_2$. Then X has an A_1-singularity type and no smooth rational point.

Proof. It easy to check that $X(\mathbb{F}_2) = \{[1 : 0 : 0 : 0]\}$. By computing partial derivatives

$$\frac{\partial F}{\partial x} = zy^2, \quad \frac{\partial F}{\partial y} = wz + x^2z + xz^2, \quad \frac{\partial F}{\partial z} = yz + x^2y, \quad \frac{\partial F}{\partial w} = y^2 + yz + z^2,$$

it is clear that $[1 : 0 : 0 : 0]$ is the only singular points on $\bar{X}$. Using Magma we see that when we blow-up this point the exceptional divisor is a single irreducible curve E of self intersection -2, hence E is isomorphic to $\mathbb{P}^1$. We can then deduce that X has an A_1 singularity type and with no smooth rational point. □

Lemma 6.10.4 ($3A_1$-singularity)**.** *Consider the surface*

$$X : w^2 + w(x^2) + (x^4 + x^2yz + xy^2z + xyz^2 + y^2z^2) \subset \mathbb{P}(1, 1, 1, 2)$$

over $\mathbb{F}_2$. Then X has a $3A_1$-singularity type and no smooth rational point.

Proof. Note that $X(\mathbb{F}_2) = \{[0:1:0:0], [0:0:1:0], [0:1:1:1]\}$. By computing partial derivatives

$$\frac{\partial F}{\partial x} = y^2z + yz^2, \quad \frac{\partial F}{\partial y} = x^2z + xz^2, \quad \frac{\partial F}{\partial z} = x^2y + xy^2 \quad \frac{\partial F}{\partial w} = x^2,$$

we deduce that $[0:1:0:0], [0:0:1:0]$ and $[0:1:1:1]$ are the only singular points on $\bar{X}$. Using Magma, we see that when we resolve these singularities the exceptional divisor contains three disjoint irreducible curves E_1, E_2, E_3 each of self intersection -2, hence E_i is isomorphic to $\mathbb{P}^1$ for $i = 1, 2, 3$. We can then deduce that X has three singular points each of type A_1. □

Lemma 6.10.5 (D_4-singularity). *Consider the surface*

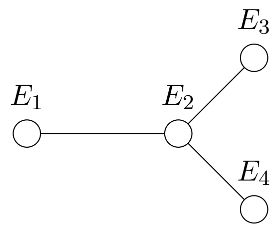
$$X : w^2 + w(y^2 + yz + z^2) + (xy^2z + xyz^2 + y^4 + y^2z^2 + z^4) \subset \mathbb{P}(1, 1, 1, 2)$$

over $\mathbb{F}_2$. Then X has a D_4-singularity type and no smooth rational point.

Proof. We can check that $X(\mathbb{F}_2) = \{[1:0:0:0]\}$. By computing partial derivatives

$$\frac{\partial F}{\partial x} = y^2z + yz^2, \quad \frac{\partial F}{\partial y} = wz + xz^2, \quad \frac{\partial F}{\partial z} = wz + xy^2 \quad \frac{\partial F}{\partial w} = y^2 + yz + z^2,$$

it easy to see that the only singular point on $\bar{X}$ is $[1:0:0:0]$. Using Magma we see that when we resolve these singularities the exceptional divisor contains one connected component with the following intersection graph:



and each E_i has self intersection -2 for $i = 1, 2, 3, 4$, hence X has a D_4-singularity type. □

Lemma 6.10.6. *Let X be a surface of Type 2.79 (i.e. $7A_1$) over $\mathbb{F}_2$. Then X has no smooth rational point.*

Proof. The graph of negative curves as shown in Appendix A on $\bar{X}'$ consists of seven skew (-1)-curves each intersecting three (-2)-curves. Moreover, all (-2)-curves are skew, hence all (-1)-curves intersect (-2)-curves at distinct points. By Proposition 2.1.3 we deduce that the (-2)-curves on $\bar{X}'$ are fixed by the action of $\mathrm{Gal}(\bar{\mathbb{F}}_2/\mathbb{F}_2)$ as all seven singular rational points of X are defined over $\mathbb{F}_2$. Hence, all (-1)-curves are also fixed by the action of $\mathrm{Gal}(\bar{\mathbb{F}}_2/\mathbb{F}_2)$. We now deduce by Lemma 2.2.26 that all the negative curves on $\bar{X}'$ are defined over $\mathbb{F}_2$, i.e. X' is split. As all rational points on (-1)-curves lie on (-2)-curves, the image of any rational point on (-1)-curve under the morphism $\pi : X' \to X$ is a rational double point singularity. Hence, if X has a smooth rational point x then the fibre $x' := \pi^{-1}(x)$ must not lie on a (-1)-curve. Suppose there exists such a point $x' \in X'(\mathbb{F}_2)$. As all seven (-1)-curves are skew and rational we can blow these curves down to a weak del Pezzo surface W of degree 9 [Dol12, Prop. 8.1.23]. As $\bar{W} \cong \mathbb{P}^2_{\bar{\mathbb{F}}_2}$ we know that W is a Galois twist of $\mathbb{P}^2_{\mathbb{F}_2}$. The classes of twists of $\mathbb{P}^2_{\mathbb{F}_2}$ up to isomorphism are classified by the

Galois cohomology group $\mathrm{H}^1(\mathbb{F}_2, \mathrm{PGL}_3(\bar{\mathbb{F}}_2))$ [Poo17, Thm. 4.5.2]. There is an injection [Poo17, Remark 1.5.10]

$$\mathrm{H}^1(\mathbb{F}_2, \mathrm{PGL}_3(\bar{\mathbb{F}}_2)) \hookrightarrow \mathrm{Br}\,\mathbb{F}_2.$$

However, by Proposition 2.5.9 $\mathrm{Br}\,\mathbb{F}_2 = 0$, hence we deduce $W = \mathbb{P}^2_{\mathbb{F}_2}$. The image of each (-1)-curve on W will be a rational point and the image of x' under this blow-down map will also be a rational point on W. However, this is a contradiction as this would imply that $\mathbb{P}^2_{\mathbb{F}_2}$ has at least eight $\mathbb{F}_2$-rational points. □

Remark 6.10.7. In the proof of Lemma 6.10.6, one can also deduce $W \cong \mathbb{P}^2_{\mathbb{F}_2}$ by using the fact that the image of a (-1)-curve under the blow-down map is a rational point and any Galois twist of projective space with a rational is trivial [Poo17, Prop. 4.5.10].

Proof of Theorem 6.1.4. The statement follows from Lemmas 6.10.3, 6.10.4, 6.10.5 and 6.10.6. □

Appendix A

Appendix for Chapter 6

A.1 Intersection matrices

Here we give the graph of negative curves for the surfaces in Proposition 6.8.14. We give the graph of negative curves in terms of its adjacency matrix.

Graph of negative curves of Type 2.31, i.e. $[A_3 + 2A_1]''$.

$$
\begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
$$

Graph of negative curves of Type 2.45, i.e. $2A_2 + A_1$.

$$\begin{pmatrix}
-2 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -2 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 2 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

Graph of negative curves of Types 2.50, 2.52, 2.53, i.e. $6A_1$.

$$\begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

Graph of negative curves of Type 2.61, i.e. $A_3 + 3A_1$.

$$\begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

Graph of negative curves of Types 2.77 and 2.78, i.e. $7A_1$.

$$\begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}$$

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